Reminders:

- HW 4 due today
- Doodle poll on class time during Ramadan
- Project proposals

Convergence and stability for IBVPs

Last time:

$$u_t = u_{xx}$$

discretized in space:

$$U'(t) = A_h U(t)$$

Where $A_h = \frac{1}{h^2}\text{tridiag}(1,-2,1)$

Then discretize in time: (using a 1-step method)

$$U^{n+1} = R(kA_h)U^n$$

Where $R(z)$ is the stability function of the method.

For example:

Explicit Euler: $R(z) = 1+z$

$$\Rightarrow U^{n+1} = (I + kA_h)U^n$$

Implicit trapezoidal Method:

$$R(z) = \left(1-\frac{z}{2}\right)^{-1}\left(1+\frac{z}{2}\right)$$

$$U^{n+1} = \left(I - \tfrac{1}{2}kA_h\right)^{-1}\left(I + \tfrac{1}{2}kA_h\right)U^n$$

Let $B_{k,h} = R(kA_h)$

So $$U^{n+1} = B_{k,h} U^n \quad (1)$$

Let $$u^n = \begin{bmatrix} u(x_1, t_n) \\ u(x_2, t_n) \\ \vdots \\ u(x_m, t_n) \end{bmatrix}$$

Then $$u^{n+1} = B_{k,h} u^n + k\tau^n \quad (2)$$

Let $E^n = U^n - u^n$

Subtract (2) from (1):

$$E^{n+1} = B_{k,h} E^n - k\tau^n$$

$$E^n = B_{k,h} E^{n-1} - k\tau^{n-1}$$

$$\Rightarrow E^{n+1} = (B_{k,h})^2 E^{n-1} - k B_{k,h} \tau^{n-1} - k\tau^n$$

$$E^N = (B_{k,h})^N E^0 - k \sum_{j=0}^{N-1} B_{k,h}^{N-1-j} \tau^j,$$

$T = kN$

We want to show that

$$\lim_{\substack{k,h\to 0 \\ N=\frac{T}{k}}} \|E^N\| = 0$$ Convergence

We say the discretization is consistent if

$$\underbrace{\lim_{h\to 0} \|E^0\| = 0}_{\text{Usually } E^0=0} \quad \text{and} \quad \lim_{k,h\to 0} \|\tau^i\| = 0$$

We say the method is Lax-Richtmeyer stable if

$$\|(B_{k,h})^n\| < C(T) \quad \forall n$$

where C is independent of k and h.

Assuming consistency and stability:

$$E^N = B^N E^0 + k\sum_j B^{N-1-j}\tau^j$$

So $\|E^N\| \le \|B^N\|\|E^0\| + k\sum_j \|B^{N-1-j}\|\cdot\|\tau^j\|$

$$\le C(T)\|E^0\| + \underbrace{kN}_{T} C(T)\max_j \|\tau^j\|$$

So $\lim\limits_{k,h\to 0} \|E^N\| = 0.$

Examples:

Explicit Euler

$$B_{k,h} = I + kA_h$$

We want $\|B_{k,h}^n\|_2 < C(T)$

A sufficient condition is that all eigenvalues of B have modulus ≤ 1.

$\mu_p = 1 + k\lambda_p$ where $-\frac{4}{h^2} \le \lambda_p \le 0$

so we need

$$-1-\alpha k \le 1 + k\lambda_p \le 1 + \alpha k$$

$$-\frac{2}{k} \le \lambda_p \le 0$$

$$\Rightarrow \quad -\frac{2}{k} \le \frac{-4}{h^2}$$

$$\frac{1}{k} \ge \frac{2}{h^2}$$

$$k \le \frac{h^2}{2}$$

(absolute stability)

Trapezoidal method: $R(z) = \dfrac{1+z/2}{1-z/2}$

$$B_{k,h} = \left(I - \tfrac{1}{2}kA_h\right)^{-1}\left(I + \tfrac{1}{2}kA_h\right)$$

We want $\|B^n_{k,h}\|_2 \le C(T)$

A sufficient condition is

$$|\mu_p| = \left| \frac{1+\frac{k\lambda_p}{2}}{1-\frac{k\lambda_p}{2}} \right| \le 1$$

Always holds since $k\lambda_p < 0$

No restriction on how $k, h \to 0$.

$$-\frac{2}{k} - \alpha \le \lambda_p \le \alpha$$

$$-\frac{2}{k} - \alpha \le \frac{-4}{h^2} \Rightarrow \frac{1}{k} \ge \frac{2}{h^2} - \frac{\alpha}{2} \Rightarrow k \le \frac{1}{\frac{2}{h^2} - \frac{\alpha}{2}} = \frac{h^2}{2}\left(\frac{1}{1 - \frac{h^2\alpha}{4}}\right)$$

with α fixed, as $h \to 0$ this gives the same result.

A slightly weaker condition can be imposed to prove L-R stability:

$$\|B_{k,h}\| \leq 1 + \alpha k$$

Then you can show that:

$$\|B_{k,h}^N\| \leq (1+\alpha k)^N \leq e^{\alpha T}$$