PDE Convergence and Stability

Homework 5 due April 18th

$u_t = u_{xx}$ → "infinitely stiff"

semi-discrete: $U'(t) = AU(t)$ → finite stiffness $|\lambda_{max}| \approx \frac{4}{h^2}$

Absolute stability will be necessary for convergence.

Forward Euler: $U^{n+1} = U^n + kAU^n = \underbrace{(I+kA)}U^n$

Trapezoidal: $U^{n+1} = U^n + \frac{k}{2}A(U^n + U^{n+1})$

$\Rightarrow U^{n+1} = \underbrace{(I-\frac{k}{2}A)^{-1}(I+\frac{k}{2}A)}U^n$

$$A_h = \frac{1}{h^2}\begin{bmatrix} -2 & 1 & & \\ 1 & \ddots & \ddots & \\ & \ddots & \ddots & 1 \\ & & 1 & -2 \end{bmatrix}$$

Both methods yield an iteration of the form $U^{n+1} = \underbrace{R(kA_h)}_{B_{k,h}}U^n$

$$\boxed{U^{n+1} = B_{k,h}U^n}$$

$U^n = (B_{k,h})^n U^0$

Exact solution vector:

$u^n = [u(x_1,t^n), u(x_2,t^n), \ldots, u(x_m,t^n)]^T$

$$\boxed{u^{n+1} = B_{k,h}u^n + k\tau^n}$$

$U^{n+1} - u^{n+1} = B_{k,h}(U^n - u^n) - k\tau^n$

$E^{n+1} = B_{k,h}E^n - \underline{k\tau^n}$

$$E^n = B_{k,h}^n E^0 - k B_{k,h}^{n-1} \tau^0 - k B^{n-2} \tau^1 - \cdots \; k \tau^{n-1}$$

$$\boxed{E^n = B_{k,h}^n E^0 - k \sum_{j=0}^{n-1} B_{k,h}^{n-1-j} \tau^j}$$

$$N = \frac{T}{k} \leftarrow \text{final time}$$

Dfn. We say a discretization

$$U^{n+1} = BU^n \quad (*)$$

is Lax-Richtmeyer stable if $\exists$ $C(T)$ independent of k, h such that

$$\|B^n\| < C(T) \quad \forall n.$$

Theorem. A consistent discretization $(*)$ is convergent if and only if it is Lax-Richtmeyer stable.

__Proof.__ Assume $\|B_{k,h}^n\| < C(T)$ for all k, h, n, and that $\overbrace{\boxed{\lim_{k,h\to} \tau^n = 0}}^{\text{Consistency}}$ and $\boxed{\lim_{k,h\to 0} \|E^0\| = 0}$

$$\|E^N\| \le \|B_{k,h}^N\| \cdot \|E^0\| + k\sum_{j=0}^{N-1} \|B^{N-1-j}\| \cdot \|\tau^j\|$$

$$\|E^N\| \le C(T)\|E^0\| + \underbrace{kN}_{T}\, C(T) \max_j \|\tau^j\|$$

$$\lim_{k,h\to 0} \|E^N\| \le \lim_{k,h\to 0}\left(C(T)\underline{\|E^0\|} + T\,C(T)\max_j \underline{\|\tau^j\|} \right) = 0$$

What does $\|B_{k,h}^n\|_2 < C(T)$ imply?

Forward Euler: $B_{k,h} = I + kA_h$

Eigenvalues of A_h: $\lambda_p = \frac{2}{h^2}(\cos(p\pi h) - 1)$

$p = 1, \ldots, m$

$h = \frac{1}{m+1}$

$$\|(I + kA_h)^n\|_2 < C(T)$$

$$\Downarrow$$

$$|1 + k\lambda_p|^n < C(T)$$

$$\Longleftrightarrow$$

Abs. Stability $\rightarrow$ $|1 + k\lambda_p| \leq 1$

Since $\lambda_p \in \left[-\frac{4}{h^2}, 0\right]$ $\Rightarrow$ A sufficient condition:

$$-\frac{4}{h^2} \leq \lambda_p \leq 0$$

$$1 - \frac{4k}{h^2} \leq 1 + k\lambda_p \leq 1$$

So we need: $1 - \frac{4k}{h^2} \geq -1$

$$-\frac{4k}{h^2} \geq -2$$

$$\frac{k}{h^2} \leq \frac{1}{2} \Rightarrow k \leq \frac{h^2}{2}$$

Trapezoidal method:

$$U^{n+1} = \underbrace{\left(I - \frac{k}{2}A_h\right)^{-1}\left(I + \frac{k}{2}A_h\right)}_{B_{k,h}} U^n$$

$$\|B_{k,h}^n\|_2 < C(T)$$

$$\left|\frac{1+\frac{k}{2}\lambda_p}{1-\frac{k}{2}\lambda_p}\right|^n < C(T)$$

$$\Longleftrightarrow$$

$$\left|\frac{1+\frac{k}{2}\lambda_p}{1-\frac{k}{2}\lambda_p}\right| \leq 1$$

Since $\lambda_p < 0$, for numerator is always smaller than denominator.

So we can choose k independent of h.

This will be true for any A-stable method.

Von Neumann Stability analysis

Given a linear PDE that is first order in time:

$$u_t = F(u, u_x, u_{xx}, \ldots)$$

$-\infty < x < \infty$

We introduce the ansatz

$$\boxed{u(x,t) = \hat{u}(t)e^{i\xi x}}$$

eigenfunction of $\frac{\partial}{\partial x}$

$$\frac{\partial}{\partial x} e^{i\xi x} = i\xi e^{i\xi x}$$

Example: $u_t = u_{xx}$

$$\hat{u}'(t)\cancel{e^{i\xi x}} = \hat{u}(t)(-\xi^2)\cancel{e^{i\xi x}}$$

$$\hat{u}'(t) = -\xi^2 \hat{u}(t) \Rightarrow \boxed{\hat{u}(t) = e^{-\xi^2 t}\hat{u}(t)}$$

Dispersion relation

$$u(x,t) = e^{-\xi^2 t} e^{i\xi x}$$

We can represent any solution in terms of these solutions

$$\hat{u}_0(\xi) = \int_{-\infty}^{\infty} u(x,0) e^{-i\xi x}\, dx$$

$$u(x,t) = \int_{-\infty}^{\infty} \hat{u}_0(\xi) e^{-\xi^2 t} e^{i\xi x}\, d\xi$$

Von Neumann analysis

$u_t = u_{xx}$

$0 \le x \le 1$

$u(0,t) = u(1,t)$

$u_x(0,t) = u_x(1,t)$

$x_j = hj \quad j = 0, 1, \ldots, m$

Forward Euler + C.D.:

$$U_j^{n+1} = U_j^n + \frac{K}{h^2}\left(U_{j+1}^n - 2U_j^n + U_{j-1}^n\right)$$

Ansatz: $U_j^n = \boxed{g(\xi)^n e^{i\xi hj}}$

Substitute: $$g(\xi)^{n+1} e^{i\xi hj} = g(\xi)^n \left[e^{i\xi hj} + \frac{K}{h^2}\left(e^{i\xi h(j+1)} - 2e^{i\xi hj} + e^{i\xi h(j-1)}\right)\right]$$

$$\boxed{g(\xi) = 1 + \frac{K}{h^2}\left(e^{i\xi h} - 2 + e^{-i\xi h}\right)}$$

$$g(\xi) = 1 + \frac{K}{h^2}\left(2\cos(\xi h) - 2\right) = 1 + 2\frac{K}{h^2}\left(\cos(\xi h) - 1\right)$$

Stability: $|g(\xi)| \le 1$

$$\left|1 + 2\frac{k}{h^2}\underline{(\cos(\xi h) - 1)}\right| \le 1$$

$\cos(\xi h) = -1$:

$$\left|1 - 4\frac{k}{h^2}\right| \le 1 \quad \text{requires} \quad 1 - 4\frac{k}{h^2} \ge -1$$

$$\Rightarrow k \le \frac{h^2}{2}$$

With periodic BCs:

$$A_h = \frac{1}{h^2}\begin{bmatrix} -2 & 1 & & & 1 \\ 1 & \ddots & \ddots & & \\ & \ddots & \ddots & \ddots & \\ & & \ddots & \ddots & 1 \\ 1 & & & 1 & -2 \end{bmatrix}$$

Circulant matrix

Every Circulant matrix has the same eigenvectors.

They have entries

$$e^{i\xi h j}$$

(discrete Fourier modes)

See appendix E of LeVeque.