Poisson's Equation (2D)

$$\nabla^2 u(x,y) = f(x,y) \quad x \in \Omega$$

$$u(x,y) = g(x,y) \quad x \in \partial\Omega$$

Applications:

u	f
Temperature	Heat source
Electrical potential	Charge distribution
Pressure potential	Source/sink of fluid
Gravitational potential	Mass distribution

More generally we could have

$$\nabla \cdot (K(x,y)\nabla u) = f(x,y)$$

Heat diffusion in a square plate

$$\nabla^2 u = f(x,y) \quad \begin{matrix} 0<x<1 \\ 0<y<1 \end{matrix}$$

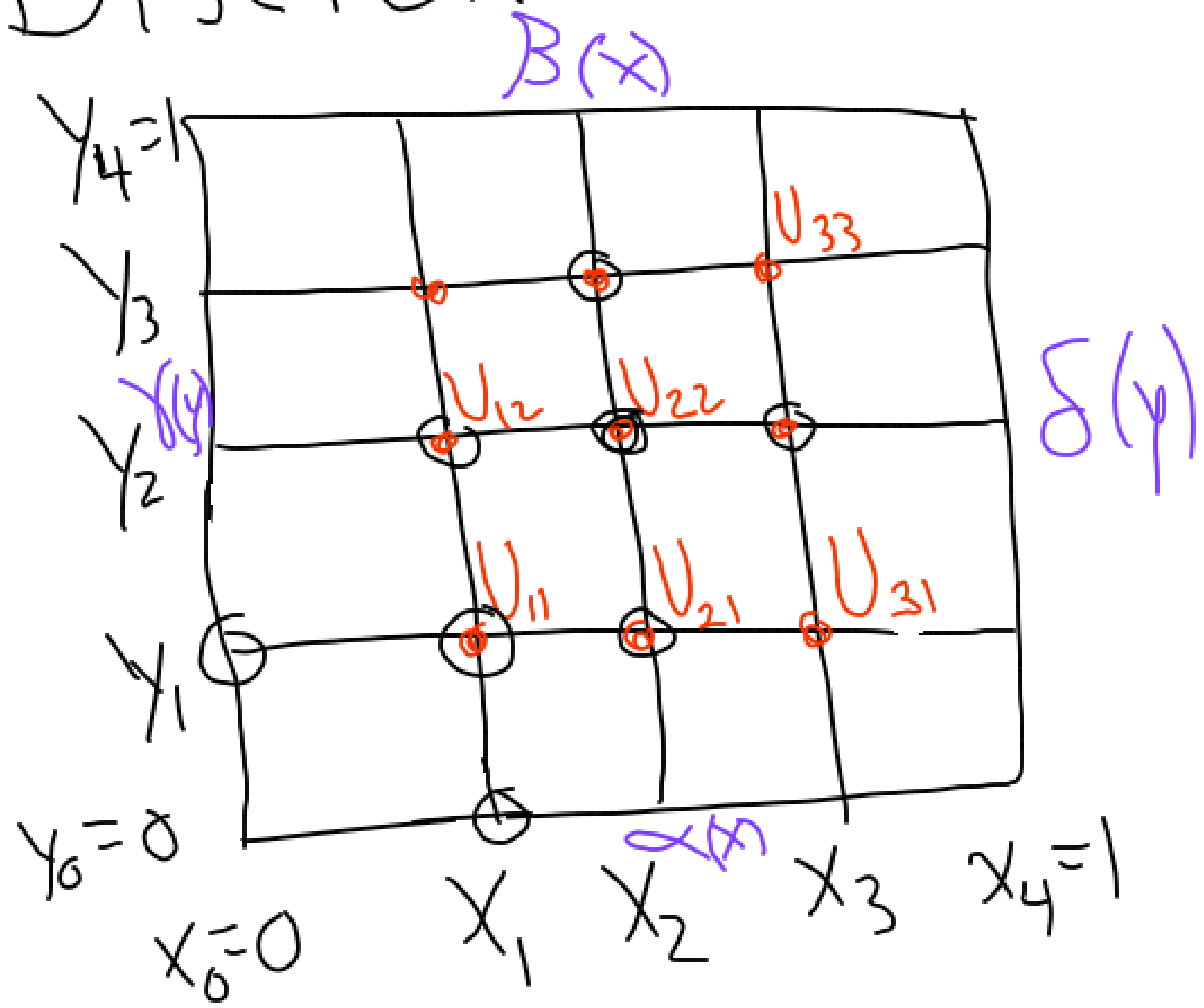
$$u(x,0) = \alpha(x)$$

$$u(x,1) = \beta(x)$$

$$u(0,y) = \gamma(y)$$

$$u(1,y) = \delta(y)$$

Discretize:



$$U_{ij} \approx u(x_i, y_j) \qquad 1 \le i,j \le m$$

$$u_{xx}(x_i, y_j) \approx \frac{U_{i+1,j} - 2U_{ij} + U_{i-1,j}}{(\Delta x)^2}$$

$$u_{yy}(x_i, y_j) = \frac{U_{i,j+1} - 2U_{ij} + U_{i,j-1}}{(\Delta y)^2}$$

Assume $\Delta x = \Delta y = h$ $\qquad AU = F$

$$\frac{1}{h^2}\left[U_{i+1,j} + U_{i,j+1} + U_{i-1,j} + U_{i,j-1} - 4U_{ij}\right] = f(x_i, y_j)$$

5-point stencil

Row-wise ordering:

$$U = \begin{bmatrix} U_{11} \\ U_{21} \\ U_{31} \\ U_{12} \\ U_{22} \\ U_{32} \\ U_{13} \\ U_{23} \\ U_{33} \end{bmatrix} \qquad A = \frac{1}{h^2}\begin{bmatrix} -4 & 1 & 0 & 1 & & & & & \\ 1 & -4 & 1 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & -4 & 0 & 0 & 1 & & & \\ 1 & 0 & 0 & -4 & 1 & 0 & 1 & & \\ 0 & 1 & 0 & 1 & -4 & 1 & 0 & 1 & \\ & & & & & & 0 & 1 & \\ & & & \ddots & & & & \ddots & \\ & & & & & & & 1 & -4 \end{bmatrix}$$

A is sparse

$A \in \mathbb{R}^{m^2 \times m^2}$

$U \in \mathbb{R}^{m^2}$

Only $\sim 5m^2$ entries of A are non-zero

$$F = \begin{bmatrix} f(x_1, y_1) \\ f(x_2, y_1) \\ \vdots \\ f(x_m, y_m) \end{bmatrix}$$

Consistency

Substute: $U_{ij} \leftarrow u(x_i, y_j)$

$$\frac{1}{h^2}\Big[u(x_{i+1}, y_j) + u(x_{i-1}, y_j) + u(x_i, y_{j+1}) + u(x_i, y_{j-1}) - 4u(x_i, y_j)\Big] = f(x_i, y_j) + \tau_{ij}$$

local trunc. error

$$\tau_{ij} = \frac{h^2}{12}\big(u_{xxxx}(x_i, y_j) + u_{yyyy}(x_i, y_j)\big) + O(h^4)$$

$$\hat{U} = \begin{bmatrix} u(x_1, y_1) \\ u(x_2, y_1) \\ \vdots \\ u(x_m, y_m) \end{bmatrix}$$

$$AU = F$$

$$A\hat{U} = F + \tau$$

$$A(\underbrace{U - \hat{U}}_{E}) = -\tau$$

$$AE = -\tau \Rightarrow E = -A^{-1}\tau$$

$$\|E\| \leq \|A^{-1}\| \cdot \|\tau\|$$

We know $\|\tau\| = O(h^2)$

We will show that $\overbrace{\|A^{-1}\| < C}^{\text{stability}}$

so that

$$\lim_{h\to 0} \|E\| = 0 \quad \text{(convergence)}$$

Take $\|\cdot\| = \|\cdot\|_2$. What is $\|A^{-1}\|_2$?

Let's find the eigenvalues of A.

$$AV = \lambda V$$

$$(AV)_{ij} = \frac{V_{i+1,j} - 2V_{ij} + V_{i-1,j}}{(\Delta x)^2} + \frac{V_{i,j+1} - 2V_{ij} + V_{i,j-1}}{(\Delta y)^2} = \lambda V_{ij}$$

Assume $V_{ij} = R_i S_j$ (separable)

Then $\frac{S_j}{(\Delta x)^2}\left(R_{i+1} - 2R_i + R_{i-1}\right) + \frac{R_i}{(\Delta y)^2}\left(S_{j+1} - 2S_j + S_{j-1}\right) = \lambda R_i S_j$

Divide by $R_i S_j$:

$$\underbrace{\frac{1}{R_i(\Delta x)^2}\left(R_{i+1} - 2R_i + R_{i-1}\right)}_{\substack{\text{only depends on } i \\ = C_1}} + \underbrace{\frac{1}{S_j(\Delta y)^2}\left(S_{j+1} - 2S_j + S_{j-1}\right)}_{\substack{\text{only depends on } j \\ = C_2}} = \lambda$$

$$\lambda = C_1 + C_2$$

$$R_{i+1} - \left(2 + C_1(\Delta x)^2\right) R_i + R_{i-1} = 0$$

System of linear difference equations

$$R_0 = 0 \qquad R_{m+1} = 0$$

$$R_i = \rho^i \quad \text{(Ansatz)}$$

$$\rho^{i+1} - \left(2 + C_1(\Delta x)^2\right)\rho^i + \rho^{i-1} = 0$$

Divide by ρ^{i-1}

$$\alpha = \frac{1}{2}\left(2 + C_1(\Delta x)^2\right)$$

$$\rho^2 - 2\alpha\rho + 1 = 0$$

$$\rho_\pm = \alpha \pm \sqrt{\alpha^2 - 1}$$

General Solution: $R_i = R_+ \rho_+^i + R_- \rho_-^i$

$R_0 = 0$: $R_+ + R_- = 0$

$$R_- = -R_+$$

$$R_i = R_+\left(\rho_+^i - \rho_-^i\right)$$

$i = m+1$:

$$\left(\rho_+^{m+1} - \rho_-^{m+1}\right) = 0$$

$$\rho_+^{m+1} = \rho_-^{m+1}$$

$$\rho_+\rho_- = \left(\alpha + \sqrt{\alpha^2-1}\right)\left(\alpha - \sqrt{\alpha^2-1}\right)$$

$$= \alpha^2 - (\alpha^2 - 1) = 1$$

$$\rho_+^{2m+2} = \rho_-^{m+1}\rho_+^{m+1} = 1$$

So ρ_+ is a $(2m+2)$-root of unity:

$$\rho_+ = e^{i2\pi\frac{p}{2m+2}} \qquad p=1,2,\dots,m$$

$$\rho_+ = e^{i\pi\frac{p}{m+1}} = e^{i\pi p\Delta x}$$

$$\rho_- = e^{-i\pi p\Delta x}$$

$$\alpha = \frac{1}{2}(\rho_+ + \rho_-) = \cos(\pi p\Delta x)$$

$$C_1 = \frac{2(\alpha-1)}{(\Delta x)^2} = \frac{2}{(\Delta x)^2}(\cos(p\pi\Delta x)-1)$$

$$p=1,2,\dots,m_x$$

Similarly

$$C_2 = \frac{2}{(\Delta y)^2}(\cos(q\pi\Delta y)-1)$$

$$q=1,2,\dots,m_y$$

$$\boxed{\lambda_{p,q} = 2\left[\frac{\cos(p\pi\Delta x)-1}{(\Delta x)^2} + \frac{\cos(q\pi\Delta y)-1}{(\Delta y)^2}\right]}$$

We care about the smallest (in magnitude)

Take $p=q=1$

$$\cos(x) = 1 - \frac{x^2}{2} + \mathcal{O}(x^4)$$

$$\lambda_{1,1} = 2\left[\frac{1-\frac{(\pi\Delta x)^2}{2}-1}{(\Delta x)^2} + \frac{1-\frac{(\pi\Delta y)^2}{2}-1}{(\Delta y)^2}\right] + \mathcal{O}(\Delta x^2) + \mathcal{O}(\Delta y^2)$$

$$\lambda_{1,1} = 2\left[-\frac{\pi^2}{2} - \frac{\pi^2}{2}\right] + \mathcal{O}(\Delta x^2, \Delta y^2)$$

$$\lim_{\Delta x, \Delta y \to 0} \lambda_{1,1} = -2\pi^2$$

So $\|A^{-1}\| \approx \frac{-1}{2\pi^2}$ as $\Delta x, \Delta y \to 0$.