Time stepping for PDEs of mixed type

$$u_t = \underbrace{f(u)}_{\text{Non-stiff (nonlinear)}} + \underbrace{L(u)}_{\text{Stiff (Linear)}}$$

Examples:

- Burgers: $u_t = -uu_x + u_{xx}$
- KdV: $u_t = -uu_x - u_{xxx}$
- Navier-Stokes

Lie-Trotter Splitting

① Solve $u_t = f(u)$ over one step

② Solve $u_t = L(u)$ over one step

Repeat

For part ①: use any standard (explicit) method

For part ②: Use exact solution or implicit method

Advantage:
Avoid nonlinear algebraic solves and take large time steps

Disadvantage:
splitting error

To analyze the error, consider:

$$u_t = Au + Bu = (A+B)u$$

Exact solution: $u(t+k) = e^{k(A+B)}\, u(t)$

$$e^{k(A+B)} = I + k(A+B) + \frac{k^2}{2}(A+B)^2 + O(k^3)$$

$$= I + k(A+B) + \frac{k^2}{2}(A^2 + B^2 + \underline{AB + BA}) + O(k^3)$$

1st-order accurate

Error is $O(k^2)$ unless A, B commute

Splitting: $u_{n+1} = e^{kB} e^{kA} u_n$

(Assume ① and ② solved exactly)

$$e^{kB} e^{kA} = \left(I + kB + \frac{k^2}{2}B^2 + O(k^3)\right)\left(I + kA + \frac{k^2}{2}A^2 + O(k^3)\right)$$

$$= I + k(A+B) + \frac{k^2}{2}(B^2 + A^2 + \underline{2BA}) + O(k^3)$$

$$\frac{K^3}{6}(A+B)^3 = \frac{K^3}{6}\left(A^3 + B^3 + A^2B + AB^2 + BAB + ABA + B^2A + BA^2\right)$$

$$= \frac{K^3}{6}\left(A^3 + B^3 + 3A^2B + 3AB^2\right)$$

vs.

$$\left(I + \underline{KB} + \underline{\frac{K^2}{2}B^2} + \frac{K^3}{6}B^3\right)\left(I + \underline{KA} + \underline{\frac{K^2}{2}A^2} + \frac{K^3}{6}A^3\right) + O(K^4)$$

$$\frac{K^3}{6}\left(B^3 + A^3 + 3A^2B + 3AB^2\right)$$

If A, B commute: no splitting error

Higher-order operator splitting

2nd-order: Strang splitting

$$U_{n+1} = e^{\frac{k}{2}A} e^{kB} e^{\frac{k}{2}A} U_n$$

$$U_{n+2} = e^{\frac{k}{2}A} e^{kB} \underbrace{e^{\frac{k}{2}A} e^{\frac{k}{2}A}} e^{kB} e^{\frac{k}{2}A} U_n$$

$$U_{n+2} = e^{\frac{k}{2}A} e^{kB} e^{kA} e^{kB} e^{\frac{k}{2}A} U_n$$

It's possible to find splitting methods of any order. (Baker-Campbell-Hausdorff)

ImEx methods

(Implicit-Explicit)