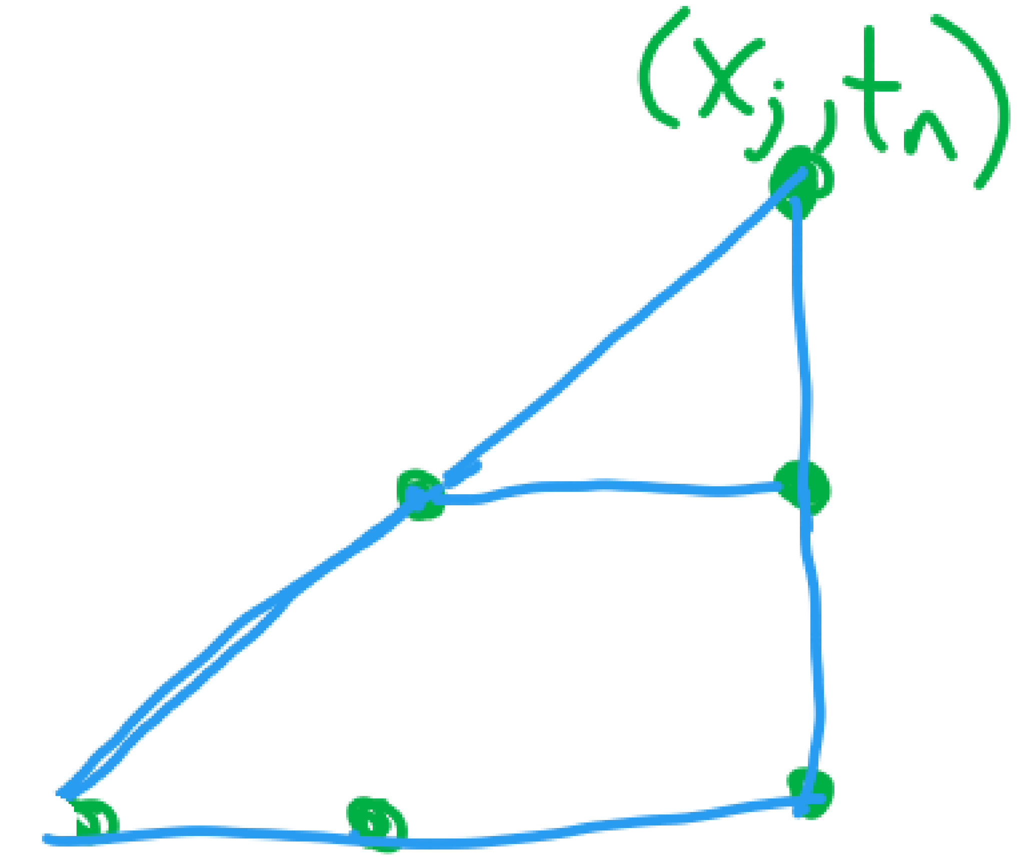
The Upwind Method

$$U_t + aU_x = 0 \qquad a > 0$$

One-sided differences:

$$\frac{U_j^{n+1} - U_j^n}{k} = -a\,\frac{U_j^n - U_{j-1}^n}{h}$$

1st-order in space and time.



The CFL condition says:

$ka \le h$ Necessary condition

$$\frac{ka}{h} \le 1$$

If $a < 0$: we can't satisfy the CFL condition.

Method of lines
Stability analysis

$0 < x < 1$

$u(x=0,t) = 0$

$U_j \approx u(jh,t)$

Semi-discretization:

$$U_j'(t) = -\frac{a}{h}\left(U_j(t) - U_{j-1}(t)\right)$$

Then use Euler in time. $U^{n+1} = (I + kL)U^n$

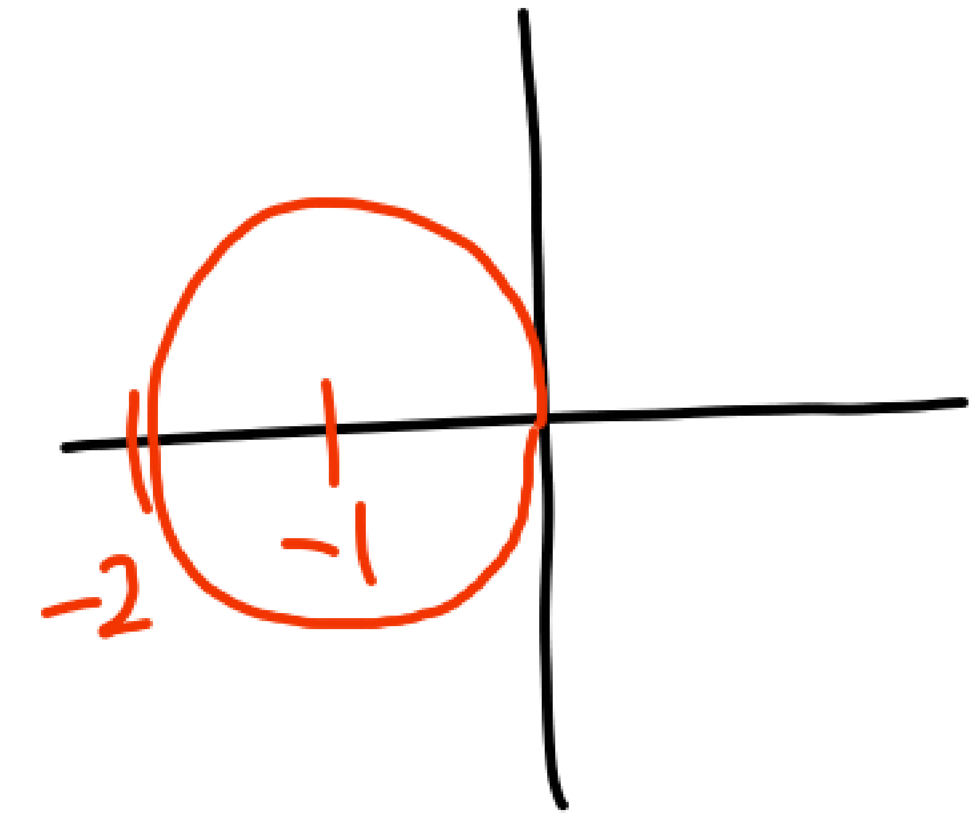
$$U'(t) = \underbrace{-\frac{a}{h}\begin{bmatrix} 1 & & & \\ -1 & 1 & & \\ & \ddots & \ddots & \\ & & -1 & 1 \end{bmatrix}}_{L}\begin{bmatrix} U_1 \\ U_2 \\ \vdots \\ U_m \end{bmatrix}$$

Doesn't blow up if $\|I + kL\| \le 1$

~~Equivalent to $\|1 + k\lambda\| \le 1$~~

Eigenvalues of L: $\lambda = -\frac{a}{h}$

We want $k\lambda \in S$



$-2 \le k\lambda \le 0$

$-2 \le -\frac{ka}{h} \le 0$

$0 \le \frac{ka}{h} \le 2$

Weaker than CFL.

Toeplitz matrices

$$\begin{bmatrix} a & d & \ddots & & \\ b & a & \ddots & \ddots & \\ c & \ddots & \ddots & \ddots & d \\ \vdots & \ddots & \ddots & \ddots & d \\ \vdots & & c & b & a \end{bmatrix}$$

Circulant Matrices

$$\begin{bmatrix} a & \ddots & & c & b \\ b & \ddots & \ddots & & c \\ c & \ddots & \ddots & \ddots & \\ & \ddots & \ddots & \ddots & \\ & & c & b & a \end{bmatrix}$$

Any FD discretization with periodic BCs yields a circulant matrix.

All circulant matrices of a given size have the same eigenvectors.

Von Neumann Stability analysis

(Technique for analyzing stability w/periodic BCs)

$$\frac{ka}{h} = \nu$$

Idea: use the ansatz

$$U_j^n = g^n e^{ijh\xi}$$

Substitute into our discretization:

$$U_j^{n+1} = U_j^n - \frac{ka}{h}\left(U_j^n - U_{j-1}^n\right)$$

$$g^{n+1} e^{ijh\xi} = g^n e^{ijh\xi} - \frac{ka}{h} g^n \left(e^{ijh\xi} - e^{i(j-1)h\xi}\right)$$

$$g = 1 - \frac{ka}{h}\left(1 - e^{-ih\xi}\right)$$

We want $|g| \le 1 + \alpha k$

So that $|g|^n \le e^{\alpha T}$ $\quad T = nk$

$$|g|^2 = \left(1 - \nu + \nu e^{-ih\xi}\right)\left(1 - \nu + \nu e^{ih\xi}\right)$$

$$= (1-\nu)^2 + \nu^2 + 2(1-\nu)\nu\cos(h\xi)$$

$$= (1-\nu)\left(1 + 2\nu\cos(h\xi)\right) + \nu^2$$

$\cos(h\xi) = +1$: $(1-\nu)(1+2\nu) + \nu^2 = 1 + \nu - \nu^2$

$\cos(h\xi) = -1$: $(1-\nu)(1-2\nu) + \nu^2 = 1 - 3\nu + 3\nu^2$

$|g|^2 \le 1 \iff 0 \le \nu \le 1$

g is the eigenvalue of our circulant matrix.

$$\underbrace{\rho(A)}_{\text{Spectral radius}} = \max_{\lambda \in \sigma(A)} |\lambda|$$

Is $\|A\|_2 = \rho(A)$?

Suppose A is diagonalizable:

$$Ar_j = \lambda_j r_j \qquad R = \begin{bmatrix} r_1 & | & \cdots & | & r_m \end{bmatrix}$$

$$AR = R\Lambda \qquad \Lambda = \operatorname{diag}(\lambda_1, \lambda_2, \ldots, \lambda_m)$$

$$A = R\Lambda R^{-1}$$

$$\|A\|_2 = \|R\Lambda R^{-1}\|_2 \le \|R\|_2 \|\Lambda\|_2 \|R^{-1}\|_2$$

$$\le \rho(A)\operatorname{cond}(R)$$

$\mathrm{Cond}(M) = \|M^{-1}\| \cdot \|M\| \geq 1$

So $\|A\|_2 \leq \rho(A)$

with equality iff R is unitary. i.e. if A has a complete set of orthogonal eigenvectors.

Matrices with this property are said to be normal.

Equivalently: $AA^* = A^*A$

For a normal matrix, if $\rho(A) \leq 1$, then $\|A^n\|_2 \leq 1$

For a non-normal matrix if $\rho(A) \leq 1$ then

$$\lim_{n\to\infty} \|A^n\| < \infty$$