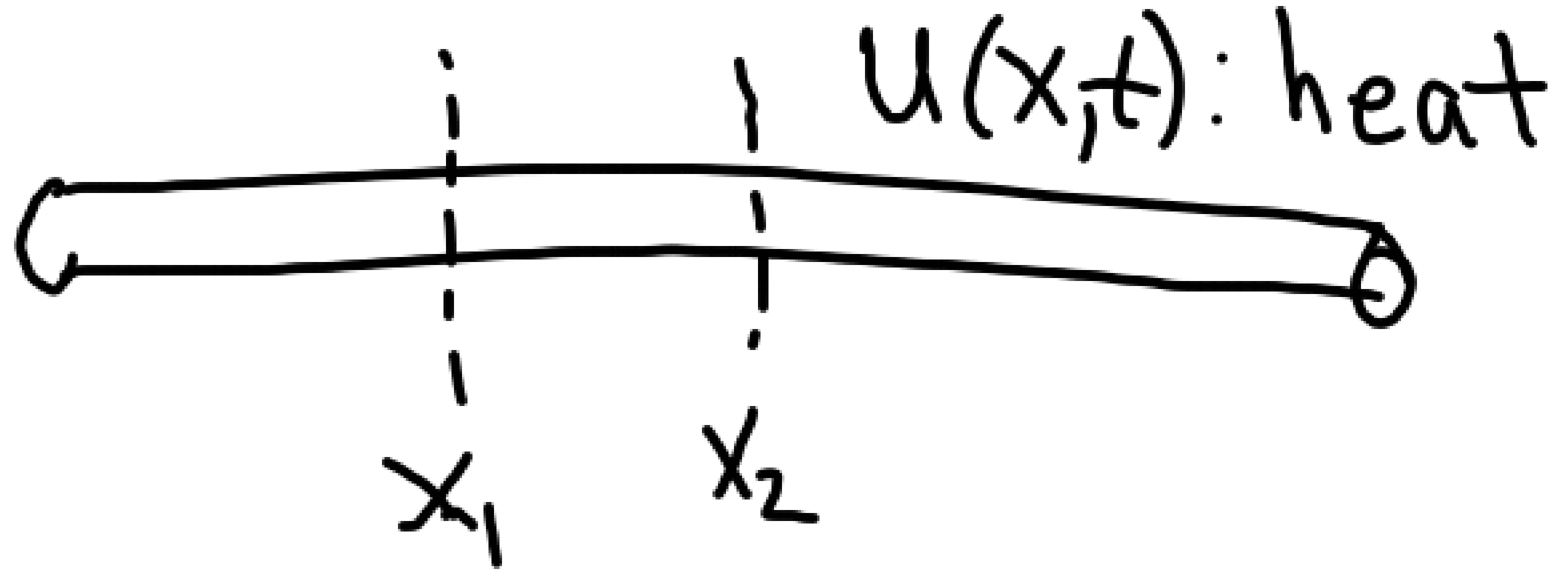
Distribution of heat in a rod



Total heat in (x_1,x_2): $\int_{x_1}^{x_2} u(x,t)\,dx$

This changes only due to flux through the endpoints x_1, x_2

$$\frac{d}{dt}\int_{x_1}^{x_2} u(x,t)\,dx = f(u(x_1,t)) - f(u(x_2,t))$$

$$\int_{x_1}^{x_2} \frac{\partial}{\partial t} u(x,t)\,dx = -\int_{x_1}^{x_2} \frac{\partial}{\partial x} f(u(x,t))\,dx$$

$$\int_{x_1}^{x_2} \left(\frac{\partial}{\partial t} u(x,t) + \frac{\partial}{\partial x} f(u(x,t))\right) dx = 0$$

This integrand must vanish everywhere:

$$u_t + f(u)_x = 0 \qquad (1)$$

This is the general form of a conservation law.

Fick's law of diffusion:

$$f(u) = -Ku_x$$

Substitute into (1):

$$u_t = Ku_{xx}$$ Heat equation

Let's include a heat source/sink along the length of the rod:

$$u_t = Ku_{xx} + \psi(x) \qquad \begin{matrix} 0<x<1 \\ t>0 \end{matrix}$$

Fix the temperature at each end:

$$u(0,t) = \alpha \qquad u(1,t) = \beta$$ Dirichlet Boundary Conditions

As $t \to \infty$, $u(x,t)$ will tend to a steady state.

$$Ku_{xx} + \psi(x) = 0$$

$$u_{xx} = \frac{-\psi(x)}{K} = f(x)$$ (not the flux)

$$\boxed{\begin{matrix} u''(x) = f(x) \\ u(0) = \alpha \quad u(1) = \beta \end{matrix}}$$

Discretize:

$$\vdash h \dashv$$

$$0 = x_0 \quad x_1 \quad x_2 \quad x_3 \quad \cdots \quad x_{m+1} = 1$$

$$U = \begin{bmatrix} U_0 \\ U_1 \\ \vdots \\ \vdots \\ U_{m+1} \end{bmatrix} \qquad U_j \approx u(x_j) \qquad h = \frac{1}{m+1}$$

$$U_0 = \alpha \qquad U_{m+1} = \beta$$

\# of unknowns $= m$

$$u''(x_j) = \frac{U_{j+1} - 2U_j + U_{j-1}}{h^2}$$

So we have:

$$\frac{U_{j+1} - 2U_j + U_{j-1}}{h^2} = f(x_j) \quad \text{for } j = 1, \dots, m$$

$$\underbrace{\frac{1}{h^2}\begin{bmatrix} -2 & 1 & & & & \\ 1 & -2 & 1 & & & \\ & 1 & -2 & 1 & & \\ & & \ddots & \ddots & \ddots & \\ & & & 1 & -2 & 1 \\ & & & & 1 & -2 \end{bmatrix} \begin{bmatrix} U_1 \\ \vdots \\ \vdots \\ \vdots \\ U_m \end{bmatrix} = \begin{bmatrix} f(x_1) - \frac{\alpha}{h^2} \\ f(x_2) \\ \vdots \\ \vdots \\ f(x_m) - \frac{\beta}{h^2} \end{bmatrix}}_{AU=F}$$

$$j=1: \quad \frac{U_2 - 2U_1 + \overset{\alpha}{\cancel{U_0}}}{h^2} = f(x_1)$$

$$\frac{U_2 - 2U_1}{h^2} = f(x_1) - \frac{\alpha}{h^2}$$

$$\hat{U} = \begin{bmatrix} u(x_1) \\ u(x_2) \\ \vdots \\ u(x_m) \end{bmatrix}$$

Global error:

$$E = U - \hat{U}$$

Definition

Let U^h denote the numerical solution obtained with mesh width h, and let $\hat{U}^h$ denote the exact solution restricted to the same mesh. We say the solution is convergent if

$$\lim_{h \to 0} \| U^h - \hat{U}^h \| = 0$$

To prove convergence, we need 2 things:

1. Consistency
2. Stability

Consistency

The exact solution does not satisfy the numerical scheme:

Local Truncation Error

$$\underbrace{\frac{u(x_{j+1}) - 2u(x_j) + u(x_{j-1})}{h^2}} = f(x_j) + \tau_j$$

$$\cancel{u''(x_j)} + \frac{1}{12} h^2 u^{(4)}(x_j) + O(h^4) = \cancel{f(x_j)} + \tau_j$$

$$\tau_j = \frac{1}{12} h^2 u^{(4)}(x_j) + O(h^4)$$

We say a discretization is <u>consistent</u> if $\lim_{h\to 0} \tau_j = 0$.

We have

$$AU = F$$
$$A\hat{U} = F + \tau$$

$$\Rightarrow A(U - \hat{U}) = -\tau$$
$$AE = -\tau$$

For discretizations of linear differential equations, the error satisfies the same equation as the numerical solution, but with the LTE as RHS.

$$\|AE\| = \|\tau\|$$
$$\|AE\| \le \|A\| \cdot \|E\| \quad \text{(submultiplicativity)}$$

Here $\|A\|$ is the induced matrix norm:

$$\|A\| = \sup_{\|x\| \neq 0} \frac{\|Ax\|}{\|x\|}$$

$$E = -A^{-1}\tau$$
$$\|E\| = \|A^{-1}\tau\| \le \|A^{-1}\| \cdot \|\tau\|$$

We just need to show that

$\|A^{-1}\| < C$ for small enough h

C is independent of h.

Stability

So we have

$$\|E\| \le C\underbrace{\|\tau\|}_{\text{here we should use a grid norm (see text appendix)}} = C\,\mathcal{O}(h^2) \Rightarrow \text{convergence}$$

Consider $\|A^{-1}\|_2$:

Since A is symmetric (normal),

$$\|A\|_2 = \max_{\lambda_p \in \sigma(A)} |\lambda_p| = \rho(A) \quad \text{spectral radius}$$

Let λ_p, $p=1,2,\dots,m$ denote eigenvalues of A.

What are the eigenvalues of A^{-1}?

$$\frac{1}{\lambda_p} \Rightarrow \|A^{-1}\|_2 = \frac{1}{\min|\lambda_p|}$$

$$h^2 A = \begin{bmatrix} -2 & 1 & & & \\ 1 & \ddots & \ddots & & \\ & \ddots & \ddots & \ddots & \\ & & \ddots & \ddots & 1 \\ & & & 1 & -2 \end{bmatrix} = \hat{A}^{m\times m}$$

$$\hat{A}v = \lambda v$$

$$v_{j-1} - 2v_j + v_{j+1} = \lambda v_j \quad j=1,\dots,m$$

$$v_0 = 0 \quad v_{m+1} = 0$$

Linear difference equations

Ansatz: $v_j = \xi^j$

$$\rho^{j-1} - 2\rho^{j} + \rho^{j+1} = \lambda\rho^{j}$$

$$\rho^2 - (2+\lambda)\rho + 1 = 0$$

$$\rho_{\pm} = \frac{2+\lambda}{2} \pm \frac{\sqrt{(2+\lambda)^2 - 4}}{2}$$

$$\rho_{\pm} = 1 + \frac{\lambda}{2} \pm \frac{\sqrt{\lambda^2 + 4\lambda}}{2}$$] Fundamental Solutions

$$V_j = a\rho_+^{j} + b\rho_-^{j}$$

$$V_0 = a + b = 0$$

$$b = -a$$

So $V_j = a(\rho_+^{j} - \rho_-^{j})$

$$V_{m+1} = a(\rho_+^{m+1} - \rho_-^{m+1}) = 0$$

$$\rho_+^{m+1} = \rho_-^{m+1}$$

$$\rho_+\rho_- = 1$$

$$\rho_+^{m+1}\rho_+^{m+1} = (\rho_+\rho_-)^{m+1} = 1$$

$$\rho_+^{2m+2} = 1 \Rightarrow \rho_+ = e^{i\pi\left(\frac{p}{m+1}\right)}$$ $p = 1, 2, \ldots, m$

$$\rho_- = \frac{1}{\rho_+} = e^{-i\pi\left(\frac{p}{m+1}\right)}$$

$$\zeta_+ + \zeta_- = 2 + \lambda$$

$$e^{i\frac{p\pi}{m+1}} + e^{-i\frac{p\pi}{m+1}} = 2 + \lambda$$

$$2\cos\left(\frac{p\pi}{m+1}\right) = 2 + \lambda$$

$$\lambda_p = 2\left(\cos\left(\frac{p\pi}{m+1}\right) - 1\right) \quad p = 1, \dots, m$$

Eigenvalues of A:

$$\frac{2}{h^2}\left(\cos\left(\frac{p\pi}{m+1}\right) - 1\right) < 0$$

The smallest one is $\approx -\pi^2 + O(h^2)$

$$\|A^{-1}\| = \frac{1}{\pi^2} + O(h^2) \Rightarrow \|A^{-1}\| < C$$