Homework 1 due
Thursday at midnight
Submit through Blackboard

Review of last class:

$$u''(x) = f(x) \quad 0 < x < 1$$

$$u(0) = \alpha \quad u(1) = \beta$$

$$u''(x_j) \approx \frac{U_{j+1} - 2U_j + U_{j-1}}{h^2}$$

$$AU = F$$

Substitute $u(x_j)$ for $U_j$:

$$A\hat{U} = F + \tau \qquad E = U - \hat{U}$$

$$AE = -\tau \qquad \text{Consistency: } \lim_{h \to 0} \|\tau\| = 0$$

$$\|E\|_2 \le \|A^{-1}\|_2 \|\tau\|_2$$

Eigenvalues of $h^2 A$: $\hat{\lambda}_p = 2\left(\cos\left(\frac{p\pi}{m+1}\right) - 1\right)$, $p = 1, 2, \ldots, m$

Eigenvalues of $A$: $\lambda_p = \frac{2}{h^2}\left(\cos(p\pi h) - 1\right)$

Smallest $|\lambda_p|$: $p = 1$ $\quad \lambda_1 = \frac{2}{h^2}\left(\cos(\pi h) - 1\right)$

$$\cos(x) = 1 - \frac{x^2}{2} + O(x^4)$$

So $\lambda_1 = \frac{2}{h^2}\left(1 - \frac{\pi^2 h^2}{2} + O(h^4) - 1\right)$

$$\lambda_1 = -\pi^2 + O(h^2) \qquad \|A^{-1}\| \le C$$

$$\left.\|A^{-1}\|_2 = \frac{1}{|\pi^2 + O(h^2)|} \approx \frac{1}{\pi^2}\right\} \text{Stability.}$$

$$\tau_j = \frac{h^2}{12}u''''(x_j) + O(h^4)$$

$$\|E\|_2 \lesssim \frac{1}{\pi^2}\|\tau\|_2$$

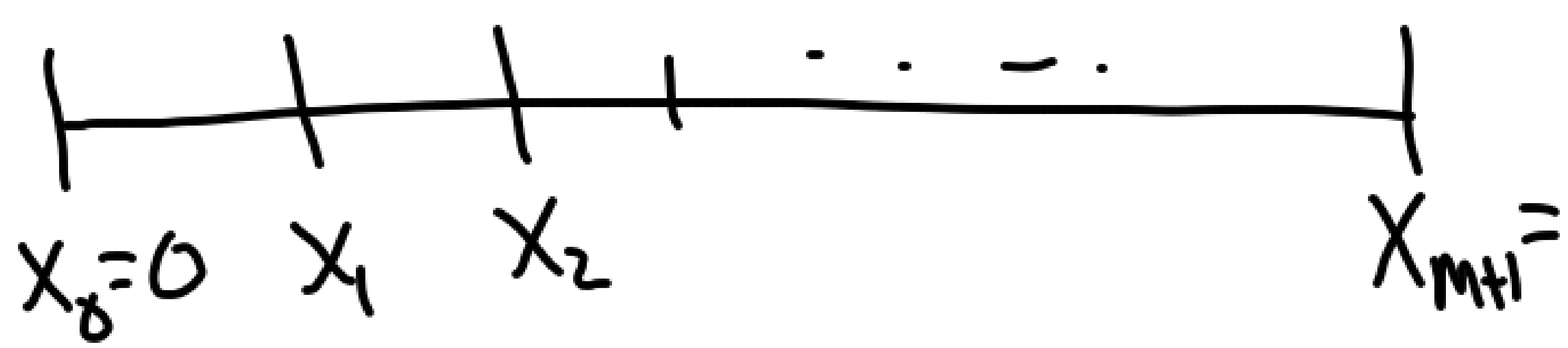
$$\|E\|_2 \lesssim \frac{h^2}{12\pi^2}\|f''\|_2$$

$$\lim_{h\to 0}\|E\|_2 = 0 \quad \text{Convergence}$$

Consistency + Stability $\Rightarrow$ convergence.

Today: Stability in the max norm.

We want to show that $\|A^{-1}\|_\infty \le C$ so that $\lim_{h\to 0}\|E\|_\infty = 0$.



$$\underbrace{\frac{1}{h^2}\begin{bmatrix} h^2 & & & & & \\ 1 & -2 & 1 & & & \\ & \ddots & \ddots & \ddots & & \\ & & \ddots & \ddots & \ddots & \\ & & & 1 & -2 & 1 \\ & & & & & h^2 \end{bmatrix}}_{A^{(m+2)\times(m+2)}} \underbrace{\begin{bmatrix} U_0 \\ U_1 \\ \vdots \\ U_m \\ U_{m+1} \end{bmatrix}}_{U} = \underbrace{\begin{bmatrix} \alpha \\ f(x_1) \\ \vdots \\ f(x_m) \\ \beta \end{bmatrix}}_{F}$$

$U_0 = \alpha$

$U_{m+1} = \beta$

Let $B = A^{-1}$.

$$B = \begin{bmatrix} B_0 \,\Big|\, B_1 \,\Big|\, \cdots \,\Big|\, B_m \,\Big|\, B_{m+1} \end{bmatrix}$$

$U = BF$

This means that $U$ is a linear combination of columns of $B$:

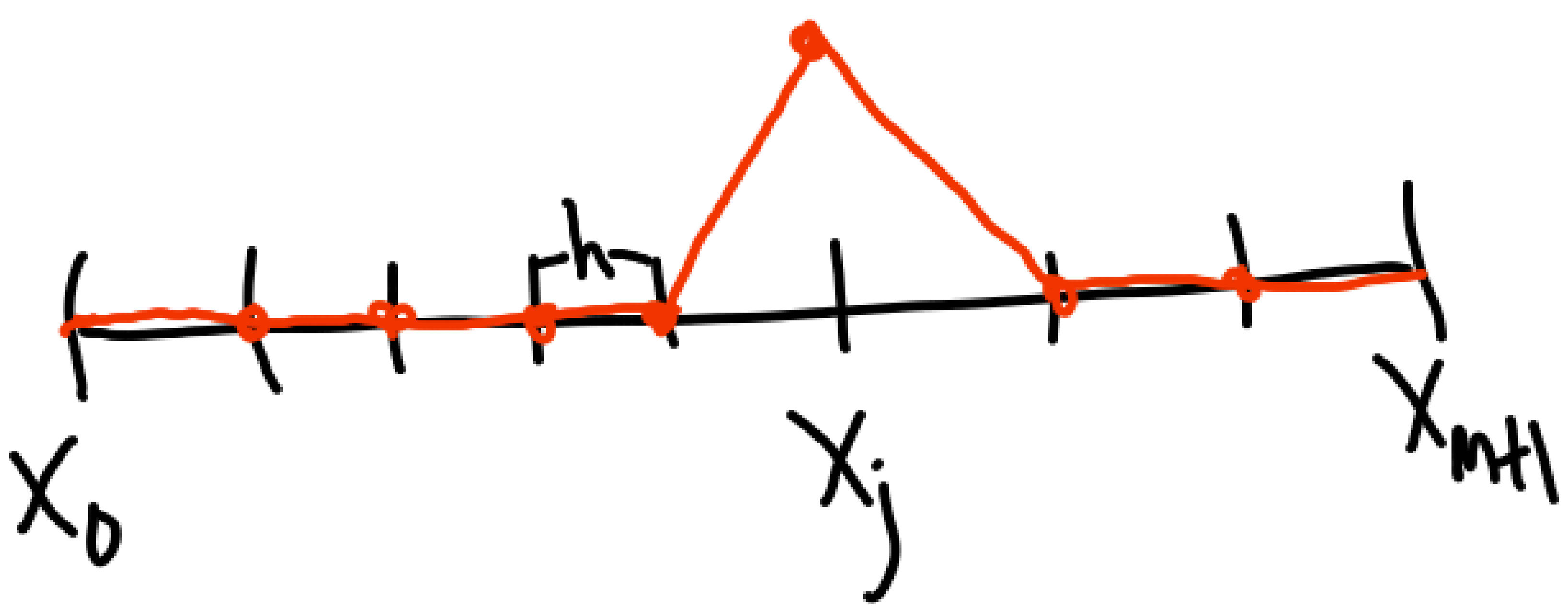
$$U = \sum_j F_j B_j$$

Suppose

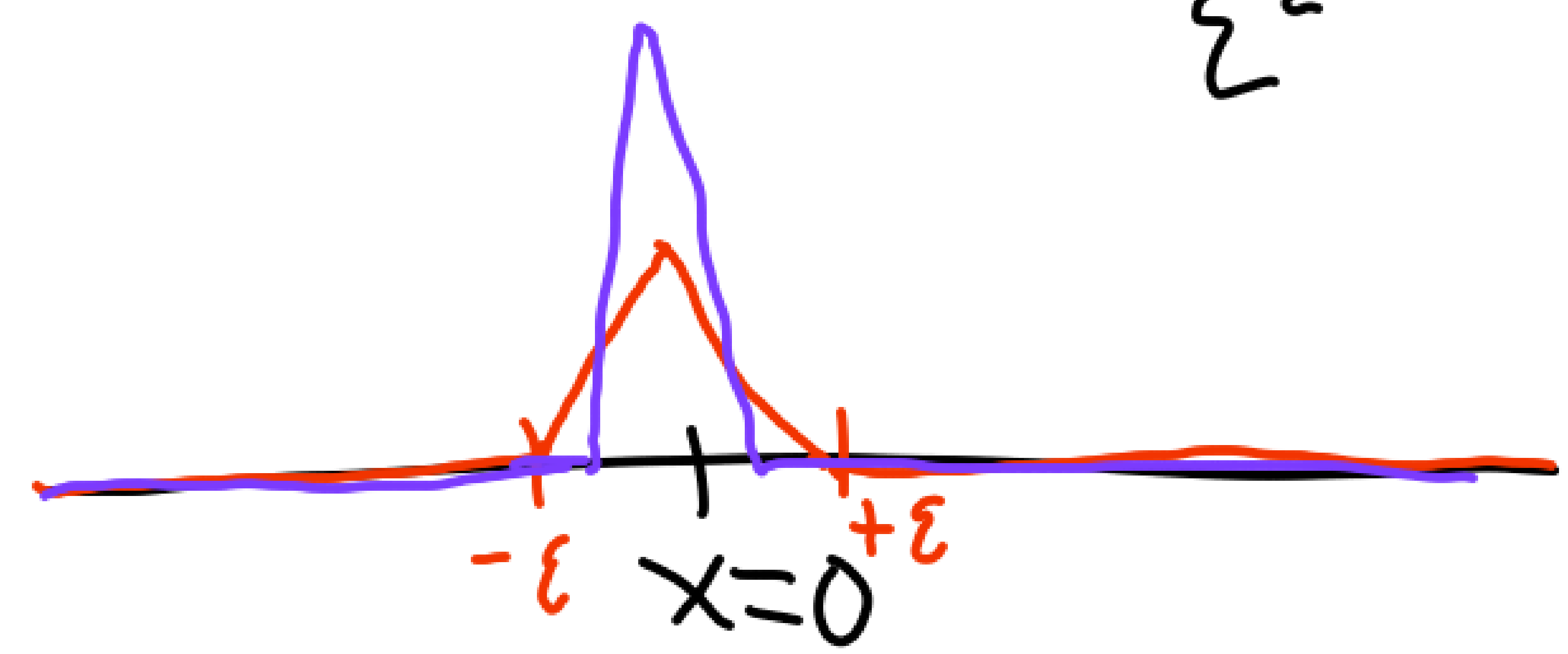
$$F = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \leftarrow j$$

Then $U = B_j$

So $B_j$ is the solution when $\alpha = \beta = 0$ and $f(x_i) = \begin{cases} 0 & i \neq j \\ 1 & i = j \end{cases}$




Consider $\phi_\varepsilon(x) \begin{cases} \dfrac{\varepsilon + x}{\varepsilon^2} & -\varepsilon \le x \le 0 \\ \dfrac{\varepsilon - x}{\varepsilon^2} & 0 \le x \le \varepsilon \end{cases}$




As $\varepsilon \to 0$ this becomes the Dirac delta function $\delta(x)$.

What is $\int_{-\infty}^{\infty} \phi_\varepsilon(x)\,dx$?

$$\text{Area} = \frac{1}{2} b \times h = \frac{1}{2}(2\varepsilon)\frac{1}{\varepsilon} = 1.$$

What is the solution of

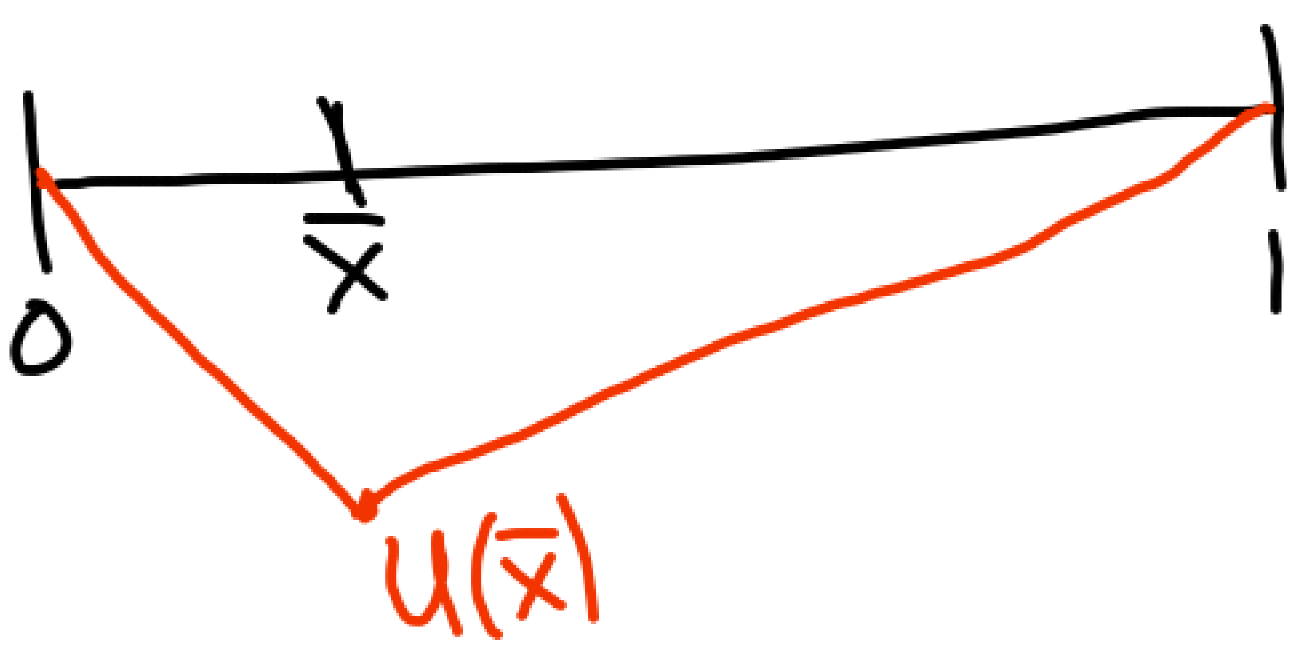
$$u''(x) = \delta(x-\bar{x})$$

$$u(0) = u(1) = 0 \quad ?$$

Physically, we have a singular heat sink at $\bar{x}$.

Away from $\bar{x}$: $u''(x) = 0$

So $u(x)$ is linear.




$$u'(\bar{x}+\varepsilon) - u'(\bar{x}-\varepsilon) = \int_{\bar{x}-\varepsilon}^{\bar{x}+\varepsilon} u''(x)\,dx = \int_{\bar{x}-\varepsilon}^{\bar{x}+\varepsilon} \delta(x-\bar{x})\,dx = 1$$

$$u(x) = \begin{cases} u_1(x) & x < \bar{x} \\ u_2(x) & x > \bar{x} \end{cases}$$

$$u_1(x) = ax$$

$$u_2(x) = b(x-1)$$

$$u'(\bar{x}+\varepsilon) = b \qquad b - a = 1$$

$$u'(\bar{x}-\varepsilon) = a \qquad b = 1 + a$$

$$u_2 = (1+a)(x-1)$$

Continuity: $u_1(\bar{x}) = u_2(\bar{x})$

$$\cancel{a\bar{x}} = (1+a)(\bar{x}-1) = \bar{x} + \cancel{a\bar{x}} - a - 1$$

$$a = \bar{x} - 1$$

$$b = \bar{x}$$

$$G(x;\bar{x}) = \begin{cases} (\bar{x}-1)x & x<\bar{x} \\ \bar{x}(x-1) & x>\bar{x} \end{cases}$$

This is a Green's function

A Green's function gives the solution of a D.E. when the inhomogeneous term is a delta function.

## Superposition

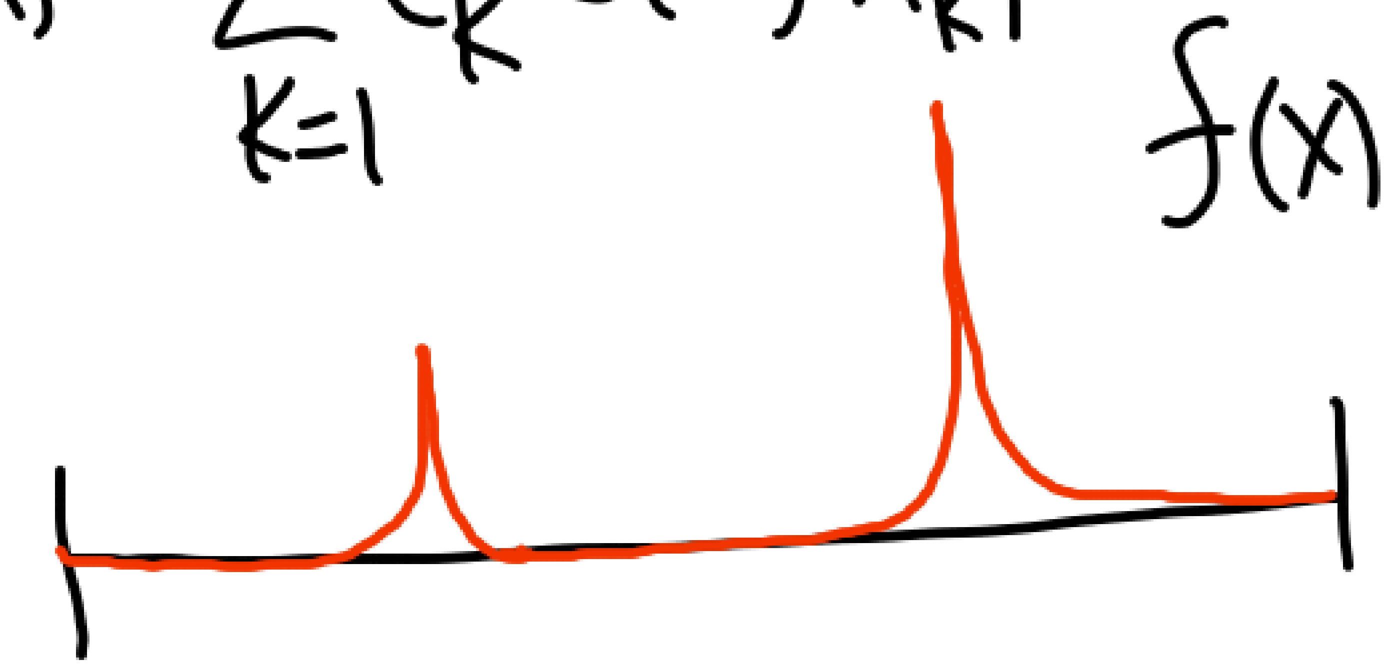
Suppose $u''(x) = \sum_{k=1}^{n} c_k \delta(x-x_k)$

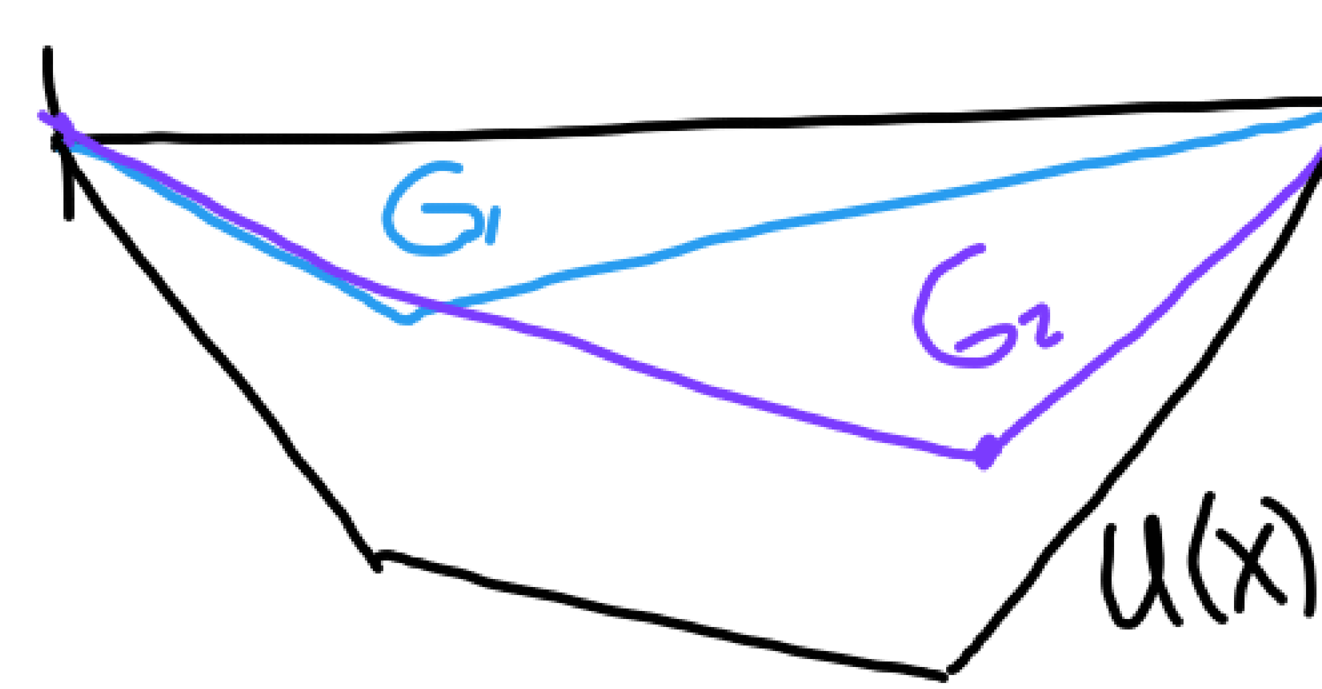
$\alpha=\beta=0$

Then

$$u(x) = \sum_{k=1}^{n} c_k G(x;x_k)$$

Example:




For any $f(x)$, we have

$$f(x) = \int_0^1 f(\bar{x})\,\delta(x-\bar{x})\,d\bar{x}$$

So the solution

$$U''(x) = f(x)$$
$$U(0) = u(1) = 0$$
$$U(x) = \int_0^1 f(\bar{x}) G(x;\bar{x})\, d\bar{x}.$$

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$$U''(x) = 0$$
$$U(0) = 1 \quad u(1) = 0$$
$$U(x) = 1 - x = G_0(x)$$

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$$U''(x) = 0$$
$$U(0) = 0 \quad u(1) = 1$$
$$U(x) = x = G_1(x)$$

$$U''(x) = f(x)$$
$$U(0) = \alpha \quad u(1) = B$$
$$U(x) = \int_0^1 f(\bar{x}) G(x;\bar{x})\, d\bar{x} + \alpha G_0(x) + B G_1(x).$$

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It can be shown that

$$(B_j)_i = B_{ij} = h\, G(x_i; x_j) \qquad 1 \le j \le m, \quad 0 \le i \le m+1$$

$$(B_0)_i = G_0(x_i)$$
$$(B_{m+1})_i = G_1(x_i)$$

So our numerical solution is the exact solution of

$$U''(x) = h\sum_{j=1}^{m} \delta(x-x_j)f(x_j)$$

$U = BF$ means

$$U_i = \alpha G_0(x_i) + \beta G_1(x_i) + h\sum_{j=1}^{m} f(x_j)G(x_i;x_j)$$

What is $\|A^{-1}\|_\infty = \|B\|_\infty$?

$\|B\|_\infty$ is the maximum abs. row sum:

$$\|B\|_\infty = \max_{0\le i\le m+1} \sum_{j=0}^{m+1} |B_{ij}|$$

$B_{i0} = G_0(x_i) = 1 - x_i \Rightarrow \max_i |B_{i0}| \le 1$

$B_{i,m+1} = G_1(x_i) = x_i \Rightarrow \max_i |B_{i,m+1}| \le 1$

$B_{ij} = hG(x_i;x_j) = hx_i(1-x_j)$

So $\max_i |B_{ij}| \le h$

So $\|B\|_\infty \le 1 + 1 + mh$

$$\le 1 + 1 + \frac{m}{m+1} < 3$$

$\|A^{-1}\|_\infty < 3$

So $\|E\|_\infty \le \|A^{-1}\|_\infty \cdot \|\tau\|_\infty$

$$< 3 \left\| \frac{h^2}{12} u''''(x) + O(h^4) \right\|_\infty$$

$$\|E\|_\infty \lesssim 3 \frac{h^2}{12} \|f''(x)\|_\infty$$

$\lim_{h \to 0} \|E\|_\infty = 0$ Converges at second-order rate (quadratically)