Initial Value Problems

Examples:

Rigid pendulum

$$\theta''(t) = -\sin\theta$$

$$\theta(t_0) = \theta_0$$

$$\theta'(t_0) = \Omega_0$$

Linearized pendulum

$$\theta''(t) = -\theta$$

N-body problem

$$r_{12} = \begin{bmatrix} x_2 - x_1 \\ y_2 - y_1 \end{bmatrix} \qquad \vec{x} = \begin{bmatrix} x_1 \\ y_1 \end{bmatrix}$$

$$F_{12} = -\frac{G m_1 m_2}{\|\vec{r}_{12}\|_2^2} \cdot \frac{\vec{r}}{\|\vec{r}_{12}\|_2} = -\frac{G m_1 m_2}{\|r_{12}\|_2^3} \vec{r}_{12}$$

(force exerted by m_2 on m_1)

Newton's law: $m_1 \vec{x}_1'' = -\frac{G m_1 m_2}{\|r\|^3} \vec{r}_{12}$

$$\begin{bmatrix} x_1(t) \\ y_1(t) \end{bmatrix}'' = -\frac{G m_2}{\|r_{12}\|^3} \begin{bmatrix} x_2 - x_1 \\ y_2 - y_1 \end{bmatrix}$$

$$\begin{bmatrix} x_i \\ y_i \end{bmatrix}'' = -G \sum_{j \neq i} \frac{1}{\|r_{ij}\|^3} \begin{bmatrix} x_i - x_j \\ y_i - y_j \end{bmatrix}$$

2N ODEs of 2nd order $\rightarrow$ 4N ODEs of first order

Any ODE of higher order can be rewritten as a first-order system:

$$\theta''(t) = -\sin(\theta(t)) \rightarrow \quad \omega(t) = \theta'(t)$$

$$\theta''(t) = \omega'(t) = -\sin(\theta(t))$$

$$\begin{bmatrix} \theta(t) \\ \omega(t) \end{bmatrix}' = \begin{bmatrix} \omega(t) \\ -\sin(\theta(t)) \end{bmatrix}$$

We say an ODE

$$u'(t) = f(u(t), t)$$

is autonomous if f does not depend explicitly on t: $f = f(u(t))$.

We can make any ODE autonomous:

$$u'(t) = f(u(t), t) \rightarrow \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}' = \begin{bmatrix} f(u_1, u_2) \\ 1 \end{bmatrix}$$

$$u_2(t_0) = t_0$$

Simple ODE examples

$$u'(t) = g(t) \qquad u(t) = \eta + \int_{t_0}^{t} g(\tau)\,d\tau$$
$$u(t_0) = \eta$$

(quadrature)

$$u'(t) = \lambda u \qquad u(t) = e^{\lambda(t-t_0)}\eta$$
$$u(t_0) = \eta$$

$$u'(t) = Au \qquad u(t) = e^{(t-t_0)A}\eta$$
$$u(t_0) = \eta$$

matrix exponential

$$e^A = I + A + \frac{1}{2!}A^2 + \frac{1}{3!}A^3 + \cdots = \sum_{j=0}^{\infty} \frac{1}{j!}A^j$$

$$u'(t) = \lambda u + g(t)$$
$$u(t_0) = \eta$$

Solution: $u(t) = e^{\lambda(t-t_0)}\eta + \int_{t_0}^{t} e^{\lambda(t-\tau)} g(\tau)$

$$u'(t) = Au + g(t)$$
$$u(t_0) = \eta$$

Solution: $\underbrace{u(t) = e^{(t-t_0)A}\eta + \int_{t_0}^{t} e^{(t-\tau)A} g(\tau)\,d\tau}_{\text{Duhamel's principle}}$

Linear ODEs: Always have a unique solution

Nonlinear ODEs

$u' = u^2 \quad u(t_0) = \eta$

$d\tau$

$$\frac{du}{dt} = u^2$$

$$\int u^{-2}\,du = \int_0^t d\tau$$

$$\left.\frac{-1}{u(\tau)}\right|_0^t = t$$

$$\frac{-1}{u(t)} + \frac{1}{\eta} = t$$

$$\frac{-1}{u(t)} = t - \frac{1}{\eta}$$

$$u(t) = \frac{-1}{t - 1/\eta}$$

Say $\eta = 1$

$t_0 = 0$

No solution for $t \geq 1$

$u' = \sqrt{u} \quad u(0) = 0$

$$\left.\begin{array}{l} u(t) = 0 \\ u(t) = \dfrac{t^2}{4} \end{array}\right\} \text{multiple solutions}$$

Lipschitz continuity

We say $f(u)$ is L.C. on D if there exists $L > 0$ such that

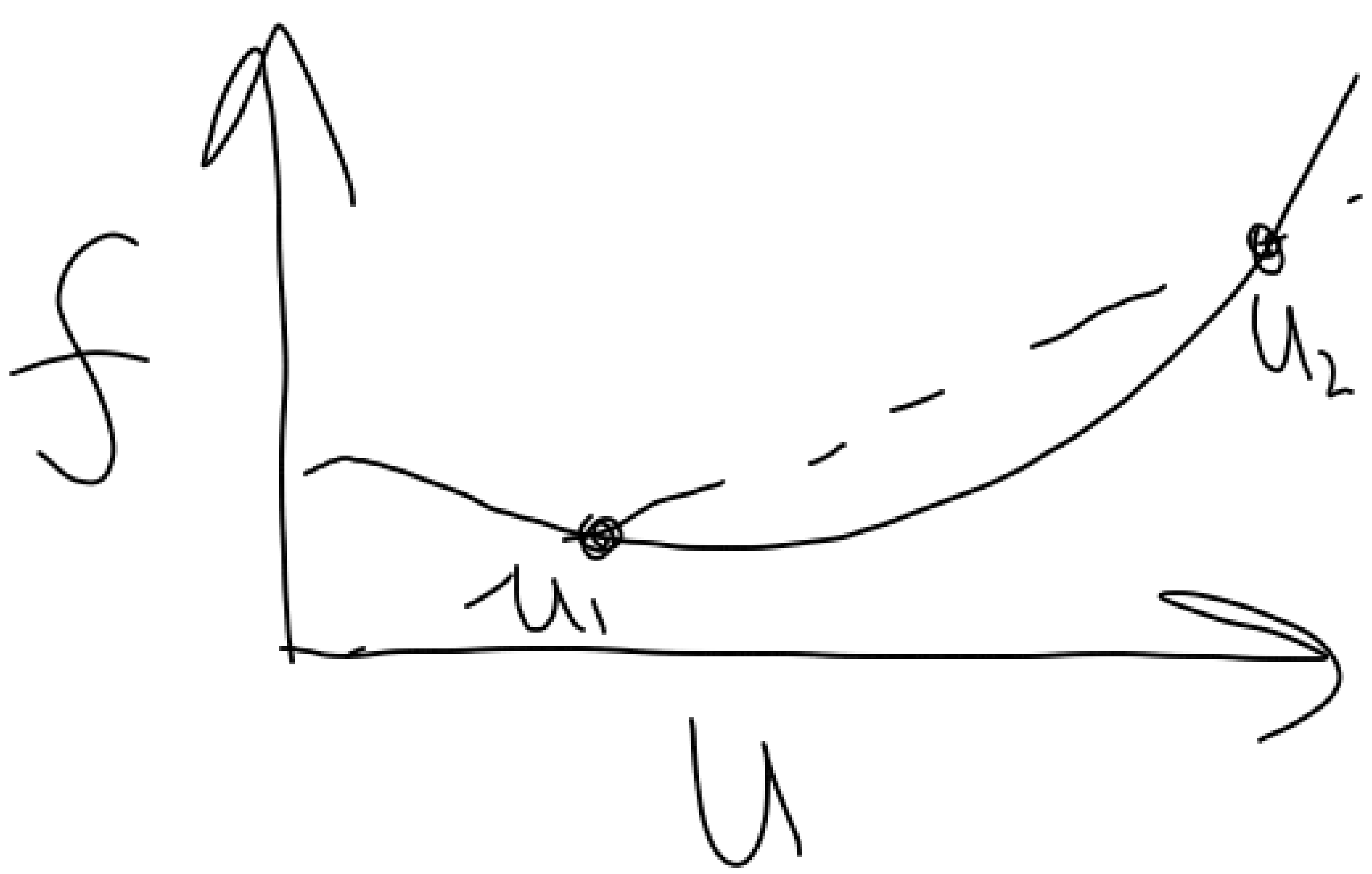
$$\frac{\|f(u_1) - f(u_2)\|}{\|u_1 - u_2\|} \leq L$$

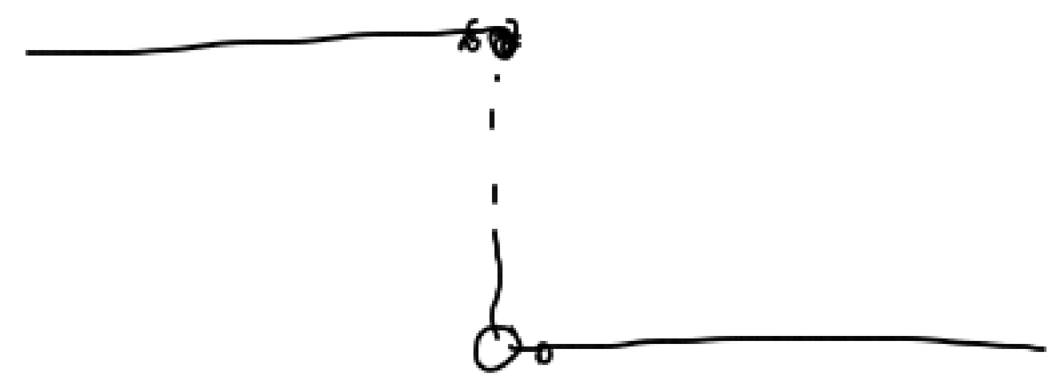
for all $u_1, u_2 \in D$. L is called a Lipschitz constant

If $f: \mathbb{C} \to \mathbb{C}$

then $\dfrac{|f(u_1)-f(u_2)|}{|u_1-u_2|}$

is the secant slope:





$f(x) = x^k \quad k>1$

L.C. on any bounded domain.

Not L.C. on $D=\mathbb{R}$

If $f:\mathbb{R}\to\mathbb{R}$ is differentiable

we can take

$$L = \sup_{x\in D} f'(x)$$

$f(x) = x^{1/k} \quad k>1$

$D=(0,1]$

$$f'(x) = \frac{1}{k}x^{\frac{1}{k}-1} = \frac{1}{k}\cdot\frac{1}{x^{1-1/k}}$$

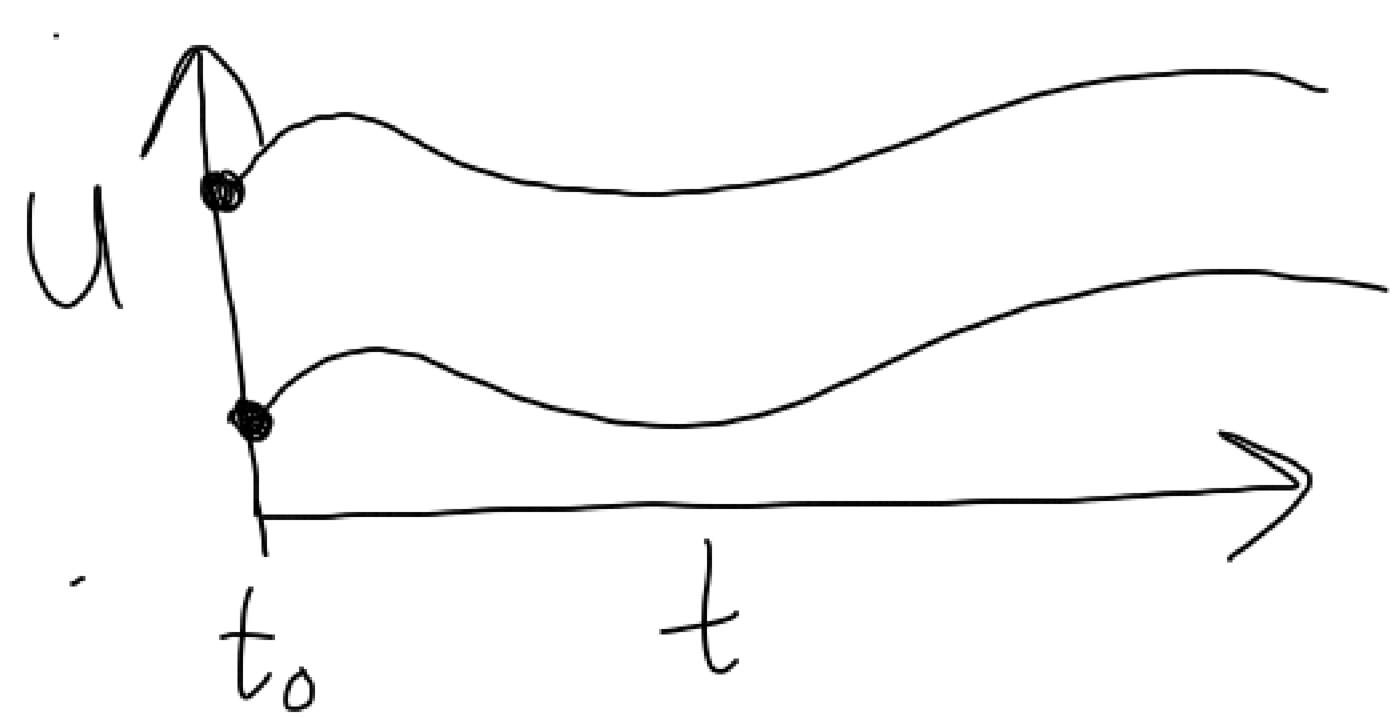
Not L.C.

Let $u'(t) = f(u(t), t)$ $u(t_0) = \eta \in D$
and suppose f is L.C. on $D \times [t_0, t_1]$
w.r.t. u. Then there exists a
unique solution $u(t)$ for $t_0 \le t \le T$
Where $T = \min\left(t_1, t_0 + \frac{a}{\max|f|}\right)$
where "a" is the minimum dist. from η to
the boundary of D.

Meaning of the Lipschitz constant

① $u'(t) = g(t)$ $\quad L=0$

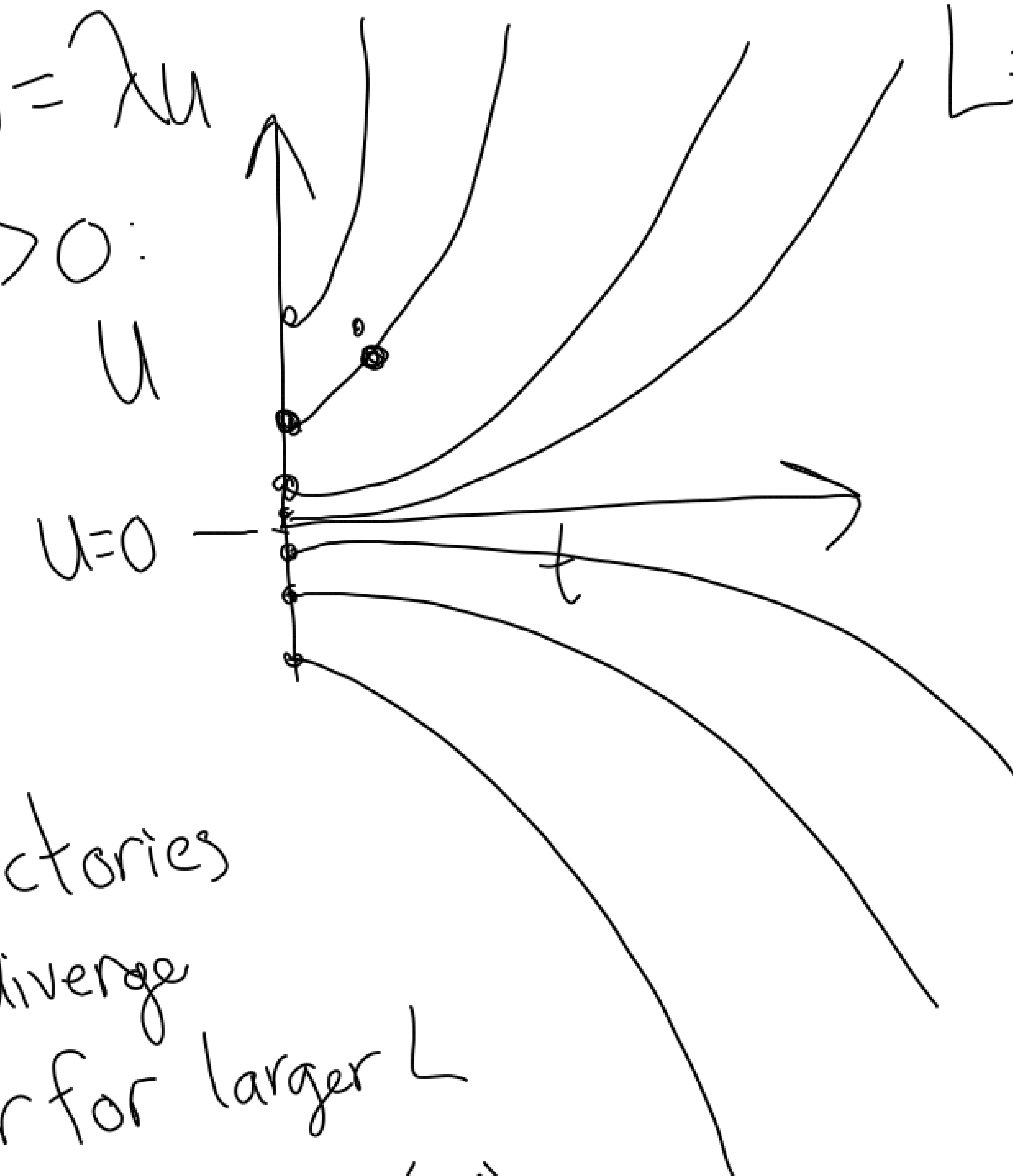
(f doesn't depend on u)



All trajectories are parallel

② $u'(t) = \lambda u$ $\quad L = |\lambda|$

$\lambda > 0$:



Trajectories diverge

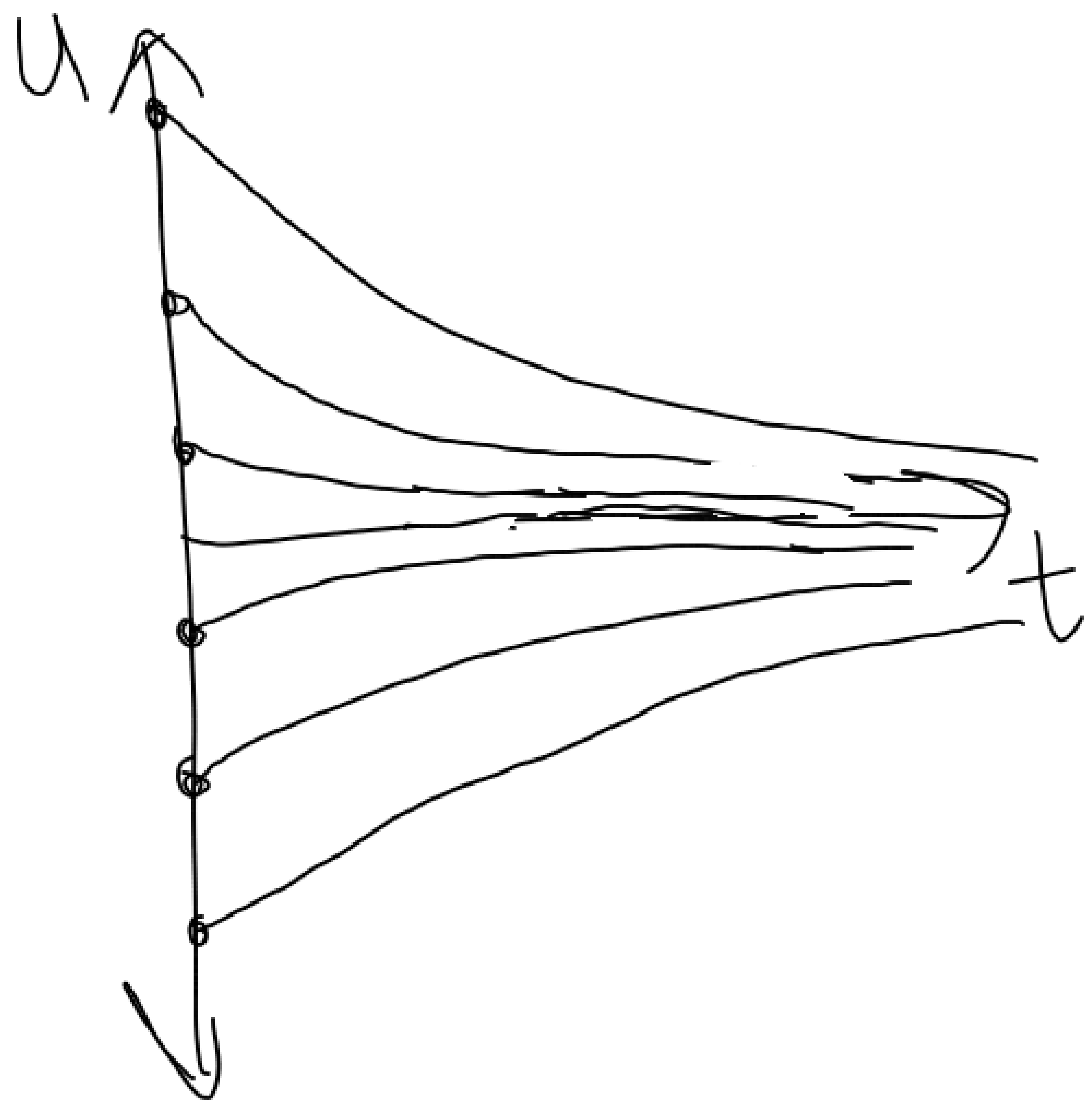
Faster for larger L

$$u(t) = e^{\lambda(t-t_0)}\eta$$

$u(t;\eta)$

$$u(t;\eta_1) - u(t;\eta_2) = e^{\lambda(t-t_0)}(\eta_1 - \eta_2)$$

$\lambda < 0$: $\quad L = |\lambda|$

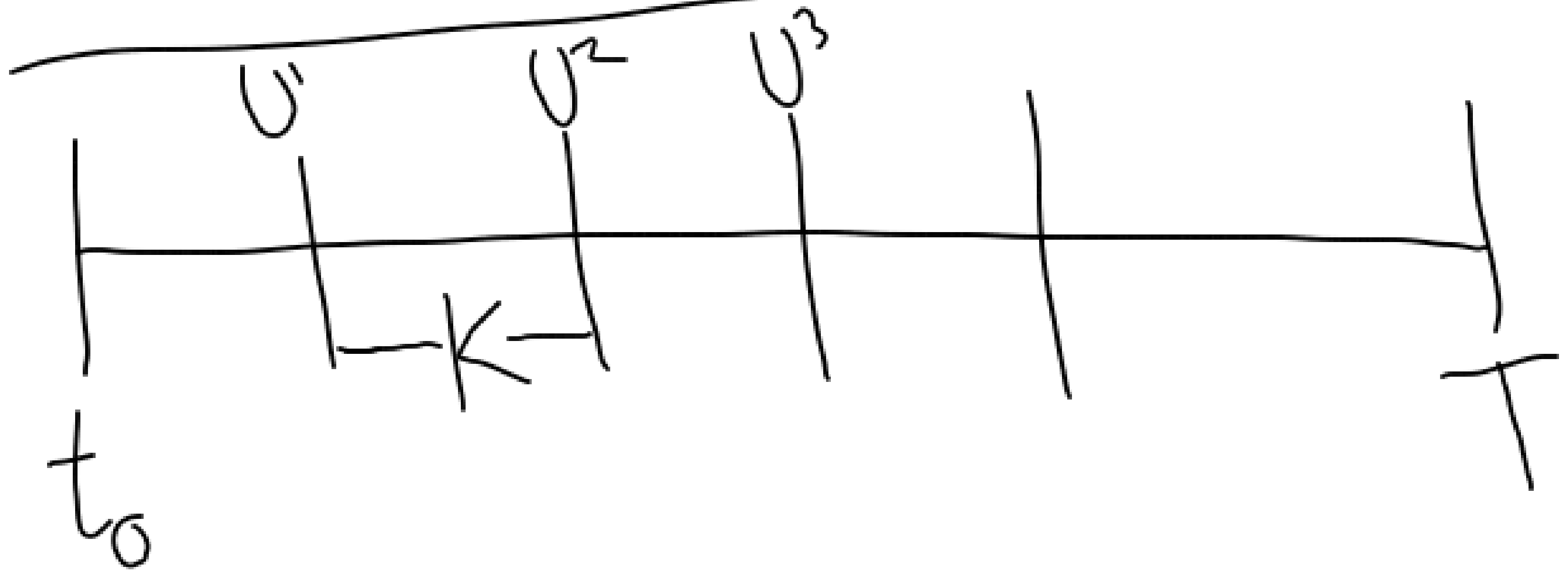


N-body Lipschitz constant

$$\boxed{\begin{aligned} x_i'(t) &= v_i(t) \\ y_i'(t) &= w_i(t) \\ \begin{bmatrix} v_i \\ w_i \end{bmatrix}' &= G \sum_{\substack{j=1 \\ j\neq i}}^{N} \frac{m_j}{\|r_{ij}\|^3} \begin{bmatrix} x_i - x_j \\ y_i - y_j \end{bmatrix} \end{aligned}}$$

L.C. if $\|r_{ij}\|$ bounded away from zero.

Numerical methods



$$U^0 = \eta$$

$$U'(t) = f(u)$$

Forward diff.: $U'(t_{n+1}) \approx \dfrac{U^{n+1} - U^n}{K} = f(U^n)$

$$\frac{U^{n+1} - U^n}{K} = f(U^n) \rightarrow U^{n+1} = U^n + Kf(U^n)$$

Euler's method

Backward diff.: $\dfrac{U^{n+1} - U^n}{K} = f(U^{n+1})$

$$\underbrace{U^{n+1} = U^n + Kf(U^{n+1})}_{}$$

Implicit Euler method

Backward Euler method