# Nonlinear Problems

## Sources of error:

1. Discretization (truncation)
2. Rounding
3. Iteration

Linear

$$\mathcal{L}u = f$$

$\downarrow$ discretize   Truncation error

$$AU = F$$

$\downarrow$ LAPACK ← Rounding error
np.linalg.solve
MATLAB "\"

$$U$$

$$\mathcal{N}u = f$$

$\downarrow$ disc.

$$G(U) = 0$$

$$J(U) = G'(U)$$

$\downarrow$ linearize

$$J(\hat{U})\hat{U} = V$$

3rd source of error: Terminating Newton iteration

$\downarrow$ Linear Solver

$$\tilde{U}$$

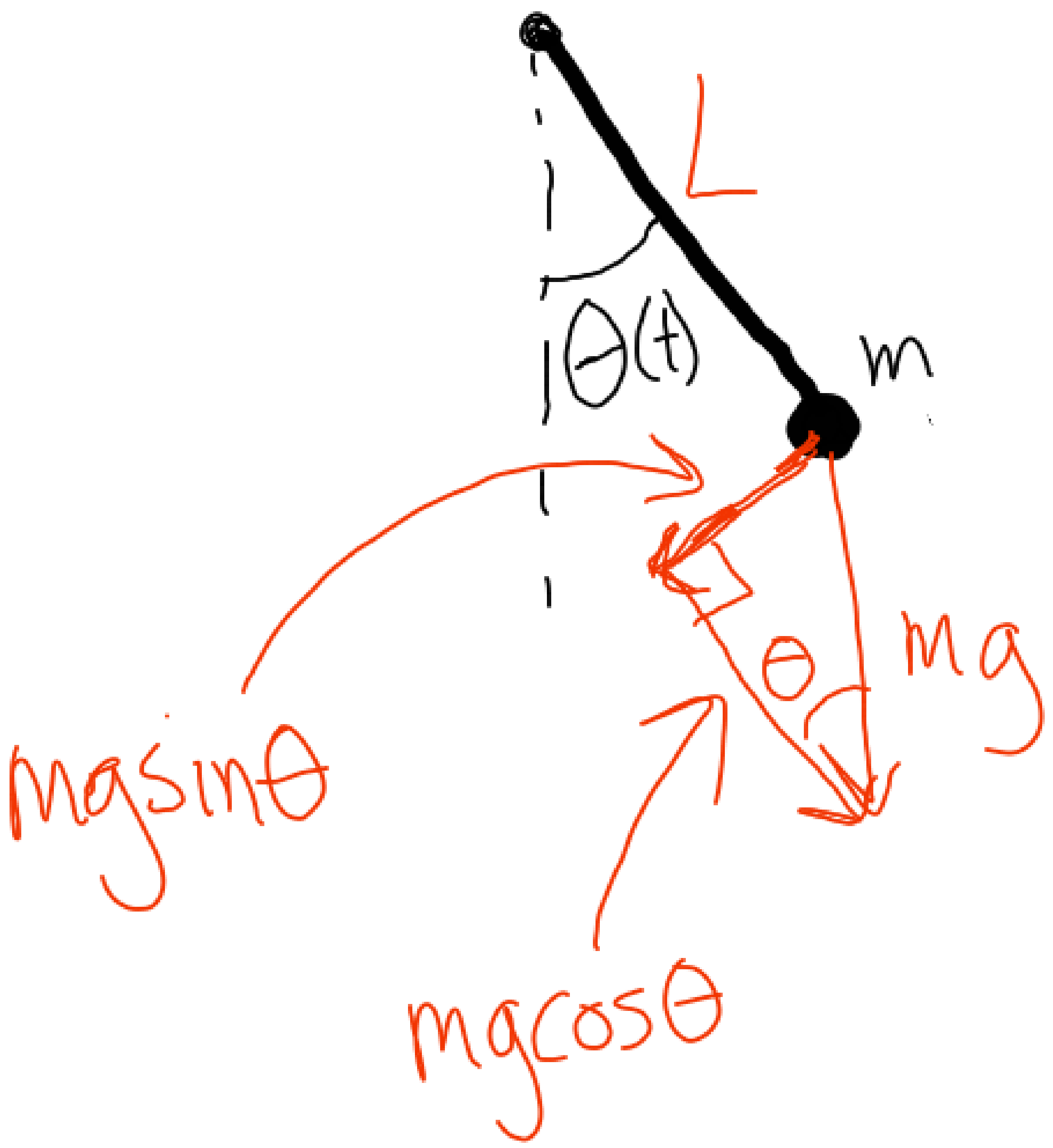
Solution? Converged?

no

$\downarrow$ yes

done

# Pendulum




Newton: $F = ma$

$$L\theta''(t) = a = \frac{F}{m} = \frac{-mg\sin\theta}{m}$$

$$\theta''(t) = -\frac{g}{L}\sin\theta$$

Let $\frac{g}{L} = 1$: $\theta''(t) = -\sin\theta$

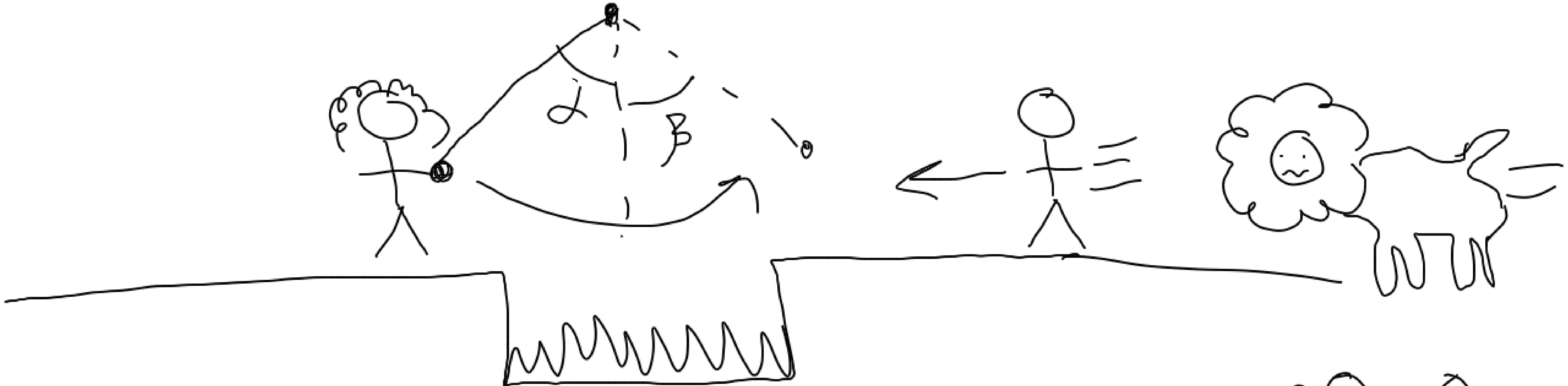
If $\theta \ll 1$:

$$\sin\theta = \theta - \frac{\theta^3}{3!} + O(\theta^5)$$

$$\theta''(t) \approx -\theta(t) \quad \text{(linear approximation)}$$

Solution: $\theta(t) = A\sin(t) + B\cos(t)$

For larger $\theta$, this is not a good approximation.

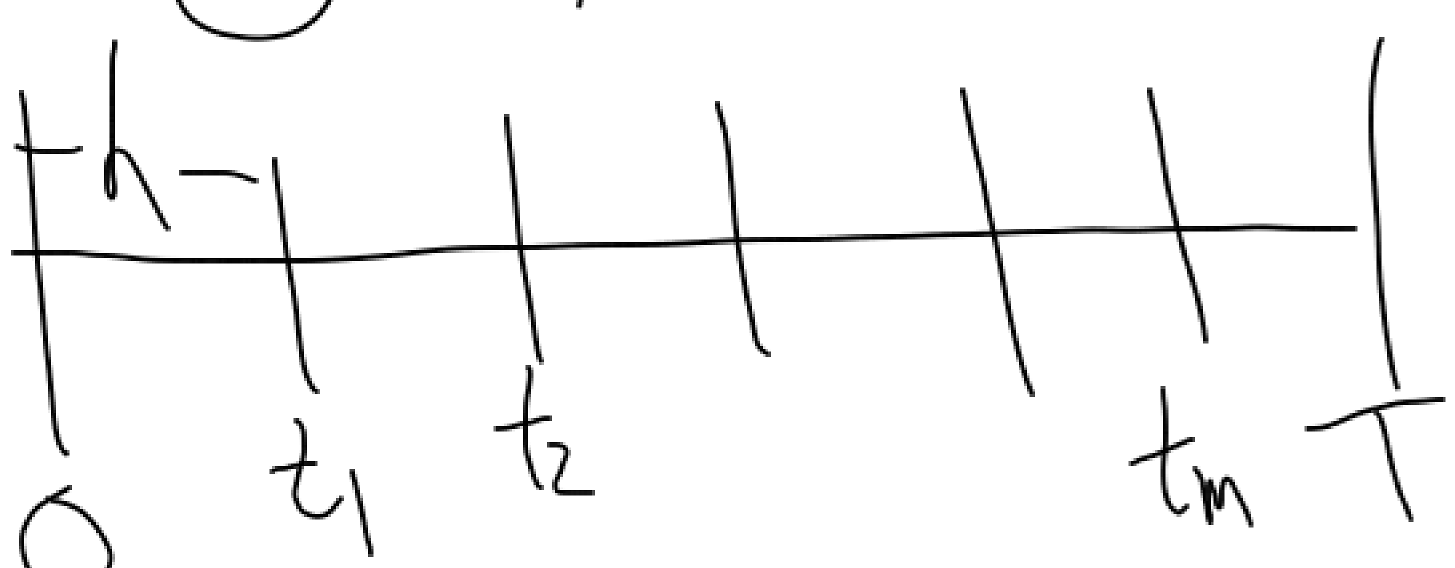



# Boundary Value problem

$$\theta''(t) = -\sin(\theta(t))$$

$$\theta(0) = \alpha$$

$$\theta(T) = \beta$$




$$\theta''(t_i) \approx \frac{\theta_{i-1} - 2\theta_i + \theta_{i+1}}{h^2}$$

$$h = \frac{T}{m+1}$$

$$\boxed{\begin{aligned} \theta_0 &= \alpha \\ -\sin(\theta_j) &= \frac{1}{h^2}\left(\theta_{j-1} - 2\theta_j + \theta_{j+1}\right) \\ \theta_{m+1} &= \beta \end{aligned}}$$

$$\theta = \begin{bmatrix} \theta_0 \\ \theta_1 \\ \vdots \\ \theta_{m+1} \end{bmatrix}$$

$$\downarrow$$

$$G(\theta) = 0$$

## Review: Taylor series for Vector-valued functions

$x, \bar{x} \in \mathbb{R}^n$

$f: \mathbb{R}^n \to \mathbb{R}^n$

$$f_i(x) = f_i(\bar{x}) + \overbrace{\sum_j \frac{\partial f_i(\bar{x})}{\partial x_j}(x_j - \bar{x}_j)}^{(J(\bar{x})(x-\bar{x}))_i} + \cdots$$

Jacobian matrix:

$$J(x) = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} & \cdots & \frac{\partial f_1}{\partial x_n} \\ \vdots & & & \vdots \\ \frac{\partial f_n}{\partial x_1} & \cdots & & \frac{\partial f_n}{\partial x_n} \end{bmatrix}$$

## Newton's method

$$f(x) = \underline{f(\bar{x}) + J(\bar{x})(x-\bar{x})} + \underline{O(\|x-\bar{x}\|^2)}$$

To find $f(x) = 0$:

$x^{[0]}$ = initial guess

$$f(x^{[0]}) = -J(x^{[0]})(x^{[1]} - x^{[0]})$$

Iterate:

$$\underline{f(x^{[k]})} = -J(\underline{x^{[k]}})(\underline{x^{[k+1]}} - x^{[k]})$$ Linear algebra

(Solve for $x^{[k+1]}$)

Back to Tarzan: $G(\Theta) = 0$

$$\boxed{G_i = \frac{1}{h^2}(\Theta_{i-1} - 2\Theta_i + \Theta_{i+1}) + \sin\Theta_i}$$

$$J = \frac{1}{h^2}\begin{bmatrix} -2 & 1 & & \\ 1 & \ddots & \ddots & \\ & \ddots & \ddots & 1 \\ & & 1 & -2 \end{bmatrix} + \begin{bmatrix} \cos\Theta_1 & & & \\ & \cos\Theta_2 & & \\ & & \ddots & \\ & & & \cos\Theta_n \end{bmatrix}$$

At each iteration, solve

$$G(\Theta^{[k]}) = -J(\Theta^{[k]})(\Theta^{[k+1]} - \Theta^{[k]})$$

This will converge to a solution if $\Theta^{[0]}$ is close enough.

Local truncation error

Substitute exact solution into discretization:

$$\frac{\Theta(t_{i-1}) - 2\Theta(t_i) + \Theta(t_{i+1})}{h^2} + \sin(\Theta(t_i)) = \tau_i$$

$$\Theta''(t_i) + \frac{1}{12}h^2\Theta^{(4)}(t_i) + O(h^4) + \sin(\Theta(t_i)) = \tau_i$$

$$\tau_i = \frac{1}{12}h^2\Theta^{(4)}(t_i) + O(h^4)$$

Global error

Let $\hat{\Theta} = \begin{bmatrix} \Theta(t_1) \\ \vdots \\ \vdots \\ \Theta(t_m) \end{bmatrix}$ $\quad E = \Theta - \hat{\Theta}$

$$G(\hat{\theta}) = \tau$$

$$G(\theta) = 0$$

$$G(\theta) - G(\hat{\theta}) = -\tau$$

Recall: $G(\theta) = G(\hat{\theta}) + J(\hat{\theta})\overbrace{(\theta - \hat{\theta})}^{E} + \underline{O(\|E\|^2)}$

$$= G(\hat{\theta}) + J(\hat{\theta})E + O(\|E\|^2)$$

$$G(\theta) - G(\hat{\theta}) = J(\hat{\theta})E + \underline{O}(\|E\|^2)$$

$$J(\hat{\theta})E \approx -\tau$$

Need to bound this as $h \to 0$

$$E \approx -(J(\hat{\theta}))^{-1}\tau \Rightarrow \|E\| \leq \|J^{-1}(\hat{\theta})\| \, \|\underset{O(h^2)}{\tau}\|$$

As $h \to 0$, $J \to A$

So we can show

$$\|J^{-1}\| < C$$

i.e., the method is stable.

Thus $\|E\| < C\|\tau\|$ as $h \to 0$

So $\lim_{h \to 0} \|E\| = 0$

(the method is convergent)

So as long as Newton's method converges, $\theta$ will approximate a solution of $\theta'' = -\sin\theta$.

(but this eqn. also has multiple solutions)

$$G_1 = \frac{\alpha - 2\theta_1 + \theta_2}{h^2} + \sin\theta_1$$

$$G_m = \frac{\theta_{m-1} - 2\theta_m + \beta}{h^2} + \sin\theta_m$$

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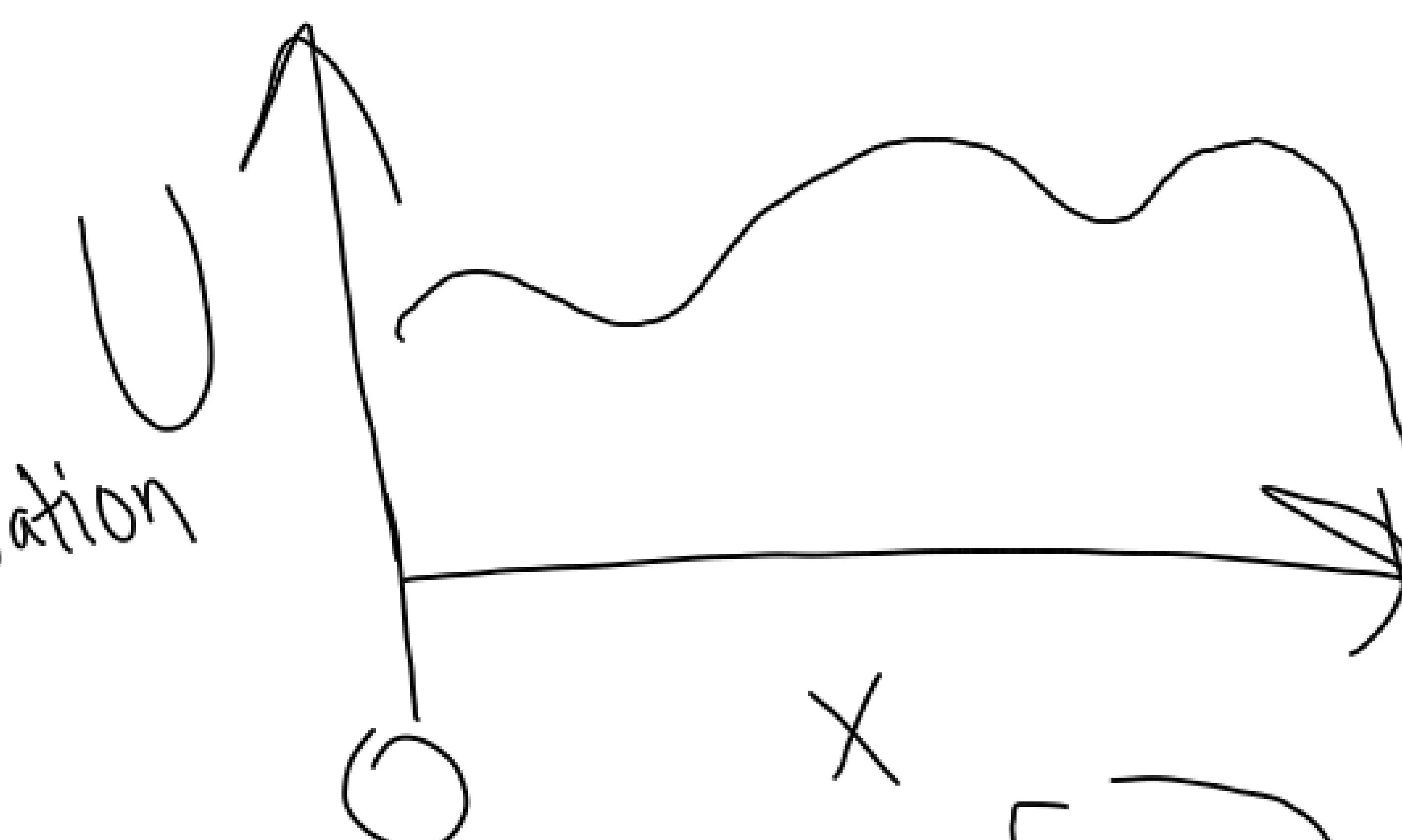
Boundary layers




$u(0) = \alpha$

$u(1) = \beta$

Singular perturbation

$$\varepsilon \downarrow$$

$$u_x = \frac{K}{a} u_{xx}$$

$$u_x = 0$$

$$\cancel{u_t} + \boxed{a u_x} - \boxed{K u_{xx}} = \psi(x)$$

Advection-diffusion