$$u_t + a u_x = 0$$

$$U_j^{n+1} = U_j^{n-1} - \frac{ak}{h}\left(U_{j+1}^n - U_{j-1}^n\right)$$

$$v_t + a v_x = C\, v_{xxx} + \text{H.O.T.}$$

Vanishes when

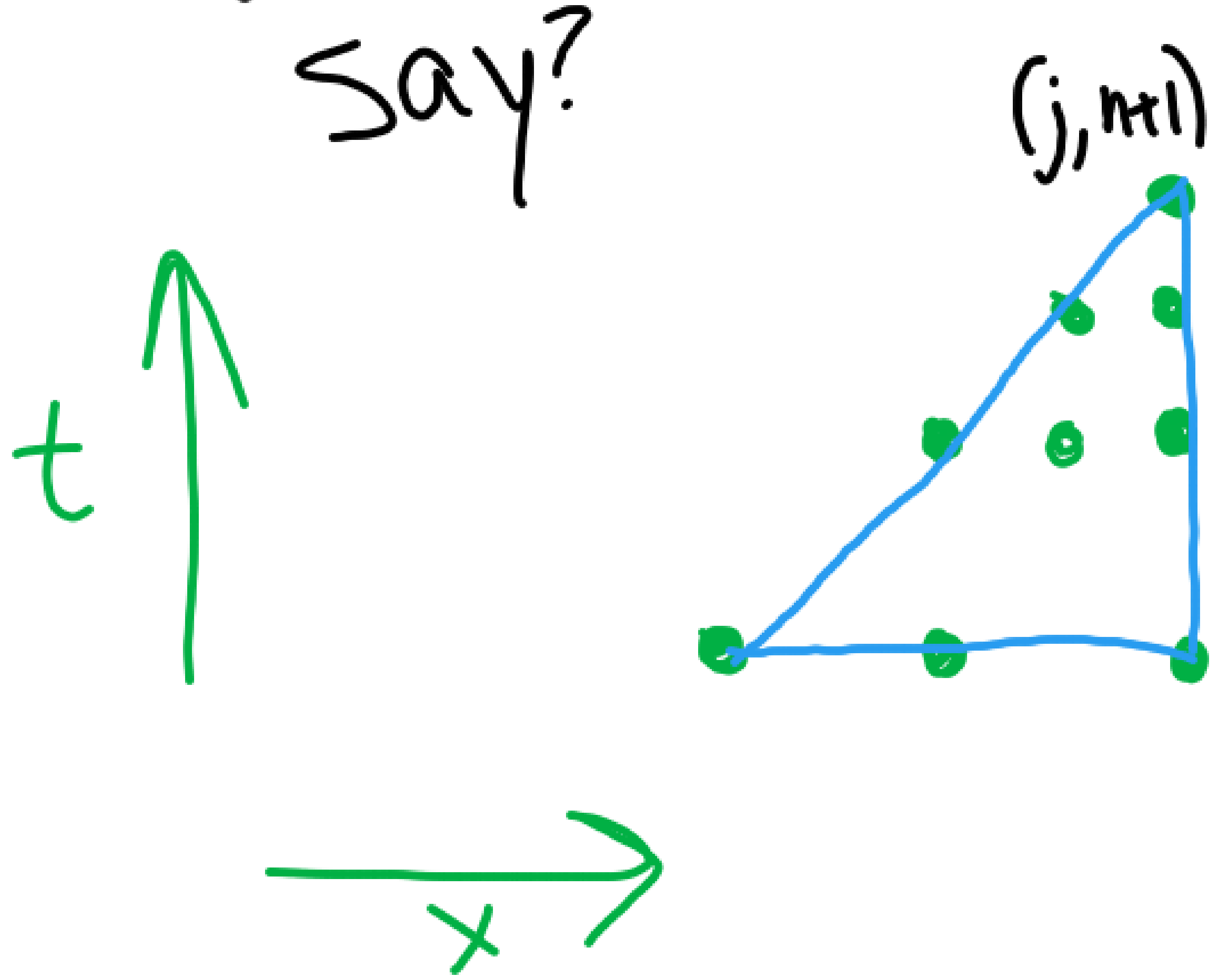
$$\nu = \frac{ka}{h} = 1$$

The upwind method

$$U_j^{n+1} = U_j^n - \frac{ka}{h}\left(U_j^n - U_{j-1}^n\right)$$

1st order in time and space

What does the CFL condition say?



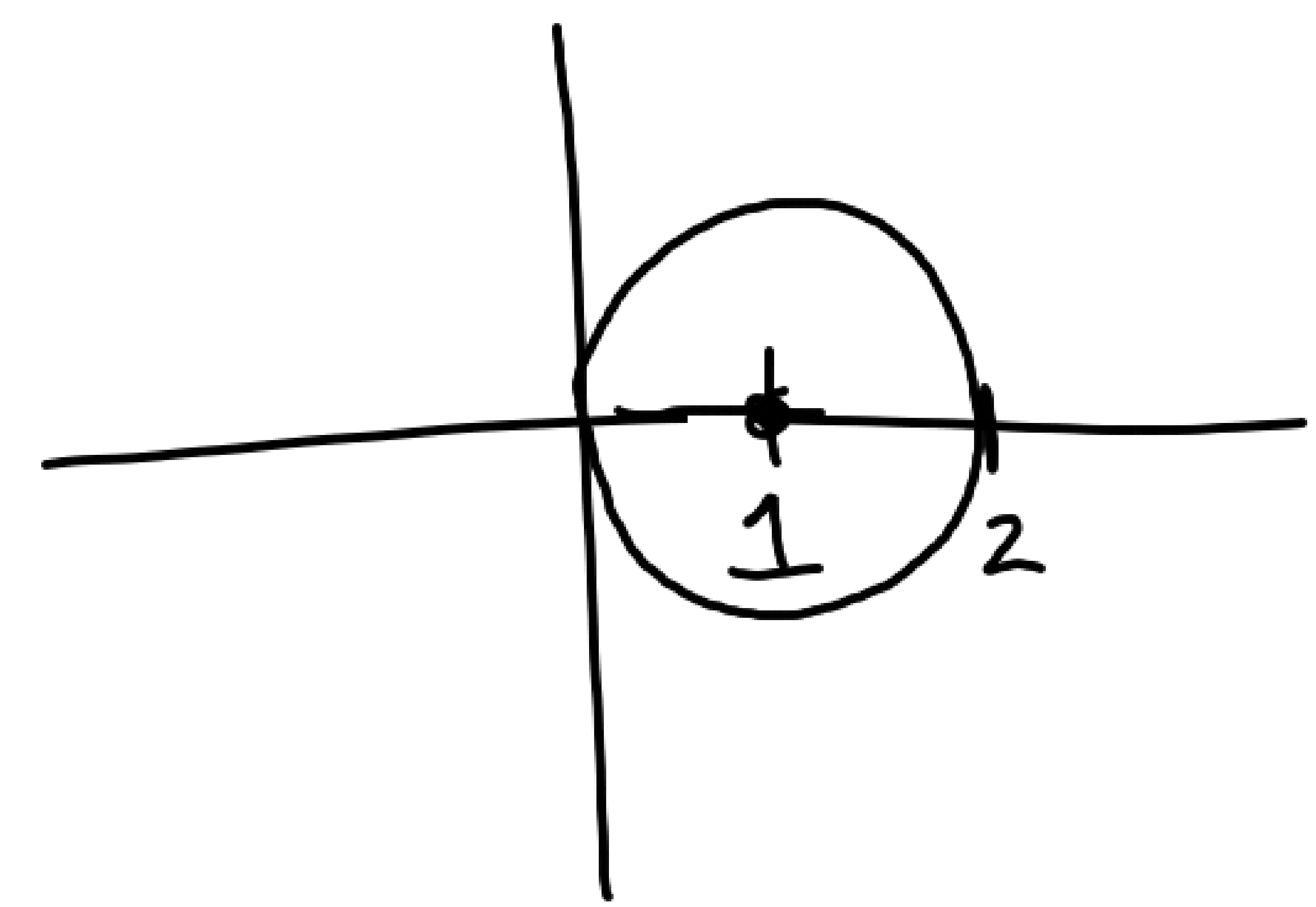
The characteristic passing through (j,n+1) must be inside the triangle.

$a \geq 0$

$$0 \leq ka \leq h$$

or $0 \leq \frac{ka}{h} \leq 1$

This is a necessary condition for stability.



$$U^{n+1} = U^n - \frac{ka}{h} \underbrace{\begin{bmatrix} 1 & & & & 1 \\ -1 & \ddots & & & \\ & \ddots & \ddots & & \\ & & \ddots & \ddots & \\ & & & -1 & 1 \end{bmatrix}}_{L} U^n$$

$$\nu = \frac{ka}{h}$$

$$U^{n+1} = \underbrace{\begin{bmatrix} 1-\nu & & & & \\ \nu & \ddots & & & \\ & \ddots & \ddots & & \\ & & \ddots & \ddots & \\ & & & \nu & 1-\nu \end{bmatrix}}_{M_\nu} U^n$$

$$U^N = M_\nu^N U^0$$

$$\|M_\nu^N\| < C$$

C independent of N.

So we need $\|M_\nu\| \le 1$.

Eigenvalues of M_ν: $\lambda = 1-\nu$

$$-1 \le 1-\nu \le 1$$

$$-2 \le -\nu \le 0$$

$$0 \le \nu \le 2$$

$$\|M^N\| \le \|M\|^N$$

We assumed

$$\|M\|_2 = \rho(M) = \max_{\lambda \in \sigma(M)} |\lambda|$$

Not true!

Von Neumann Analysis

$$U_j^{n+1} = U_j^n - \nu\left(U_j^n - U_{j-1}^n\right)$$

$$U_j^n \rightarrow g^n e^{ijh\xi}$$

$$g^{n+1} e^{ijh\xi} = g^n\left(e^{ijh\xi} - \nu\left(e^{ijh\xi} - e^{i(j-1)h\xi}\right)\right)$$

$$g = 1 - \nu\left(1 - e^{-ih\xi}\right)$$

$$= 1 - \nu\left(1 - \cos(h\xi) + i\sin(h\xi)\right)$$

$$= 1 - \nu\left(1 - \cos(h\xi)\right) + i\nu\sin(h\xi)$$

$$|g|^2 = \left(1 - \nu - \nu\cos(h\xi)\right)^2 + \nu^2\sin^2(h\xi)$$

$$|g|^2 = \underline{1} + \underline{\nu^2} + \underline{\nu^2\cos^2(h\xi)} - \underline{2\nu} - \underline{2\nu\cos(h\xi)}$$
$$+ \underline{2\nu^2\cos(h\xi)} + \underline{\nu^2\sin^2(h\xi)}$$

$$= 1 + 2\nu^2 - 2\nu + 2\cos(h\xi)\,\nu(\nu - 1)$$

$\cos(h\xi) = +1$:

$$1 + 2\nu^2 - 2\nu + 2\nu^2 - 2\nu$$

$$= 1 + 4\nu^2 - 4\nu = (1 - 2\nu)^2 \leq 1$$

$$0 \leq \nu \leq 1$$

In general, $\|A\|_2 \neq \rho(A)$.

Why?

Let $A = R\Lambda R^{-1}$

Then $\|A\|_2 = \|R\Lambda R^{-1}\| \leq \|R\|_2 \|\Lambda\|_2 \|R^{-1}\|_2$

$= \rho(A)\,\text{cond}(R)$

$\text{cond}(R) = 1$ iff R is unitary

Unitary matrices satisfy $RR^T = R^TR = I$.

Dfn. We say $A \in \mathbb{R}^{m\times m}$ is a normal matrix if

$$A^TA = AA^T.$$

Thm. $A \in \mathbb{R}^{m\times m}$ has orthogonal eigenvectors iff A is normal.

Corollary:

$\rho(A) = \|A\|_2$ iff A is normal.

Normal matrices:

- Symmetric
- Skew-symmetric
- Circulant
- Others

Non-normal example:

$$\begin{bmatrix} 0.8 & 100 \\ 0 & 0.9 \end{bmatrix}$$

$\lambda = 0.8, 0.9$

$Av_1 = 0.8 v_1$

$Av_2 = 0.9 v_2$

$U^0 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

$A^3 U^0$

$$U^0 = c_1 v_1 + c_2 v_2$$

$$AU^0 = c_1 A v_1 + c_2 A v_2$$

$$= \frac{4}{5} c_1 v_1 + \frac{9}{10} c_2 v_2$$

$$\|AU^0\|_2^2 = \frac{16}{25} c_1^2 \|v_1\|_2^2 + \frac{81}{100} c_2^2 \|v_2\|^2$$

$$\frac{72}{50} c_1 c_2 \langle v_1, v_2 \rangle$$

If $\rho(A) < 1$ then

$$\lim_{n \to \infty} \|A^n v\| = 0$$

But this does not imply that

$$\|A^n v\|_2 \leq \|v\| \quad \forall n.$$

For non-normal matrices, we can study the pseudospectra:

$$\{\lambda \in \sigma(A + M_\varepsilon) : \|M_\varepsilon\| \leq \varepsilon\}$$

Topics to review

- Linear multistep methods
- Runge-Kutta methods
- Zero-stability
- Absolute stability, stability region
- Choosing the step size
- Stiffness
- A-stability, A(α)-stability, L-stability
- Discretizations of the heat equation, advection equation
- Method of lines stability analysis
- Von Neumann analysis
- Modified equation analysis
- The CFL condition