Poisson's Equation in 2D

$$\nabla^2 u = f$$

Special case: $\nabla^2 = 0$

"Laplace's Equation"

More generally we can have

$$\nabla \cdot (K(x,y)\nabla u) = f$$

Homework 2 due next Thursday

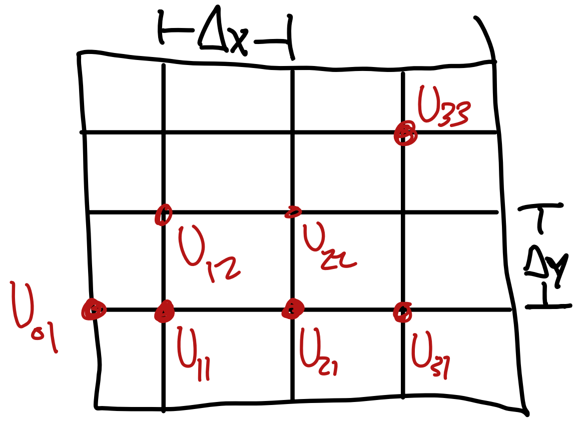
Applications:

u	f	K
Elec. Pot.	Charge	Permittivity
Temperature	heat source	Heat conductivity
Grav. Pot.	mass	—
Concentration	source	

In 2D:

$$u_{xx} + u_{yy} = f(x,y)$$

$$0 < x < 1$$
$$0 < y < 1$$



$$u(x,0) = \alpha(x) \quad u(0,y) = \gamma(x)$$

$$u(x,1) = \beta(x) \quad u(1,y) = \mu(x)$$

$$u_{xx} \approx \frac{U_{i+1,j} - 2U_{i,j} + U_{i-1,j}}{(\Delta x)^2} \qquad 1 \le i \le m \qquad \Delta x = \frac{1}{m+1}$$

$$u_{yy} \approx \frac{U_{i,j+1} - 2U_{i,j} + U_{i,j-1}}{(\Delta y)^2} \qquad 1 \le j \le n \qquad \Delta y = \frac{1}{n+1}$$

$$\Delta x = \Delta y = h$$

$$\frac{1}{h^2}\left[U_{i+1,j} + U_{i-1,j} + U_{i,j+1} + U_{i,j-1} - 4U_{ij}\right] = f(x_i, y_j) = f_{ij}$$

5-point stencil

$$1 \le i,j \le m$$

m^2 equations

Linear algebra

We must choose an "ordering" for U. $AU=F$

Row-wise ordering:

$$U=\begin{bmatrix} U_{11}\\ U_{21}\\ U_{31}\\ U_{12}\\ \vdots\\ U_{mm}\end{bmatrix} \qquad F=\begin{bmatrix} f_{11}\\ f_{21}\\ f_{31}\\ \vdots\\ f_{mm}\end{bmatrix} \qquad A=\frac{1}{h^2}\begin{bmatrix} \boxed{\begin{matrix}-4 & 1 & & \\ 1 & \ddots & \ddots & \\ & \ddots & \ddots & 1\\ & & 1 & -4\end{matrix}} & 0 & \ddots & \\ 0 & \boxed{\begin{matrix}-4 & 1 & & \\ 1 & \ddots & \ddots & \\ & \ddots & \ddots & 1\\ & & 1 & -4\end{matrix}} & \ddots & 1\\ 1 & \ddots & \ddots & \\ & \ddots & 1 & \boxed{\begin{matrix}-4 & 1 & & \\ 1 & \ddots & \ddots & \\ & \ddots & \ddots & 1\\ & & 1 & -4\end{matrix}}\end{bmatrix}$$

1 row $m \times m$

Sparse $m^2 \times m^2$ matrix

$\sim \underline{5}m^2$ non-zero entries

Consistency

Substitute $U_{ij} \to u(x_i, y_j)$

$$\underline{\frac{1}{h^2}\left[u(x_{i+1},y_j)+u(x_{i-1},y_j)+u(x_i,y_{j+1})+u(x_i,y_{j-1})-4u(x_i,y_j)\right]=f(x_i,y_j)+\tau_{ij}}$$

$$\cancel{u_{xx}}+\frac{h^2}{12}u_{xxxx}+\cancel{u_{yy}}+\frac{h^2}{12}u_{yyyy}+\mathcal{O}(h^4)=\cancel{f(x_i,y_j)}+\tau_{ij}$$

$$\tau_{ij}=\frac{h^2}{12}(u_{xxxx}+u_{yyyy})+\mathcal{O}(h^4)$$

2nd-order accurate locally

Global Error

$$\hat{U}_{ij}=u(x_i,y_j) \qquad E=U-\hat{U}$$

$$AU=F$$
$$\underline{A\hat{U}=F+\tau}$$
$$AE=-\tau$$
$$E=-A^{-1}\tau \Rightarrow \|E\|\le\|A^{-1}\|\cdot\|\tau\|$$

We know $\|\tau\| = \mathcal{O}(h^2)$.

We need to bound $\|A^{-1}\|$. (stability)

Take $\|\cdot\| = \|\cdot\|_2$. We need to show

max. eig. of A^{-1} is bounded as $h \to 0$.

i.e. min. eig. of A does not vanish as $h \to 0$.

What are the eigenvalues of $A \in \mathbb{R}^{m^2 \times m^2}$?

Suppose $AV = \lambda V$:

$$\frac{V_{i+1,j} - 2V_{ij} + V_{i-1,j}}{\underline{h^2}} + \frac{V_{i,j+1} - 2V_{ij} + V_{i,j-1}}{h^2} = \lambda V_{ij} \quad 1 \le i,j \le m$$

Assume separability: $V_{ij} = R_i S_j$

$$\underline{S_j} \frac{R_{i+1} - 2R_i + R_{i-1}}{h^2} + \underline{R_i} \frac{S_{j+1} - 2S_j + S_{j-1}}{h^2} = \lambda R_i S_j$$

Divide by $R_i S_j$:

$$\underbrace{\frac{\overbrace{R_{i+1} - 2R_i + R_{i-1}}^{= C_1}}{R_i h^2}}_{\substack{\text{depends only on } i \\ \Rightarrow \text{must be constant}}} + \underbrace{\frac{\overbrace{S_{j+1} - 2S_j + S_{j-1}}^{= C_2}}{S_j h^2}}_{\substack{\text{depends only on } j \\ \Rightarrow \text{must be constant}}} = \lambda$$

$$\frac{R_{i+1}-2R_i+R_{i-1}}{R_i h^2} = C_1$$

$$R_{i+1}+(-2-C_1h^2)R_i+R_{i-1}=0$$

$1 \le i \le m$

$R_i = \xi^i \quad \xi \in \mathbb{C} \Rightarrow$ solve

$$\Rightarrow C_1 = \frac{2}{h^2}(\cos(p\pi h)-1) \quad p=1,2,\dots,m$$

Similarly

$$S_{j+1}+(-2-C_2h^2)S_j+S_{j-1}=0$$

$1 \le j \le m$

$$C_2 = \frac{2}{h^2}(\cos(q\pi h)-1)$$

$q=1,2,\dots,m$

$$\lambda_{pq} = \frac{2}{h^2}\left(\cos(\underline{p\pi h})+\cos(\underline{q\pi h})-2\right) \qquad 1 \le p,q \le m$$

$\lambda_{pq} < 0$

$$-\infty < \lambda_{pq} < \underline{-2\pi^2}$$

$$\Rightarrow \|A^{-1}\| < \frac{1}{2\pi^2} \Rightarrow \|E\| < \frac{1}{2\pi^2}\mathcal{O}(h^2)$$

$\lim_{h\to 0}\|E\| = 0$ 2nd-order convergence