Initial Value ODEs

Examples:

① Rigid pendulum

$$y'(t) = f(y(t))$$

$$\Theta''(t) = \underbrace{-\underbrace{\sin(\Theta(t))}_{\text{autonomous}} + \cos(\omega t)}_{\text{non-autonomous}}$$

$$\Theta(t_0) = \Theta_0$$

$$\Theta'(t_0) = \Omega_0$$

$$y'(t) = f(t, y(t))$$

② SIR model

$S(t)$: Susceptible

$I(t)$: Infectious

$R(t)$: Removed

$$S'(t) = -\beta S I$$

$$I'(t) = \beta S I - \gamma I$$

$$R'(t) = \gamma I$$

$$(S(0), I(0), R(0)) = (S_0, I_0, R_0)$$

$$S + I + R = 1$$

$$\frac{d}{dt}(S+I+R) = -\beta SI + \beta SI - \gamma I + \gamma I$$
$$= 0$$

---

③ Lorenz system

$$y_1'(t) = -\sigma y_1 + \sigma y_2$$
$$y_2'(t) = -y_1 y_3 + r y_1 - y_2$$
$$y_3'(t) = y_1 y_2 - b y_3$$

We can write a higher order ODE as a system of first-order ODEs:

$$\Theta''(t) = -\sin(\Theta(t))$$

Let $\phi(t) = \Theta'(t)$

Then $\phi'(t) = -\sin(\Theta(t))$

We can write any non-autonomous ODE as a system of autonomous ODEs:

$$\theta''(t) = -\sin(\theta(t)) + \cos(\omega t)$$

$y_1(t) = \theta(t)$ $\quad y_2(t) = \theta'(t)$ $\quad y_3(t) = t$

$$\boxed{\begin{aligned} y_2'(t) &= -\sin(y_1(t)) + \cos(\omega y_3(t)) \\ y_1'(t) &= y_2(t) \\ y_3'(t) &= 1 \end{aligned}}$$

So we can (WLOG) restrict our attention to First-order autonomous systems

We will write

$$u'(t) = f(u)$$
$$u(t_0) = \eta$$

Scalar linear IVP

$$u'(t) = \lambda u + g(t) \qquad \lambda \in \mathbb{C}$$
$$u(t_0) = \eta$$

$$u(t) = e^{(t-t_0)\lambda}\eta + \int_{t_0}^{t} e^{(t-\tau)\lambda} g(\tau)\, d\tau$$

Linear System IVP

$$u'(t) = Au(t) + g(t)$$
$$u(t_0) = \eta$$

$$u(t) \in \mathbb{C}^N$$

Solution:

$$\boxed{u(t) = e^{(t-t_0)A}\eta + \int_{t_0}^{t} e^{(t-\tau)A} g(\tau)\, d\tau}$$

Duhamel's principle

$$e^M = I + M + \frac{1}{2!}M^2 + \frac{1}{3!}M^3 + \cdots$$
$$= \sum_{k=0}^{\infty} \frac{1}{k!} M^k$$

Example: Linearized pendulum

$$|\theta| \ll 1: \qquad \theta''(t) = -\sin(\theta) \approx -\theta(t)$$
$$\phi(t) = \theta'(t)$$

$$\begin{bmatrix} \theta \\ \phi \end{bmatrix}' = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} \theta \\ \phi \end{bmatrix}$$

Existence and Uniqueness

Linear IVPs: Unique solution exists for all t.

Nonlinear IVPs: ?

$$u'(t) = (u(t))^2$$
$$u(0) = \eta > 0$$
$$\int_0^t u^{-2}\,du = \int_0^t dt$$
$$-u^{-1}\Big|_0^t = t$$

$$\frac{-1}{u(t)} - \frac{-1}{u(0)} = t$$
$$\frac{-1}{u(t)} + \frac{1}{\eta} = t$$
$$\frac{1}{u} = \eta^{-1} - t$$
$$u(t) = \frac{1}{\eta^{-1} - t}$$

Solution exists for $0 \le t < \eta^{-1}$

$$u'(t) = (u(t))^{1/2}$$
$$u(0) = \eta \ge 0$$
$$\int_0^t u^{-1/2}\,du = \int_0^t dt$$
$$2u^{1/2}\Big|_0^t = t$$
$$2\sqrt{u(t)} - 2\sqrt{\eta} = t$$
$$2\sqrt{u} = t + 2\sqrt{\eta}$$
$$u = \left(\frac{t}{2} + \sqrt{\eta}\right)^2$$

Solution exists for all time

We assumed $u \neq 0$

What if $\eta = 0$?

$$u = \frac{t^2}{4}$$

But we can also choose: $u(t) = 0$

Solution not unique

Lipschitz constant

Given $f(u)$ and domain D, we say L is a L.C. for f on D if

$$\|f(u_1) - f(u_2)\| \leq L\|u_1 - u_2\|$$

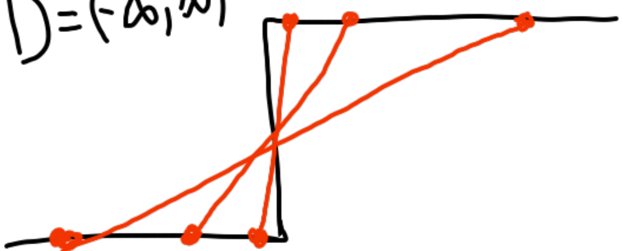
for all $u_1, u_2 \in D$. $(0 \leq L < \infty)$

We say f is Lipschitz continuous on D if such an L exists.

Examples:

$$H(u) = \begin{cases} -1 & u<0 \\ 1 & u>0 \end{cases}$$

$D=(-\infty,\infty)$



Not Lipschitz.

Are all continuous functions Lipschitz continuous? No

For any continuously differentiable function, if

$$\sup_{u\in D}\|f'(u)\| < \infty$$

Then

$$L = \sup_{u\in D}\|f'(u)\|$$

$f(u)=u^2 \quad D=(-\infty,\infty)$

$\dfrac{|f(u_1)-f(u_2)|}{|u_1-u_2|}$ can be made arbitrarily large by taking large enough u_1, u_2

Given the IVP

$$u'(t) = f(u)$$
$$u(t_0) = \eta$$

Suppose $f(u)$ is Lipschitz continuous

for $\eta - a \le u \le \eta + a$

Then a unique solution exists

for $t_0 \le t \le t_0 + \dfrac{a}{\sup_{u \in D} |f|}$

$(D = [\eta - a, \eta + a])$

$$u(t) = \eta + \int_{t_0}^{t} f(\tau)\, d\tau$$