Today we will discuss how to solve $AU=F$ coming from the 1D BVP

$$u''(x)=f(x) \quad 0<x<1$$
$$u(0)=\alpha$$
$$u(1)=\beta$$

Discretization:

$$\frac{U_{i+1}-2U_i+U_{i-1}}{h^2}=F_i \quad 1\le i\le m$$

$$U_0=U_{m+1}=0$$

Jacobi's method

Let

$$G=\begin{bmatrix} 0 & 1 & & & \\ 1 & \ddots & \ddots & & \\ & \ddots & \ddots & \ddots & \\ & & \ddots & \ddots & 1 \\ & & & 1 & 0 \end{bmatrix}$$

and

$$A=\frac{1}{h^2}\begin{bmatrix} -2 & 1 & & & \\ 1 & \ddots & \ddots & & \\ & \ddots & \ddots & \ddots & \\ & & \ddots & \ddots & 1 \\ & & & 1 & -2 \end{bmatrix}$$

Then $h^2A+2I=G$

$$A=\frac{1}{h^2}(G-2I)$$

$$AU = F$$

$$\frac{1}{h^2}(G-2I)U = F$$

$$\frac{1}{h^2}GU - \frac{2}{h^2}U = F$$

$$U = \frac{1}{2}(GU - h^2F)$$

Fixed-point iteration:

Initial guess: $U^{[0]}$

$$U^{[k+1]} = \frac{1}{2}(GU^{[k]} - h^2F)$$

(repeat)

Let's show that $U^{[k]} \to U$ as $k \to \infty$.

$$e^{[k]} = U^{[k]} - U$$

$$U^{[k+1]} - U = \frac{1}{2} G(U^{[k]} - U)$$

$$e^{[k+1]} = \frac{1}{2} G e^{[k]}$$

$$e^{[k]} = \left(\frac{G}{2}\right)^k e^{[0]}$$

Since G is symmetric, it has a complete set of orthogonal eigenvectors.

Claim: A and G have the same eigenvectors.

$$Av = \lambda v$$

$$\frac{1}{h^2}(G - 2I)v = \lambda v$$

$$Gv - 2v = h^2 \lambda v$$

$$Gv = (h^2\lambda + 2)v$$

Eigenvalues of A: $\frac{2}{h^2}\left(\cos(p\pi h) - 1\right)$ $\quad p = 1, 2, \ldots, m$

$h = \frac{1}{m+1}$

Eigenvalues of G: $\gamma_p = h^2\lambda_p + 2 = 2\cos(p\pi h)$

Let $\tilde{G} = \frac{G}{2}$. $\tilde{\gamma}_p = \cos(p\pi h)$. Notice that

$|\tilde{\gamma}_p| < 1 \quad \forall p.$

Write $e^{[0]}$ in the basis of eigenvectors of $\tilde{G}$:

$$e^{[0]} = \sum_{p=1}^{m} c_p v_p$$

Then

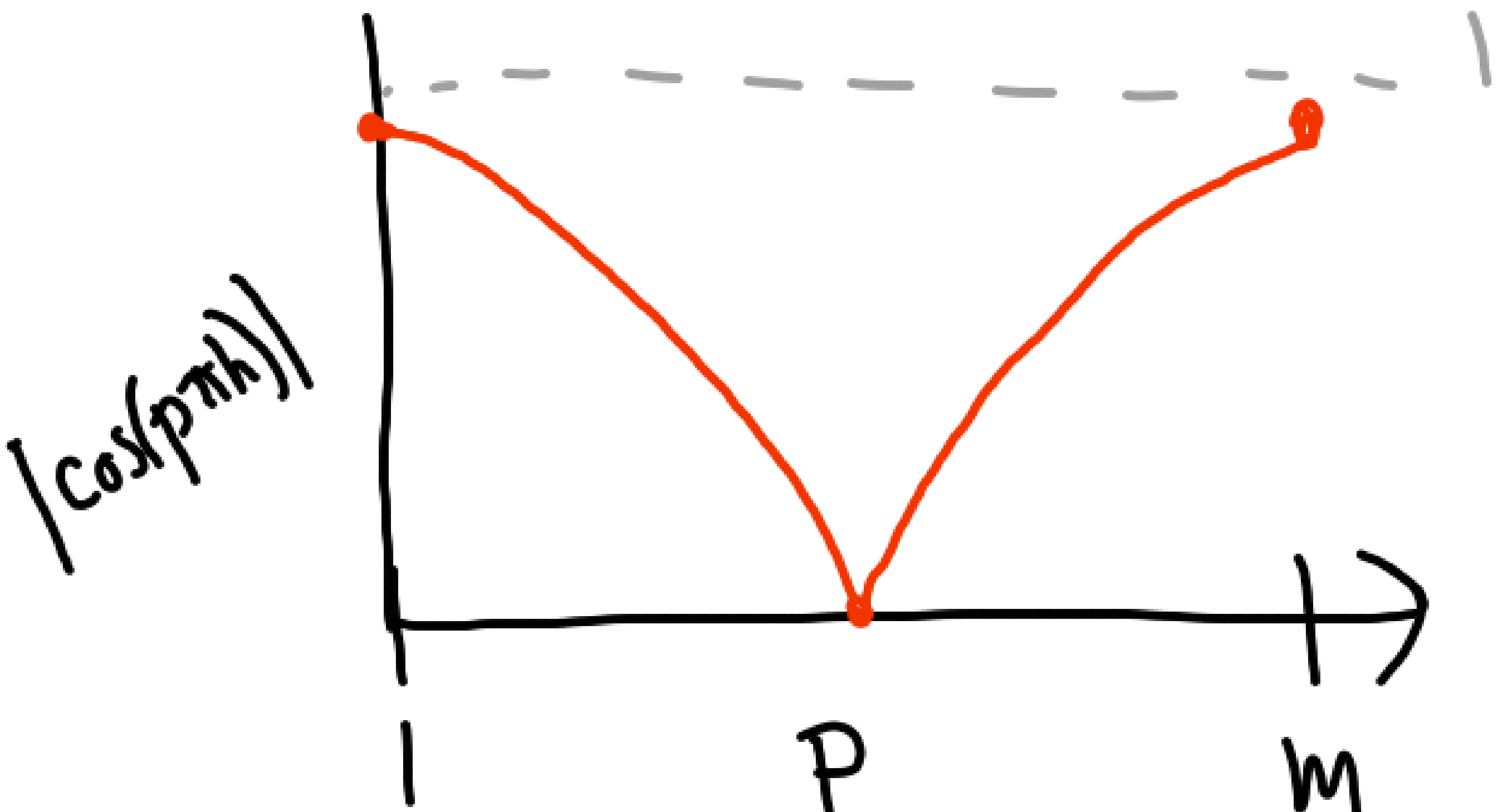
$$\tilde{G}e^{[0]} = \sum_{p=1}^{m} c_p \tilde{\gamma}_p v_p$$

$$\tilde{G}^k e^{[0]} = \sum_{p=1}^{m} c_p \tilde{\gamma}_p^k v_p$$

$$\lim_{k\to\infty} \tilde{\gamma}_p^k = 0 \Rightarrow \lim_{k\to\infty} \|\tilde{G}^k e^{[0]}\| = 0.$$

so $\|e^{[k]}\| \to 0$ as $k \to \infty$

How quickly will Jacobi converge?



Convergence will get slower as $h \to 0$.

Recall that the eigenvectors are $(v_p)_j = \sin(p\pi j h) = \sin(p\pi x_j)$

So the dominant error corresponds to the lowest and highest frequencies.

Under-relaxation

$$\hat{U}^{[k+1]} = \frac{1}{2}\left(GU^{[k]} - h^2 F\right)$$

$$U^{[k+1]} = U^{[k]} + \omega\left(\hat{U}^{[k+1]} - U^{[k]}\right)$$

$$U^{[k+1]} = (1-\omega)U^{[k]} + \omega \hat{U}^{[k+1]}$$

$$= (1-\omega)U^{[k]} + \frac{\omega}{2}\left(GU^{[k]} - h^2 F\right)$$

$$= \underbrace{\left(1-\omega+\frac{\omega}{2}G\right)}_{\hat{G}}U^{[k]} - \frac{\omega h^2}{2}F$$

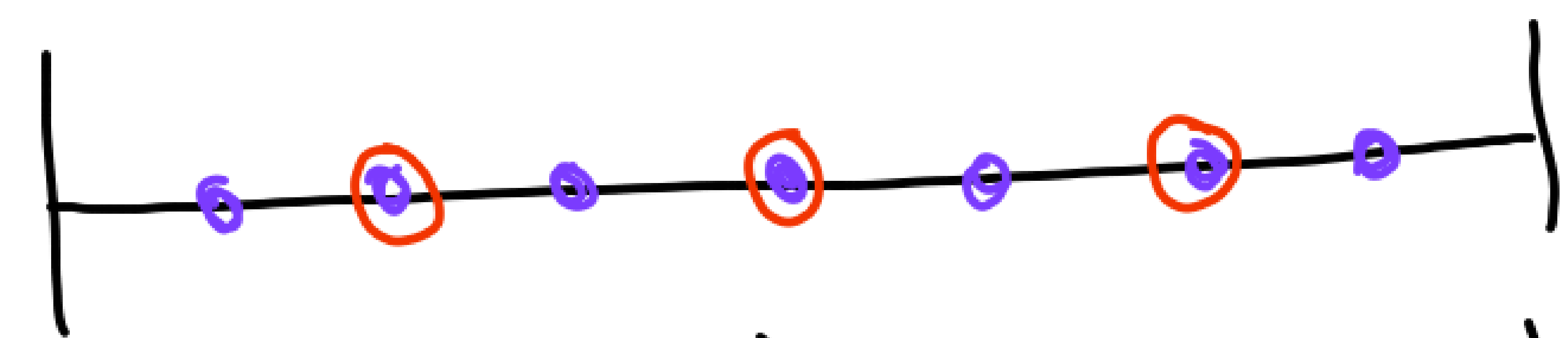
We can minimize

$$\max_{\frac{m}{2} \le p \le m} |\hat{\gamma}_p| \quad \text{where } \hat{\gamma}_p \text{ are}$$

the eigenvalues of $\hat{G}$, by taking $\omega = \frac{2}{3}$.

Multigrid

Start on a grid with m points and use underrelaxed Jacobi iterations.



On the finest grid, we solve $AU = F$. After ν iterations, we have an approximation U_ν.

Define $e_\nu = U_\nu - U$

and $AU_\nu - F = -r$ (residual)

$AU - F = 0$

$$\boxed{Ae_\nu = -r}$$

This is is the equation we solve on the coarse grid.

We iterate with underrelaxed Jacobi and then move to an even coarser grid.

Eventually we reach a system small enough to apply a direct solver.

Then we subtract all the corrections e_ν to get our solution.

<u>Mesh width</u>