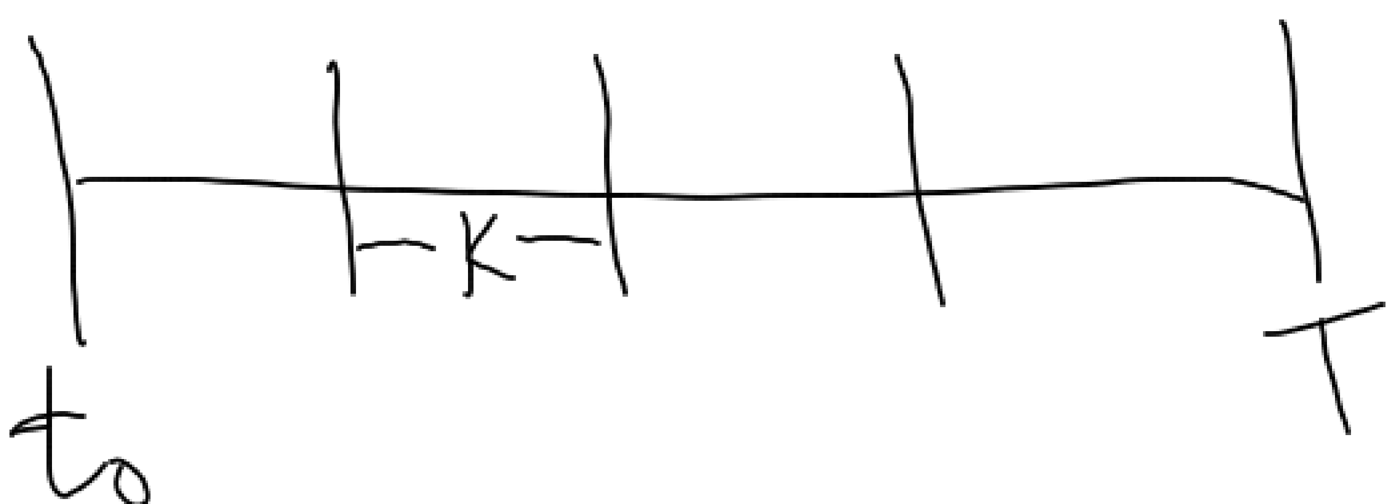
Numerical methods for initial value ODEs

$$u'(t) = f(u,t) \qquad t_0 \le t \le T$$
$$u(t_0) = \eta$$



Explicit Euler: $\dfrac{U^{n+1} - U^n}{k} = f(U^n)$

Implicit Euler: $\dfrac{U^{n+1} - U^n}{k} = f(U^{n+1})$

Truncation error:

$$\frac{u(t_{n+1}) - u(t_n)}{k} = f(u(t_n)) + \tau_n$$

$$\frac{\cancel{u(t_n)} + \cancel{ku'(t_n)} + \frac{k^2}{2}u''(t_n) + O(k^3) - \cancel{u(t_n)}}{k} = \cancel{u'(t_n)} + \tau_n$$

$$\tau_n = \frac{k}{2}u''(t_n) + O(k^2)$$

$\underline{1\text{st}}$ order accurate

One step error

$$U^{n+1} = U^n + kf(U^n)$$
$$u(t_{n+1}) = u(t_n) + kf(u(t_n)) + \underline{\mathcal{L}_n}$$
$$\mathcal{L}_n = k\tau_n \qquad \mathcal{L}_n = O(k^2)$$

We commit $\mathcal{O}(1/k)$ errors, each of size $\mathcal{O}(k^2)$.

The error at time T will be $\mathcal{O}(k)$.

(if the problem and the method are stable).

To achieve higher order, use

- Multiple derivatives
- Multiple steps
- Multiple evaluations of f (stages)

Multi-derivative methods (Taylor series)

$$u(t_{n+1}) = u(t_n + k) = u(t_n) + k u'(t_n) + \frac{k^2}{2} u''(t_n) + \mathcal{O}(k^3)$$

$$U^{n+1} = U^n + k f(U^n) + \frac{k^2}{2} \underline{\underline{f_u(U^n)}} f(U^n)$$

$$\frac{d}{dt} f(u(t)) = \frac{\partial f}{\partial u} \frac{du}{dt} = \underline{f_u(u(t)) f(u(t))}$$

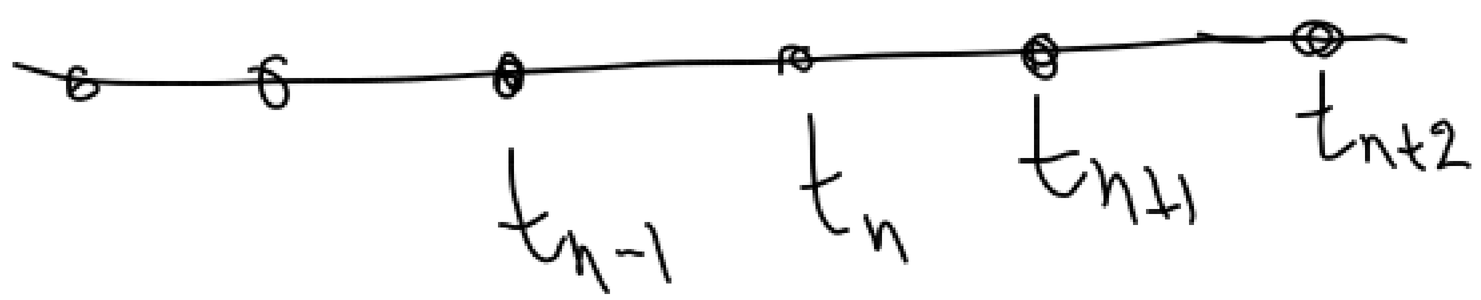
Drawbacks:
- Need to compute derivatives for each problem
- # of terms grows exponetially with order

Multistep methods

$$\frac{U^{n+1} - U^{n-1}}{2k} = f(U^n)$$

Leapfrog
Midpoint

2nd order accurate



Drawback:

- Require multiple starting values
- Harder to change step size k

Runge-Kutta methods

Use multiple evaluations of f

$$U^* = U^n + \frac{k}{2} f(U^n)$$

$$U^{n+1} = U^n + k f(U^*)$$

2nd order accurate

$$\frac{U^{n+1} - U^n}{k} = f\left(U^n + \frac{k}{2} f(U^n)\right)$$

$$\frac{u(t_{n+1}) - u(t_n)}{k} = \underline{f\left(u(t_n) + \frac{k}{2} f(u(t_n))\right)} + \tau_n$$

$U^n = u(t_n)$

$$f\left(U^n + \frac{k}{2}f(U^n)\right) = f(U^n) + \frac{k}{2}\underbrace{f_u(U^n)f(U_n)}^{u''(t_n)} + \frac{k^2}{8}f_{uu}\left(f(U^n), f(U^n)\right) + O(k^3)$$

$$= u'(t_n) + \frac{k}{2}u''(t_n) + \frac{k^2}{8}\underbrace{f_{uu}\left(f(U^n, f(U^n)\right)}_{u'''(t_n)} + O(k^3)$$

$$\cancel{u'(t_n)} + \cancel{\frac{k}{2}u''(t_n)} + \frac{k^2}{6}u'''(t_n) + O(k^3) = \cancel{u'(t_n)} + \cancel{\frac{k}{2}u''(t_n)} + \frac{k^2}{8}u'''(t_n) + O(k^3) + \tau_n$$

$$\tau_n = \frac{k^2}{24}u'''(t_n) + O(k^3)$$

Drawback:

- Multiple evaluations of f per step

Advantages:

- Self-starting
- Easy to change k

$$\frac{U^{n+1} - U^n}{k} = \frac{1}{2}\left(f(U^{n+1}) + f(U^n)\right)$$ Trapezoidal method

$$U^{n+1} = U^n + \frac{k}{2}\left(f(U^{n+1}) + f(U^n)\right)$$

$$\theta''(t) = -\sin\theta$$

$\downarrow$ linearize

$$\theta''(t) = -\theta(t)$$

$$\left.\begin{aligned} u'(t) &= \lambda u(t) \\ u(0) &= \eta \end{aligned}\right\} \text{Dahlquist test problem}$$

$$u(t) = e^{\lambda t}\eta$$

$|u(t)| \to \infty$ if $\mathrm{Re}(\lambda) > 0$

$|u(t)| \to 0$ if $\mathrm{Re}(\lambda) < 0$

$|u(t)| = |\eta|$ if $\mathrm{Re}(\lambda) = 0$

Explicit Euler

$$U^{n+1} = U^n + kf(U^n)$$

$$\Rightarrow U^{n+1} = U^n + k\lambda U^n$$

$$U^{n+1} = (1+k\lambda)U^n$$

$$U^{n+1} = (1+k\lambda)^{n+1}\eta$$

What is $\lim_{n\to\infty} |U^n|$?

Depends on whether $|1+k\lambda|$ is

$>1 \to \infty$

$=1 \to |\eta|$

$<1 \to 0$

$T = Nk$

$\uparrow$ total # of steps

$k = T/N$