Quick Review

BVP: $u''(x) = f(x)$

$0 < x < 1 \quad u(0) = \alpha$
$\qquad\qquad\quad u(1) = \beta$

$$\frac{U_{j+1} - 2U_j + U_{j-1}}{h^2} = f(x_j)$$

$$AU = F$$

$$AE = -\tau \Rightarrow \|E\|_2 \leq \|A^{-1}\|_2 \|\tau\|_2$$

We bounded E by showing that

$$\|A^{-1}\|_2 < C$$

Computing the eigenvalues

$$\hat{A} = h^2 A = \begin{bmatrix} -2 & 1 & & & \\ 1 & \ddots & \ddots & & \\ & \ddots & \ddots & \ddots & \\ & & \ddots & \ddots & 1 \\ & & & 1 & -2 \end{bmatrix}$$

$$Av = \lambda v$$

$$v_{j+1} - 2v_j + v_{j-1} = \lambda v_j \quad j = 1, 2, \ldots, m$$

$$v_0 = 0 \quad v_{m+1} = 0$$

Ansatz: $V_j = \xi^j \quad \xi \in \mathbb{C}$

$$\xi^{j+1} - (2+\lambda)\xi^j + \xi^{j-1} = 0$$

$$\xi^2 - (2+\lambda)\xi + 1 = 0$$

$$\left.\xi_\pm = \frac{2+\lambda}{2} \pm \frac{\sqrt{\lambda^2+4\lambda}}{2}\right\}$$ Fundamental solutions

Look for a linear combination:

$$V_j = a\xi_+^j + b\xi_-^j \Rightarrow V_j = a(\xi_+^j - \xi_-^j)$$

$V_0 = a + b = 0$
$\Rightarrow b = -a$

$$V_{m+1} = a(\xi_+^{m+1} - \xi_-^{m+1}) = 0$$

$$\xi_+^{m+1} = \xi_-^{m+1}$$

$$\xi_+^{m+1}\xi_+^{m+1} = \xi_-^{m+1}\xi_+^{m+1} = 1 \Rightarrow \xi_-\xi_+ = 1$$

$$\xi_-\xi_+ = \frac{1}{4}\left(2+\lambda-\sqrt{\lambda^2+4\lambda}\right)\left(2+\lambda+\sqrt{\lambda^2+4\lambda}\right)$$

$$= \frac{1}{4}\left((2+\lambda)^2 - (\lambda^2+4\lambda)\right)$$

$$= \frac{1}{4}\left(4 + \cancel{4\lambda} + \cancel{\lambda^2} - \cancel{\lambda^2} - \cancel{4\lambda}\right) = 1$$

$$\xi_+^{2m+2} = 1 \Rightarrow \xi_+ = e^{\frac{\pi i}{m+1}p} \qquad p = 1, 2, \dots, m$$

$$\xi_- = \xi_+^{-1} = e^{-\frac{\pi i}{m+1}p}$$

$$\xi_+ + \xi_- = 2 + \lambda = e^{\frac{\pi i p}{m+1}} + e^{-\frac{\pi i p}{m+1}}$$

$$= 2\cos\left(\frac{p\pi}{m+1}\right)$$

$$2+\lambda = 2\cos\left(\frac{p\pi}{m+1}\right)$$

$$\lambda = 2\left(\cos\left(\frac{p\pi}{m+1}\right)-1\right)$$

Eigenvalues of A:

$$\lambda = \frac{2}{h^2}\left(\cos\left(\frac{p\pi}{m+1}\right)-1\right)$$

$$p = 1, 2, \ldots, m$$

Stability in the max norm

$$\|E\|_\infty \le \|A^{-1}\|_\infty \|\tau\|_\infty$$

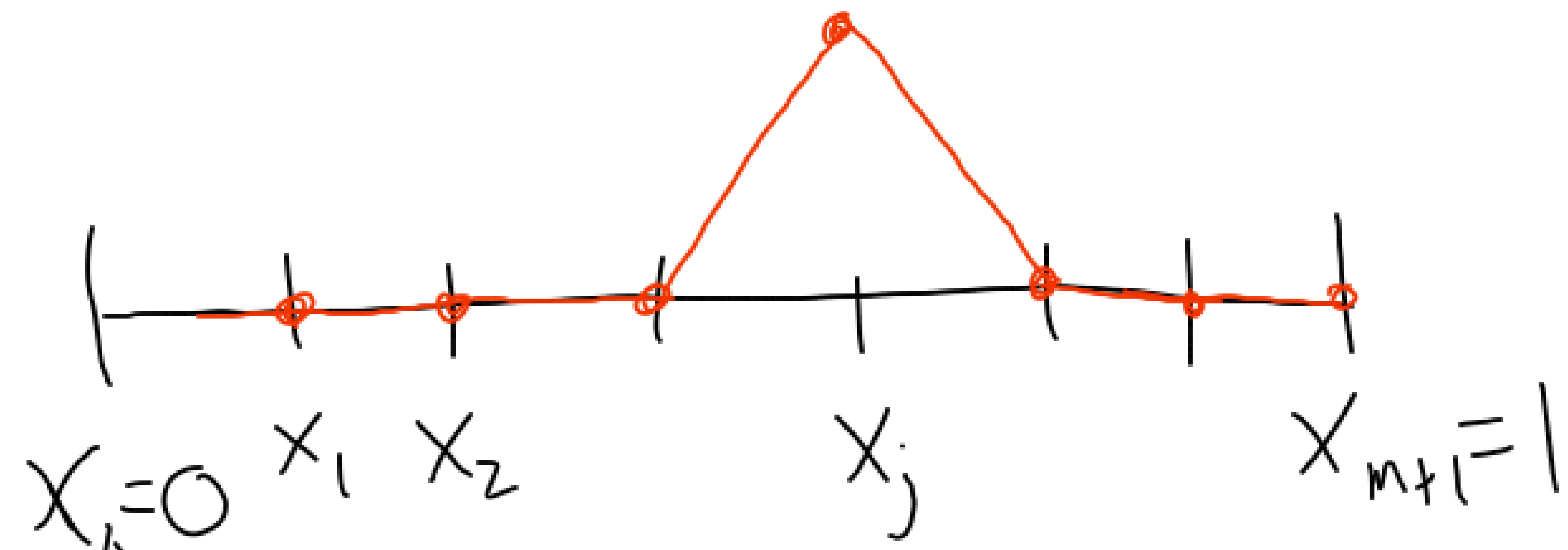
Need to show that $\|A^{-1}\|_\infty < C$

$$\underbrace{\frac{1}{h^2}\begin{bmatrix} h^2 & & & & & \\ 1 & -2 & 1 & & & \\ & \ddots & \ddots & \ddots & & \\ & & \ddots & \ddots & \ddots & \\ & & & 1 & -2 & 1 \\ & & & & & h^2 \end{bmatrix}}_{A^{(m+2)\times(m+2)}} \underbrace{\begin{bmatrix} U_0 \\ U_1 \\ \vdots \\ \vdots \\ U_m \\ U_{m+1} \end{bmatrix}}_{U} = \underbrace{\begin{bmatrix} \alpha \\ f(x_1) \\ \vdots \\ \vdots \\ f(x_m) \\ \beta \end{bmatrix}}_{F}$$

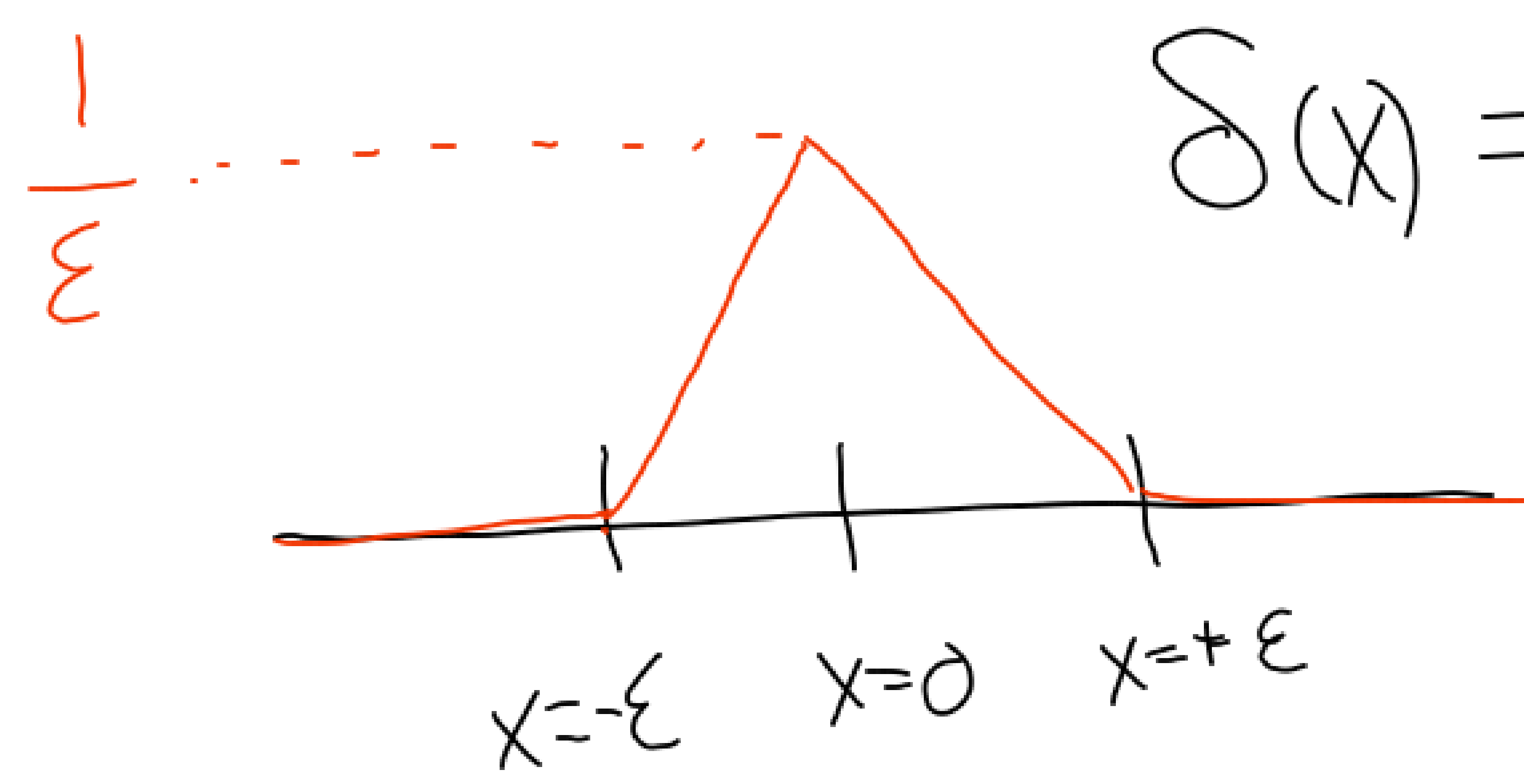
$$B = A^{-1}$$

$$U = BF \iff U = \sum_{j=0}^{m+1} B_j F_j$$

$$B = \left[\begin{array}{c|c|c|c|c} B_0 & B_1 & \cdots & B_m & B_{m+1} \end{array} \right]$$



Dirac-delta (δ) function



$$\delta(x) = \begin{cases} \dfrac{\varepsilon + x}{\varepsilon^2} & -\varepsilon \le x \le 0 \\ \dfrac{\varepsilon - x}{\varepsilon^2} & 0 \le x \le \varepsilon \end{cases}$$

$$\int_{-\infty}^{\infty} \delta(x)\,dx = \frac{2\varepsilon}{2\varepsilon} = 1$$

Suppose: $\alpha = \beta = 0$

and $f(x_i) = \begin{cases} 0 & i \ne j \\ 1 & i = j \end{cases}$

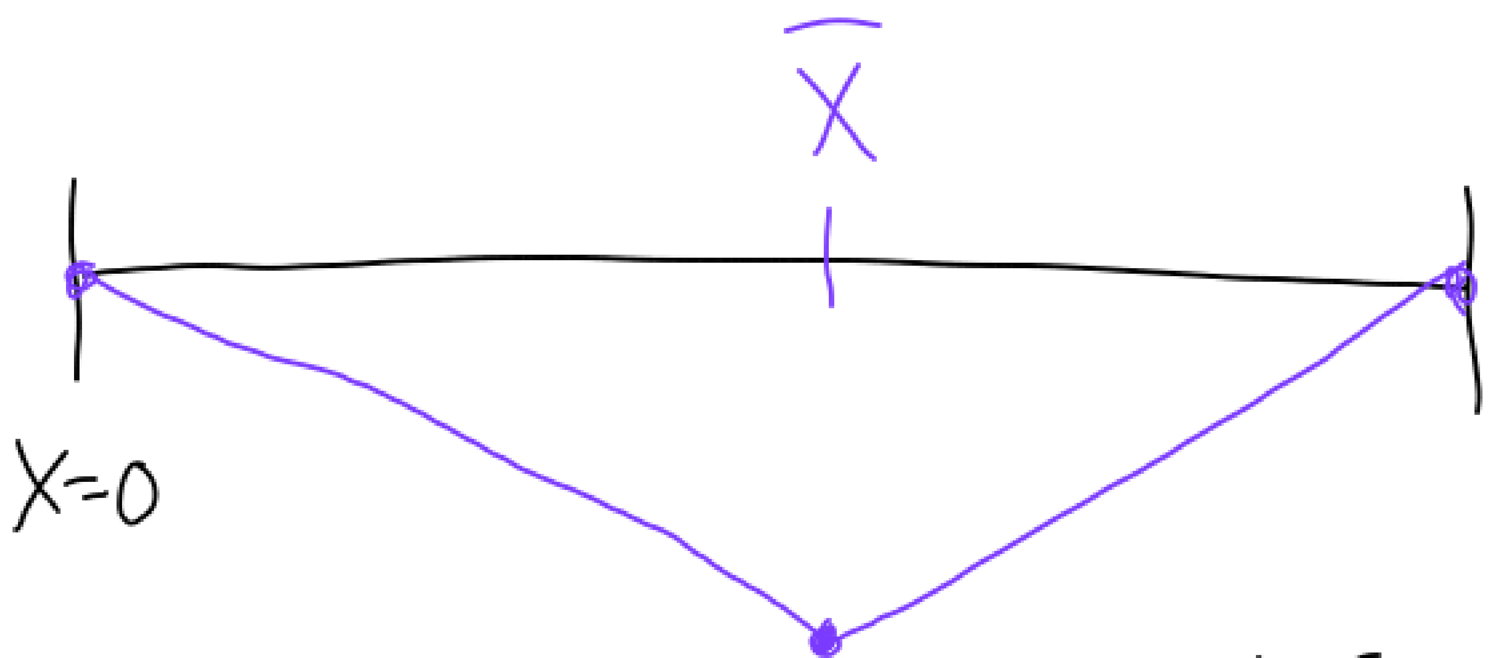
then $U = B_j$

Consider the BVP

$$u''(x) = \delta(x-\bar{x})$$

$$u(0) = u(1) = 0$$

$\Rightarrow u(x)$ must be linear except at $x = \bar{x}$.



$$u'(\bar{x}+\varepsilon) - u'(\bar{x}-\varepsilon) = \int_{x-\varepsilon}^{x+\varepsilon} u''(x)\,dx$$

$$= \int_{x-\varepsilon}^{x+\varepsilon} \delta(x-\bar{x})\,dx = 1$$

Let

$$u(x) = \begin{cases} a_1 x & x < \bar{x} \\ a_2(x-1) & x > \bar{x} \end{cases}$$

Continuity: $a_1\bar{x} = a_2(\bar{x}-1)$

$$a_2 - a_1 = 1 \Rightarrow a_2 = 1 + a_1$$

$$a_1\bar{x} = (1+a_1)(\bar{x}-1)$$

$$\cancel{a_1\bar{x}} = \bar{x} - 1 + \cancel{a_1\bar{x}} - a_1$$

$$a_1 = \bar{x} - 1 \qquad a_2 = \bar{x}$$

$$u(x) = \left.\begin{cases} (\bar{x}-1)x & x < \bar{x} \\ \bar{x}(x-1) & x > \bar{x} \end{cases}\right\} G(x;\bar{x})$$

Green's function

Any function $f(x)$ can be written as

$$f(x) = \int f(\bar{x})\delta(x-\bar{x})\,d\bar{x}$$

So the general solution to

$$u''(x) = f(x) \quad u(0)=u(1)=0$$

is

$$u(x) = \int_0^1 f(\bar{x})G(x;\bar{x})\,d\bar{x}$$

All of this suggests that the columns of B should approximate G:

$$B_j \sim hG(x;x_j)$$

in fact $B_{ij} = hG(x_i;x_j) \quad 1 \leq j \leq m$

What about B_0, B_{m+1}?

Consider

$$u''(x)=0 \quad u(0)=0$$
$$u(1)=1 \qquad B_{i,m+1} = x$$
$$u(x) = x$$

$$u''(x)=0 \quad u(1)=0$$
$$u(0)=1 \qquad B_{i,0} = 1-x_i$$
$$u(x) = 1-x$$

∞-norm of B:

$$\|B\|_\infty = \max_{0\leq i\leq m+1} \sum_{j=0}^{m+1} |B_{ij}| \leq 1+1+mh$$
$$\leq 2+\frac{m}{m+1} < 3$$

$$\frac{1}{h^2}\left(\begin{bmatrix} 0 & & & & \\ & -2 & & & \\ & & \ddots & & \\ & & & -2 & \\ & & & & 0 \end{bmatrix} + \begin{bmatrix} 0 & & & \\ 1 & \ddots & & \\ & \ddots & \ddots & \\ & & 1 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 1 & & \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ & & & 0 \end{bmatrix} \right)$$

$$+ \begin{bmatrix} 1 & & & & \\ & 0 & & & \\ & & \ddots & & \\ & & & 0 & \\ & & & & 1 \end{bmatrix}$$

$$\frac{1}{h^2} = (m+1)^2$$