We've studied:

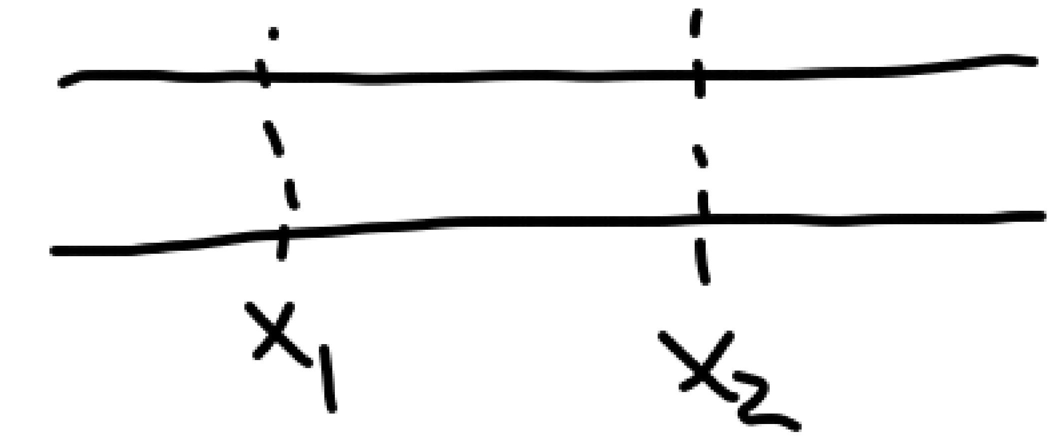
Elliptic PDE: $\nabla^2 u = f(\vec{x})$

Parabolic PDE: $u_t = \nabla^2 u$

Today: Hyperbolic PDE

Hyperbolic PDEs model waves:

- Water
- Pressure
- Electromagnetic
- Fluid motions



Chemical concentration: $u(x,t)$

Total amount of u in $[x_1, x_2]$:

$$\int_{x_1}^{x_2} u(x,t)\,dx$$

This changes in time only due to flux through the endpoints:

$$\frac{d}{dt}\int_{x_1}^{x_2} u(x,t)\,dx = f(u(x_1,t)) - f(u(x_2,t))$$

$$\int_{x_1}^{x_2} \frac{\partial}{\partial t} u(x,t)\,dx = -\int_{x_1}^{x_2} \frac{\partial}{\partial x} f(u(x,t))\,dx$$

$$\int_{x_1}^{x_2} \underbrace{\left(u_t + f(u)_x\right)}_{\text{must vanish ptwise}} dx = 0$$

$$u_t + f(u)_x = 0$$ Conservation law

Fluid flowing with velocity a: $f(u) = au$

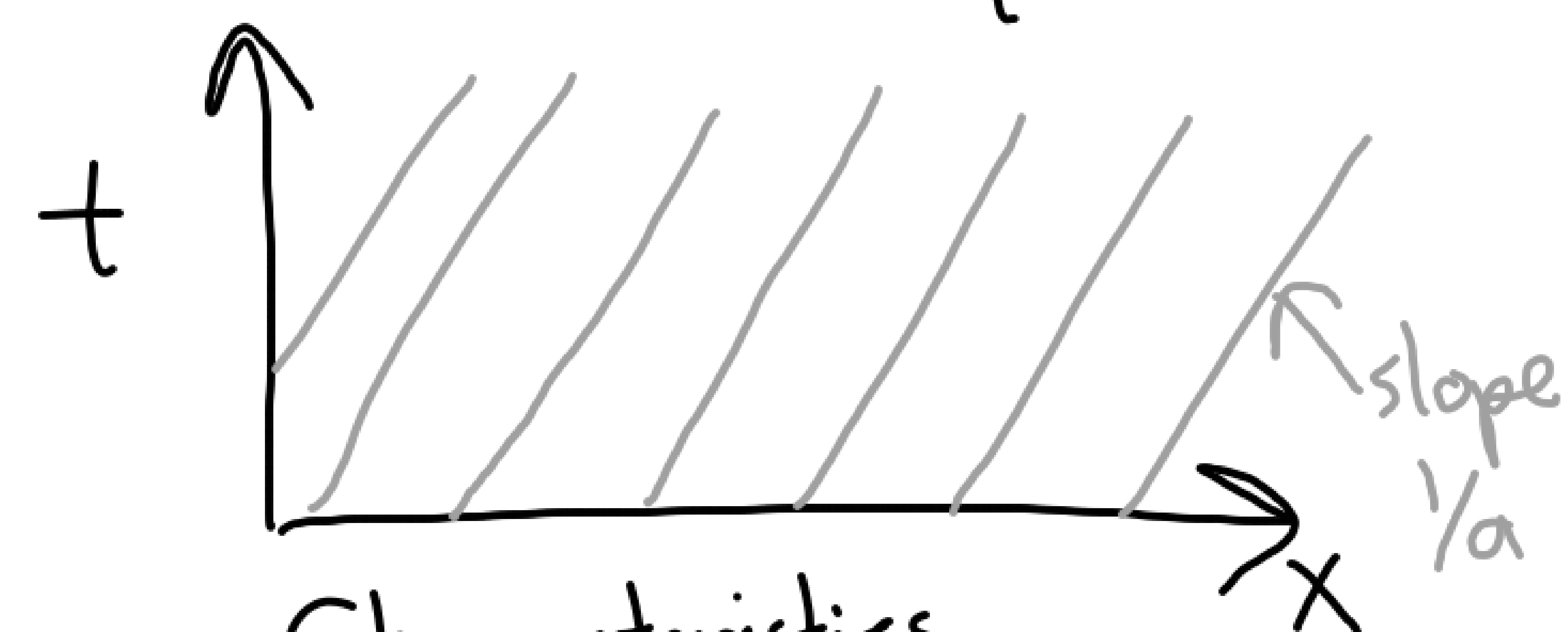
$$u_t + a u_x = 0$$ Advection equation

Initial data: $u(x,0) = \eta(x)$

Consider the Cauchy problem:

$-\infty < x < \infty$

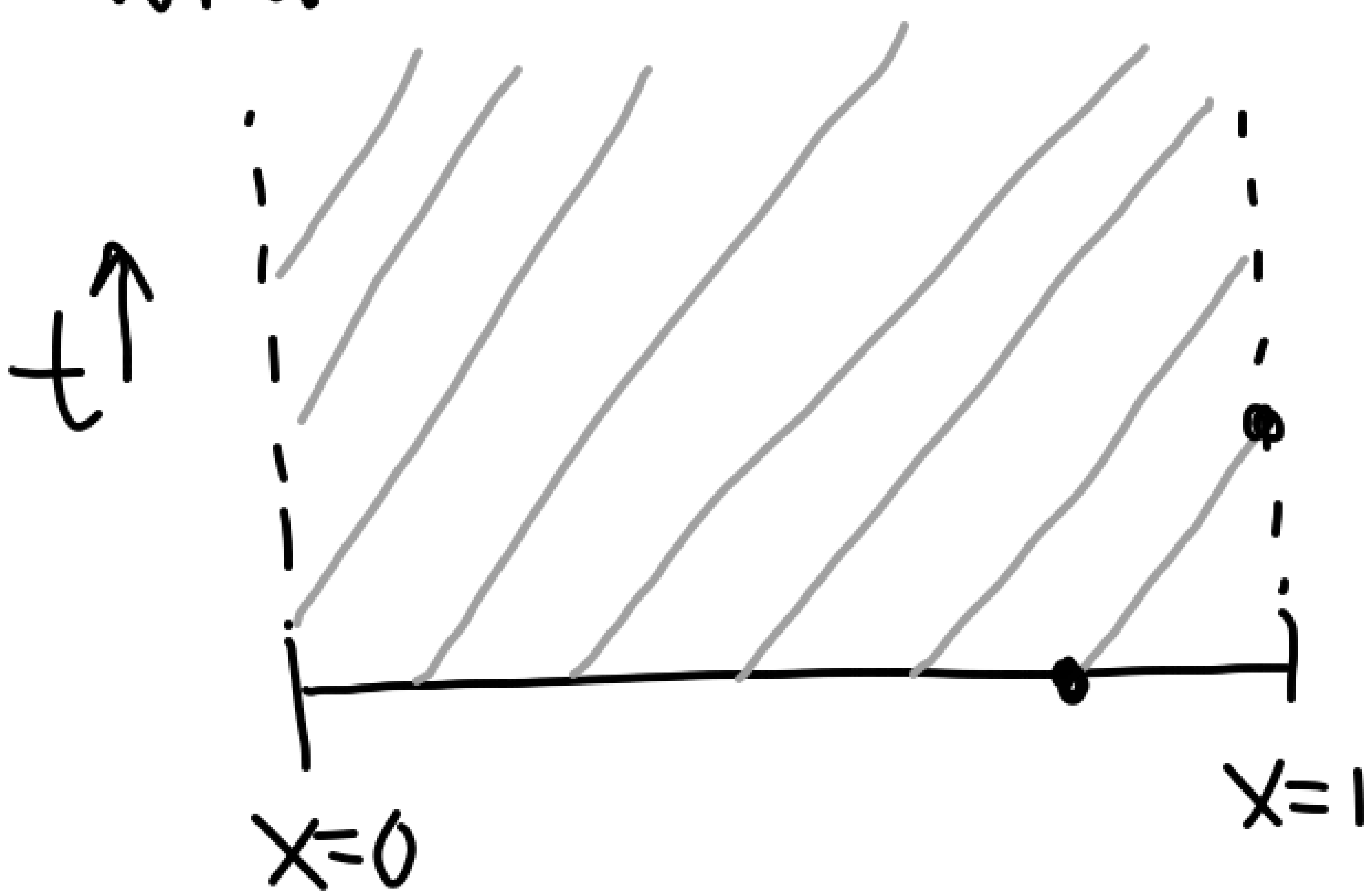
Solution: $u(x,t) = \eta(x - at)$



Characteristics

Solution is constant along each characteristic.

What if $0<x<1$?



We need boundary values only at the left.

Discretization

C.D. in space:

$$U_j'(t) = -\frac{a}{2h}\left(U_{j+1}(t) - U_{j-1}(t)\right)$$

Euler in time:

$$U_j^{n+1} = U_j^n - \frac{ka}{2h}\left(U_{j+1}^n - U_{j-1}^n\right)$$

2nd order in space, 1st order in time.

Is it stable?

- Von Neumann analysis
- Method of lines analysis

Von Neumann analysis

$$U_j^n \to g^n e^{ijh\xi}$$

$$g^{n+1} e^{ijh\xi} = g^n e^{ijh\xi}\left(1 - \frac{ka}{2h}\left(e^{ih\xi} - e^{-ih\xi}\right)\right)$$

$$g = 1 - \frac{ka}{2h}\left(2i\sin(h\xi)\right)$$

$$g = 1 - i\frac{ka}{h}\sin(h\xi)$$

$$|g|^2 = 1 + \left(\frac{ka}{h}\sin(h\xi)\right)^2 > 1$$

$$|g| > 1 \Rightarrow \text{unstable}$$

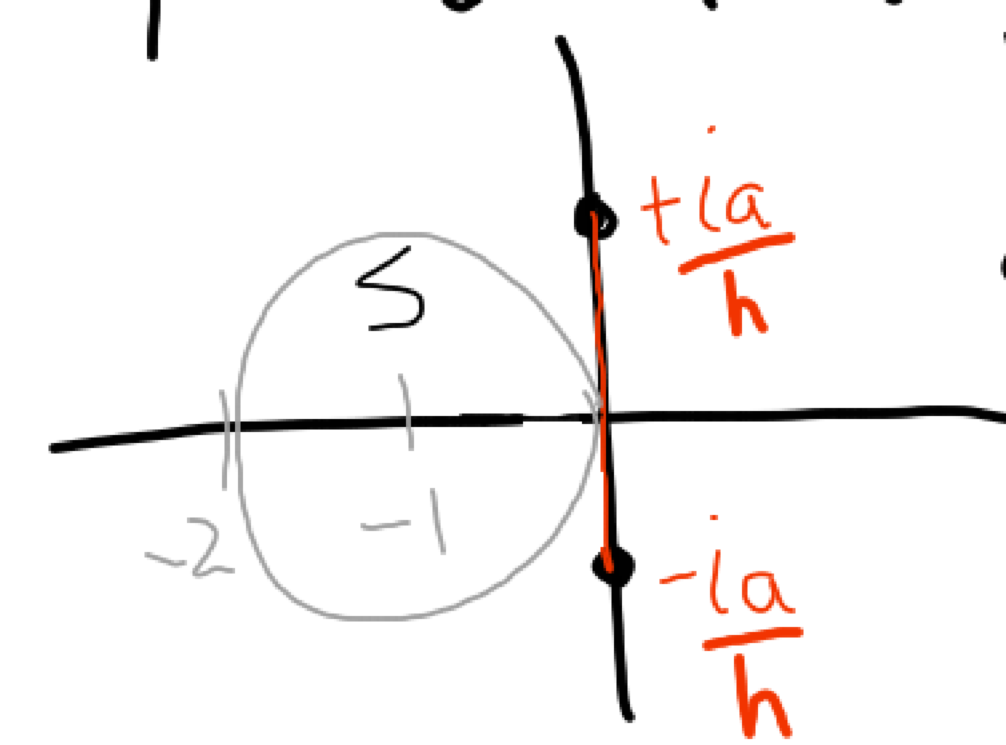
MOL Analysis

$$U_j'(t) = -\frac{a}{2h}\left(U_{j+1} - U_{j-1}\right)$$

$$U'(t) = -\frac{a}{2h}\begin{bmatrix} 0 & 1 & & & \\ -1 & 0 & 1 & & \\ & \ddots & \ddots & \ddots & \\ & & -1 & 0 & 1 \\ & & & -1 & 0 \end{bmatrix}$$

Skew-symmetric matrix

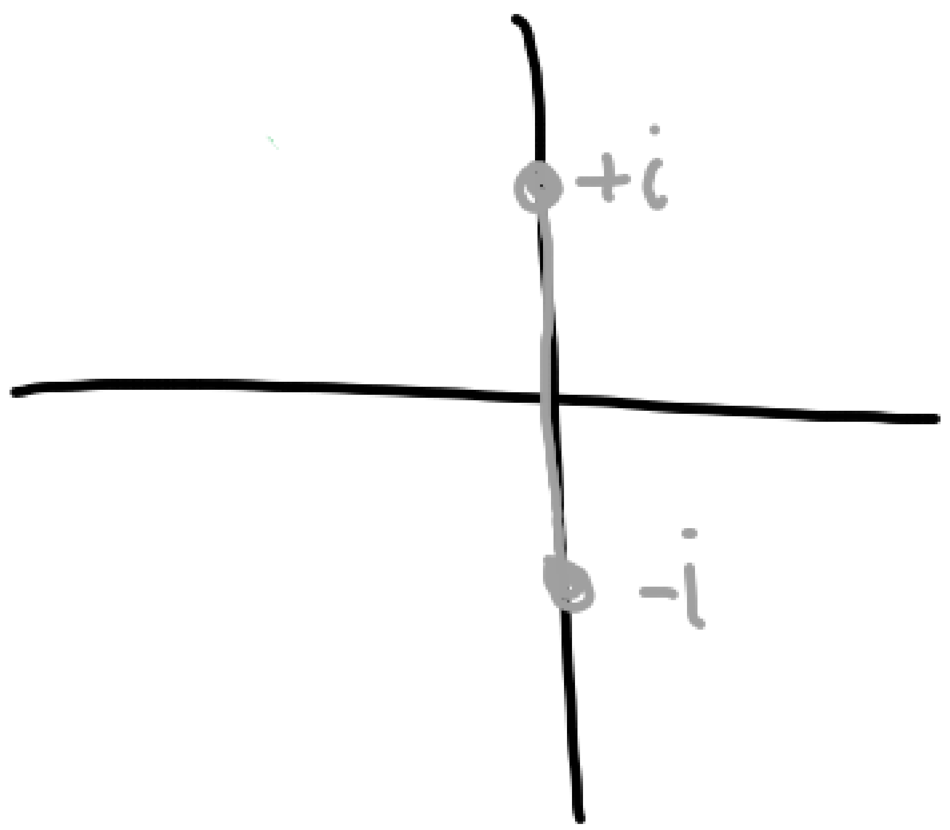
Purely imaginary eigenvalues



No way to choose k so that $k\lambda \in S$

Leapfrog + CD

$$U_j^{n+1} = U_j^{n-1} - \frac{ka}{h}\left(U_{j+1}^n - U_{j-1}^n\right)$$



$\left|\frac{ka}{h}\right| \leq 1$ for stability

Lax-Friedrichs

$$U_j^{n+1} = \frac{U_{j+1}^n + U_{j-1}^n}{2} - \frac{ka}{2h}\left(U_{j+1}^n - U_{j-1}^n\right)$$

Since: $\frac{1}{2}\left(U_{j+1}^n + U_{j-1}^n\right) = U_j^n + \frac{1}{2}\left(U_{j+1}^n - 2U_j^n + U_{j-1}^n\right)$

We can write this a

$$U_j^{n+1} = -\frac{ka}{2h}\left(U_{j+1}^n - U_{j-1}^n\right) + U_j^n + \frac{1}{2}\left(U_{j+1}^n - 2U_j^n + U_{j-1}^n\right)$$

$$u_t + au_x = 0$$

$$\frac{U_j^{n+1}-U_j^n}{K}+a\frac{U_{j+1}^n-U_{j-1}^n}{2h}=\frac{h^2}{2K}\cdot\frac{U_{j+1}^n-2U_j^n+U_{j-1}^n}{h^2}$$

Seems like an approximation of

$$u_t+au_x=\underbrace{\frac{h^2}{2K}u_{xx}}_{\substack{\text{Stabilizes}\\\text{the method}\\\text{"artificial viscosity"}}}$$