# Neumann Boundary Conditions

Review: $u''(x) = f(x) \quad u(0) = \alpha \quad u(1) = \beta$

Discretize:

$x \in [0,1] \Rightarrow x_j = jh \quad j = 0, 1, \ldots, m+1$

$u(x) \Rightarrow [U_0, U_1, \ldots, U_{m+1}]$

$$\frac{d^2}{dx^2} = \frac{1}{h^2}\begin{bmatrix} h & 0 & \cdots & & 0 \\ 1 & -2 & 1 & & \\ & \ddots & \ddots & \ddots & \\ & & 1 & -2 & 1 \\ 0 & \cdots & & 0 & h \end{bmatrix}$$

$u''(x) = f(x) \Rightarrow AU = F$

What if the left end is insulated?

$u'(0) = 0$

In general we could impose a heat flux:

$u'(0) = \sigma_0$

or $\quad u'(1) = \sigma_1$.

How to discretize this?

Methods:

1a One-sided finite difference

1b Defect correction

2 Ghost point

1a One-sided FD

$$U'(0) = \sigma_0$$

$$U'(0) \approx D_+(U(x_0)) = \frac{U_1 - U_0}{h} = \sigma_0$$

$$\frac{1}{h^2}\begin{bmatrix} -h & h & & & & \\ 1 & -2 & 1 & & & \\ & \ddots & \ddots & \ddots & & \\ & & \ddots & \ddots & \ddots & \\ & & & 1 & -2 & 1 \\ 0 & \cdots & \cdots & \cdots & 0 & h^2 \end{bmatrix} \begin{bmatrix} U_0 \\ U_1 \\ \vdots \\ \vdots \\ U_{m+1} \end{bmatrix} = \begin{bmatrix} \sigma_0 \\ f(x_1) \\ \vdots \\ \vdots \\ \beta \end{bmatrix}$$

Local truncation error:

$$D_+(u(\bar{x})) = u'(\bar{x}) + \frac{h}{2}u''(\bar{x}) + \mathcal{O}(h^2)$$

so $$\frac{u(x_1)-u(x_0)}{h} = \sigma_0 + \tau$$

$$\underline{u'(x_0)} + \frac{h}{2}u''(x_0) = \underline{\sigma_0} + \tau + \mathcal{O}(h^2)$$

$$\tau = \frac{h}{2}u''(x_0) + \mathcal{O}(h^2)$$

(1b) Defect correction

We know $u''(x) = f(x)$,

so we can subtract the leading error term:

$$\underbrace{\frac{U_1-U_0}{h} = \sigma_0 + \frac{h}{2}f(x_0)}_{\text{2nd-order approximation.}}$$

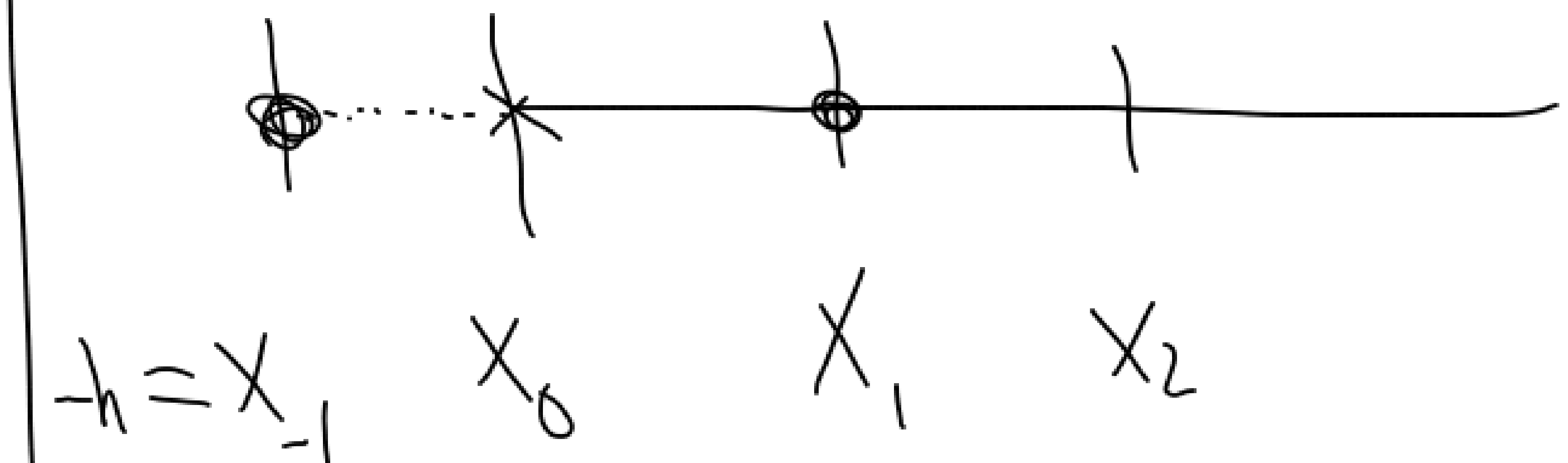
Alternatively, we can use a 3-point one-sided FD formula

$$u'(x_0) = aU_0 + bU_1 + cU_2$$

We find: $u'(x_0) \approx \frac{1}{h}\left(-\frac{3}{2}U_0 + 2U_1 - \frac{1}{2}U_2\right)$

$\Rightarrow$ 2nd-order accurate

## ② Ghost-point method




$$u'(x_0) = \frac{U_1 - U_{-1}}{2h} = \sigma_0$$

Impose $u''(x) = f(x)$ also at $x_0$:

$$\frac{U_1 - 2U_0 + U_{-1}}{h^2} = f(x_0)$$

Solve for $U_{-1}$:

$$U_{-1} = h^2 f(x_0) + 2U_0 - U_1$$

Substitute:

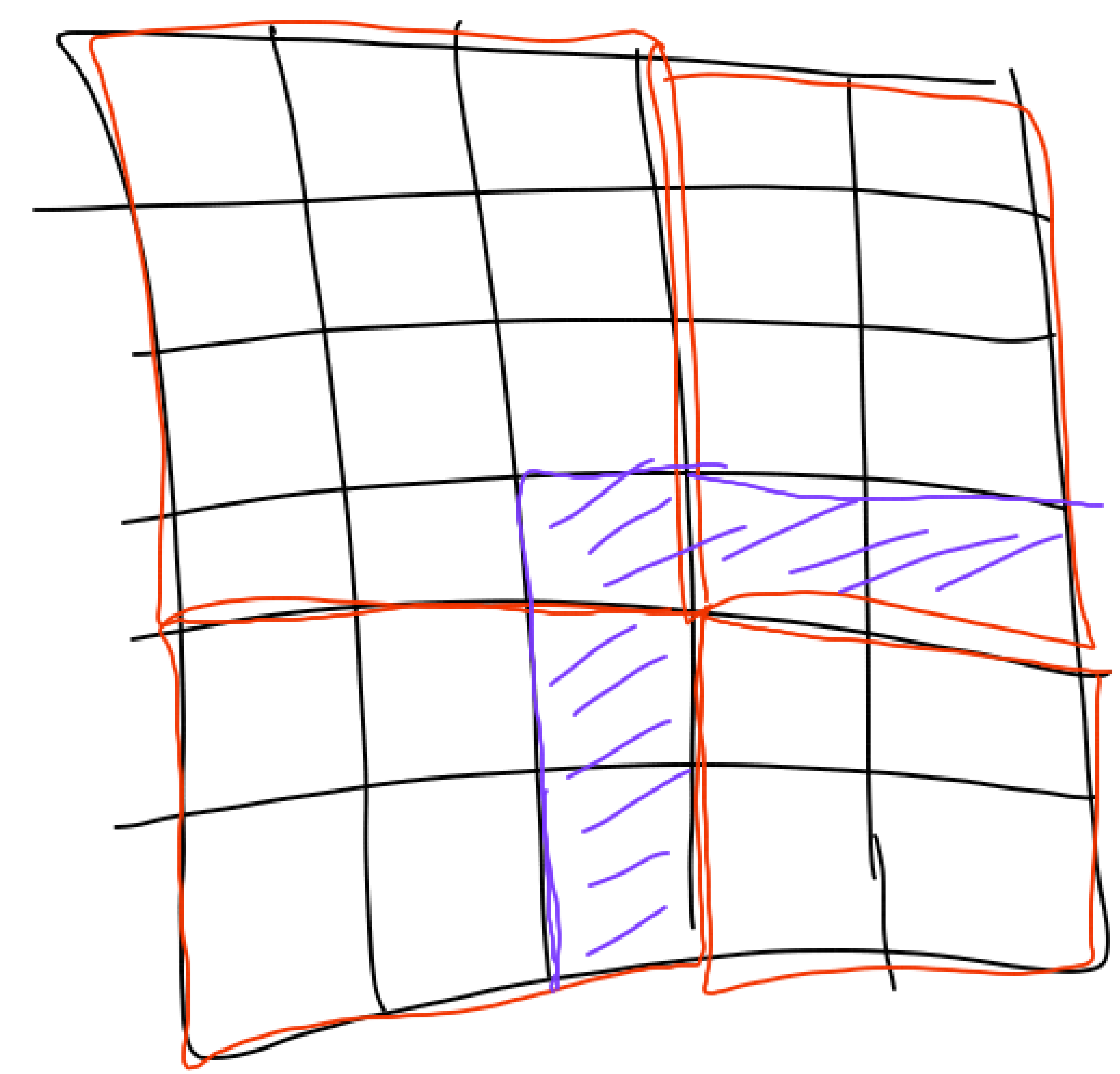
$$\underline{U_1} - h^2 f(x_0) - \underline{2U_0} + \underline{U_1} = 2h\sigma_0$$

$$2(U_1 - U_0) = 2h\sigma_0 + h^2 f(x_0)$$

$$\frac{U_1 - U_0}{h} = \sigma_0 + \frac{h}{2} f(x_0)$$

→ same as the defect correction formula.

Ghost points also play a role in domain decomposition:



What if both ends are <u>insulated</u>?

$u'(0)=0 \qquad u''(x)=f(x)$

$u'(1)=0$

Only conditions on derivatives!

If $u^*(x)$ is a solution, so is $u^*(x)+C$ for $C\in\mathbb{R}$.

If $f(x)=0$, there are $\infty$-ly many solutions.

$$\int_0^1 u''(x)\,dx = \underbrace{u'(1)-u'(0)=\int_0^1 f(x)\,dx}_{\text{Condition for existence of solutions}}$$

e.g. $f(x)=1-2x$.

In general:

$u'(0)=\sigma_0 \quad u'(1)=\sigma_1$

$$\Rightarrow \int_0^1 f(x)\,dx=\sigma_1-\sigma_0$$

Discretization:

$$\underbrace{\frac{1}{h^2}\begin{bmatrix} -h & h & & & & \\ 1 & -2 & 1 & & & \\ & \ddots & \ddots & \ddots & & \\ & & \ddots & \ddots & \ddots & \\ & & & \ddots & \ddots & \ddots \\ & & & 1 & -2 & 1 \\ & & & & -h & h \end{bmatrix}}_{A}\begin{bmatrix} U_0 \\ \vdots \\ \vdots \\ U_{m+1} \end{bmatrix} = \begin{bmatrix} \sigma_0 \\ f(x_1) \\ \vdots \\ f(x_m) \\ \sigma_1 \end{bmatrix}$$

is it singular?

$$A\begin{bmatrix} 1 \\ \vdots \\ \vdots \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ \vdots \\ \vdots \\ 0 \end{bmatrix}$$