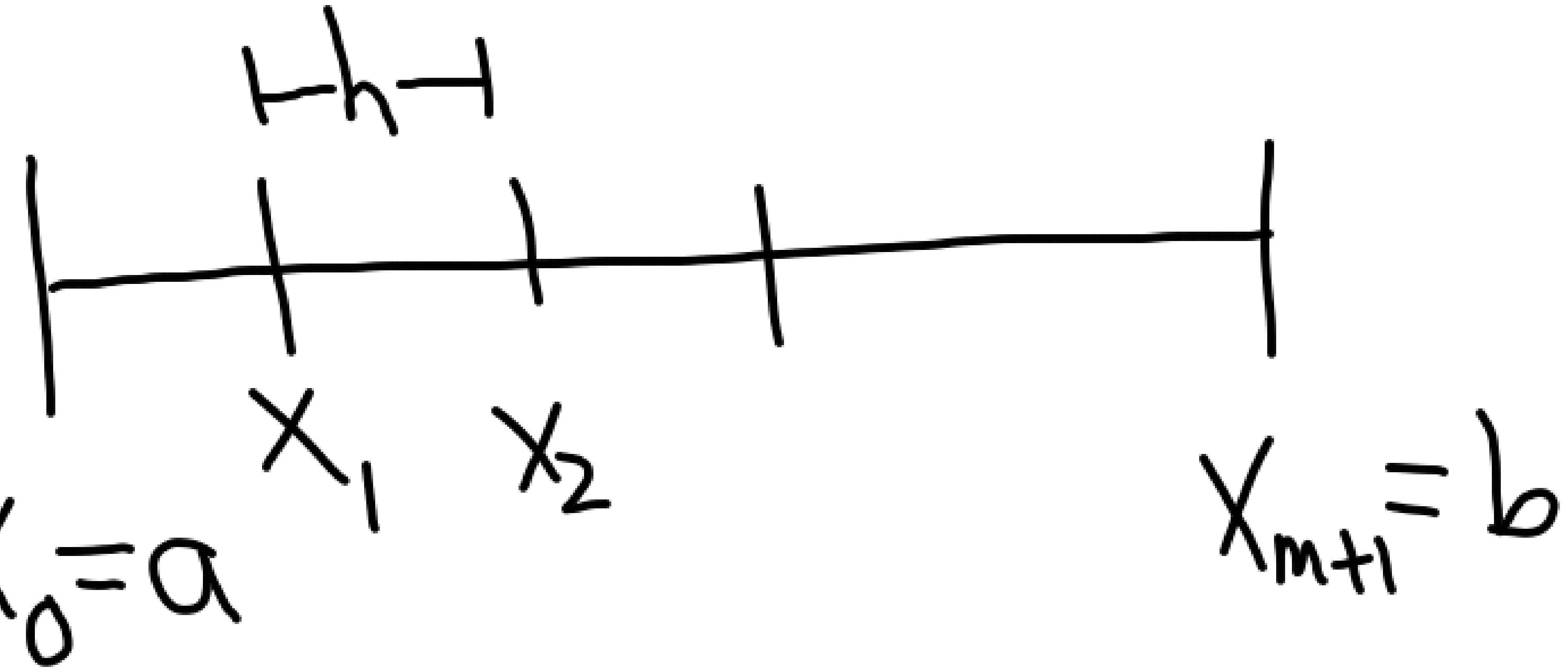
Homework 4 due March 31st

1 a)



$$h = \frac{b-a}{m+1} \qquad x_j = a + hj$$

$$\frac{U_{j-1} - 2U_j + U_{j-1}}{h^2} + U_j = f(x_j)$$

$$1 \le j \le m$$

$$U_0 = \alpha \qquad U_{m+1} = \beta$$

b) $AU = F \qquad p = 1, 2, \dots, m$

$$A = \frac{1}{h^2}\,\text{tridiag}(1,-2,1) + I$$

$$\lambda_A = \frac{2}{h^2}\left(\cos\left(\frac{p\pi}{m+1}\right) - 1\right) + 1$$

Since A is symmetric

$$\|A\|_2 \|A^{-1}\|_2 = \frac{|\lambda_{max}|}{|\lambda_{min}|}$$

$$|\lambda_{max}| = \frac{4+\mathcal{O}(h^2)}{h^2} + 1 \approx \frac{4}{h^2}$$

for $p=1$: $\cos\left(\frac{p\pi}{m+1}\right) = 1 - \frac{1}{2}\frac{\pi^2}{(m+1)^2} + \mathcal{O}\left(\frac{1}{(m+1)^4}\right)$

$$\lambda_A \approx \frac{2}{h^2}\left(\frac{-\pi^2}{2(m+1)^2}\right) + 1$$

$$\lambda_A \approx \frac{-\pi^2}{(b-a)^2} + 1$$

$$|\lambda_{min}| = \left|1 - \frac{\pi^2}{(b-a)^2}\right| + \mathcal{O}(h^2)$$

c) If $b-a=\pi$,

$$|\lambda_{min}| = \mathcal{O}(h^2)$$

So $\|A^{-1}\|_2 = \frac{1}{\mathcal{O}(h^2)}$

So there is no constant C such that

$$\|A^{-1}\|_2 \leq C \text{ as } h \to 0.$$

a)

$x_{j-1} \qquad x_j \qquad x_{j+1}$

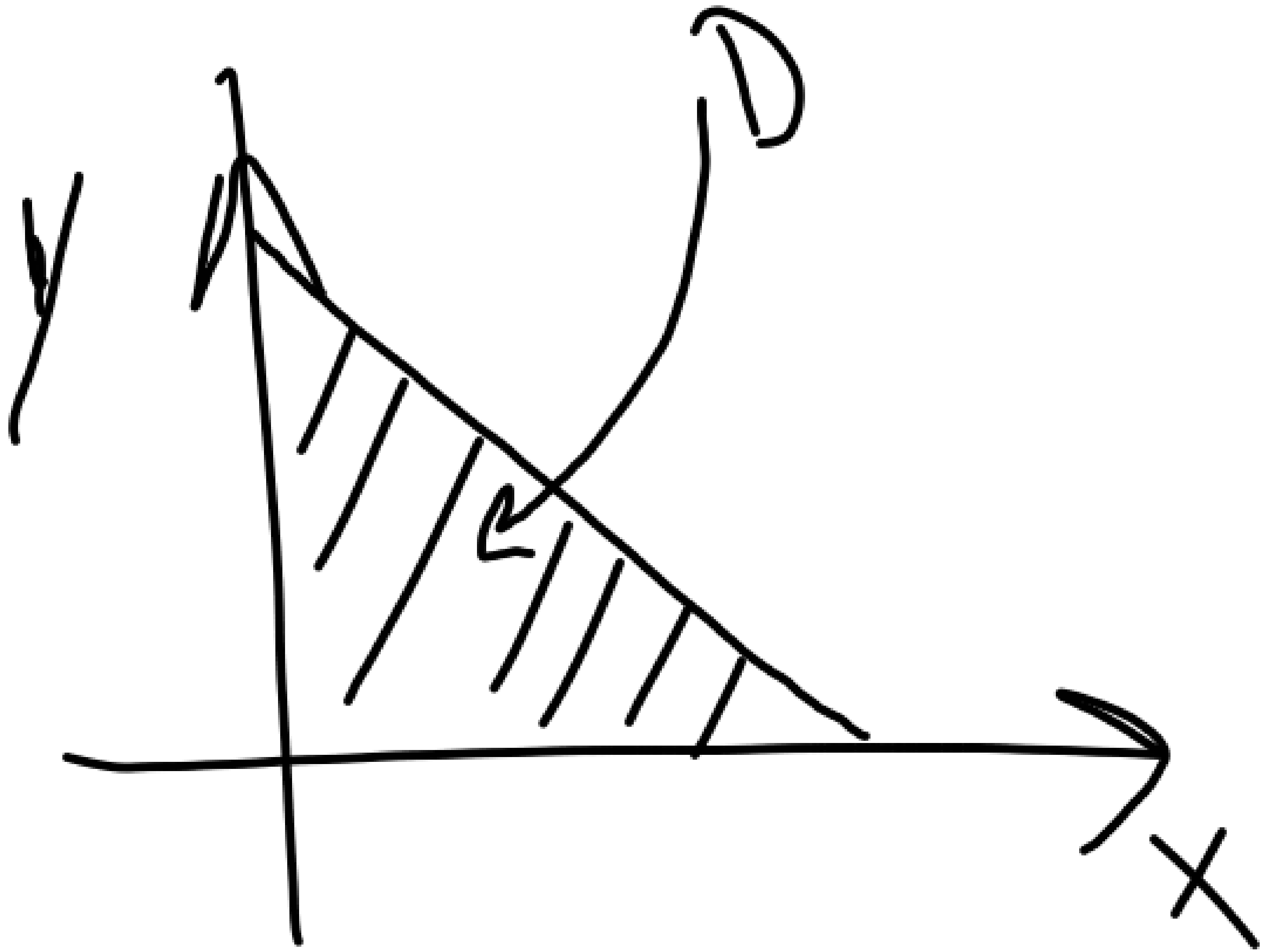
$$u'(x_j) \approx \frac{U_{j+1} - U_{j-1}}{2h}$$

$$\frac{U_{j+1} - 2U_j + U_{j-1}}{h^2} + U_j \frac{U_{j+1} - U_{j-1}}{2h} = f(x_j)$$

$$1 \leq j \leq m$$

$$U_0 = U_{m+1} = 0$$

b) $\frac{1}{12} h^2 u^{(4)}(x_j) + \frac{1}{6} u(x_j) u'''(x_j) h^2 + \mathcal{O}(h^4)$



$$y'(t) = (\beta x - \gamma)y$$

$$y(t) = y(0)e^{\int_0^t (\beta x(t) - \gamma)dt} \geq 0$$

$$x'(t) + y'(t) = -\gamma y \leq 0$$

$$\Rightarrow x(t) + y(t) \leq 1 \quad \forall\, t > 0.$$

$x'(t) = -\beta xy \leq 0$ for $x, y \geq 0$

Assume $\exists\, t > 0$ s.t. $x(t) < 0$.

Then by continuity, $\exists\, t_* \quad 0 \leq t_* < t$

s.t. $x(t_*) = 0$

Then $x'(t_*) = 0 \Rightarrow x'(t) = 0 \quad \forall\, t \geq t_*$.

$\Rightarrow x(t) = 0 \quad \forall\, t \geq t_* \quad \Rightarrow\!\Leftarrow$

$$u'(t) = \lambda(u(t) - \cos(t)) - \sin(t)$$

$$\lambda = -10$$

$$\lambda = -250$$

Euler: $U^{n+1} = U^n + k f(U^n, t_n)$

$$u'(t) = \underbrace{\lambda u(t) + g(t)}_{f(u,t)}$$

$$U^{n+1} = U^n + k(\lambda U^n + g(t_n))$$

$$U^{n+1} = (1 + k\lambda)U^n + k g(t_n)$$

We previously worked out that

$$E^{n+1} = (1 + k\lambda)E^n - k\tau^n$$

Both U^n and E^n will grow without bound if

$$|1 + k\lambda| > 1.$$

In our example:

$k = \frac{1}{100}$

$\lambda = -10 : |1 + k\lambda| = 0.9$

$\lambda = -250 : |1 + k\lambda| = 1.5$

For any one-step method applied to this problem, we have

$$E^{n+1} = R(k\lambda)E^n - k\tau^n$$

We call $R(z)$ the stability function of the method.

For Euler's method: $R(z) = 1+z$

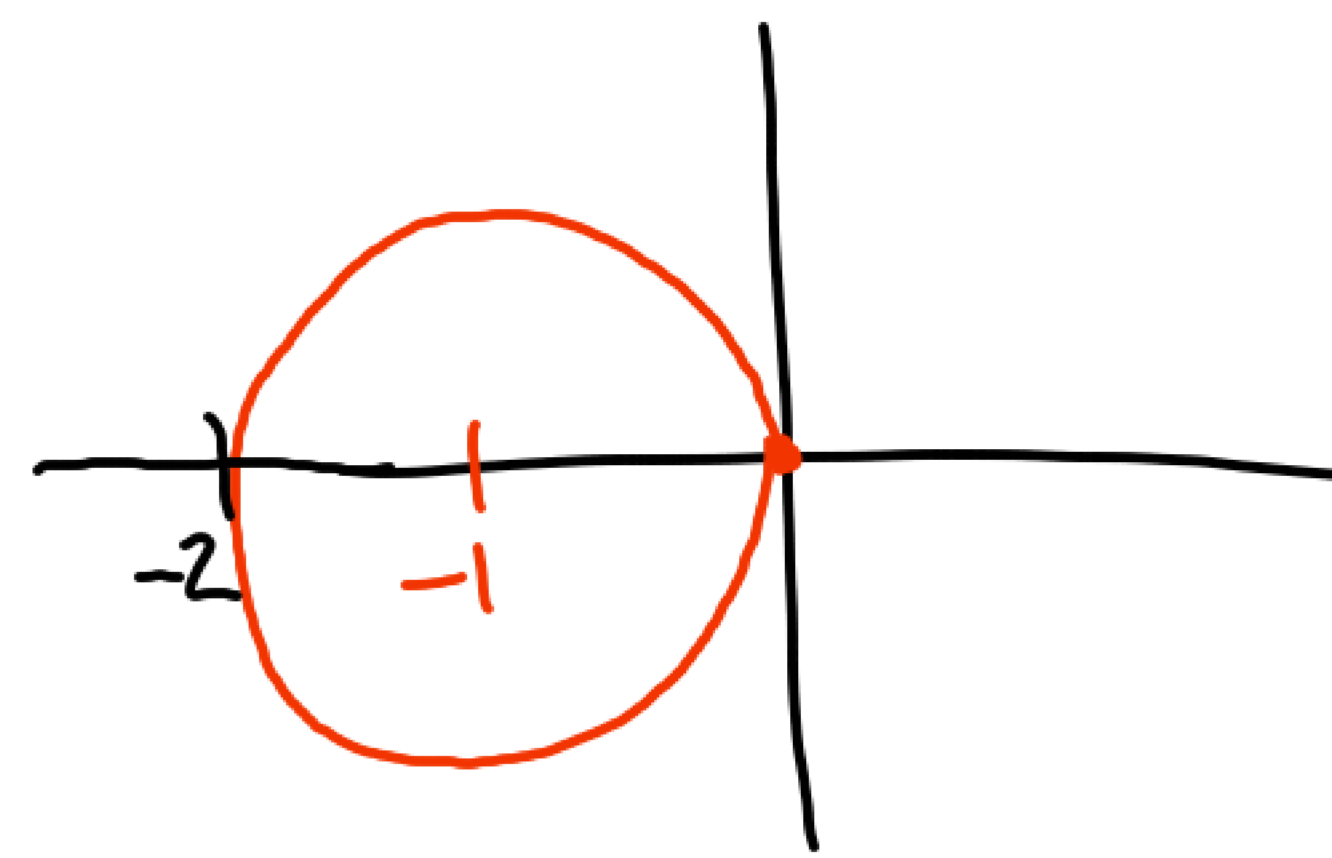
We have that $|E^n|$ remains bounded if $|R(k\lambda)| < 1$

The set of $z \in \mathbb{C}$ for which

$$|R(z)| \le 1$$

is called the region of absolute stability

For Euler: $|1+z| \le 1$



We need $-2 \le k\lambda \le 0$