Homework 3

Exercise 1 *(Multigrid)*

For this problem, you may use the code from the multigrid notebook. Modify the V-cycle setup to answer the following questions. Try to explain your results.

(a) How does the accuracy change as we change the number of Jacobi iterations performed at each step?

(b) Is it better to use a finer grid, or more Jacobi iterations if we want to improve the solution accuracy?

(c) What happens if we don't perform any Jacobi iterations in the "up" part of the V-cycle?

(d) What happens if we don't recurse all the way down to the 1-point grid?

(e) What happens if we use the original Jacobi method, or some other value of ω?

Exercise 2 *(Lipschitz constant for an ODE)*

Let $f(u) = \log(u)$.

(a) Determine the best possible Lipschitz constant for this function over $2 \leq u < \infty$.

(b) Is $f(u)$ Lipschitz continuous over $0 < u < \infty$?

(c) Consider the initial value problem

$$\begin{aligned} u'(t) &= \log(u(t)), \\ u(0) &= 2. \end{aligned}$$

Explain why we know that this problem has a unique solution for all $t \geq 0$ based on the existence and uniqueness theory described in Section 5.2.1. (Hint: Argue that f is Lipschitz continuous in a domain that the solution never leaves, though the domain is not symmetric about $\eta = 2$ as assumed in the theorem quoted in the book.)

Exercise 3 *(Lipschitz constant for a system of ODEs)*

Consider the system of ODEs

$$\begin{aligned} u_1' &= 3u_1 + 4u_2, \\ u_2' &= 5u_1 - 6u_2. \end{aligned}$$

Determine the Lipschitz constant for this system in the max-norm $\|\cdot\|_\infty$ and the 1-norm $\|\cdot\|_1$. (See Appendix A.3.)

Exercise 4 *(matrix exponential form of solution)*

The initial value problem

$$v''(t) = -4v(t), \qquad v(0) = v_0, \quad v'(0) = v_0'$$

has the solution $v(t) = v_0 \cos(2t) + \frac{1}{2}v_0' \sin(2t)$. Determine this solution by rewriting the ODE as a first order system $u' = Au$ so that $u(t) = e^{At}u(0)$ and then computing the matrix exponential using (D.30) in Appendix D.

Exercise 5 *Well-posedness of SIR model*

Consider the SIR model

$$\begin{aligned} x'(t) &= -\beta xy \\ y'(t) &= \beta xy - \gamma y \\ z'(t) &= \gamma y, \end{aligned}$$

where x, y, z represent susceptible, infected, and removed proportions of the population. Let initial conditions $x(0), y(0), z(0)$ be given such that $x(0) + y(0) + z(0) = 1$. Since $x + y + z = 1$ for all time, we can study the system by considering just the first two differential equations and setting $z(t) = 1 - x(t) - y(t)$.

Consider the domain $D = \{(x, y) : x \geq 0, y \geq 0, x + y \leq 1\}$. Show that the SIR model has a unique solution for all $t > 0$ whenever $(x(0), y(0)) \in D$, as follows:

(i) Show that if $(x(0), y(0)) \in D$, then $(x(t), y(t)) \in D$ for all time. Hint to: to show that x, y remain non-negative, consider the behavior of the SIR system when $x = 0$ or $y = 0$. Be sure to state your reasoning clearly and carefully.

(ii) Show that the function

$$f : \begin{bmatrix} x \\ y \end{bmatrix} \to \begin{bmatrix} -\beta xy \\ \beta xy - \gamma y \end{bmatrix}$$

is Lipschitz continuous for $(x, y) \in D$.