Initial boundary value problems

(IBVPs)

We'll focus on evolution equations:

$$\frac{\partial u}{\partial t} = f\left(u, \frac{\partial u}{\partial x}, \frac{\partial^2 u}{\partial x^2}, \dots\right)$$

$$\longrightarrow u_t = f(u, u_x, u_{xx}, \dots)$$

$u = u(x,t)$ $\quad x \in [0,1] \quad t \in [0,T]$

Initial values: $u(x, t=0) = \eta(x)$

Boundary Values: $u(x=0,t) = \alpha(t) \quad u(x=1,t) = \beta(t)$

Diffusion equation

$$u_t = K u_{xx} + f(x)$$

K: Heat Conductivity $\quad f(x)$: Heat source

For simplicity: $f(x) = 0$

$K = 1$

$$u(0,t) = u(1,t) = 0$$

$$u(x, t=0) = \eta(x)$$

$$u_t = u_{xx} \quad (1)$$

Exact solution:

$$\eta(x) = \sum_{p=0}^{\infty} \hat{u}_p(0) \sin(p\pi x)$$

$$u(x,t) = \sum_{p=0}^{\infty} \hat{u}_p(t) \sin(p\pi x)$$

Substitute into (1):

$$\sum_{p=0}^{\infty} \hat{u}_p'(t) \sin(p\pi x) = -p^2\pi^2 \sum_{p=0}^{\infty} \hat{u}_p(t) \sin(p\pi x)$$

Equate term-by-term:

$$\hat{u}_p'(t) = -p^2\pi^2 \hat{u}_p(t)$$ ← Linear, constant-coefficient ODE

$$\Rightarrow \hat{u}_p(t) = e^{-p^2\pi^2 t} \hat{u}_p(0)$$

So we have

$$u(x,t) = \sum_{p=0}^{\infty} e^{-p^2\pi^2 t} \hat{u}_p(0) \sin(p\pi x)$$

Each Fourier mode decays exponentially in time.

Higher wavenumbers decay faster.

$$u_t = u_{xx} \qquad x \in [0,1]$$

First discretize in space:

$$x_j = jh \qquad j = 0, 1, 2, \ldots, m+1$$

$$h = \frac{1}{m+1}$$

$$U_j(t) \approx u(x_j, t)$$

$$u_{xx}(x_j, t) \approx \frac{U_{j+1}(t) - 2U_j(t) + U_{j-1}(t)}{h^2}$$

Our PDE becomes a system of ODEs

This approach is called the method of lines

$$\underbrace{\begin{bmatrix} U_1'(t) \\ U_2'(t) \\ \vdots \\ U_m'(t) \end{bmatrix}}_{U'(t)} = \frac{1}{h^2} \underbrace{\begin{bmatrix} -2 & 1 & & \\ 1 & \ddots & \ddots & \\ & \ddots & \ddots & 1 \\ & & 1 & -2 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} U_1(t) \\ U_2(t) \\ \vdots \\ U_m(t) \end{bmatrix}}_{U(t)}$$

$$U'(t) = AU(t)$$ Semi-discretization

Solution: $U(t) = e^{tA} U(0)$

Discretize in time using RK, LM, etc.

Semi-discrete Fourier Analysis

$Ar_p = \lambda_p r_p$ where $\lambda_p = \frac{2}{h^2}(\cos(p\pi h) - 1)$

$(r_p)_j = \sin(p\pi jh) = \sin(p\pi x_j)$

$AR = R\Lambda$ where $R = [r_1 | r_2 | \cdots | r_m]$

$$\Lambda = \begin{bmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_m \end{bmatrix}$$

$A = R\Lambda R^{-1}$

$\Rightarrow U'(t) = R\Lambda R^{-1}U(t)$

$R^{-1}U'(t) = \Lambda R^{-1}U(t)$

Define $\hat{U}(t) = R^{-1}U(t)$

$\hat{U}'(t) = \Lambda \hat{U}(t)$

$\hat{U}_p'(t) = \lambda_p \hat{U}_p(t)$

Solution: $\hat{U}_p(t) = e^{\lambda_p t}\hat{U}_p(0)$

Comparing with the exact solution, we should have $\lambda_p \approx -p^2\pi^2$.

In fact:

$\lambda_p = \frac{2}{h^2}\left(1 - \frac{1}{2}(p\pi h)^2 + O(p^4h^4) - 1\right)$

$\lambda_p = -p^2\pi^2 + O(p^4h^2)$ ✓

So this approximates the exact solution well for small values of ph.

Absolute stability

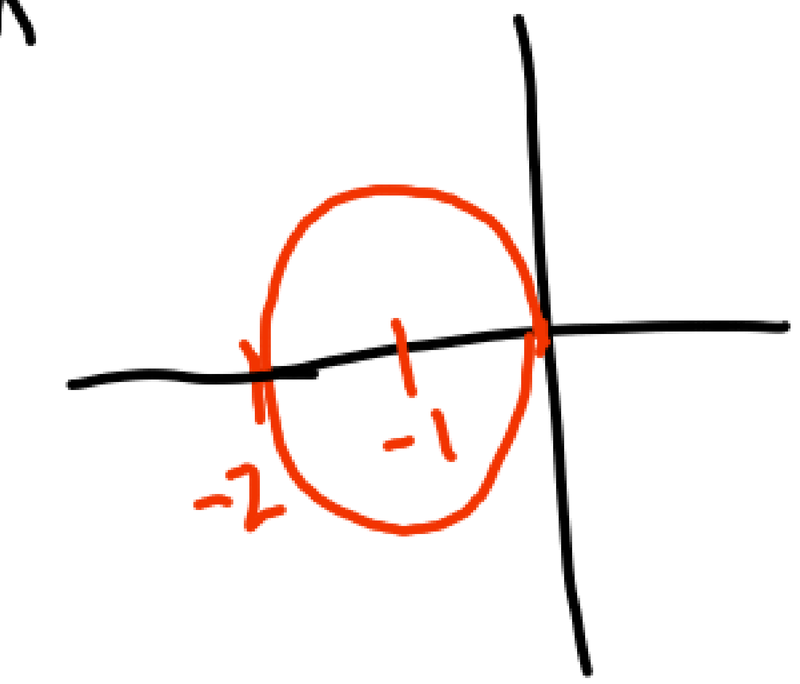
We need

$$k\lambda_p \in S \quad \forall\, \lambda_p \in \sigma(A)$$

where S is the stability region of our time discretization.

Use Euler's method in time:

$$U^{n+1} = U^n + kAU^n$$



For abs. stability we need

$$-2 \le k\lambda_p \le 0$$

Largest magnitude eigenvalue: $\approx \dfrac{-4}{h^2}$

$$-2 \le \frac{-4k}{h^2} \le 0 \Rightarrow 0 \le k \le \frac{h^2}{2}$$

Very small time step required

The original problem with eigenvalues $-p^2\pi^2$, $0 \le p < \infty$ is infinitely stiff.

By discretizing in space, we make the stiffness finite.

For small h, λ_m gets very large. So we should use an A-stable or $A(\alpha)$-stable method.

Diagonally implicit Runge-Kutta methods

$$Y_i = U^n + k\sum_{j=1}^{i} a_{ij} f(Y_j) \qquad 1 \le i \le s$$

$$U^{n+1} = U^n + k\sum_{j=1}^{s} b_j f(Y_j)$$

$$\begin{array}{c|c} c & A \\ \hline & b^T \end{array}$$

A is lower triangular

More efficient than fully-implicit RK since we can solve each stage equation sequentially.

For example:

$$\begin{array}{c|ccc} 0 & 0 & 0 & 0 \\ 1/2 & 1/4 & 1/4 & \\ 1 & 1/3 & 1/3 & 1/3 \\ \hline & 1/3 & 1/3 & 1/3 \end{array}$$

TR-BDF2

A-stable

L-stable $\quad (R(z) \to 0 \text{ as } z \to -\infty)$

2nd-order accurate