$u_t = K u_{xx} + \psi(x)$ Heat equation

$\Big\downarrow$ Steady state

$u''(x) = f(x)$ Poisson equation

Discretize:

Continuous	Discrete
$0 < x < 1$	$x_j = jh \quad j = 0, 1, \dots, m+1$
$u(x)$	$U = [U_0, \dots, U_{m+1}]$
$\dfrac{d^2}{dx^2}$	$A = \dfrac{1}{h^2}\begin{bmatrix} h^2 & & & & \\ 1 & -2 & 1 & & \\ & \ddots & \ddots & \ddots & \\ & & 1 & -2 & 1 \\ & & & & h^2 \end{bmatrix}$
$u''(x) = f(x)$	$AU = F$

Neumann BCs

For example, if the left end of the rod is insulated:

$$u'(0) = 0$$

(no heat flux through end of rod)

More generally we could have

$$u'(0) = \sigma \qquad u(1) = \beta$$

How can we discretize the Neumann condition?

Method 1: One-sided FD

$$u'(0) \approx D_+ u(0) = \frac{U_1 - U_0}{h} \Rightarrow \frac{U_1 - U_0}{h} = \sigma$$

Recall that

$$D_+ u(\bar{x}) = u'(\bar{x}) + \underbrace{\frac{h}{2} u''(\bar{x})}_{\text{Leading trunc. error}} + O(h^2)$$

So this method is
$\underline{1}$st-order accurate.

We can use one more point to get a more accurate formula:

$$U'(0) = \frac{-\frac{3}{2}U_0 + 2U_1 - \frac{1}{2}U_2}{h} + O(h^2)$$

$$\frac{-\frac{3}{2}U_0 + 2U_1 - \frac{1}{2}U_2}{h} = \sigma$$

This is 2nd-order accurate.

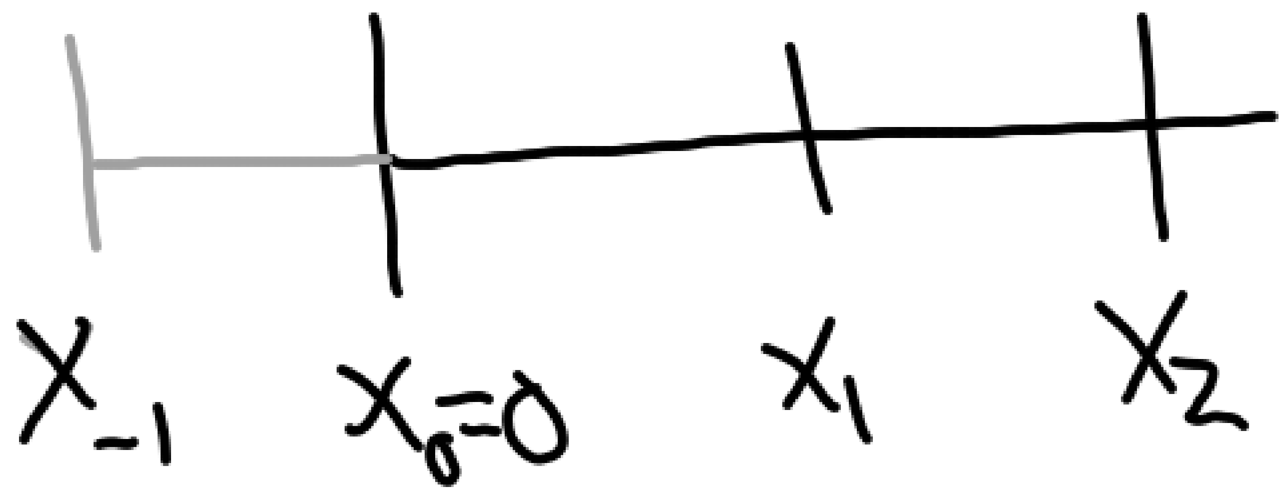
Alternatively: we know

$$\frac{h}{2}u''(0) = \frac{h}{2}f(0)$$

$$\Rightarrow \frac{U_1 - U_0}{h} - \frac{h}{2}f(0) = \sigma$$

Both 2nd-order accurate

Method 2: Ghost point method



We can impose $u''(x)=f(x)$ at $x=0$:

$$\frac{U_1-2U_0+U_{-1}}{h^2}=f(0)$$

$$u'(0)=\sigma \Rightarrow \underbrace{\frac{U_1-U_{-1}}{2h}=\sigma}_{\text{2nd-order accurate}}$$

$$U_{-1}=h^2f(0)-U_1+2U_0$$

$$\frac{U_1-\left(h^2f(0)-U_1+2U_0\right)}{2h}=\sigma$$

$$\frac{2U_1-2U_0}{2h}-\frac{h}{2}f(0)=\sigma$$

$$\frac{U_1-U_0}{h}-\frac{h}{2}f(0)=\sigma$$

$$\frac{U_1-U_0}{h}=\sigma+\frac{h}{2}f(0)$$

$$u'(0) = \sigma \quad u(1) = \beta$$

$$u''(x) = f(x)$$

$$AU = F$$

$$\frac{1}{h^2}\begin{bmatrix} -h & h & & & \\ 1 & -2 & 1 & & \\ & \ddots & \ddots & \ddots & \\ & & 1 & -2 & 1 \\ & & & & h^2 \end{bmatrix} \begin{bmatrix} U_0 \\ U_1 \\ \vdots \\ U_{m+1} \end{bmatrix} = \begin{bmatrix} \sigma + \frac{h}{2} f(0) \\ f(x_1) \\ \vdots \\ f(x_m) \\ \beta \end{bmatrix}$$

$$u''(x) = f(x)$$

$$u'(0) = 0 \quad u'(1) = 0$$

If $f(x) = 0$,

one solution is

$$u(x) = 0$$

In fact $u(x) = c$ is a solution for every c.

$$\int_0^1 U''(x)\,dx = U'(1) - U'(0)$$

$$= \int_0^1 f(x)\,dx$$

So if $\boxed{\int_0^1 f(x)\,dx = U'(1) - U'(0)} = 0$

there will be infinitely many solutions. Otherwise, there are no solutions.

$$U''(x) = f(x)$$

$$U'(0) = \sigma_0 \qquad U'(1) = \sigma_1$$

$$\frac{U_1 - U_0}{h} = \sigma_0 \qquad \frac{U_{m+1} - U_m}{h} = \sigma_1$$

$$\underbrace{\frac{1}{h^2}\begin{bmatrix} -h & h & & & \\ 1 & -2 & 1 & & \\ & \ddots & \ddots & \ddots & \\ & & 1 & -2 & 1 \\ & & & -h & h \end{bmatrix}}_{A} U = \begin{bmatrix} \sigma_0 \\ f(x_1) \\ \vdots \\ f(x_m) \\ \sigma_1 \end{bmatrix}$$

$v = \begin{bmatrix} | \\ \vdots \\ \vdots \\ | \end{bmatrix}$ $Av = 0$

A is singular

Either:

- no solution
- infinitely many solutions

(if $AU = F$, then $A(U + cV) = F$)