$$u_t = K u_{xx} + \psi(x)$$ Heat equation

$\downarrow$ steady state

$$u''(x) = f(x)$$ Poisson Equation

$u(0)=\alpha \quad u(1)=\beta$

$$f(x) \longrightarrow \begin{bmatrix} \alpha \\ f(x_1) \\ \vdots \\ f(x_m) \\ \beta \end{bmatrix}$$

Discretize:

$0<x<1 \longrightarrow x_j = jh \quad j=0,1,\dots,m+1$

$u(x) \longrightarrow U = [U_0, U_1, \dots, U_{m+1}]$

$$\frac{d^2}{dx^2} \longrightarrow \frac{1}{h^2}\begin{bmatrix} h^2 & & & & \\ 1 & -2 & 1 & & \\ & \ddots & \ddots & \ddots & \\ & & 1 & -2 & 1 \\ & & & & h^2 \end{bmatrix}$$

$$u''(x) = f(x)$$

$$\downarrow$$

$$AU = F$$

Neumann BCs

$$u''(x) = f(x) \quad 0 < x < 1$$

$$\underbrace{u'(0) = 0}_{\text{no flux through left end of rod (insulated)}} \quad u(1) = \beta$$

More generally we could have $u'(0) = \sigma \quad u(1) = \beta$

How to discretize this?

Method 1:

One-sided FD

$$u'(0) \approx D_+ u(0) = \boxed{\frac{U_1 - U_0}{h} = \sigma}$$

$$D_+ u(\bar{x}) = u'(\bar{x}) + \underbrace{\frac{h}{2} u''(\bar{x})}_{\text{LTE}} + \mathcal{O}(h^2)$$

This 1st-order accurate.

How to improve it?

We Know $u''(0) = f(0)$

So we use

$$\frac{U_1 - U_0}{h} - \frac{h}{2} f(0) = \sigma$$

$$\frac{1}{h^2}\begin{bmatrix} -\frac{3h}{2} & 2h & -\frac{h}{2} & & & \\ 1 & -2 & 1 & & & \\ & \ddots & \ddots & \ddots & & \\ & & & 1 & -2 & 1 \\ & & & & & h^2 \end{bmatrix}\begin{bmatrix} U_0 \\ U_1 \\ \vdots \\ U_m \\ U_{m+1} \end{bmatrix} = \begin{bmatrix} \sigma \\ f_1 \\ \vdots \\ f_m \\ \beta \end{bmatrix}$$

Alternatively, use a 3-point FD formula:

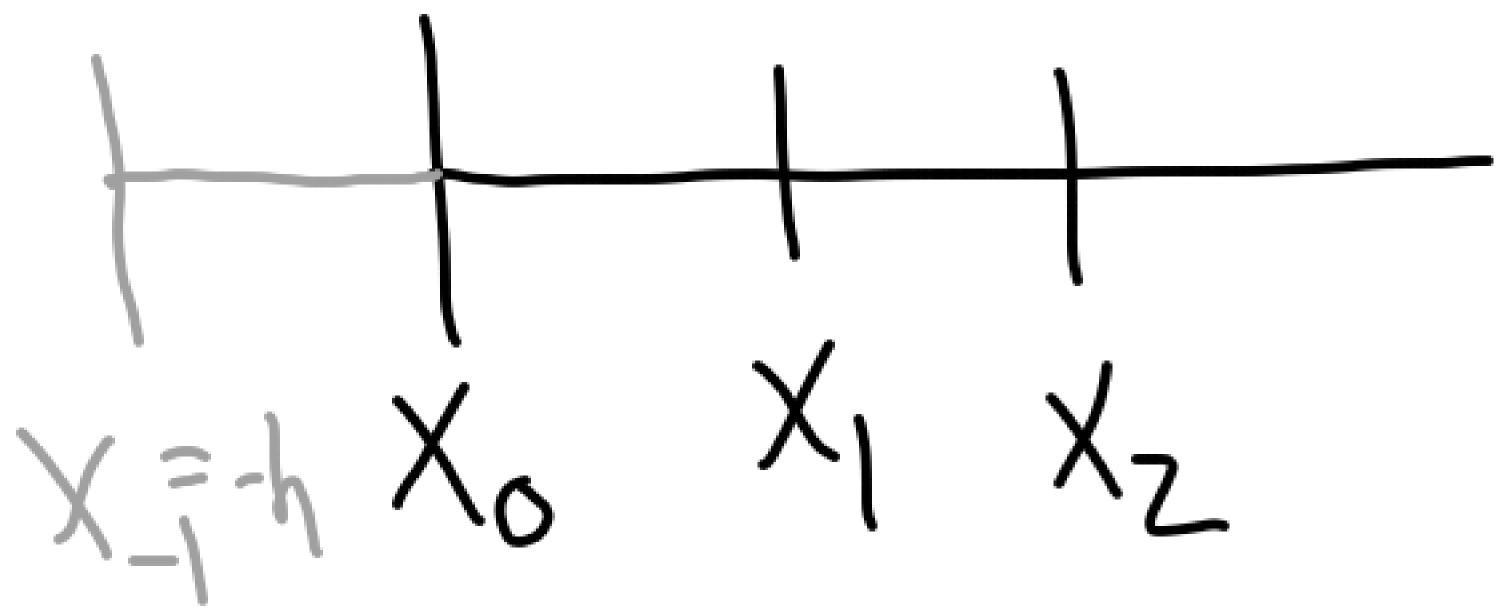
$$u'(0) \approx \frac{-\frac{3}{2}U_0 + 2U_1 - \frac{1}{2}U_2}{h} = \sigma$$

Both are 2nd-order accurate.

What if we have $u(0) = \alpha \quad u'(1) = \sigma$?

(Same approaches)

Method 2: Ghost point method



We can use a CD:

$$\frac{U_1 - U_{-1}}{2h} = \sigma$$

To solve for U_{-1}, we apply $u''(x) = f(x)$ at $x_0 = 0$

$$\frac{U_1 - 2U_0 + U_{-1}}{h^2} = f(0)$$

$$U_{-1} = h^2 f(0) - U_1 + 2U_0$$

Substitute:

$$\frac{U_1 - h^2 f(0) + U_1 - 2U_0}{2h} = \sigma$$

$$\frac{2U_1 - 2U_0}{2h} - \frac{h}{2} f(0) = \sigma$$

$$\frac{U_1 - U_0}{h} - \frac{h}{2} f(0) = \sigma$$

Same formula as before.

What if we have

$$U''(x) = f(x)$$

$$U'(0) = 0 \qquad U'(1) = 0$$

Integrate:

$$\int_0^1 U''(x)\,dx = \underbrace{U'(1) - U'(0) = \int_0^1 f(x)\,dx}_{\text{Necessary condition to have a solution.}}$$

If $f(x) = 0$, what is the solution? $U(x) = C$

(infinitely many solutions)

If $f(x) \neq 0$ but $\int_0^1 f(x)\,dx = 0$ (example: $f(x) = \sin(2\pi x)$)

then we'll have infinitely many solutions.

$$U''(x) = f(x)$$

$$U'(0) = \sigma_0 \qquad U'(1) = \sigma_1$$

$$\frac{U_1 - U_0}{h} = \sigma_0 \qquad \frac{U_{m+1} - U_m}{h} = \sigma_1$$

$$\underbrace{\frac{1}{h^2}\begin{bmatrix} -h & h & & & & \\ 1 & -2 & 1 & & & \\ & \ddots & \ddots & \ddots & & \\ & & \ddots & \ddots & \ddots & \\ & & & 1 & -2 & 1 \\ & & & & -h & h \end{bmatrix}}_{A} \begin{bmatrix} U_0 \\ U_1 \\ \vdots \\ U_m \\ U_{m+1} \end{bmatrix} = \begin{bmatrix} \sigma_0 \\ f(x_1) \\ \vdots \\ f(x_m) \\ \sigma_1 \end{bmatrix}$$

Is A non-singular? No

$$A \underbrace{\begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix}}_{V} = \begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix}$$

If $AU = F$

then $A(U + cV) = F$

So either we have no solution, or infinitely many.