Revision of SINUM manuscript #088420

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The authors thank the reviewers for their constructive comments. We have revised the paper according to the comments and suggestions. During our revisions, we discovered a pattern for the coefficients of some effective order four, classical order two SSP methods. These are included in a Section 5.1.3. We also simplified the proof of Theorem 4.2 in Section 4. Responses to the comments of the reviewers have been included below.

Referee #1

Remark 1:

First of all, I think the title should be slightly modified. The concept of effective order is different from the concept of order but the terms should be used in a similar way (effective order of a RK method, a RK method with effective order p, . . .).

Thus, the title "effective order SSP RK methods" is nonsense. In a similar way, in the third paragraph of the introduction section, the sentence ". . . SSP properties of explicit RK methods of effective order. Effective order methods use . . . ", is incorrect. The same comment is valid for the title of section 3.

The title has been changed to "Strong stability preserving explicit Runge–Kutta methods of maximal effective order". Also incorrect instances throughout the paper of "effective order methods" and similar are now corrected.

Remark 2:

The introduction should be redone. In the introduction section I expect to have an introduction to the problem, the limitations of the known results, and a brief explanation of the new approach proposed. These lines are not followed by the authors, and after reading the introduction, the reader does not get the ideas of the problem and how it is going to be solved.

Following your suggestions we have rewritten the introduction. We emphasized the main problems of our study and how these are being addressed.

Remark 3:

A spatial discretization of (2.1) gives rise to autonomous ODEs, but the ODE in (2.2) is non autonomous. On the other hand, in [4] the study is done for autonomous problems. Hence, as the paper is based on the theory in [4], I think it is more convenient to deal with autonomous problems (ODE (2.2), equations for RK methods, (2.3)).

Observe too that in the paper, t is used to denote two different concepts: the independent variable in the ODE (2.2) and the trees (see e.g., (3.1)).

The equations (2.2), (2.3) and the definition of the RK methods in page 2 have been changed to the autonomous form. Also, in Section 2, line 6 the following phrase is added: "Particularly, $\boldsymbol{F}$ may be time-dependent, but we can always make a transformation to an autonomous form."

We agree that the symbol t is overloaded. However, we think that changing it would cause more confusion due to its strong traditional use in both senses. We have kept the current notation since it is consistent with the literature and we hope the meaning is clear from context.

Remark 4:

Section 3: The authors try to summarize in few pages the nontrivial theory of effective order for RK methods. Some definitions are given with few rigor. On the other hand, the reader gets lost with the sequence of definitions and notations, e.g., in the first paragraph we find: group G, rooted trees, elements of G as equivalence classes, elementary weights, ...

Example 3.1 and Table 3.2 do not seem to be relevant for the explanation of effective order. Example 3.3 does not seem to be relevant for the paper either.

On the other hand, it is not easy for the reader to understand how Table 3.3 should be used. It is not clear either why in Table 3.4, methods with classical order e.g. 3, have $\beta_0 = \beta_1 = 0$. *As the aim of the authors is to construct methods with effective order p, the authors should begin with the material in Section 3.3.1, and include a previous section to explain as brief as possible the idea of effective order (composition of methods, starting and stopping procedures, and so on). Tables 3.1 and 3.3 can be included, but the authors should stress that,* α_i *in Table 3.3 is the elementary weight associated to the index* i *in Table 3.1.*

The interested reader can read [4] so see all the details.

Following the referee's suggestions, we start with the material from (the old) Section 3.3.1 and concentrate on deriving the effective order conditions. Examples 3.1 and 3.3 and Table 3.2 from the original submission have been deleted. Table 3.3 (now Table 3.2) has been more clearly integrated with the text. The rest of section 3 has been shortened and tightly focused in order to provide the elements of effective order theory most essential to the discussion at hand. To clarify the significance of Table 3.1, we added the following: "The ordering of trees given in Table 3.1 is used throughout the remainder of this work; thus t_9 refers to the tree with elementary weight $\boldsymbol{b}^T\boldsymbol{c}^4$."

On page 6, line 7 we added a phrase to explain how the table is used in practice: "For a specified classical and effective order, these are the equality constraints $\Phi(K)$ in the optimization problem (2.5) for method M"

Later on that page, we added "We note that increasing the classical order of the main method requires $\alpha_i = 1/\gamma(t_i)$ and thus by Table 3.2 requires more of the β_i to be zero."

Remark 5:

The fact that, for $q \leq 5$, methods with effective order q have order q for linear problems is important for inequality (5.1). This information is given at the end of Section 3.1.1, but I think it should be given as a Remark.

Thanks for the comment. We stated this with Remark 3.2 in Section 3.2:

"The effective order conditions of the main method for the "tall" trees $t_1, t_2, t_4, t_8, t_{17}, \ldots$ match the classical order conditions and these are precisely the order conditions for linear problems. This follows from inductive application of the group product on the tall trees. Therefore, methods of effective order q have classical order at least q for linear problems."

Remark 6:

Section 4 is a bit messy, and I propose to change the order of the material in it. I propose to state Lemma 4.3, then state Lemma 4.2, and then state Theorem 4.1. The authors should explain that the proofs will be done later on. The authors should stress the beginning of proof of Lemma 4.2 (in a quick reading, it is difficult to find it).

We rearranged the material in this section as suggested. The proof is much simpler and now directly follows the theorem.

Remark 7:

It is not clear to me the relevance of Remark 4.4 in this paper.

The remark now reads:

"It is worth noting here an additional result that follows directly from what we have proved. Using Theorem 4.2 and [6, Theorem 4.1], it follows that any irreducible explicit Runge–Kutta method with positive radius of circle contractivity has effective order at most four."

Remark 8:

Proof of Lemma 4.2, Case 2: Just after defining the set I, the authors should stress that $I \neq \emptyset$ because $v_1 = 0$ and $\beta_2 \neq 0$. The explanation of why $v_1 = 0$ should be included here.

Lemma 4.2 has been omitted since the proof of Theorem 4.2 is now simplified.

Remark 9:

Proof of Lemma 4.2, Case 3: The authors should explain, that, as $v_1 = 0$, the indexes i, j are in $\{2, \dots, s\}$.

Similarly with Remark 8, the Lemma 4.2 has been omitted.

Remark 10:

The authors should explain the inequality $\mathcal{C}_{s,q,p} \leq \mathcal{C}_{s,q}^{\mathrm{lin}}$ in formula (5.1). For $q \leq 5$ inequalities (5.1) are true because for $q \leq 5$ methods with effective order q have order q for linear problems. For $q \geq 5$, they are also true because by Theorem 4.1 $\mathcal{C}_{s,q,p} = 0$.

We note that Remark 3.2 is sufficient for the inequality $\mathcal{C}_{s,q,p} \leq \mathcal{C}_{s,q}^{\mathrm{lin}}$ to hold and we now say so before the inequalities (5.1):

"From Remark 3.2 and the fact that the ESSPRK(s, q, p) methods form a super class of the SSPRK(s, q) methods, we have ..."

Remark 11:

Legend in Table 5.1: Please, include the value of $\mathcal{C}_{6,4}^{\mathrm{lin}}$.

We now explicitly stated that $\mathcal{C}_{6,4}^{\mathrm{lin}} = 0.44$.

Minor remarks:

Remark 1:

Page 1, line 10 in Section 1: the authors say that SSP "time discretizations (formerly TVD discretizatons . . . ". In some contexts, the term TVD discretizations are also used nowadays. I would suggest to change the term "formely" and use "also known as TVD discretizations" (or something similar).

We followed your suggestion.

Remark 2:

Section 2: Please, explain the meaning of A, b and c.

The coefficients A, b and c of an RK method are explained after its definition in page 2. We clarified this in Table 3.1 as well.

Remark 3:

Equations (2.4): Please, explain that I is the identity matrix. Observe that I is used to denote the identity matrix and certain set in the proof of Lemma 4.2 (Case 2).

We defined that I is the $s \times s$ identity matrix in (2.4).

Remark 4:

Table 5.2, first row: I think is better to write $\tau(t) = (\alpha\beta^{-1})(t)$.

We followed your suggestion.

Referee #2

Remark 1:

On page 3, where the effective SSP coefficient is defined, it would be clearer to say "To allow a fair comparison of explicit methods with different number of stages, we consider ...".

We followed your suggestion.

Remark 2:

On page 5, in Example 3.3, the use of the phrase "backward Euler" has been used (ambiguously) to refer to the backward-in-time Euler in the past, and the clause "time-step of $-\Delta t$*" makes it sound like it is suggesting that this is indeed the "backward Euler" the authors are talking about. However, clearly the authors in this case mean the "implicit Euler with a time-step of* $-\Delta t$*", so perhaps it would remove the ambiguity to say it this way.*

We agree with your comment, however we decided to remove the particular example as suggested by the other referee.