Exercise 1 *Diffusion-reaction*

Consider the PDE

$$u_t = \kappa u_{xx} - \gamma u, \tag{Ex1a}$$

which models a diffusion with decay provided $\kappa > 0$ and $\gamma > 0$. Consider methods of the form

$$U_j^{n+1} = U_j^n + \kappa \frac{k}{2h^2}[U_{j+1}^n - 2U_j^n + U_{j-1}^n + U_{j+1}^{n+1} - 2U_j^{n+1} + U_{j-1}^{n+1}] - k\gamma[(1-\theta)U_j^n + \theta U_j^{n+1}] \tag{Ex1b}$$

where θ is a parameter. In particular, if $\theta = 1/2$ then the decay term is modeled with the same centered-in-time approach as the diffusion term and the method can be obtained by applying the Trapezoidal method to the MOL formulation of the PDE. If $\theta = 0$ then the decay term is handled explicitly. For more general reaction-diffusion equations it may be advantageous to handle the reaction terms explicitly since these terms are generally nonlinear, so making them implicit would require solving nonlinear systems in each time step (whereas handling the diffusion term implicitly only gives a linear system to solve in each time step).

(a) By computing the local truncation error, show that this method is $\mathcal{O}(k^p + h^2)$ accurate, where $p = 2$ if $\theta = 1/2$ and $p = 1$ otherwise.

(b) Using von Neumann analysis, show that this method is unconditionally stable if $\theta \geq 1/2$.

(c) Show that if $\theta = 0$ then the method is stable provided $k \leq 2/\gamma$, independent of h.

Exercise 2 *(trapezoid method for advection)*

Consider the method

$$U_j^{n+1} = U_j^n - \frac{ak}{2h}(U_j^n - U_{j-1}^n + U_j^{n+1} - U_{j-1}^{n+1}). \tag{Ex2a}$$

for the advection equation $u_t + au_x = 0$ on $0 \leq x \leq 1$ with periodic boundary conditions.

(a) This method can be viewed as the trapezoidal method applied to an ODE system $U'(t) = AU(t)$ arising from a method of lines discretization of the advection equation. What is the matrix A? Don't forget the boundary conditions.

(b) Suppose we want to fix the Courant number ak/h as $k,\ h \to 0$. For what range of Courant numbers will the method be stable if $a > 0$? If $a < 0$? Justify your answers in terms of eigenvalues of the matrix A from part (a) and the stability regions of the trapezoidal method.

(c) Apply von Neumann stability analysis to the method (Ex2a). What is the amplification factor $g(\xi)$?

(d) For what range of ak/h will the CFL condition be satisfied for this method (with periodic boundary conditions)?

(e) Suppose we use the same method (Ex2a) for the initial-boundary value problem with $u(0,t) = g_0(t)$ specified. Since the method has a one-sided stencil, no numerical boundary condition is needed at the right boundary (the formula (Ex2a) can be applied at x_{m+1}). For what range of ak/h will the CFL condition be satisfied in this case? What are the eigenvalues of the A matrix for this case and when will the method be stable?

Exercise 3 *(Exact modified equation for upwind)*

Recall the upwind method:

$$U_j^{n+1} = U_j^n - \nu \left(U_j^n - U_{j-1}^n \right)$$

We observed that this method gives the exact solution to the advection equation when $\nu = 1$. Derive the modified equation for this method to infinite order in k, h and show that it reduces to the advection equation when $\nu = 1$.