Revision of ”Effective order strong stability preserving Runge–Kutta methods”

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The authors would like to thank the reviewers for their constructive comments and the editor for the time invested. We have revised the paper according to the comments and suggestions. In addition we were able to find a pattern for the coefficients of effective order four, classical order two ESSPRK methods. These are included in Section 5.1.3. Answers to the comments of the reviewers have been included below.

Referee #1

Remark 1:

First of all, I think the title should be slightly modified. The concept of effective order is different from the concept of order but the terms should be used in a similar way (effective order of a RK method, a RK method with effective order p, . . .).

Thus, the title “effective order SSP RK methods” is nonsense. In a similar way, in the third paragraph of the introduction section, the sentence “. . . SSP properties of explicit RK methods of effective order. Effective order methods use . . . ”, is incorrect. The same comment is valid for the title of section 3.

The title has been changed to “Strong stability preserving explicit Runge–Kutta methods of maximal effective order". Also all instances of "effective order methods" are now corrected. Particularly, the title of section 3 now reads "The effective order of Runge–Kutta methods". The introduction is redone and care has been taken so that "effective order" is correctly used.

Remark 2:

The introduction should be redone. In the introduction section I expect to have an introduction to the problem, the limitations of the known results, and a brief explanation of the new approach proposed. These lines are not followed by the authors, and after reading the introduction, the reader does not get the ideas of the problem and how it is going to be solved.

Following your suggestions we have rewritten the introduction. We emphasized the main problems of our study and how these are being addressed.

Remark 3:

A spatial discretization of (2.1) gives rise to autonomous ODEs, but the ODE in (2.2) is non autonomous. On the other hand, in [4] the study is done for autonomous problems. Hence, as the paper is based on the theory in [4], I think it is more convenient to deal with autonomous problems (ODE (2.2), equations for RK methods, (2.3)).

Observe too that in the paper, t is used to denote two different concepts: the independent variable in the ODE (2.2) and the trees (see e.g., (3.1)).

The equations (2.2), (2.3) and the definition of the RK methods in page 2 have been changed to the autonomous form. Also, in Section 2, line 6 the following phrase is added: "Particularly, $\boldsymbol{F}$ may be time-dependent, but we can always make a transformation to an autonomous form."

We agree that the symbol t is overloaded. However, we think that changing it would cause more confusion due to its strong traditional use in both senses. We have kept the current notation since it is consistent with the literature and the meaning is generally clear from context.

Remark 4:

Section 3: The authors try to summarize in few pages the nontrivial theory of effective order for RK methods. Some definitions are given with few rigor. On the other hand, the reader gets lost with the sequence of definitions and notations, e.g., in the first paragraph we find: group G, rooted trees, elements of G as equivalence classes, elementary weights, ...

Example 3.1 and Table 3.2 do not seem to be relevant for the explanation of effective order. Example 3.3 does not seem to be relevant for the paper either.

On the other hand, it is not easy for the reader to understand how Table 3.3 should be used. It is not clear either why in Table 3.4, methods with classical order e.g. 3, have $\beta_0 = \beta_1 = 0$*. As the aim of the authors is to construct methods with effective order p, the authors should begin with the material in Section 3.3.1, and include a previous section to explain as brief as possible the idea of effective order (composition of methods, starting and stopping procedures, and so on). Tables 3.1 and 3.3 can be included, but the authors should stress that,* α_i *in Table 3.3 is the elementary weight associated to the index* i *in Table 3.1.*

The interested reader can read [4] so see all the details.

Section 3 has been rearranged. Based on your suggestions we briefly review J. Butcher theory about the algebraic representation of RK methods, focusing on the structure of the RK group which yields the concepts of classical and effective order. Then a section on how to construct RK methods of effective order follows. Tables 3.1 and 3.2 are included and we added the following line in Table 3.2: "Recall that α_i and β_i are the elementary weights associated with the index i in Table 3.1", as mentioned. Examples and relevant tables in the first part of this section are omitted. Finally in page 6, line 6 we added: "For a specified classical and effective order, these are the equality constraints $\Phi(K)$ in the optimization problem (2.5) for method M" to explain how Table 3.3 is used in practice. In page 10, line 11 we added the following: "We note that increasing the classical order of the main method results in setting more of the β_i to zero." This clarifies the relation between the classical order and β_i.

Remark 5:

The fact that, for $q \leq 5$, methods with effective order q have order q for linear problems is important for inequality (5.1). This information is given at the end of Section 3.1.1, but I think it should be given as a Remark.

Thanks for the comment. We stated this with Remark 3.2 in Section 3.2:

"The effective order conditions of the main method for the "tall" trees $t_1, t_2, t_4, t_8, t_{17}, \ldots$ match the classical order conditions and these are precisely the order conditions for linear problems. This follows from inductive application of the product operation (3.1) on the tall trees. Therefore, methods of effective order q have classical order at least q for linear problems."

D: Here and in one other place, the referee seems to imply that this fact does not hold for $q > 5$, but we claim that it does. We should make sure of this.

Y: Our claim as stated in this remark is correct.

Remark 6:

Section 4 is a bit messy, and I propose to change the order of the material in it. I propose to state Lemma 4.3, then state Lemma 4.2, and then state Theorem 4.1. The authors should explain that the proofs will be done later on. The authors should stress the beginning of proof of Lemma 4.2 (in a quick reading, it is difficult to find it).

We rearranged the material in this section as suggested. We also stated that the proof of Theorem 4.2 can be found at the end of the section.

Remark 7:

It is not clear to me the relevance of Remark 4.4 in this paper.

The remark now reads:

"It is worth noting here an additional result that follows directly from what we have proved. Using Theorem 4.2 and [6, Theorem 4.1], it follows that any irreducible explicit Runge–Kutta method with positive radius of circle contractivity has effective order at most four."

Remark 8:

Proof of Lemma 4.2, Case 2: Just after defining the set I, the authors should stress that $I \neq \emptyset$ because $v_1 = 0$ and $\beta_2 \neq 0$. The explanation of why $v_1 = 0$ should be included here.

The explanation why $v_1 \neq 0$ is now after set I is defined. The text reads:

"Let the set $J = \{i : v_i = \beta_2\}$. Note that $v_1 = 0$ because the first row of matrix A is identically zero. Since $\beta_2 \neq 0$ and $\boldsymbol{v} \neq \mathbf{0}$, the set J is not empty. Then (4.4) yields . . . ".

Note that set I is changed to J.

Remark 9:

Proof of Lemma 4.2, Case 3: The authors should explain, that, as $v_1 = 0$, the indexes i, j are in $\{2, \ldots, s\}$.

Lemma 4.2, Case 3 has been changed to:

"Since $\boldsymbol{b} > \mathbf{0}$, (4.6) implies that $\boldsymbol{w}$ contains both positive and negative elements. Furthermore, $v_1 = 0$ for any explicit method, thus $w_1 = 0$. Then, we can choose $i, j \in \{2, \ldots, s\}$ such that $w_i < 0 < w_j$. By (4.3) $v_i \neq 0$, $v_j \neq 0$, and $v_i \neq v_j$. Application of Lemma 4.5 reveals that all v_k for $k \in \{1, 2, \ldots, s\}$ must be equal except for one, which is a contradiction since $v_1 = 0$."

Remark 10:

The authors should explain the inequality $\mathcal{C}_{s,q,p} \leq \mathcal{C}^{\mathrm{lin}}_{s,q}$ in formula (5.1). For $q \leq 5$ inequalities (5.1) are true because for $q \leq 5$ methods with effective order q have order q for linear problems. For $q \geq 5$, they are also true because by Theorem 4.1 $\mathcal{C}_{s,q,p} = 0$.

Please note that Remark 3.2 is sufficient for the inequality $\mathcal{C}_{s,q,p} \leq \mathcal{C}^{\mathrm{lin}}_{s,q}$ to hold. We added the following line before inequalities (5.1):

"From Remark 3.2 and the fact that the ESSPRK(s, q, p) methods form a super class of the SSPRK(s, q) methods, we have . . . "

D: Perhaps we don't want to emphasize the difference between our explanation and that of the referee.

Y: I removed the redundant text so we dont emphasize this difference.

Remark 11:

Legend in Table 5.1: Please, include the value of $\mathcal{C}_{6,4}^{\text{lin}}$.

We now explicitly stated that $\mathcal{C}_{6,4}^{\text{lin}} = 0.44$.

Minor remarks:

Remark 1:

Page 1, line 10 in Section 1: the authors say that SSP "time discretizations (formerly TVD discretizatons ...". In some contexts, the term TVD discretizations are also used nowadays. I would suggest to change the term "formely" and use "also known as TVD discretizations" (or something similar).

We followed your suggestion.

Remark 2:

Section 2: Please, explain the meaning of A, b *and* c.

The coefficients A, b and c of an RK method are explained after its definition in page 2:
"Such methods are characterized by the coefficient matrix $A = (a_{ij}) \in \mathbb{R}^{s\times s}$, the weight vector $\boldsymbol{b} = (b_i) \in \mathbb{R}^s$ and the abscissa $\boldsymbol{c} = (c_i) \in \mathbb{R}^s$, where $c_i = \sum_{j=1}^{i-1} a_{ij}$. The accuracy and stability of the method depend on the coefficients of the Butcher tableau $(A, \boldsymbol{b}, \boldsymbol{c})$ [4]."
We clarified this in Table 3.1 as well.

Remark 3:

Equations (2.4): Please, explain that I is the identity matrix. Observe that I is used to denote the identity matrix and certain set in the proof of Lemma 4.2 (Case 2).

We defined that I is the $s \times s$ identity matrix in (2.4). In the proof of Theorem 4.2, we also changed the set I to J to remove any ambiguity.

Remark 4:

Table 5.2, first row: I think is better to write $\tau(t) = (\alpha\beta^{-1})(t)$.

We followed your suggestion.

Referee #2

Remark 1:

On page 3, where the effective SSP coefficient is defined, it would be clearer to say "To allow a fair comparison of explicit methods with different number of stages, we consider . . . ".

We followed your suggestion.

Remark 2:

On page 5, in Example 3.3, the use of the phrase "backward Euler" has been used (ambiguously) to refer to the backward-in-time Euler in the past, and the clause "time-step of $-\Delta t$*" makes it sound like it is suggesting that this is indeed the "backward Euler" the authors are talking about. However, clearly the authors in this case mean the "implicit Euler with a time-step of* $-\Delta t$*", so perhaps it would remove the ambiguity to say it this way.*

We agree with your comment, however we decided to remove the particular example as suggested by the other referee.