$$u_t = k u_{xx} + \psi(x)$$

Steady state:

$$u_{xx} = -\frac{\psi(x)}{k} = f(x)$$

$$\left.\begin{aligned} u(0) &= \alpha \\ u(1) &= \beta \end{aligned}\right\} \text{Dirichlet}$$

Today:

$$\left.\begin{aligned} u'(0) &= \sigma_0 \\ u'(1) &= \sigma_1 \end{aligned}\right\} \text{Neumann}$$

Continuous

$0 \le x \le 1$

$u(x)$

$f(x)$

$$\frac{d^2}{dx^2}$$

$$\frac{d^2}{dx^2} u = f(x)$$

Discrete

$0 \quad x_1 \quad x_2 \quad \cdots \quad x_m \quad 1$

$$x_j = jh = \frac{j}{m+1}$$

$$U = [U_0, U_1, \dots, U_m, U_{m+1}]$$

$$F = [\alpha, f(x_1), \dots, f(x_m), \beta]$$

$$A = \frac{1}{h^2}\begin{bmatrix} h^2 & & & & \\ 1 & -2 & 1 & & \\ & \ddots & \ddots & \ddots & \\ & & 1 & -2 & 1 \\ & & & & h^2 \end{bmatrix}$$

$$AU = F$$

$u''(x) = f(x) \qquad 0 \le x \le 1$

$$\left.\begin{aligned} u'(0) &= 0 \\ u'(1) &= 0 \end{aligned}\right\}$$

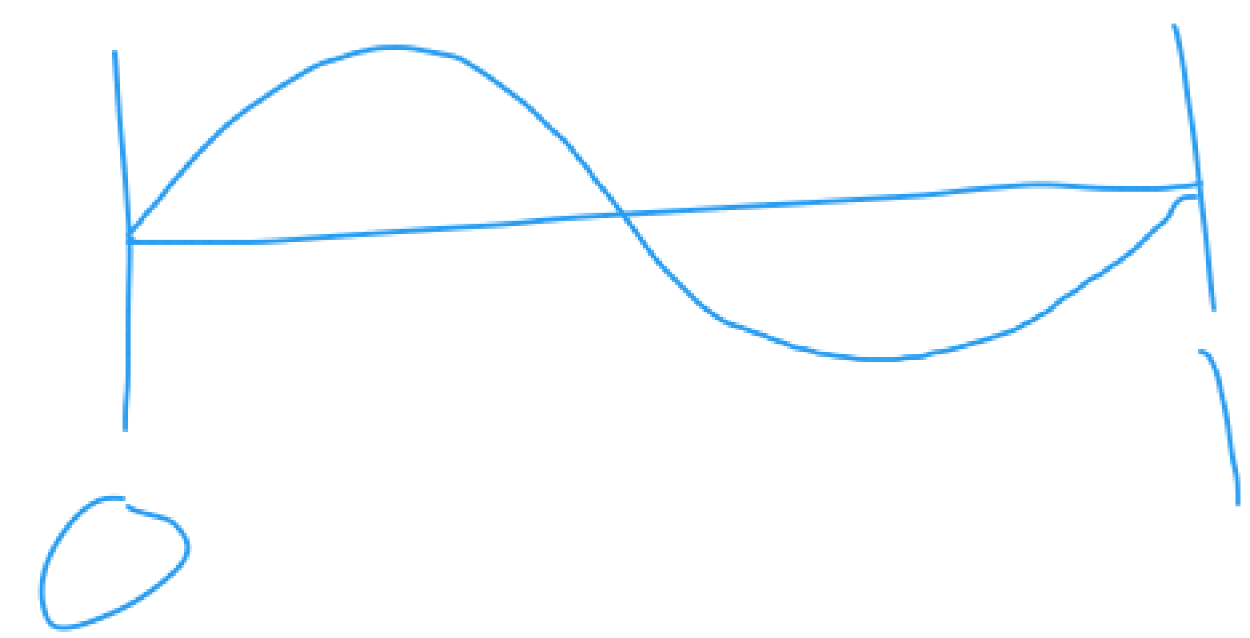
(Think about $\underline{u_t = K u_{xx} + \psi(x)}$)

If $\psi(x) > 0 \quad \forall x$

$u(t,x) \to +\infty$

If $\psi(x) = 0 \to u(x,t) = 0$

What if $\psi(x) = \sin(2\pi x)$



$\int_0^1 \psi(x)\,dx = 0$

$$\int_0^1 u_t\,dx = K\int_0^1 u_{xx}\,dx + \int_0^1 \psi(x)\,dx$$

$$\frac{d}{dt}\int_0^1 u(x,t)\,dx = K\left[u_x(1,t) - u_x(0,t)\right] + \int_0^1 \psi(x)\,dx$$

Suppose there exists a steady state

$$0 = K(U'(1) - U'(0)) + \int_0^1 \psi(x)\,dx$$

$$\underline{\int_0^1 \psi(x)\,dx} = -K\underline{(U'(1) - U'(0))}$$

A steady state exists iff this is satisfied.

$$\int_0^1 -\frac{\psi}{K}\,dx = U'(1) - U'(0)$$

$$\boxed{\int_0^1 f(x)\,dx = U'(1) - U'(0)}$$

Condition for Neumann BVP to be well posed.

$U'(0) = \sigma_0 \leftarrow$ How to discretize?

$U(1) = \beta$

Two methods:

1. Use a one-sided FD
2. Use a ghost point

$x_0 = 0 \quad x_1 = h$

$$U'(0) \approx \frac{U_1 - U_0}{h} = D_+ u(x_0)$$

$$\frac{U_1 - U_0}{h} = \sigma_0$$

$$A = \frac{1}{h^2}\begin{bmatrix} -h & h & & & \\ 1 & -2 & 1 & & \\ & \ddots & \ddots & \ddots & \\ & & 1 & -2 & 1 \\ & & & & h^2 \end{bmatrix} \quad U = \begin{bmatrix} \sigma_0 \\ f(x_1) \\ \vdots \\ \vdots \end{bmatrix}$$

The centered FD approximation in the interior has $\tau = O(h^2)$

The BC formula has $\tau = O(h)$

Recall: $D_+ u(\bar{x}) = u'(\bar{x}) + \boxed{\frac{h}{2} u''(\bar{x})} + O(h^2)$

Defect correction: Subtract the error term

$$\boxed{\frac{U_1 - U_0}{h} = \sigma_0 + \frac{h}{2} f(x_0)}$$

Claim: this gives an error $O(h^2)$

Another way to get 2nd-order accuracy: use U_0, U_1, U_2 in our FD formula

$$U'(x_0) = \frac{-\frac{3}{2}U_0 + 2U_1 - \frac{1}{2}U_2}{h} + O(h^2)$$

② Ghost point

Centered difference: $\boxed{U'(x_0) \approx \frac{U_1 - U_{-1}}{2h}}$

Impose ODE at x_0: $\frac{U_{-1} - 2U_0 + U_1}{h^2} = f(x_0)$

$$U_{-1} = h^2 f(x_0) + 2U_0 - U_1$$

$$U'(x_0) \approx \frac{U_1}{2h} - \frac{1}{2h}\left(h^2 f(x_0) + 2U_0 - U_1\right)$$

$$= \frac{U_1}{2h} - \frac{h}{2} f(x_0) - \frac{U_0}{h} + \frac{U_1}{2h}$$

$$U'(x_0) = \boxed{\frac{U_1 - U_0}{h} - \frac{h}{2} f(x_0) = \sigma_0}$$

Same as defect correction

$u''(x) = f(x)$

$u'(0) = \sigma_0 \rightarrow \dfrac{U_1 - U_0}{h} = \sigma_0$

$u'(1) = \sigma_1 \rightarrow \boxed{\dfrac{U_{m+1} - U_m}{h} = \sigma_1}$

$AU = F$:

$$\frac{1}{h^2}\begin{bmatrix} -h & h & & & \\ 1 & -2 & 1 & & \\ & \ddots & \ddots & \ddots & \\ & & 1 & -2 & 1 \\ & & & -h & h \end{bmatrix} \begin{bmatrix} U_0 \\ \vdots \\ \vdots \\ U_{m+1} \end{bmatrix} = \begin{bmatrix} \sigma_0 \\ f(x_1) \\ \vdots \\ f(x_m) \\ \sigma_1 \end{bmatrix}$$

Does this system have a solution?

Does A^{-1} exist? No

$$A \underline{\begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix}} = \begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix}$$

Since $Av = 0v$ (zero eigenvalue)

We have either no solution or ∞ many.