Absolute Stability

$$u'(t) = -u$$
$$u(0) = 1$$
$$\Rightarrow u(t) = e^{-t}$$

Recall: $\|E^n\| \le T e^{|\lambda| T} \|\tau\|_\infty$

When will the error really grow exponentially?

For Euler's method applied to

$$u'(t) = \lambda u$$

we found:

$$E^{n+1} = (1 + k\underline{\lambda})E^n - k\tau^n$$

The error will grow at each step if $|1 + k\lambda| > 1$.

More examples

$$u'(t) = -\sin(t)$$
$$u(0) = 1$$
$$\Rightarrow u(t) = \cos(t)$$

$$u'(t) = \lambda(u - \cos(t)) - \sin(t)$$
$$u(0) = 1$$
$$\Rightarrow u(t) = \cos(t)$$

Prothero – Robinson Problem

Lipschitz constant: $|\lambda|$

$$|1 + k\lambda| \leq 1$$
$$-1 \leq 1 + k\lambda \leq 1$$
$$-2 \leq k\lambda \leq 0 \quad (\lambda < 0)$$
$$0 \leq k \leq \frac{-2}{\lambda}$$

We say the method is "Absolutely stable" if this condition is satisfied.

For the problem:

$$U'(t) = Au + g(t)$$

$A \in \mathbb{R}^{m \times m}$

We have (for Euler's method)

$$E^{n+1} = (I + kA)E^n - k\tau^n$$

The error will grow w/o bound if $\|I + kA\| > 1$.

For abs. stability we need $\|I + kA\| \leq 1$.

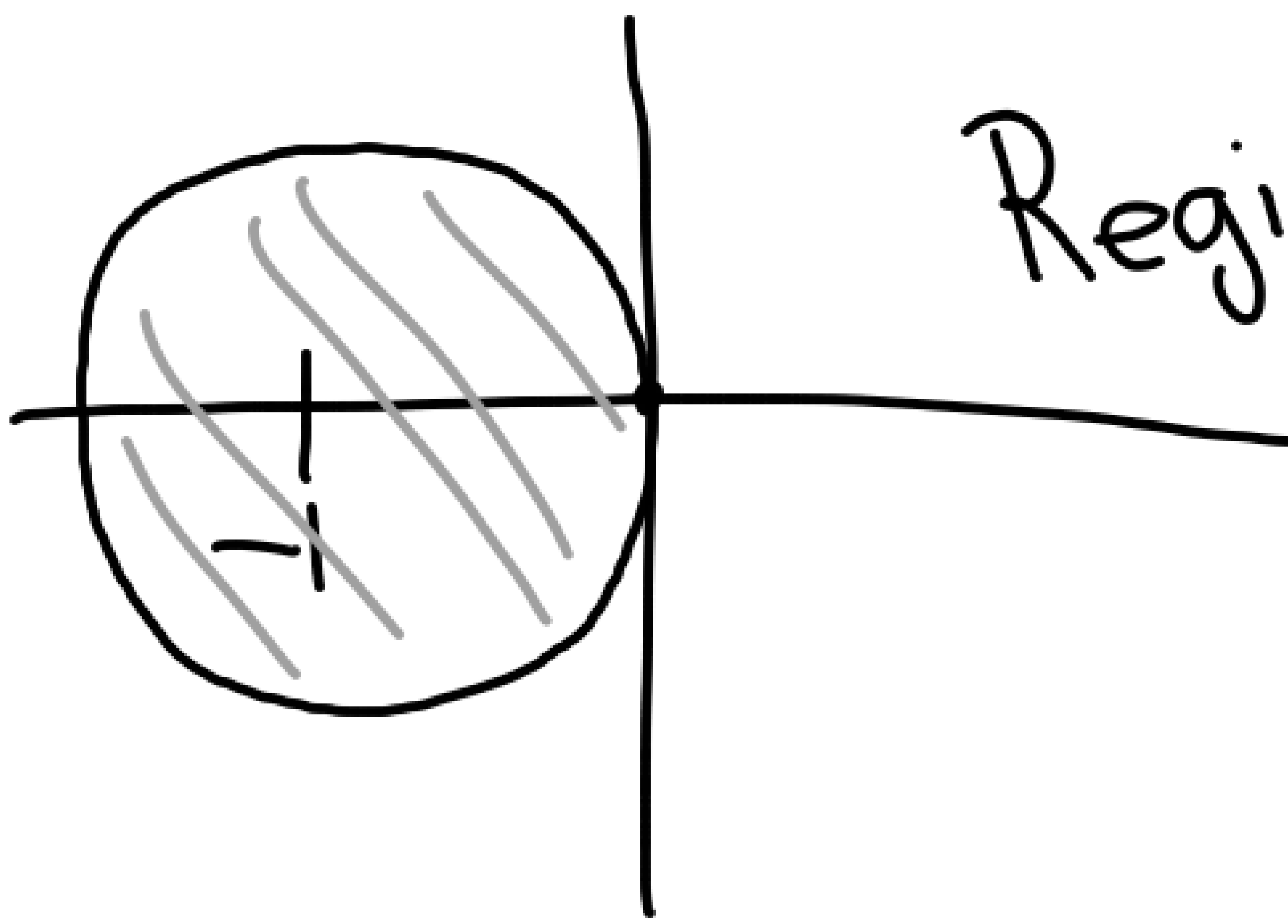
Taking the 2-norm, we need the eigenvalues of $I + kA$ to be smaller than 1 in magnitude.

If λ is an e.v. of A, then $1 + k\lambda$ is an e.v. of $I + kA$

So we need $|1 + k\lambda| \leq 1 \quad \forall \lambda \in \sigma(A)$

Let $S=\{z\in\mathbb{C}: |1+z|\le 1\}$

$(z=k\lambda)$



Region of absolute stability.

Implicit Euler method

$$U^{n+1}=U^n+kf(U^{n+1})$$

$f(u)=\lambda u$

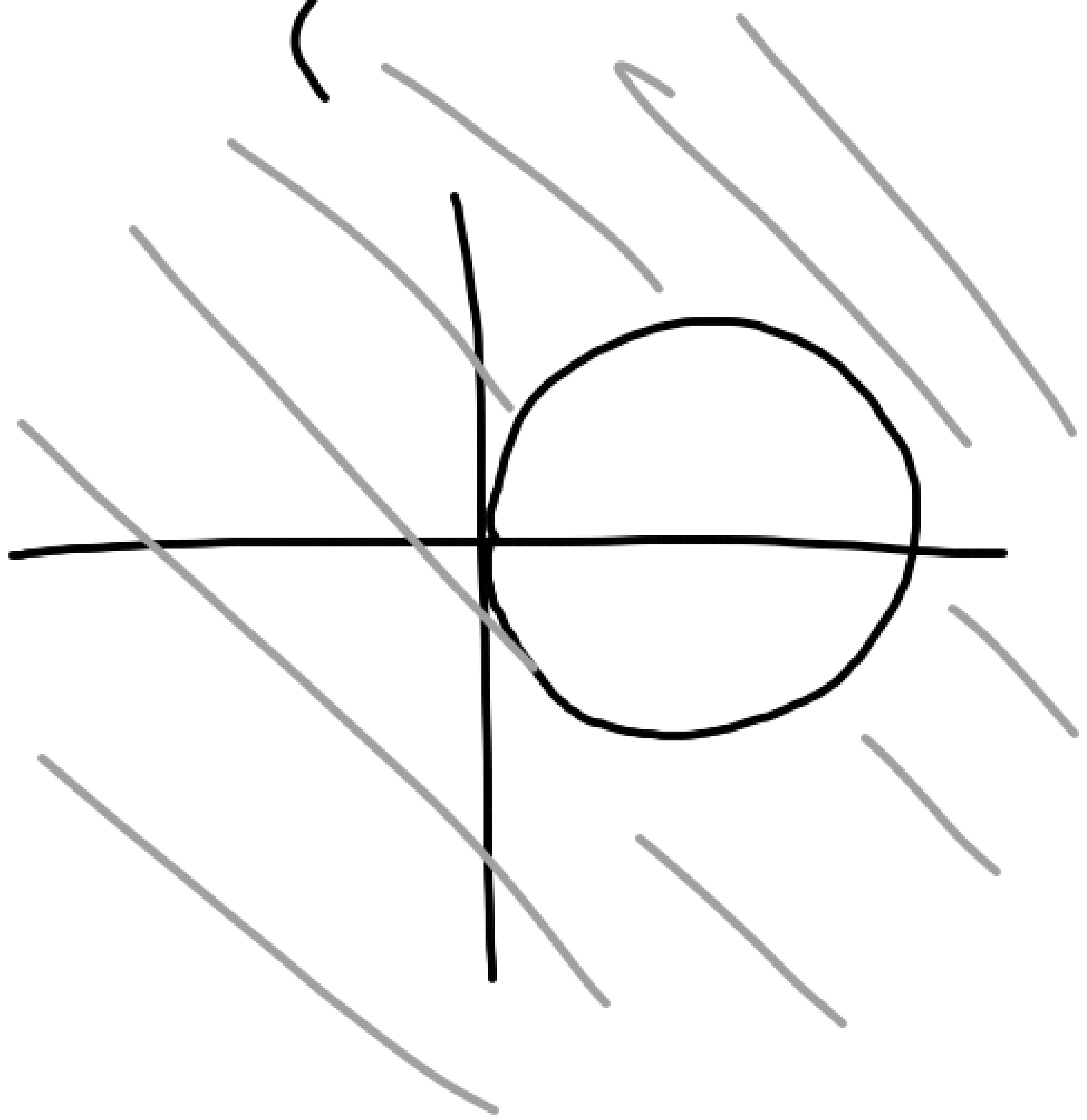
$$U^{n+1}=U^n+k\lambda U^{n+1}$$

$$U^{n+1}=\frac{1}{1-k\lambda}U^n$$

$$E^{n+1}=\frac{1}{1-k\lambda}E^n-k\tau^n$$

We need $\left|\frac{1}{1-k\lambda}\right|\le 1$

$$S = \left\{ z \in \mathbb{C} : \left| \frac{1}{1-z} \right| \leq 1 \right\}$$



Linearized Pendulum

$$u''(t) = -\sin(u) \approx -u$$

$$u'(t) = v(t) \qquad \begin{bmatrix} u(0) \\ v(0) \end{bmatrix} = \eta$$

$$v'(t) = -u(t)$$

$$\begin{bmatrix} u \\ v \end{bmatrix}' = \underbrace{\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}}_{A} \begin{bmatrix} u \\ v \end{bmatrix} \Rightarrow u(t) = e^{At}\eta$$

Eigenvalues: $\lambda = \pm i$

Explicit Euler: not abs. stable for any K

Implicit Euler: Abs. stable for any K.

Any one-step applied to

$$U'(t) = \lambda u \qquad R(z) \approx e^{z}$$

yields

$$U^{n+1} = \underbrace{R(z)}_{\substack{\text{stability}\\ \text{function!}}} U^n \qquad z = k\lambda$$

EE: $R(z) = 1 + z$

IE: $R(z) = \dfrac{1}{1-z}$

Region of abs. stability:

$$S = \{\lambda \in \mathbb{C} : |R(z)| \le 1\}$$

Trapezoidal Method

$$U^{n+1} = U^n + \frac{k}{2}\left(f(U^n) + f(U^{n+1})\right)$$

$$\Rightarrow U^{n+1} = U^n + \frac{1}{2}\left(k\lambda U^n + k\lambda U^{n+1}\right)$$

$$\left(1 - \frac{z}{2}\right)U^{n+1} = \left(1 + \frac{z}{2}\right)U^n$$

$$U^{n+1} = \frac{1 + z/2}{1 - z/2} U^n$$

$$R(z) = \frac{1 + z/2}{1 - z/2}$$

What is S for this method?

$$\left|\frac{1+z/2}{1-z/2}\right| \le 1$$

$$|1+z/2| \le |1-z/2|$$

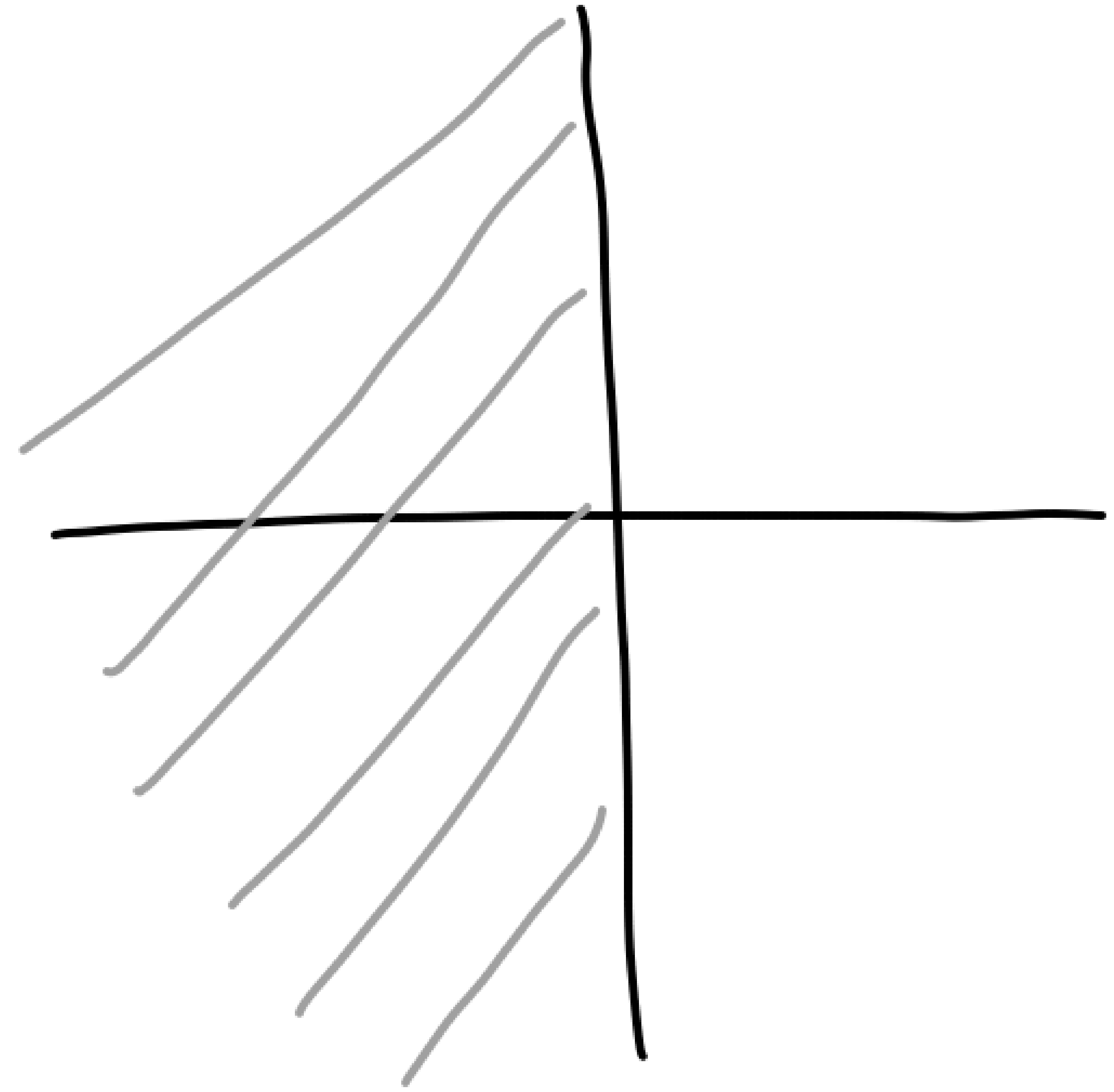
$$z = x + iy$$

$$\left|1+\frac{x}{2}+i\frac{y}{2}\right|^2 \le \left|1-\frac{x}{2}-\frac{iy}{2}\right|^2$$

$$\left(1+\frac{x}{2}\right)^2 + \cancel{\left(\frac{y}{2}\right)^2} \le \left(1-\frac{x}{2}\right)^2 + \cancel{\left(\frac{y}{2}\right)^2}$$

$$1 + x + \frac{x^2}{4} \le 1 - x + \frac{x^2}{4}$$

$$x \le -x$$



Note that for $z = iy$

$$|R(z)| = 1.$$

So with this method, the pendulum will oscillate forever.

Abs. Stability for LMMs

$$\sum_{j=0}^{r} \alpha_j U^{n+j} = k \sum_{j=0}^{r} \beta_j f(U^{n+j})$$

Application to $u'(t) = \lambda u(t)$:

$$\sum_{j=0}^{r} \alpha_j U^{n+j} = k\lambda \sum_{j=0}^{r} \beta_j U^{n+j}$$

$$\sum_{j=0}^{r} (\alpha_j - z\beta_j) U^{n+j} = 0$$

$$U^n = \zeta^n : \underbrace{\sum_{j=0}^{r} (\alpha_j - z\beta_j) \zeta^j}_{\text{2nd characteristic polynomial}}$$

$\pi(\zeta; z)$

The global error satisfies

$$\sum_{j=0}^{r} (\alpha_j - z\beta_j) E^{n+j} = k\tau^{n+j}$$

We have absolute stability if the roots of $\pi(\zeta; z)$ are ≤ 1 in magnitude.

The boundary locus method

$$S = \{ z \in \mathbb{C} : |\zeta| \le 1 \text{ for all roots of } \pi \}$$

On the boundary of S, we have $|\zeta| = 1$ for some root.

Set $\zeta = e^{i\theta}$:

$$\pi = \sum_{j=0}^{r} (\alpha_j - z\beta_j) e^{i\theta j} = 0$$

So $\sum_{j=0}^{r} \alpha_j e^{i\theta j} = z \sum_{m=0}^{r} \beta_j e^{i\theta m}$

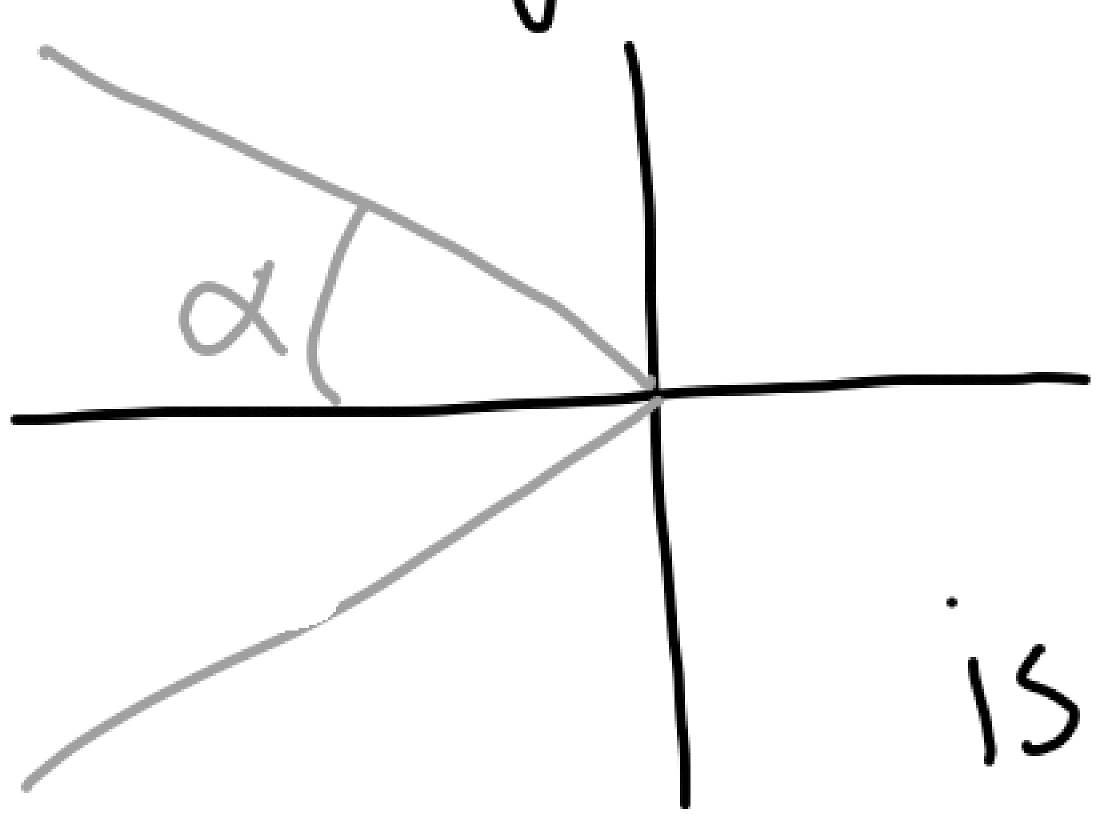
So $$z = \frac{\sum_{j=0}^{r} \alpha_j e^{i\theta j}}{\sum_{m=0}^{r} \beta_j e^{i\theta m}}$$

Evalute the RHS for $0 \le \theta \le 2\pi$

We say a method is A-stable if (e.g. Imp. Euler, Trapezoidal)

$$\mathbb{C}^- \subset S$$

We say a method is $A(\alpha)$-stable if the wedge (sector)



is in S.

The region of stability of any explicit method is stable.