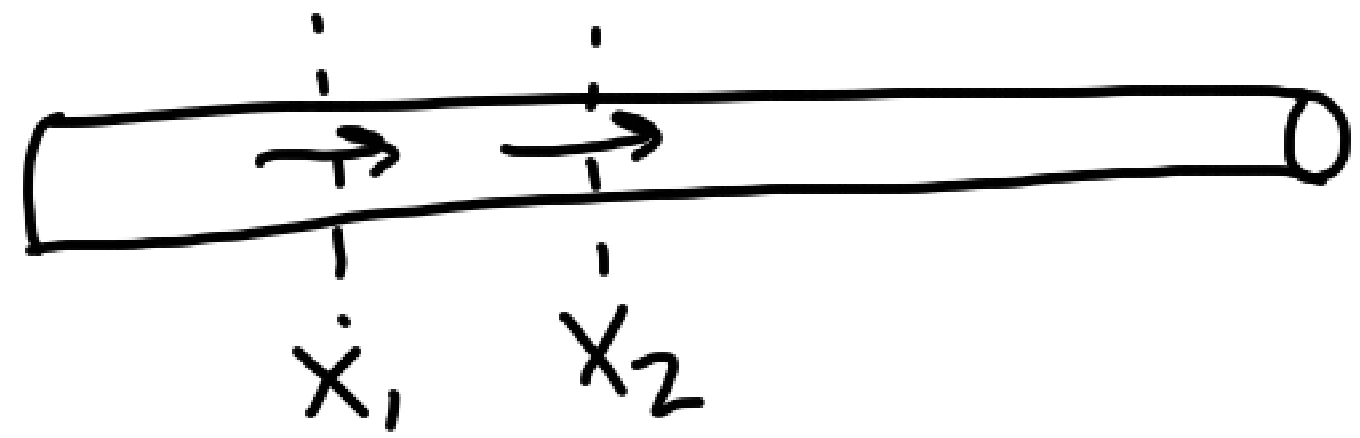
Flow of heat in a rod



$u(x,t)$: heat

$$\frac{d}{dt}\int_{x_1}^{x_2} u(x,t)\,dx = f(u(x_1,t)) - f(u(x_2,t))$$

$$\int_{x_1}^{x_2} \frac{\partial}{\partial t} u(x,t)\,dx = -\int_{x_1}^{x_2} \frac{\partial}{\partial x} f(u(x,t))\,dx$$

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$$\underbrace{\int_{x_1}^{x_2} (u_t + f(u)_x)\,dx = 0}_{\text{Conservation law}}$$

The integrand must vanish for every x,t:

$$u_t + f(u)_x = 0$$

Fick's law of diffusion:

$$f(u) = -K u_x$$

$$\Rightarrow u_t = K u_{xx}$$

Heat equation

If there is a distributed source of heat along the rod:

$$u_t = Ku_{xx} + \psi(x)$$

What happens after a long time?

$$u_t = 0$$

$$Ku_{xx} + \psi(x) = 0$$

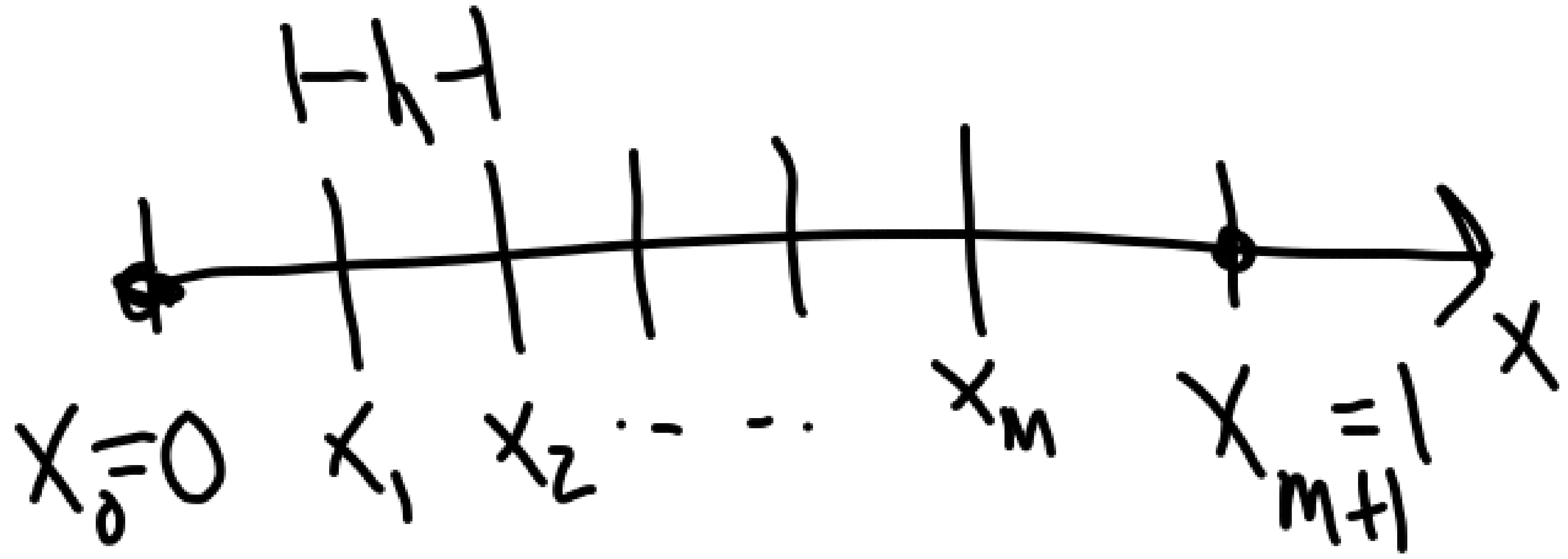
$$u_{xx} = -\frac{\psi(x)}{K} = f(x)$$

We have the 1D BVP:

$$u''(x) = f(x); \quad 0 < x < 1$$

Assume the ends are kept at fixed temperature: $u(0) = \alpha$
$u(1) = \beta$

Now let's discretize:



$U_j \approx u(x_j) \quad 0\le j\le m+1$

$$\underbrace{\frac{U_{j+1}-2U_j+U_{j-1}}{h^2}}_{D^2U_j}=f(x_j) \quad \text{for } 1\le j\le m$$

$U_0=\alpha$

$U_{m+1}=B$

We have

$$\underbrace{\frac{1}{h^2}\begin{bmatrix} -2 & 1 & & & & \\ 1 & -2 & 1 & & & \\ & 1 & -2 & 1 & & \\ & & \ddots & \ddots & \ddots & \\ & & & 1 & -2 & 1 \\ & & & & 1 & -2 \end{bmatrix}}_{A_h}\underbrace{\begin{bmatrix} U_1 \\ U_2 \\ \vdots \\ U_m \end{bmatrix}}_{U^h}=\underbrace{\begin{bmatrix} f(x_1)-\frac{\alpha}{h^2} \\ f(x_2) \\ \vdots \\ f(x_m)-\frac{B}{h^2} \end{bmatrix}}_{F}$$

$$\frac{U_2-2U_1+\overset{\alpha}{U_0}}{h^2}=f(x_1)$$

$$\frac{U_2-2U_1}{h^2}=f(x_1)-\frac{\alpha}{h^2}$$

We have $A_h U^h = F$ (1)

$\Rightarrow U^h = A_h^{-1} F$

Convergence

Let U^h denote the solution of (1).

We say the method is convergent if

$$\lim_{h \to 0} \|U^h - \hat{U}^h\| = 0$$

Where $\hat{U}^h = [u(x_1), u(x_2), \ldots, u(x_m)]^T$

Note that here $\|\cdot\|$ is a grid norm, e.g.

$$\int u\,dx \approx \|U\|_1 = h \sum_{j=1}^{m} |U_j|$$

(See appendix A of LeVeque)

We call $U - \hat{U} = E$ the global error.

To bound the global error:
Substitute the exact solution into our numerical scheme:

$$D^2 u(x_j) = \frac{u(x_{j+1}) - 2u(x_j) + u(x_{j-1})}{h^2} = f(x_j) + \underbrace{\tau_j}_{\text{local truncation error}}$$

From the text: $D^2 u(x_j) = u''(x_j) + \frac{1}{12}h^2 u^{(4)}(x_j) + O(h^4)$

So: $\cancel{u''(x_j)} + \frac{1}{12}h^2 u^{(4)}(x_j) + O(h^4) = \cancel{f(x_j)} + \tau_j$

So $\tau_j = \frac{1}{12}h^2 u^{(4)}(x_j) + O(h^4)$

Consistency

We say a discretization is consistent if $\lim_{h\to 0} \tau_j = 0$.

Notice that

$$A\hat{U} = F + \tau$$

$$AU = F$$

Subtract:

$$A(U - \hat{U}) = -\tau$$

So $AE = -\tau$.

$$E = -A^{-1}\tau$$

$$\|E\| = \|A^{-1}\tau\| \leq \|A^{-1}\| \cdot \|\tau\|$$

Here the induced matrix norm is

$$\|M\| = \sup_{\|x\| \neq 0} \frac{\|Mx\|}{\|x\|}$$

Since $\|\tau\| = O(h^2)$, if we can bound $\|A^{-1}\|$, we can prove convergence.

The idea that $\|A^{-1}\|$ is bounded as $h \to 0$ is stability.

Let $\|\cdot\| = \|\cdot\|_2$.

Then $\|M\|_2 = \max_{\lambda \in \sigma(M)} |\lambda|$

What are the eigenvalues of A^{-1}?

$$Av = \lambda v \Rightarrow v = \lambda A^{-1} v$$

$$\Rightarrow \frac{1}{\lambda} v = A^{-1} v$$

$$h^2 A = \begin{bmatrix} -2 & 1 & & & \\ 1 & \ddots & \ddots & & \\ & \ddots & \ddots & \ddots & \\ & & \ddots & \ddots & 1 \\ & & & 1 & -2 \end{bmatrix} = \hat{A}$$

$\hat{A} v = \lambda v$

$$\boxed{\begin{array}{l} v_{j+1} - 2v_j + v_{j-1} = \lambda v_j \\ v_0 = 0 \quad v_{M+1} = 0 \end{array}} \quad 1 \le j \le m$$

System of linear difference equations

Ansatz: $v_j = \xi^j$

$$\xi^{j+1} - 2\xi^j + \xi^{j-1} = \lambda \xi^j$$

$$\xi^2 - (2+\lambda)\xi + 1 = 0$$

$$\xi_\pm = \frac{2+\lambda}{2} \pm \frac{\sqrt{(2+\lambda)^2 - 4}}{2}$$

$$\xi_\pm = 1 + \frac{\lambda}{2} \pm \frac{\sqrt{4\lambda + \lambda^2}}{2}$$

General solution: $v_j = a\xi_+^j + b\xi_-^j$

$$v_0 = 0 \Rightarrow a + b = 0 \Rightarrow b = -a$$

$$V_j = a(\xi_+^j - \xi_-^j)$$

$$\xi_+^{m+1} = \xi_-^{m+1}$$

$$\xi_+^{m+1}\xi_+^{m+1} = \xi_+^{m+1}\xi_-^{m+1} = 1$$

$$\xi_+^{2m+2} = 1$$

$$\xi_+ = e^{\pi i\left(\frac{p}{m+1}\right)}$$

$$\xi_- = e^{-\pi i\left(\frac{p}{m+1}\right)}$$

$p = 1, 2, \ldots, m$

Note $\xi_+\xi_- = \left(1+\frac{\lambda}{2}+\frac{\sqrt{4\lambda+\lambda^2}}{2}\right)\left(1+\frac{\lambda}{2}-\frac{\sqrt{4\lambda+\lambda^2}}{2}\right)$

$$= \left(1+\frac{\lambda}{2}\right)^2 - \frac{4\lambda+\lambda^2}{4}$$

$$= 1+\lambda+\frac{\lambda^2}{4}-\lambda-\frac{\lambda^2}{4}$$

$$= 1$$

$$\xi^+ + \xi^- = e^{\pi i p/m+1} + e^{-\pi i p/m+1}$$

$$2\cos\left(\frac{p\pi}{m+1}\right) = 2+\lambda$$

$$\lambda_p = 2\left(\cos\left(\frac{p\pi}{m+1}\right)-1\right) \quad p=1,2,\dots,m$$

Eigenvalues of A^{-1}: $\frac{1}{2}\left(\cos\left(\frac{p\pi}{m+1}\right)-1\right)^{-1}$

What happens as $h\to 0$?

Which is the smallest $|\lambda_p|$?

$$\cos\left(\frac{p\pi}{m+1}\right) \approx 1 -$$