Stiff ODEs
(Chapter 8)

$$u'(t) = \lambda(u(t) - \cos(t)) - \sin(t)$$
$$u(0) = 1$$

$$u(t) = \cos(t)$$

For other initial data $u(t_0) = \eta$,

$$u(t) = e^{\lambda(t-t_0)}(\eta - \cos(t_0)) + \cos(t)$$

For Euler: $-2 \le k\lambda \le 0$

$\lambda = -10$

$0 \le k \le \frac{1}{5}$

Forward Euler:

$$U^{n+1} = U^n + kf(U^n)$$

Backward Euler

$$U^{n+1} = U^n + kf(U^{n+1})$$

We say a problem is stiff if the stable step size is much smaller than the step size needed to obtain sufficient local accuracy.

L-stability

When $\lambda \to -\infty$ the solution of

$$u'(t) = \lambda u(t)$$

goes to zero arbitrarily fast.

Can implicit methods capture this behavior?

$$U^{n+1} = R(z) U^n$$

What is $\lim_{z \to -\infty} |R(z)|$? Should be zero.

BE: $R(z) = \frac{1}{1-z} \to 0$ as $z \to -\infty$
(L-stable)

Imp. Trap. RK: $|R(z)| = \left|\frac{1+\frac{z}{2}}{1-\frac{z}{2}}\right| \to 1$ as $z \to -\infty$
(Not L-stable)

ODE Solvers in MATLAB and ODEPACK (Scipy)

- Use error estimation and step size adaptation
- Continuous/dense output

$$Y_i = U^n + k\sum_{j=1}^{r} a_{ij} f(Y_j)$$

$$U^{n+1} = U^n + k\sum_{j=1}^{r} b_j f(Y_j)$$

$$u(t_n+\theta k) \approx U^{n+\theta} = U^n + k\sum_{j=1}^{r} \underbrace{b_j(\theta)}_{\text{Interpolant weights.}} f(Y_j)$$

Matlab

ode23: 3rd-order
explicit RK method
w/embedded 2nd-order
error estimator.
(4 stages)

ode45: 5th-order explicit
RK w/4th order
error estimator
(7 stages)

Non-stiff problems

ode113: Multistep methods
of order 1 to 13.
Automatically adapts
k and order.
Predictor-corrector
(A.B. + AM)

Mildly stiff problems

ode15s: Multistep BDF
order 1-5
Uses Newton

Stiff problems

ode23s: Rosenbrock method

Like implicit RK, but only performs 1 Newton iteration per stage.

2nd-order L-Stable

w/3rd-order A-stable error estimator.

A-stable multistep method are at most 2nd order accurate.

There exist A-stable (and L-stable) implicit RK methods of any order.

Homework 5 due April 14th.