$$u''(x) = f(x) \quad 0 \le x \le 1$$

$$u(0) = \alpha \qquad u(1) = \beta$$

$$\frac{1}{h^2}\left(U_{i-1} - 2U_i + U_{i+1}\right) = f(x_i) \quad 1 \le i \le m$$

$$\underline{U_0 = \alpha} \qquad \underline{U_{m+1} = \beta}$$

$$x_0 = 0 \qquad x_{m+1} = 1$$

$$\frac{1}{h^2}\begin{bmatrix} h^2 & & & & & \\ 1 & -2 & 1 & & & \\ & \ddots & \ddots & \ddots & & \\ & & \ddots & \ddots & \ddots & \\ & & & 1 & -2 & 1 \\ & & & & & h^2 \end{bmatrix} \begin{bmatrix} U_0 \\ U_1 \\ \vdots \\ \vdots \\ U_m \\ U_{m+1} \end{bmatrix} = \begin{bmatrix} \alpha \\ f(x_1) \\ \vdots \\ \vdots \\ f(x_m) \\ \beta \end{bmatrix}$$

Now we'll eliminate U_0, U_1.

$$\frac{1}{h^2}\begin{bmatrix} -2 & 1 & & & \\ 1 & \ddots & \ddots & & \\ & \ddots & \ddots & \ddots & \\ & & \ddots & \ddots & 1 \\ & & & 1 & -2 \end{bmatrix} \begin{bmatrix} U_1 \\ \vdots \\ \vdots \\ \vdots \\ U_m \end{bmatrix} = \begin{bmatrix} f(x_1) - \frac{\alpha}{h^2} \\ f(x_2) \\ \vdots \\ f(x_{m-1}) \\ f(x_m) - \frac{\beta}{h^2} \end{bmatrix}$$

$i=1$: $\dfrac{U_0-2U_1+U_2}{h^2}=f(x_1)$

$$\frac{\alpha-2U_1+U_2}{h^2}=f(x_1)$$

$$\frac{-2U_1+U_2}{h^2}=f(x_1)-\frac{\alpha}{h^2}$$

$i=m+1$: $\dfrac{U_{m-1}-2U_m+\beta}{h^2}=f(x_m)$

$$\frac{U_{m-1}-2U_m}{h^2}=f(x_m)-\frac{\beta}{h^2}$$

$U_{xx}+U_{yy}=f(x,y)$ $\quad 0\le x\le x^*$, $0\le y\le y^*$

$$\boxed{\frac{U_{i-1,j}-2U_{ij}+U_{i+1,j}}{\Delta x^2}+\frac{U_{i,j-1}-2U_{ij}+U_{i,j+1}}{\Delta y^2}=f(x_i,y_j)} \quad (*)$$

$$AU=F$$

$$\boxed{\begin{array}{l} u(x,0)=u(x,1)=0 \\ u(0,y)=u(1,y)=0 \end{array}}$$

We have $(*)$ with $1\le i,j\le m$.

Let (V,λ) be an eigenpair of A.

$$\boxed{AV=\lambda V} \qquad \lambda^{p,q} \qquad V^{p,q}$$

$$\frac{\frac{V_{i-1,j}-2V_{ij}+V_{i+1,j}}{\Delta x^2}}{} + \frac{V_{ij-1}-2V_{ij}+V_{ij+1}}{\Delta y^2} = \lambda V_{ij}$$

Assume $V_{ij}^{p,q}$ is separable:

$$\underline{V_{ij}} = R(i)S(j) = R_i S_j$$

$$\frac{R(i-1)S(j)-2R(i)S(j)+R(i+1)S(j)}{\Delta x^2}$$

$$+R(i)\frac{S(j-1)-2S(j)+S(j+1)}{\Delta y^2} = \lambda R(i)S(j)$$

Divide by $R_i S_j = V_{ij}$

$$\underbrace{\frac{R_{i-1}-2R_i+R_{i+1}}{R_i\Delta x^2}}_{=C_1} + \underbrace{\frac{S_{j-1}-2S_j+S_{j+1}}{S_j\Delta y^2}}_{=C_2} = \lambda$$

$$C_1+C_2=\lambda$$

$$\frac{R_{i-1}-2R_i+R_{i+1}=R_i\Delta_x^2 C_1}{}$$

$$R_{i+1}-\underbrace{(2+C_1\Delta_x^2)}_{2\alpha}R_i+R_{i-1}=0 \quad 1\le i\le m$$

Ansatz: $R_i=\xi^i$

$$\xi^{i+1}-2\alpha\xi^i+\xi^{i-1}=0$$

Divide by ρ^{i-1}: $\rho^2 - 2\alpha\rho + 1 = 0$

$$\boxed{\rho_\pm = \alpha \pm \sqrt{\alpha^2 - 1}}$$

General solution: $\underline{R_i = R_+\rho_+^i + R_-\rho_-^i}$

Now we need BCs: $R_0 = 0$ $R_{m+1} = 0$

$R_0 = R_+ + R_- = 0 \Rightarrow R_- = -R_+$

$\Rightarrow R_i = R_+(\rho_+^i - \rho_-^i)$

$R_{m+1} = R_+(\underline{\rho_+^{m+1} - \rho_-^{m+1}}) = 0 \Rightarrow \rho_+^{m+1} = \rho_-^{m+1}$

$$\left(\frac{\rho_+}{\rho_-}\right)^{m+1} = 1$$

$$\rho_+\rho_- = (\alpha + \sqrt{\alpha^2-1})(\alpha - \sqrt{\alpha^2-1})$$

$$= \alpha^2 - (\alpha^2 - 1) = 1$$

$$\rho_+^{2m+2} = 1 \Rightarrow \rho_+ = e^{\frac{ip\pi}{m+1}} = e^{ip\pi\Delta x}$$

$$\rho_- = e^{-ip\pi\Delta x}$$

$$\alpha = \frac{\rho_+ + \rho_-}{2} = \cos(p\pi\Delta x)$$

$$2\alpha = 2 + C_1 \Delta x^2$$

$$2\cos(p\pi\Delta x) = 2 + C_1 \Delta x^2$$

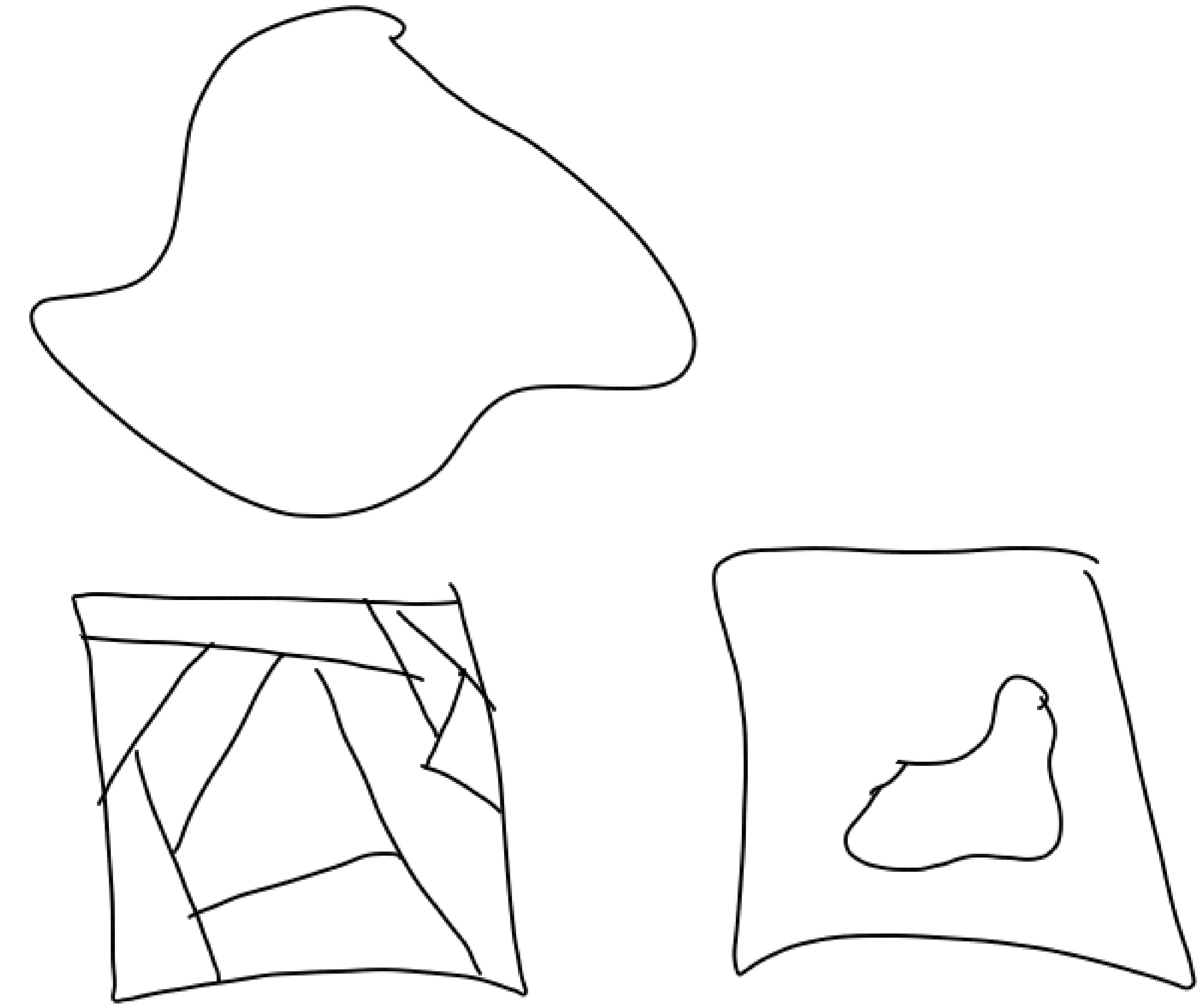
$$C_1 = 2\,\frac{\cos(p\pi\Delta x)-1}{\Delta x^2}$$

$$C_2 = 2\,\frac{\cos(q\pi\Delta y)-1}{\Delta y^2}$$

$$\lambda^{p,q} = 2\left[\underbrace{\frac{\cos(p\pi\Delta x)-1}{(\Delta x)^2}} + \underbrace{\frac{\cos(q\pi\Delta y)-1}{\Delta y^2}}_{-\pi^2}\right]$$

For $p=1$: $\rightarrow \approx -\pi^2$

Smallest $|\lambda| \approx 2\pi^2$



$$\lim_{h\to 0} \|A^{-1}\| = \left(\min|\lambda^{p,q}|\right)^{-1} = \frac{1}{2\pi^2} < C$$

$\Rightarrow$ Stability

Kronecker Product Structure of the 5-point Laplacian

$$(AU)_{ij} = \frac{1}{h^2}\left[U_{i,j-1} + U_{i,j+1} + U_{i-1,j} + U_{i+1,j} - 4U_{ij}\right] \approx u_{xx}(x_i,y_j) + u_{yy}(x_i,y_j)$$

5-pt. Laplacian

Let $T^{m\times m} = \text{tridiag}(1,-2,1)$

Then

Kronecker product

Kronecker sum

$$A = \frac{1}{h^2}\left(\underline{I^{m\times m} \otimes T} + \underline{T \otimes I^{m\times m}}\right) = \frac{1}{h^2}\left(T \oplus T\right)$$

Dfn. of Kronecker product

$$B \otimes C = \begin{bmatrix} b_{11}C & & b_{1m}C \\ \vdots & & \\ b_{m1}C & \cdots & b_{mm}C \end{bmatrix}$$

$$I \otimes T = \begin{bmatrix} \boxed{T} & 0 & & \\ 0 & \boxed{T} & & \\ & & T & \\ & & & \ddots & \\ & & & & T \end{bmatrix} \qquad T \otimes I = \begin{bmatrix} -2I & I & & & \\ I & \ddots & \ddots & & \\ & \ddots & \ddots & \ddots & \\ & & \ddots & \ddots & I \\ & & & I & -2I \end{bmatrix}$$

Let $\sigma(B) = \{\lambda_1, \ldots \lambda_m)$

$\sigma(C) = \{\mu_1, \ldots, \mu_m)$

$$\sigma(B \otimes C) = \{\lambda_i \mu_j : 1 \leq i, j \leq m\}$$

$$\sigma(B \oplus C) = \{\lambda_i + \mu_j : 1 \leq i, j \leq m\}$$

If $\{v_1, \ldots, v_m\}$ are eigenvectors of B

$\{w_1, \ldots, w_m\}$ are " " C

Then $\{v_i \otimes w_j\}$ are " " $B \otimes C$

$\{v_i \otimes w_j\}$ are eigenvectors of $B \oplus C$

Proof is based on

$$(A \otimes B)(C \otimes D) = (AB) \otimes (CD)$$