Advection

$$u_t + f(u)_x = 0 \quad \text{(conservation law)}$$

$$f(u) = au : \boxed{u_t + a u_x = 0}$$

Advection – simplest first-order hyperbolic PDE

Hyperbolic conservation laws

$$u_t + A u_x = 0 \quad (1) \qquad u(x,t) \in \mathbb{R}^n, \quad A \in \mathbb{R}^{n \times n}$$

Suppose $AR = R\Lambda$

Where Λ: diagonal with real entries. (eigenvalues of A)

R is matrix of eigenvectors of A.

Let $w = R^{-1}u$

$$u_t + R\Lambda R^{-1} u_x = 0$$

$$R^{-1}u_t + \Lambda R^{-1} u_x = 0$$

$$w_t + \Lambda w_x = 0 \leftarrow \text{system of decoupled PDEs, each equivalent to advection}$$

We say (1) is hyperbolic if A is diagonalizable with real eigenvalues.

Example: $A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ $u = \begin{bmatrix} p \\ q \end{bmatrix}$

$\lambda^2 - 1 = 0$

$\lambda = \pm 1$

$p_t + q_x = 0$ $\quad p_{tx} = -q_{xx}$

$q_t + p_x = 0$ $\quad q_{tt} = -p_{xt}$

$\Rightarrow q_{tt} = q_{xx}$ Second-order wave equation

$u_t + a u_x = 0 \quad -\infty < x < \infty$

$u(x,0) = \eta(x)$

Exact solution:

$$\boxed{u(x,t) = \eta(x - at)}$$

Check it:

$u_t = -a\eta'$

$u_x = \eta'$

$-a\eta' + a\eta' = 0$ ✓

Nonlinear hyperbolic PDEs:

(2) $u_t + f(u)_x = 0$

We say (2) is hyperbolic if $f'(u)$ is diagonalizable with real eigenvalues.

Examples

- Waves (water, acoustic, electromagnetic)
- Traffic flow
- Fluid dynamics (inviscid)

Discretizations

$$\underbrace{u_t}_{\text{Forward diff.}} + a\underbrace{u_x}_{\text{Centered difference}} = 0$$

$$\boxed{\frac{U_j^{n+1}-U_j^{n}}{k} + a\frac{U_{j+1}^{n}-U_{j-1}^{n}}{2h} = 0}$$

$$U_j^{n+1} = U_j^{n} - \frac{ka}{h}\frac{U_{j+1}^{n}-U_{j-1}^{n}}{2}$$

Von Neumann: $U_j^n = g^n e^{ijh\xi}$

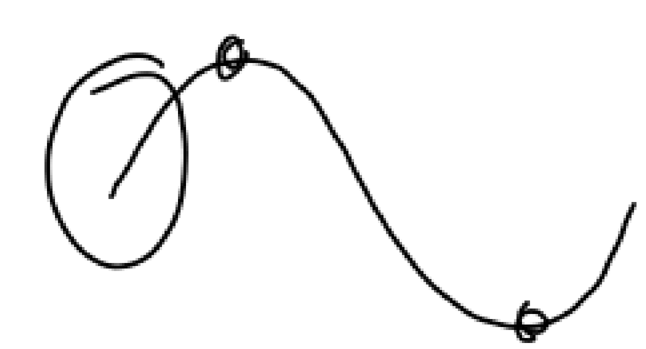
$$g^{n+1}e^{ijh\xi} = g^n\left(e^{ijh\xi} - \frac{ka}{h}\frac{e^{ih\xi(j+1)}-e^{ih\xi(j-1)}}{2}\right)$$

$$g = 1 - \frac{ka}{h}\frac{e^{ih\xi}-e^{-ih\xi}}{2}$$

Use: $e^{\pm i\theta} = \cos\theta \pm i\sin\theta$

$$g = 1 - \frac{ka}{h} i\sin(h\xi)$$

$$|g|^2 = 1 + \left(\frac{ka}{h}\right)^2 \sin^2(h\xi) > 1$$



MOL Analysis

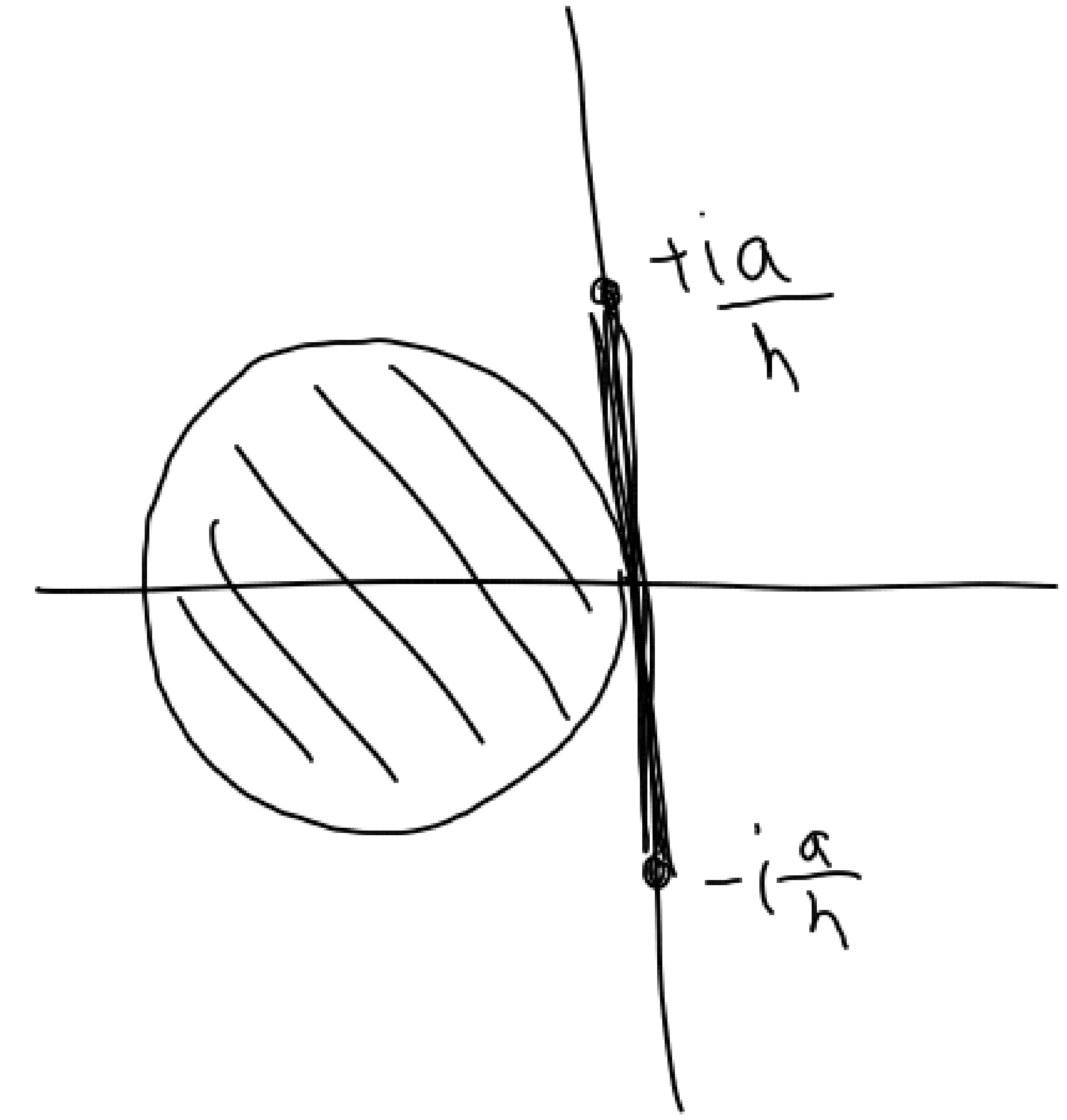
$$U'(t) = AU$$

assume periodic BCs

$$A = -\frac{a}{2h}\begin{bmatrix} 0 & 1 & & & -1 \\ -1 & 0 & 1 & & \\ & \ddots & \ddots & \ddots & \\ & & -1 & 0 & 1 \\ 1 & & & -1 & 0 \end{bmatrix}$$

circulant matrix

$$\lambda = -i\frac{a}{h}\sin(2\pi p h)$$

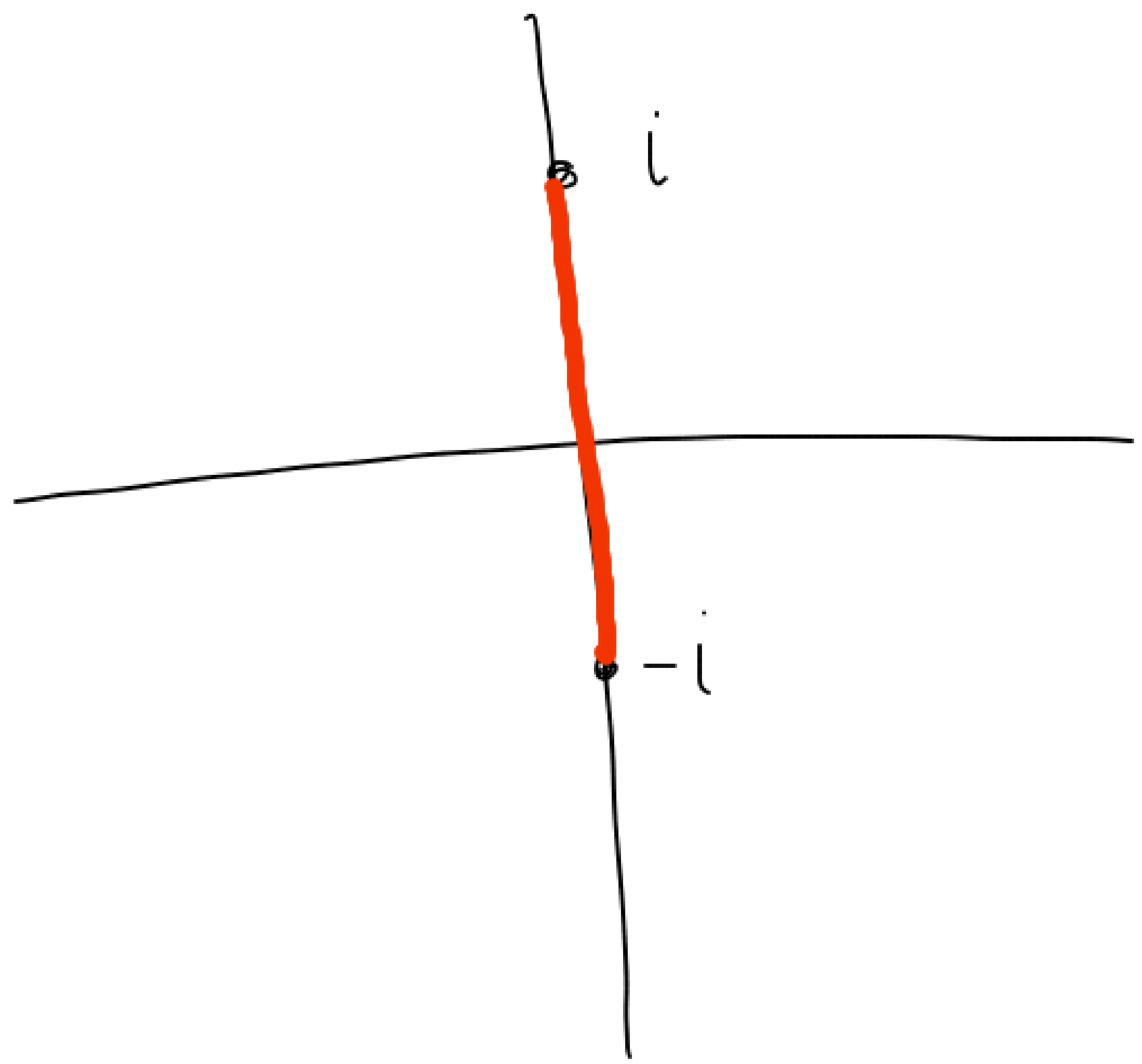


We need a method that includes part of the imaginary axis in its stability region.

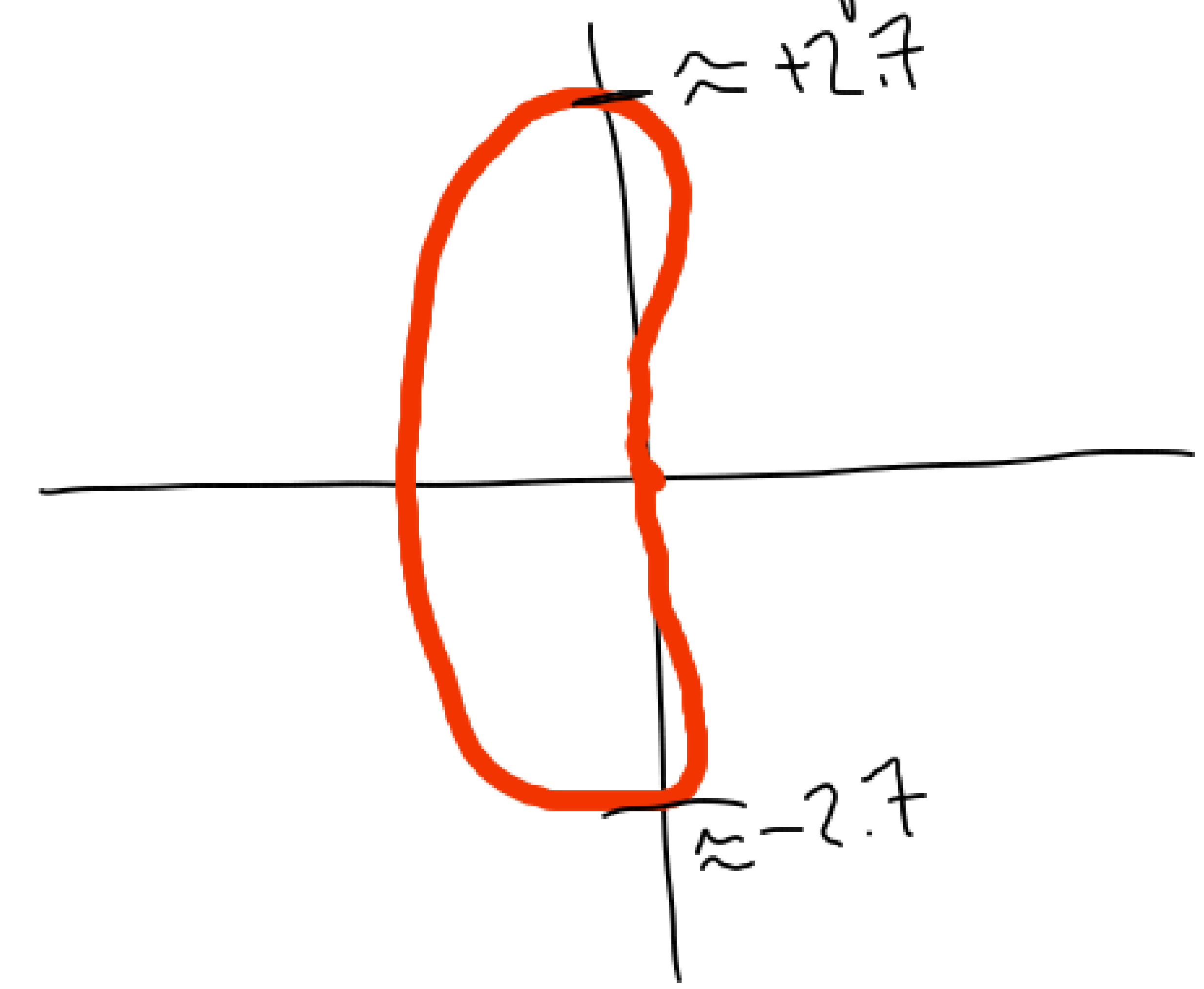
Should we use an implicit method?

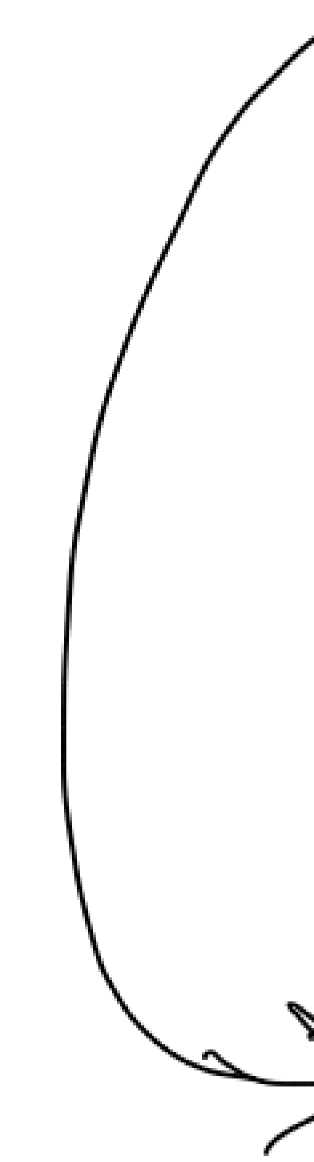
No.

Leapfrog: $U^{n+1} = U^{n-1} + 2kAU^n$



We could also use e.g. 4th-order Runge-Kutta.





We need $\left|\underbrace{k\frac{a}{h}}\right| \le 1$

$\left|k\frac{a}{h}\right| \lesssim 2.7$

$\frac{ka}{h}$ is referred to as the CFL number (Courant-Friedrichs-Lewy)

Lax-Friedrichs Method

$$U_j^{n+1} = \underbrace{\frac{U_{j+1}^n + U_{j-1}^n}{2}} - \frac{ak}{2h}\left(U_{j+1}^n - U_{j-1}^n\right)$$

$$\frac{1}{2}\left(U_{j+1} + U_{j-1}^n\right) = U_j^n + \frac{h^2}{2}\left[\frac{U_{j+1}^n - 2U_j^n + U_{j-1}^n}{h^2}\right]$$

Diffusion!

$$\Rightarrow \frac{U_j^{n+1} - U_j^n}{k} + a\frac{U_{j+1}^n - U_{j-1}^n}{2h} = \frac{h^2}{2k}\frac{U_{j+1}^n - 2U_j^n + U_{j-1}^n}{h^2}$$

Looks like a discretization of:

$$u_t + a u_x = \frac{h^2}{2k} u_{xx}$$ Advection-diffusion

$\frac{h^2}{2k} \to 0$ as $h, k \to 0$