Linear Multistep Methods (LMMs)

$$u'(t) = f(u)$$
$$u(t_0) = \eta$$

A LMM takes the form:

$$\sum_{j=0}^{r} \alpha_j U^{n+j} = k \sum_{j=0}^{r} \beta_j f(U^{n+j})$$

This is a formula for U^{n+r} based on $U^n, U^{n+1}, \ldots, U^{n+r-1}$

The method is

Explicit if $\beta_r = 0$

Implicit if $\beta_r \neq 0$.

Local truncation error

$$\sum_{j=0}^{r} \alpha_j u(t_n + kj) - k \sum_{j=0}^{r} \beta_j u'(t_n + kj) = k \tau^{n+r}$$

$$u(t_n+kj)=\sum_{i=0}^{\infty}\frac{(kj)^i}{i!}u^{(i)}(t_n)$$

$$u'(t_n+kj)=\sum_{i=1}^{\infty}\frac{(kj)^{i-1}}{(i-1)!}u^{(i)}(t_n)$$

$$\tau^{n+r}=\frac{1}{k}\left[\sum_{j=0}^{r}\alpha_j u(t_n)+\sum_{j=0}^{r}\sum_{i=1}^{\infty}\left(\alpha_j\frac{(kj)^i}{i!}-k\beta_j\frac{(kj)^{i-1}}{(i-1)!}\right)u^{(i)}(t_n)\right]$$

We could find conditions for higher order accuracy from this expression.

$$\tau^{n+r}=\frac{1}{k}\sum_{j=0}^{r}\alpha_j u(t_n)+\sum_{i=1}^{\infty}u^{(i)}(t_n)k^{i-1}\quad\sum_{j=0}^{r}\left(\alpha_j\frac{j^i}{i!}-\beta_j\frac{j^{i-1}}{(i-1)!}\right)$$

To have $\tau^{n+r}=\mathcal{O}(k)$, we need

$$\sum_{j=0}^{r}\alpha_j=0\quad\text{and}\quad\sum_{j=0}^{r}(j\alpha_j-\beta_j)=0$$

Examples

2-step Adams-Bashforth: $U^{n+2} = U^{n+1} + \frac{k}{2}\left(-f(U^n) + 3f(U^{n+1})\right)$

2nd-order

Leapfrog: $U^{n+2} = U^n + kf(U^{n+1})$

Backward Differentiation Formula:

$$U^{n+2} = \frac{4}{3}U^{n+1} - \frac{1}{3}U^n + 2kf(U^{n+2})$$

A 2-step, 1st-order:

$$U^{n+2} = 3U^{n+1} - 2U^n + kf(U^n)$$

Let's test these with

$$u'(t) = 0 \quad (*)$$
$$u(0) = 0$$

$$U^0 = 0 \quad U^1 = k$$

Zero-stability

If we apply a LMM to (*) we get

$$\underbrace{\sum_{j=0}^{r} \alpha_j U^{n+j} = 0}_{\text{Linear difference equation}} \quad (**)$$

Ansatz: $U^n = \zeta^n$

$$\sum_{j=0}^{r} \alpha_j \zeta^{n+j} = 0 \Rightarrow \rho(\zeta) = \sum_{j=0}^{r} \alpha_j \zeta^j = 0$$

First Characteristic Polynomial

$\rho(\zeta)$ is a polynomial of degree r, with roots $\zeta_1, \zeta_2, \ldots, \zeta_r$.

If they are distinct, the general solution is

$$U^n = \sum_{j=1}^{r} c_j \zeta_j^n$$

The values c_j are determined by $U^0, U^1, \ldots, U^{r-1}$.

Notice that $\rho(1) = \sum_{j=0}^{r} \alpha_j = 0$

So 1 is a root of ρ for any consistent method.

What if some roots are equal?

For example: $U^{n+2}-2U^{n+1}+U^n=0$

$$\rho(\xi)=\xi^2-2\xi+1$$
$$=(\xi-1)^2$$

$$\xi_1=\xi_2=1 \Rightarrow U^n=1^n=1$$

The other fundamental solution is $U^n=n\xi_1^n=n1^n=n.$

Check: $n+2-2(n+1)+n=0.$

In general, a root ξ_j of multiplicity m leads to the fundamental solutions

$$U^n=\xi_j^n,\ U^n=n\xi_j^n,\ U^n=n^2\xi_j^n,$$

$$\ldots U^n=n^{m-1}\xi_j^n$$

We want to determine whether the solution of (**) remains bounded as $n\to\infty$ (i.e., converges to zero as $k\to 0$).

The solution of (**) is bounded as $n \to \infty$ iff the roots of $\rho(\xi)$ satisfy the following condition:

$$\left.\begin{array}{l} |\xi_j| \le 1 \quad \forall j \\ |\xi_j| < 1 \text{ if } \xi_j \text{ is a multiple root.} \end{array}\right\} \text{The root condition}$$

We say the LMM is zero-stable if this condition is satisfied.

In fact, any zero-stable and consistent LMM is convergent (when f is Lipschitz).

We can prove this by writing the LMM as a one-step method.

$$V^n = \begin{bmatrix} U^n \\ U^{n+1} \\ \vdots \\ U^{n+r-1} \end{bmatrix} \quad V^{n+1} = \begin{bmatrix} U^{n+1} \\ \vdots \\ U^{n+r} \end{bmatrix}$$

We can write the LMM

$$\sum_{j=0}^{r} \alpha_j U^{n+j} = k \sum_{j=0}^{r} \beta_j f(U^{n+j})$$

Let $\alpha_r = 1$

as

$$V^{n+1} = \begin{bmatrix} 0 & 1 & & & \\ & \ddots & 1 & & \\ & & \ddots & \ddots & \\ & & & \ddots & 1 \\ -\alpha_0 & -\alpha_1 & \cdots & \cdots & -\alpha_{r-1} \end{bmatrix} V^n + \begin{bmatrix} 0 \\ \vdots \\ 0 \\ k \sum_{j=0}^{r} \beta_j f(V_{j-1}^n) + k\beta_r f(V_r^{n+1}) \end{bmatrix}$$

Companion matrix C

If we follow our earlier analysis, we find this method is stable if the eigenvalues of C satisfy the root condition.

For one-step methods we have

$$U^{n+1} = U^n + \Psi(kf)$$

$$U^{n+1} - U^n = \Psi(kf)$$

$$\rho(\zeta) = \zeta - 1$$

So all one-step methods are zero-stable.

$u'(t) = -u$

$u(0) = 1$

$u(t) = e^{-t}$

$U^0 = 1$ $\qquad$ $U^0 = 1$

$U^1 = e^{-k}$ $\qquad$ $U^1 = 1$

Consistency of a LMM also requires that the starting values converge to the exact solution as $k \to 0$.