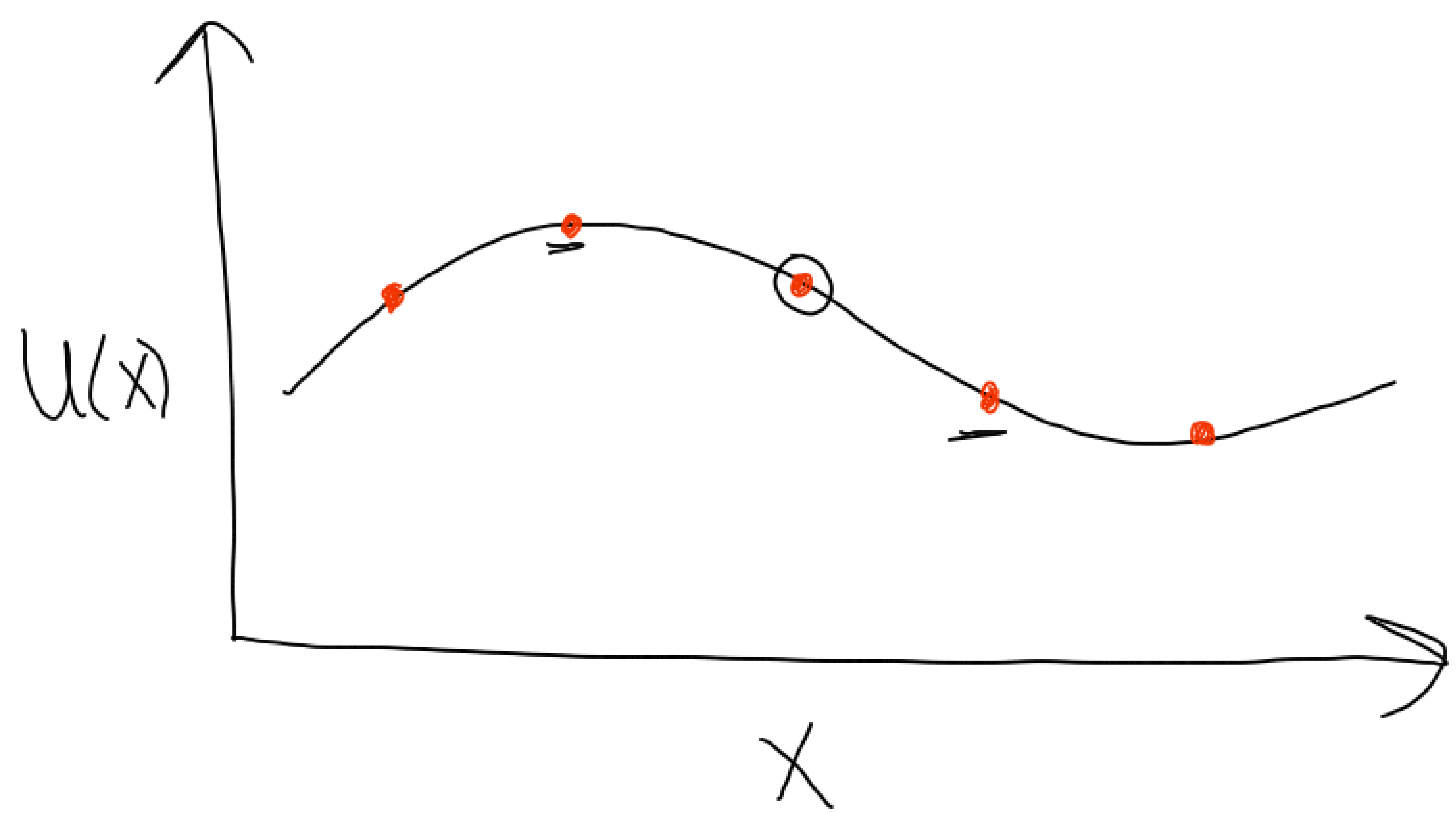
Finite difference Approximation



How to approximate the derivative given a finite set of values?

Dfn. of the derivative:

$$u'(x) = \lim_{h \to 0} \frac{u(x+h) - u(x)}{h}$$

This suggests:

$$u'(\bar{x}) \approx D_+(u) = \frac{u(\bar{x}+h) - u(\bar{x})}{h}$$

or $$D_-(u) = \frac{u(\bar{x}) - u(\bar{x}-h)}{h}$$

$$D_0(u) = \frac{u(\bar{x}+h) - u(\bar{x}-h)}{2h}$$

Taylor's Theorem says:

$$u(x) = \sum_{j=0}^{\infty} u^{(j)}(\bar{x}) \frac{(x-\bar{x})^j}{j!}$$

$$= \sum_{j=0}^{p} u^{(j)}(\bar{x}) \frac{(x-\bar{x})^j}{j!} + \overbrace{\mathcal{O}\left((x-\bar{x})^{p+1}\right)}^{\text{Truncation error}}$$

$f(h) = \mathcal{O}(h^p)$ means $\exists\, C>0,\ h_0>0$

such that

$$f(h) < Ch^p \quad \forall\, h<h_0.$$

$$u(x+h) = u(x) + hu'(x) + \frac{h^2}{2}u''(x) + \frac{h^3}{6}u'''(x) + O(h^4)$$

$$u(x-h) = u(x) - hu'(x) + \frac{h^2}{2}u''(x) - \frac{h^3}{6}u'''(x) + O(h^4)$$

$$u(x) = u(x)$$

$$D_+(u(x)) = \frac{u(x+h) - u(x)}{h} = u'(x) + \frac{h}{2}u''(x) + \frac{h^2}{6}u'''(x) + O(h^3)$$

First-order accurate (error is $O(h)$)

$$D_-(u(x)) = u'(x) - \frac{h}{2}u''(x) + \frac{h^2}{6}u'''(x) + O(h^3)$$ Again first-order

$$D_0(u(x)) = \frac{u(x+h) - u(x-h)}{2h} = u'(x) + \frac{h^2}{6}u'''(x) + O(h^4)$$

2nd-order accurate

Deriving finite diff formulas

Suppose we know $u(x)$, $u(x+h)$, $u(x+2h)$ and we want to approximate $u''(x)$.

$$u''(x) \approx a u(x) + b u(x+h) + c u(x+2h)$$

We need

$$u(x+2h) = u(x) + 2hu'(x) + 2h^2 u''(x) + \frac{4}{3}h^3 u'''(x) + O(h^4)$$

so

$$u''(x) \approx au + b\left(u + hu' + \frac{h^2}{2}u'' + \frac{h^3}{6}u'''\right) + c\left(u + 2hu' + 2h^2u'' + \frac{4}{3}h^3u'''\right) + O(h^4)$$

Collecting terms:

$$(a+b+c)u = 0$$

$$(b+2c)hu' = 0$$

$$h^2\left(\frac{1}{2}b + 2c\right)u'' = u'' \Rightarrow h^2\left(\frac{1}{2}b + 2c\right) = 1$$

$$h^3\left(\frac{b}{6} + \frac{4}{3}c\right)u''' = 0$$

$$a + b + c = 0 \Rightarrow a = -b - c$$

$$b + 2c = 0 \Rightarrow b = -2c$$

$$\left(\frac{1}{2}b + 2c\right)h^2 = 1 \Rightarrow ch^2 = 1$$

$$c = \frac{1}{h^2} \quad b = \frac{-2}{h^2} \quad a = \frac{1}{h^2}$$

$$u''(x) \approx \frac{u(x) - 2u(x+h) + u(x+2h)}{h^2}$$

First-order approximation

A general approach to find FD coefficients

Given values $u(x_1), u(x_2), \ldots, u(x_n)$ find a FD formula for $u^{(k)}(\bar{x})$ (as accurate as possible).

Taylor series:

$$u(x_i) = u(\bar{x}) + (x_i - \bar{x})u'(\bar{x}) + \frac{(x_i-\bar{x})^2}{2}u''(\bar{x}) + \cdots$$

$$= \sum_{j=0}^{\infty} \frac{(x_i-\bar{x})^j}{j!} u^{(j)}(\bar{x})$$

We want a formular of the form

$$u^{(k)}(\bar{x}) = \sum_{i=1}^{n} c_i u(x_i) + O(h^p)$$

Where $h = \max_j |x_j - \bar{x}|$.

We have

$$u^{(k)}(\bar{x}) = \sum_{i=1}^{n} c_i \sum_{j=0}^{\infty} \frac{(x_i-\bar{x})^j}{j!} u^{(j)}(\bar{x})$$

We get

$$\sum_{i=1}^{n} c_i u(\bar{x}) = 0 \Rightarrow \sum_i c_i = 0$$

$$\sum_{i=1}^{n} c_i (x_i - \bar{x}) u'(\bar{x}) = 0 \Rightarrow \sum_i (x_i - \bar{x}) c_i = 0$$

$$
\begin{bmatrix}
1 & 1 & \cdots & 1 \\
x_1-\bar{x} & x_2-\bar{x} & \cdots & x_n-\bar{x} \\
\frac{(x_1-\bar{x})^2}{2} & \cdots & \cdots & \frac{(x_n-\bar{x})^2}{2} \\
\vdots & & & \\
\frac{(x_1-\bar{x})^{n-1}}{(n-1)!} & \cdots & \cdots & \frac{(x_n-\bar{x})^{n-1}}{(n-1)!}
\end{bmatrix}
\begin{bmatrix}
c_1 \\ c_2 \\ \vdots \\ c_n
\end{bmatrix}
=
\begin{array}{c}
\begin{bmatrix}
0 \\ \vdots \\ 0 \\ 1 \\ 0 \\ \vdots \\ 0
\end{bmatrix}
\end{array}
\begin{array}{l}
\\ \\ \\ \leftarrow k+1 \\ \text{entry} \\ \\ \\
\end{array}
$$

Vandermonde

Non-singular if $x_i \neq x_j$
$\forall\ i,j$.

We get a solution if $n > k$.

Order of approximation: $\mathcal{O}(h^{n-k})$.

Homework: finite difference notebook
bold questions

Due one week from Thursday.