$$f'(x_1) = a f(x_0) + b f(x_1) + c f(x_2)$$

$$f(x_0) = f(x_1) + (x_0 - x_1) f'(x_1) + \frac{(x_0 - x_1)^2}{2} f''(x_1) + \frac{(x_0 - x_1)^3}{6} f'''(x_1) + O((x_0 - x_1)^4)$$

$$f(x_2) = f(x_1) + (x_2 - x_1) f'(x_1) + \frac{(x_2 - x_1)^2}{2} f''(x_1) + \frac{(x_2 - x_1)^3}{6} f'''(x_1) + O((x_2 - x_1)^4$$

$$(a + b + c) = 0$$

$$LTE = \left(a \frac{(x_0 - x_1)^3}{6} + c \frac{(x_2 - x_1)^3}{6}\right) f'''(x_1)$$

$$(a(x_0 - x_1) + c(x_2 - x_1)) = 1$$

$$a \frac{(x_0 - x_1)^2}{2} + b \frac{(x_0 - x_1)^2}{2} = 0$$

$$a = \frac{x_1 - x_2}{(x_0 - x_1)(x_0 - x_2)}$$

$$c = \frac{x_0 - x_1}{(x_0 - x_2)(x_2 - x_1)}$$

$$b = \frac{(x_1 - x_2)^2 - (x_0 - x_1)^2}{(x_0 - x_1)(x_0 - x_2)(x_2 - x_1)}$$

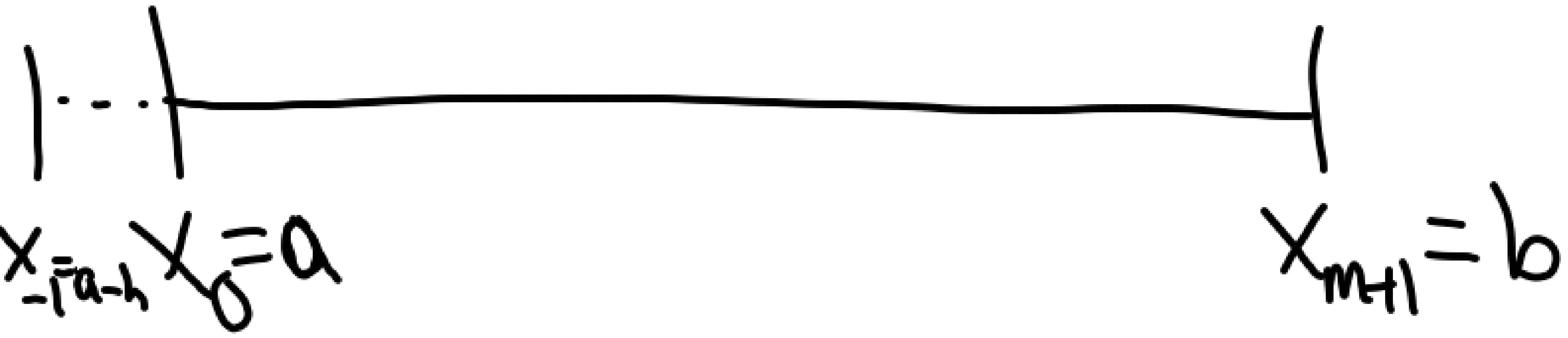
$$D^2 u(x_j) = \frac{u(x_{j-1}) - 2u(x_j) + u(x_{j+1})}{h^2}$$

$$D^2 D^2 u(x_j) = \frac{D^2 u(x_{j-1}) - 2D^2 u(x_j) + D^2 u(x_{j+1})}{h^2}$$

$$D^2 D^2 u(x_j) = \frac{1}{h^4}\Big[u(x_{j-2}) - 2u(x_{j-1}) + u(x_j) - 2\big(u(x_{j-1}) - 2u(x_j) + u(x_{j+1})\big)$$
$$+ u(x_j) - 2u(x_{j+1}) + u(x_{j+2})\Big]$$

$$= \frac{1}{h^4}\Big[u(x_{j-2}) - 4u(x_{j-1}) + 6u(x_j) - 4u(x_{j+1}) + u(x_{j+2})\Big]$$

$$2 \leq j \leq m-1$$



$$U_0 = 0$$
$$U_{m+1} = 0$$

$$u'(a) \approx \frac{U_1 - U_0}{h} = 0 \Rightarrow U_1 = 0$$

$$u'(b) \approx \frac{U_{m+1} - U_m}{h} = 0 \Rightarrow U_m = 0$$

First-order accurate.

$$\frac{U_{j+2} - 4U_{j+1} + 6U_j - 4U_{j-1} + U_{j-2}}{h^4}$$

$$+ \frac{U_{j+1} - 2U_j + U_{j-1}}{h^2} = f(x_j)$$

$$2 \leq j \leq m-1$$

$$U_0 = U_1 = U_m = U_{m+1} = 0$$

b) $A = \text{tridiag}(1, -2, 1)/h^2$

$$\lambda_A = \frac{2}{h^2}\left(\cos(p\pi h) - 1\right) \qquad p = 1, 2, \ldots, m$$

$$h = \frac{1}{m+1}$$

$$A^2 U + AU = F$$

$$\underbrace{(A^2 + A)}_{M} U = F$$

Eigenvalues of M: $\lambda_A^2 + \lambda_A = \lambda_m$

$$\lambda_m = \frac{4}{h^4}\left(\cos(p\pi h) - 1\right)^2 + \frac{2}{h^2}\left(\cos(p\pi h) - 1\right)$$

$$\|M^{-1}\|_2 = \frac{1}{\min_p |\lambda_m|}$$

$$p = 1: \cos(p\pi h) = \cos(\pi h) \approx 1 - \frac{\pi^2 h^2}{2} + O(h^4)$$

$$\min |\lambda_m| \approx \left| \frac{4}{h^4}\left(\frac{-\pi^2 h^2}{2}\right)^2 + \frac{2}{h^2}\left(\frac{-\pi^2 h^2}{2}\right) \right|$$

$$= |\pi^4 - \pi^2| = \pi^4 - \pi^2$$

$$\lim_{h \to 0} \|M^{-1}\| = \frac{1}{\pi^4 - \pi^2} < \infty$$

So the method is stable.