Poisson's Equation

$$\nabla^2 u = f(x,y)$$

Applications:

u	f
Temperature	Heat source
Electrical potential	Dist. of charge
Probability	Potential
Concentration	Source
Grav. potential	Mass

In some cases we have

$$\nabla \cdot (K(x,y)\nabla u) = f(x,y)$$

where K is heat conductivity, permittivity, etc.

2D Heat diffusion

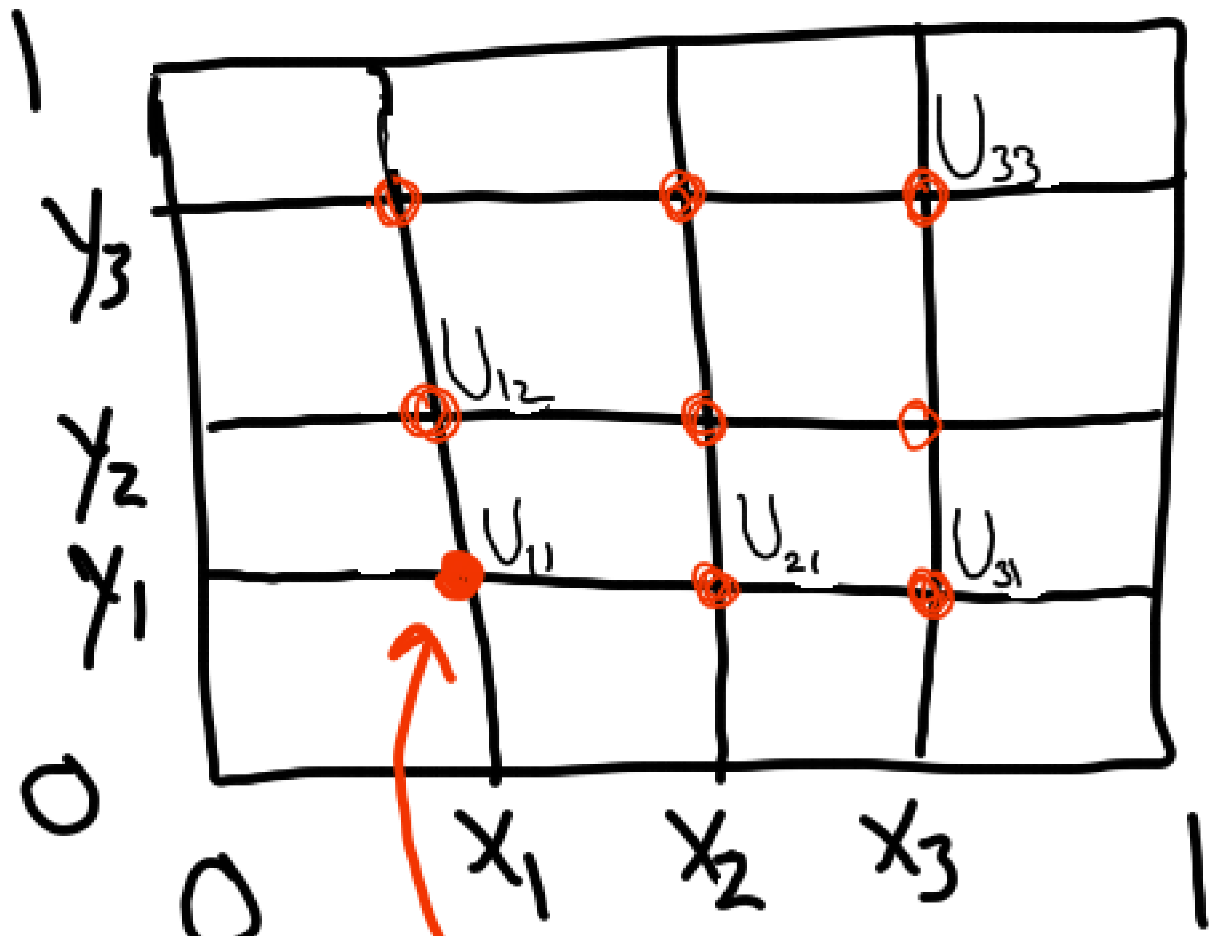
$$u_{xx} + u_{yy} = f(x,y) \qquad \begin{matrix} 0<x<1 \\ 0<y<1 \end{matrix}$$

$$u(x,0) = \alpha(x)$$

$$u(x,1) = \beta(x)$$

$$u(0,y) = \gamma(y)$$

$$u(1,y) = \delta(y)$$



$U_{ij} \approx u(x_i, y_j)$

$\Delta x = \Delta y = h$

$AU = F$

$$\frac{1}{h^2}\left[U_{i+1j} + U_{i-1j} + U_{ij+1} + U_{ij-1} - 4U_{ij}\right] = f_{ij} = f(x_i, y_j)$$

$1 \le i, j \le m$

$$u_{xx}(x_i, y_j) \approx \frac{U_{i+1j} - 2U_{ij} + U_{i-1j}}{\Delta x^2}$$

$$u_{yy}(x_i, y_j) \approx \frac{U_{ij+1} - 2U_{ij} + U_{ij-1}}{\Delta y^2}$$

Row-wise ordering:

$$U = \begin{bmatrix} U_{11} \\ U_{21} \\ U_{31} \\ U_{12} \\ \vdots \\ U_{33} \end{bmatrix}, \quad A = \frac{1}{h^2}\begin{bmatrix} -4 & 1 & & 1 & & & & & \\ 1 & -4 & 1 & & 1 & & & & \\ & 1 & -4 & & & 1 & & & \\ 1 & & & -4 & 1 & & 1 & & \\ & 1 & & 1 & -4 & 1 & & 1 & \\ & & 1 & & 1 & -4 & & & 1 \\ & & & 1 & & & -4 & 1 & \\ & & & & 1 & & 1 & -4 & 1 \\ & & & & & 1 & & 1 & -4 \end{bmatrix}$$

A is sparse

$U \in \mathbb{R}^{m^2}$

A is $m^2 \times m^2$

Only $\sim 5m^2$ of the entries of A are non-zero.

Consistency

$$\frac{1}{h^2}\left[u(x_{i+1}, y_j) + u(x_{i-1}, y_j) + u(x_i, y_{j+1}) + u(x_i, y_{j-1}) - 4u(x_i, y_j)\right] - f(x_i, y_j)$$

$$= \tau_{ij}$$

$$\tau_{ij} = \frac{h^2}{12}\left(u(x_i, y_j)_{xxxx} + u(x_i, y_j)_{yyyy}\right) + O(h^4)$$

$$\hat{U} = \begin{bmatrix} u(x_1, y_1) \\ \vdots \\ u(x_m, y_m) \end{bmatrix}$$

$AU = F$

$A\hat{U} = F + \tau$

$A(U - \hat{U}) = -\tau$

$AE = -\tau$

$\|E\| \leq \|A^{-1}\| \|\tau\|$

We need $\|A^{-1}\| < C$ as $h \to 0$.

Let's show that $\|A^{-1}\|_2 < C$.

We need to show that the eigenvalues of A are bounded away from zero as $h \to 0$.

Let $AV = \lambda V$.

$$AV = \boxed{\frac{V_{i+1\,j} - 2V_{ij} + V_{i-1\,j}}{\Delta x^2} + \frac{V_{i\,j+1} - 2V_{ij} + V_{i\,j-1}}{\Delta y^2} = \lambda V_{ij}}$$

Assume: $V_{ij} = R_i S_j$

$$\frac{R_{i+1}S_j - 2R_iS_j + R_{i-1}S_j}{\Delta x^2} + R_i\frac{S_{j+1} - 2S_j + S_{j-1}}{\Delta y^2} = \lambda R_i S_j$$

$$\underbrace{\frac{R_{i+1} - 2R_i + R_{i-1}}{R_i \Delta x^2}}_{=C_1} + \underbrace{\frac{S_{j+1} - 2S_j + S_{j-1}}{S_j \Delta y^2}}_{=C_2} = \lambda$$

$$R_{i+1} + (-2 - C_1\Delta x^2)R_i + R_{i-1} = 0 \qquad 1 \le i \le m$$

$$R_0 = R_{m+1} = 0$$

Ansatz: $R_i = \xi^i$

$$C_1 = \frac{2}{\Delta x^2}\left(\cos(p\pi\Delta x)-1\right) \quad p = 1, 2, \ldots, m$$

$$C_2 = \frac{2}{\Delta y^2}\left(\cos(q\pi\Delta y)-1\right) \quad q = 1, 2, \ldots, m$$

$$\lambda_{pq} = 2\left[\frac{\cos(p\pi\Delta x)-1}{\Delta x^2} + \frac{\cos(q\pi\Delta y)-1}{\Delta y^2}\right]$$

$$\lambda_{pq} = 2\left[-\frac{\pi^2\Delta x^2}{2\Delta x^2} - \frac{\pi^2\Delta y^2}{2\Delta y^2}\right] + O(\Delta x^4, \Delta y^4)$$

$$\lambda_{11} = -2\pi^2 + O(\Delta x^4, \Delta y^4)$$

$$\|A^{-1}\|_2 < \frac{1}{2\pi^2}$$

for small h.

Grid-function norms

$V_j \approx v(x_j)$

What does $\|V_j\|_p$ mean?

We want $\|V_j\|_p \approx \|v(x)\|_p = \left(\int |v(x)|^p \, dx \right)^{1/p}$

as $h \to 0$

$$\|V\|_p = \left(h \sum V_j^p \right)^{1/p}$$