Initial Boundary Value Problems

$$\frac{\partial u}{\partial t} = f\left(u, \frac{\partial u}{\partial x}, \frac{\partial^2 u}{\partial x^2}, \ldots\right)$$

Evolution equation

$$u_t = f(u, u_x, u_{xx}, \ldots)$$

$$u = u(x,t) \quad x \in [0,1] \quad t \in [0,T]$$

$$u(x,0) = \eta(x)$$

$$u(0,t) = \alpha(t)$$

$$u(1,t) = \beta(t)$$

Diffusion equation

$$u_t = \kappa u_{xx} + f(x)$$

$$\downarrow$$

$$u_t = u_{xx}$$

$$u(0,t) = 0$$

$$u(1,t) = 0$$

Exact solution:

$$u(x,t) = \sum_{p=0}^{\infty} \hat{u}_p(t)\sin(p\pi x)$$

$$u(x,0) = \sum_{p=0}^{\infty} \hat{u}_p(0)\sin(p\pi x) = \eta(x)$$

Substitution into the PDE gives:

$$\sum_p \hat{u}_p'(t)\sin(p\pi x) = \sum_p \hat{u}_p(t)\cdot(-p^2\pi^2)\sin(p\pi x)$$

$$\hat{u}_p'(t) = -p^2\pi^2\hat{u}_p(t)$$

$$\hat{u}_p(t) = e^{-p^2\pi^2 t}\hat{u}_p(0) \Rightarrow \sum_{p=0}^{\infty}\hat{u}_p(0)e^{-p^2\pi^2 t}\sin(p\pi x)$$

Discretization (method of lines)

First discretize in space:

$$x_j = jh \qquad j = 0, 1, \dots, m+1$$

$$h = \frac{1}{m+1}$$

$$U_j(t) \approx u(x_j, t)$$

$$u_{xx}(x_j, t) = \frac{u_{j+1}(t) - 2u_j(t) + u_{j-1}(t)}{h^2}$$

$$U_j'(t) = \frac{U_{j+1}(t) - 2U_j(t) + U_{j-1}(t)}{h^2}$$

Now discretize in time using RK, LM, etc.

$$\begin{bmatrix} U_1 \\ \vdots \\ U_m \end{bmatrix}' = \underbrace{\frac{1}{h^2}\begin{bmatrix} -2 & 1 & & \\ 1 & \ddots & \ddots & \\ & \ddots & \ddots & 1 \\ & & 1 & -2 \end{bmatrix}}_{A} \begin{bmatrix} U_1 \\ \vdots \\ U_m \end{bmatrix}$$

$$U'(t) = AU$$

$$U(t) = e^{tA} U(0).$$

$A = R\Lambda R^{-1}$ where $\lambda_p = \frac{2}{h^2}(\cos(p\pi h)-1)$

$r_{jp} = \sin(p\pi jh)$

$U'(t) = R\Lambda R^{-1}U(t)$

$R^{-1}U'(t) = \Lambda R^{-1}U(t)$ $\quad \hat{U}(t) = R^{-1}U(t)$

$\hat{U}'(t) = \Lambda \hat{U}(t)$

$\hat{U}_p'(t) = \lambda_p \hat{U}_p(t)$

$\hat{U}_p(t) = e^{t\lambda_p}\hat{U}_p(0)$ analogous to $\hat{u}_p(t) = e^{-p^2\pi t}\hat{u}_p(0)$

The factor $-p^2\pi^2$ has been replaced by

$$\lambda_p = \frac{2}{h^2}(\cos(p\pi h)-1)$$

$$= \frac{2}{h^2}\left(-\frac{p^2\pi^2h^2}{2} + O(h^4)\right)$$

$$= -p^2\pi^2 + O(h^2)$$

(for small ph)

So the numerical "modes" decay at almost the same rate as in the exact solution, for small ph.

Stability

Use Euler's method in time:

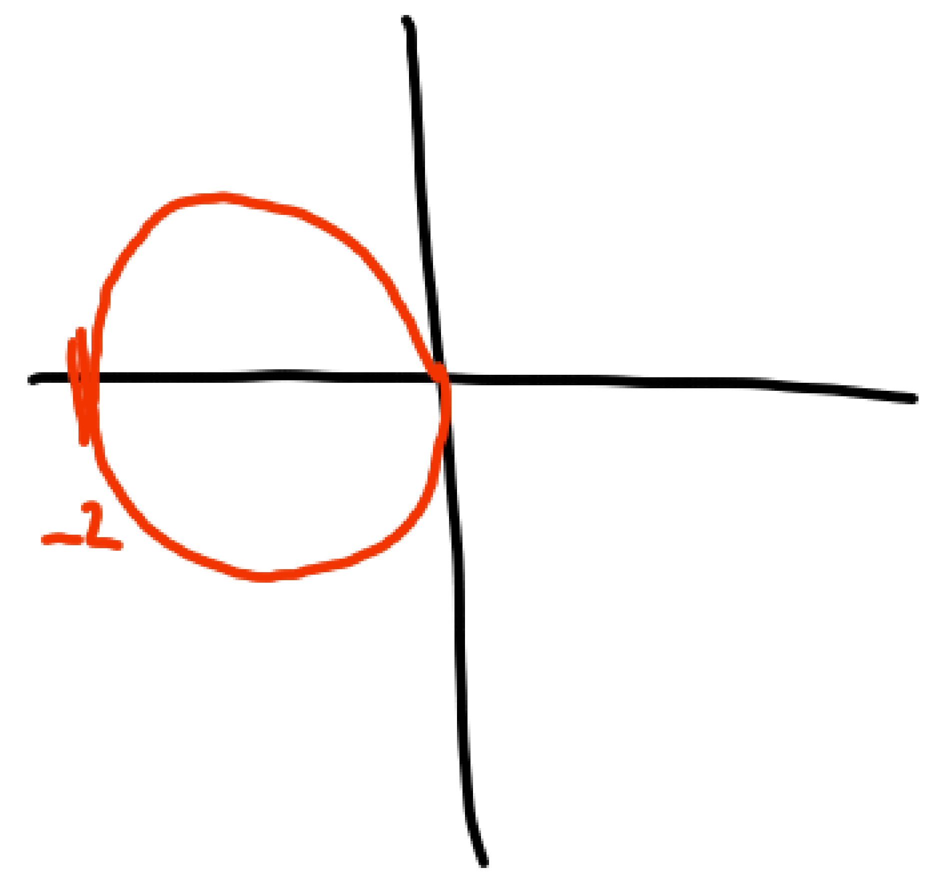
$$U^{n+1} = U^n + KAU^n$$

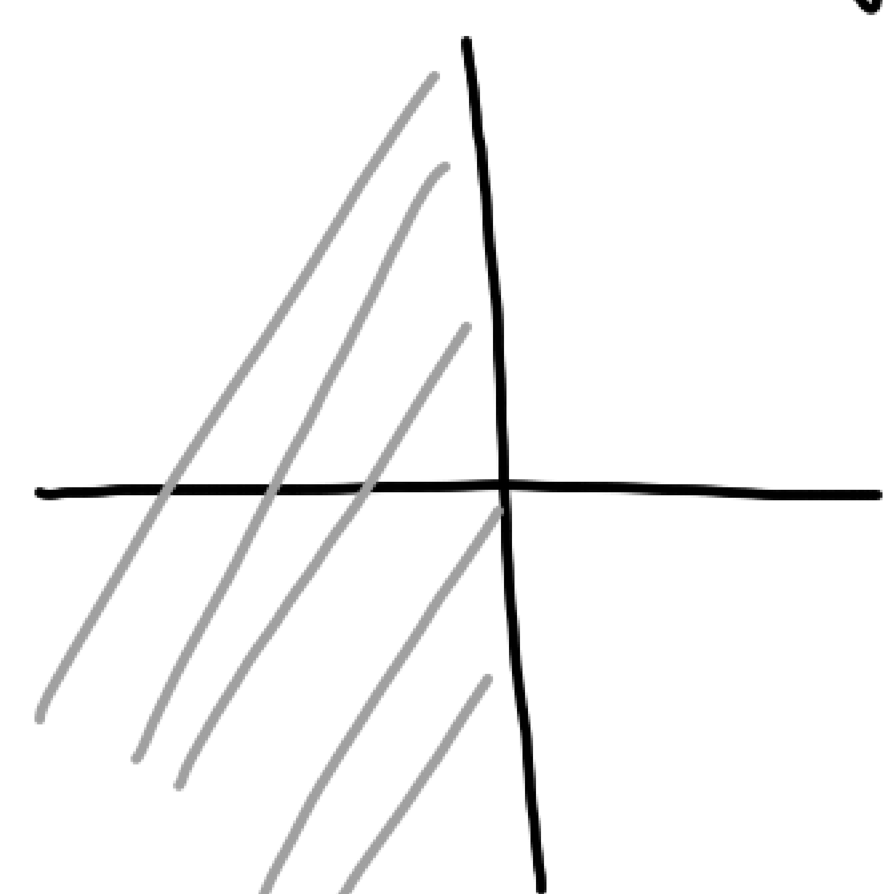
For absolute stability, we need:

$$-2 \le K\lambda_p \le 0$$

The largest magnitude eigenvalue is $\lambda_p \approx \frac{-4}{h^2}$

$$-2 \le \frac{-4K}{h^2} \Rightarrow K \le \frac{h^2}{2}$$





The original IBVP is infinitely stiff since $\lim_{p\to\infty} p^2\pi^2 = \infty$, but our discretized problem has finite stiffness.

We should use an A-stable or $A(\alpha)$-stable method, so we have absolute stability with any step size.

Diagonally Implicit RK (DIRK)

$$Y_i = U^n + k\sum_{j=1}^{i} a_{ij} f(Y_j)$$

$$U^{n+1} = U^n + k\sum_{j=1}^{r} b_j f(Y_j)$$

$$\begin{array}{c|c} c & A \\ \hline & b^T \end{array}$$

A is lower-triangular

More efficient than fully implicit RK methods since we can solve each stage sequentially.

For example:

$$\begin{array}{c|ccc} 0 & 0 & 0 & 0 \\ 1/2 & 1/4 & 1/4 & 0 \\ 1 & 1/3 & 1/3 & 1/3 \\ \hline & 1/3 & 1/3 & 1/3 \end{array}$$

A-stable

L-stable

2nd-order accurate

TR-BDF2

Fully-discrete scheme (Euler):

$$U_j^{n+1} = U_j^n + \frac{K}{h^2}\left(U_{j+1}^n - 2U_j^n + U_{j-1}^n\right)$$

To get LTE:

$$u(x_j, t_{n+1}) = u(x_j, t_n) + \frac{K}{h^2}\left(u(x_{j+1}, t_n) - 2u(x_j, t_n) + u(x_{j-1}, t_n)\right) + \tau_{j+1}^n$$

After using Taylor series we get

$$\tau_j^n = \frac{K}{2}u_{tt} - \frac{h^2}{12}u_{xxx} + O(K^2) + O(h^4)$$

1st order in time

2nd order in space