Lax-Wendroff

Cauchy-Kovalevsky (generalization to higher order, other PDEs)

$$u_t = -au_x$$

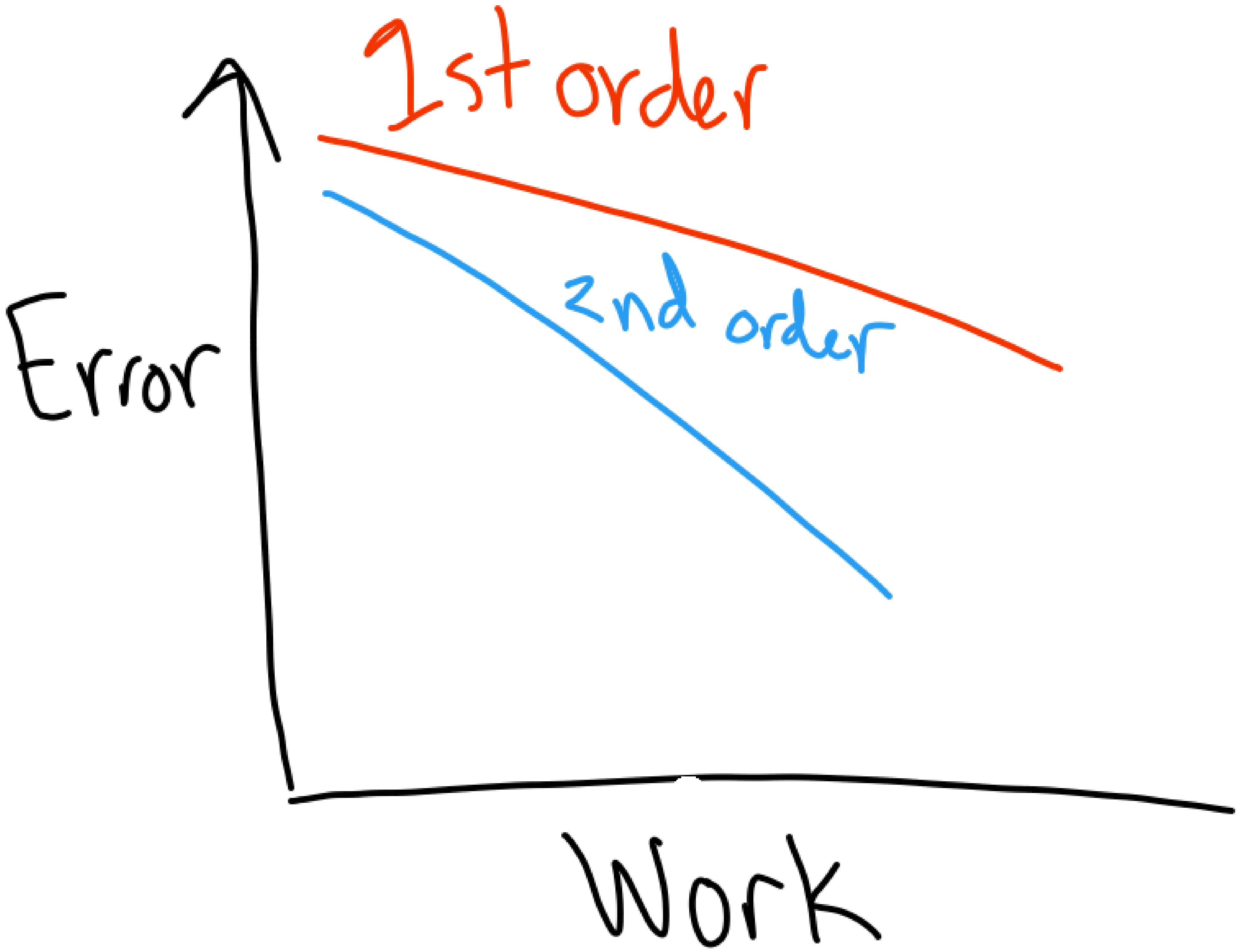
$$u(x,t+k) = u(x,t) + k u_t(x,t) + \frac{k^2}{2} u_{tt}(x,t) + O(k^3)$$

$$u_{tx} = -au_{xx} \qquad u_{tt} = -au_{xt}$$

$$u_{tt} = a^2 u_{xx}$$

$$U_j^{n+1} = U_j^n - ka \frac{U_{j+1}^n - U_{j-1}^n}{2h} + \frac{k^2 a^2}{2}\left(\frac{U_{j+1}^n - 2U_j^n + U_{j-1}^n}{h^2}\right)$$

2nd order in space and time.


1st order
2nd order
Error
Work

Hyperbolic Systems of PDEs

Examples:

- Euler equations (compressible gas dynamics)
- Maxwell's equations
- Acoustics
- Elasticity

$p(x,t)$: pressure

$u(x,t)$: velocity

$$p_t = -K u_x$$

Bulk modulus

density

$$\rho u_t = -p_x$$

$ma = F$ (Newton)

$$p_t + K u_x = 0$$

$$u_t + \frac{1}{\rho} p_x = 0$$

$$q = \begin{bmatrix} p \\ u \end{bmatrix}$$

$$q_t + \begin{bmatrix} 0 & K \\ 1/\rho & 0 \end{bmatrix} q_x = 0$$

$$q_t + A q_x = 0$$

Linear hyperbolic system

$$A r_{1,2} = \lambda r_{1,2} \qquad R = \left[r_1 \,|\, r_2 \right]$$

$$AR = R\Lambda \qquad \Lambda = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}$$

$$A = R \Lambda R^{-1}$$

$$q_t + R \Lambda R^{-1} q_x = 0$$

$$w = R^{-1} q \quad \text{(Characteristic Variables)}$$

$$R^{-1} q_t + \Lambda R^{-1} q_x = 0$$

$$w = \begin{bmatrix} w^1 \\ w^2 \end{bmatrix}$$

$$w_t + \Lambda w_x = 0$$

$$\boxed{\begin{aligned} w^1_t + \lambda_1 w^1_x &= 0 \\ w^2_t + \lambda_2 w^2_x &= 0 \end{aligned}}$$

If $\lambda_1, \lambda_2 \in \mathbb{R}$ then Solutions behave like those of the advection equation.

Dfn. We say

$$q_t + A q_x = 0$$

is hyperbolic if A is diagonalizable with real eigenvalues.

$$A = \begin{bmatrix} 0 & K \\ 1/\rho & 0 \end{bmatrix}$$

$$\lambda^2 - \frac{K}{\rho} = 0$$

$$\lambda_{\pm} = \pm\sqrt{\frac{K}{\rho}} \in \mathbb{R}$$

Since $K>0$, $\rho>0$

Eigenvectors: $r = \begin{bmatrix} r_1 \\ r_2 \end{bmatrix}$

$$Ar = \lambda r$$

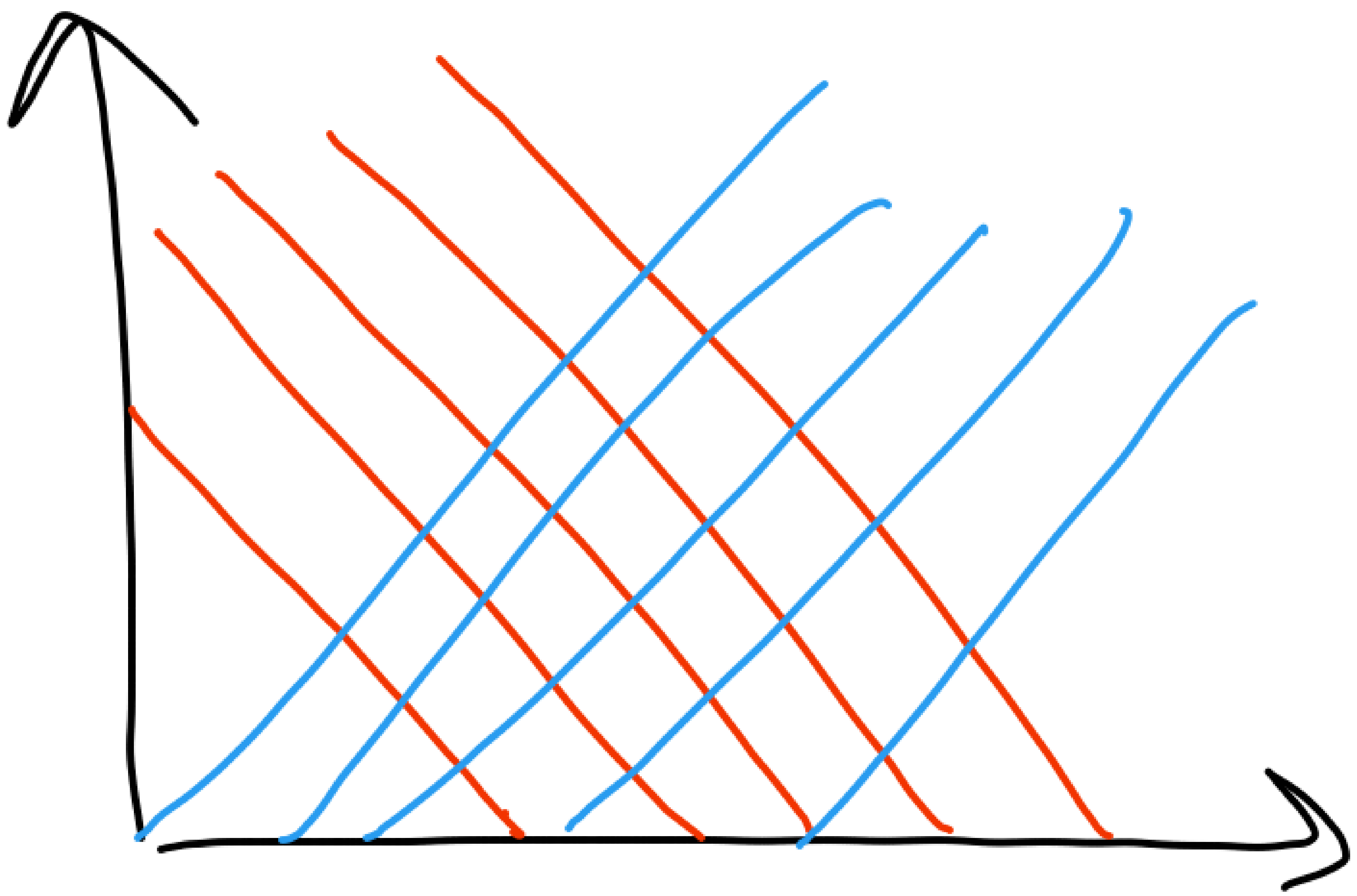
$$\frac{1}{\rho} r_1 = \pm\sqrt{\frac{K}{\rho}}\, r_2$$

$$r_1 = \pm\sqrt{K\rho}\, r_2$$

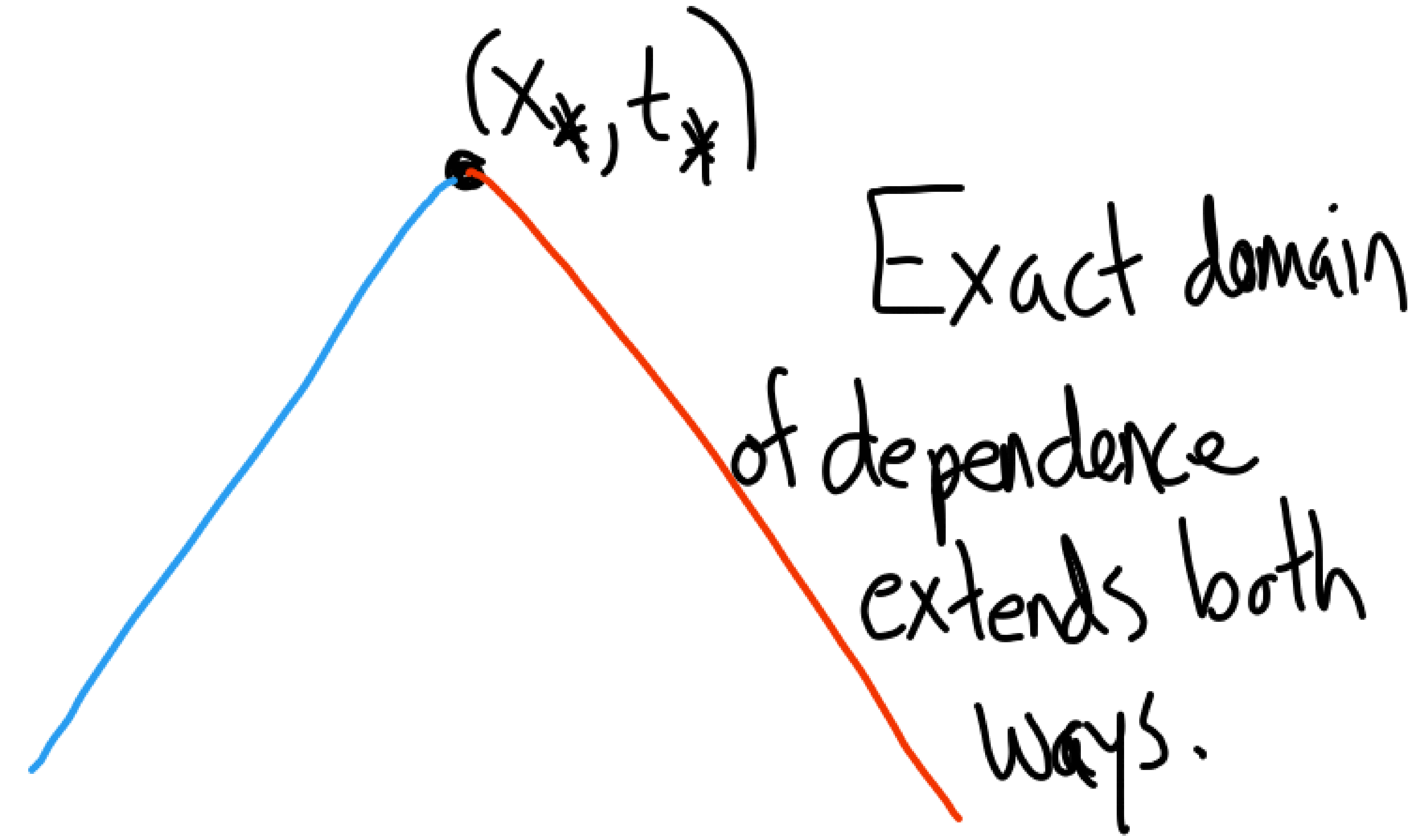
$Z = \sqrt{K\rho}$: Impedance

$$R = \begin{bmatrix} -Z & Z \\ 1 & 1 \end{bmatrix}$$

We have 2 families of characteristics:



The CFL condition:



The upwind method can't satisfy the CFL condition.

Lax-Wendroff for systems

$$U_j^{n+1} = U_j^n - \frac{ka}{2h}\left(U_{j+1}^n - U_{j-1}^n\right) + \frac{k^2a^2}{h^2}\left(U_{j+1}^n - 2U_j^n + U_{j-1}^n\right)$$

$$\Rightarrow Q_j^{n+1} = Q_j^n - \frac{k}{2h}A\left(Q_{j+1}^n - Q_{j-1}^n\right) + \frac{k^2}{h^2}A^2\left(Q_{j+1}^n - 2Q_j^n + Q_{j-1}^n\right)$$

Nonlinear hyperbolic systems

$$q_t + f(q)_x = 0$$

Quasilinear form:

$$q_t + f'(q) q_x = 0$$

We say this is hyperbolic if $f'(q)$ is diagonalizable with real eigenvalues (for all q).

Simplest example:

$$u_t + \left(\tfrac{1}{2}u^2\right)_x = 0$$

(Inviscid Burgers eqn.)

$$u_t + u u_x = 0$$

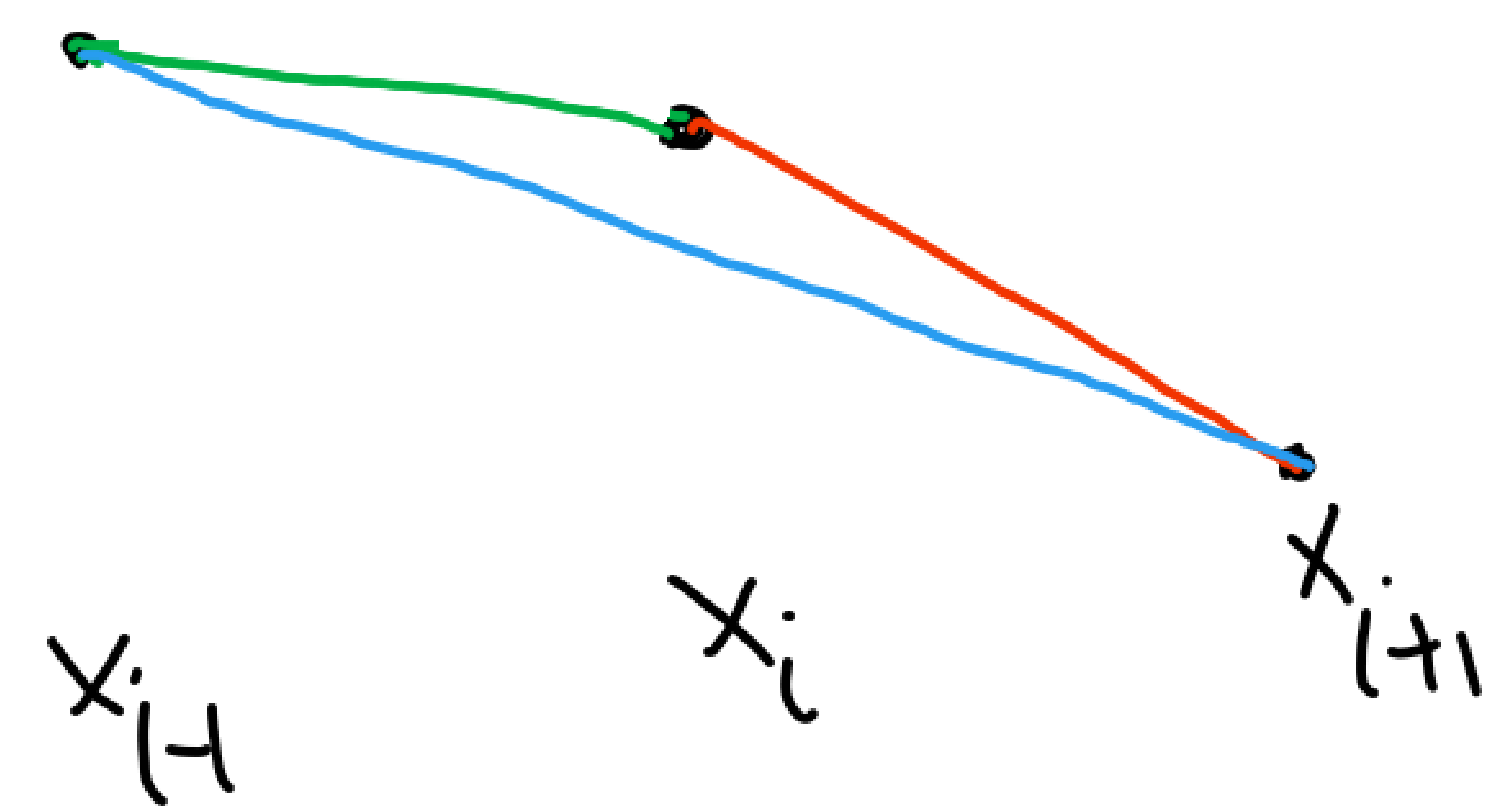
Challenge: solution derivative (in x) blows up in finite time. (Shock formation)

Often we wish to compute weak solutions (with discontinuities).

To deal with this, specialized numerical methods have been developed to avoid oscillations. (limiters)

The minmod limiter

We want approximate u_x:



Minmod: use the one-sided slope that has least modulus.

$$\rho_t + (\rho(1-\rho))_x = 0$$

$$\rho - \rho^2$$

$$\rho_t + \underbrace{(1-2\bar{\rho})}_{\text{char. speed}}\rho_x = 0$$

$$a(\rho) = 1 - 2\rho$$

$$\frac{Ka}{h} \le 1$$

$$\max_{\rho_0(x)} \frac{Ka(\rho)}{h} \le 1$$

$$TV(u) = \int |u_x|\,dx$$