Norms

Vector norms

$$v = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_m \end{bmatrix}$$

$$\|v\|_1 = \sum_{j=1}^{m} |v_i|$$

$$\|v\|_2 = \left(\sum_{j=1}^{m} v_i^2 \right)^{1/2}$$

$$\|v\|_\infty = \max_i |v_i|$$

Function norms

$v(x)$

$$\|v\|_1 = \int |v(x)|\,dx$$

$$\|v\|_2 = \left(\int (v(x))^2\,dx \right)^{1/2}$$

$$\|v\|_\infty = \max_x |v(x)|$$

Grid-function norms

$$\|v\|_1 = \sum_{j=1}^{m} h_j |v_j|$$

$$\|v\|_2 = \left(\sum_{j=1}^{m} h\, v_j^2 \right)^{1/2}$$

$$\|v\|_\infty = \max_j |v_j|$$

Diffusion of heat in a rod

Let $f(x) = \frac{-\psi(x)}{K}$

$$\frac{d}{dt}\int_0^1 u(x,t)\,dx = K\left(\frac{\partial}{\partial x}u(x=1,t) - \frac{\partial}{\partial x}u(x=0,t)\right) + \int_0^1 \psi(x)\,dx$$

heat

heat capacity

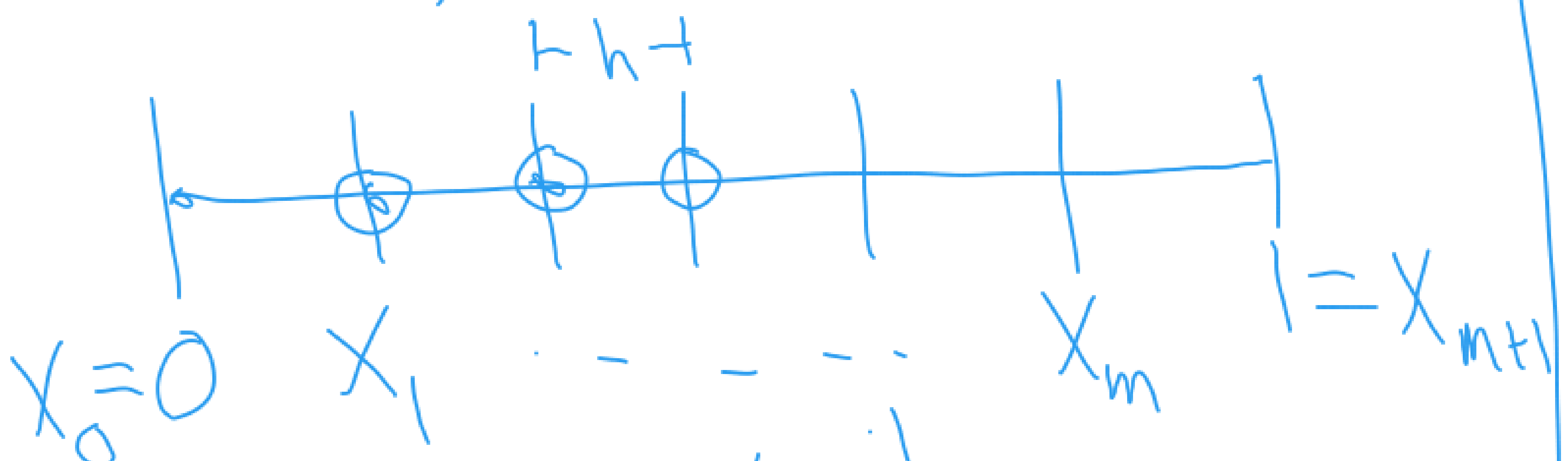
$$\int_0^1 \frac{\partial}{\partial t}u(x,t)\,dx = K\int_0^1 \frac{\partial^2 u}{\partial x^2}\,dx + \int_0^1 \psi(x)\,dx \Rightarrow \boxed{u_t = Ku_{xx} + \psi(x)}$$

Poisson

Let $t \to \infty$: $Ku_{xx} + \psi(x) = 0 \Rightarrow \boxed{u_{xx} = f(x)}$

$$u_{xx} = f(x) \qquad 0 \le x \le 1$$

$$u(0) = \alpha \qquad u(1) = \beta$$



$$x_j = jh$$

$$h = \frac{1}{m+1}$$

$$U = \begin{bmatrix} U_1 \approx u(x_1) \\ \vdots \\ \vdots \\ U_m \approx u(x_m) \end{bmatrix}$$

$$u_{xx}(x_i) \approx \boxed{\frac{U_{i+1} - 2U_i + U_{i-1}}{h^2}}$$

$$\frac{1}{h^2} \underbrace{\begin{bmatrix} -2 & 1 & & & \\ 1 & -2 & 1 & & \\ & 1 & -2 & 1 & \\ & & \ddots & \ddots & \ddots \\ & & & 1 & -2 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} U_1 \\ U_2 \\ \vdots \\ \vdots \\ U_m \end{bmatrix}}_{U} = \underbrace{\begin{bmatrix} f(x_1) - \frac{\alpha}{h^2} \\ f(x_2) \\ \vdots \\ \vdots \\ f(x_m) - \frac{\beta}{h^2} \end{bmatrix}}_{F}$$

First equation: $\dfrac{U_2 - 2U_1 + \alpha}{h^2} = f(x_1)$

$$\frac{U_2 - 2U_1}{h^2} = f(x_1) - \frac{\alpha}{h^2}$$

$$AU^h = F$$
$$U^h = A^{-1}F$$

$$\hat{U} = \begin{bmatrix} u(x_1) \\ u(x_2) \\ \vdots \\ u(x_m) \end{bmatrix}$$ Exact solution on the grid.

Global error: $U^h - \hat{U} = E^h$

Dfn. We say a sequence of solutions U^h converges to the exact solution $\hat{U}$ if

$$\lim_{h\to 0} \|U^h - \hat{U}\| = 0$$
$$\|E^h\|$$

$$A\hat{U} = F + \tau$$

Local truncation error

Substitute the exact solution into the discretization:

$$\boxed{\frac{u(x_{i+1}) - 2u(x_i) + u(x_{i-1})}{h^2}} = u''(x_i) + \frac{1}{12}h^2 u^{(4)}(x_i) + O(h^4)$$

$$u''(x) = f(x)$$

$$\boxed{\frac{u(x_{i+1}) - 2u(x_i) + u(x_{i-1})}{h^2} = f(x_i) + \tau_i}$$ LTE

$$\tau_i = \frac{1}{12}h^2 u^{(4)}(x_i) + O(h^4) \qquad \tau = \begin{bmatrix} \tau_1 \\ \vdots \\ \tau_m \end{bmatrix}$$

Dfn. We say a discretization is consistent if $\lim_{h\to 0} \|\tau\| = 0$

$AU^h = F$

$A\hat{U} = F + \tau$

$A(\underline{U^h - \hat{U}}) = -\tau$

$AE = -\tau$

For linear diff. eqs., the global error satisfies the same eqn. but with a different RHS.

$E = -A^{-1}\tau$

$\|E\| = \|A^{-1}\tau\| \le \underline{\|A^{-1}\|}\,\underline{\|\tau\|}$

$\underline{\lim_{h\to 0} \|\tau\| = 0}$

So we just need to show that $\|A^{-1}\|$ doesn't blow up as $h \to 0$.

Induced matrix norm

$$\|M\| = \sup_{x \neq 0} \frac{\|Mx\|}{\|x\|}$$

Stability means that the LTEs are not excessively amplified.

$$\boxed{\text{Stability + consistency implies } \underline{\text{convergence}}}$$

We will show that $\|A^{-1}\| < C$ as $h \to 0$, so

$$\lim_{h\to 0} \|E\| \le \lim_{h\to 0} \|A^{-1}\|\,\|\tau\| = \lim_{h\to 0} C\,\mathcal{O}(h^2) = 0$$

$$\|A\|_2 = \max_p |\lambda_p| = \rho(A) \quad \text{(spectral radius)}$$

$\lambda_1, \lambda_2, \ldots, \lambda_m$: eigenvalues of A

$$A v_p = \lambda_p v_p \qquad A \approx \frac{d^2}{dx^2}$$

$$v_p = \lambda_p A^{-1} v_p$$

$$\frac{1}{\lambda_p} v_p = A^{-1} v_p$$

Eigenvalues of A^{-1}: $\left\{ \frac{1}{\lambda_p} \right\}$

$$\|A^{-1}\|_2 = \frac{1}{\min_p |\lambda_p|}$$

$$\lambda_p = \frac{2}{h^2}\left(\cos(p\pi h) - 1\right) \le 0 \quad p = 1, 2, \ldots, m$$

$$u_j^p = \sin(p\pi j h)$$

Which λ_p is closest to zero?

$$p = 1: \ \lambda_1 = \frac{2}{h^2}\left(\cos(\pi h) - 1\right)$$

$$\cos(x) = 1 - \frac{x^2}{2} + O(x^4)$$

$$\lambda_1 = \frac{2}{h^2}\left(\cancel{1} - \frac{\pi^2 h^2}{2} + O(h^4) - \cancel{1}\right)$$

$$\lambda_1 = -\pi^2 + O(h^2)$$

$$\lim_{h \to 0} \|A^{-1}\| = \frac{1}{\pi^2} \quad \text{So } \|E\| = O(h^2)$$

Eigenvalues of A

$$h^2 A = \begin{bmatrix} -2 & 1 & & & \\ 1 & \ddots & \ddots & & \\ & \ddots & \ddots & \ddots & \\ & & \ddots & \ddots & 1 \\ & & & 1 & -2 \end{bmatrix} = \hat{A}$$

$$\hat{A} v = \lambda v$$

$$\underline{v_{j-1} - 2v_j + v_{j+1} = \lambda v_j} \qquad 1 \le j \le m$$

$$\underline{v_0 = v_{m+1} = 0}$$

$$v_{j-1} - (2+\lambda) v_j + v_{j+1} = 0 \qquad \text{Ansatz: } v_j = \xi^j$$

$$\xi^{j+1} - (2+\lambda)\xi^j + \xi^{j-1} = 0 \qquad \xi \neq 0$$

$$\xi^2 - (2+\lambda)\xi + 1 = 0$$

$$\xi_\pm = \frac{2+\lambda \pm \sqrt{(2+\lambda)^2 - 4}}{2} = 1 + \frac{\lambda}{2} \pm \frac{\sqrt{\lambda^2 + 4\lambda}}{2}$$

General solution: $c_1 \xi_-^j + c_2 \xi_+^j = v_j$

Use $v_0 = 0$ and $v_{m+1} = 0$ to find c_1, c_2

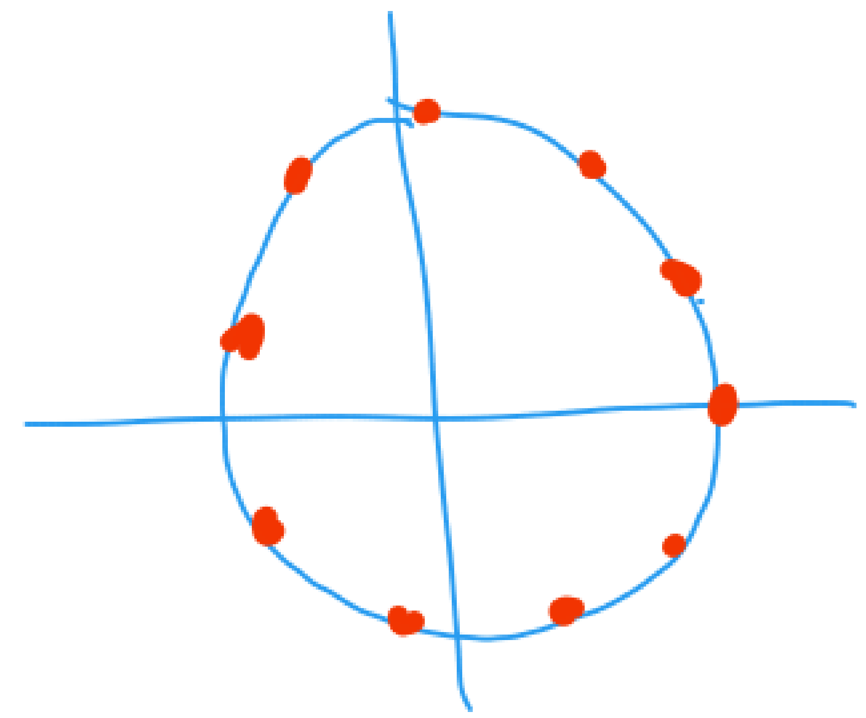
$$c_1 + c_2 = 0 \qquad v_{m+1} = c_1 \xi_-^{m+1} - c_1 \xi_+^{m+1} = 0$$

$$c_2 = -c_1 \qquad \cancel{c_1}\left(\xi_-^{m+1} - \xi_+^{m+1}\right) = 0$$

$$\rho_-^{m+1} = \rho_+^{m+1}$$

$$\underbrace{\rho_+^{m+1}\rho_-^{m+1}}_{=1} = \rho_+^{m+1}\rho_+^{m+1}$$

$$\rho_+^{2m+2} = 1$$



$$\rho_+ = e^{\pi i\left(\frac{p}{m+1}\right)}$$

$$\rho_- = \frac{1}{\rho_+} = e^{-\pi i\left(\frac{p}{m+1}\right)}$$

$$\rho_+ + \rho_- = 2 + \lambda$$

$$\lambda = 2\cos\left(\frac{p\pi}{m+1}\right) - 2$$

$$= 2\left(\cos(p\pi h) - 1\right)$$