Linear Multistep Methods

$$u'(t) = f(u)$$
$$u(t_0) = \eta$$

A LMM takes the form

$$\sum_{j=0}^{r} \alpha_j U^{n+j} = k \sum_{j=0}^{r} \beta_j f(U^{n+j})$$

This is a formula to compute U^{n+r} from $U^n, U^{n+1}, \ldots, U^{n+r-1}$. (An r-step method)

This method is
implicit if $\beta_r \neq 0$
explicit if $\beta_r = 0$.

The coefficients α_j, β_j determine the accuracy and stability of the method.

Examples

2-step Adams-Bashforth:

$$U^{n+2} = U^{n+1} + \frac{k}{2}\left(-f(U^n) + 3f(U^{n+1})\right)$$

Leapfrog: $U^{n+2} = U^n + kf(U^{n+1})$

Backward differentiation formula: $U^{n+2} = \frac{4}{3}U^{n+1} - \frac{1}{3}U^n + 2kf(U^{n+2})$

A 2-step 1st-order:

$$U^{n+2} = 3U^{n+1} - 2U^n + kf(U^n)$$

A 3-step method: $U^{n+3} = 3U^{n+2} - 2U^{n+1} + kf(U^n)$

Let's Solve

$$u'(t) = 0$$
$$u(t_0) = 0$$

Consistency of initial Values:

$U^0 \to u(t_0)$
$U^1 \to u(t_0 + k)$ as $k \to 0$

$\rho(\zeta) = \zeta^3 - 3\zeta^2 + 2\zeta$

$\zeta^2 - 3\zeta + 2 = (\zeta - 1)(\zeta - 2)$

$\zeta_1 = 0$
$\zeta_2 = 1$
$\zeta_3 = 2$

Local Truncation Error $t_{n+j} = t_n + jk$

$$\sum_{j=0}^{r} \alpha_j U^{n+j} = k\sum_{j=0}^{r} \beta_j f(U^{n+j})$$

$$\sum_{j=0}^{r} \alpha_j u(t_{n+j}) = k\sum_{j=0}^{r} \beta f(u(t_{n+j})) + k\tau^{n+r}$$

$$u(t_{n+j}) = \sum_{i=0}^{\infty} \frac{(jk)^i}{i!} u^{(i)}(t_n)$$

$$f(u(t_{n+j})) = u'(t_{n+j}) = \sum_{i=0}^{\infty} \frac{(jk)^i}{i!} u^{(i+1)}(t_n)$$

$$= \sum_{i=1}^{\infty} \frac{(jk)^{i-1}}{(i-1)!} u^{(i)}(t_n)$$

Substitution gives

$$\sum_{j=0}^{r} \alpha_j \sum_{i=0}^{\infty} \frac{(jk)^i}{i!} u^{(i)}(t_n) = k\sum_{j=0}^{r} \beta_j \sum_{i=1}^{\infty} \frac{(jk)^{i-1}}{(i-1)!} u^{(i)}(t_n) + k\tau^{n+r}$$

$$\sum_{j=0}^{r} \sum_{i=1}^{\infty} k^{i-1} u^{(i)}(t_n) \left[\alpha_j \frac{j^i}{i!} - \frac{j^{i-1}}{(i-1)!}\beta_j\right] + \frac{1}{k}\sum_{j=0}^{r} \alpha_j u(t_n) = \tau^{n+r}$$

We want $\tau^{n+r} = O(k^p)$

We need:

$$\frac{1}{k} u(t_n) \sum_{j=0}^{r} \alpha_j = 0$$

$$\Rightarrow \boxed{\sum_{j=0}^{r} \alpha_j = 0}$$

$i=1$ (k^0): $\boxed{\sum_{j=0}^{r} (j\alpha_j - \beta_j) = 0}$

This ensures that $\tau^{n+r} = O(k^1)$, so the method is consistent.

For 2nd order we need:

$$\sum_{j=0}^{r} \left(\frac{j^2}{2} \alpha_j - j\beta_j \right) = 0$$

etc.

Zero-stability

If we apply a LMM to

$$u'(t) = 0$$

we get $\underbrace{\sum_{j=0}^{r} \alpha_j U^{n+j} = 0}_{\text{Linear difference equations}}$ $(*)$

Ansatz: $U^n = \zeta^n$ for some $\zeta \in \mathbb{C}$

$$\sum_{j=0}^{r} \alpha_j \zeta^{n+j} = 0 \quad \text{or} \quad \sum_{j=0}^{r} \alpha_j \zeta^j = 0$$

Let $\rho(\zeta) = \sum_{j=0}^{r} \alpha_j \zeta^j$

We call $\rho(\xi)$ the "first characteristic polynomial" of the LMM.

It has roots $\xi_1, \xi_2, \dots, \xi_r$.

If these are distinct, the general solution of (*) is

$$U^n = c_1 \xi_1^n + c_2 \xi_2^n + \cdots + c_r \xi_r^n.$$

The values c_i are determined by the initial values $U^0, U^1, \dots, U^{r-1}$.

Note that $\rho(1) = \sum_{j=0}^{r} \alpha_j = 0$

So 1 is always a root.

What if we have a multiple root of ρ?

For example:

$$U^{n+2} - 2U^{n+1} + U^n = 0$$

$$\xi^2 - 2\xi + 1 = 0$$

$$(\xi - 1)^2 = 0$$

$\xi_1 = \xi_2 = 1.$

The general solution is

$$U^n = c_1 \xi_1^n + c_2 n \xi_1^n$$

Check that $n 1^n$ is a solution:

$$n+2 - 2(n+1) + n = 0$$

In general if ζ_j is a root of multiplicity m, then we have fundamental solutions

$$\zeta_j^n, n\zeta_j^n, n^2\zeta_j^n, \ldots n^{m-1}\zeta_j^n$$

The solution of (*) is bounded as $n\to\infty$ iff the roots of $\rho(\zeta)$ satisfy:

$$\left.\begin{array}{ll} |\zeta_j|\leq 1 & \forall j \\ |\zeta_j|<1 & \text{if } \zeta_j \text{ is repeated} \end{array}\right\} \text{"The root condition."}$$

<u>Thm.</u>

For any IVP

$$u'(t)=f(u)$$
$$u(t_0)=\eta$$

with f Lipschitz continuous, a consistent LMM gives a convergent solution

$$\lim_{\substack{K\to 0 \\ NK=T}} \|U^N-u(t_N)\|=0$$

if $\rho(\zeta)$ satisfies the root condition.

We could prove this by writing the LMM as a 1-step method.

$$V^n = \begin{bmatrix} U^n \\ U^{n+1} \\ \vdots \\ U^{n+r-1} \end{bmatrix} \quad V^{n+1} = \begin{bmatrix} U^{n+1} \\ \vdots \\ U^{n+r} \end{bmatrix}$$

We can write the LMM

$$\sum_{j=0}^{r} \alpha_j U^{n+j} = k \sum_{j=0}^{r} \beta_j f(U^{n+j})$$

in vector form

$$V^{n+1} = \underbrace{\begin{bmatrix} 0 & 1 & & & \\ 0 & 0 & 1 & & \\ & & & \ddots & \\ & & & & 1 \\ -\alpha_0 & -\alpha_1 & -\alpha_2 & \cdots & -\alpha_{r-1} \end{bmatrix}}_{C \text{ (companion matrix)}} V^n + \begin{bmatrix} 0 \\ \vdots \\ 0 \\ k\sum_{j=0}^{r} \beta_j f(U^{n+j}) \end{bmatrix}$$

$$E^{n+1} = CE^n + k\tau^n$$

We need to bound $\|C^n\|$.

This is bounded iff the eigenvalues of C satisfy the root condition.

These are just the roots of $\rho(\zeta)$.

Why didn't we discuss zero-stability for one-step methods?

We have

$$U^{n+1} = U^n + \Psi(kf)$$

$$\underbrace{U^{n+1} - U^n}_{\sum \alpha_j U^{n+j}} = \Psi$$

$$\rho(\zeta) = \zeta - 1 \Rightarrow \zeta = 1.$$

All one-step methods are zero-stable.