Midterm 2 Review

Linear Multistep Methods

$$u'(t) = f(u,t)$$

$$\sum_{j=0}^{r} \alpha_j U^{n+j} = k \sum_{j=0}^{r} \beta_j f(U^{n+j}, t_n + kj)$$

Consistency conditions

$$\sum_{j=0}^{r} \alpha_j = 0 \qquad \sum_{j=0}^{r} (j\alpha_j - \beta_j) = 0$$

$$\frac{1}{k} \sum_{j=0}^{r} \alpha_j u(t_n + kj) = \sum_{j=0}^{r} \beta_j u'(t_n + jk)$$

$$\frac{1}{k} \sum_{j=0}^{r} \alpha_j \sum_{i=0}^{\infty} \frac{u^{(i)}(t_n)(kj)^i}{i!}$$

$$= \sum_{j=0}^{r} \beta_j \sum_{i=0}^{\infty} \frac{u^{(i+1)}(t_n)(kj)^i}{i!}$$

Zero stability

$$u'(t) = 0$$

$$\sum_{j=0}^{r} \alpha_j U^{n+j} = 0$$

$$\rho(\zeta) = \sum_{j=0}^{r} \alpha_j \zeta^j$$

Roots of $\rho(\zeta)$ must satisfy the root condition

Absolute Stability

$$U'(t) = \lambda u$$

$$\sum_{j=0}^{r} \alpha_j U^{n+j} = k\lambda \sum_{j=0}^{r} \beta_j U^{n+j}$$

$$U^{n+j} \to \zeta^j : \quad \pi(\zeta; z) = \sum_{j=0}^{r} (\alpha_j - z\beta_j)\zeta^j$$

$$z = k\lambda$$

Roots of π must satisfy the root condition:

(a) $|\zeta| \le 1$

(b) if ζ is a repeated root, $|\zeta| < 1$.

Runge-Kutta Methods

$$Y_i = U^n + k\sum_{j=1}^{r} a_{ij} f(Y_j, t_n + c_j k)$$

$$U^{n+1} = U^n + k\sum_{j=1}^{r} b_j f(Y_j, t_n + c_j k)$$

Consistency: $c_i = \sum_{j=1}^{r} a_{ij}$

$$\sum_{j=1}^{r} b_j = 1$$

$$\begin{array}{c|c} c & A \\ \hline & b^T \end{array}$$

Explicit

Diagonally implicit

Fully implicit

Zero-Stability: $U^{n+1} = U^n$

(all RKMs are zero-stable)

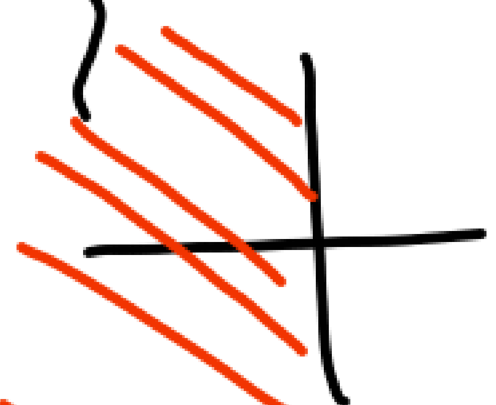
Abs. stability: $U'(t) = \lambda u$

$U^{n+1} = R(z) U^n$ $\quad U'(t) = Lu$

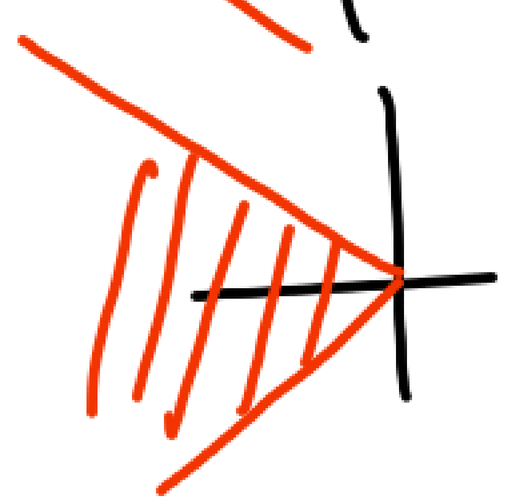
Stability region:

$$S = \{ z \in \mathbb{C} : |R(z)| \le 1 \}$$

A-stability



A(α)-stability



L-stability: $\lim_{z \to -\infty} |R(z)| < 1$.

Stiffness

Characterization

Stiffness ratio

$$\frac{|\lambda_{max}|}{|\lambda_{min}|} >> 1$$

How to deal with stiffness

Choice of step size

Local accuracy:

$$K\tau^n < \varepsilon$$

Estimate local error by comparing different numerical solutions

Step size can be limited by accuracy or stability.

The Heat Equation

$$u_t = u_{xx}$$

Stiffness

Discretizations and truncation error

$$U^{n+1} = B_{k,h} U^n$$

Lax-Richtmeyer Stability:

$$\|B^n_{k,h}\| \le C_T$$

C_T independent of k,h.

This implies

$$\|B_{k,h}\| \le 1 + k\alpha \Rightarrow \|B_{k,h}^n\| \le (1+k\alpha)^n \le e^{\alpha T}$$

for $nk \le T$.

In practice we require $\|B_{k,h}\| \le 1$.

Von Neumann analysis

$$U_j^n \to g^n e^{ijh\xi} \qquad |g| \le 1$$

MOL analysis:

$$k\lambda \in S$$

The Advection Equation

Boundary conditions

Characteristics, exact solution

The CFL condition

Modified equation analysis

↳ dissipation, dispersion