Absolute Stability

$$u'(t) = -u \qquad U^{n+1} = U^{n-1} + 2kf(U^n)$$

$$u(0) = 1$$

$$u(t) = e^{-t}$$

$$U^0 = 1 \quad U^1 = e^{-k}$$

$u'(t) = -\sin(t)$

$u(0) = 1$

$u(t) = \cos(t)$

$u'(t) = -\sin(t) + \lambda(u(t) - \cos(t))$

$u(0) = 1$ $\qquad L = |\lambda|$

$u(t) = \cos(t)$

$k = \frac{1}{100}$ $\quad \lambda = -10$

$k = \frac{1}{100}$ $\quad \lambda = -250$

$k = \frac{1}{1000}$ $\quad \lambda = -250$

$$u'(t) = \lambda u$$

$$E^{n+1} = \underbrace{(1+k\lambda)}_{\text{E^n grows if this has magnitude >1.}} E^n - k\tau^n$$

We need

$$|1+k\lambda| \leq 1$$

$$|1-250k| \leq 1$$

$$-1 \leq 1-250k \leq 1$$

$$-2 \leq -250k \leq 0$$

$$0 \leq k \leq \frac{2}{250}$$

Apply backward Euler to

$$u'(t) = \lambda u$$

$$U^{n+1} = U^n + k\lambda U^{n+1}$$

$$(1-k\lambda)U^{n+1} = U^n$$

$$U^{n+1} = \frac{1}{1-k\lambda} U^n$$

$$E^{n+1} = \frac{1}{1-k\lambda} E^n - k\tau^n$$

We need $\left|\frac{1}{1-k\lambda}\right| \leq 1$.

Satisfied for any k if $\lambda \leq 0$.

For any 1-step method applied to

$$u'(t) = \lambda u$$

We get

$$U^{n+1} = \underbrace{R(k\lambda)}_{\text{stability function}} U^n$$

Absolute stability means

$$|R(k\lambda)| \le 1.$$

Let

$$S_R = \{ z \in \mathbb{C} : |R(z)| \le 1 \}$$

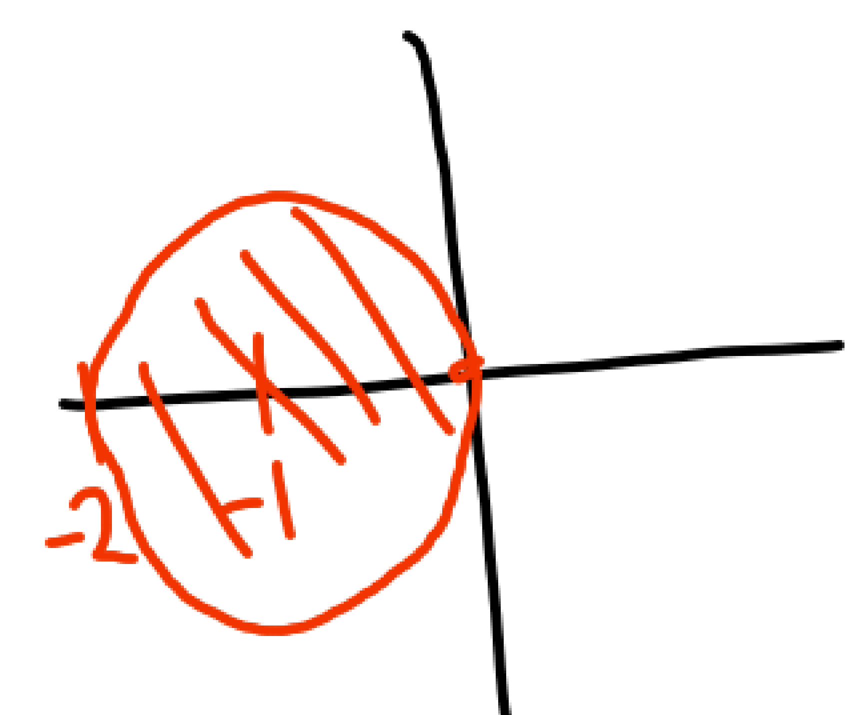
Then abs. stability means

$$k\lambda \in S_R.$$

We call S_R the "region of absolute stability."

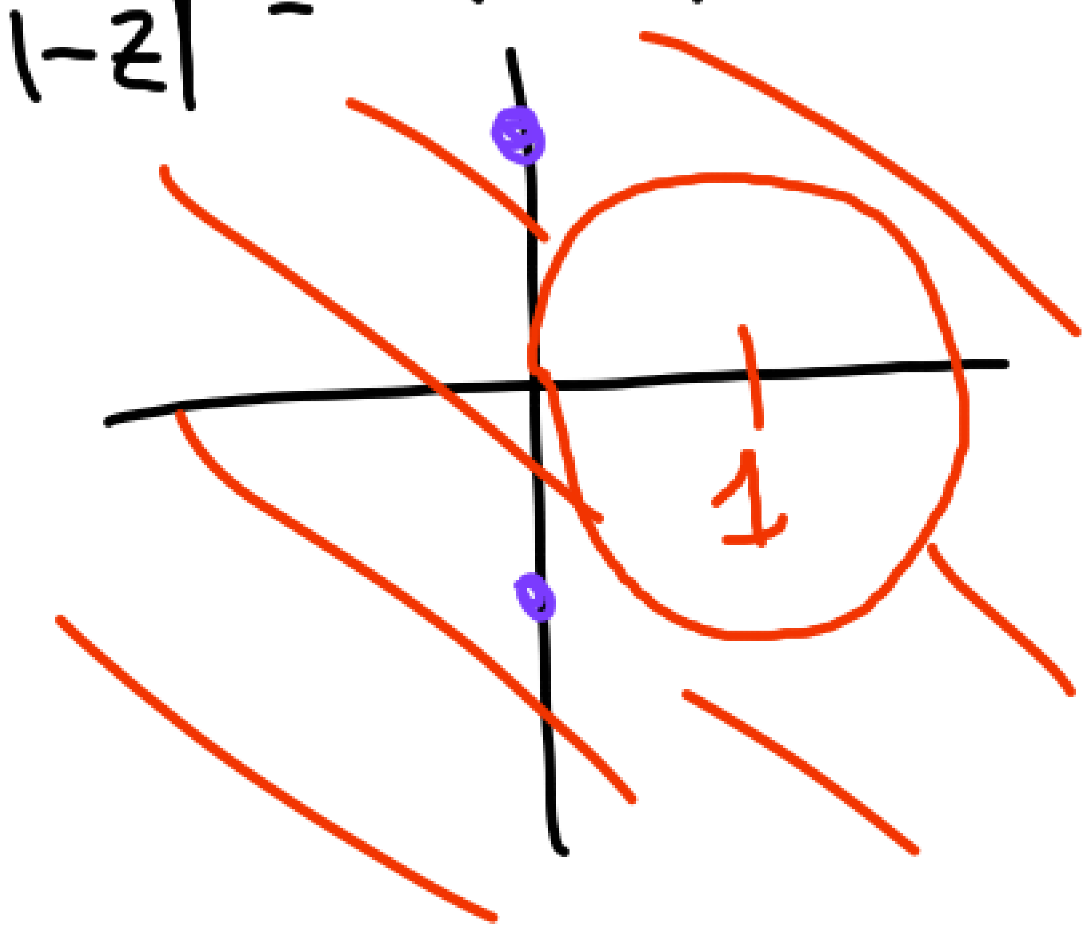
Forward Euler

$$|1+z| \le 1$$

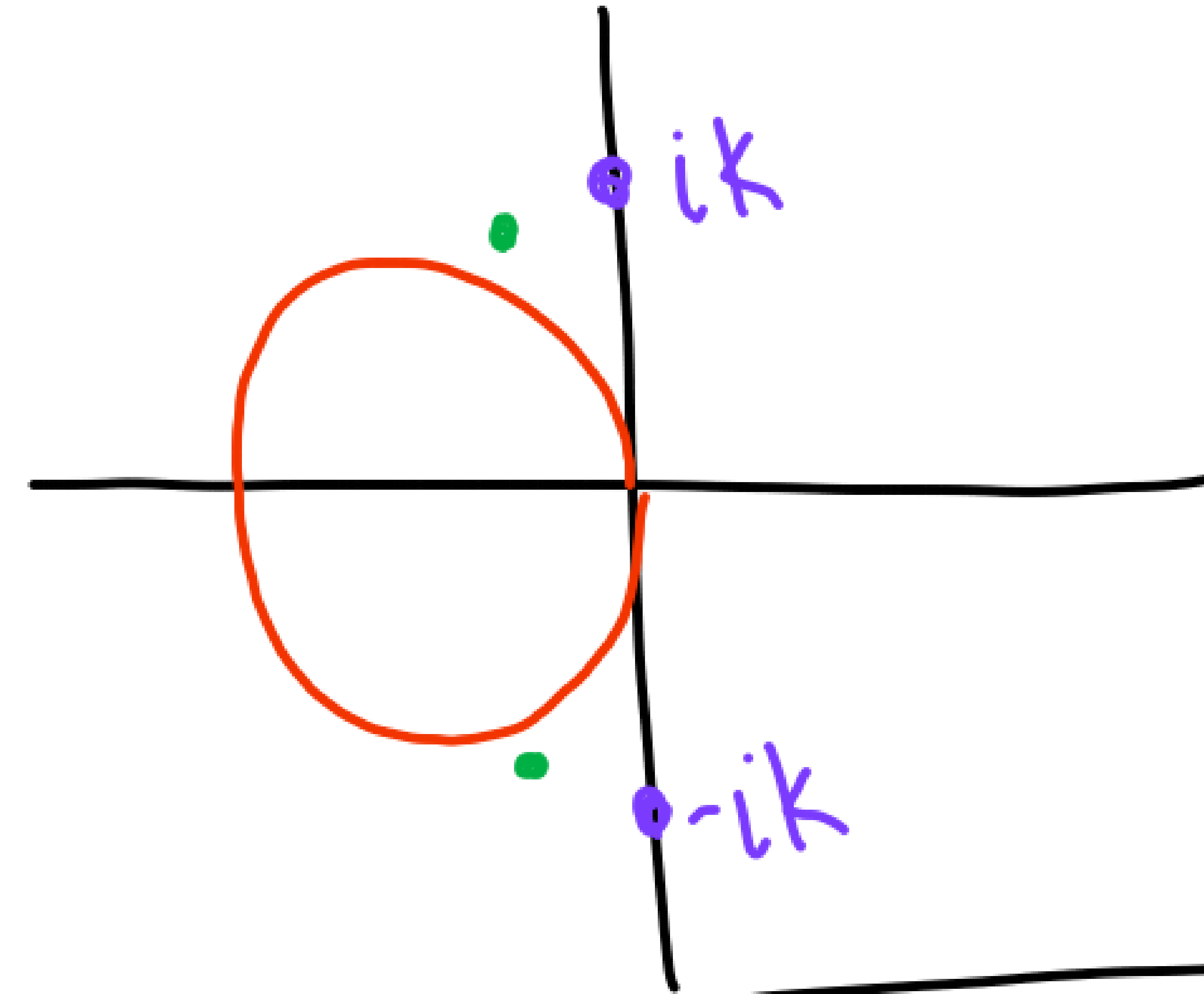


Backward Euler

$$\frac{1}{|1-z|} \le 1 \iff |1-z| \ge 1$$



Linear Systems of ODEs



Linearized pendulum:

$$\theta''(t) = -\theta(t)$$

$$u_1(t) = \theta(t)$$
$$u_2(t) = \theta'(t)$$

$$\begin{bmatrix} u_1 \\ u_2 \end{bmatrix}' = \underbrace{\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}}_{M} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$$

$$u'(t) = Mu$$

$$\det(\lambda I - L) = 0$$
$$\lambda^2 + 1 = 0$$
$$\lambda = \pm i$$

$$\Rightarrow U^{n+1} = R(kM)U^n$$

Forward Euler: $U^{n+1} = (I + kM)U^n$

Absolute Stability: $\|I + kM\|_2 \leq 1$

Equivalently: $k\lambda \in S_R$

Damped Pendulum:

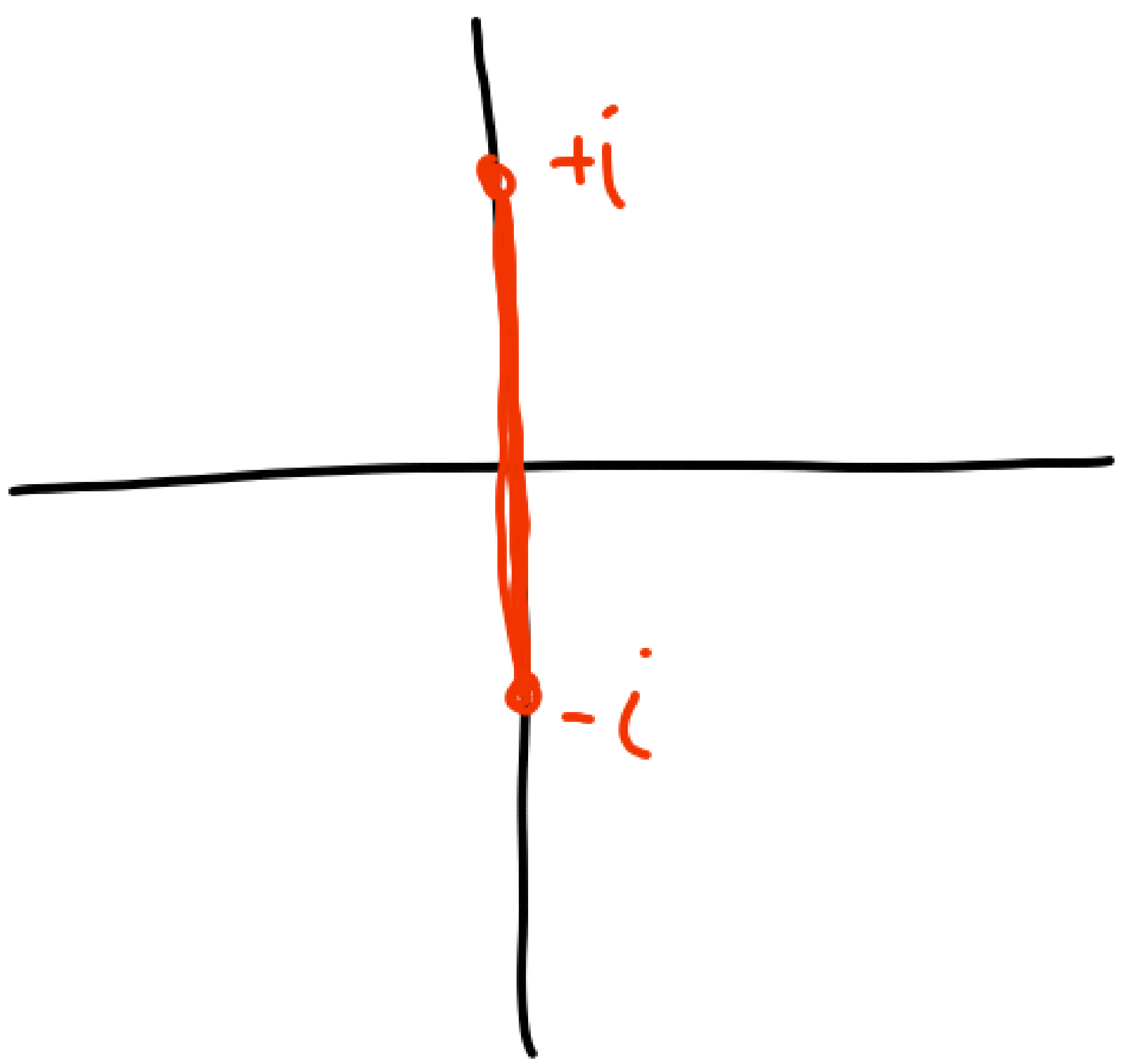
$$\theta''(t) = -\theta(t) - \varepsilon\,\theta'(t)$$

$$\begin{bmatrix} u_1 \\ u_2 \end{bmatrix}' = \underbrace{\begin{bmatrix} 0 & 1 \\ -1 & -\varepsilon \end{bmatrix}}_{M} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$$

$$\lambda(\lambda + \varepsilon) + 1 = 0$$
$$\lambda^2 + \varepsilon\lambda + 1 = 0 \qquad -\frac{\varepsilon}{2} \pm \frac{\sqrt{\varepsilon^2 - 4}}{2} = \lambda$$

Leapfrog

$$U^{n+2} = U^n + kf(U^{n+1})$$



Boundary locus method

(for finding the abs. stability region of a LMM)

$$\sum_{j=0}^{r} \alpha_j U^{n+j} = k \sum_{j=0}^{r} \beta_j f(U^{n+j})$$

$u'(t) = \lambda u$

$$\sum_{j=0}^{r} (\alpha_j - z\beta_j) U^{n+j} = 0 \quad \Big| \quad U^{n+j} \to \zeta^{n+j}$$

$$\Rightarrow \sum_{j=0}^{r} (\alpha_j - z\beta_j) \zeta^j = \pi(\zeta; z)$$

How to find values z s.t. $\pi(\zeta;z)$ has a root with magnitude 1.

Let $\zeta = e^{i\theta}$

$$\sum_{j=0}^{r} (\alpha_j - z\beta_j) e^{ij\theta} = 0$$

$$\sum_j \alpha_j e^{ij\theta} = z \sum_j \beta_j e^{ij\theta}$$

$$z(\theta) = \frac{\sum_j \alpha_j e^{ij\theta}}{\sum_m \beta_m e^{im\theta}}$$

Now evaluate $z(\theta)$ for $0 \le \theta \le 2\pi$.

This set is the boundary of the stability region.