Today:
- Brief review
- Max-norm stability
- Relation to solution by Green's function

Review:

$$u''(x) = f(x) \quad 0<x<1$$
$$u(0) = \alpha$$
$$u(1) = \beta$$

Discretize: $$\frac{U_{j+1} - 2U_j + U_{j-1}}{h^2} = f(x_j)$$

$\Rightarrow$ Linear algebra: $AU = F$

also: $A\underset{\substack{\uparrow \\ \text{global} \\ \text{error}}}{E} = -\underset{\substack{\uparrow \\ \text{local truncation} \\ \text{error}}}{\tau}$

$$\|E\|_2 \leq \|A^{-1}\|_2 \cdot \|\tau\|_2$$

Consistent: $\|\tau\| = \mathcal{O}(h^2)$

Stable: $\|A^{-1}\|_2 < C$

(C is indepent of h)

$$\lim_{h \to 0} \|E\|_2 = 0$$

(Convergence)

$$\|E\|_\infty \leq \|A^{-1}\|_\infty \|\tau\|_\infty$$

$$\|\tau\|_\infty = \max_j |\tau_j| = \mathcal{O}(h^2)$$

We want $\|A^{-1}\|_\infty < C$

$$U_0 = \alpha$$

$$U_{m+1} = \beta$$

$$\underbrace{\frac{1}{h^2}\begin{bmatrix} h^2 & & & & \\ 1 & -2 & 1 & & \\ & \ddots & \ddots & \ddots & \\ & & \ddots & \ddots & \ddots \\ & & 1 & -2 & 1 \\ & & & & h^2 \end{bmatrix}}_{A^{(m+2)\times(m+2)}} \underbrace{\begin{bmatrix} U_0 \\ U_1 \\ \vdots \\ \vdots \\ U_m \\ U_{m+1} \end{bmatrix}}_{U} = \underbrace{\begin{bmatrix} \alpha \\ f(x_1) \\ f(x_2) \\ \vdots \\ f(x_m) \\ \beta \end{bmatrix}}_{F}$$

$$A^{-1} = B$$

$$U = BF$$

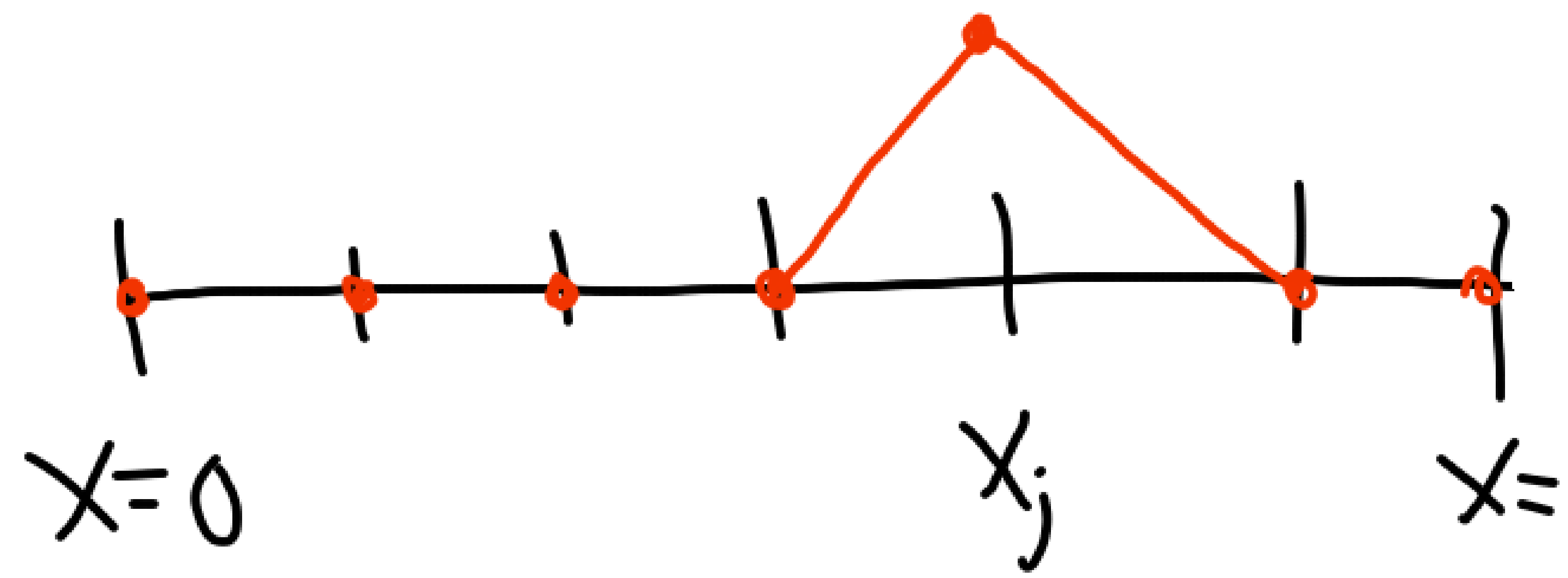
$$\begin{bmatrix} B_0 & B_1 & \cdots & B_m & B_{m+1} \end{bmatrix} \begin{bmatrix} F_0 \\ \vdots \\ F_{m+1} \end{bmatrix} = U$$

$$U = \sum_{j=0}^{m+1} F_j B_j$$

Suppose: $\alpha = \beta = 0$

$f(x_i) = 0 \quad \forall\, i \neq j$

$f(x_j) = 1$

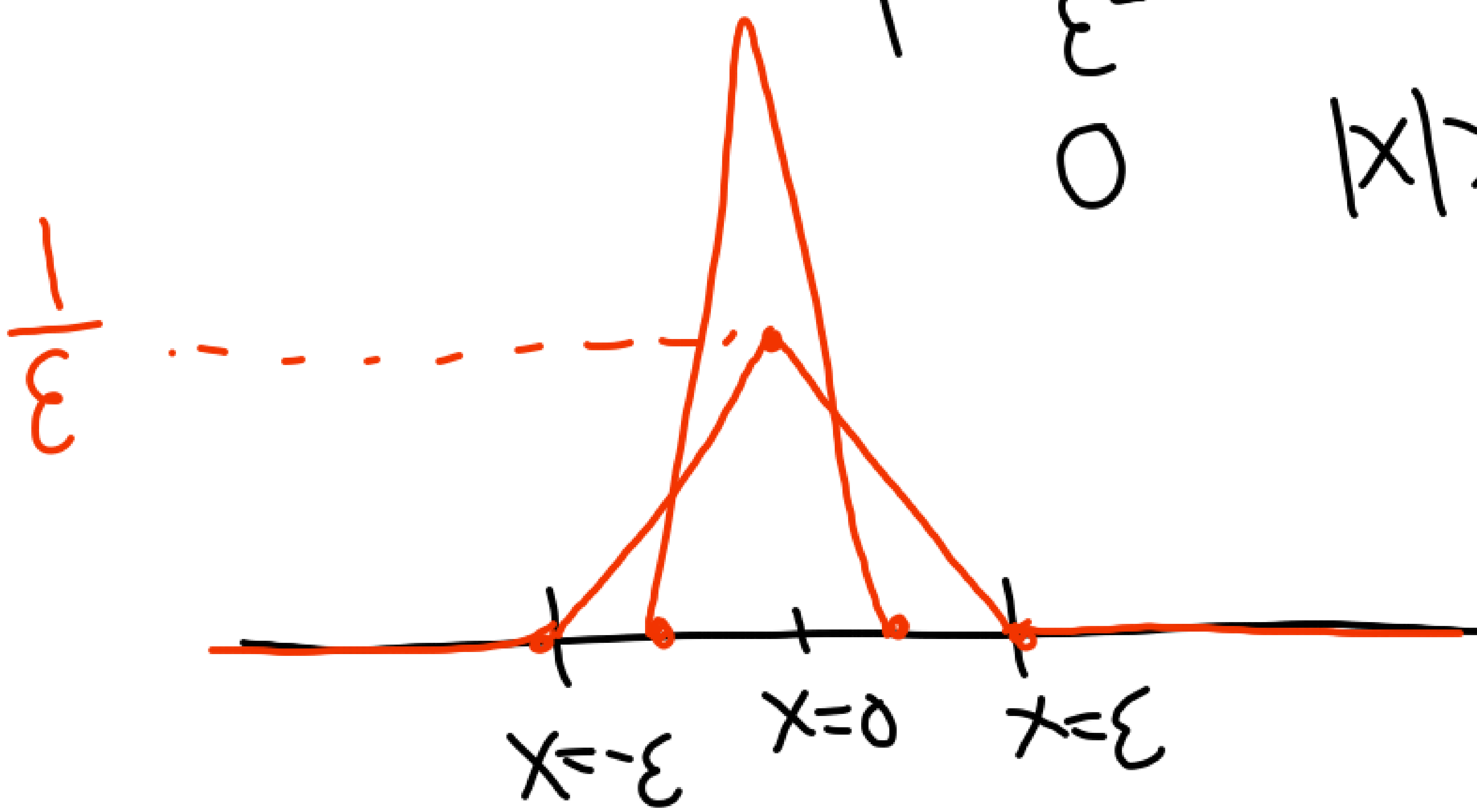



In this case $U = B_j$.

# $\delta$ functions

Consider the family of functions:

$$\phi_\varepsilon(x) = \begin{cases} \dfrac{\varepsilon + x}{\varepsilon^2} & -\varepsilon \le x \le 0 \\[2ex] \dfrac{\varepsilon - x}{\varepsilon^2} & 0 \le x \le \varepsilon \\[2ex] 0 & |x| > \varepsilon \end{cases}$$




$$\int_{-\infty}^{\infty} \phi_\varepsilon(x)\,dx = \tfrac{1}{2}\, b \cdot h = \tfrac{1}{2} 2\varepsilon \cdot \tfrac{1}{\varepsilon}$$
$$= 1$$

In the limit $\varepsilon \to 0$ we call this the Dirac $\delta$ function

$$\delta(x)$$

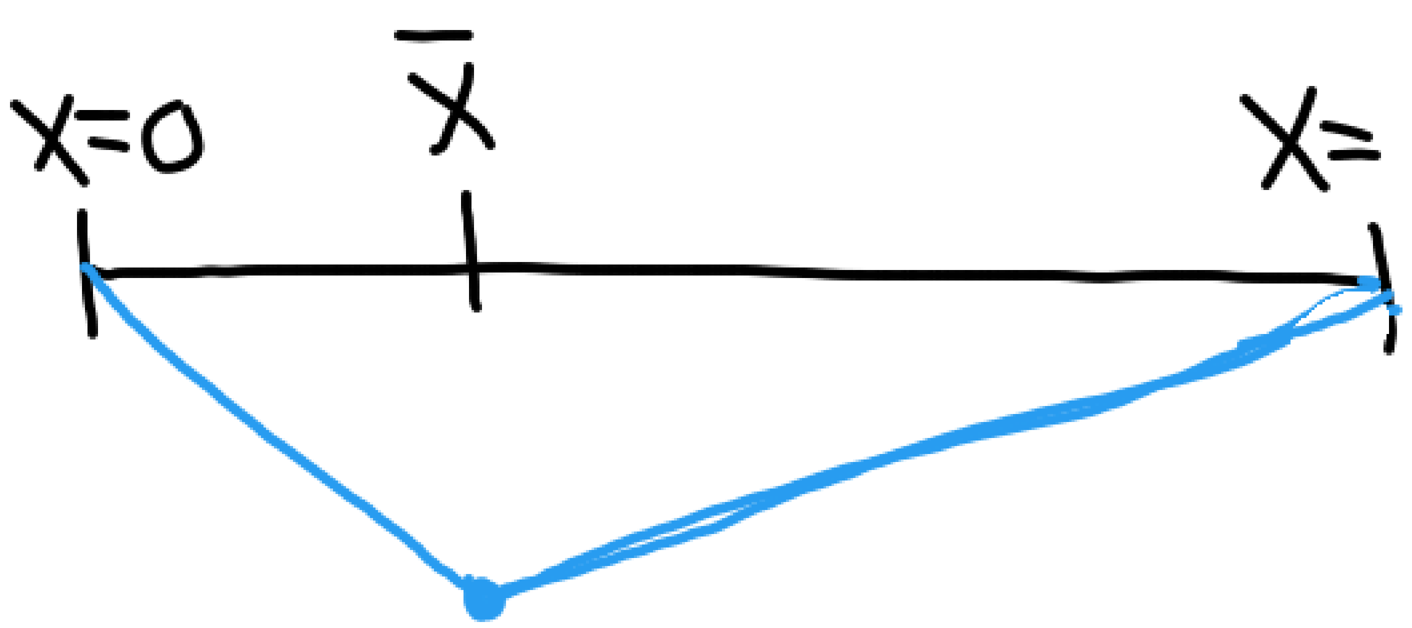
One important property:

$$\int f(x)\, \delta(x - \bar{x})\, dx = f(\bar{x})$$

$$u''(x) = f(x) = \delta(x-\bar{x})$$
$$u(0) = u(1) = 0$$

Away from $\bar{x}$, $u(x)$ is linear.

$u(x)$ should have a local minimum at $\bar{x}$.




$$u'(\bar{x}+\varepsilon) - u'(\bar{x}-\varepsilon) = \int_{\bar{x}-\varepsilon}^{\bar{x}+\varepsilon} u''(x)\,dx = 1$$

Why a minimum? heat source

$$u_t = Ku_{xx} + \psi(x)$$
$$0 = Ku_{xx} + \psi(x)$$
$$u''(x) = \frac{-\psi(x)}{K} = f(x)$$

$$u(x) = \begin{cases} u_1(x) & 0 \le x \le \bar{x} \\ u_2(x) & \bar{x} \le x \le 1 \end{cases}$$

$u_1(x) = a_1 x$ Continuity: $a_1\bar{x} = a_2(\bar{x}-1)$

$u_2(x) = a_2(x-1)$ $a_2 - a_1 = 1$ $a_1 = \bar{x} - 1$

So $\cancel{a_2}\bar{x} = \cancel{a_2}\bar{x} - a_2 + \bar{x} \Rightarrow a_2 = \bar{x}$

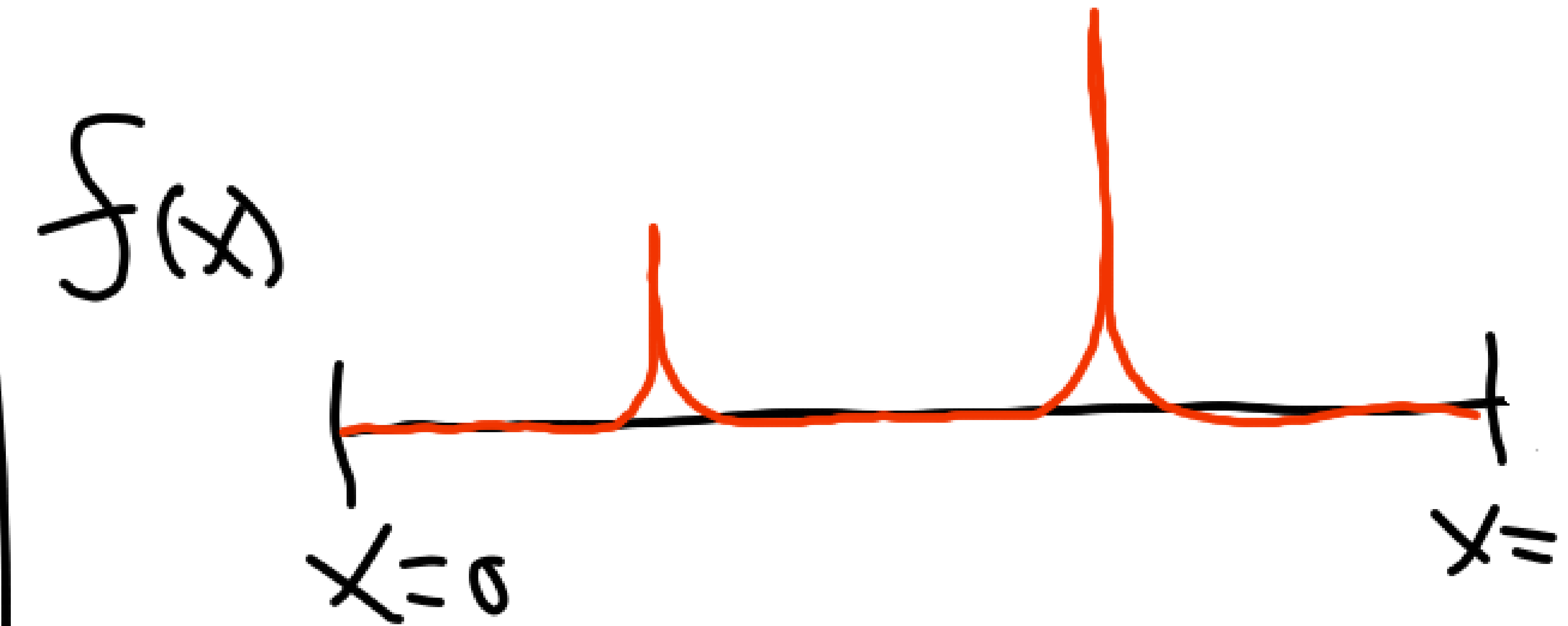
$$G(x,\bar{x}) = \begin{cases} x(\bar{x}-1) & 0 \le x \le \bar{x} \\ \bar{x}(x-1) & \bar{x} \le x \le 1 \end{cases}$$

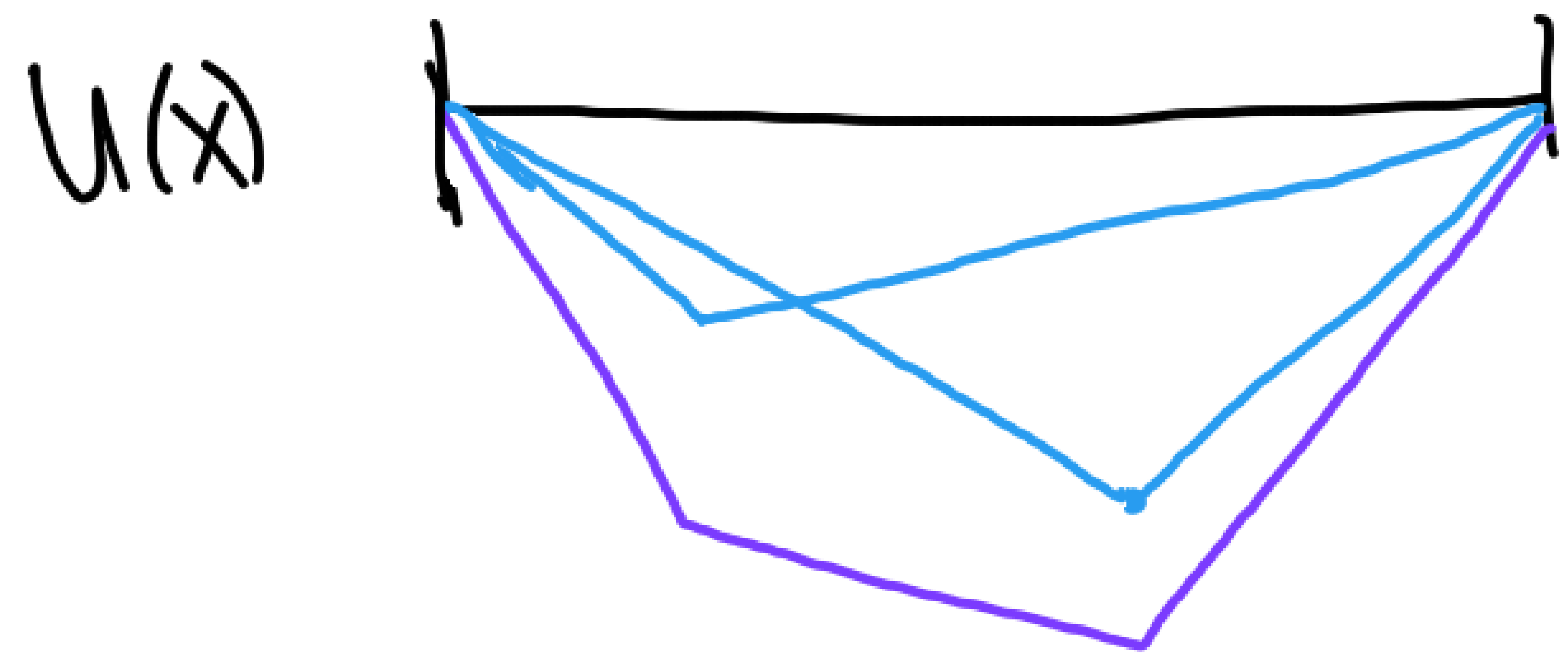
This is the solution of

$$u''(x) = \delta(x-\bar{x})$$
$$u(0)=u(1)=0$$

By linearity, we have the solution for $u''(x) = \sum_{k=1}^{n} c_k \delta(x-x_k)$

$$u(x) = \sum_{k=1}^{n} c_k G(x,x_k)$$




For any $f(x)$, we have

$$f(x) = \int_0^1 f(\bar{x})\,\delta(x-\bar{x})\,d\bar{x}$$

Then the solution of

$$u''(x) = f(x)$$
$$u(0) = u(1) = 0$$

is

$$u(x) = \int_0^1 f(\bar{x})\, G(x,\bar{x})\, d\bar{x}$$

We call $G$ a Green's function.

What about

$$u''(x) = 0$$
$$u(0) = 1 \quad u(1) = 0 \quad ?$$

$$u(x) = 1 - x = G_0(x)$$

$$u''(x) = 0 \quad u(0) = 0 \quad u(1) = 1$$
$$u(x) = x = G_1(x)$$

Then the solution of

$$u''(x) = f(x)$$
$$u(0) = \alpha \quad u(1) = \beta$$

is

$$\boxed{u(x) = \int_0^1 f(\bar{x})\, G(x,\bar{x})\, d\bar{x} + \alpha G_0(x) + \beta G_1(x)}$$

(*)

$(B_j)_i = h\,G(x_i, x_j)$ $\quad U = BF$

$B_{ij}$

What about the boundary conditions?

$$U''(x) = 0$$

$$U(0) = 1 \quad u(1) = 0$$

$$F = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ \vdots \\ 0 \end{bmatrix} \quad U = B_0$$

In fact $(B_0)_j = B_{j0} = G_0(x_j)$

and $B_{j,m+1} = G_1(x_j)$

$$U = \sum_{j=0}^{m+1} F_j B_j$$

So the numerical solution of

$$U''(x) = f(x)$$

$$U(0) = \alpha \quad u(1) = \beta$$

is

$$\boxed{U_i = \alpha G_0(x_i) + \beta G_1(x_i) + h\sum_{j=1}^{m} f(x_j)\, G(x_i, x_j)}$$

$U_i$ is the exact solution $u(x_i)$ of the problem:

$$U''(x) = \sum_{j=1}^{m} f(x_j)\,\delta(x - x_j)$$

$$U(0) = \alpha \quad u(1) = \beta$$

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So what is $\|A^{-1}\|_\infty = \|B\|_\infty$?

$\|B\|_\infty$ is the maximum absolute row sum:

$$\|B\|_\infty = \max_i \sum_{j=0}^{m+1} |B_{ij}|$$

$B_{i0} = G_0(x_i) = 1 - x_i \Rightarrow \max_i |B_{i0}| = 1$

$B_{i,m+1} = G_1(x_i) = x_i \Rightarrow \max_i |B_{i,m+1}| = 1$

$B_{ij} = h\,x_i(1 - x_j)$ for $1 \le j \le m$

so $\max_i |B_{ij}| = h = \frac{1}{m+1}$

$\|B\|_\infty < 1 + 1 + 1 = 3$

How fast will $\|E\|$ go to zero as $h \to 0$?