So far we've studied:

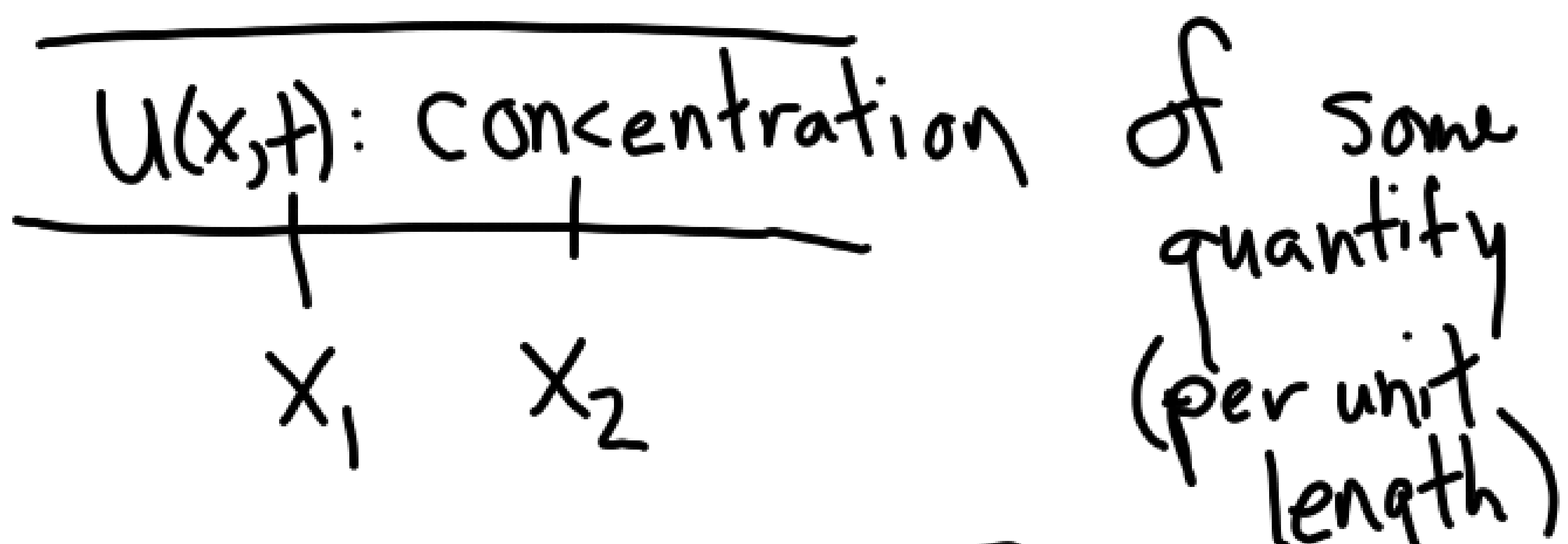
- An elliptic PDE:
  $$\nabla^2 u = f$$
- A parabolic PDE:
  $$u_t = \nabla^2 u$$

Today: a Hyperbolic PDE

Hyperbolic PDEs model waves:

- Water waves
- Sound (pressure waves)
- EM waves
- Fluid dynamics

Consider flow of a fluid in a channel:



Total amount in $[x_1, x_2]$:

$$\int_{x_1}^{x_2} u(x,t)\,dx$$

This should only change due to flow through the endpoints:

$$\frac{d}{dt}\int_{x_1}^{x_2} u\,dx = f(u(x_1,t)) - f(u(x_2,t))$$

Here $f(u)$ is the flux (rate of flow)

If u is smooth enough:

$$\int_{x_1}^{x_2} \frac{\partial}{\partial t} u(x,t)\,dx = -\int_{x_1}^{x_2} \frac{\partial}{\partial x} f(u)\,dx$$

$$\int_{x_1}^{x_2} \left(\frac{\partial u}{\partial t} + \frac{\partial f}{\partial x}\right) dx = 0 \quad \text{Conservation law}$$

This implies: $u_t + f(u)_x = 0$ (pointwise)

Let's just take a constant velocity: $f(u) = a\,u(x,t)$

$$u_t + a u_x = 0$$

Advection equation

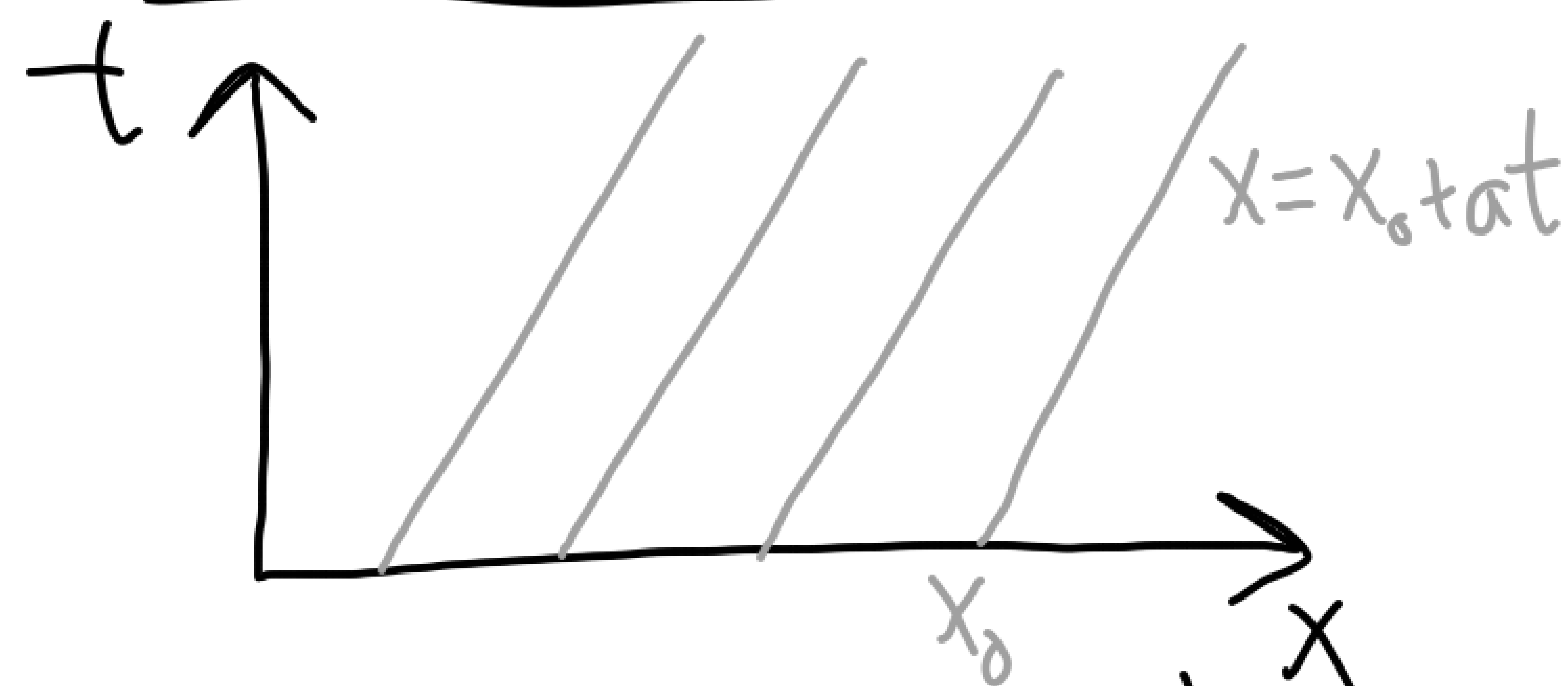
$$u(x, t=0) = \eta(x)$$

Solution: $u(x,t) = \eta(x - at)$

Check it: $-a\eta' + a\eta' = 0$

(For the Cauchy problem)

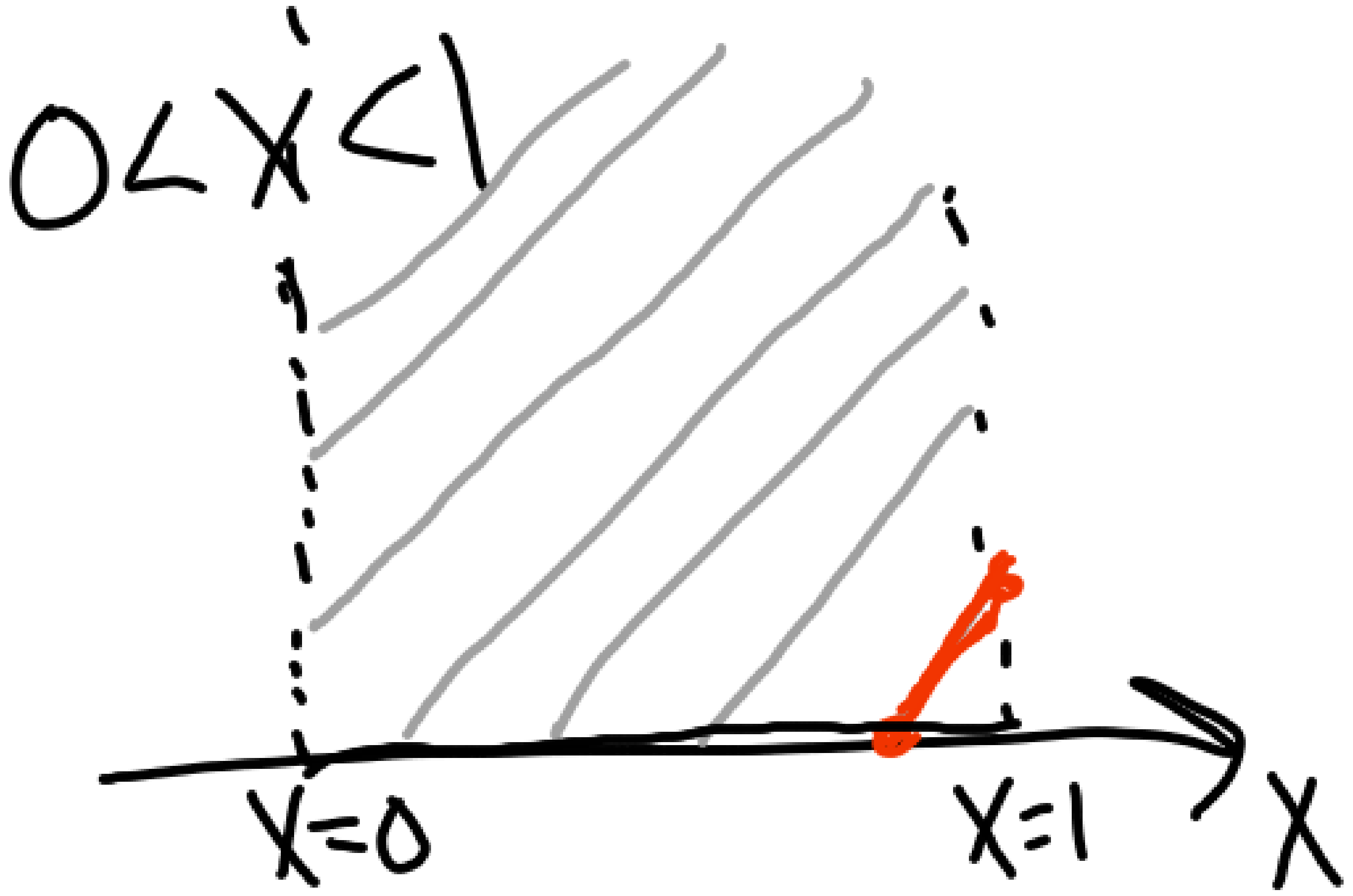
Characteristics



The solution is constant along these lines.

What about bounded domains?

$0 < x < 1$



We must specify the solution along each characteristic.

Thus we need a boundary condition

$$u(x=0,t) = \alpha(t) \quad \text{if } a>0$$

$$u(x=1,t) = \beta(t) \quad \text{if } a<0.$$

Discretization

Centered in space, Euler in time:

$$U_j^{n+1} = U_j^n - ka\,\frac{U_{j+1}^n - U_{j-1}^n}{2h}$$

Is this stable?

Von Neumann analysis

$$U_j^n = g^n e^{ijh\xi}$$

$$g^{n+1} e^{ijh\xi} = g^n e^{ijh\xi} - \frac{ka}{2h} g^n e^{ijh\xi}\left(e^{ih\xi} - e^{-ih\xi}\right)$$

$$g = 1 - \frac{ka}{2h}\underbrace{\left(e^{ih\xi} - e^{-ih\xi}\right)}_{2i\sin(h\xi)}$$

$$e^{\pm ih\xi} = \cos(h\xi) \pm i\sin(h\xi)$$

$$g = 1 - \frac{ka}{h} i\sin(h\xi)$$

We want $|g| \leq 1$ for all ξ.

$$|g|^2 = 1 + \left(\frac{ka}{h}\sin(h\xi)\right)^2$$

Unstable

Method of lines analysis

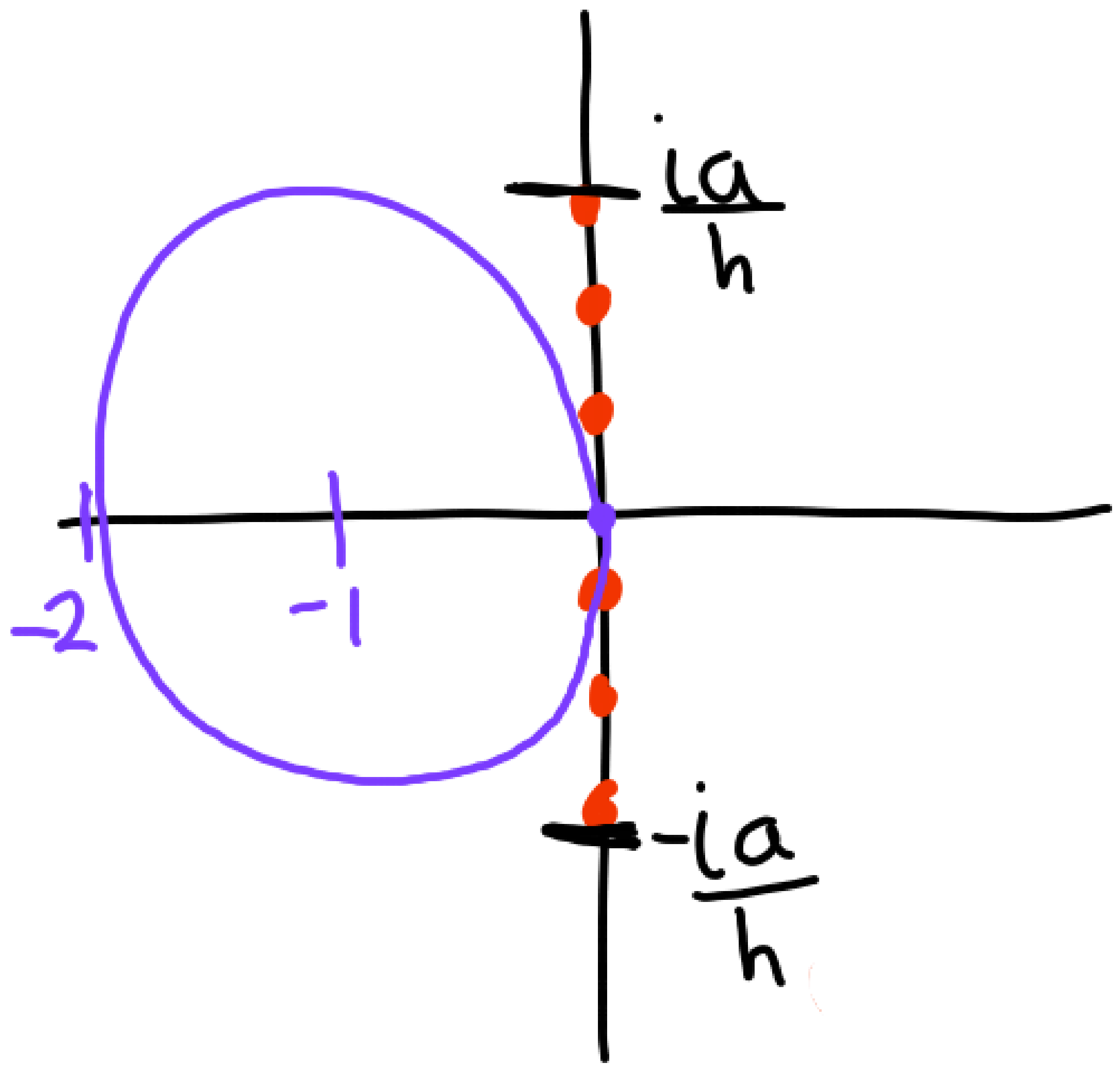
If we discretize only in space:

$$U_j'(t) = -\frac{a}{2h}\left(U_{j+1}(t) - U_{j-1}(t)\right)$$

$$U'(t) = -\frac{a}{2h}\begin{bmatrix} 0 & 1 & & & \\ -1 & \ddots & \ddots & & \\ & \ddots & \ddots & \ddots & \\ & & \ddots & \ddots & 1 \\ & & & -1 & 0 \end{bmatrix} U(t)$$

Do the eigenvalues lie inside the absolute stability region?

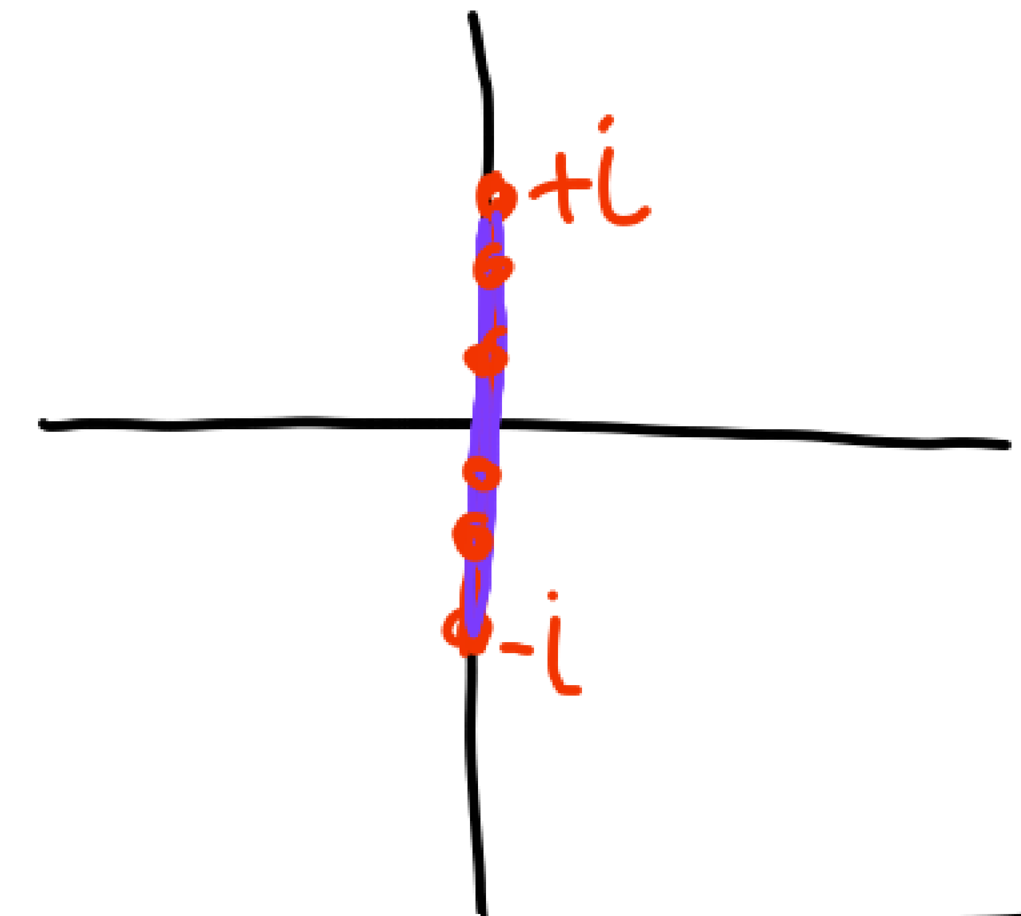
Skew-symmetric $\Rightarrow$ imag. eigenvalues



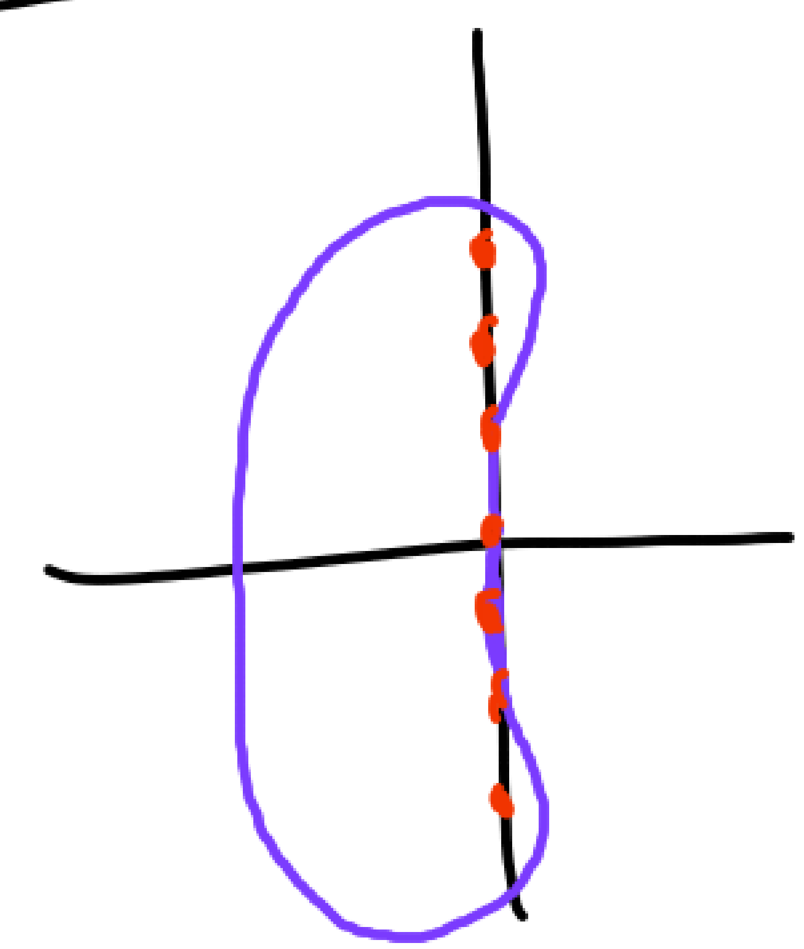
We should use a method that includes part of the imaginary axis!

For instance, Leapfrog:

$$U_j^{n+1} = U_j^{n-1} - \frac{ka}{h}\left(U_{j+1}^n - U_{j-1}^n\right)$$



RK4



Lax-Friedrichs

$$U_j^{n+1} = U_j^n - \frac{ka}{2h}\left(U_{j+1}^n - U_{j-1}^n\right)$$

approximates

$$u_t + au_x = -\varepsilon u_{xx}$$

$$U_j^{n+1} = \frac{U_{j+1}^n + U_{j-1}^n}{2} - \frac{ka}{2h}\left(U_{j+1}^n - U_{j-1}^n\right)$$

Observe: $$\frac{U_{j+1}^n + U_{j-1}^n}{2} = U_j^n + \frac{1}{2}\left(U_{j+1}^n - 2U_j^n + U_{j-1}^n\right)$$

So the LF method can be written:

$$\frac{U_j^{n+1} - U_j^n}{k} = -\frac{a}{2h}\left(U_{j+1}^n - U_{j-1}^n\right) + \frac{h^2}{2k}\,\frac{U_{j+1}^n - 2U_j^n + U_{j-1}^n}{h^2}$$

This looks like an approximation of

$$u_t + au_x = \frac{h^2}{2k} u_{xx}$$

Advection-diffusion