So far we have studied:

- An elliptic PDE: $\nabla^2 u = f(\vec{x})$
- A parabolic PDE: $u_t = \nabla^2 u$

Today: A hyperbolic PDE

Hyperbolic PDEs model waves:

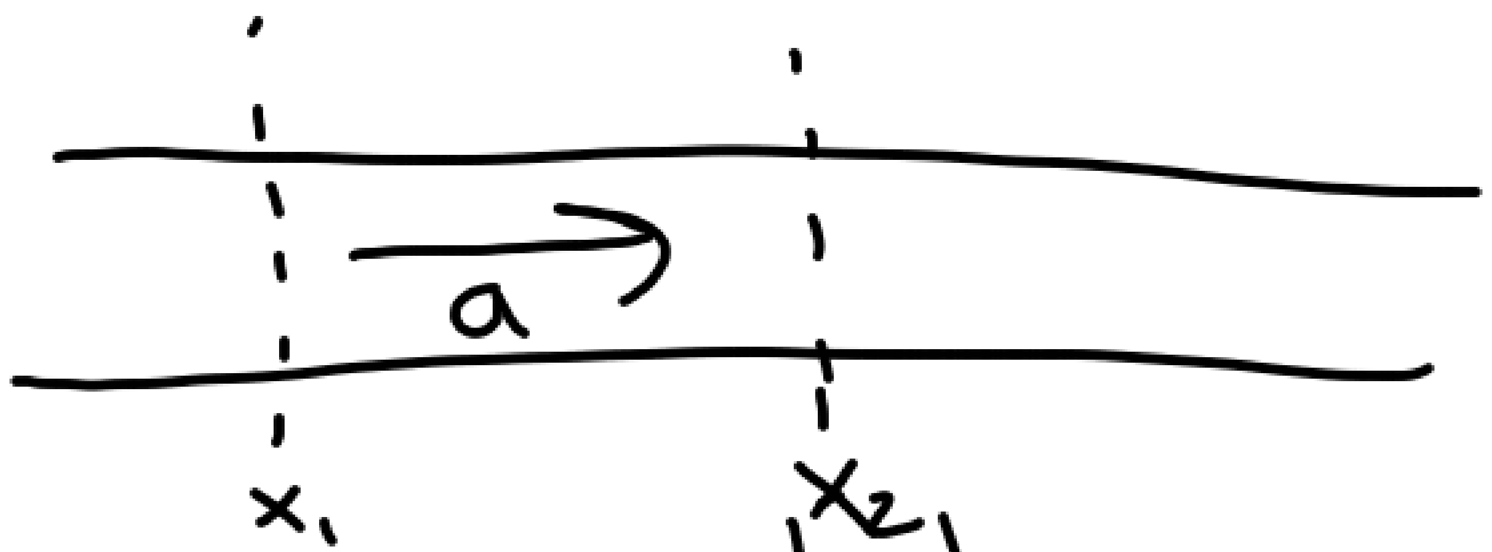
- water waves
- electromagnetic
- pressure (sound)
- motions of fluids

The simplest hyperbolic PDE:

$$u_t + a u_x = 0$$

(Advection)

$u = u(x,t)$ a: constant (speed)



$u(x,t)$: concentration

Total of u in $[x_1, x_2]$:

$$\int_{x_1}^{x_2} u(x,t)\,dx$$

This should change only due to flux through the boundaries:

$$\frac{d}{dt}\int_{x_1}^{x_2} u(x,t)\,dx = f(u(x_1,t)) - f(u(x_2,t))$$

In our case: $f(u) = au$

Assuming sufficient smoothness:

$$\int_{x_1}^{x_2} \frac{\partial}{\partial t} u(x,t)\,dx = -\int_{x_1}^{x_2} \frac{\partial}{\partial x} f(u(x,t))\,dx$$

$$\int_{x_1}^{x_2} \left[\frac{\partial}{\partial t} u(x,t) + \frac{\partial}{\partial x} f(u(x,t))\right] dx = 0$$

The integrand must vanish pt.wise:

$$u_t + f(u)_x = 0$$

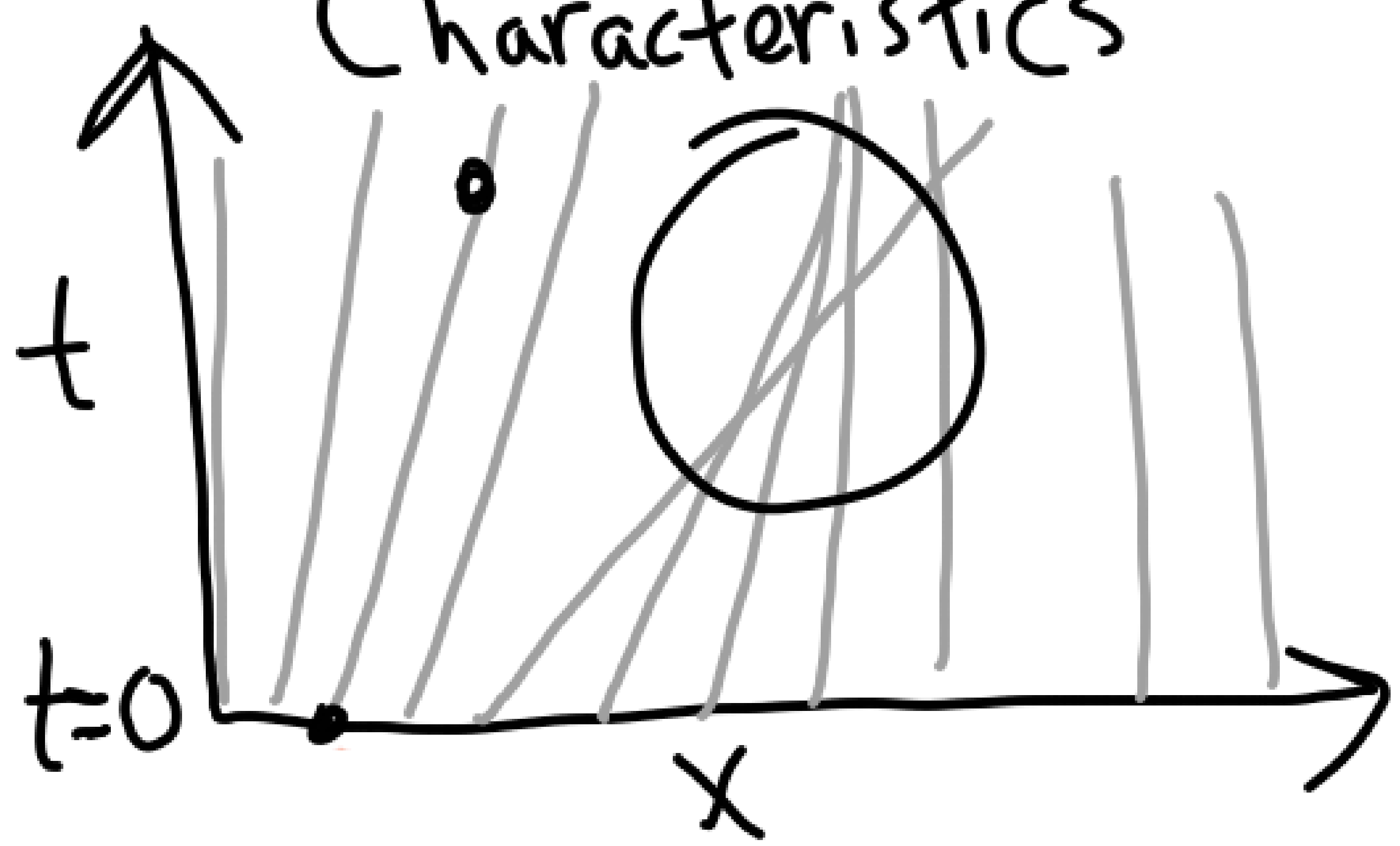
With constant velocity a:

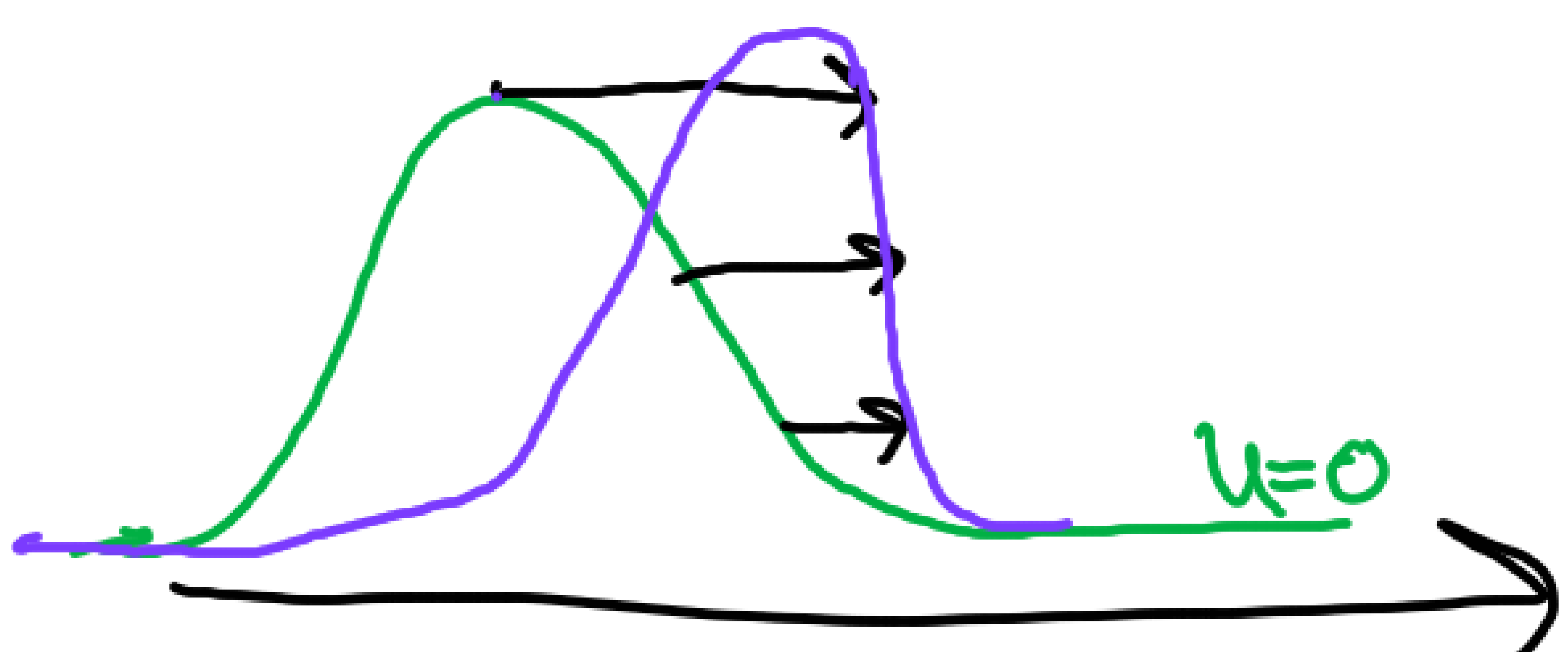
$$u_t + a u_x = 0.$$

Burgers equation:

$$u_t + uu_x = 0$$

Characteristics





Advection equation:

$$u_t + au_x = 0$$

Cauchy problem: $-\infty < x < \infty$

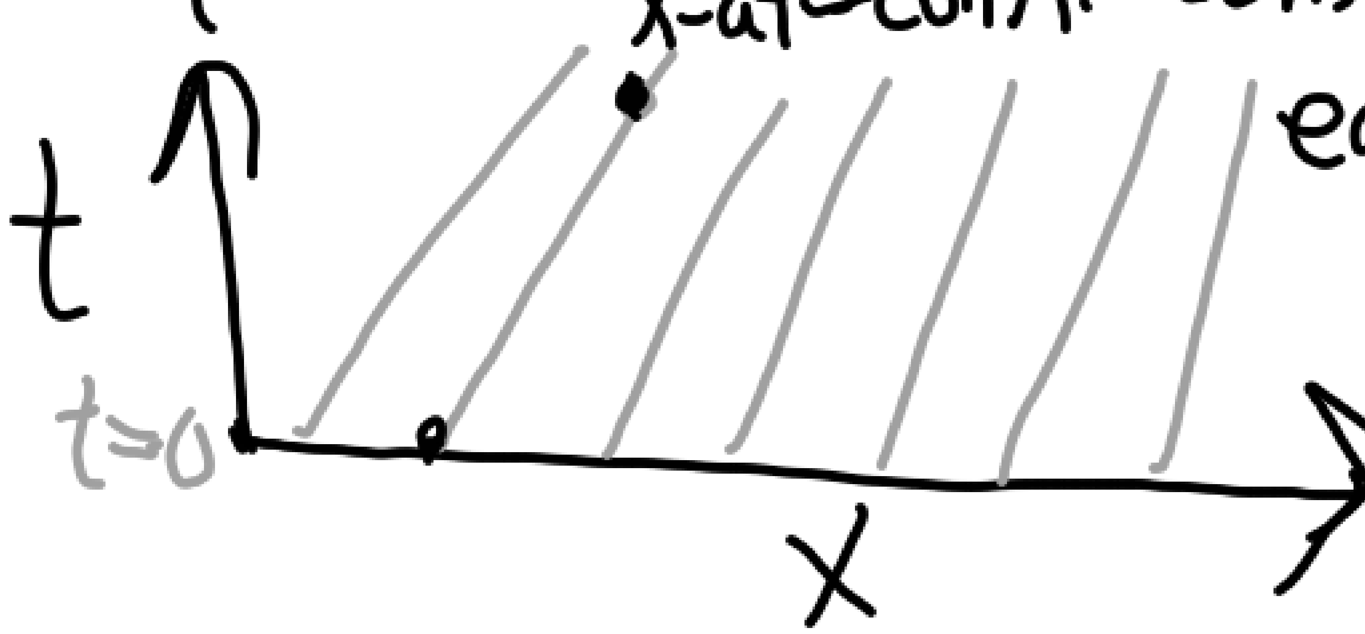
$$u(x, t=0) = \eta(x)$$

$$u(x,t) = \eta(x-at)$$

Check: $u_t = -a\eta'$ $\quad u_x = \eta'$

$\Rightarrow -a\eta' + a\eta' = 0$ ✓

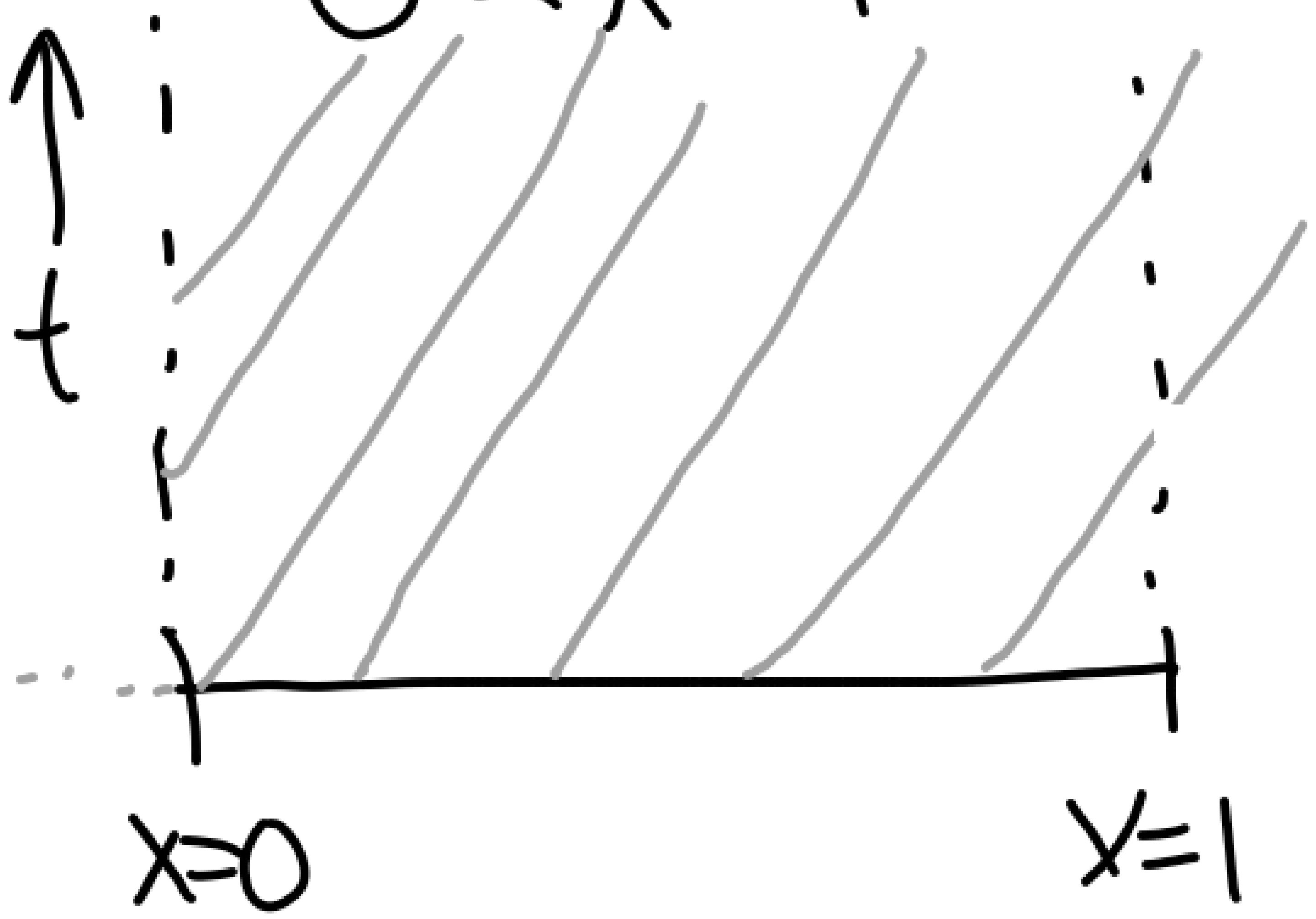
Characteristics:



$u(x,t)$ is constant along each line

What about a bounded domain?

$0 < x < 1$



We need one value for the solution along each characteristic.

No BC allowed at $x=1$.
BC required at $x=0$.
$\Bigg\}\ a>0$

Discretization

$u_t = -a u_x$

CD in space: $u_x(x_i, t) \approx \dfrac{U_{i+1} - U_{i-1}}{2h}$

(Explicit) Euler in time

$$U_j^{n+1} = U_j^n - \frac{ka}{2h}\left(U_{j+1}^n - U_{j-1}^n\right)$$

$$\frac{U_j^{n+1} - U_j^n}{k} = -\frac{a}{2h}\left(U_{j+1}^n - U_{j-1}^n\right)$$

$$u(x=1,t) = u(x=0,t) \Rightarrow U_{m+1}^n = U_0^n$$

Method of lines
Stability analysis

Semi-discretization:

$$U'(t) = \underbrace{-\frac{a}{2h}\begin{bmatrix}0 & 1 & & \\ -1 & \ddots & \ddots & \\ & \ddots & \ddots & 1 \\ & & -1 & 0\end{bmatrix}}_{} U(t)$$

Eigenvalues $\times$ K
Should be in abs. stability region

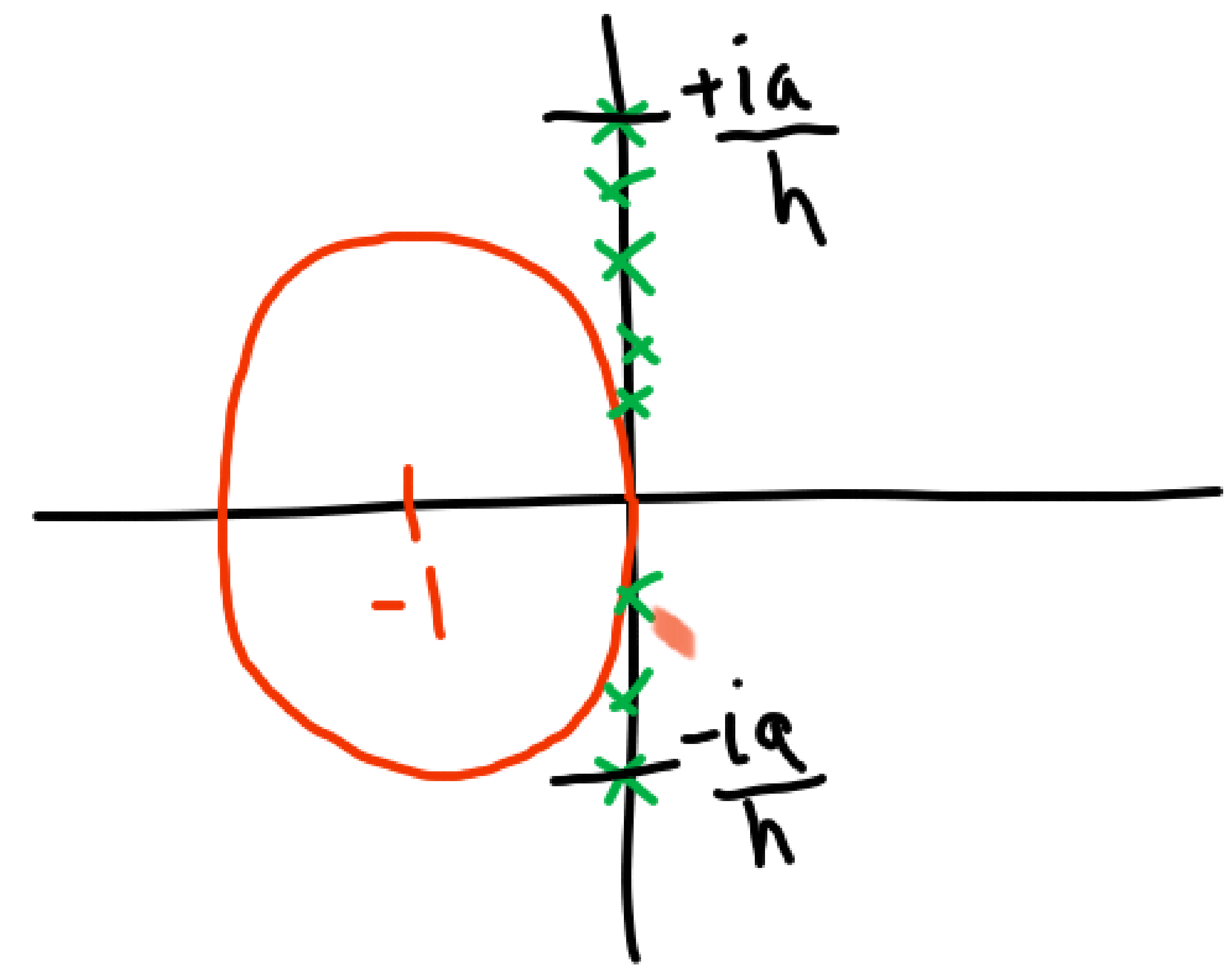
$\to$ not satisfied for any $K>0$.

Explicit Euler was a bad choice for this problem.

⋮

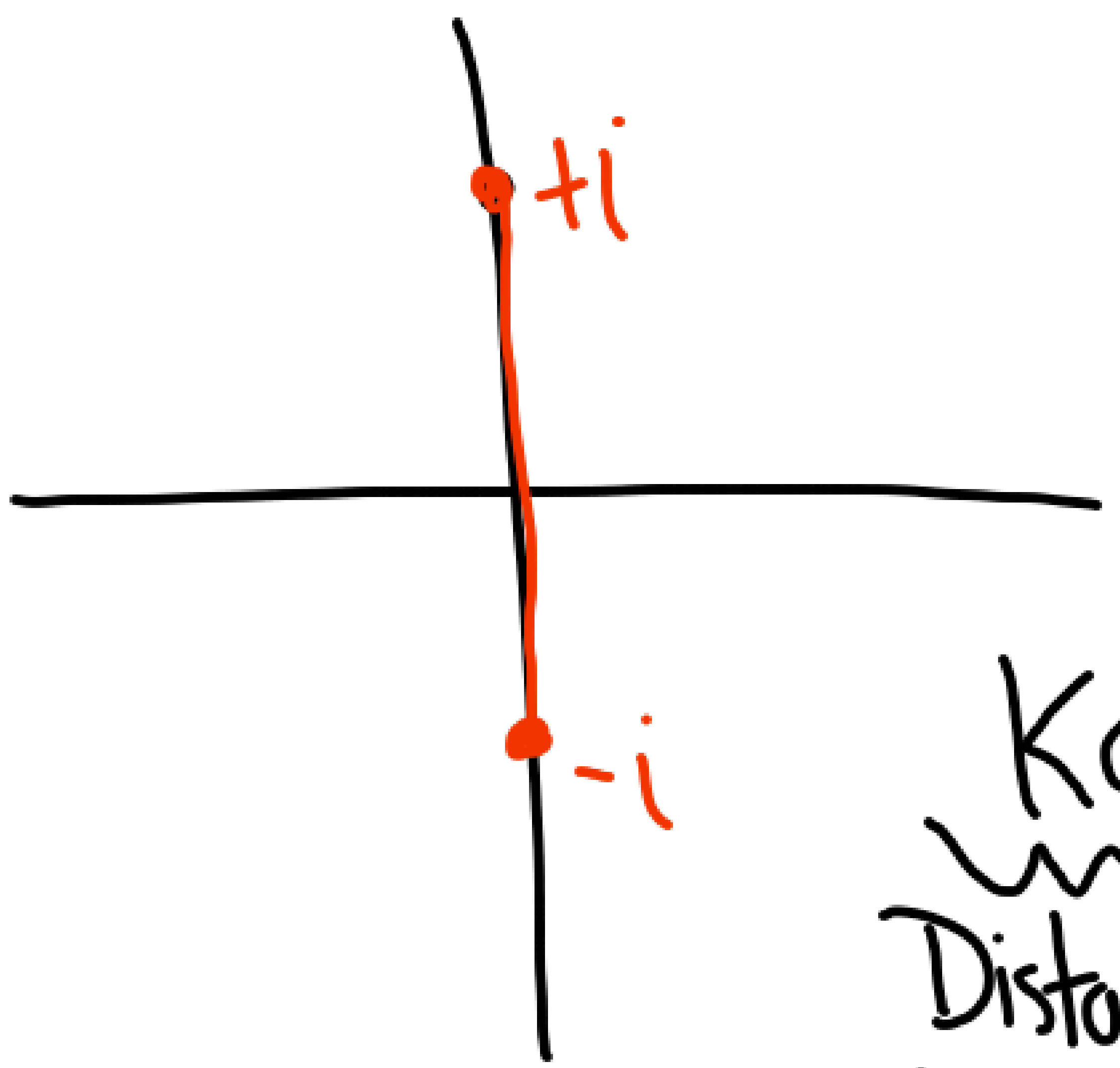
Skew-symmetric matrix

$\Rightarrow$ imaginary eigenvalues



Instead: Leapfrog
(centered in time)

$$U_j^{n+1} = U_j^{n-1} - \frac{ka}{h}\left(U_{j+1}^n - U_{j-1}^n\right)$$



$$k\frac{a}{h} \leq 1$$

$$\underbrace{ka}_{\text{Distance traveled in 1 time step}} \leq h \qquad X_2 - X_1$$

Lax-Friedrichs (1st-order)

$$U_j^{n+1} = \frac{1}{2}\left(U_{j+1}^n + U_{j-1}^n\right) - \frac{ka}{2h}\left(U_{j+1}^n - U_{j-1}^n\right)$$

Notice that

$$\frac{1}{2}\left(U_{j+1}^n + U_{j-1}^n\right) = U_j^n + \frac{h^2}{2}\cdot\overbrace{\frac{U_{j+1}^n - 2U_j^n + U_{j-1}^n}{h^2}}^{\approx u_{xx}}$$

$$U^{n+1} = U_j^n - \frac{ka}{2h}\left(U_{j+1}^n - U_j^n\right) + \frac{h^2}{2}\,\frac{U_{j+1}^n - 2U_j^n + U_{j-1}^n}{h^2}$$

Look like an approximation of $u_t + au_x = \frac{h^2}{k}u_{xx}$