Modified Equation Analysis

Idea: Given a discretization, find a modified PDE such the numerical solution of the original PDE is the exact solution of the modified PDE.

Example $u_t + a u_x = 0$

Forward time, centered space

$$\frac{U_j^{n+1} - U_j^n}{k} + \frac{a}{2h}\left(U_{j+1}^n - U_{j-1}^n\right) = 0 \qquad (1)$$

Suppose there exists $v(x,t)$ such that if we replace U_j^n by $v(x_j, t_n)$, then (1) is satisfied exactly.

$$\frac{v(x,t+k)-v(x,t)}{k}+\frac{a}{2h}\left(v(x+h,t)-v(x-h,t)\right)=0$$

$$v(x,t\pm k)=v(x,t)\pm k v_t(x,t)+\frac{k^2}{2}v_{tt}(x,t)\pm\frac{k^3}{6}v_{ttt}+O(k^4)$$

$$v(x\pm h,t)=v\pm h v_x+\frac{h^2}{2}v_{xx}+O(h^3)$$

We get

$$v_t+\frac{k}{2}v_{tt}+O(k^2)+a\left(v_x+O(h^2)\right)=0$$

$$v_t+av_x=-\frac{k}{2}v_{tt}+O(k^2,h^2)$$

$$v_t+av_x=O(k,h^2)$$

$$v_{tt}=-av_{xt}+O(k,h^2)$$

$$v_{tx}=-av_{xx}+O(k,h^2)$$

$$v_{tt}=a^2v_{xx}+O(k,h^2)$$

$$v_t+av_x=-\frac{ka^2}{2}v_{xx}+O(k^2,h^2)$$

Anti-diffusive

Solutions blow up.

Lax-Friedrichs

$$U_j^{n+1} = \frac{1}{2}\left(U_{j+1}^n + U_{j-1}^n\right) - \frac{ka}{2h}\left(U_{j+1}^n - U_{j-1}^n\right)$$

$$U_j^n \to v(x,t)$$

$$\cancel{v} + \cancel{k}v_t + \frac{k^{\cancel{2}}}{2}v_{tt} + O(k^{\cancel{3}\,2}) = \cancel{v} + \frac{h^2}{2k}v_{xx} + O(k^{\cancel{3}\,2}) - \cancel{k}a\left(v_x + O(h^2)\right)$$

$$v_t + av_x = -\frac{k}{2}v_{tt} + \frac{h^2}{2k}v_{xx} + O(k^2, h^2)$$

$$= -\frac{k}{2}a^2 v_{xx} + \frac{h^2}{2k}v_{xx} + O(k^2, h^2)$$

$$v_t + av_x = \left(\frac{h^2}{2k} - \frac{ka^2}{2}\right)v_{xx} + O(k^2, h^2)$$

We need $\frac{h^2}{2k} \geq \frac{ka^2}{2}$

$$h^2 \geq k^2 a^2$$

$$\underbrace{\frac{ka}{h}}_{\text{CFL Number}} \leq 1$$

Lax-Friedrichs is stable iff $\frac{ka}{h} \leq 1$.

$$ka \leq h$$

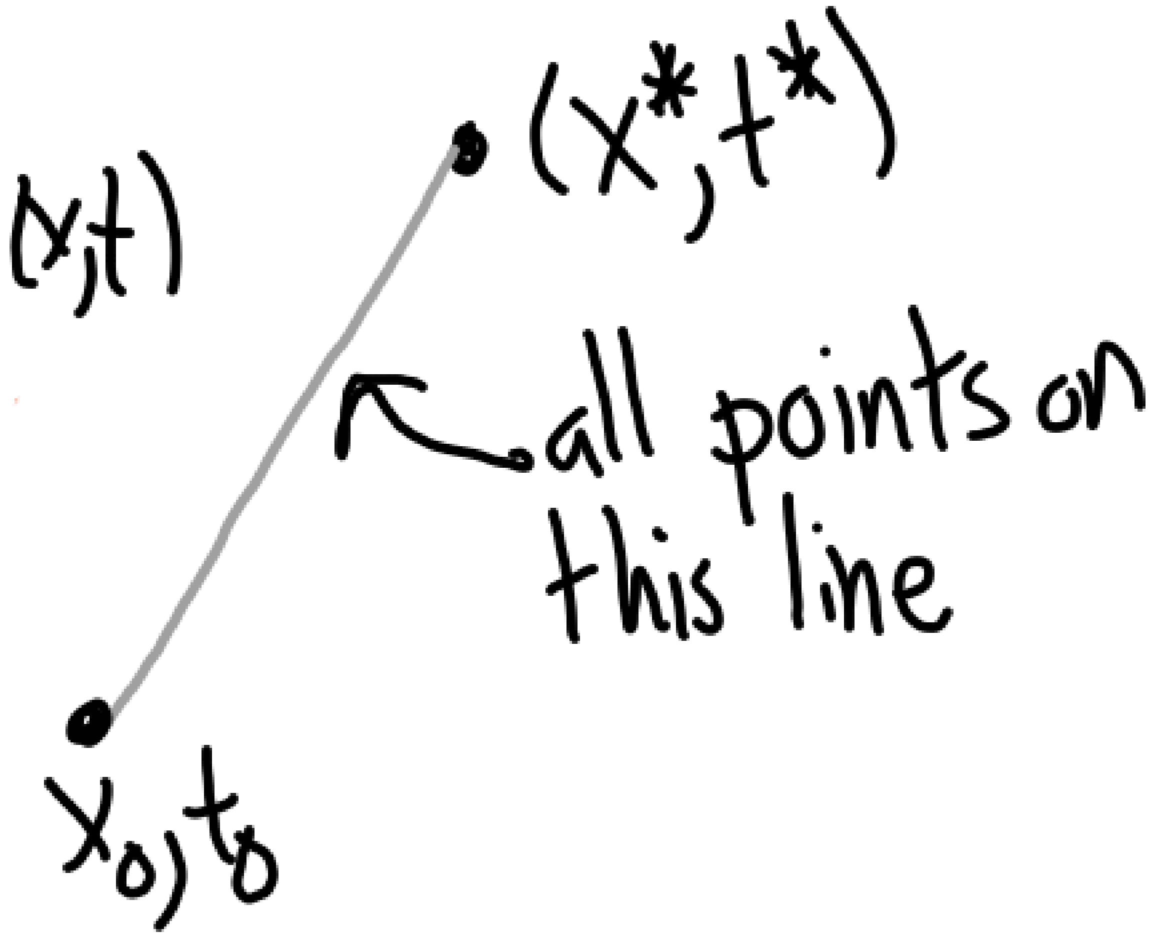
The CFL condition

(Courant, Friedrichs, Lewy 1927)

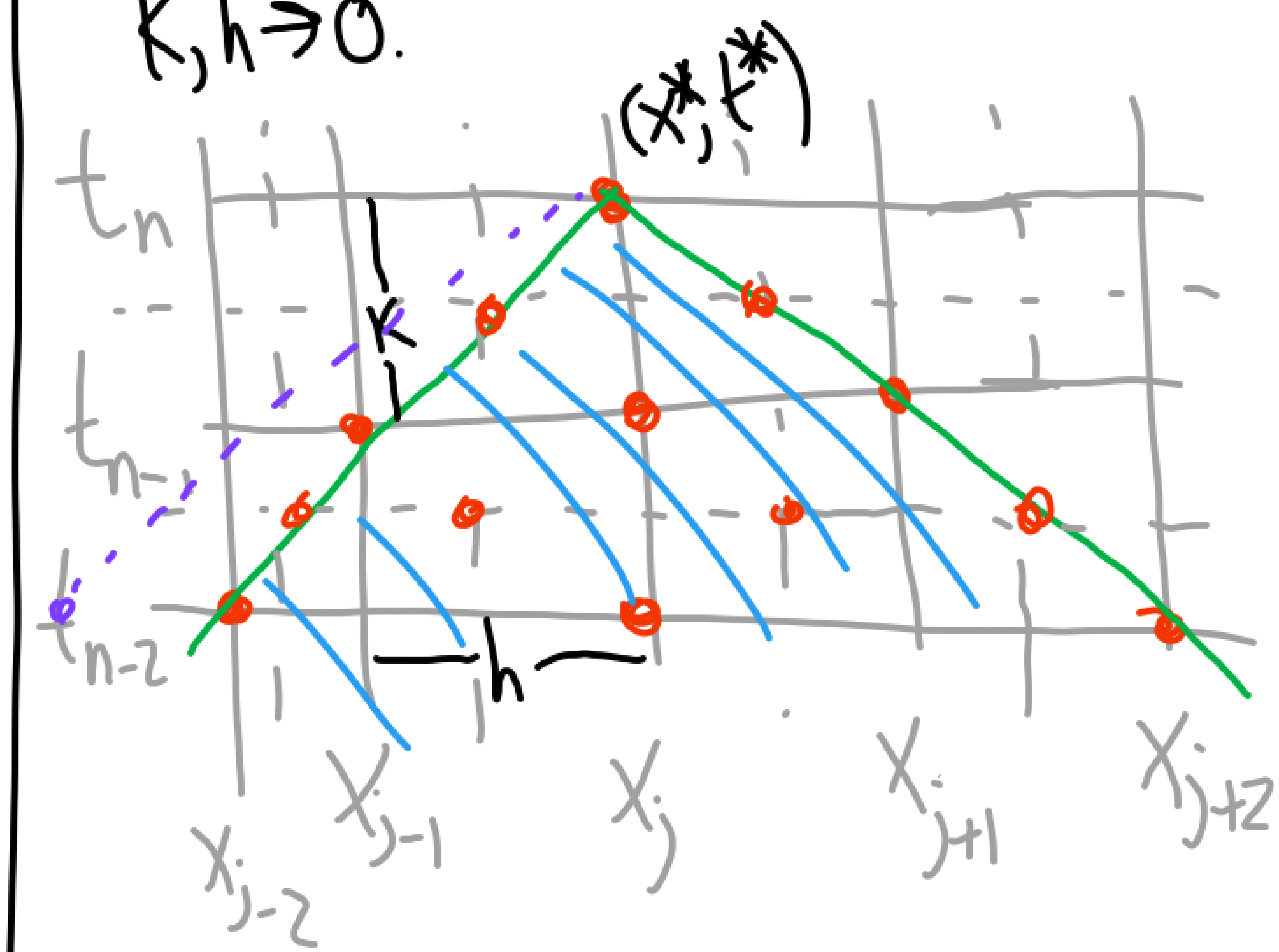
The domain of dependence of $u(x,t)$ is the set of (x,t) values on which $u(x,t)$ depends.

For $u_t = u_{xx}$: $D(x^*,t^*) = \{(x,t) : t < t^*\}$

For $u_t + a u_x = f(x,t)$



CFL says: The D.o.D of the exact solution must be contained in the D.o.D. of the numerical solution, as $k,h \to 0$.



We need the characteristic passing through (x^*, t^*) to lie inside the triangle.

Information can't propagate more than one grid point per time step.

$$\underbrace{ka}_{\text{dist travelled in 1 step.}} \leq \underbrace{h}_{\text{dist. between grid pts.}}$$

If the CFL Condition is not satisfied, the method cannot be convergent.

Centered in time + space (Leapfrog)

$$\frac{U_j^{n+1}-U_j^{n-1}}{2k}+a\,\frac{U_{j+1}^n-U_{j-1}^n}{2h}=0$$

$$\frac{v(x,t+k)-v(x,t-k)}{2k}+a\,\frac{v(x+h,t)-v(x-h,t)}{2h}=0$$

$$v_t+\frac{k^2}{6}v_{ttt}+\mathcal{O}(k^3)+a\left(v_x+\frac{h^2}{6}v_{xxx}+\mathcal{O}(h^3)\right)=0$$

$$v_t+av_x=\left(a^3\frac{k^2}{6}-a\frac{h^2}{6}\right)v_{xxx}+\mathcal{O}(h^3,k^3,k^2h^2)$$

$$v_{tt}=a^2v_{xx}+\mathcal{O}(k^2,h^2)$$

$$v_{ttt}=a^2v_{xxt}+\mathcal{O}(k^2,h^2)$$

$$v_t=-av_x+\mathcal{O}(k^2,h^2)$$

$$v_{txx}=-av_{xxx}+\mathcal{O}(k^2,h^2)$$

$$v_{ttt}=-a^3v_{xxx}+\mathcal{O}(k^2,h^2)$$

$$v_t + a v_x = C v_{xxx}$$

Ansatz: $v(x,t) = e^{i(\xi x - \omega t)}$

$$v_t = -i\omega v$$

$$v_x = i\xi v$$

$$v_{xxx} = -i\xi^3 v$$

$$v_{xx} = -\xi^2 v$$

$$v_{xxxx} = \xi^4 v$$

$$-i\omega v + a i\xi v = -i\xi^3 v C$$

Assume $v \neq 0$

$$\omega = a\xi + C\xi^3$$

$$v(x,t) = e^{i(\xi x - t(a\xi + C\xi^3))}$$

if $C = 0$: $v = e^{i\xi(x - at)}$

if $C \neq 0$: $v = e^{i\xi(x - at - Ct\xi^2)}$

$$= e^{i\xi(x - \underbrace{(a + C\xi^2)}_{\text{velocity depends on } \xi!}t)}$$

Numerical dispersion