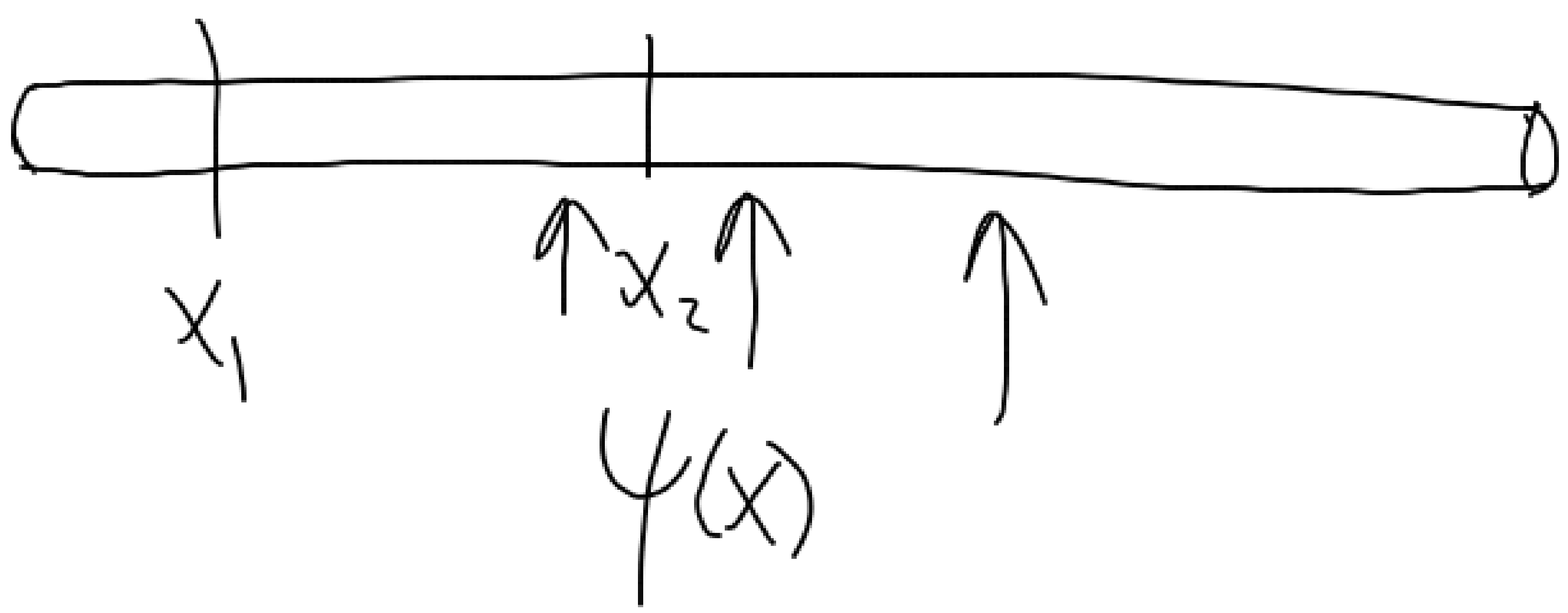
Boundary Value Problems



Heat flow in a thin rod

$$\frac{d}{dt}\int_{x_1}^{x_2} u(x,t)\,dx = \int_{x_1}^{x_2}\psi(x)\,dx + f(x_1) - f(x_2)$$

$$\int_{x_1}^{x_2}\frac{\partial}{\partial t}u(x,t)\,dx = \int_{x_1}^{x_2}\psi(x)\,dx - \int_{x_1}^{x_2}\frac{\partial}{\partial x}f(x)\,dx$$

Fick's Law of Diffusion: $f = -K u_x$

heat conductivity

$$\int_{x_1}^{x_2}\underbrace{\left[u_t - K u_{xx} - \psi(x)\right]}_{\text{must vanish } \forall x}\,dx = 0$$

Conservation Law

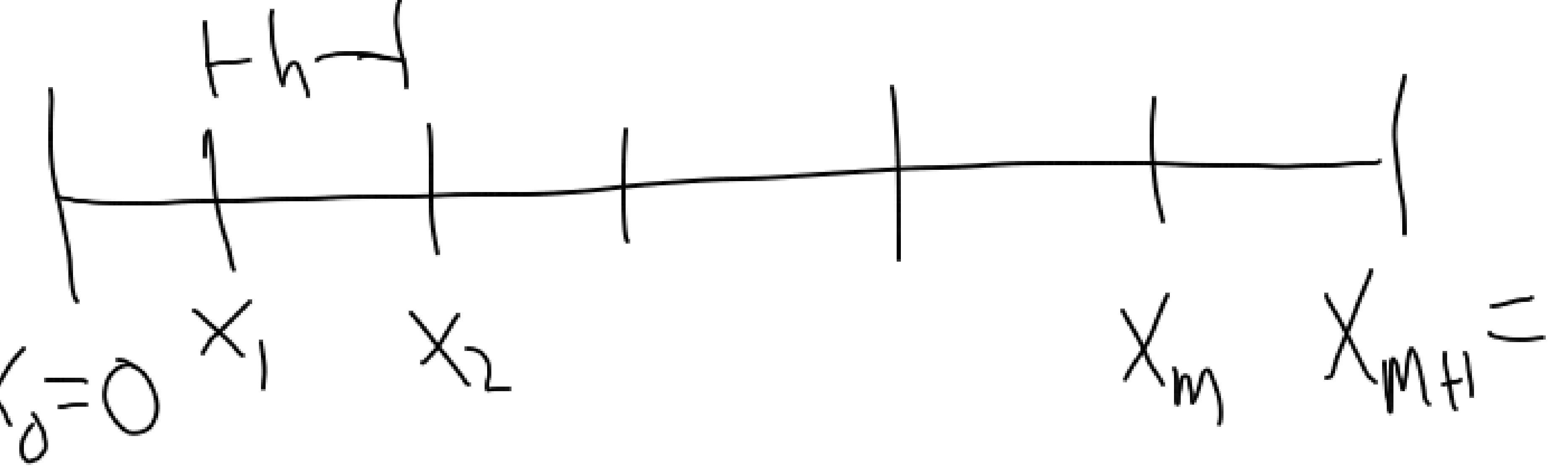
$u_t = K u_{xx} + \psi(x)$ Heat equation

As $t \to \infty$, $u_t \to 0$

$K u_{xx} = -\psi(x)$

$$\boxed{\begin{array}{l} u_{xx} = \dfrac{-\psi(x)}{K} = f(x) \quad \text{Poisson's equation} \\ 0 < x < 1 \\ u(0) = \alpha \quad u(1) = \beta \end{array}}$$

Discretization



$h = \dfrac{1}{m+1}$ $\quad x_j = jh$

$$U = \begin{bmatrix} U_1 \\ U_2 \\ \vdots \\ U_m \end{bmatrix}$$

$U_0 = \alpha$

$U_{m+1} = \beta$

$U_j \approx u(x_j)$

We'll use $U_0 = \alpha \quad U_{m+1} = \beta$ $\qquad AU = F$

$$u''(x_j) \approx \frac{U_{j+1} - 2U_j + U_{j-1}}{h^2} = f(x_j) \quad j = 1, 2, \ldots, m$$

Centered FD approximation
2nd-order accurate

$$\underbrace{\frac{1}{h^2}\begin{bmatrix} -2 & 1 & & & \\ 1 & -2 & 1 & & 0 \\ & \ddots & \ddots & \ddots & \\ & & \ddots & \ddots & \ddots \\ 0 & & & \ddots & 1 \\ & & & 1 & -2 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} U_1 \\ U_2 \\ \vdots \\ \vdots \\ U_m \end{bmatrix}}_{U} = \overset{F}{\begin{bmatrix} f(x_1) - \frac{\alpha}{h^2} \\ f(x_2) \\ \vdots \\ \vdots \\ f(x_m) - \frac{\beta}{h^2} \end{bmatrix}}$$

$$\frac{U_2 - 2U_1 + \alpha}{h^2} = f(x_1)$$

$$\frac{U_2 - 2U_1}{h^2} = f(x_1) - \frac{\alpha}{h^2}$$

$$\frac{\beta - 2U_m + U_{m-1}}{h^2} = f(x_m)$$

How accurate is our solution?

$$\hat{U} = \begin{bmatrix} u(x_1) \\ u(x_2) \\ \vdots \\ \vdots \\ u(x_m) \end{bmatrix}$$

Global error

$$E = U - \hat{U}$$

We would like

$$\|\hat{U}\| \to \|u\| \text{ as } m \to \infty$$

Norms

Vector norms:

$$\|v\|_1 = \sum_j |v_j|$$

$$\|v\|_2 = \left(\sum_j |v_j|^2\right)^{1/2}$$

$$\|v\|_\infty = \max_j |v_j|$$

Function norms

$w(x)$

$$\|w\|_1 = \int |w(x)| dx$$

$$\|w\|_2 = \left(\int |w(x)|^2 dx\right)^{1/2}$$

$$\|w\|_\infty = \max_x |w(x)|$$

Grid-function norms

$$\|U\|_1 = h \sum_j |U_j|$$

$$\|U\|_2 = \left(h \sum_j |U_j|^2\right)^{1/2}$$

$$\|U\|_\infty = \max_j |U_j|$$

We want to bound $\|E\|$.

How?

Substitute: $U_j \to u(x_j)$

$$\frac{U_{j+1}-2U_j+U_{j-1}}{h^2} = f(x_j)$$

$$\downarrow$$

$$\frac{u(x_{j+1})-2u(x_j)+u(x_{j-1})}{h^2} = f(x_j) + \underbrace{\tau_j}_{\text{Local truncation error}}$$

We can show that

$$\frac{u(x_{j+1})-2u(x_j)+u(x_{j-1})}{h^2} = u''(x_j) + \frac{1}{12}h^2 u^{(4)}(x_j) + \mathcal{O}(h^4)$$

$$\cancel{u''(x_j)} + \frac{1}{12}h^2 u^{(4)}(x_j) + \mathcal{O}(h^4) = \cancel{f(x_j)} + \tau_j$$

So $\tau_j = \frac{1}{12}h^2 u^{(4)}(x_j) + \mathcal{O}(h^4)$

$$A\hat{U} = F + \tau \qquad \tau = \begin{bmatrix} \tau_1 \\ \vdots \\ \vdots \\ \tau_m \end{bmatrix}$$

$$AU = F$$

$$A(U-\hat{U}) = -\tau$$

$$AE = -\tau \qquad \|\tau\| = \mathcal{O}(h^2)$$

$$E = -A^{-1}\tau$$

$$\|E\| = \|A^{-1}\tau\| \le \|A^{-1}\| \, \|\tau\|$$

Consistency

We say a discretization is consistent if

$$\lim_{h\to 0} \|\tau\| = 0.$$

Convergence

We say a discretization is convergent if

$$\lim_{h\to 0} \|E\| \to 0.$$

We have

$$\|E\| \le \|A^{-1}\| \, \underbrace{\|\tau\|}_{\text{Vanishes as } h\to 0}$$

So we can prove convergence if

$$\underbrace{\|A^{-1}\| < C \text{ as } h\to 0}_{\text{Stability}}$$

Convergence = consistency + stability

Fundamental Theorem of Numerical Analysis

2-Norm Convergence

We need to show that $\|A^{-1}\|_2 < C$ as $h \to 0$.

$$\|A\|_2 = \underbrace{\max_{1 \le j \le m} |\lambda_j|}_{\rho(A)} \quad \lambda_j \in \sigma(A)$$

$\rho(A)$ spectral radius

What are the eigenvalues of A^{-1}?

$$Av = \lambda v \Rightarrow v = \lambda A^{-1} v$$

$$A^{-1} v = \frac{1}{\lambda} v$$

We need to show that the eigenvalues of A are bounded away from zero as $h \to 0$.

$$\lambda_p = \frac{2}{h^2}(\cos(p\pi h) - 1) \quad p = 1, \dots, m$$

$$\cos(x) = 1 - \frac{x^2}{2} + \frac{x^4}{4!} \cdots$$

$$\cos(p\pi h) = 1 - \frac{p^2\pi^2 h^2}{2} + \mathcal{O}(h^4)$$

$$\lambda_p \approx \frac{2}{h^2}\left(-\frac{p^2\pi^2 h^2}{2}\right) = -p^2\pi^2$$

$$\lambda_p < -\pi^2 \quad \forall\, p, h \qquad \text{So } \|A^{-1}\|_2 < \frac{1}{\pi^2}$$

So $\|E\|_2 \leq \frac{1}{\pi^2} O(h^2)$

$$\lim_{h \to 0} \|E\|_2 = 0$$