Stability and convergence of one-step methods for Initial-value problems

$$u'(t) = f(u,t)$$

$$u(t_0) = \eta$$

Euler's method:

$$\frac{U^{n+1} - U^n}{k} = f(U^n, t_n)$$

We want to show that

$$\lim_{k \to 0} \| \underbrace{U^N - u(t_N)}_{E^N} \| = 0$$

where $T = Nk$ is fixed.

Model problem:

$$u'(t) = \lambda u(t) + g(t)$$

$$\frac{U^{n+1} - U^n}{k} = \lambda U^n + g(t_n)$$

$$\frac{u(t_{n+1}) - u(t_n)}{k} = \lambda u(t_n) + g(t_n) + \tau^n$$

$$\frac{U^{n+1}-u(t_{n+1})-(U^{n}-u(t_{n}))}{k}=\lambda(U^{n}-u(t_{n}))-\tau^{n}$$

$$\frac{E^{n+1}-E^{n}}{k}=\lambda E^{n}-\tau^{n}$$

$$\boxed{E^{n+1}=(1+k\lambda)E^{n}-k\tau^{n}}$$

$$U^{n+1}=(1+k\lambda)U^{n}+kg(t_{n})$$

$$E^{N}=(1+k\lambda)E^{N-1}-k\tau^{N-1}$$

$$E^{N}=(1+k\lambda)\left((1+k\lambda)E^{N-2}-k\tau^{N-2}\right)-k\tau^{N-1}$$

$$E^{N}=(1+k\lambda)^{2}E^{N-2}-k(1+k\lambda)\tau^{N-2}-k\tau^{N-1}$$

$$E^{N}=(1+k\lambda)^{N}\underbrace{E^{0}}_{\text{usually zero}}-k\sum_{m=1}^{N}(1+k\lambda)^{N-m}\tau^{m-1}$$

$$|E^{N}|=\left|-k\sum_{m=1}^{N}(1+k\lambda)^{N-m}\tau^{m-1}\right|$$

$$\leq k\sum_{m=1}^{N}|1+k\lambda|^{N-m}|\tau^{m-1}|$$

$$|1+k\lambda|\leq 1+k|\lambda|\leq\sum_{k=0}^{\infty}\frac{(k|\lambda|)^{j}}{j!}=e^{k|\lambda|}$$

$$|E^{N}|\leq k\sum_{m=1}^{N}e^{k|\lambda|(N-m)}|\tau^{m-1}|$$

$$\leq k\sum e^{Nk|\lambda|}|\tau^{m-1}|=k\sum_{m=1}^{N}e^{T|\lambda|}|\tau^{m-1}|$$

$$|E^N| \le kNe^{T|\lambda|} \max_{1\le m\le N} |\tau^{m-1}|$$

$$\tau = \begin{bmatrix} \tau^0 \\ \tau^1 \\ \vdots \\ \tau^{N-1} \end{bmatrix}$$

$$|E^N| \le \underbrace{Te^{T|\lambda|}}_{\text{indep. of } k.} \underbrace{\|\tau\|_\infty}_{O(k)}$$

$$\lim_{k\to 0} |E^N| = 0$$

What if $T=10, |\lambda|=10$?

$$|E^N| \le 10e^{100}\|\tau\|_\infty$$

Convergence of Euler's method for general IVPs

$$u'(t) = f(u)$$
$$u(t_0) = \eta$$

assume:

$$\|f(v)-f(w)\| \le L\|v-w\|$$

$$\frac{U^{n+1}-U^n}{k} = f(U^n)$$

$$\frac{u(t_{n+1})-u(t_n)}{k} = f(u(t_n)) + \tau^n$$

$$\frac{E^{n+1}-E^n}{k} = f(U^n) - f(u(t_n)) - \tau^n$$

$$E^{n+1} = E^n + k\left(f(U^n) - f(u(t_n))\right) - k\tau^n$$

$$\|E^N\| \le \|E^{N-1}\| + k\|f(U^{N-1}) - f(u(t_{N-1}))\| + k\|\tau^{N-1}\|$$

$$\|E^N\| \le \|E^{N-1}\| + kL\|E^{N-1}\| + k\|\tau^{N-1}\|$$

$$\|E^N\| \le (1+kL)\|E^{N-1}\| + k\|\tau^{N-1}\|$$

$$\|E^N\| \le (1+kL)^N \underbrace{\|E^0\|}_{=0} + k\sum_{m=1}^{N}(1+kL)^{N-m}\|\tau^{m-1}\|$$

$$\|E^N\| \le \underbrace{Te^{TL}}_{\text{Indep. of } k} \underbrace{\max_{1\le m\le N}\|\tau^{m-1}\|}_{O(k)}$$

So $\lim_{k\to\infty}\|E^N\| = 0$

Convergence of Runge-Kutta methods for general IVPs

$$U^* = U^n + \frac{1}{2}kf(U^n)$$

$$U^{n+1} = U^n + kf(U^*)$$

$$\frac{U^{n+1} - U^n}{k} = f\left(U^n + \frac{1}{2}kf(U^n)\right) = \Psi(U^n$$

Claim: if L is a Lipschitz constant for f, then

$$L + \frac{1}{2}kL^2$$

is a Lipschitz constant for Ψ.

$$\|\Psi(v)-\Psi(w)\| = \|f(v+\tfrac{1}{2}kf(v)) - f(w+\tfrac{1}{2}kf(w))\|$$

$$\le L\|v+\tfrac{1}{2}kf(v) - (w+\tfrac{1}{2}kf(w))\|$$

$$\le L\|v-w\| + \tfrac{1}{2}kL\|f(v)-f(w)\|$$

$$\le L\|v-w\| + \tfrac{1}{2}kL^2\|v-w\|$$

$$\underline{\|\Psi(v)-\Psi(w)\| \le (L+\tfrac{1}{2}kL^2)\|v-w\|}$$

$$u(t_{n+1}) = u(t_n) + k\Psi(u(t_n)) + k\tau^n$$

$$U^{n+1} = U^n + k\Psi(U^n)$$

$$E^{n+1} = E^n + k(\Psi(U^n)-\Psi(u(t_n))) - k\tau^n$$

$$\|E^N\| \le \|E^{N-1}\| + k\|\Psi(U^{N-1})-\Psi(u(t_{N-1}))\| + k\|\tau^{N-1}\|$$

$$\|E^N\| \le \|E^{N-1}\| + k(L+\tfrac{1}{2}kL^2)\|E^{N-1}\| + k\|\tau^{N-1}\|$$

$$\|E^N\| \le (1+kL+\tfrac{1}{2}k^2L^2)\|E^{N-1}\| + k\|\tau^{N-1}\|$$

$$\|E^N\| \le (1+kL+\tfrac{1}{2}k^2L^2)^N\|E^0\| + k\sum_{m=1}^{N}(1+kL+\tfrac{1}{2}k^2L^2)^{N-m}\|\tau^{m-1}\|$$

$$\Rightarrow \|E^N\| \le Te^{TL}\max_{1\le m\le N}\|\tau^{m-1}\|$$