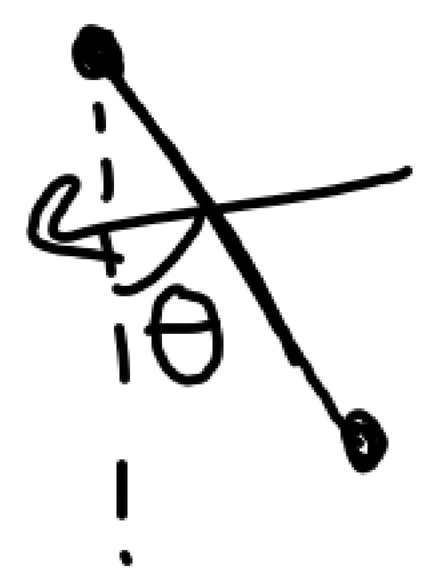
Initial value problems

Examples:

① Rigid pendulum



$$\theta''(t) = -\sin(\theta(t))$$ Autonomous

$$\theta(t_0) = \theta_0$$

$$\theta'(t_0) = \Omega_0$$

Driven pendulum: $\theta''(t) = -\sin(\theta) + \varepsilon\cos(\omega t)$

2nd-order

Non-autonomous

② SIR model (epidemiology)

$S(t)$: Susceptible

$I(t)$: Infectious

$R(t)$: Removed

β: contact rate

γ: removal rate

$$\boxed{\begin{aligned} S'(t) &= -\beta S I \\ I'(t) &= \beta S I - \gamma I \end{aligned}}$$

$$R'(t) = \gamma I$$

$$\frac{d}{dt}(S+I+R)=0$$

$$S+I+R=1$$

$$R=1-S-I$$

$$S(t_0)=S_0$$

$$I(t_0)=I_0$$

Any ODE system can be written as a first-order autonomous ODE system.

$$\theta''(t)=-\sin(\theta(t))$$

High-order ODE transformed to 1st-order system

Let
$$\boxed{\begin{aligned}\phi&=\theta'(t)\\ \phi'(t)&=-\sin(\theta(t))\end{aligned}}$$

Transform non-autonomous to autonomous

$$\theta'(t)=\phi(t)$$

$$\phi'(t)=-\sin(\theta(t))+\varepsilon\cos(\omega\tau(t))$$

Let $\tau'(t)=1,\ \tau(t_0)=t_0$

We will write our IVP in the form

$$u'(t) = f(u(t), t)$$
$$u(t_0) = \eta$$

Linear, scalar IVP:

$$u'(t) = \lambda u(t)$$
$$u(t_0) = \eta$$

$$u(t) = e^{\lambda(t-t_0)}\eta$$

$$u'(t) = g(t)$$
$$u(t_0) = \eta$$

$$u(t) = \eta + \int_{t_0}^{t} g(\tau)\,d\tau$$

Linear system:

$$u'(t) = Au(t)$$
$$u(t_0) = \eta$$

$A \in \mathbb{R}^{m\times m}$

$u: \mathbb{R} \to \mathbb{R}^m$

$$u(t) = e^{(t-t_0)A}\eta$$

$$e^M = I + M + \frac{1}{2!}M^2 + \cdots = \sum_{j=0}^{\infty} \frac{1}{j!}M^j$$

Inhomogeneous linear scalar IVP

$$u'(t) = \lambda u(t) + g(t)$$
$$u(t_0) = \eta$$

$$u(t) = e^{\lambda(t-t_0)}\eta + \int_{t_0}^{t} e^{\lambda(t-\tau)} g(\tau)\,d\tau$$

Inhomogeneous

$$u'(t) = Au(t) + g(t)$$
$$u(t_0) = \eta$$

$$u(t) = e^{(t-t_0)A}\eta + \int_{t_0}^{t} e^{(t-\tau)A} g(\tau)\,d\tau$$

Duhamel's principle

Existence and uniqueness

Linear IVPs: unique solution exists for all time.

Nonlinear IVPs: ???

$$u'(t) = (u(t))^2$$
$$u(0) = \eta > 0$$

$$\frac{du}{dt} = u^2$$

$$u^{-2}\,du = dt$$

$$\int_0^t u^{-2}\,du = \int_0^t d\tau$$

$$\left.\frac{-1}{u}\right|_0^t = t$$

$$\frac{-1}{u(t)} + \frac{1}{\eta} = t$$

$$\frac{1}{u(t)} = \frac{1}{\eta} - t$$

$$u(t) = \frac{1}{\frac{1}{\eta} - t}$$

exists until $t = 1/\eta$

$$u'(t) = (u(t))^{1/2} \qquad u(0)=\eta$$

$$u^{-1/2}\,du = dt$$

$$\int_0^t u^{-1/2}\,du = t$$

$$2u^{1/2}\Big|_0^t = t$$

$$2\sqrt{u(t)} - 2\sqrt{\eta} = t$$

$$\sqrt{u(t)} = \frac{t}{2} + \sqrt{\eta}$$

$$u(t) = \left(\frac{t}{2} + \sqrt{\eta}\right)^2$$

What if $\eta = 0$?

$$u(t) = \frac{t^2}{4}$$

$$u(0) = 0$$

Solution not unique

Lipschitz constant

Given $f(u)$ and domain D, we say L is a Lipschitz constant for f on D if

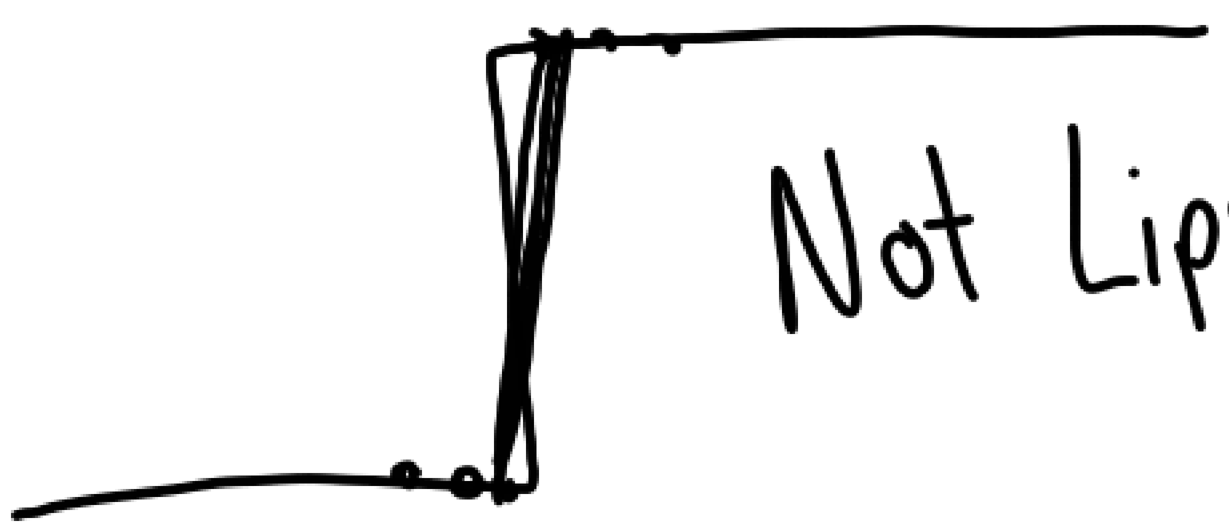
$$\|f(u_1) - f(u_2)\| \leq L\|u_1 - u_2\|$$

for all $u_1, u_2 \in D$. $\quad 0 < L < \infty$

If such an L exists, we say f is Lipschitz continuous on D.

Examples:

$$H(x) = \begin{cases} -1 & x<0 \\ 1 & x \geq 0 \end{cases}$$



Not Lipschitz

$f(x) = x^2 \quad D=[0,\infty)$

Not Lipschitz

$f(x) = x^{1/2} \quad D=[0,1]$

Not Lipschitz

If f is differentiable we can take

$$L = \sup_{u \in D} \|f'(u)\|$$

Given the IVP

$$u'(t) = f(u)$$
$$u(t_0) = \eta$$

Suppose $f(u)$ is Lipschitz continuous for $\eta - a \le u \le \eta + a$.

Then a unique solution exists for $t \le t_0 + \dfrac{a}{\sup\limits_{u \in D} |f|}$

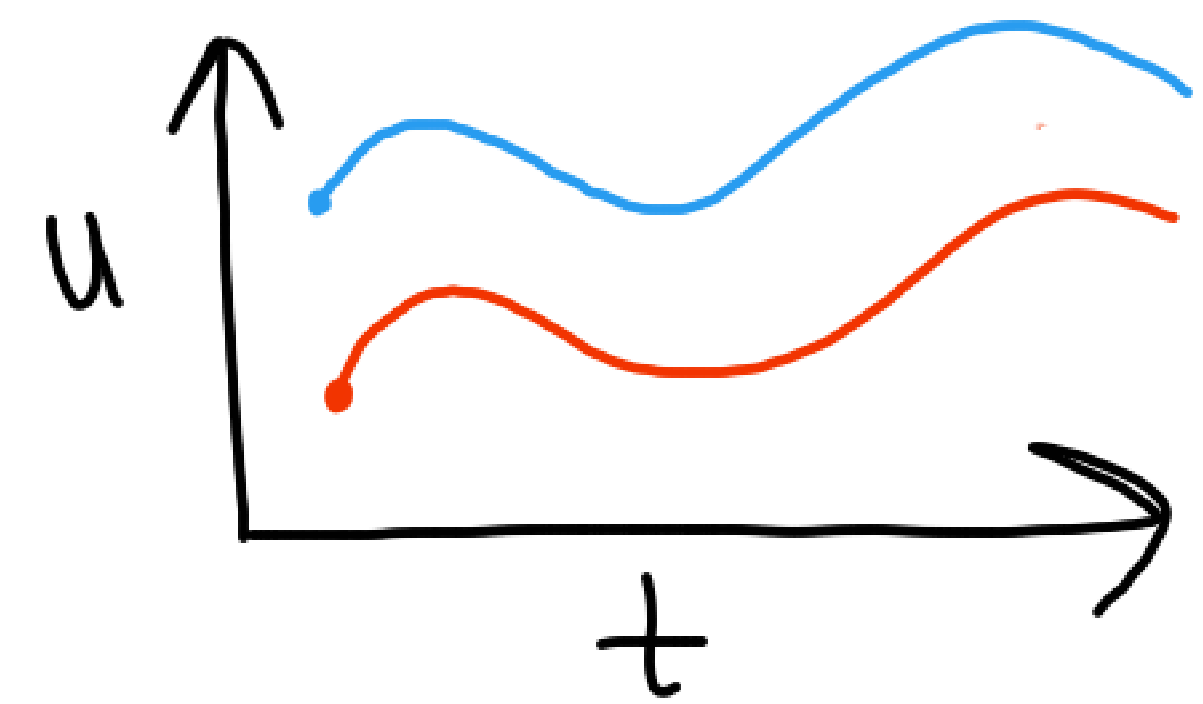
$$D = [\eta - a, \eta + a]$$

Meaning of the Lipschitz Constant

Examples:

① $u'(t) = g(t) \quad u(t_0) = \eta$

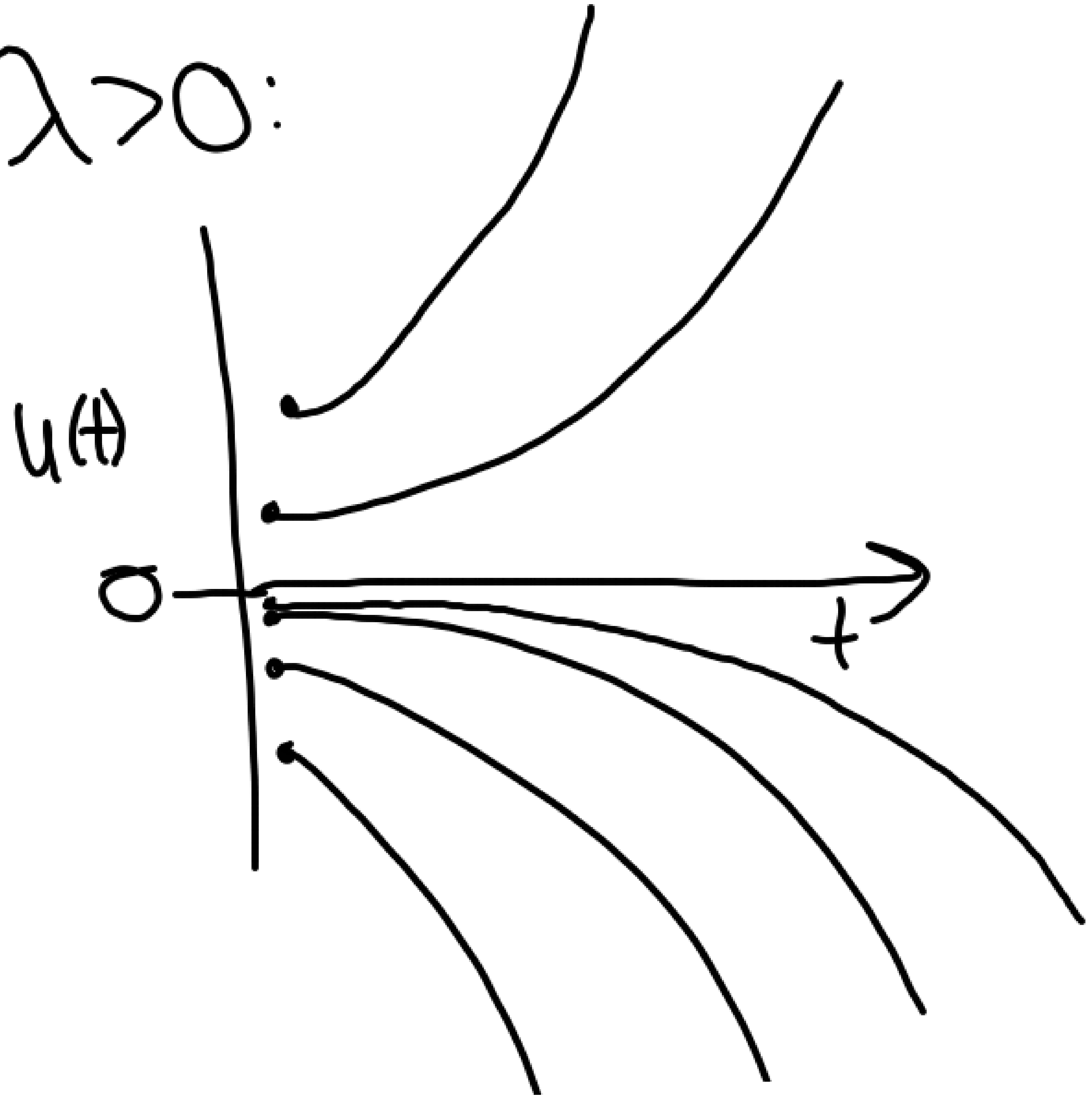
$L = 0$



② $u'(t) = \lambda u(t) \quad u(t_0) = \eta$

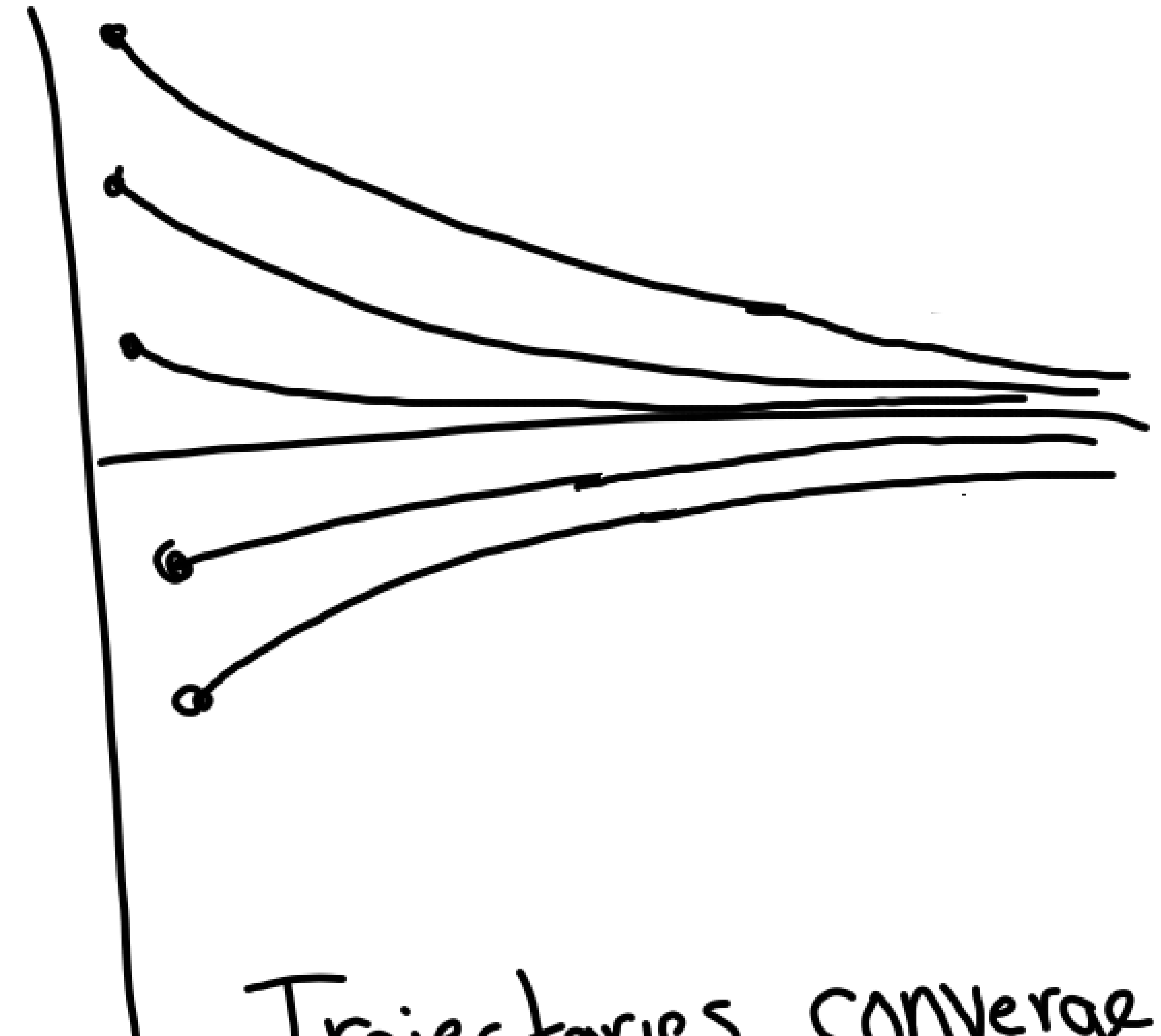
$L = |\lambda|$

$\lambda > 0$:



Trajectories diverge exponentially with rate L

$\lambda < 0$:



Trajectories converge exponentially with rate L