# Absolute Stability

(Ch. 7 of LeVeque)

$$u'(t) = \lambda u(t) \qquad \lambda \in \mathbb{C}$$
$$u(0) = \eta$$

Exact solution: $u(t) = e^{\lambda t}\eta$

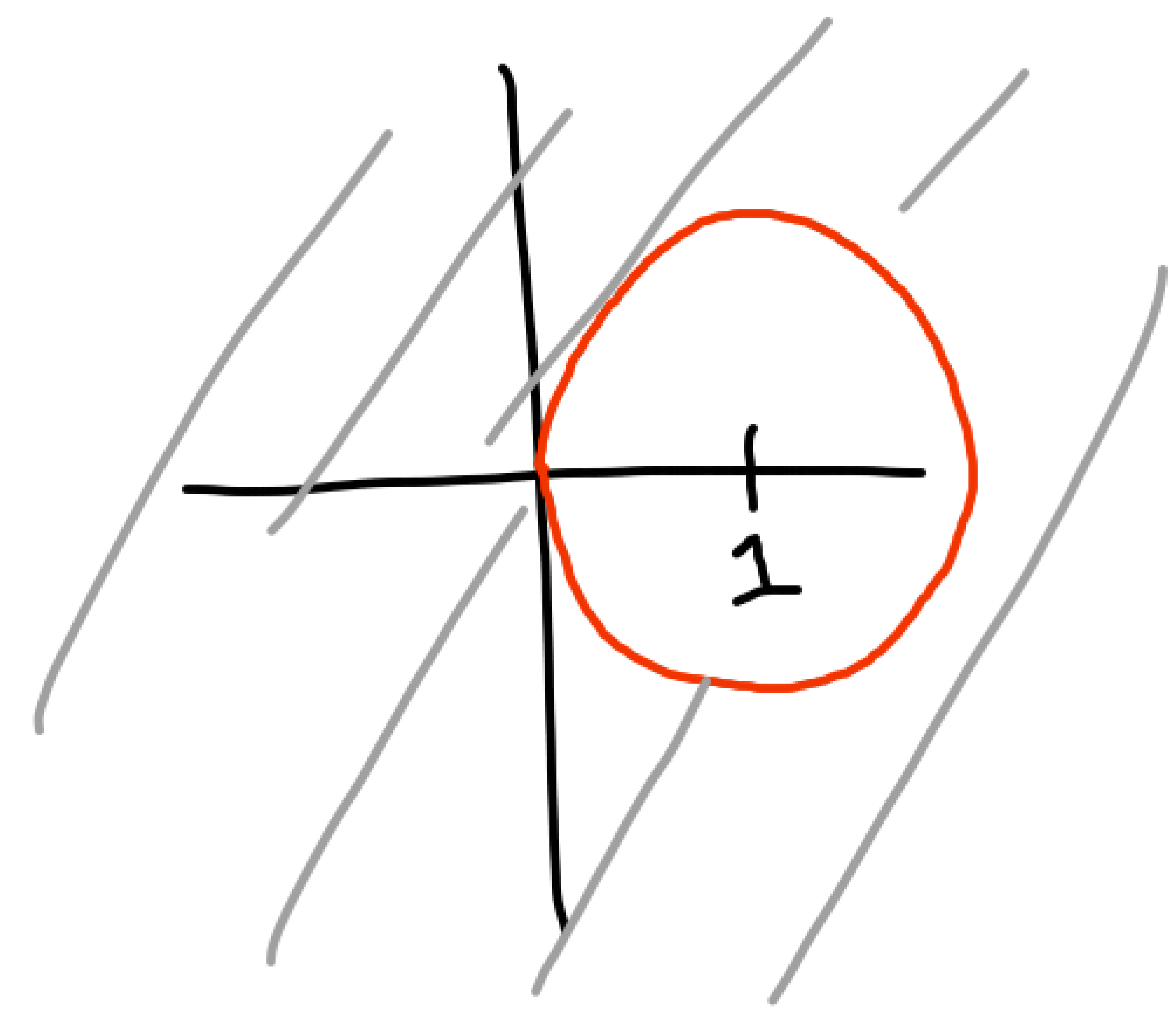
## Backward Euler

$$U^{n+1} = U^n + kf(U^{n+1})$$
$$U^{n+1} = U^n + k\lambda U^{n+1}$$
$$R(k\lambda) \qquad U^{n+1} = \boxed{\frac{1}{1-k\lambda}}U^n$$

$$E^{n+1} = \frac{1}{1-k\lambda}E^n + k\tau^n$$

$$S = \{z \in \mathbb{C} : |R(z)| \le 1\}$$

$$\left|\frac{1}{1-z}\right| \le 1 \iff |1-z| \ge 1$$




$z = k\lambda$

# Trapezoidal Method

$$U^{n+1} = U^n + \frac{k}{2}\left(f(U^n) + f(U^{n+1})\right)$$

$$U^{n+1} = U^n + \frac{k\lambda}{2}\left(U^n + U^{n+1}\right)$$

$$\left(1-\frac{z}{2}\right)U^{n+1} = \left(1+\frac{z}{2}\right)U^n$$

$$U^{n+1} = \frac{1+\frac{z}{2}}{1-\frac{z}{2}}U^n$$

$$R(z) = \frac{1+\frac{z}{2}}{1-\frac{z}{2}}$$

$$\left|1+\frac{z}{2}\right|^2 \le \left|1-\frac{z}{2}\right|^2$$

$$z = x + iy$$

$$\left(1+\frac{x}{2}\right)^2 + \cancel{y^2} \le \left(1-\frac{x}{2}\right)^2 + \cancel{y^2}$$

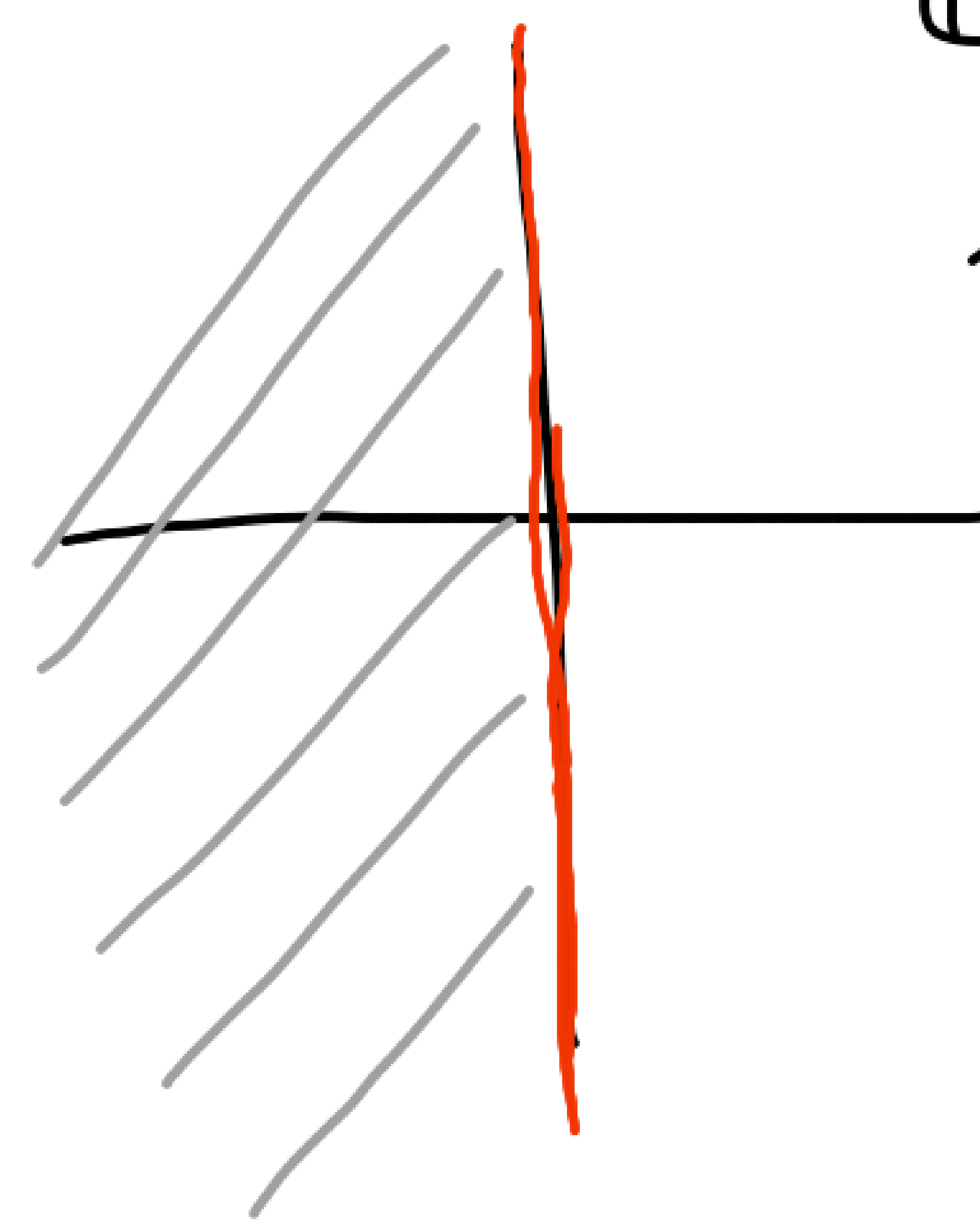
$$x \le -x$$

$$2x \le 0$$

$$x \le 0$$

We say a method is A-stable if

$$\mathbb{C}^- = \{x+iy : x \le 0\} \subseteq S$$



Both Backward Euler and the implicit trapezoidal method are A-stable.

Consider the linear system

$$u'(t) = Lu(t)$$

where $L = \underbrace{R\Lambda R^{-1}}_{\text{Eigenvalue decomposition}}$

$$u'(t) = R\Lambda R^{-1}u(t)$$

$$R^{-1}u'(t) = \Lambda R^{-1}u(t) \qquad w(t) = R^{-1}u(t)$$

$$w'(t) = \Lambda w(t)$$

If we apply a numerical method to this problem, we will have absolute stability iff

$$\sigma(L) \in$$

# Absolute Stability for LMMs

$$\sum_{j=0}^{r} \alpha_j U^{n+j} = k \sum_{j=0}^{r} \beta_j f(U^{n+j})$$

$f(u) = \lambda u$:

$$\sum_{j=0}^{r} (\alpha_j - z\beta_j) U^{n+j} = 0$$

Ansatz: $U^n = \zeta^n$:

$$\pi(\zeta; z) = \sum_{j=0}^{r} (\alpha_j - z\beta_j) \zeta^j = 0$$

We have absolute stability for a given $z \in \mathbb{C}$ if the roots $\{\zeta_1, \dots, \zeta_r\}$ of $\pi(\zeta; z)$ satisfy:

$|\zeta_j| \leq 1$ and

$|\zeta_j| < 1$ if $\zeta_j$ is a multiple root.

## Boundary locus method

$\zeta = e^{i\theta}$

$$\sum_{j=0}^{r} (\alpha_j - z\beta_j) e^{i\theta j} = 0$$

$$\sum \alpha_j e^{i\theta j} = z \sum_j \beta_j e^{i\theta j}$$

$$z = \frac{\sum \alpha_j e^{i\theta j}}{\sum \beta_j e^{i\theta j}}$$

$0 \leq \theta \leq 2\pi$

## Leapfrog

$$U^{n+2} = U^n + 2kf(U^{n+1})$$

$$U^{n+2} = U^n + 2zU^{n+1}$$

$$\zeta^2 - 2z\zeta - 1 = 0$$

$$\zeta_\pm = z \pm \frac{\sqrt{4z^2+4}}{2}$$

$$\zeta_\pm = z \pm \sqrt{z^2+1}$$

$$e^{2i\theta} - 2ze^{i\theta} - 1 = 0$$

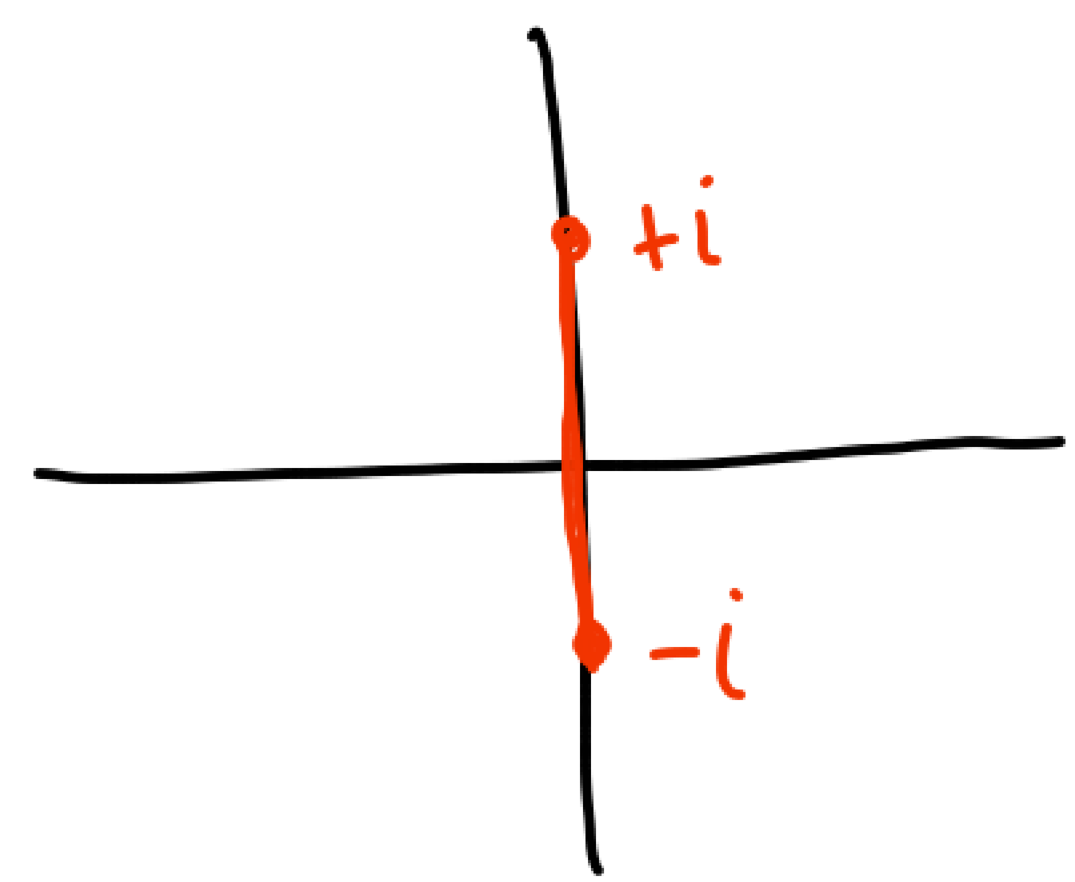
$$z = \frac{e^{i\theta} - e^{-i\theta}}{2}$$

$$e^{2i\theta} - 1 = 2ze^{i\theta}$$

$$e^{i\theta} - e^{-i\theta} = 2z$$

$$z = \frac{e^{i\theta} - e^{-i\theta}}{2}$$

$$z = i\sin\theta$$

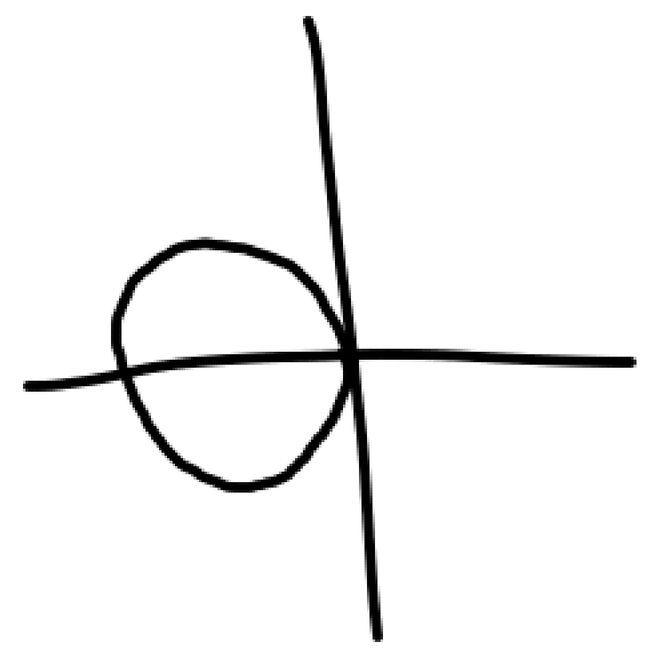


# Linearized Pendulum

$$\theta''(t) = -a\theta(t)$$

$u_1 = \theta(t)$

$u_2 = \theta'(t)$



$u_1'(t) = u_2(t)$

$u_2'(t) = -a u_1(t)$

$$u'(t) = \underbrace{\begin{bmatrix} 0 & 1 \\ -a & 0 \end{bmatrix}}_{L} u(t)$$

$\det(\lambda I - L) = 0$

$\lambda^2 + a = 0 \qquad \lambda = \pm i\sqrt{a}$

Which methods could we use for this problem?

- Leapfrog $\quad k\lambda \in [-i, i]$

  $\pm ik\sqrt{a} \in [-i, i]$

  $0 \le k\sqrt{a} \le 1$

  $0 \le k \le \frac{1}{\sqrt{a}}$

- Backward Euler: Abs. stable $\forall k$

  Pendulum will be damped

- Trapezoidal: Abs. stable $\forall k$

# Runge-Kutta Methods

$$Y_i = U^n + k\sum_{j=1}^{r} a_{ij} f(Y_j, t_n + c_j k) \qquad i = 1, 2, \dots, r$$

$$U^{n+1} = U^n + k\sum_{j=1}^{r} b_j f(Y_j, t_n + c_j k)$$

$$\begin{array}{c|c} c & A \\ \hline & b^T \end{array}$$

Butcher Tableau

For example:

$$\begin{array}{c|cc} 0 & 0 & 0 \\ 1/2 & 1/2 & 0 \\ \hline & 0 & 1 \end{array}$$

$$Y_1 = U^n$$

$$Y_2 = U^n + \frac{k}{2} f(Y_1, t_n)$$

$$U^{n+1} = U^n + k f\left(Y_2, t_n + \frac{k}{2}\right)$$

$$Y = \begin{bmatrix} Y_1 \\ \vdots \\ Y_r \end{bmatrix} \qquad u' = \lambda u(t) \qquad \mathbb{1} = \begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix}$$

$$Y = \mathbb{1} U^n + k\lambda A Y$$

$$(I - zA) Y = \mathbb{1} U^n$$

$$Y = (I - zA)^{-1} \mathbb{1} U^n$$

$$U^{n+1} = U^n + z b^T Y$$

$$U^{n+1} = U^n + z b^T (I - zA)^{-1} \mathbb{1} U^n$$

$$U^{n+1} = \big(\underbrace{1 + z b^T (I - zA)^{-1} \mathbb{1}}_{R(z)}\big) U^n$$

$$(I - zA)^{-1} = I + zA + z^2A^2 + \cdots = \sum_{m=0}^{\infty} (zA)^m$$

If $A$ is strictly lower triangular: $A^r = O_{r \times r}$

$$(I - zA)^{-1} = \sum_{m=0}^{r-1} (zA)^m$$

So $R(z)$ is a polynomial of degree at most $r$.
(Explicit methods)

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For implicit methods

$$R(z) = \frac{P(z)}{Q(z)}$$

Where $P, Q$ are polynomials of degree at most $r$.