Systems of Hyperbolic PDEs

Acoustic waves (1 dimension)

$$p_t = -K u_x$$ (kinematic)

Bulk modulus

$$\rho u_t = -p_x$$ (Newton: F=ma)

density

$K > 0$

$\rho > 0$

$$p_t + K u_x = 0$$

$$u_t + \frac{1}{\rho} p_x = 0$$

$$p_{tt} = -K u_{xt}$$

$$u_{tx} = \frac{-1}{\rho} p_{xx}$$

$$\Rightarrow p_{tt} = \frac{K}{\rho} p_{xx}$$

(wave equation)

$$q = \begin{bmatrix} p \\ u \end{bmatrix} \qquad A = \begin{bmatrix} 0 & K \\ 1/\rho & 0 \end{bmatrix}$$

Linear Hyperbolic PDE

$$q_t + A q_x = 0 \qquad (1)$$

We say that (1) is hyperbolic if A is diagonalizable with real eigenvalues.

$$A = R \Lambda R^{-1}$$

$$q_t + R \Lambda R^{-1} q_x = 0$$

$$R^{-1} q_t + \Lambda R^{-1} q_x = 0$$

$$w = R^{-1} q$$ Characteristic Variables

$$\underbrace{w_t + \Lambda w_x = 0}_{\text{System of decoupled advection equations}}$$

$$\lambda^2 - \frac{K}{\rho} = 0$$

$$\lambda_\pm = \pm\sqrt{\frac{K}{\rho}} = \pm c$$

$$r = \begin{bmatrix} r_1 \\ r_2 \end{bmatrix}$$

$$A r = \lambda_\pm r$$

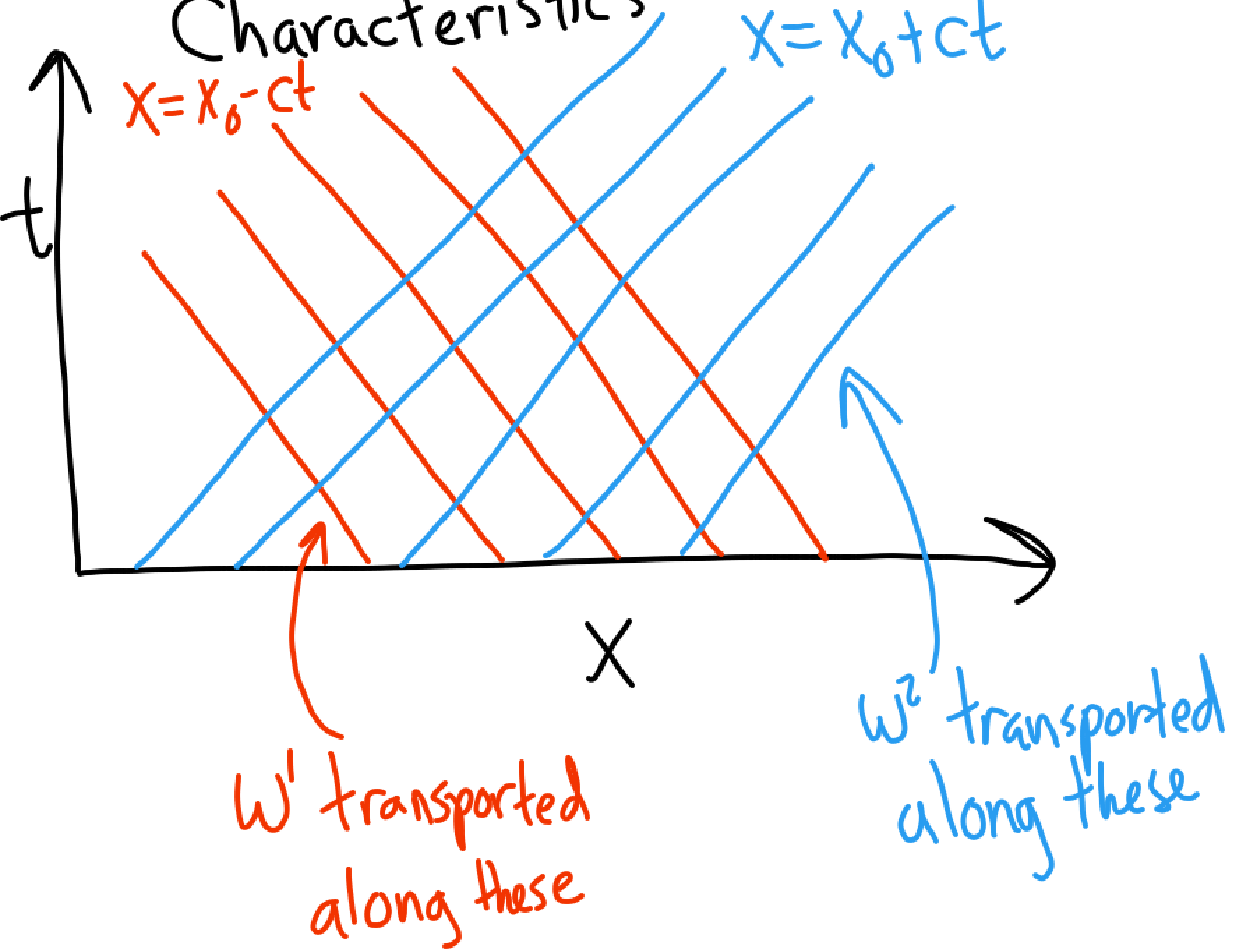
$$K r_2 = \pm\sqrt{\frac{K}{\rho}}\, r_1$$

$$r_1 = \pm\sqrt{K\rho}\, r_2$$

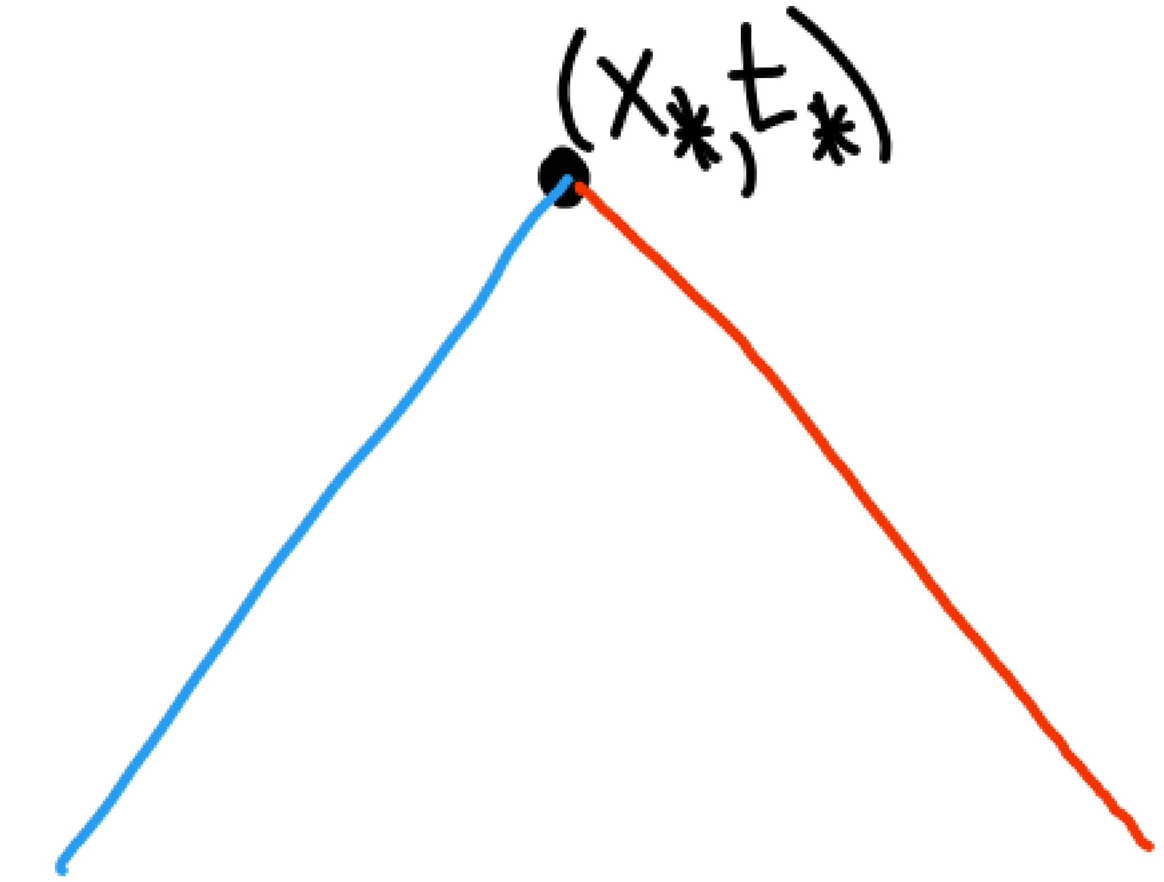
$$\sqrt{K\rho} = Z$$ Impedance

$$R = \begin{bmatrix} -Z & Z \\ 1 & 1 \end{bmatrix} \qquad \Lambda = \begin{bmatrix} -c & 0 \\ 0 & c \end{bmatrix}$$

Characteristics



Domain of dependence:



Numerical discretization must include info. from both directions.

For a 3-point stencil:

$$\frac{kc}{h} \le 1$$

time step size

Lax-Wendroff:

Scalar advection: $U_j^{n+1} = U_j^n - \frac{ka}{2h}\left(U_{j+1}^n - U_{j-1}^n\right) + \frac{k^2a^2}{h^2}\left(U_{j+1}^n - 2U_j^n + U_{j-1}^n\right)$

($u_t + au_x = 0$)

hyp. system: $Q_j^{n+1} = Q_j^n - \frac{k}{2h}A\left(Q_{j+1}^n - Q_{j-1}^n\right) + \frac{k^2}{h^2}A^2\left(Q_{j+1}^n - 2Q_j^n + Q_{j-1}^n\right)$

($q_t + Aq_x = 0$)

Nonlinear Hyperbolic Systems

$$q_t + f(q)_x = 0$$

flux (the f)

$q(x,t) \in \mathbb{R}^m$

$f(q): \mathbb{R}^m \to \mathbb{R}^m$

$f'(q)$: Jacobian ($m \times m$ matrix)

Quasilinear form:

$$q_t + f'(q) q_x = 0 \qquad (2)$$

We say (2) is hyperbolic if $f'(q)$ is diagonalizable with real eigenvalues.

Main difficulty:

Formation of singularities (discontinuities (shocks)) after finite time.

Standard finite difference methods generate oscillations near discontinuities.