Modified Equation Analysis

$$u_t + a u_x = 0$$

Forward-time, centered space:

$$(1) \qquad \frac{U_j^{n+1} - U_j^n}{k} + \frac{a}{2h}\left(U_{j+1}^n - U_{j-1}^n\right) = 0$$

We suppose there exists a smooth function $v(x,t)$ such that v satisfies (1) exactly.

$$\frac{v(x,t+k)-v(x,t)}{k} + a\frac{v(x+h,t)-v(x-h,t)}{2h} = 0$$

$$v_t = -av_x + \mathcal{O}(k,h^2)$$

$$v_{tt} \approx -av_{xt}$$

$$v_{tx} \approx -av_{xx}$$

$$v_{tt} = a^2 v_{xx} + \mathcal{O}(k,h^2)$$

$$v(x,t+k) = v + kv_t + \frac{k^2}{2}v_{tt} + \mathcal{O}(k^3)$$

$$v(x\pm h,k) = v \pm hv_x + \frac{h^2}{2}v_{xx} + \mathcal{O}(h^3)$$

$$\Rightarrow \frac{\cancel{v} + kv_t + \frac{k^2}{2}v_{tt} - \cancel{v}}{k} + a\frac{\cancel{v} + hv_x + \cancel{\frac{h^2}{2}v_{xx}} - (\cancel{v} - hv_x + \cancel{\frac{h^2}{2}v_{xx}})}{2h} = \mathcal{O}(h^2,k^2)$$

$$v_t + av_x = -\frac{k}{2}v_{tt} + \mathcal{O}(h^2,k^2) \Rightarrow v_t + av_x = \underbrace{-\frac{ka}{2}v_{xx}}_{\text{anti-diffusive}} + \mathcal{O}(k^2,h^2)$$

Centered in time and space (Leapfrog)

$$\frac{U_j^{n+1} - U_j^{n-1}}{2K} + a\frac{U_{j+1}^{n} - U_{j-1}^{n}}{2h} = 0$$

$$\frac{v(x,t+K) - v(x,t-K)}{2K} + a\frac{v(x+h,t) - v(x-h,t)}{2h} = 0$$

$$= \mathcal{O}(h^4, K^4)$$

$$\frac{\cancel{v} + Kv_t + \cancel{\frac{K^2}{2}v_{tt}} + \frac{K^3}{6}v_{ttt} - (\cancel{v} - Kv_t + \cancel{\frac{K^2}{2}v_{tt}} - \frac{K^3}{6}v_{ttt})}{2K} + a\frac{\cancel{v} + hv_x + \cancel{\frac{h^2}{2}v_{xx}} + \frac{h^3}{6}v_{xxx} - (\cancel{v} - hv_x + \cancel{\frac{h^2}{2}v_{xx}} + \frac{h^3}{6}v_{xxx})}{2h}$$

$$v_t + av_x = -a\frac{h^2}{6}v_{xxx} - \frac{K^2}{6}v_{ttt} + \mathcal{O}(h^3, K^3)$$

$$V_t + aV_x = \frac{1}{6}\left(-ah^2 V_{xxx} - k^2 V_{ttt}\right)$$

$$V_t + aV_x = \underbrace{\frac{a}{6}\left(k^2a^2 - h^2\right)V_{xxx}}_{\text{How does this term affect the solution?}} + O(h^4, k^4, k^2h^2)$$

$$V_t + aV_x = O(h^2, k^2)$$

$$V_{tt} = a^2 V_{xx} + O(h^2, k^2)$$

$$V_{ttx} = a^2 V_{xxx} + O(h^2, k^2)$$

$$V_{ttt} = -aV_{xtt} + O(h^2, k^2)$$

$$V_{ttt} = -a^3 V_{xxx}$$

Ansatz: $V(x,t) = e^{i(\xi x - \omega t)}$ $\quad \omega = \omega(\xi)$

$$V_t = -i\omega V \qquad V_x = i\xi V$$

$$-i\omega V + i\xi a V = \frac{a}{6}\left(k^2a^2 - h^2\right)\left(-i\xi^3\right)V$$

$$\omega - \xi a = \frac{a}{6}\left(k^2a^2 - h^2\right)\xi^3$$

$$\omega(\xi) = \xi a\left(1 + \frac{k^2a^2 - h^2}{6}\xi^2\right)$$

$$V(x,t) = e^{i\xi\left(x - \frac{\omega}{\xi}t\right)} \qquad \frac{\omega}{\xi}: \text{phase velocity}$$

$$\frac{\omega}{\xi} = a\left(1 + \frac{k^2a^2 - h^2}{6}\xi^2\right)$$

Exact solution of advection equation:

$$u(x,t) = \eta(x-at)$$

We have

$$v(x,t) = \eta\left(x - \frac{\omega}{\xi}t\right)$$

(for a fixed value of ξ)

We see that the speed depends on ξ. This is called <u>numerical dispersion</u>.

What if

$$k^2a^2 - h^2 = 0$$

i.e. $\left|\frac{ka}{h}\right| = 1$

For an evolution equation:

$$u_t = \sum_{j=0}^{m} \alpha_j \frac{\partial^j}{\partial x^j} u(x,t)$$

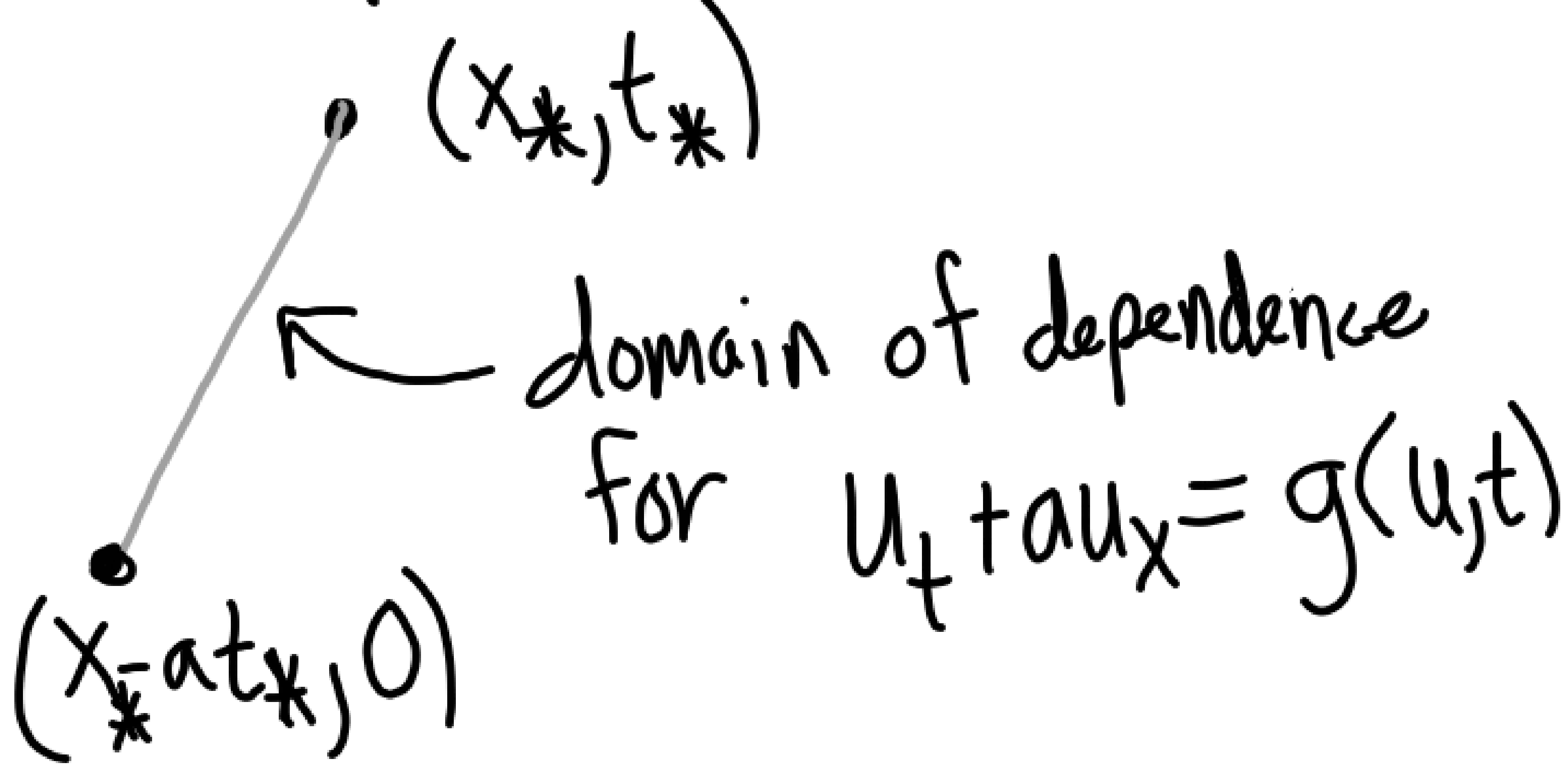
- Odd-order derivatives are dispersive (modify phase)
- Even-order derivatives are diffusive (modify amplitude)

The CFL Condition

Courant-Friedrichs-Lewy (1927)

Domain of dependence:

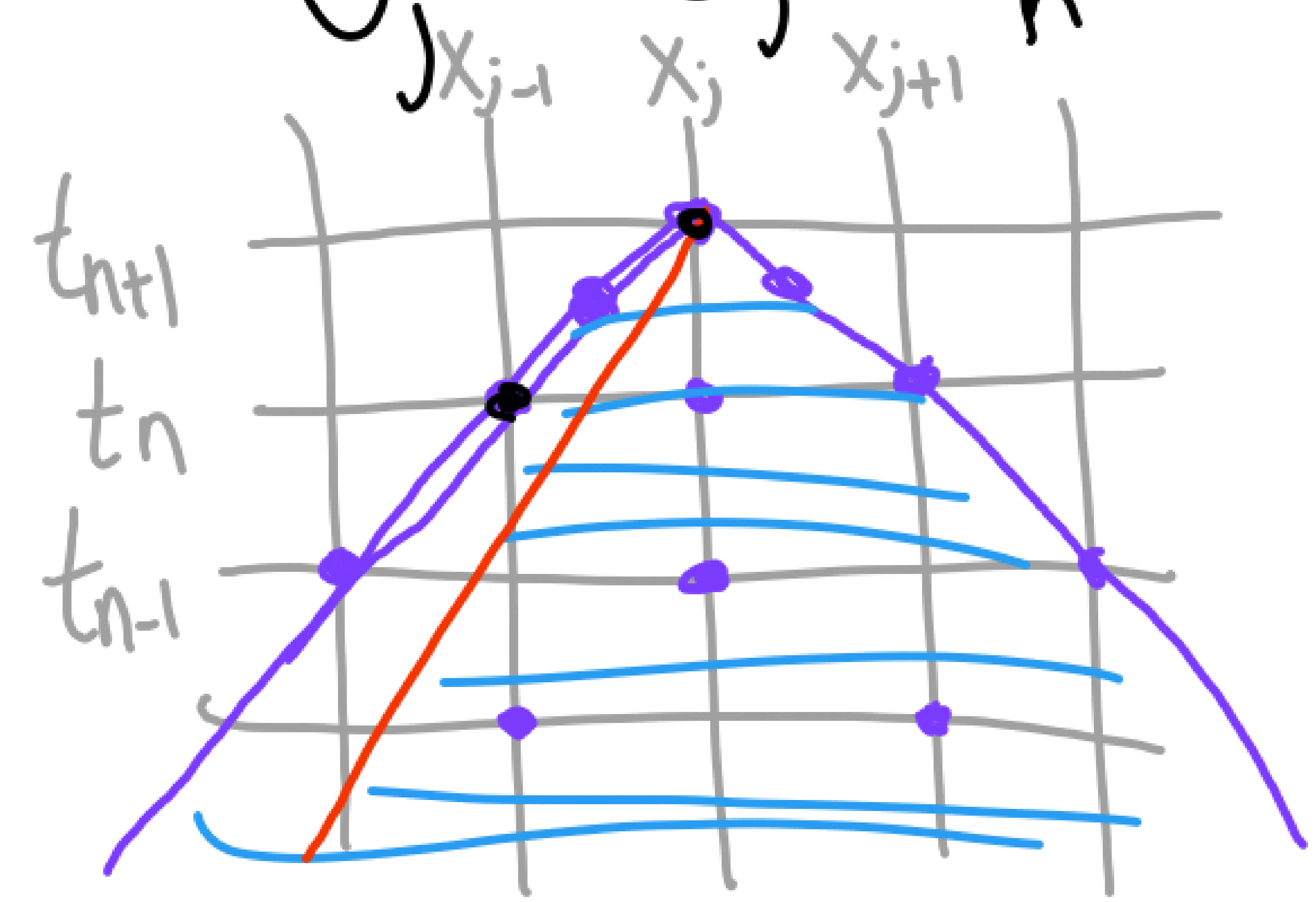
The set of points (in (x,t)) that can influence the solution at some prescribed point:



The CFL condition says that a numerical scheme cannot be convergent unless the numerical DoD contains the true DoD, in the limit $k, h \to 0$.

Numerical D.o.D.

$$U_j^{n+1} = U_j^{n-1} - \frac{ka}{h}\left(U_{j+1}^n - U_{j-1}^n\right)$$



We need the characteristic to lie inside the numerical D.o.D.

dist between grid pts: h

dist. traveled by characteristic in one time step: ka

So we need $|ka| \leq h$

i.e. $\left|\frac{ka}{h}\right| \leq 1$

$\frac{ka}{h}$ is called the CFL number.