Spectral Method

(for linear, constant-coeff. PDEs)

$$
\begin{tikzcd}
u(x,t=0) \arrow[d, "\text{Fourier Transform}"'] & & u(x,t=T) \\
\hat{u}(\xi,t=0) \arrow[rr, "\text{(exact) time evolution}"] & & \hat{u}(\xi,t=T) \arrow[u, "\text{inverse FT}"']
\end{tikzcd}
$$

Pseudospectral method

(for nonlinear/variable-coeff. PDEs)

Compute derivatives in Fourier space:

$$u_x \approx \mathcal{F}^{-1}\left(D[i\xi]\,\mathcal{F}(u)\right)$$

This gives a semi-discretization.

Then integrate in time with RK or multistep.

2 Challenges:

1. Aliasing instabilities
2. Time stepping
   High-order derivatives cause stiffness.

$$\frac{d^3}{dx^3} e^{i\xi x} = (i\xi)^3 e^{i\xi x} = -i\xi^3 e^{i\xi x}$$

$$u_t + \underbrace{\left(\tfrac{1}{2}u^2\right)_x}_{\text{Nonlinear}} + \underbrace{u_{xxx}}_{\text{Dispersion}} = 0$$

Korteweg-de Vries (KdV)

$\frac{2}{3}$ Filter