Today:

- Exam review
- Zero stability

$$S'(t) = -\beta S I$$

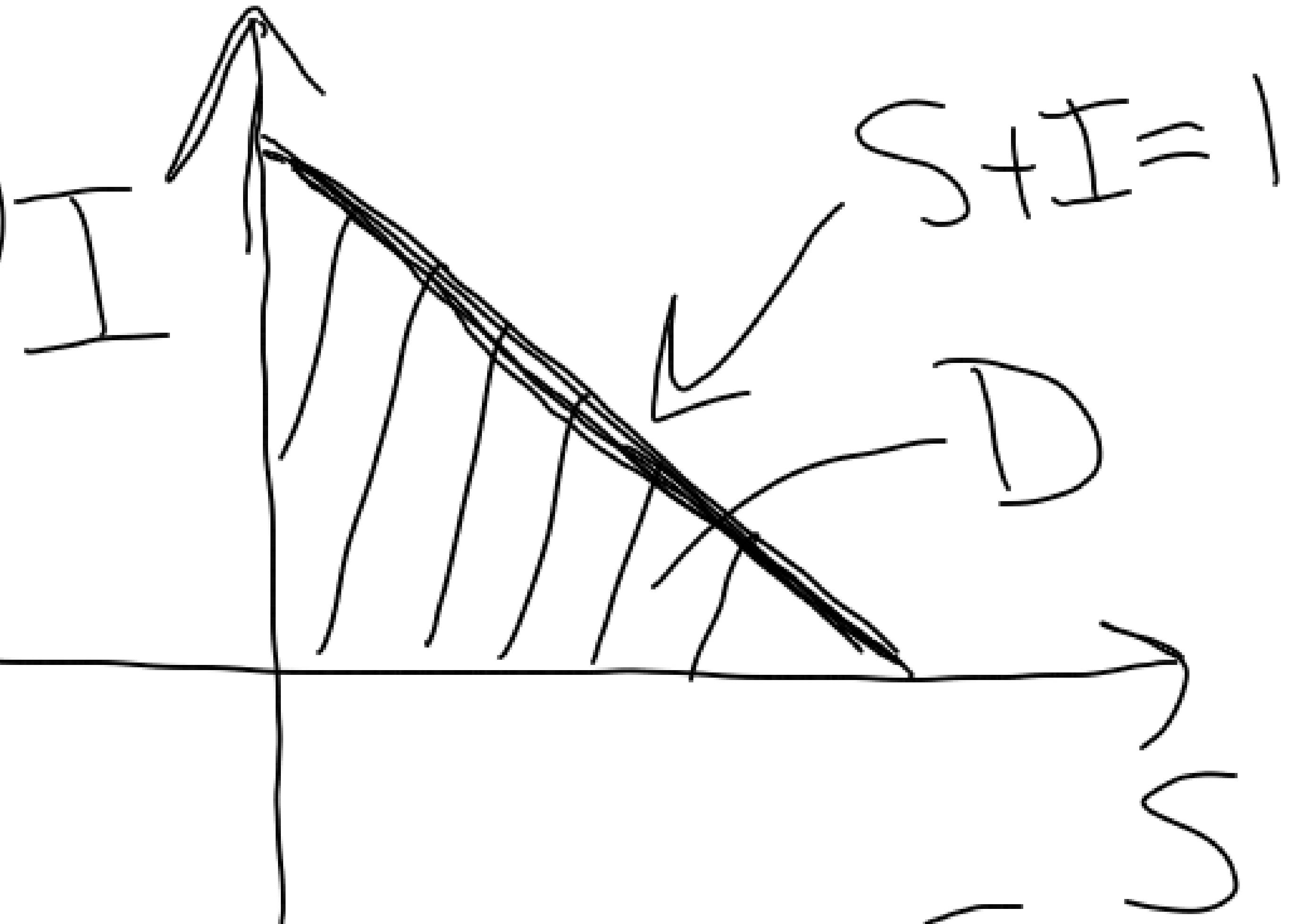
$$I'(t) = \beta S I - \gamma I$$

$$R'(t) = \gamma I$$

1 $S' + I' + R' = (S+I+R)'$

$$= 0$$

(2)



$[S(0), I(0)] \in D \Rightarrow [S(t), I(t)] \in D$

Let $u(t) = \begin{bmatrix} S(t) \\ I(t) \end{bmatrix}$ and $u(0) \in D$.

Let $t_1 = \inf\{t : u(t) \notin D\}$

By continuity, $u(t_1) \in \partial D$.

We have $(S+I)' = -\gamma I < 0$ for $0 \le t < t_1$

So $S(t_1) + I(t_1) < S(0) + I(0) < 1$.

Therefore either $S(t_1) = 0$ or $I(t_1) = 0$ (or both).

$(0 \le t < t_1)$

Now $S'(t) = -\beta S(t) I(t) \ge -\beta S(t)$

So $S(t) \ge e^{-\beta t} S_0 > 0$ $\quad S(t_1) > 0$

$I'(t) = \beta S I - \gamma I \ge -\gamma I$ $\quad (0 \le t < t_1)$

$I(t) \ge e^{-\gamma t} I_0 > 0$ $\quad I(t_1) > 0$

$\Rightarrow$ contradiction.

③ $f(u) = \begin{bmatrix} -\beta S I \\ \beta S I - \gamma I \end{bmatrix}$ $\quad f'(u) = \begin{bmatrix} -\beta I & -\beta S \\ \beta I & \beta S - \gamma \end{bmatrix}$

$$\frac{\|f(u_1) - f(u_2)\|}{\|u_1 - u_2\|} \le L$$

$u_1 \neq u_2$

$u_1, u_2 \in D$

$$\sup_{u \in D} \|f'(u)\|_\infty = \sup_{u \in D} \max\left\{|\beta I| + |\beta S|, |\beta I| + |\beta S - \gamma|\right\}$$

$$= \sup_{u \in D} \max\left\{2\beta, \beta + |\beta S| + \gamma\right\}$$

$$= \max\left\{2\beta, 2\beta + \gamma\right\}$$

Leapfrog: $\underline{U^{n+2}} = \underline{U^n} + \underline{2kf(U^{n+1})} \rightarrow U^{n+2} - U^n = 2kf(U^{n+1})$

Backward differentiation: $\underline{U^{n+2} = \frac{4}{3}U^{n+1} - \frac{1}{3}U^n + \frac{2}{3}kf(U^{n+2})}$

General form: (r-step method)

$$\sum_{j=0}^{r} \alpha_j U^{n+j} = \underline{k}\sum_{j=0}^{r} \beta_j f(U^{n+j}, t_{n+j})$$

$t_{n+j} = t_n + kj$

$$u(t_{n+j}) = u(t_n) + kj\,u'(t_n) + \frac{(kj)^2}{2!}u''(t_n) + \dots = \sum_{i=0}^{\infty} \frac{(jk)^i}{i!} u^{(i)}(t_n)$$

$$f(u(t_{n+j})) = \sum_{i=0}^{\infty} \frac{(jk)^i}{i!} u^{(i+1)}(t_n) = \sum_{i=1}^{\infty} \frac{(jk)^{i-1}}{(i-1)!} u^{(i)}(t_n)$$

Substitution:

$$\sum_{j=0}^{r} \alpha_j \sum_{i=0}^{\infty} \frac{(jk)^i}{i!} u^{(i)}(t_n) = \sum_{j=0}^{r} k\beta_j \sum_{i=1}^{\infty} \frac{(jk)^{i-1}}{(i-1)!} u^{(i)}(t_n) + k\underline{\tau^{n+r}}$$

Equating Coefficients of $u^{(0)}(t_n)$: $\boxed{\sum_{j=0}^{r} \alpha_j = 0}$

" " $u^{(1)}(t_n)$: $\sum_{j=0}^{r} \alpha_j j\cancel{k} = \cancel{k}\sum_{j=0}^{r} \beta_j$

$$\boxed{\sum_{j=0}^{r} j\alpha_j = \sum_{j=0}^{r} \beta_j}$$

These two conditions guarantee $\tau = O(k)$
(consistency)

Stability of LMMs

If we apply a LMM to $u'(t)=0$:

$$(*) \quad \underbrace{\sum_{j=0}^{r} \alpha_j U^{n+j} = 0}_{\text{Linear system of difference equations}}$$

$$\rho(1)=0$$

Ansatz: $U^n = \zeta^n \longleftarrow$ exponent

$$\zeta \in \mathbb{C}$$

$$\sum_{j=0}^{r} \alpha_j \zeta^{n+j} = 0 \qquad \zeta \neq 0$$

$$\rho(\zeta) = \sum_{j=0}^{r} \alpha_j \zeta^j = 0$$ polynomial of degree r.

$\rho(\zeta)$ is called the first characteristic polynomial of the method.

Let $\{\zeta_1, \ldots, \zeta_r\}$ denote the roots of ρ. Assume they are distinct.

Then all solutions of $(*)$ are of the form

$$\boxed{U^n = a_1 \rho_1^n + a_2 \rho_2^n + \cdots + a_r \rho_r^n}$$

The values $a_1, \ldots, a_r$ depend on $U^0, \ldots, U^{r-1}$.

We want $|U^n| < \infty$ as $n \to \infty$.

This is equivalent to

$$|\rho_j| \leq 1 \quad \forall j.$$

What if we have a repeated root?

Consider: $U^{n+2} - 2U^{n+1} + U^n = 0$

$$U^n = \rho^n \Rightarrow \rho^2 - 2\rho + 1 = 0$$

$$(\rho - 1)^2 = 0$$

$\rho = 1 \Rightarrow$ one solution is $U^n = 1^n = 1$

The other fundamental solution is $U^n = n$:

$$n+2 - 2(n+1) + n = 0$$

So the general soln. is

$$U^n = a_1 + a_2 n \quad \leftarrow \text{blows up as } n \to \infty$$

$$a_1 \rho^n + a_2 n \rho^n$$

We say a LMM is zero-stable if the root of $\rho(\zeta)$ satisfy:

(1) $|\zeta_j| \le 1$

(2) If ζ_j is a repeated root, $|\zeta_j| < 1$.

The root condition.

Thm. Given an IVP

$$u'(t) = f(u)$$
$$u(t_0) = u_0$$

with f Lipschitz, any consistent and zero-stable LMM is convergent.