Jacobi's Method and Multigrid

$$u''(x) = f(x) \quad 0 < x < 1$$

$$u(0) = \alpha \quad u(1) = \beta$$

$$\Rightarrow AU = F$$

Jacobi's Method

Let $G = \begin{bmatrix} 0 & 1 & & & \\ 1 & \ddots & \ddots & & \\ & \ddots & \ddots & \ddots & \\ & & \ddots & \ddots & 1 \\ & & & 1 & 0 \end{bmatrix}$

and $A = \frac{1}{h^2}\begin{bmatrix} -2 & 1 & & & \\ 1 & -2 & \ddots & & \\ & \ddots & \ddots & \ddots & \\ & & \ddots & \ddots & 1 \\ & & & 1 & -2 \end{bmatrix}$

$$A = \frac{1}{h^2}(G - 2I)$$

$$AU = F$$

$$\frac{1}{h^2}(G - 2I)U = F$$

$$GU - 2U = h^2 F$$

$$U = \frac{1}{2}(GU - h^2 F) \qquad (*)$$

Jacobi iteration:

① Pick an initial guess $U^{[0]}$

② Repeat: $U^{[k+1]} = \frac{1}{2}(GU^{[k]} - h^2 F)$

U is a fixed point of this iteration.

If we start a $U^{[0]} \neq U$, will

$U^{[k]} \to U$ as $k \to \infty$?

Let $e^{[k]} = U^{[k]} - U$

$$U^{[k+1]} - U = \frac{1}{2}(GU^{[k]} - h^2 F) - U$$

$$e^{[k+1]} = \frac{1}{2} G e^{[k]} \qquad (\text{use } *)$$

Let $\tilde{G} = \frac{1}{2} G$. $e^{[k]} = \tilde{G}^k e^{[0]}$

$\tilde{G}$ is symmetric $\to$ its eigenvectors form an orthogonal basis for $\mathbb{R}^m$.

What are the eigenvalues and eigenvectors of $\tilde{G}$?

$$\lambda_p = \frac{2}{h^2}\left(\cos(p\pi h) - 1\right)$$

$$|\cos(p\pi h)|$$

Let $Av = \lambda v$

$p = 1, 2, \ldots, m$

$$\frac{1}{h^2}(G - 2I)v = \lambda v$$

$$h = \frac{1}{m+1}$$

$$Gv - 2v = \lambda h^2 v$$

$$\left.\begin{aligned} Gv = (\lambda h^2 + 2)v \end{aligned}\right] \text{same eigenvectors}$$

Eigenvalues of G: $\lambda_p h^2 + 2 = 2\cos(p\pi h) = \gamma_p$

" of $\tilde{G}$: $\cos(p\pi h) = \tilde{\gamma}_p$

Notice: $|\tilde{\gamma}_p| < 1$ so $\|e^{[k]}\|_2 = \|\tilde{G}^k e^{[0]}\|_2 \leq \|G^k\|_2 \|e^{[0]}\|_2$

$\nearrow 0$ as $k \to \infty$

Write $e^{[0]}$ in the basis of eigenvectors of $\tilde{G}$

$$e^{[0]} = \sum_{p=1}^{m} c_p \tilde{v}_p \leftarrow \text{eigenvectors of } \tilde{G}$$

$$G^k e^{[0]} = \sum_{p=1}^{m} c_p G^k \tilde{v}_p$$

$$= \sum_{p=1}^{m} c_p \tilde{\gamma}_p^k \tilde{v}_p$$

$$\lim_{k\to\infty} G^k e^{[0]} = \sum_{p=1}^{m} c_p \lim_{k\to\infty} \tilde{\gamma}_p^k \tilde{v}_p = 0$$

The rate of convergence depends on how close $|\tilde{\gamma}_p|$ is to 1

Under-relaxed Jacobi

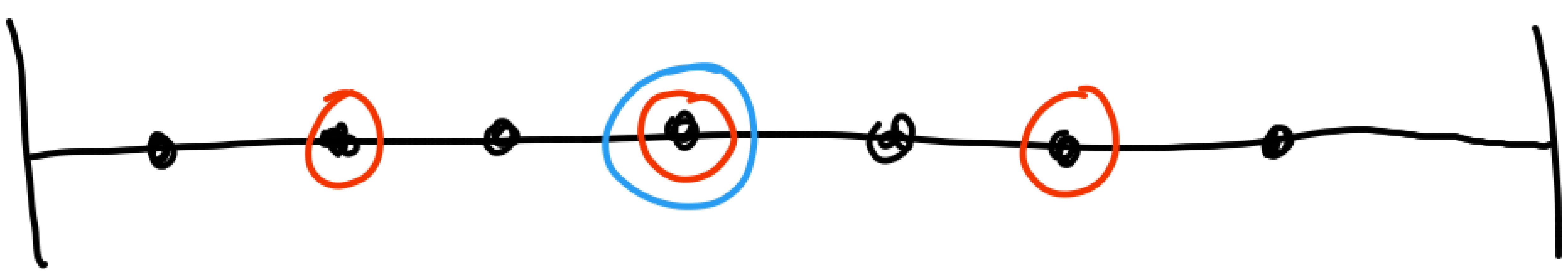
$$\hat{U}^{[k+1]} = \frac{1}{2}\left(GU^{[k]} - h^2 F\right)$$

$$U^{[k+1]} = U^{[k]} + \omega\left(\hat{U}^{[k+1]} - U^{[k]}\right)$$

$$= \underbrace{\left((1-\omega)I + \frac{\omega}{2}G\right)}_{\hat{G}} U^{[k]} - \omega\frac{h^2}{2}F$$

Multigrid

Start on a grid with m points and apply under-relaxed Jacobi with $\omega = \frac{2}{3}$ to solve $AU = F$. After ν iterations, restrict/coarsen the grid:



Define $e_\nu = U_\nu - U$

soln. after ν iterations

$$AU_\nu - F = -r_\nu$$

$$AU - F = 0$$

$$\underbrace{Ae_\nu = -r_\nu}_{}$$

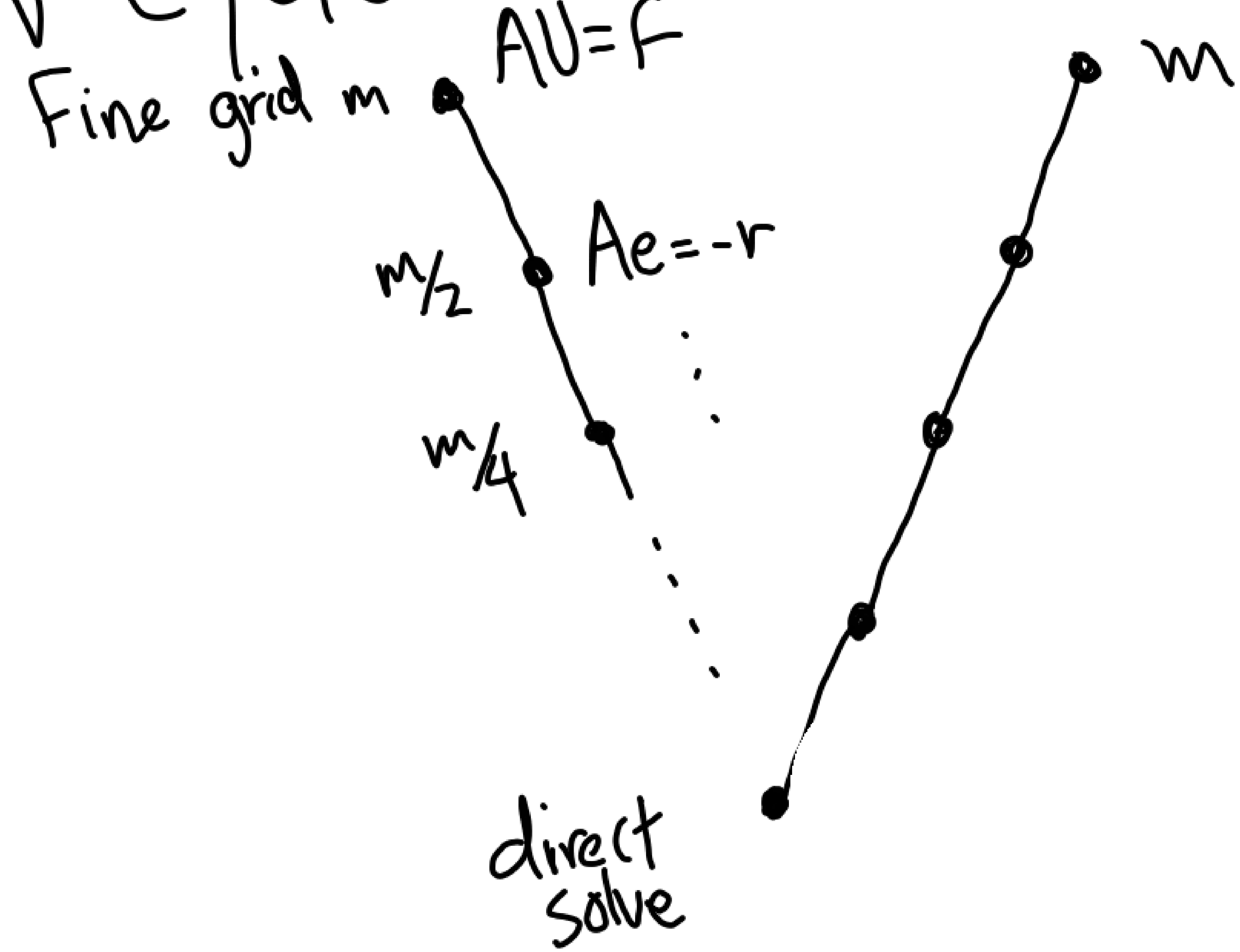
We solve this equation on the coarse grid.

Then correct the solution:

$$U_\nu - e_\nu$$

We do this iteratively, coarsening until we can apply a direct solver. Then subtract all of the corrections.

V-cycle



Complexity: $\mathcal{O}(m \log m)$