1. I will drop your lowest homework grade

2. Exams will be take-home.

Multigrid

Jacobi's method

$u''(x) = f(x)$

$\downarrow$

$$\boxed{\frac{1}{h^2}\left(U_{i-1} - 2U_i + U_{i+1}\right) = F_i} \quad AU = F$$

$$U_i = \frac{U_{i+1} + U_{i-1}}{2} - \frac{h^2}{2} F_i$$

$$U_i^{[k+1]} = \frac{U_{i+1}^{[k]} + U_{i-1}^{[k]}}{2} - \frac{h^2}{2} F_i$$ Jacobi iteration

U (the exact solution) is a fixed point of this iteration.

Jacobi's method is an example of fixed-point iteration.

Think of it as a map $g : U^{[k]} \to U^{[k+1]}$

$g(U) = U$

Does the iteration converge to U from any initial guess $U^{[0]}$? Yes.

Let $G = \begin{bmatrix} 0 & \frac{1}{2} & & \\ \frac{1}{2} & \ddots & \ddots & \\ & \ddots & \ddots & \frac{1}{2} \\ & & \frac{1}{2} & 0 \end{bmatrix}$

Then $U = GU - \frac{h^2}{2}F$

$U^{[k+1]} = GU^{[k]} - \frac{h^2}{2}F \qquad e^{[k]} = U^{[k]} - U$

$U^{[k+1]} - U = G(U^{[k]} - U)$

$$\boxed{e^{[k+1]} = Ge^{[k]} = G^k e^{[0]}}$$

We want $\lim_{k\to\infty} \|e^{[k]}\|_2 = 0$

This is equivalent to $\boxed{\|G^k\|_2 < 1}$

Submultiplicativity

$\|G^k\|_2 \le \|G\|_2^k$

$= \left(\max_{\mu\in\sigma(G)} |\mu|\right)^k$

Theorem. Let A be diagonalizable with eigenvalues $\{\lambda_1, \lambda_2, \dots, \lambda_m\}$ and let $G = f(A)$ where f is analytic. Then the eigenvalues of G are $\{f(\lambda_1), f(\lambda_2), \dots, f(\lambda_m)\}$.

$A = \frac{1}{h^2}\,\text{tridiag}(1,-2,1)$

$G = \text{tridiag}(\frac{1}{2}, 0, \frac{1}{2})$

$p\pi h \in (0,\pi)$

$$G = \frac{h^2 A + 2I}{2} = \frac{h^2}{2}A + I$$

$G = f(A)$ where $f(x) = \frac{h^2}{2}x + 1$

$$\lambda_p = \frac{2}{h^2}\left(\cos(p\pi h) - 1\right)$$

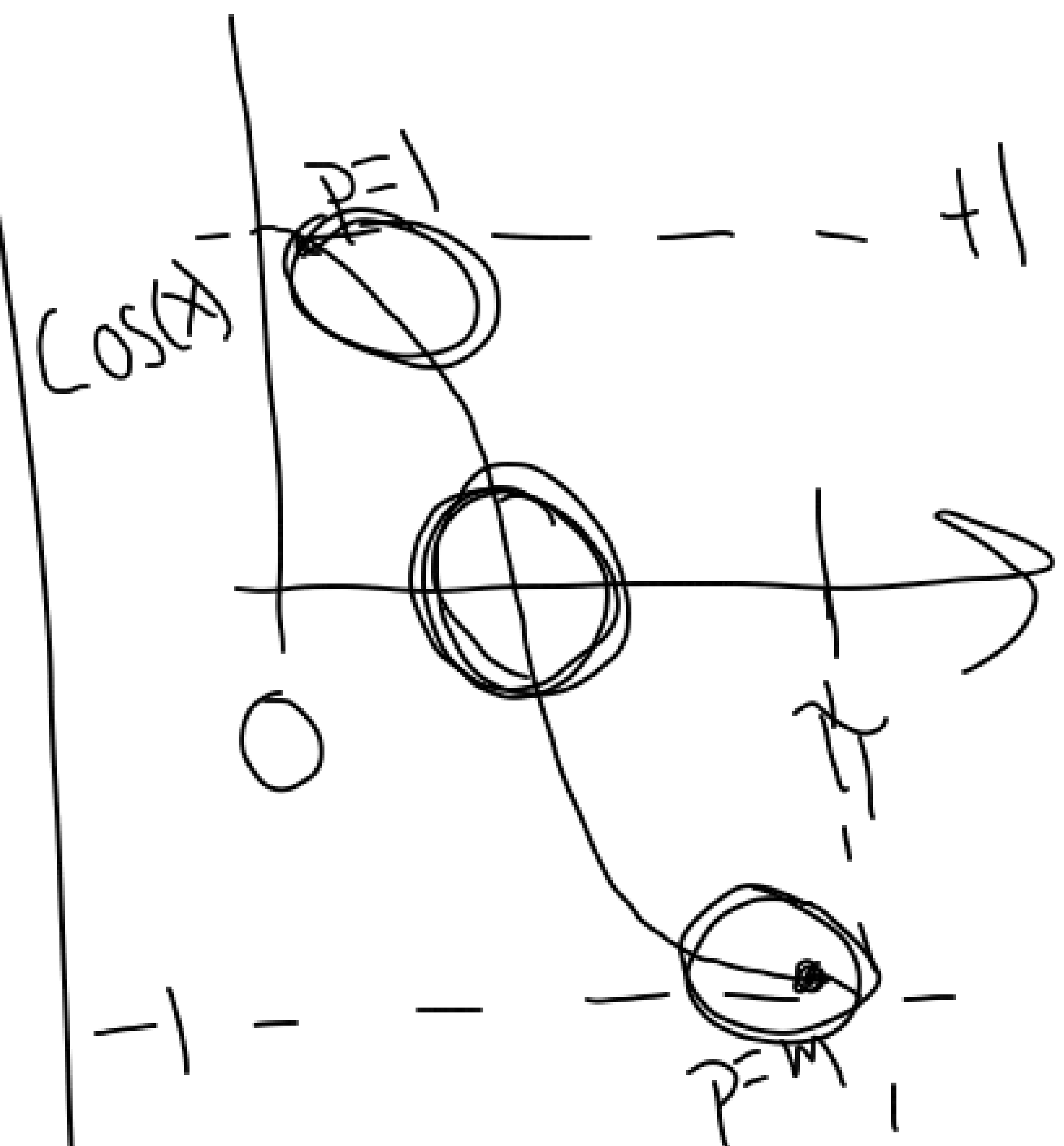
$A v_p = \lambda_p v_p$

$$\mu_p = f(\lambda_p) = \cos(p\pi h)$$

$p = 1, 2, \dots, m$

$h = \frac{1}{m+1}$

$$\max_p |\mu_p| < 1$$



$\cos(\pi h) = 1 - \frac{1}{2}(\pi h)^2 + O(h^4)$

$\cos(\pi h \frac{m}{m+1}) = -1 + \frac{1}{2}(\pi h)^2 + O(h^4)$

$G v_p = \mu_p v_p$

We can write $e^{[0]} = \boxed{\sum_{p=1}^{m} c_p v_p}$

Where v_p are the eigenvectors of G.

Then $e^{[k+1]} = G^k e^{[0]} = G^k \sum_{p=1}^{m} c_p v_p$

$= G^{k-1} \sum_{p=1}^{m} c_p G v_p = G^{k-1} \sum_{p=1}^{m} c_p \mu_p v_p$

$$= \boxed{\sum_{p=1}^{m} c_p \underline{\mu_p^k} v_p}$$

Problem: Slow decay of components with $|\mu_p| \hat{\approx} 1$.

$$U^{[k+1]} = G_\omega U^{[k]} - \omega \frac{h^2}{2} F$$

$$G_\omega = (1-\omega) I + \omega G \qquad 0 \le \omega \le 1$$

Eigenvalues of G_ω: $\hat{\mu}_p = (1-\omega) + \omega \cos(p\pi h)$

$$\hat{U}^{[k+1]} = G U^{[k]} - \frac{h^2}{2} F$$

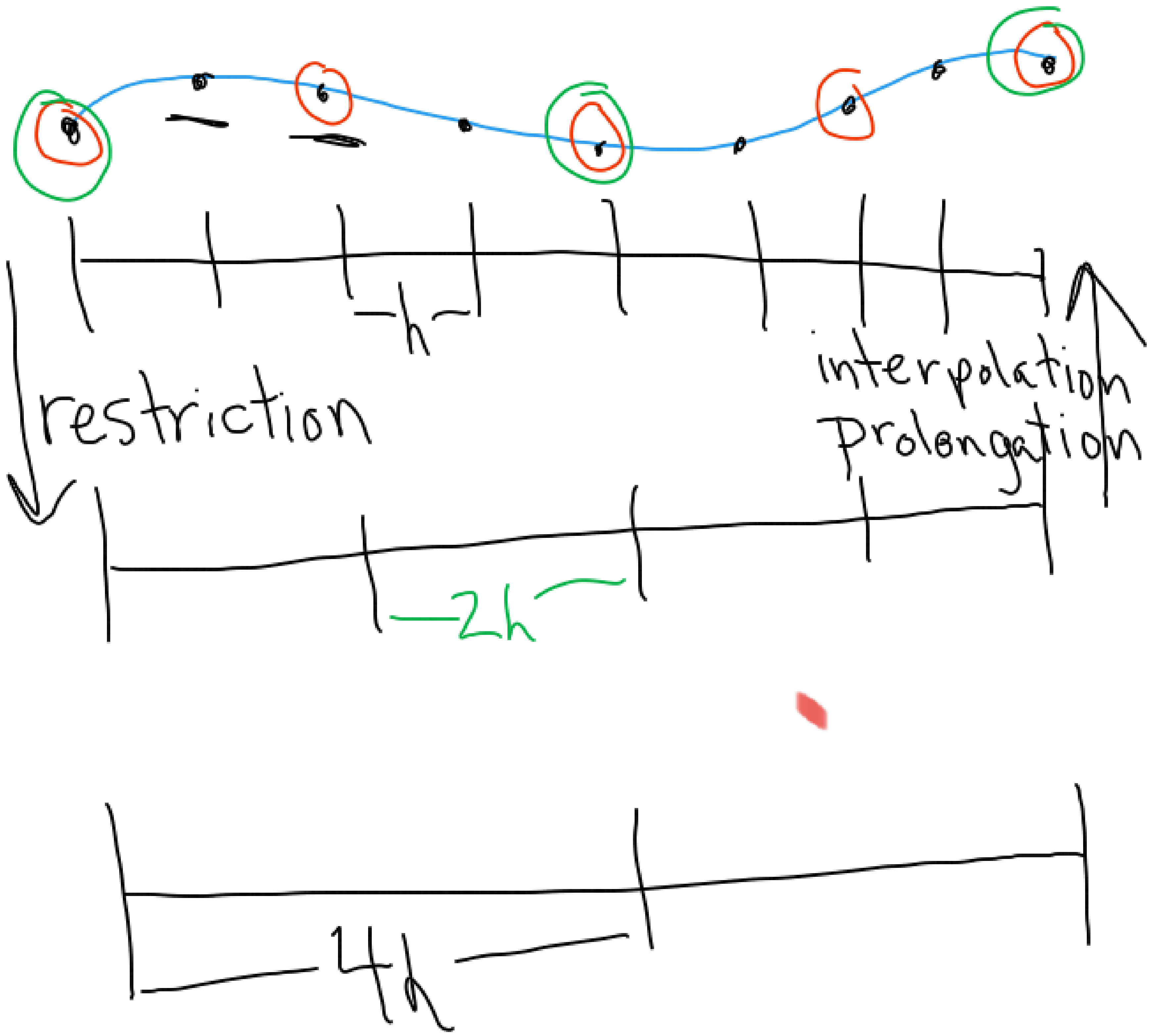
$$U^{[k+1]} = (1-\omega) U^{[k]} + \omega \hat{U}^{[k+1]}$$

Under-relaxed Jacobi (URJ)

Multigrid

1. Start with grid spacing h and iterate with URJ until high freq. errors are small.
2. Now compute on grid with spacing $2h$. Some "low" frequencies are now "high frequencies.
3. $4h$

etc.



Going back up:

$$e = U_h - U \to U = U_h - e$$

We replace U_h with $U_h - e$.

Fine grid

Solve $AU = F$

$\to U_h$

$$e = U_h - U$$

$$\boxed{r = F - AU_h}$$

$$0 = F - AU$$

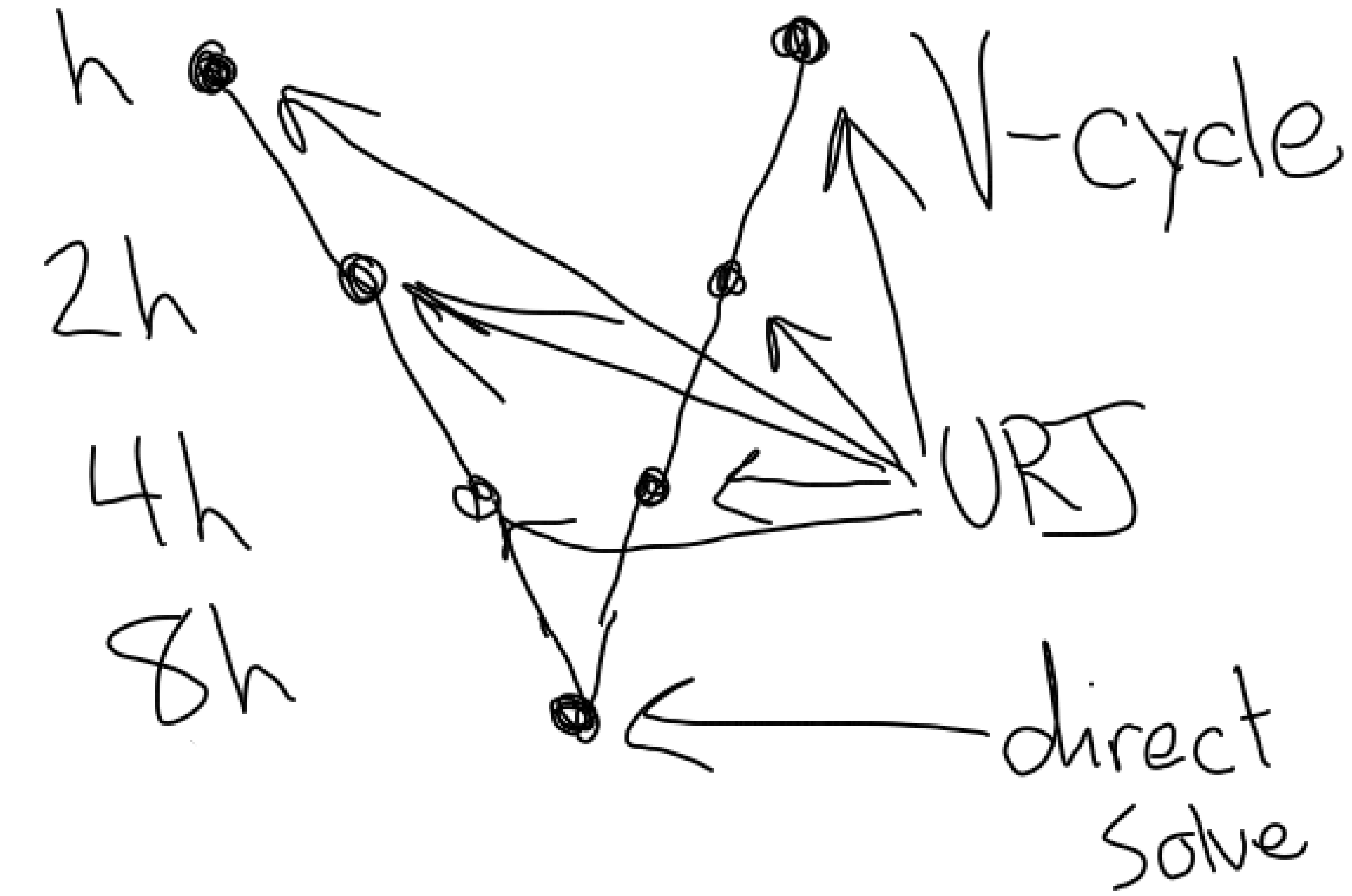
$$-r = A(U_h - U)$$

$$\boxed{Ae = -r}$$

Solve on coarser grid.

BCs: $e_0 = 0$

$e_{m+1} = 0$



How much work is this? Say URJ on a grid with m points takes Cm flops. If we do ν iterations at each step:

$$\nu Cm + \nu C\frac{m}{2} + \nu C\frac{m}{4} + \cdots$$

$$= \nu Cm\left(1 + \frac{1}{2} + \frac{1}{4} + \cdots\right)$$

$$\le 2\nu Cm \to 4\nu Cm$$

Total work: $O(m)$

Conventional solvers: $O(m^3)$ or maybe $O(m^2)$