So far:

$$\text{Linear Diff. Eqn.} \xrightarrow{\text{Discretize}} \text{Linear Algebra} \xrightarrow{\text{Solve}} \text{Approximate solution}$$

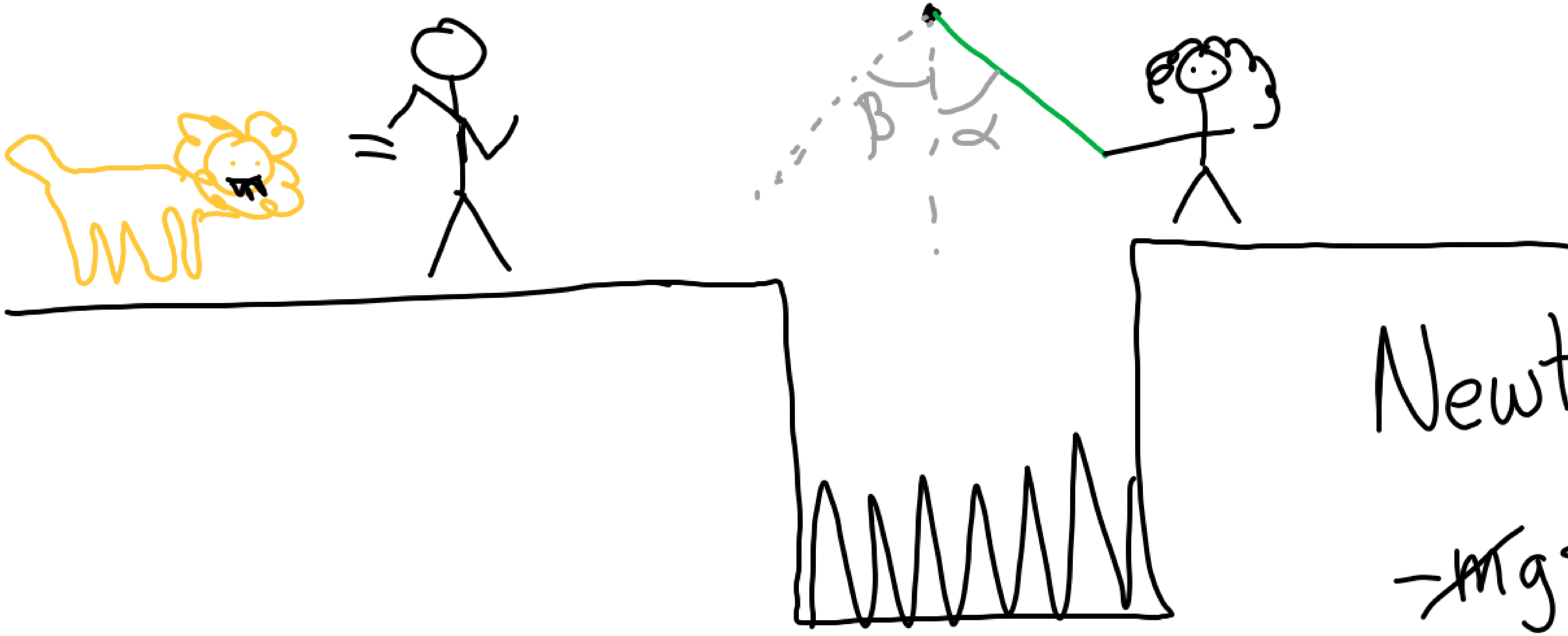
Today:

$$\begin{tikzcd}
\text{Nonlinear D.E.} \arrow[r, "\text{Discretize}"] & \text{Nonlinear Algebra} \arrow[r, "\text{Linearize}", "\text{(Newton)}"'] & \text{Linear Algebra} \arrow[r, "\text{solve}"] & \text{Check for convergence} \arrow[lll, bend right, "\text{No}"'] \arrow[d, "\text{Yes}"] \\
 & & & \text{Approximate solution}
\end{tikzcd}$$

May have multiple solutions (or none)

Sources of error:

1. Discretization
2. Rounding
3. Linearization

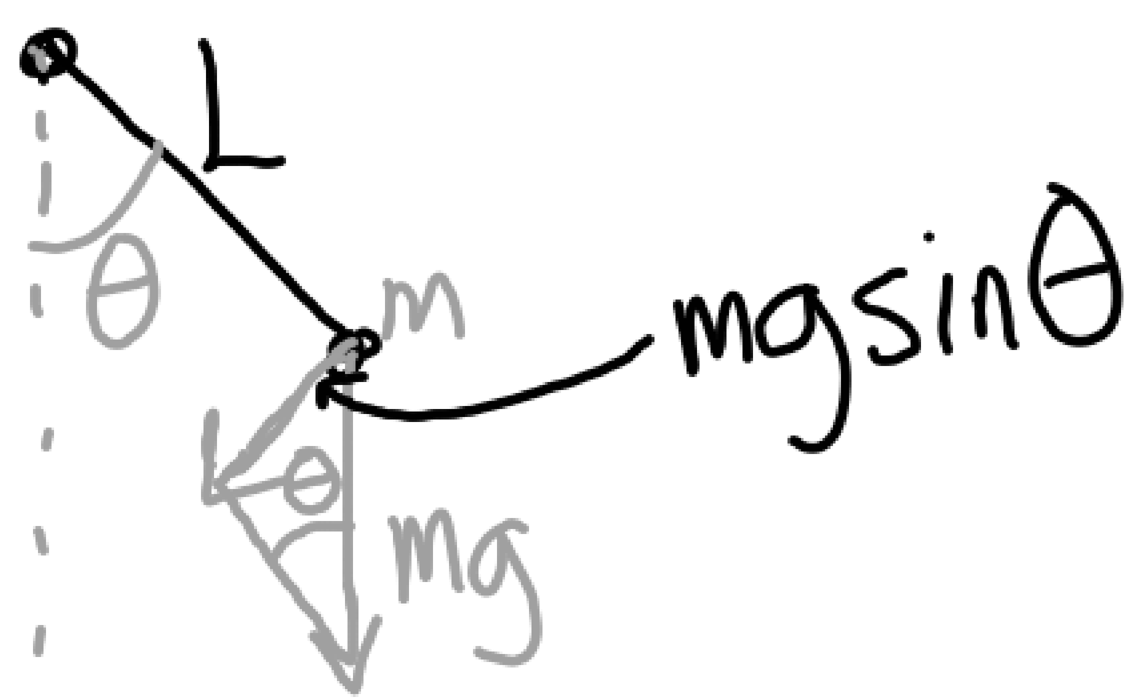


Newton: $F = ma$

$$-\cancel{m}g\sin\theta = \cancel{m}\,\theta''(t)L$$

$$\theta''(t) = -\frac{g}{L}\sin(\theta(t))$$

Choose units so $\frac{g}{L} = 1$



$$\boxed{\begin{aligned} \theta''(t) &= -\sin(\theta(t)) \\ \theta(0) &= \alpha \\ \theta(T) &= \beta \end{aligned}}$$

Nonlinear BVP

$$h = \frac{1}{m+1}$$

$$\theta''(t_i) \approx \frac{\theta_{i+1} - 2\theta_i + \theta_{i-1}}{h^2} = -\sin(\theta_i) \qquad 1 \le i \le m$$

$$\theta_0 = \alpha \qquad \theta_{m+1} = \beta$$

First we pick some initial guess $\theta^{[0]}$.

$$G_i(\theta) = \frac{\theta_{i+1} - 2\theta_i + \theta_{i-1}}{h^2} + \sin(\theta_i) = 0$$

We want to find θ_* such that $G(\theta_*) = 0$.

We can write

$$G(\theta_*) = G(\theta^{[0]}) + G'(\theta^{[0]})(\theta_* - \theta^{[0]}) + \mathcal{O}(\|\theta_* - \theta^{[0]}\|^2)$$

$$G'(\theta) = J(\theta) = \begin{bmatrix} \frac{\partial G_1}{\partial \theta_1} & \frac{\partial G_1}{\partial \theta_2} & \cdots & \frac{\partial G_1}{\partial \theta_m} \\ \frac{\partial G_2}{\partial \theta_1} & & & \\ \vdots & & & \\ \frac{\partial G_m}{\partial \theta_1} & \cdots & \cdots & \frac{\partial G_m}{\partial \theta_m} \end{bmatrix}$$

Let $\delta^{[k]} = \theta_* - \theta^{[k]}$

Discarding the error term gives

$$G(\theta_*) = 0 \Rightarrow$$

$$G(\theta^{[0]}) + J(\theta^{[0]})\delta^{[0]} = 0$$

$$\underbrace{J(\theta^{[0]})\,\delta^{[0]} = -G^{[0]}}_{\text{Linear Algebra}}$$

Newton's method:

1. Start with initial guess $\theta^{[0]}$. Set $k=0$.
2. Solve
$$J(\theta^{[k]})\delta^{[k]} = -G(\theta^{[k]})$$
3. $$\theta^{[k+1]} = \theta^{[k]} + \delta^{[k]}$$
4. Check $\|G(\theta^{[k+1]})\| < \varepsilon$
If not converged, increment k and repeat 2-4.

$$G_i(\theta) = \frac{1}{h^2}(\theta_{i+1} - 2\theta_i + \theta_{i-1}) + \sin(\theta_i)$$

$$J_{ij}(\theta) = \begin{cases} \frac{1}{h^2} & j = i \pm 1 \\ \frac{-2}{h^2} + \cos(\theta_i) & j = i \\ 0 & \text{otherwise} \end{cases}$$

Consistency

Substitute exact solution: $\hat{\theta} = \begin{bmatrix} \theta(t_1) \\ \vdots \\ \theta(t_m) \end{bmatrix}$

$$G_i(\hat{\theta}) = \cancel{\theta''(t_i)} + \frac{1}{12}h^2\theta^{(4)}(t_i) + \cancel{\sin(\theta(t_i))} + O(h^4)$$

$$G_i(\hat{\theta}) = \frac{1}{12}h^2\theta^{(4)}(t_i) + O(h^4) = \tau_i$$

Method is consistent and locally 2nd-order

Stability

$$G(\hat{\theta}) = \tau$$

$$G(\theta_*) = 0$$

$$\tau = G(\hat{\theta}) - G(\theta_*)$$

$$E = \hat{\theta} - \theta_*$$

Write: $G(\theta_*) = G(\hat{\theta}) + J(\hat{\theta})E + \mathcal{O}(\|E\|^2)$

Then $\tau = -J(\hat{\theta})E + \mathcal{O}(\|E\|^2)$

We want to discard the error term and write } We haven't justified this. It requires more work.

$$E = -(J(\hat{\theta}))^{-1}\tau \rightarrow \|E\| \leq \|J(\hat{\theta})^{-1}\| \cdot \|\tau\|$$

This holds because $J \rightarrow A$ as $h \rightarrow 0$. (more precisely, $h^2 J \rightarrow h^2 A$)

Then if we can show $\|J(\hat{\theta})^{-1}\| < C$ as $h \rightarrow 0$, we have convergence:

$$\lim_{h \to 0} \|E\| = 0.$$