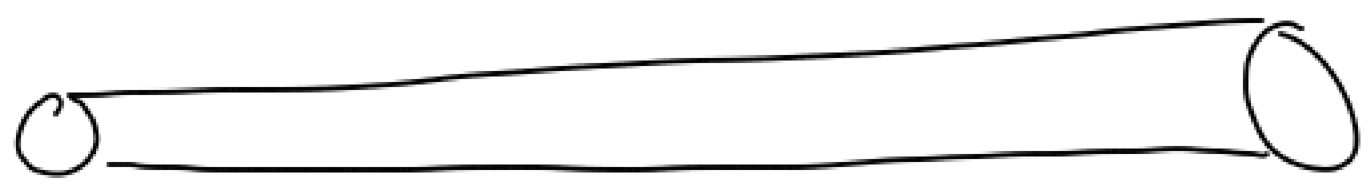
Homework

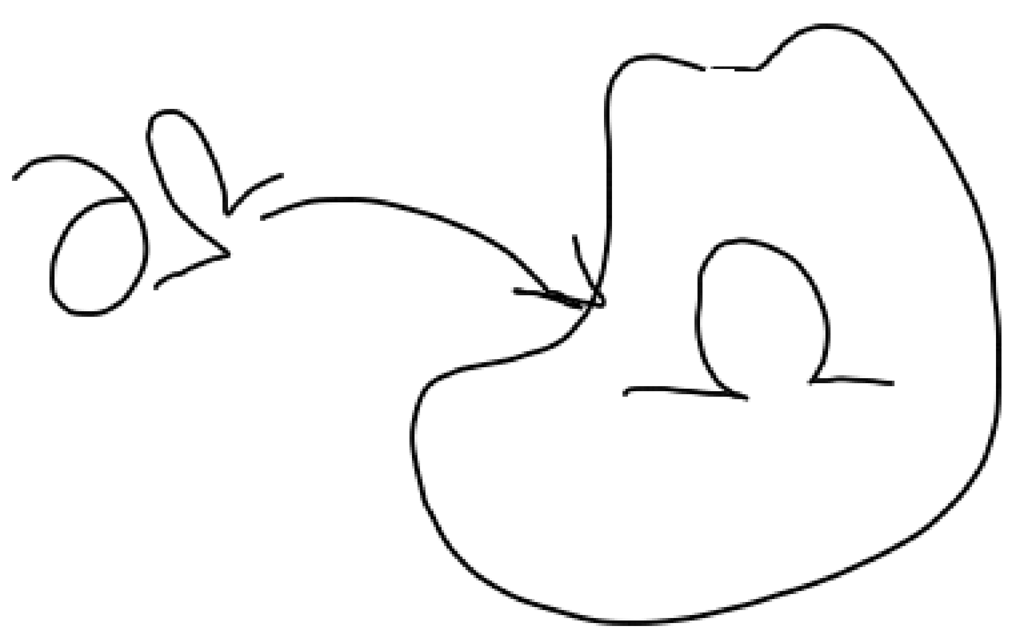
- Put all your solutions in one file
- Long blocks of code: separate file or in an appendix
- Make sure solutions are in order and labeled.

Past lectures:



Flow of heat in a rod (1D)

Today: multiple dimensions



Fick's law of diffusion:
Heat flux is proportional to the gradient ∇u

$$u_t = \nabla \cdot \left(K(x,y) \nabla u \right) + \psi(x,y)$$

$u(x,y,t)$

If $K(x,y) = K$:

$$u_t = K \nabla^2 u + \psi(x,y) \quad \text{Poisson's Eqn.}$$

Steady state: $\boxed{\nabla^2 u = \frac{-\psi(x,y)}{K} = f(x,y)}$

If $f=0$: $\nabla^2 u = 0$ Laplace's eqn.

$$\nabla^2 u = f(x,y)$$

$$u = g(x,y) \quad (x,y) \in \partial\Omega$$

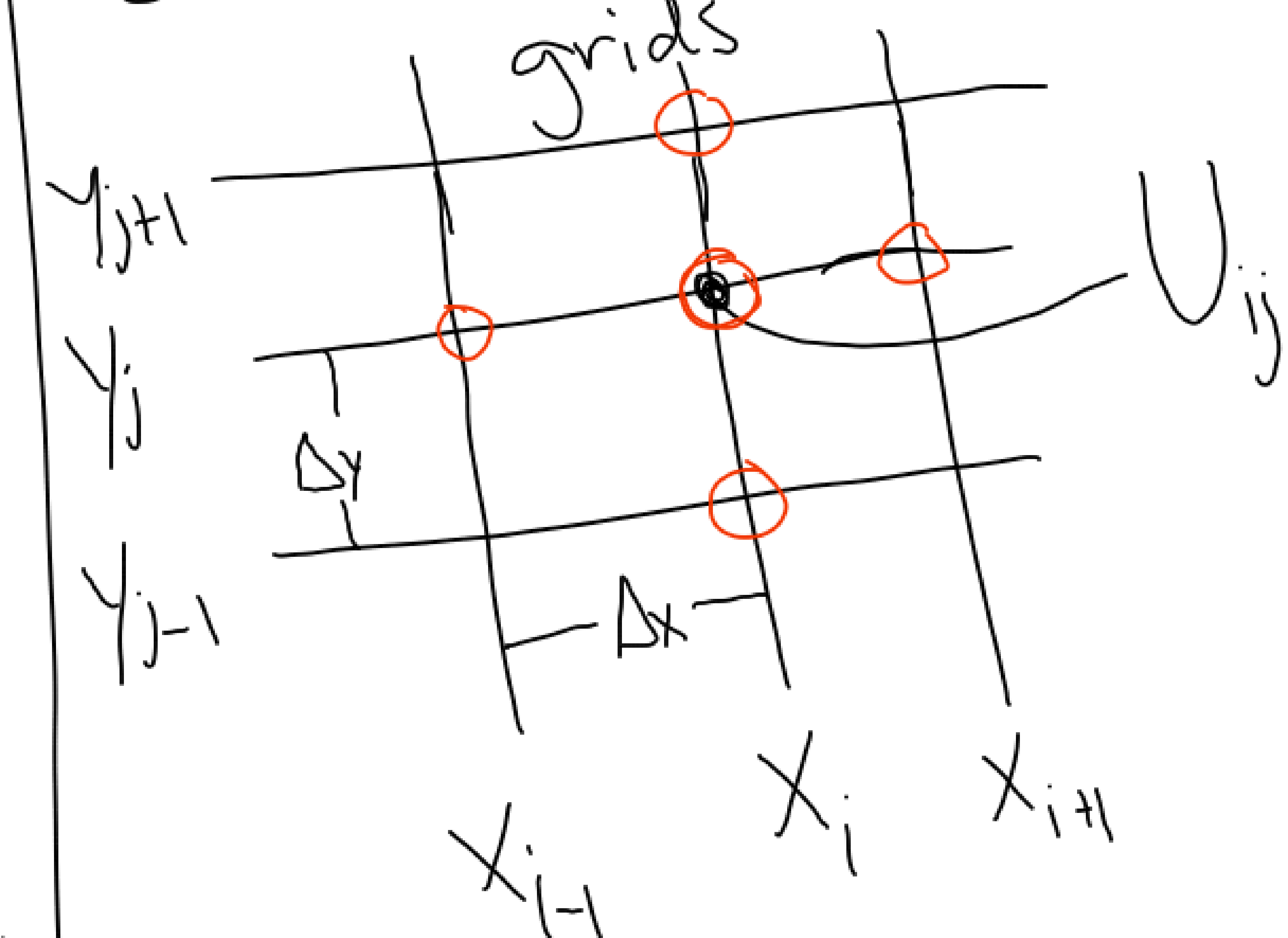
BVP (elliptic PDE)

Applications:

- Heat u = Temperature, $-f$: Heat source
- u = grav. potential, f = mass
- u = electrical potential, f = charge

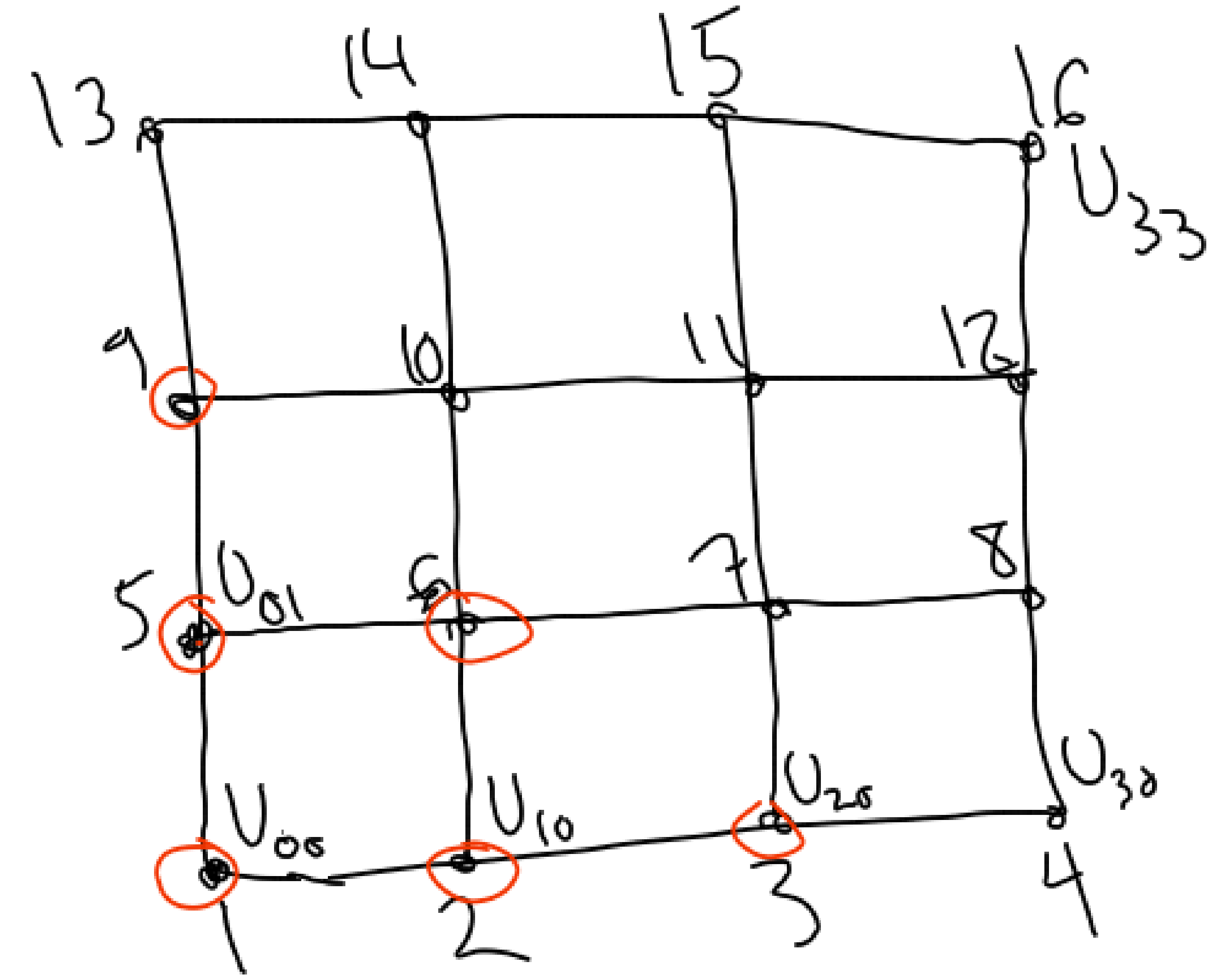
Discretization in 2D

Structured vs. unstructured grids



$$u_{xx} + u_{yy} = f(x,y)$$

$$U_{xx}(x_i,y_j) \approx \frac{U_{i+1,j} - 2U_{ij} + U_{i-1,j}}{(\Delta x)^2}$$



$$U = [U_{00}, U_{10}, U_{20}, U_{30}, U_{01}, \ldots U_{33}]^T$$

$$u_{yy}(x_i,y_j) \approx \frac{U_{i,j-1} - 2U_{ij} + U_{i,j+1}}{(\Delta y)^2}$$

Let $\Delta x = \Delta y = h$

$$\frac{1}{h^2}\left[U_{i+1,j} + U_{i-1,j} + U_{i,j+1} + U_{i,j-1} - 4U_{ij}\right] = f(x_i,y_j) \leftarrow AU = F$$

If $f=0$: $U_{ij} = \frac{1}{4}\left(U_{i+1,j} + U_{i-1,j} + U_{i,j+1} + U_{i,j-1}\right)$

What does A look like?

$$A = \frac{1}{h^2}\begin{array}{l} \\ \text{row 6} \\ \\ \end{array}\begin{bmatrix} & & & & & & & & & & & & \\ 0 & 1 & 0 & 0 & 1 & -4 & 1 & 0 & 0 & 1 & 0 & \cdots & 0 \\ & & & & & & & & & & & & \end{bmatrix}$$

Sparse

If our grid is $m \times m$ then

A is $m^2 \times m^2$ but has only $5m^2$ non-zero entries

Toeplitz matrix:

A matrix where all entries on each diagonal are equal.

$$\begin{bmatrix} 1 & 2 & \cdots & \\ 0 & \ddots & \ddots & 2 \\ & \ddots & 0 & 1 \end{bmatrix}$$

1D FD disc.

Circulant matrix:

Like Toeplitz, but diagonals "wrap around":

$$\begin{bmatrix} 1 & 2 & 3 & \cdots & 0 \\ 0 & \ddots & \ddots & \ddots & 3 \\ & \ddots & \ddots & \ddots & 2 \\ 3 & & \ddots & 0 & 1 \\ 2 & 3 & & & \end{bmatrix}$$

1D FD disc. with periodic BCs.

Block Toeplitz matrix:

$$\begin{bmatrix} A & C & & \\ B & A & C & \\ & B & A & C \\ & & B & A \end{bmatrix}$$

Each block diagonal consists only of copies of a given block

Multi-dimensional FD discretizations.

$AU = F$

$AE = -\underset{\uparrow}{\tau}$ $\quad E = U - \hat{U}$

$\mathcal{O}(h^2)$ $\quad$ exact solution vector

Stability requires that $\|A^{-1}\| < C$ as $h \to 0$

$$\|A^{-1}\|_2 = \frac{1}{2\pi^2} + \mathcal{O}(h^2) \text{ as } h \to 0$$

So this method is stable and convergent.