Modified Equation Analysis

Idea: Given a numerical method, find a modified PDE such that the numerical solution of the original PDE is the exact solution of the modified PDE.

Example: $u_t + a u_x = 0$

Centered difference in time and space:

$$\frac{U_j^{n+1} - U_j^{n-1}}{2k} + a\frac{U_{j+1}^n - U_{j-1}^n}{2h} = 0$$

Assume there exists $v(x,t)$ satisfying:

What if $k^2 a^2 = h^2$?

Suppose $a > 0$. $\Rightarrow ka = h$

$$U_j^{n+1} - U_j^{n-1} + \frac{ka}{h}\left(U_{j+1}^n - U_{j-1}^n\right) = 0$$

$$\frac{v(x,t+k)-v(x,t-k)}{2k}+a\frac{v(x+h,t)-v(x-h,t)}{2h}=0$$

$$\frac{\cancel{v}+k\underline{v_t}+\cancel{\frac{k^2}{2}v_{tt}}+\frac{k^3}{6}v_{ttt}-\left(\cancel{v}-\underline{kv_t}+\cancel{\frac{k^2}{2}v_{tt}}-\frac{k^3}{6}v_{ttt}\right)+O(k^5)}{2k}+a\frac{\cancel{v}+h\underline{v_x}+\cancel{\frac{h^2}{2}v_{xx}}+\frac{h^3}{6}v_{xxx}-\left(\cancel{v}-\underline{hv_x}+\cancel{\frac{h^2}{2}v_{xx}}-\frac{h^3}{6}v_{xxx}\right)+O(h^5)}{2h}$$

$$v_t+av_x=-\frac{k^2}{6}v_{ttt}-a\frac{h^2}{6}v_{xxx}+O(k^4,h^4)$$

The numerical solution is a 4th order approximation of this modified PDE.

$$v_t+av_x=O(k^2,h^2)$$

$$v_t\approx -av_x$$

$$v_{tx}\approx -av_{xx} \Rightarrow v_{tt}\approx a^2v_{xx}$$

$$v_{tt}\approx -av_{xt}$$

$$v_{ttt}\approx a^2v_{xxt}$$

$$v_{txx}\approx -av_{xxx}$$

$$v_{ttt}=-a^3v_{xxx}+O(k^2,h^2)$$

$$V_t + aV_x = \underbrace{\frac{a}{6}\left(K^2a^2 - h^2\right)V_{xxx}}_{\text{dispersive}} + O(K^4, h^4, h^2K^2)$$

Ansatz: $V(x,t) = e^{i(\xi x - \omega t)}$

ξ = wavenumber

ω = frequency

$V_t = -i\omega V \qquad V_x = i\xi V$

$V_{xxx} = -i\xi^3 V$

$-i\omega V + ia\xi V = \frac{-ia\xi^3}{6}\left(K^2a^2 - h^2\right)V$

$\omega - a\xi = \frac{a\xi^3}{6}\left(K^2a^2 - h^2\right)$

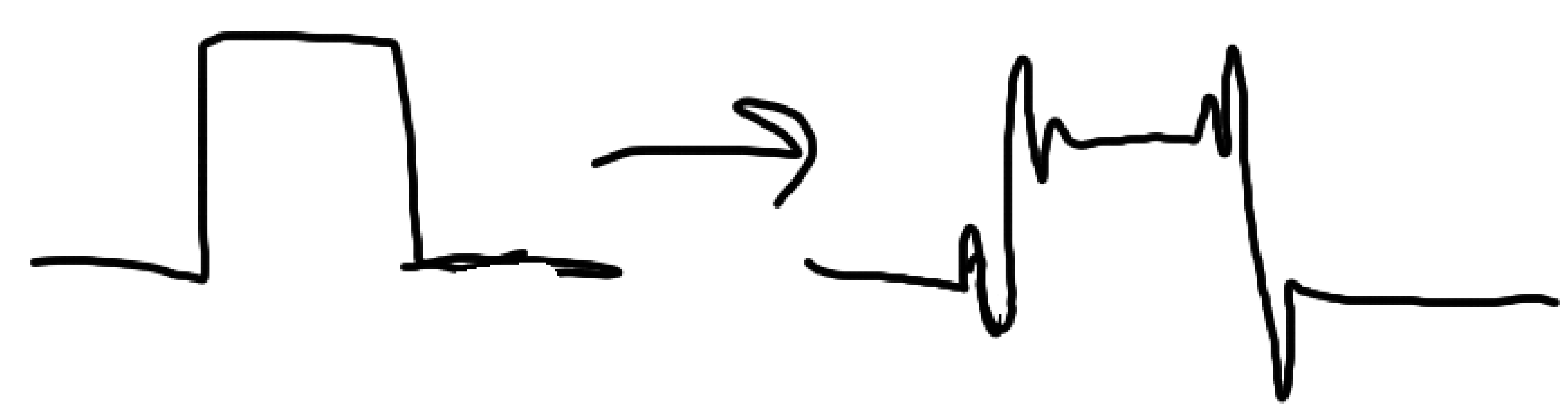
$\omega = a\xi\left(1 + \frac{K^2a^2 - h^2}{6}\xi^2\right)$ "dispersion relation"

$V(x,t) = \exp\left(i\xi\left(x - \left(a + a\xi^2\frac{K^2a^2 - h^2}{6}\right)t\right)\right)$

If $K^2a^2 = h^2$: $V(x,t) = \exp(i\xi(x - at))$

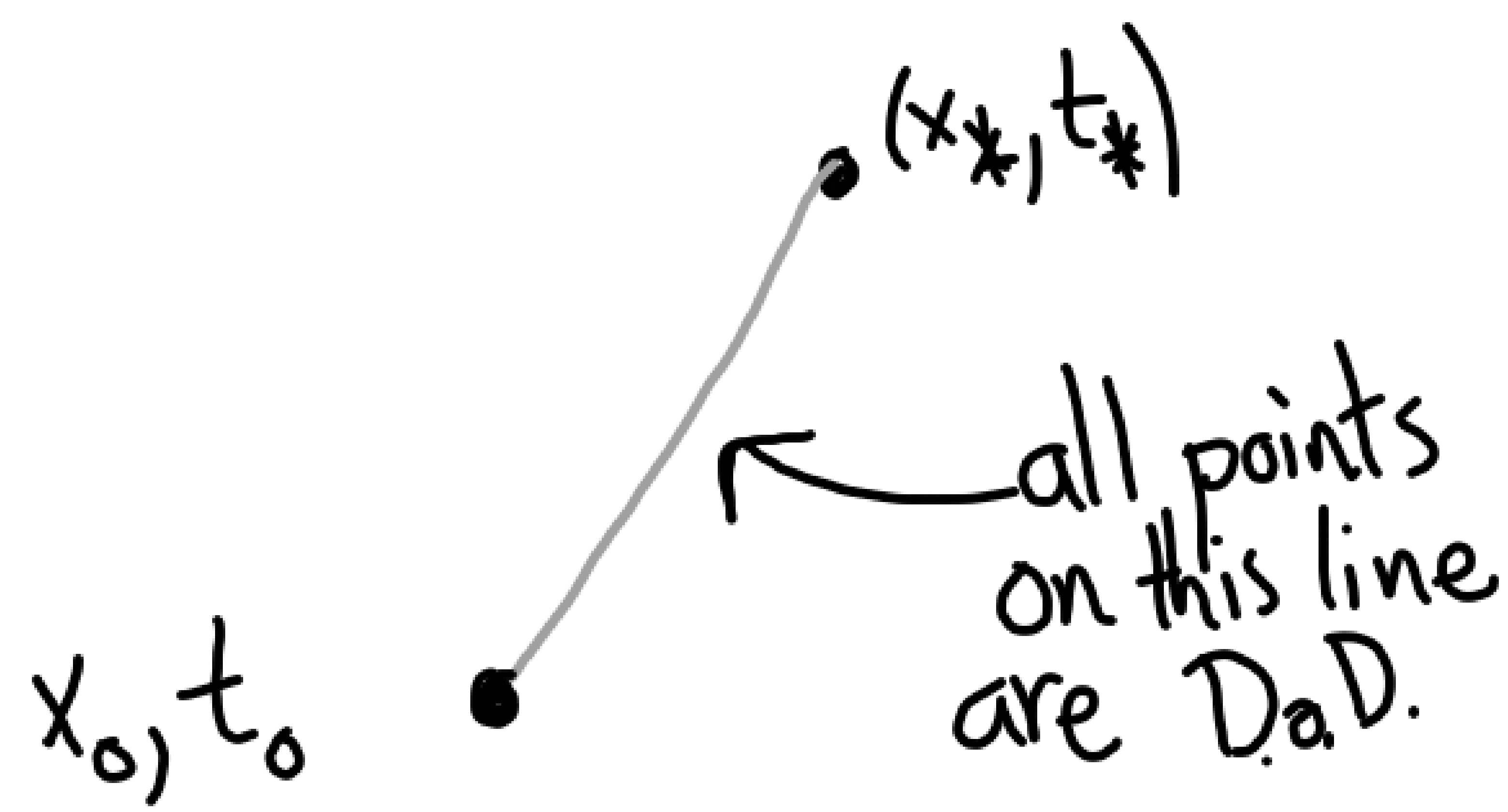
In general: $V(x,t) = \exp(i\xi(x - c(\xi)t))$

Different wavenumbers move at different speeds.



The CFL Condition

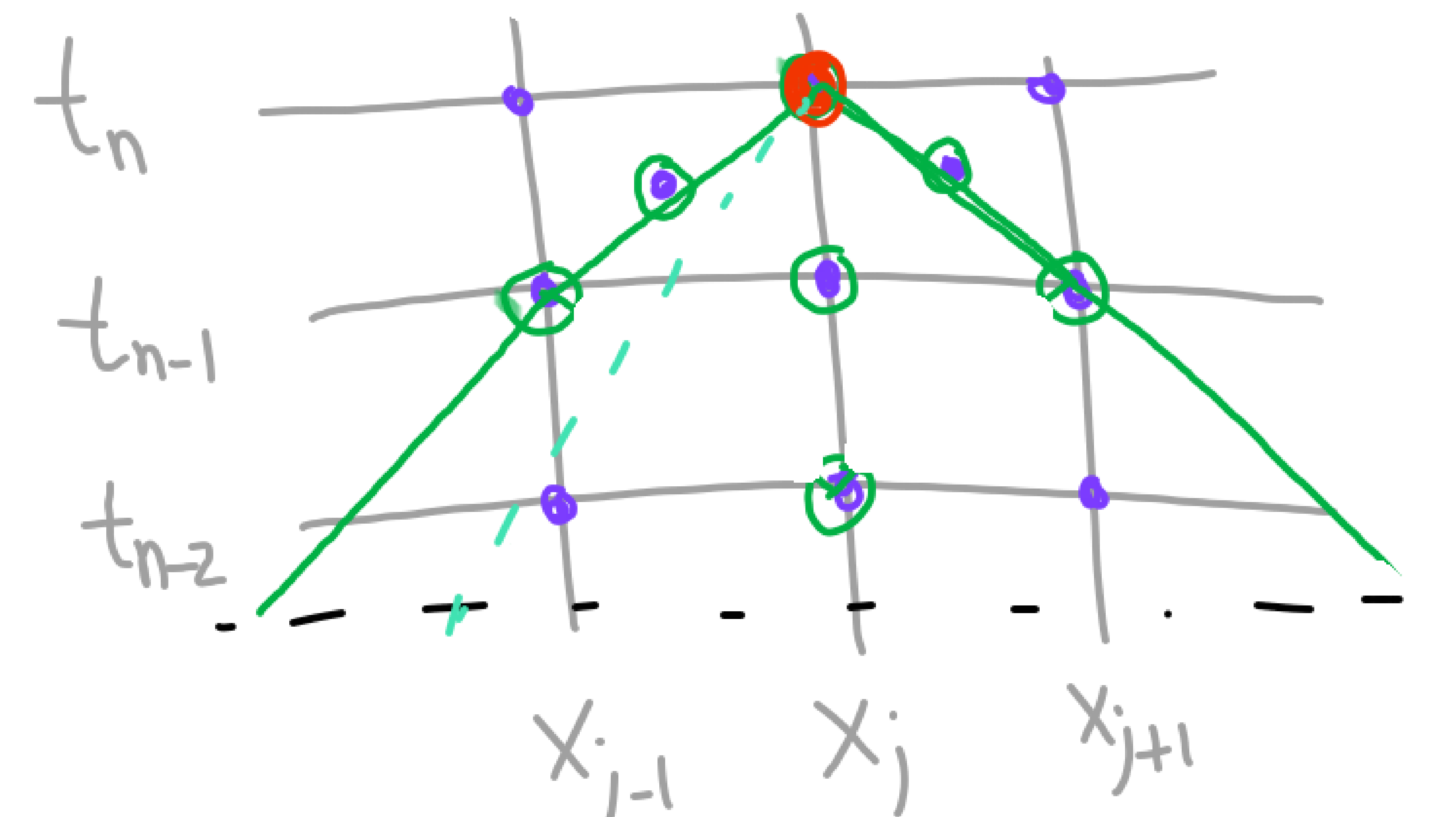
The domain of dependence is the set of points (in (x,t)) that can influence at some prescribed point



$$x_* = x_0 + a(t_* - t_0)$$

$$u_t + a u_x = g(t) \Rightarrow u(x_*, t_*) = u(x_0, t_0) + \int_{t_0}^{t_*} g(t)\,dt$$

The D.o.D. of the numerical solution must include the true D.o.D as $k, h \to 0$.



$$U_j^{n+1} = U_j^{n-1} - a\frac{k}{h}\left(U_{j+1}^n - U_{j-1}^n\right)$$

Values outside of △ cannot influence the numerical solution at ●.

Information can travel at most from x_j to x_{j+1} (distance h) in time k.

$$|ka| \leq h$$

$$\left|\frac{ka}{h}\right| \leq 1.$$

The CFL condition is necessary but not always sufficient for stability.

$\frac{ka}{h}$ is the Courant number.