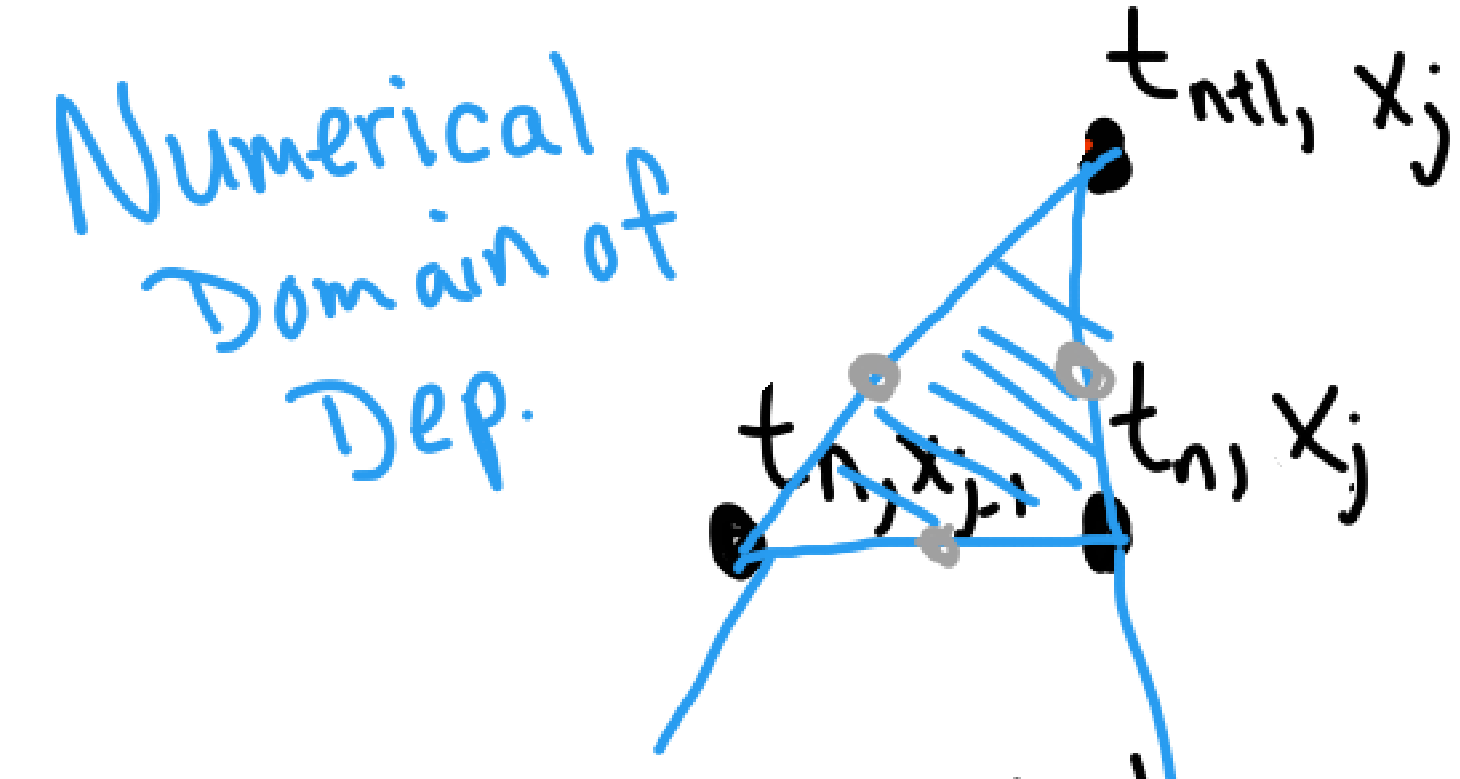
- Progress reports due Saturday
- Homework 6 now on Github
  due next Saturday

The Upwind Method

$$u_t + a u_x = 0 \qquad a > 0$$

$$\frac{U_j^{n+1} - U_j^n}{k} = -\frac{a}{h}\left(U_j^n - U_{j-1}^n\right)$$

1st-order in time and space.



CFL condition:

Characteristic speed: a

NDoD "speed": $\frac{h}{k}$

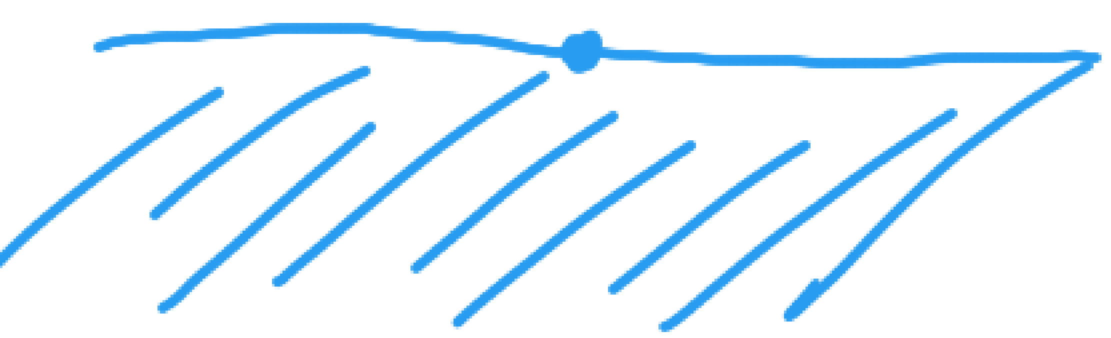
CFL: $a \le \frac{h}{k}$

$$\nu = \frac{ka}{h} \le 1$$

Parabolic vs. Hyperbolic PDEs

$$u_t = u_{xx}$$

Infinite domain of dependence



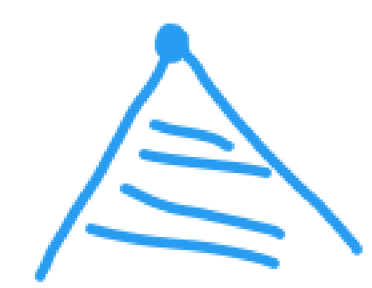
$$D(x,t) = \{(x_*, t_*) : t_* < t\}$$

CFL says:

- Use implicit methods
- Use global stencil (spectral)
- reduce k faster than h, e.g. $k \le \alpha h^2$

$$u_t + a u_x = 0$$

Finite domain of dependence



$$u_t + a u_x = u_{xx}$$

Advection-diffusion equation

Kuramoto-Sivashinsky equation

$$u_t + u u_x + u_{xx} + u_{xxxx} = 0$$

- $u u_x$: steepening (energy moves to higher wavenumbers)
- u_{xx}: exp. growth $e^{\xi^2 t}$
- u_{xxxx}: exp decay $e^{-\xi^4 t}$

Burgers:

Stability of the Upwind Method

$u_t + a u_x = 0 \quad 0<x<1$

$a>0$

$u(x,0)=\eta(x)$

$u(0,t)=0$

$$U^{n+1} = U^n - \underbrace{\frac{ka}{h}}_{\nu} \begin{bmatrix} 1 & & & \\ -1 & \ddots & & \\ & \ddots & \ddots & \\ & & -1 & 1 \end{bmatrix} U^n$$

$$U^{n+1} = \underbrace{\begin{bmatrix} 1-\nu & & & \\ \nu & 1-\nu & & \\ & \ddots & \ddots & \\ & & \nu & 1-\nu \end{bmatrix}}_{M} U^n$$

Toeplitz matrix

Highly non-normal

$K(R) \gg 1.$

$E^{n+1} = M E^n - k \tau^n$

Eigenvalues of M: $1-\nu$

We need $|1-\nu| \le 1$

$$-1 \le 1-\nu \le 1$$

$$-2 \le -\nu \le 0$$

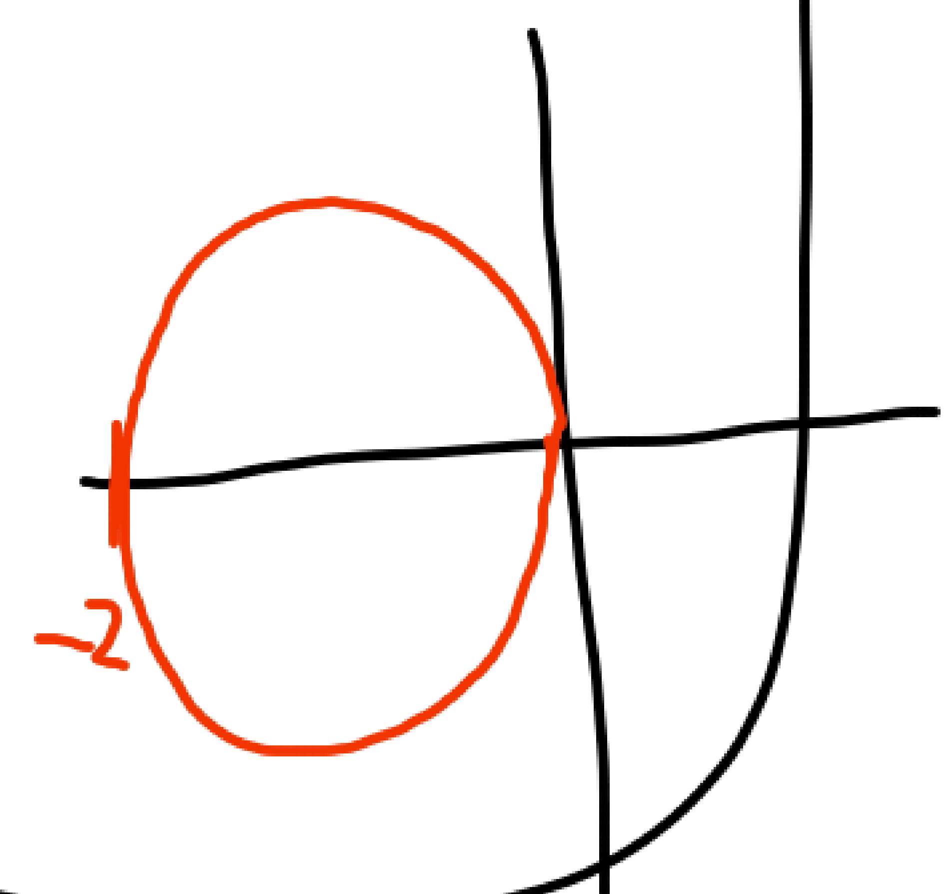
$$0 \le \nu \le 2$$

Method of Lines Analysis

$$U_j'(t) = \frac{-a}{h}\left(U_j - U_{j-1}\right)$$

$$U'(t) = -\frac{a}{h}\begin{bmatrix} 1 & & & \\ -1 & 1 & & \\ & \ddots & \ddots & \\ & & -1 & 1 \end{bmatrix} U(t)$$

Eigenvalues: $-\frac{a}{h}$



$$-2 \le -\frac{ka}{h} \le 0$$

$$\Rightarrow 0 \le \nu \le 2$$

Weaker than CFL!

Von Neumann Analysis

$$u_t + a u_x = 0 \qquad 0<x<1$$
$$a>0$$
$$u(x,0)=\eta(x)$$
$$u(0,t)=u(1,t)$$

$$U_j^{n+1} = U_j^n - \nu\left(U_j^n - U_{j-1}^n\right)$$

$$U_j^n = g^n e^{ijh\xi} \qquad U_1^{n+1} = U_1^n - \nu\left(U_1^n - U_m^n\right)$$

$$g = 1-\nu\left(1-e^{-ih\xi}\right)$$

$$g = 1-\nu+\nu\left(\cos(h\xi) - i\sin(h\xi)\right)$$

$$g = 1-\nu+\nu\cos(h\xi) - i\nu\sin(h\xi)$$

$$|g|^2 = \left(1-\nu+\nu\cos(h\xi)\right)^2 + \nu^2\sin^2(h\xi)$$

$$= 1+\nu^2+\underline{\nu^2\cos^2(h\xi)} - 2\nu + 2\nu\cos(h\xi)$$
$$-2\nu^2\cos(h\xi) + \underline{\nu^2\sin^2(h\xi)}$$

$$|g|^2 = 1+2\nu^2+2\nu\left(\cos(h\xi)-1-\nu\cos(h\xi)\right)$$

if $\cos(h\xi)=1$: $|g|^2=1$

if $\cos(h\xi)=-1$: $|g|^2 = 1+2\nu^2+2\nu(\nu-2)$

$$= 4\nu^2-4\nu+1$$

$$|g|^2 = (2\nu-1)^2$$

$|g|^2 \le 1 \iff (2\nu-1)^2 \le 1$

$-1 \le 2\nu - 1 \le 1$

$0 \le 2\nu \le 2$

$0 \le \nu \le 1$

Agrees with CFL!

(this is sufficient for $|g|^2 \le 1$ for all $h\xi$)

Here

$$M = \begin{bmatrix} 1-\nu & & & \nu \\ \nu & \ddots & & \\ & \ddots & \ddots & \\ & & \nu & 1-\nu \end{bmatrix}$$

Circulant (normal)

For stability of

$U^{n+1} = MU^n$

we need $\|M\|_2 \le 1$.

Is this equivalent to

$\rho(M) = \max_{\lambda \in \sigma(M)} |\lambda| \le 1$?

(spectral radius)

Not in general.

Let $M = R\Lambda R^{-1}$ (M is diagonalizable)

$$\|M\|_2 \le \|R\|_2 \|\Lambda\|_2 \|R^{-1}\|_2$$
$$= \underbrace{\|R\|_2 \|R^{-1}\|_2}_{\kappa(R)} \rho(M)$$

If the eigenvectors of M are not orthogonal, $\kappa(R) > 1$.

I.e. $\|M\|_2 = \rho(M)$ iff M is unitarily diagonalizable.

i.e. M is normal.

Thm. M is normal iff $M^T M = M M^T$.

Some subclasses of normal matrices:

- Unitary
- Symmetric
- Skew-symmetric
- Circulant

Toeplitz matrices:

$$\begin{bmatrix} a & b & c & \cdots & \cdots \\ z & a & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & c \\ \vdots & & \ddots & \ddots & b \\ \vdots & & & z & a \end{bmatrix}$$

Circulant matrices:

$$\begin{bmatrix} a & b & c & \cdots & \\ & \ddots & \ddots & \ddots & c \\ & & \ddots & \ddots & b \\ c & & & & a \\ b & c & & & \end{bmatrix}$$

For non-normal matrices:

- Eigenvalues are very sensitive to perturbations
- So eigenvalues don't characterize the behavior of the matrix.

For example: we can have $|\lambda| < 1$ for all $\lambda \in \sigma(M)$

but $\|M^n\|$ can grow with n.

We must still have $\lim_{n \to \infty} \|M^n\| = 0$.

Embree + Trefethen

Spectra and Pseudospectra