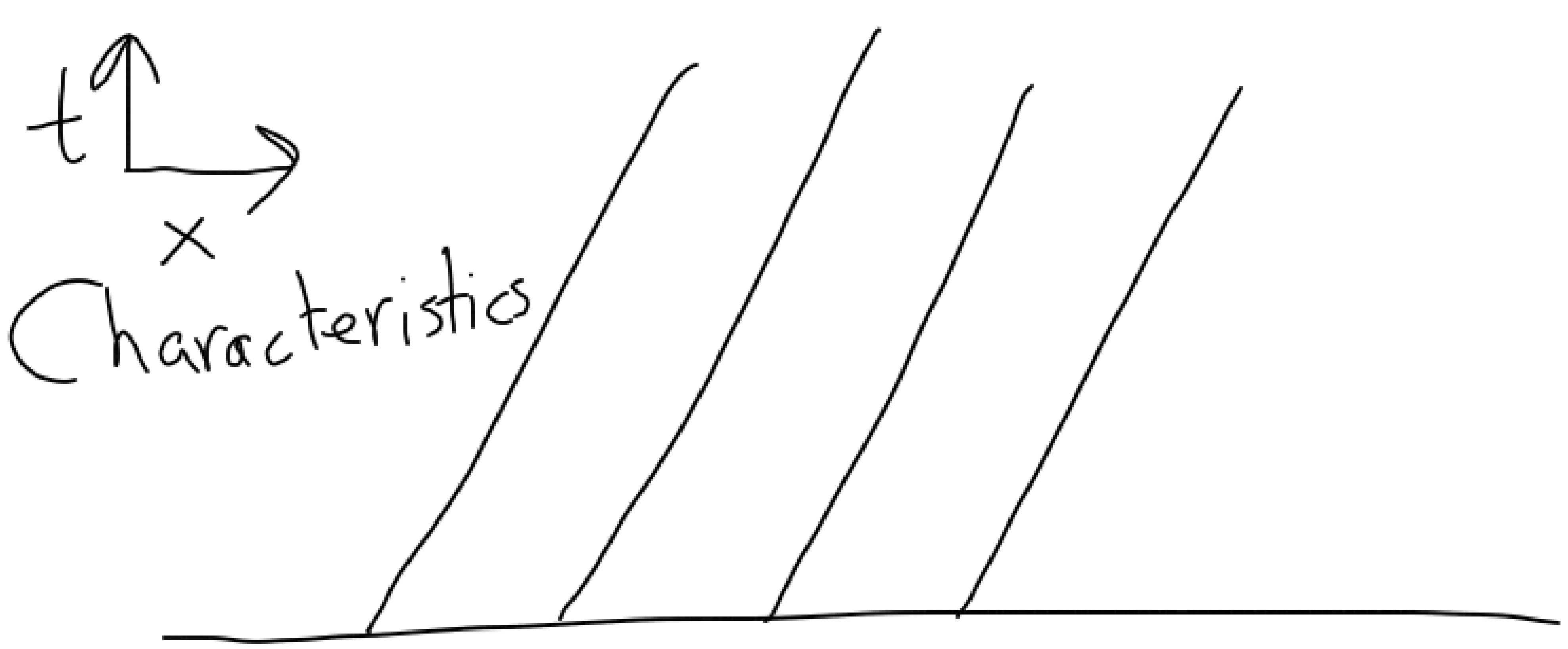
$$u_t + a u_x = 0 \qquad a > 0$$
$$-\infty < x < \infty$$
$$u(x,0) = \eta(x)$$

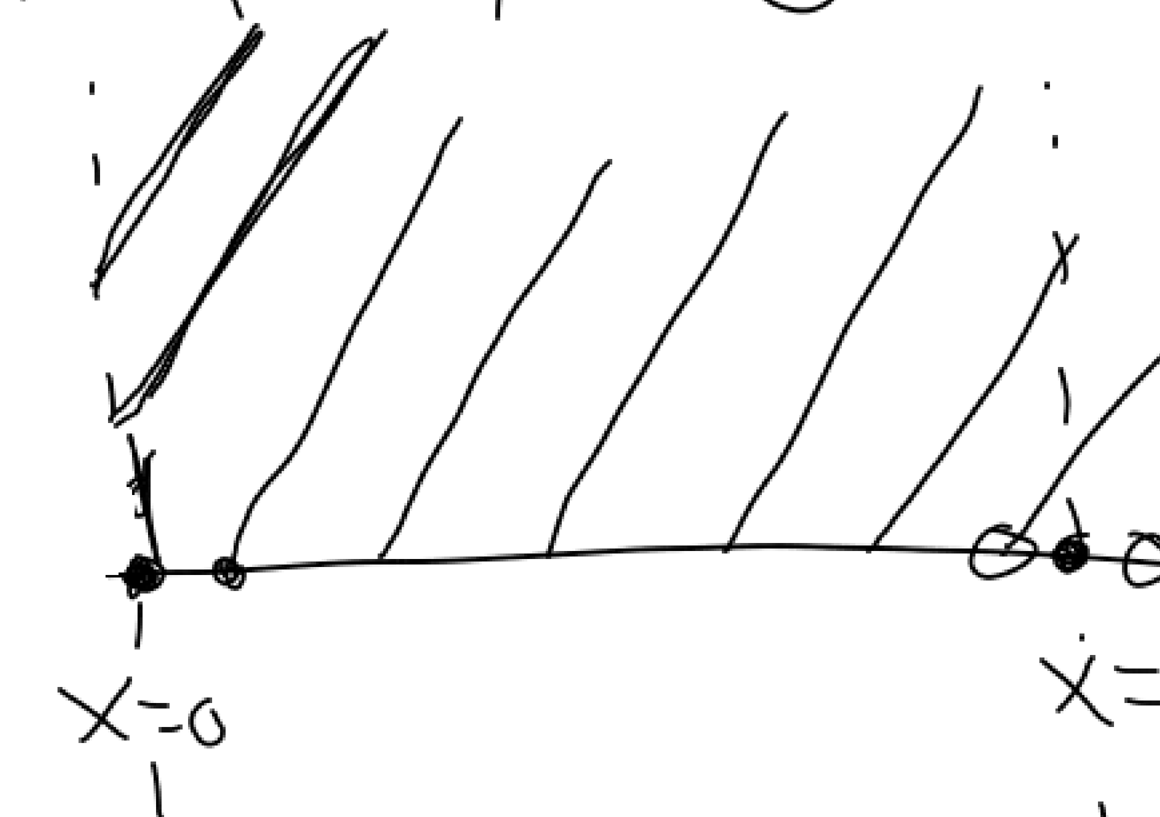
Solution: $u(x,t) = \eta(x - at)$

Constant along lines $x = x_0 + at$

Characteristics



What if $0 \le x \le 1$?



Need BC only at left edge: $u(0,t) = g(t)$

Upwind Method

$$U_j^{n+1} = U_j^n - \frac{ka}{h}\left(U_j^n - U_{j-1}^n\right)$$

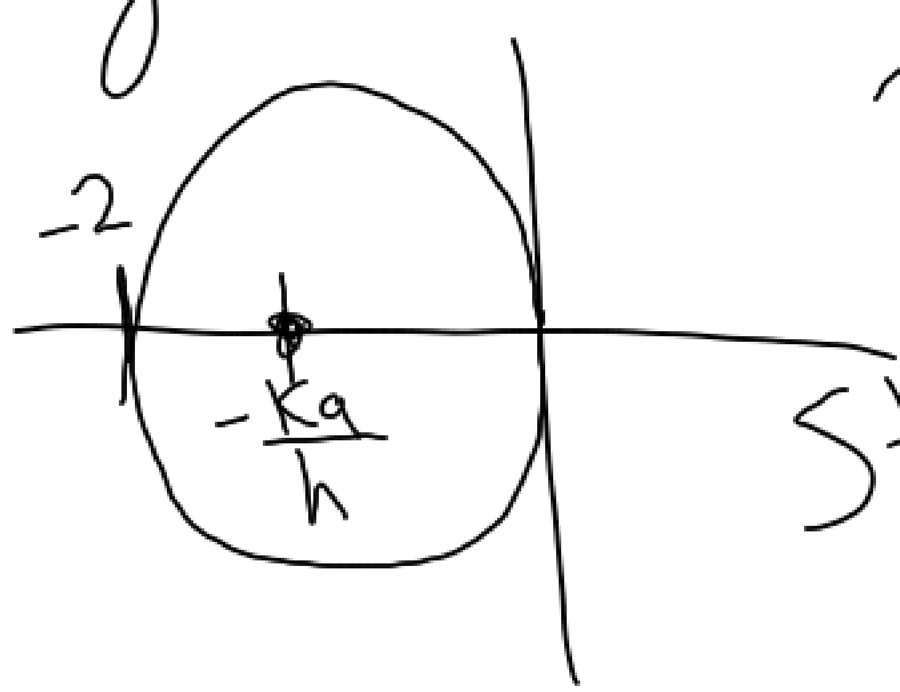
$$\frac{U_j^{n+1} - U_j^n}{k} = -\frac{a}{h}\left(U_j^n - U_{j-1}^n\right)$$

Semi-discrete: $U'(t) = \underbrace{-\frac{a}{h}\begin{bmatrix} 1 & & & & -1 \\ -1 & 1 & & & \\ & \ddots & \ddots & & \\ & & \ddots & \ddots & \\ & & & -1 & 1 \end{bmatrix}}_{A} U$

Periodic BC

Eigenvalues of A: $\lambda = \frac{-a}{h}$

$\lambda k = -\frac{ka}{h}$



Stability: $-\frac{ka}{h} \geq -2 \;\rightarrow\; \boxed{\frac{ka}{h} \leq 2}$

Von Neumann

$$U_j^{n+1} = \underline{U_j^n} - \underline{\frac{ka}{h}\left(U_j^n - U_{j-1}^n\right)}$$

$$U_j^n = g^n e^{ijh\xi}$$

$$g^{n+1} e^{ijh\xi} = g^n e^{ijh\xi} - \frac{ka}{h}\left(g^n e^{ijh\xi} - g^n e^{i(j-1)h\xi}\right)$$

$$g = 1 - \frac{ka}{h}\left(1 - e^{-ih\xi}\right)$$

$$\nu = \frac{ka}{h}$$

(CFL Number)

$$g = \underbrace{1 - \nu + \nu\cos(h\xi)}_{Re(g)} - i\underbrace{\nu\sin(h\xi)}_{Im(g)}$$

$$|g|^2 = (Re(g))^2 + (Im(g))^2$$

$$|g|^2 = (1 - \nu + \nu\cos(h\xi))^2 + \nu^2\sin^2(h\xi)$$

$$|g|^2 = 1 + \underline{\nu^2} + \underline{\nu^2\cos^2(h\xi)} - 2\nu + 2\nu\cos(h\xi) - \nu^2\cos(h\xi) + \underline{\nu^2\sin^2(h\xi)}$$

$$|g|^2 = 1 + \underline{2\nu^2 - 2\nu + (2\nu - \nu^2)\cos(h\xi)}$$

$$|g|^2 = 1 - \boxed{2\nu(1-\nu)(1-\cos(h\xi))}$$

Claim: $\underline{|g|^2 \leq 1}$ iff $\underline{0 \leq \nu \leq 1}$

Observe: $0 \le \nu(1-\nu) \le \frac{1}{4}$ for $\nu \in [0,1]$

$$0 \le 2\nu(1-\nu) \le \frac{1}{2}$$

Also: $0 \le 1 - \cos(h\xi) \le 2$

Multiply: $0 \le 1 - |g|^2 \le 2$

$$-1 \le -|g|^2 \le 1$$

$$|g|^2 \le 1$$

So we need $0 \le \frac{ka}{h} \le 1$

Normal matrices

Dfn. A square matrix is normal if it is unitarily diagonalizable.

i.e. if it has a complete set of orthogonal eigenvectors.

$A = R \Lambda R^{-1}$ (eigenvalue decomposition)

$$A^n = \underbrace{R \Lambda R^{-1} R \Lambda R^{-1} \cdots R \Lambda R^{-1}}_{n \text{ times}}$$

$$A^n = R \Lambda^n R^{-1} \Rightarrow \|A^n\|_2 \le \|R\| \|\Lambda^n\| \|R^{-1}\|$$

$$\|\Lambda^n\| = \max_j |\lambda_j|^n = (\rho(A))^n$$

$\rho(A)$ = spectral radius

$$\|R\| \cdot \|R^{-1}\| = K(R) \quad \text{(condition number)}$$

$$\|A^n\| \le (\rho(A))^n K(R)$$

If A is normal: $K(R) = 1$

$$\|A^n\|_2 = (\rho(A))^n$$

For non-normal matrices,

$$\|A\|_2 > \rho(A)$$

Upwind method:

$$U^{n+1} = (I + kA)U^n$$

$$U^n = \underbrace{(I + kA)^n U^0}$$

This can grow if

$$\|I + kA\|_2 > 1$$

If $I + kA$ is non-normal, we can have that $\underline{\rho(I+kA) < 1}$ but $\underline{\|I + kA\| > 1}$.

This occurs when we take $1 < \nu \le 2$ in the non-periodic case.

$$\begin{bmatrix} 1 & & & \\ -1 & \ddots & & \\ & \ddots & \ddots & \\ & & -1 & 1 \end{bmatrix}$$

Highly non-normal

$$\begin{bmatrix} 1 & & & -1 \\ -1 & \ddots & & \\ & \ddots & \ddots & \\ & & -1 & 1 \end{bmatrix}$$

Normal

Symmetric, skew-symmetric, and circulant matrices are normal.

Take home message:

For non-normal matrices, eigenvalues don't tell the whole story.

Easy way to check normality:

If $A^*A = AA^*$ then A is normal.

Modified equation analysis

Goal: find a PDE that our numerical solution satisfies exactly.

$$U_j^n = V(x_j, t_n)$$

Upwind: $\dfrac{U_j^{n+1} - U_j^n}{k} = -a \dfrac{U_j^n - U_{j-1}^n}{h}$

Substitute: $\dfrac{V(x_j, t_{n+1}) - V(x_j, t_n)}{k} = -a \dfrac{V(x_j, t_n) - V(x_{j-1}, t_n)}{h}$

$$V(x_j, t_{n+1}) = V(x_j, t_n) + kV_t + \frac{k^2}{2} V_{tt} + O(k^3)$$

$$V(x_{j-1}, t_n) = V(x_j, t_n) - hV_x + \frac{h^2}{2} V_{xx} + O(h^3)$$

$$\frac{V + kV_t + \frac{k^2}{2} V_{tt} - V}{k} + O(k^2)$$

$$= -a \frac{V - V + hV_x - \frac{h^2}{2} V_{xx}}{h} + O(h^3)$$

$$V_t + \frac{k}{2} V_{tt} + O(k^2) = -aV_x + a\frac{h}{2} V_{xx} + O(h^2)$$

$$V_t + aV_x = \frac{1}{2}\left(ahV_{xx} - kV_{tt}\right) + O(h^2, k^2)$$

$$V_t = -aV_x + O(h, k)$$

$$V_{tx} \approx -aV_{xx} \qquad V_{tt} \approx -aV_{xt}$$

$$V_{tt} \approx a^2 V_{xx}$$

$$V_t + aV_x = \frac{1}{2}\left(ahV_{xx} - ka^2 V_{xx}\right) + O(h^2, k^2, kh)$$

$$V_t + aV_x = \frac{a}{2} V_{xx}(h - ka) + O(h^2, k^2, kh)$$

$$V_t + aV_x = \frac{ah}{2} V_{xx}\Big(1 - \underbrace{\frac{ka}{h}}_{\nu}\Big) + O(h^2, k^2, kh)$$

Dominant error is diffusion

If $\nu = 1$, this term vanishes.

$$U_j^{n+1} = \underline{U_j^n} - \frac{ka}{h}\left(\underline{U_j^n} - U_{j-1}^n\right)$$

Set $\frac{ka}{h} = 1$:

$$U_j^{n+1} = U_{j-1}^n \quad \text{Exact solution}$$