Spectral methods

Example: Advection-diffusion

$$u_t + a u_x = \varepsilon u_{xx}$$

$$u(x, t=0) = u_0(x)$$

Ansatz: $u(x,t) = \hat{u}(t) e^{i\xi x}$

$$\hat{u}'(t) e^{i\xi x} = -ia\xi \hat{u}(t) e^{i\xi x} - \varepsilon \xi^2 \hat{u}(t) e^{i\xi x}$$

$$\hat{u}'(t) = -(ia\xi + \varepsilon\xi^2)\hat{u}(t)$$

$$\underbrace{\hat{u}(t)}_{\text{FT of solution } u(x,t)} = \exp(-(ia\xi - \varepsilon\xi^2)t)\underbrace{\hat{u}(0)}_{\text{Fourier transform of } u_0(x)}$$

Solution of any linear PDE (first order in time) (Periodic BCs)

$$\begin{tikzcd}
u(x,t=0) \arrow[d, "FT"'] & u(x,t) \\
\hat{u}(\xi, t=0) \arrow[r, "\text{solve linear ODEs}"'] & \hat{u}(\xi, t) \arrow[u, "\text{Inverse FT}"']
\end{tikzcd}$$

$U(t)$: discretized in x

F, F^{-1}: discrete FT and inverse FT

$$\vec{\xi} = \begin{bmatrix} \xi_1 \\ \xi_2 \\ \vdots \\ \xi_m \end{bmatrix}$$ wavenumber grid

$$D = \begin{bmatrix} i\xi_1 & & & \\ & i\xi_2 & & \\ & & \ddots & \\ & & & i\xi_m \end{bmatrix}$$ (differentiation matrix)

Semi-discrete spectral solution:

$$U'(t) = F^{-1}(-aD + \varepsilon D^2) F U(0)$$

(for $u_t = -a u_x + \varepsilon u_{xx}$)

$$U(t) = \exp\left(t F^{-1}(-aD + \varepsilon D^2) F\right) U(0)$$

$$\exp(F^{-1} M F) = I + F^{-1} M F + \frac{1}{2}(F^{-1} M F)^2 + \cdots$$

$$\frac{1}{2} F^{-1} M \underline{F F^{-1}} M F = \frac{1}{2} F^{-1} M^2 F$$

$$\exp(F^{-1} M F) = F^{-1} \exp(M) F$$

$$\to U(t) = F^{-1}\exp(t(-aD+\varepsilon^2 D))\underbrace{FU(0)}$$

Should take $\mathcal{O}(m^2)$ operations but can be done in $\mathcal{O}(m\log m)$ (FFT)

Pseudospectral Methods

$$u_t + a(x)u_x = 0$$

The FT leads to a convolution of $a(x)$ with u.
(ODEs are fully coupled)

Idea: Use $\hat{u}(\xi,t)$ to compute x-derivatives but use $u(x,t)$ to compute products.
Use method of lines.

$$u_x \approx F^{-1}DFU$$

So we get

$$U'(t) = AF^{-1}DFU$$

where $A = \begin{bmatrix} a(x_1) & & \\ & \ddots & \\ & & a(x_m) \end{bmatrix}$

Now discretize in time e.g. with RK.

We can do the same for nonlinear problems:

$$u_t + u u_x + u_{xxx} = 0$$

(Korteweg-de Vries) (KdV)

$$U'(t) = -\mathrm{diag}(U) F^{-1} D F U - F^{-1} D^3 F U$$

(Pseudo-) Spectral methods:

- Useful in simple geometries (rectangles, circles, etc.)
- Fourier methods: periodic BCs
- Chebyshev methods: Dirichlet BCs
- Very fast
- Very accurate: as Δx decreases, error goes to zero faster than any power of Δx.

Aliasing instability

Two cures:

- Fully resolve the solution (expensive)

- Filtering: remove/reduce high-wavenumber energy.

Soliton