Stability and convergence of Euler's method

$t_0 \le t \le T$

$E^n = U^n - u(t_n)$

Convergence: $\lim_{k \to 0} \|E^N\| = 0$

where $N = \frac{T - t_0}{k}$.

$kN = T - t_0$

$E^0 = U^0 - u(t_0) = U^0 - \eta$

$u'(t) = \lambda u(t) + g(t)$ (1)

$u(0) = \eta$

$u(t) = e^{\lambda t}\eta + \int_0^t e^{\lambda(t-\tau)} g(\tau)\, d\tau$

$U^{n+1} = U^n + k(\lambda U^n + g(t_n))$

$U^{n+1} = (1 + k\lambda)U^n + kg(t_n)$

$u(t_{n+1}) = (1 + k\lambda)u(t_n) + kg(t_n) + k\tau^n$

$E^{n+1} = (1 + k\lambda)E^n - k\tau^n$

$E^n = (1 + k\lambda)E^{n-1} - k\tau^{n-1}$

$$E^{n+1} = (1+k\lambda)\left((1+k\lambda)E^{n-1} - k\tau^{n-1}\right) - k\tau^{n}$$

$$E^{n+1} = (1+k\lambda)^2 E^{n-1} - k(1+k\lambda)\tau^{n-1} - k\tau^{n}$$

$$E^{n+1} = (1+k\lambda)^3 E^{n-2} - k(1+k\lambda)^2\tau^{n-1} - k(1+k\lambda)\tau^{n-1} - k\tau^{n}$$

$$E^{N} = \cancel{(1+k\lambda)^N E^0} - k\sum_{n=1}^{N}(1+k\lambda)^{N-n}\tau^{n-1}$$

Lemma: $|1+k\lambda| \leq e^{k|\lambda|}$

Proof: $e^{k|\lambda|} = 1 + k|\lambda| + \frac{k^2|\lambda|^2}{2!} + \dots \geq 1 + k|\lambda|$

$$\geq |1+k\lambda|$$

$$|E^N| = k\left|\sum_{n=1}^{N}(1+k\lambda)^{N-n}\tau^{n-1}\right|$$

$$\leq k\sum_{n=1}^{N}|1+k\lambda|^{N-n}|\tau^{n-1}|$$

$$\leq k\sum_{n=1}^{N} e^{k|\lambda|(N-n)}|\tau^{n-1}|$$

$$\leq k\sum_{n=1}^{N} e^{kN|\lambda|}|\tau^{n-1}|$$

$$\leq k\sum_{n=1}^{N} e^{T|\lambda|}|\tau^{n-1}|$$

$$\leq kNe^{T|\lambda|}\max_{1\leq n\leq N}|\tau^{n-1}|$$

$$|E^N| \leq Te^{T|\lambda|}\|\tau\|_\infty \quad (2)$$

Theorem. Let U^N denote the solution given by Euler's method applied to (1), after $N = \frac{T}{k}$ steps. Then

$$\lim_{k \to 0} \|U^N - u(T)\| = 0.$$

How useful is the bound (2)?

Let $T = 10$ $\lambda = 10$.

$$|E^N| \le 10 e^{100} \|\tau\|_\infty$$

$$u'(t) = f(u)$$
$$u(0) = \eta$$

$$\frac{\|f(u_1) - f(u_2)\|}{\|u_1 - u_2\|} \le L$$

$\forall\, u_1, u_2$

$$\tau^n \approx \frac{1}{2} k u''(t_n)$$

$$U^{n+1} = U^n + k f(U^n)$$

$$u(t_{n+1}) = u(t_n) + k f(u(t_n)) + k\tau^n$$

$$E^{n+1} = E^n + k\left(f(U^n) - f(u(t_n))\right) - k\tau^n$$

$$\|E^N\| = \|E^{N-1} + k\left(f(U^{N-1}) - f(u(t_{N-1}))\right) - k\tau^{N-1}\|$$

$$\|E^N\| \le \|E^{N-1}\| + k\|f(U^{N-1}) - f(u(t_{N-1}))\| + k\|\tau^{N-1}\|$$

$$\le \|E^{N-1}\| + kL\|U^{N-1} - u(t_{N-1})\| + k\|\tau^{N-1}\|$$

$$\|E^N\| \le (1+kL)\|E^{N-1}\| + k\|\tau^{n-1}\|$$

$$\|E^N\| \le |1+kL|^N \underline{\|E^0\|} + k\underline{\sum_{n=1}^{N} |1+kL|^{N-n} \|\tau^{n-1}\|}$$

$$\|E^N\| \le k\sum_{n=1}^{N} |1+kL|^{N-n} \|\tau^{n-1}\| \le ke^{TL} \sum_{n=1}^{N} \|\tau^{n-1}\|$$

$$\le Nke^{TL} \max_{0\le n<N} \|\tau^n\|$$

$$\boxed{\|E^N\| \le Te^{TL} \max_{0\le n\le N} \|\tau^n\| = O(k)}$$

Thus $\lim_{k\to 0} \|E^N\| = 0$

$$u'(t) = Au(t)$$

$$u'(t) = R\Lambda R^{-1} u(t)$$

$$R^{-1}u'(t) = \Lambda R^{-1} u(t) \qquad w = R^{-1}u$$

$$w'(t) = \Lambda w(t)$$

$u'(t) = A(t)u(t)$ Non-autonomous

$u(t) = f(u)$ Nonlinear

Midpoint Runge-Kutta

$$U^* = U^n + \frac{1}{2}kf(U^n)$$

$$U^{n+1} = U^n + kf(U^*)$$

$$U^{n+1} = U^n + \underline{kf\left(U^n + \frac{1}{2}kf(U^n)\right)}$$

$$U^{n+1} = U^n + k\Psi(U^n, k)$$

$$u(t_{n+1}) = u(t_n) + k\Psi(u(t_n), k) + k\tau^n$$

$$\underline{E^{n+1} = E^n + k\left(\Psi(U^n, k) - \Psi(u(t_n), k)\right) - k\tau^n}$$

Claim: If L is a L.C. for f then Ψ is Lipschitz continuous with

$$L_\Psi = L + \frac{1}{2}kL^2$$

Proof: $\|\Psi(U_1) - \Psi(U_2)\| =$

$$\left\|f\left(U_1 + \frac{1}{2}kf(U_1)\right) - f\left(U_2 + \frac{1}{2}kf(U_2)\right)\right\|$$

$$\leq L\left\|U_1 + \frac{1}{2}kf(U_1) - U_2 - \frac{1}{2}kf(U_2)\right\|$$

$$\leq L\left(\|U_1 - U_2\| + \frac{1}{2}k\|f(U_1) - f(U_2)\|\right)$$

$$\leq L\left(\|U_1 - U_2\| + \frac{1}{2}kL\|U_1 - U_2\|\right)$$

$$= \underline{\left(L + \frac{1}{2}kL^2\right)}\|U_1 - U_2\|$$

So $\dfrac{\|\Psi(U_1)-\Psi(U_2)\|}{\|U_1-U_2\|} \le L+\frac{1}{2}kL^2$

Now we can proceed as before, but with $L+\frac{1}{2}kL^2$ in place of L.

We can prove convergence of any consistent one-step method in the same way.