Linear Multistep Methods

$$u'(t) = f(u)$$
$$u(t_0) = \eta$$

A LMM takes the form:

$$\sum_{j=0}^{r} \alpha_j U^{n+j} = k \sum_{j=0}^{r} \beta_j f(U^{n+j})$$

This is a formula for U^{n+r}

If $\beta_r \neq 0$, the method is implicit.

If $\beta_r = 0$, the method is explicit.

Local Truncation Error

$$\sum_{j=0}^{r} \alpha_j U(\overset{t_{n+j}}{t_n + jK}) - K\sum_{j=0}^{r} \beta_j U'(t_n + jK) = K\tau^{n+r}$$

$$U(t_n + jK) = \sum_{i=0}^{\infty} \frac{(jK)^i}{i!} U^{(i)}(t_n) = U(t_n) + \sum_{i=1}^{\infty} \frac{(jK)^i}{i!} U^{(i)}(t_n)$$

$$U'(t_n + jK) = \sum_{i=0}^{\infty} \frac{(jK)^i}{i!} U^{(i+1)}(t_n) = \sum_{i=1}^{\infty} \frac{(jK)^{i-1}}{(i-1)!} U^{(i)}(t_n)$$

$$\sum_{j=0}^{r} \alpha_j U(t_n) + \sum_{i=1}^{\infty} \sum_{j=0}^{r} \left(\alpha_j \frac{(jK)^i}{i!} - K\beta_j \frac{(jK)^{i-1}}{(i-1)!} \right) U^{(i)}(t_n) = K\tau^{n+r}$$

For consistency:

$$\left.\begin{aligned} &\sum_{j=0}^{r} \alpha_j = 0 \\ &\sum_{j=0}^{r} (j\alpha_j - \beta_j) = 0 \end{aligned}\right\} \text{1st order}$$

For 2nd order accuracy:

$$\sum_{j=0}^{r} \left(\frac{j^2}{2}\alpha_j - j\beta_j \right) = 0$$

etc.

Examples:

2-step Adams-Bashforth

$$U^{n+2} = U^{n+1} + \frac{K}{2}\left(3f(U^{n+1}) - f(U^n)\right)$$

Leapfrog: $U^{n+2} = U^n + 2Kf(U^{n+1})$

Backward Differentiation Formula:

$$U^{n+2} = \frac{4}{3}U^{n+1} - \frac{1}{3}U^n + 2kf(U^{n+2})$$

A 2-step 1st-order method:

$$U^{n+2} = 3U^{n+1} - 2U^n + kf(U^n)$$

Consistency for LMMs also requires consistency of the starting values

$$\lim_{K\to 0} \|U^j - u(t_j)\| = 0$$

for $j = 0, 1, \ldots, r-1$

Test problem:

$$u'(t) = 0 \quad (*)$$
$$u(0) = 0$$

$$U^0 = 0 \quad U^1 = K$$

Zero-stability

If we apply a LMM to (*), we get

$$\underbrace{\sum_{j=0}^{r} \alpha_j U^{n+j} = 0}_{\text{Linear difference equations}} \quad (**)$$

Ansatz: $U^n = \zeta^n \quad \zeta \in \mathbb{C}$

We have: $\displaystyle\sum_{j=0}^{r} \alpha_j \zeta^j = 0$

We call $\rho(\zeta) = \displaystyle\sum_{j=0}^{r} \alpha_j \zeta^j$ the 1st characteristic polynomial of the LMM.

$\rho(\zeta)$ is a polynomial of degree r, with roots $\zeta_1, \zeta_2, \dots, \zeta_r$.

If they are distinct, then all solutions of (**) are of the form $U^n = \displaystyle\sum_{j=1}^{r} c_j \zeta_j^n$.

Notice that $\rho(1) = \displaystyle\sum_{j=0}^{r} \alpha_j = 0$.

What about repeated roots?

For example: $U^{n+2} - 2U^{n+1} + U^n = 0$

$$\rho(\zeta) = \zeta^2 - 2\zeta + 1$$
$$= (\zeta - 1)^2 \quad \zeta_1 = \zeta_2 = 1$$

One fundamental solution is
$$U^n = 1^n = 1.$$

The other is $U^n = n1^n = n$.

Check: $n+2 - 2(n+1) + n = 0$ ✓

The general soln. is: $U^n = c_1 + c_2 n$.

In general, a root ζ_j of multiplicity m leads to the fundamental solutions
$$\zeta_j^n, \; n\zeta_j^n, \; n^2\zeta_j^n, \ldots n^{m-1}\zeta_j^n$$

We want to know whether the solution of (**) remains bounded as $n \to \infty$, $k \to 0$ with nk fixed.

The solution of (**) is bounded as $n \to \infty$ iff the roots of $\rho(\zeta)$ satisfy the root condition:

$$|\zeta_j| \le 1 \quad \forall j$$

and if ζ_j is a multiple root then $|\zeta_j| < 1$.

If this holds, we say the LMM is zero-stable.

AB2: $U^{n+2} = U^{n+1} + \cdots$

$$\rho(\zeta) = \zeta^2 - \zeta \quad \text{roots: } \zeta = 0, 1$$

Leapfrog: $U^{n+2} = U^n + 2kf(U^{n+1})$

$$\rho(\zeta) = \zeta^2 - 1 \quad \text{roots: } \pm 1$$

Unstable: $U^{n+2} = 3U^{n+1} - 2U^n + kf(U^n)$

$$\rho(\zeta) = \zeta^2 - 3\zeta + 2 = (\zeta - 1)(\zeta - 2)$$

roots: $1, 2$

$U^n = c_1 1^n + c_2 2^n$ Not zero stable

Any zero-stable and consistent method for the IVP

$$u'(t) = f(u)$$
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is convergent if f is Lipschitz.

To prove this, we write the LMM as a one-step method

$$V^n = \begin{bmatrix} U^n \\ U^{n+1} \\ \vdots \\ U^{n+r-1} \end{bmatrix} \qquad V^{n+1} = \begin{bmatrix} U^{n+1} \\ \vdots \\ \vdots \\ U^{n+r} \end{bmatrix}$$

The LMM

$$\sum_{j=0}^{r} \alpha_j U^{n+j} = k \sum_{j=0}^{r} \beta_j f(U^{n+j})$$

is then

$$V^{n+1} = \frac{1}{\alpha_r} \underbrace{\begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & 0 \cdots & 0 \\ & 0 & \ddots & \ddots & \\ & 0 & & \ddots & 1 \\ -\alpha_0 & -\alpha_1 & \cdots & \cdots & -\alpha_{r-1} \end{bmatrix}}_{\text{companion matrix } C} V^n + \begin{bmatrix} 0 \\ \vdots \\ 0 \\ \frac{k}{\alpha_r} \sum_{j=0}^{r} \beta_j f(U^{n+j}) \end{bmatrix}$$

$$V^{n+1} = CV^n + Kw^n$$

If we apply this repeatedly, we get formulas with increasing powers of C.

So for convergence we will need to bound $\|C^n\|$ for all n.

This is bounded iff $\rho(\zeta)$ satisfies the root condition.

For any one-step method, $\rho(\zeta) = \zeta - 1$.

So all one-step methods are zero-stable.