Stiffness

Course project proposals:
Send to me by e-mail
by midnight today.

The step size K can be restricted in 2 ways:

1. $K \le K_{acc}$ (maximum K that satisfies error bound)

2. $K \le K_{stab}$ (maximum absolutely stable step size)

With explicit methods, usually

$$K_{stab} \gtrsim \frac{1}{L}$$

L = Lipschitz constant

If $L \gg 1$, then K_{stab} will be very small for any explicit method.

Prothero-Robinson Problem

$$u'(t) = \lambda\left(u(t) - \cos(t)\right) - \sin(t)$$

$$u(0) = 1$$

$$u(t) = \cos(t)$$

With IC: $u(t_0) = \eta$

Solution: $u(t) = e^{\lambda(t-t_0)}\left(\eta - \cos(t_0)\right) + \cos(t)$

We say this problem is stiff:

1. There is a slowly-changing solution, but nearby solutions are rapidly-changing.

2. Implicit methods give an accurate solution with much less work (compared to explicit methods) because $k_{stab} \ll k_{acc}$ for explicit methods.

Chemical Reaction Problem

$$u_1'(t) = -k_1 u_1(t)$$
$$u_2'(t) = k_1 u_1(t) - k_2 u_2(t)$$
$$u_3'(t) = k_2 u_2(t)$$

$$A \xrightarrow{k_1} B \xrightarrow{k_2} C$$

$$u = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} \qquad u'(t) = \begin{bmatrix} -k_1 & 0 & 0 \\ k_1 & -k_2 & 0 \\ 0 & k_2 & 0 \end{bmatrix} u(t)$$

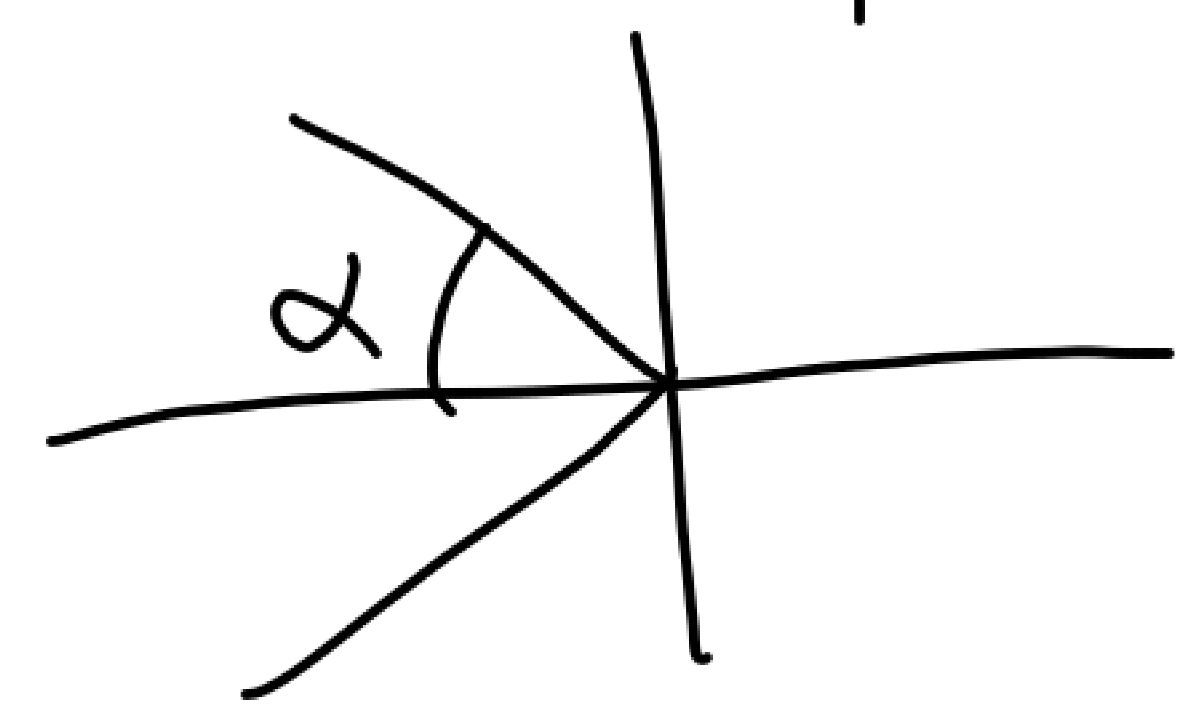
$$\lambda = 0,\ \underline{-k_1},\ -k_2$$

Absolute stability region:

$$S = \{ z\,\mathbb{C} : |R(z)| \le 1 \}$$

A method is A-stable if $\mathbb{C}^- \subset S$

$A(\alpha)$-stability



With a A-stable method,

$K_{stab} = \infty$ (for a stable problem)

So we can take $K \approx K_{acc}$.

L-Stability

For $\lambda \to -\infty$, the solution of $u' = \lambda u$ exhibits arbitrarily fast decay.

Do implicit methods show this behavior?

$$U^{n+1} = R(z) U^n \qquad z = k\lambda$$

So we should have

$$\lim_{z \to -\infty} |R(z)| = 0.$$

Backward Euler: $U^{n+1} = U^n + kf(U^{n+1})$

$$|R(z)| = \left|\frac{1}{1-z}\right| \to 0 \text{ as } z \to -\infty$$

We say this method is L-stable.

Implicit Trapezoidal: $U^{n+1} = U^n + \frac{k}{2}\left(f(U^n) + f(U^{n+1})\right)$

$$|R(z)| = \left|\frac{1+\frac{z}{2}}{1-\frac{z}{2}}\right| \longrightarrow 1$$

as $z \to -\infty$

Not L-stable.

Good methods for stiff problems:

- Diagonally-implicit Runge-Kutta
- Backward Differentiation Formulas