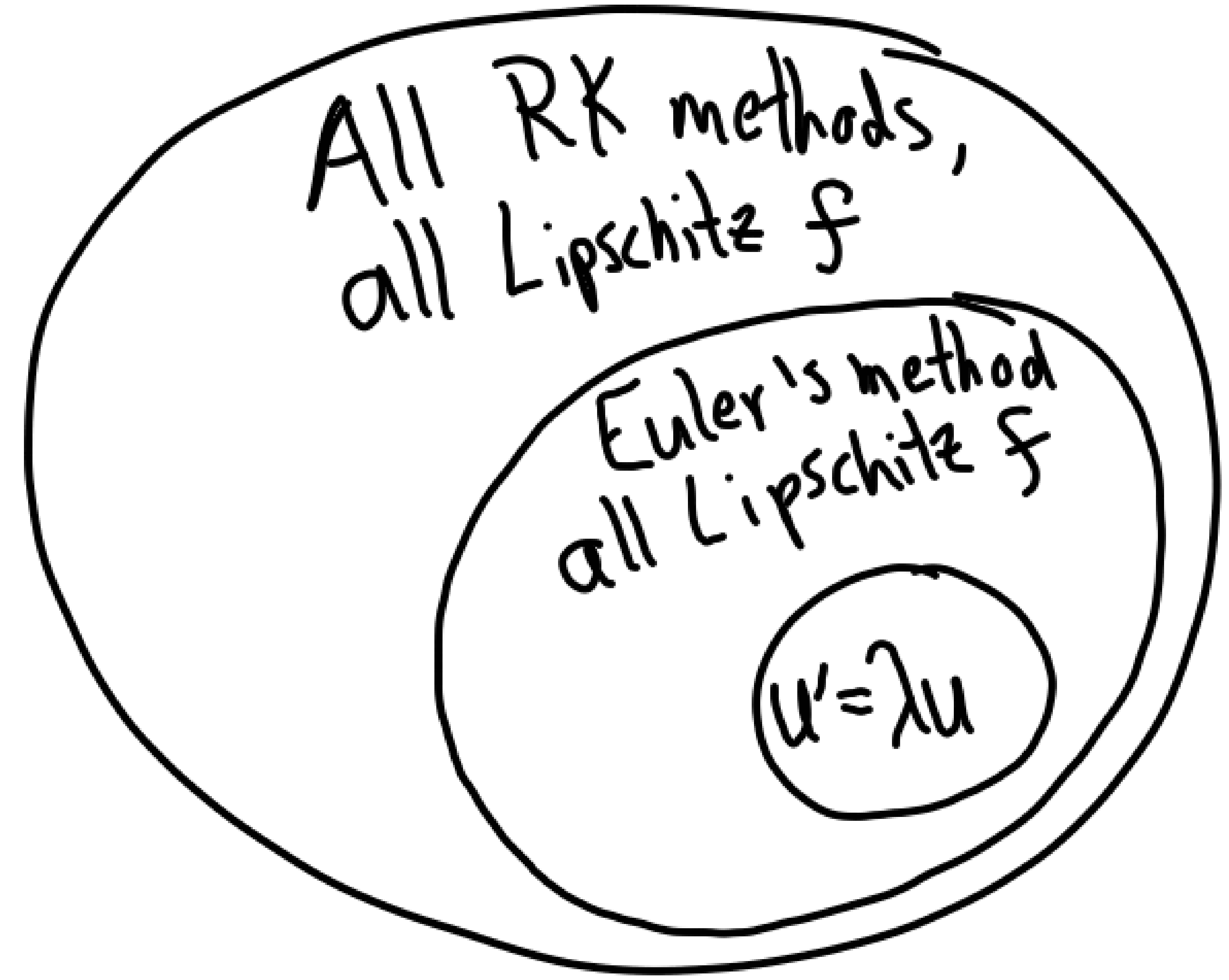
Stability and Convergence of Runge-Kutta

We want to show that any RK method applied to

$$u'(t) = f(u(t), t)$$
$$u(t_0) = \eta \qquad t_0 \le t \le T$$

gives a convergent solution if if f is Lipschitz w.r.t. u.



What do we mean by convergence?

$$\lim_{k \to 0} \| \underbrace{U^N - u(t_N)}_{E^N} \| = 0$$

Where $NK = T$ is fixed.
$(N \to \infty)$

Model problem:

$$u'(t) = \lambda u(t) + g(t)$$
$$u(t_0) = \eta$$
$$\lambda \in \mathbb{C}$$

Euler's method:

$$(1) \quad \frac{U^{n+1} - U^n}{k} = \lambda U^n + g(t_n)$$

$$(2) \quad \frac{u(t_{n+1}) - u(t_n)}{k} = \lambda u(t_n) + g(t_n) + \tau^n$$

Subtract (2) from (1)

$$\frac{E^{n+1} - E^n}{k} = \lambda E^n - \tau^n$$

$$E^{n+1} = E^n + k(\lambda E^n - \tau^n) = (1+k\lambda)E^n - k\tau^n$$

$$E^n = E^{n-1} + k(\lambda E^{n-1} - \tau^{n-1})$$

$$E^{n+1} = E^{n-1} + k(\lambda E^{n-1} - \tau^{n-1})$$
$$+ k(\lambda(E^{n-1} + k(\lambda E^{n-1} - \tau^{n-1})) - \tau^n)$$

$$E^{n+1} = E^{n-1}(1 + 2k\lambda + k^2\lambda^2) - (k + k^2\lambda)\tau^{n-1} - k\tau^n$$

$$E^{n+1} = (1+k\lambda)^2 E^{n-1} - k(1+k\lambda)\tau^{n-1} - k\tau^n$$

$$E^N = \underline{(1+k\lambda)^N E^0} - k\sum_{m=1}^{N}(1+k\lambda)^{N-m}\tau^{m-1}$$

(Analogous to Duhamel's principle:

$$u(t) = e^{\lambda(t-t_0)}\eta + \int_{t_0}^{t} e^{\lambda(t-\tau)} g(\tau)\,d\tau \quad)$$

Usually $E^0 = U^0 - \eta = 0$.

So we want to show that

$$\lim_{\substack{k\to 0\\ Nk=T}} \Big|\underbrace{k\sum_{m=1}^{N}(1+k\lambda)^{N-m}\tau^{m-1}}_{S_k = E^N}\Big| = 0$$

We can write

$$|S_k| \le k\sum_{m=1}^{N}|1+\lambda k|^{N-m}|\tau^{m-1}|$$

We have

$$|1+\lambda k| \le 1 + |\lambda| k \le \sum_{j=0}^{\infty}\frac{(k|\lambda|)^j}{j}$$

$$|1+\lambda k| \le e^{k|\lambda|}$$

$$|S_k| \le k\sum_{m=1}^{N} e^{(N-m)k|\lambda|}|\tau^{m-1}|$$

$$\le Nk e^{Nk|\lambda|}\max_{1\le m\le N}|\tau^{m-1}|$$

$$|S_k| \le T e^{T|\lambda|}\|\tau\|_\infty$$

$$\lim_{k\to 0} |E^N| = \lim_{k\to 0} T e^{T|\lambda|} \|\tau\|_\infty = 0$$

$Nk = T$

Here $\tau = \begin{bmatrix} \tau^0 \\ \tau^1 \\ \vdots \\ \tau^{N-1} \end{bmatrix}$.

How big is $Te^{T|\lambda|}$?

What if $T=10$, $\lambda=10$?

$10e^{100}$

Convergence of Euler's method for general IVPs

$$u'(t) = f(u)$$
$$u(t_0) = \eta$$

We assume $\exists L>0$ s.t.
$\|f(v) - f(u)\| \le L\|u-v\|$ $\forall u,v$

$$\frac{U^{n+1} - U^n}{k} = f(U^n)$$

$$\frac{u(t_{n+1}) - u(t_n)}{k} = f(u(t_n)) + \tau^n$$

$$\frac{E^{n+1} - E^n}{k} = f(U^n) - f(u(t_n)) - \tau^n$$

$$E^{n+1} = E^n + k\left(f(U^n) - f(u(t_n))\right) - k\tau^n$$

$$\|E^{n+1}\| \leq \|E^n\| + k\|f(U^n) - f(u(t_n))\| + k\|\tau^n\|$$

$$\|E^{n+1}\| \leq \|E^n\| + kL\|E^n\| + k\|\tau^n\|$$

Iteratively apply this inequality to obtain:

$$\|E^N\| \leq (1+kL)^N \underbrace{\|E^0\|}_{=0 \text{ since } U^0 = \eta} + k\sum_{m=1}^{N}(1+kL)^{N-m}\|\tau^{m-1}\|$$

We have $(1+kL) \leq e^{kL}$.

We get $\|E^N\| \leq Te^{TL} \max\limits_{1 \leq m \leq N}\|\tau^{m-1}\|$

So $\lim\limits_{k \to 0}\|E^N\| = 0$.

Convergence of a RK method for a general IVP

$$U^* = U^n + \frac{k}{2} f(U^n)$$

$$U^{n+1} = U^n + k f(U^*)$$

$$\frac{U^{n+1} - U^n}{k} = \underbrace{f\left(U^n + \frac{k}{2} f(U^n)\right)}_{\Psi(U^n, k)}$$

$$\frac{u(t_{n+1}) - u(t_n)}{k} = \underbrace{f\left(u(t_n) + \frac{k}{2} f(u(t_n))\right)}_{\Psi(u(t_n), k)} + \tau^n$$

$$\frac{E^{n+1} - E^n}{k} = \Psi(U^n, k) - \Psi(u(t_n), k) - \tau^n$$

Claim: if f is Lipschitz with constant L then Ψ is also Lipschitz with constant $L + \frac{1}{2} k L^2$.

$$
\begin{aligned}
\|\Psi(v)-\Psi(w)\| &= \|f(v+\tfrac{1}{2}kf(v))-f(w+\tfrac{1}{2}kf(w))\| \\
&\le L\|v+\tfrac{1}{2}kf(v)-(w+\tfrac{1}{2}kf(w))\| \\
&\le L\|v-w+\tfrac{1}{2}k(f(v)-f(w))\| \\
&\le L(\|v-w\|+\tfrac{1}{2}k\|f(v)-f(w)\|) \\
&\le L(\|v-w\|+\tfrac{1}{2}kL\|v-w\|) \\
&\le \underbrace{(L+\tfrac{1}{2}kL^2)}_{\hat{L}}\|v-w\|
\end{aligned}
$$

So $E^{n+1}=E^n+k(\Psi(U^n,k)-\Psi(u(t_n),k))-k\tau^n$

$$
\begin{aligned}
\Rightarrow \|E^{n+1}\| &\le \|E^n\|+k\hat{L}\|E^n\|+k\|\tau^n\| \\
&\le (1+k\hat{L})\|E^n\|+k\|\tau^n\|
\end{aligned}
$$

Follow the same steps as before with $\hat{L}$ in place of L:

$$\|E^N\| \le Te^{T\hat{L}}\max_{1\le m\le N}\|\tau^{m-1}\|$$

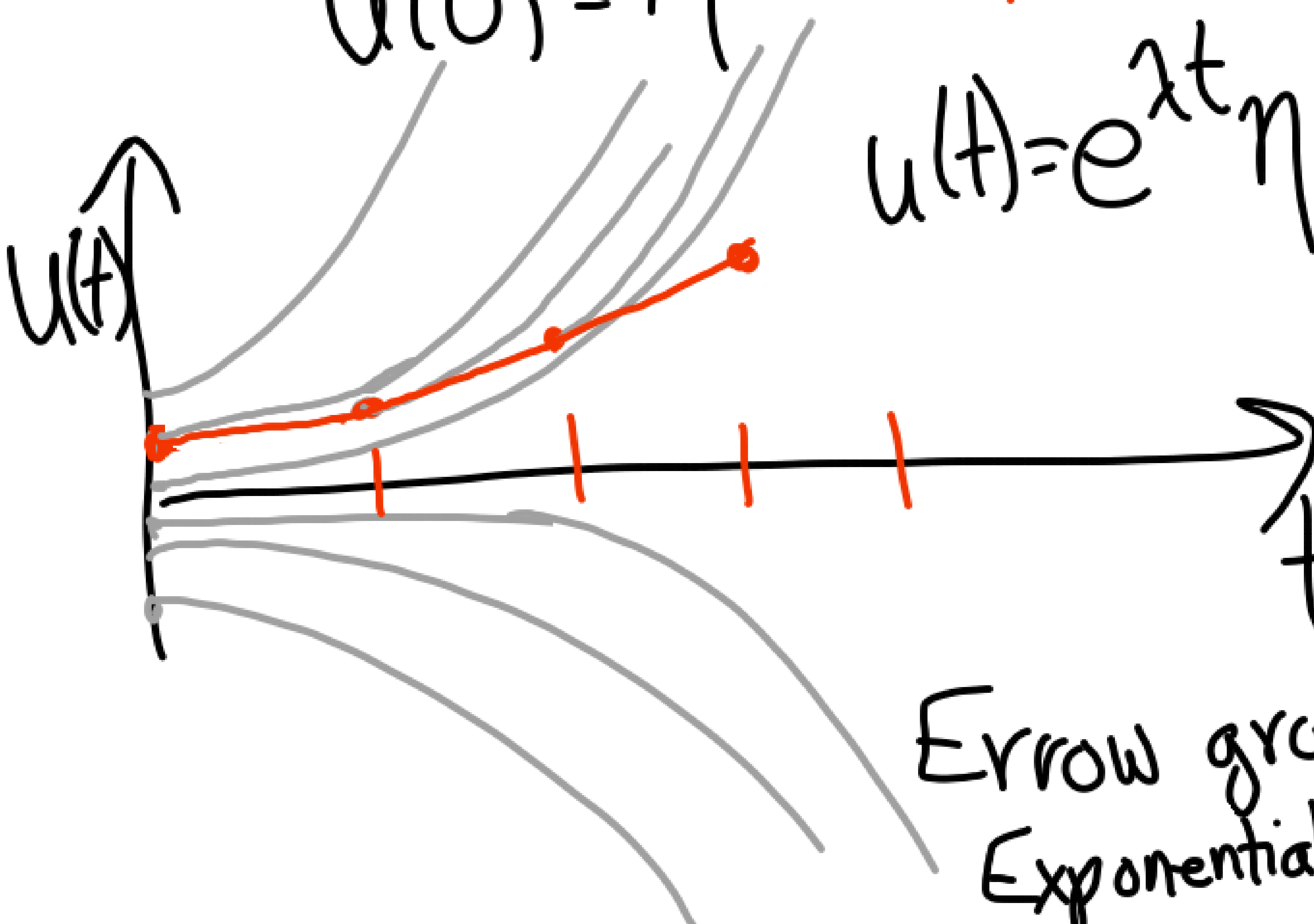
Does the global error actually grow exponentially in time?

$$u'(t) = \lambda u(t)$$
$$u(0) = \eta$$

$\lambda \in \mathbb{R}$

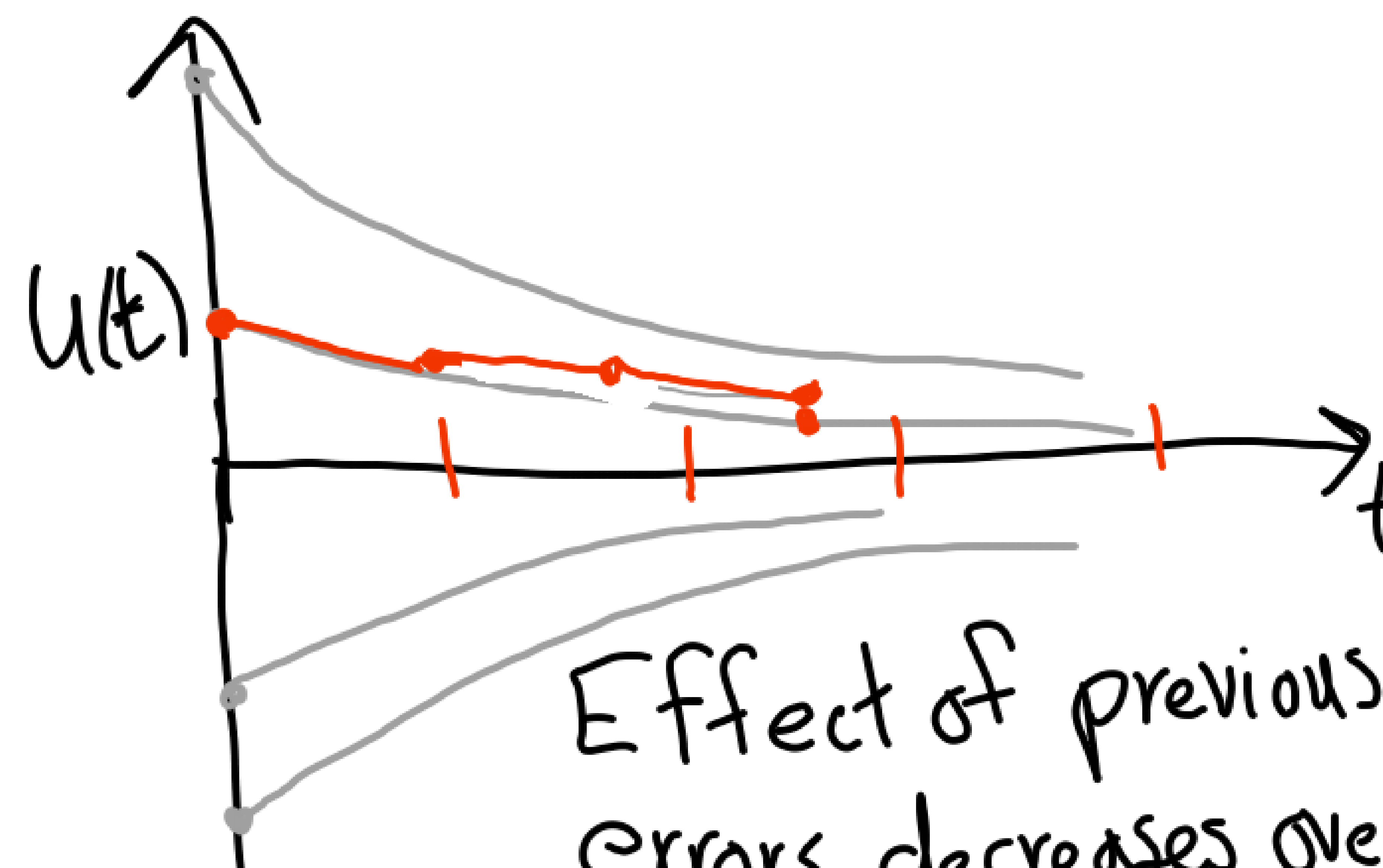
$\lambda > 0$

$$u(t) = e^{\lambda t}\eta$$



Errow grows Exponentially

$\lambda < 0$



Effect of previous errors decreases over time.