Reminder: Homework 2 due Thursday

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3 Exercises numbered 1, 2, 4

Today: Solving $AU=F$

1. Jacobi's method
2. Multigrid

$$U''(x) = f(x) \quad 0<x<1$$

$$U(0)=\alpha \quad u(1)=\beta$$

Discretize:

$$\frac{U_{i+1}-2U_i+U_{i-1}}{h^2} = F_i$$

Jacobi's method

$$A = \frac{1}{h^2}\begin{bmatrix} -2 & 1 & & \\ 1 & \ddots & \ddots & \\ & \ddots & \ddots & 1 \\ & & 1 & -2 \end{bmatrix}$$

Let

$$G = \begin{bmatrix} 0 & 1 & & \\ 1 & \ddots & \ddots & \\ & \ddots & \ddots & 1 \\ & & 1 & 0 \end{bmatrix}$$

$$G = h^2A + 2I \qquad A = \frac{1}{h^2}(G-2I)$$

$$AU=F$$

$$\frac{1}{h^2}(G-2I)U=F$$

$$\frac{1}{h^2}GU-\frac{2}{h^2}U=F$$

$$\frac{2}{h^2}U=\frac{1}{h^2}GU-F$$

$$U=\frac{1}{2}GU-\frac{h^2}{2}F$$

Jacobi:

① Guess $U^{[0]}$

② While $\|AU-F\|>\varepsilon$

$$U^{[k+1]}=\frac{1}{2}GU^{[k]}-\frac{h^2}{2}F$$

End

The true solution U is a fixed point of this iteration.

Starting from $U^{[0]}\neq U$, will $U^{[k]}$ converge to U? How quickly?

Proof that $\lim_{k\to\infty} U^{[k]}=U$

$$e^{[k]}=U^{[k]}-U$$

$$U^{[k+1]}-U=\frac{1}{2}G(U^{[k]}-U)$$

$$e^{[k+1]}=\frac{1}{2}Ge^{[k]}$$

$$e^{[k]}=\left(\frac{1}{2}G\right)^k e^{[0]}$$

$$\|e^{[k]}\|_2\leq\left\|\frac{1}{2}G\right\|_2^k\cdot\|e^{[0]}\|_2$$

G is a symmetric matrix so it has a complete set of orthonormal eigenvectors.

Let $\tilde{G} = \frac{1}{2}G$. Same goes for $\tilde{G}$.

So any $e^{[0]} \in \mathbb{R}^m$ can be expressed as

$$e^{[0]} = \sum_{p=1}^{m} c_p v_p$$

eigenvectors of $\tilde{G}$

eigenvalues of $\tilde{G}$

$$\text{So } \tilde{G}e^{[0]} = \sum_{p=1}^{m} c_p \tilde{G} v_p = \sum_{p=1}^{m} c_p \tilde{\gamma}_p v_p$$

$$\text{Similarly } \tilde{G}^k e^{[0]} = \sum_{p=1}^{m} c_p \tilde{G}^k v_p = \sum_{p=1}^{m} c_p \tilde{\gamma}_p^k v_p$$

So if $|\tilde{\gamma}_p| < 1$ for all $p = 1, 2, \ldots, m$

then $\lim_{k\to\infty} \tilde{G}^k e^{[0]} = \vec{0}$

What are $\tilde{\gamma}_p$?

Let (v_p, λ_p) be an eigenpair of A. Then

$$A v_p = \lambda_p v_p$$

$$\frac{1}{h^2}(G - 2I)v_p = \lambda_p v_p$$

$$G v_p - 2v_p = h^2 \lambda_p v_p$$

$$\frac{1}{2} G v_p = \left(\frac{h^2}{2}\lambda_p + 1\right) v_p \Rightarrow \tilde{G} v_p = \tilde{\gamma}_p v_p$$

So V_p is also an eigenvector of $\tilde{G} = \frac{1}{2}G$.

The corresponding eigenvalue is

$$\tilde{\gamma}_p = \frac{h^2}{2}\lambda_p + 1$$

Recall: $\lambda_p = \frac{2}{h^2}\left(\cos(p\pi h) - 1\right)$

So $\tilde{\gamma}_p = \cos(p\pi h)$ $\quad h = \frac{1}{m+1}$

$p = 1, 2, \ldots, m$

$$|\tilde{\gamma}_p| = |\cos(p\pi h)| < 1.$$

So $\lim_{k \to 0} \|e^{[k]}\| = 0.$

The slowest-converging parts correspond to $p = m$, $p = 1$

Recall: $(V_p)_j = \sin(p\pi h j)$ $\quad j = 1, 2, \ldots, m$

$$= \sin(p\pi x_j)$$

Under-relaxed Jacobi

$$\hat{U}^{[k+1]} = \frac{1}{2}\left(GU^{[k]} - h^2 F\right)$$

$$U^{[k+1]} = U^{[k]} + \omega\left(\hat{U}^{[k+1]} - U^{[k]}\right)$$

$\omega = 1$: Original Jacobi

$\omega < 1$: Under-relaxed Jacobi

$$U^{[k+1]} = (1-\omega)U^{[k]} + \omega \hat{U}^{[k+1]}$$

$$= (1-\omega)U^{[k]} + \frac{\omega}{2}\left(GU^{[k]} - h^2 F\right)$$

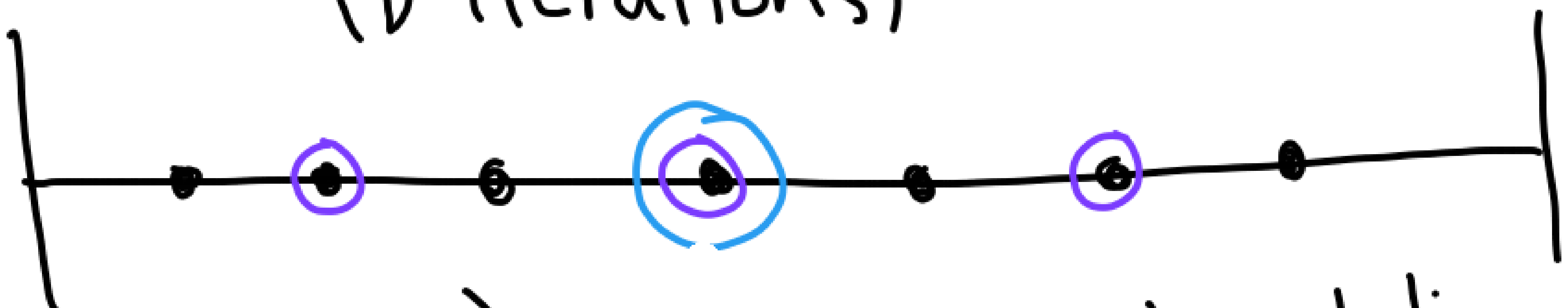
$$U^{[k+1]} = \underbrace{\left(I - \omega I + \omega \tilde{G}\right)}_{G_\omega} U^{[k]} - \omega \frac{h^2}{2} F$$

Eigenvalues of G_ω:

$$1 - \omega + \omega \tilde{\gamma}_p$$

Multigrid

Start on a (fine) grid with m points and iterate with underrelaxed Jacobi (ν iterations)



(Solve $AU=F$) $\rightarrow$ approximate solution U_ν

Define $e_\nu = U_\nu - U$

Write $AU_\nu = F - r$ residual (we can compute it)

$AU_\nu - F = -r$

So $Ae_\nu = -r$.

We want to solve this for e_ν, and then correct our solution:

$$U_\nu - e_\nu = U$$

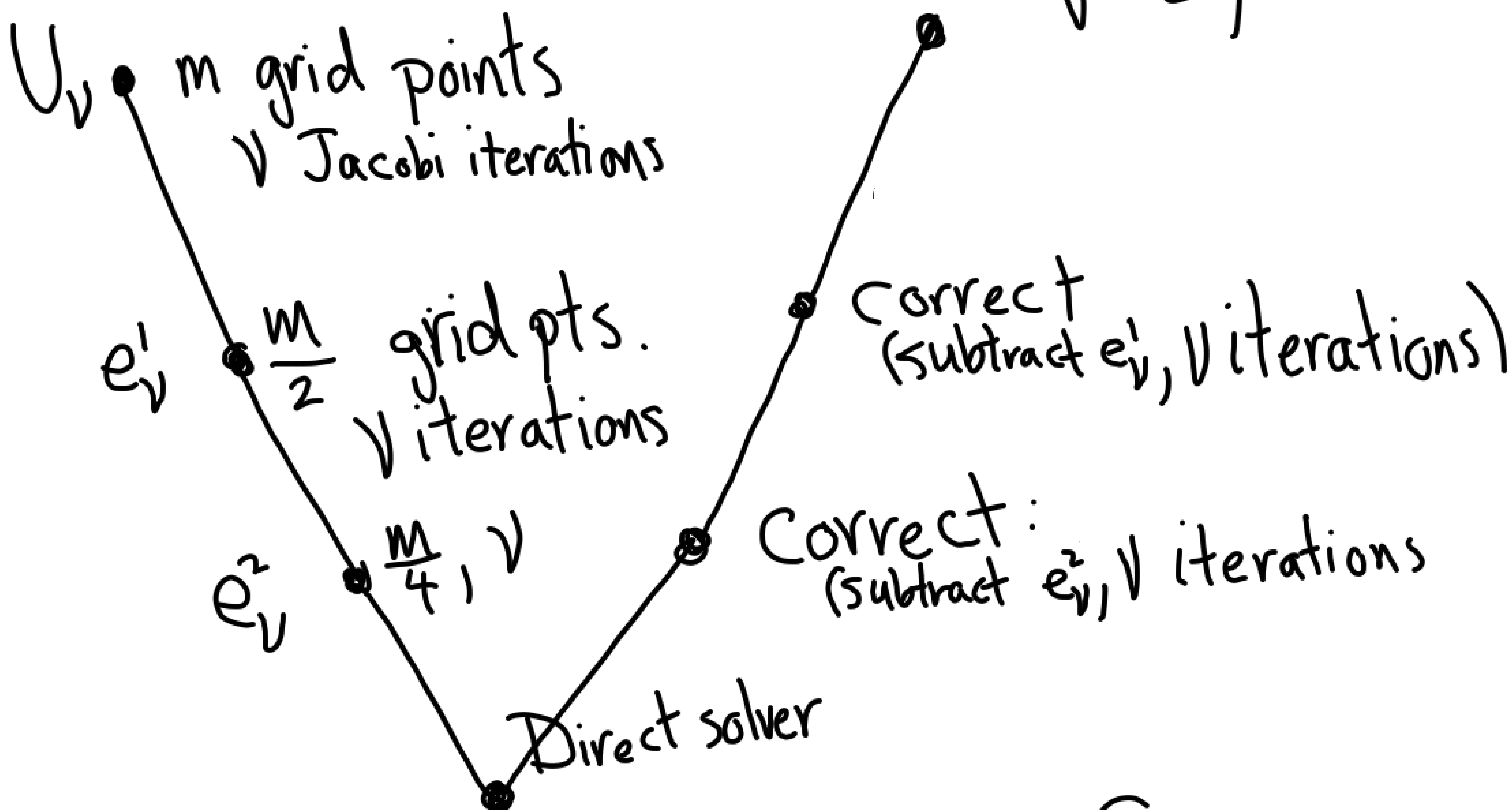
(restriction)

We use a coarsened grid by neglecting the odd points.

Iterate again with under-relaxed Jacobi.

We can repeat this on even coarser grids.

V-cycle



Cost of one Jacobi iteration: Cm

Cost of V-cycle: $2\left(C\nu m + C\nu \frac{m}{2} + C\nu \frac{m}{4} + \dots\right) \approx 4C\nu m$