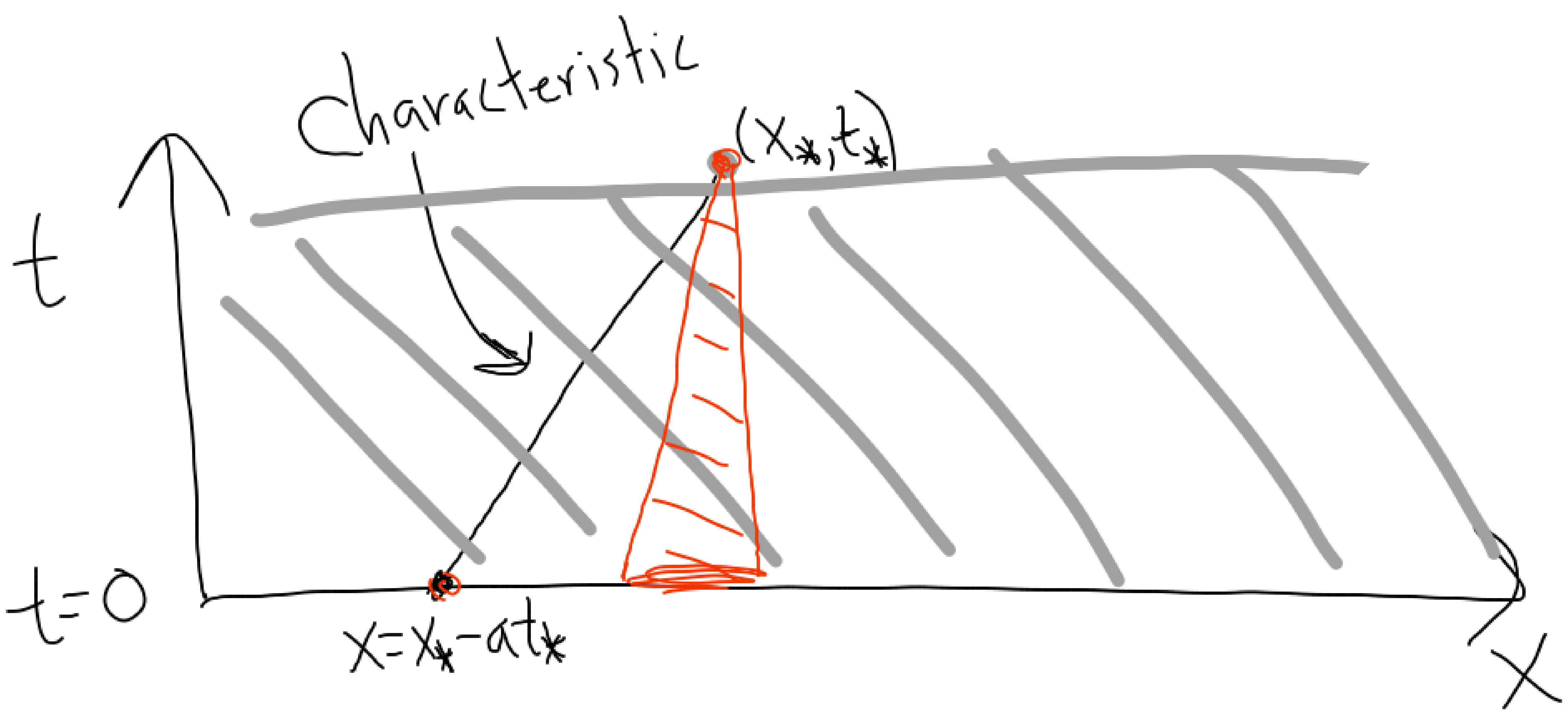
$$u_t + a u_x = 0 \qquad u(x,0) = \eta(x)$$

Exact solution: $u(x,t) = \eta(x - at)$



$$x = x_* - at_* + at$$

$$x = x_* + a(t - t_*)$$

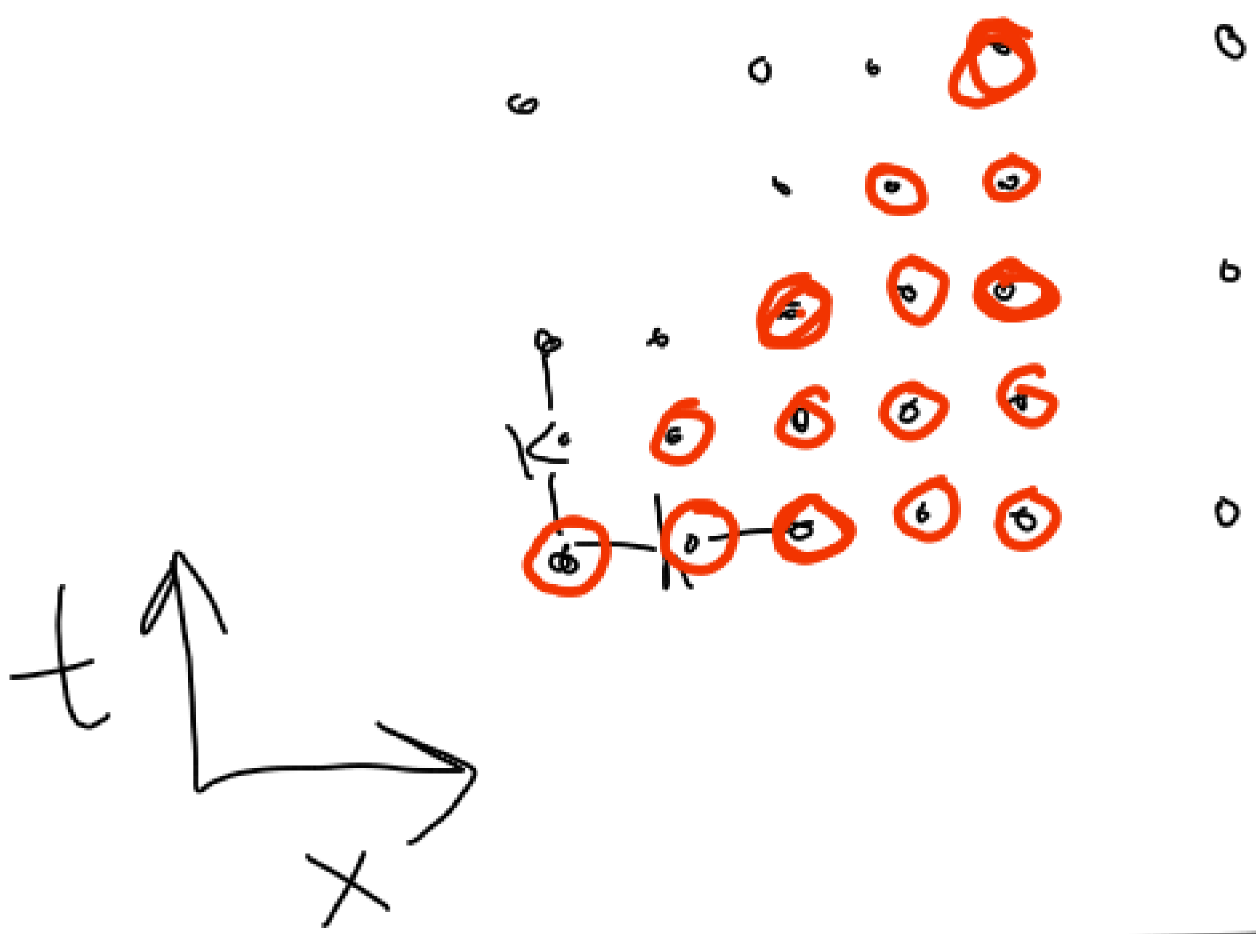
Domain of dependence

The D. of D. of a PDE Solution at a point (x_*, t_*) is the set of all points (x,t) with $t < t_*$ on which the Solution at (x_*, t_*) depends.

Numerical Domain of Dependence

Upwind method:

$$U_j^{n+1} = U_j^n - \frac{ka}{h}\left(U_j^n - U_{j-1}^n\right)$$



CFL Theorem.

A numerical method cannot be convergent unless the numerical D. of D. includes the true D. of D. in the limit as $k, h \to 0$.

For hyperbolic PDEs, the D. of D. is bounded. Information propagates at finite speed.

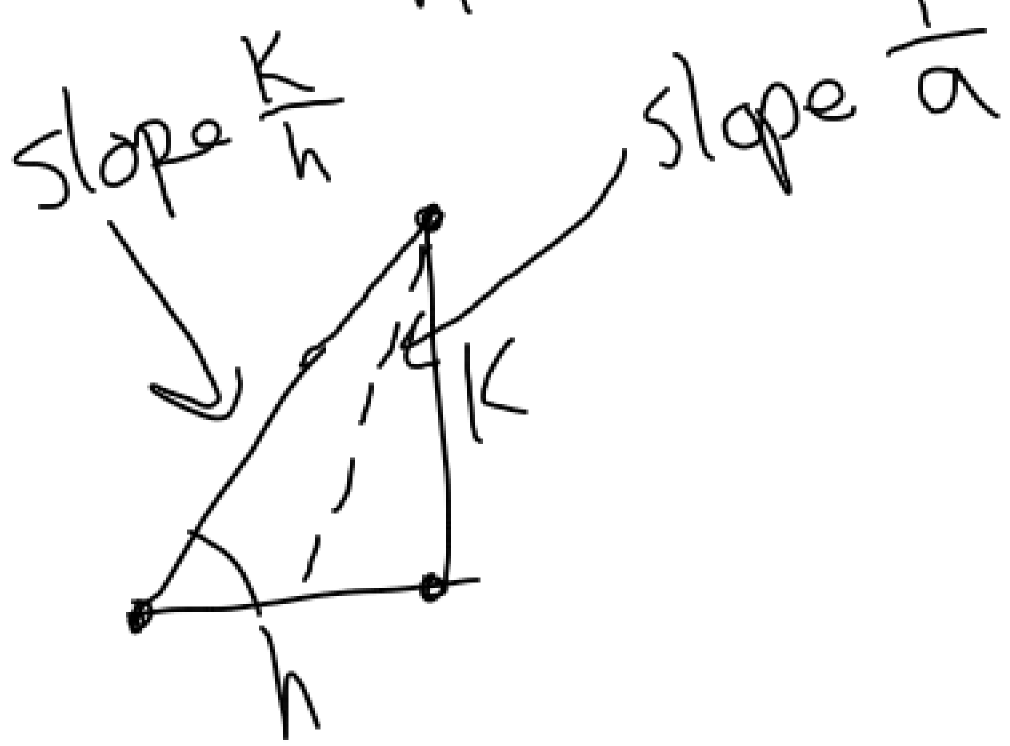
For parabolic PDEs, the D. of D. is all points at each earlier time. Information propagates instantaneously.

What does CFL imply for the upwind method?

$$0 \le \nu \le 1$$

$$\uparrow$$

$$\frac{ka}{h}$$



$$\frac{1}{a} \ge \frac{k}{h} \Rightarrow \frac{ka}{h} \le 1$$

$$u_{tt} = a^2 u_{xx}$$

$$\Downarrow$$

$$v_t + a w_x = 0$$

$$w_t + a v_x = 0$$

Information travels both directions.

Need to use points from both sides.

Lax-Wendroff

$$u(x,t+k) = u(x,t) + k u_t(x,t) + \frac{k^2}{2} u_{tt}(x,t) + O(k^3)$$

$$u_t = -a u_x$$

$$u_{tt} = a^2 u_{xx}$$

$$u(x,t+k) = u(x,t) - k a u_x(x,t) + \frac{k^2 a^2}{2} u_{xx} + O(k^3)$$

Cauchy-Kowalewski procedure

$$\boxed{U_j^{n+1} = U_j^n - \frac{ka}{2h}\left(U_{j+1}^n - U_{j-1}^n\right) + \frac{k^2 a^2}{2}\left(U_{j-1}^n - 2U_j^n + U_{j+1}^n\right)}$$

2nd-order in time and space

Modified Eqn. Analysis for L-W

$$U_j^{n+1} = U_j^n - \frac{1}{2}\left(U_{j+1}^n - U_{j-1}^n\right) + \frac{1}{2}\left(U_{j+1}^n - 2U_j^n + U_{j-1}^n\right)$$

$$U_j^{n+1} = U_{j-1}^n$$

$$U_j^{n+1} = U_j^n - \frac{ka}{2h}\left(U_{j+1}^n - U_{j-1}^n\right) + \frac{k^2a^2}{2h^2}\left(U_{j+1}^n - 2U_j^n + U_{j-1}^n\right)$$

Assume $V(x_j, t_n) = U_j^n$

$$V(x_j, t_n + k) = V(x_j, t_n) - \frac{ka}{2h}\left(V(x_j + h, t_n) - V(x_j - h, t_n)\right) + \frac{k^2a^2}{2h^2}\left(V(x_j + h, t_n) - 2V(x_j, t_n) + V(x_j - h, t_n)\right)$$

$$V + kV_t + \frac{k^2}{2}V_{tt} + \frac{k^3}{6}V_{ttt} + O(k^4) = V - \frac{ka}{2h}\left(hV_x + \frac{h^2}{2}V_{xx} + \frac{h^3}{6}V_{xxx} - \left(-hV_x + \frac{h^2}{2}V_{xx} - \frac{h^3}{6}V_{xxx}\right) + O(h^4)\right)$$
$$+ \frac{k^2a^2}{2h^2}\left(\frac{h^2}{2}V_{xx} + \frac{h^3}{6}V_{xxx} + \frac{h^2}{2}V_{xx} - \frac{h^3}{6}V_{xxx} + O(h^4)\right)$$

$$V_t + \frac{k}{2}V_{tt} + \frac{k^2}{6}V_{ttt} = -aV_x - \frac{ah^2}{6}V_{xxx} + \frac{ka^2}{2}V_{xx} + O(k^3) + O(h^3) + O(h^2k)$$

$$V_t + aV_x + \frac{k}{2}V_{tt} + \frac{k^2}{6}V_{ttt} - \frac{ka^2}{2}V_{xx} + \frac{ah^2}{6}V_{xxx} = O(h^3, k^3, kh^2)$$

$$\underline{V_t \approx -aV_x}$$

$$\boxed{V_{tt} \approx a^2 V_{xx}}$$

$$V_{ttt} \approx a^2 \underline{V_{xxt}}$$

$$\boxed{V_{ttt} \approx -a^3 V_{xxx}}$$

$$\underline{V_{txx} \approx -aV_{xxx}}$$

$$V_t + aV_x + \cancel{\frac{k}{2}a^2 V_{xx}} \cancel{- \frac{ka^2}{2}V_{xx}} - \frac{k^2a^3}{6}V_{xxx} + \frac{ah^2}{6}V_{xxx} = O(\ldots)$$

$$V_t + aV_x = \frac{ah^2}{6}\left(\frac{k^2a^2}{h^2} - 1\right)V_{xxx} + O()$$

ν^2

Advection-dispersion

High-frequency components move more slowly.