Initial Value Problems

Examples:

① Rigid Pendulum

$$\theta''(t) = -\sin(\theta(t))$$
$$\theta(t_0) = \theta_0$$
$$\theta'(t_0) = \Omega_0$$

2nd-order ODE
Can be re-written as a system of 1st-order ODEs

$$\Omega(t) = \theta'(t)$$
$$\Downarrow$$
$$\Omega'(t) = -\sin(\theta(t))$$
$$\theta'(t) = \Omega(t)$$

Because we can always do this, we will focus ONLY on solving 1st-order ODEs.

② SIR model

Compartmental epidemiology

$S(t)$: Susceptible
$I(t)$: Infectious
$R(t)$: Removed

β: contact rate
γ^{-1}: average time of infectiousness

$$\boxed{\begin{aligned} S'(t) &= -\beta S I \\ I'(t) &= \beta S I - \gamma I \end{aligned}}$$

$$R'(t) = \gamma I$$

$$\frac{d}{dt}(S+I+R) = 0$$

$$S+I+R=1$$

$$\boxed{\begin{aligned} S(t_0) &= S_0 \\ I(t_0) &= I_0 \end{aligned}}$$

Linear IVPs

Scalar:

$$u'(t) = \lambda u(t) \qquad \lambda \in \mathbb{C}$$

$$u(t_0) = \eta$$

$$u(t) = e^{\lambda(t-t_0)}\eta$$

Linear System: $u(t): \mathbb{R} \to \mathbb{C}^m$

$$u'(t) = A u(t) \qquad A \in \mathbb{C}^{m \times m}$$

$$u(t_0) = \eta$$

$$u(t) = e^{A(t-t_0)}\eta$$

$$e^M = I + M + \frac{M^2}{2!} + \cdots$$

$$= \sum_{j=0}^{\infty} \frac{M^j}{j!}$$

Inhomogeneous System:

$$u'(t) = Au(t) + g(t)$$
$$u(t_0) = \eta$$

$$\boxed{u(t) = e^{A(t-t_0)}\eta + \int_{t_0}^{t} e^{A(t-\tau)} g(\tau)\,d\tau}$$

Duhamel's Principle

Existence and Uniqueness

Linear IVPs:

A unique solution exists for all t.

What about for nonlinear IVPs?

$$u'(t) = (u(t))^2$$
$$u(0) = \eta > 0$$

$$u(t) = \frac{1}{\eta^{-1} - t}$$

Solution exists only up to $t = \eta^{-1}$.

$f(u) = u^2$ is L.C. on $[\eta - a, \eta + a]$ for any $a < \infty$, but not on $[0, \infty)$

$$u'(t) = \sqrt{u(t)}$$
$$u(0) = 0$$

$f(u) = \sqrt{u}$ is L.C. for $u \in [\varepsilon, \infty)$ for any $\varepsilon > 0$, but not at $u = 0$.

$u(t) = \frac{1}{4}t^2$ or $u(t) = 0$

$$u'(t) = \frac{1}{2}t = \sqrt{u(t)}$$

Solution is not unique

Lipschitz Continuity

Given a function f on domain D we say L is a Lipschitz constant for f on D if

$$\|f(u)-f(v)\| < L\|u-v\|$$

We say f is Lipschitz continuous if such an $L<\infty$ exists.

If f is differentiable then we can take

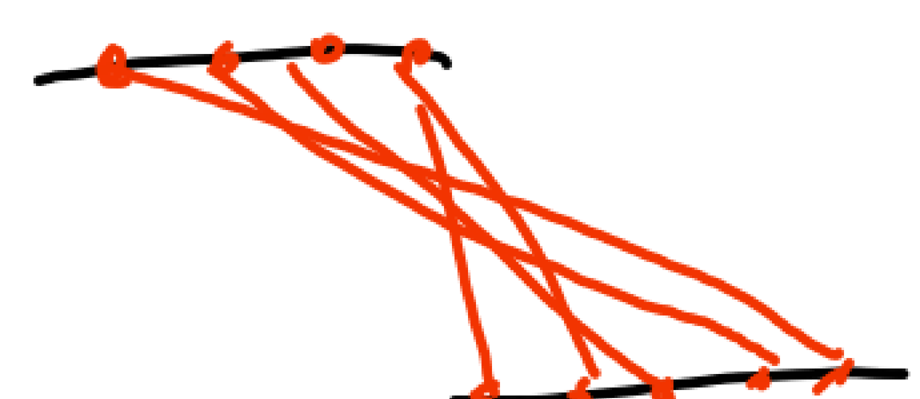
$$L = \sup_{u\in D} \|f'(u)\|$$

$\boxed{\begin{array}{l} f(u)=\lambda u \\ L=\lambda \end{array}}$

Examples

① Heaviside

$$f(u)=\begin{cases} 0 & u<0 \\ 1 & u>0 \end{cases}$$



Not L.C.

② $f(x) = \frac{1}{x}$ $D = \mathbb{R}/0$

Continuous

Not L.C.

If we take $D = [\varepsilon, \infty]$
then $f(x) = \frac{1}{x}$ is L.C.

We can always rewrite a non-autonomous ODE

$$u'(t) = f(u,t)$$

as an autonomous ODE

$$v'(t) = f(v)$$

by setting $v = \begin{bmatrix} u \\ t \end{bmatrix}$ with $t' = 1$.

Given the IVP

$$u'(t) = f(u)$$
$$u(t_0) = \eta$$

$D = [\eta - a, \eta + a]$

Suppose f is L.C.

$$\eta - a \le u \le \eta + a$$

Then a unique solution exists for $t \le t_0 + \frac{a}{\sup_{u \in D} |f(u)|}$.

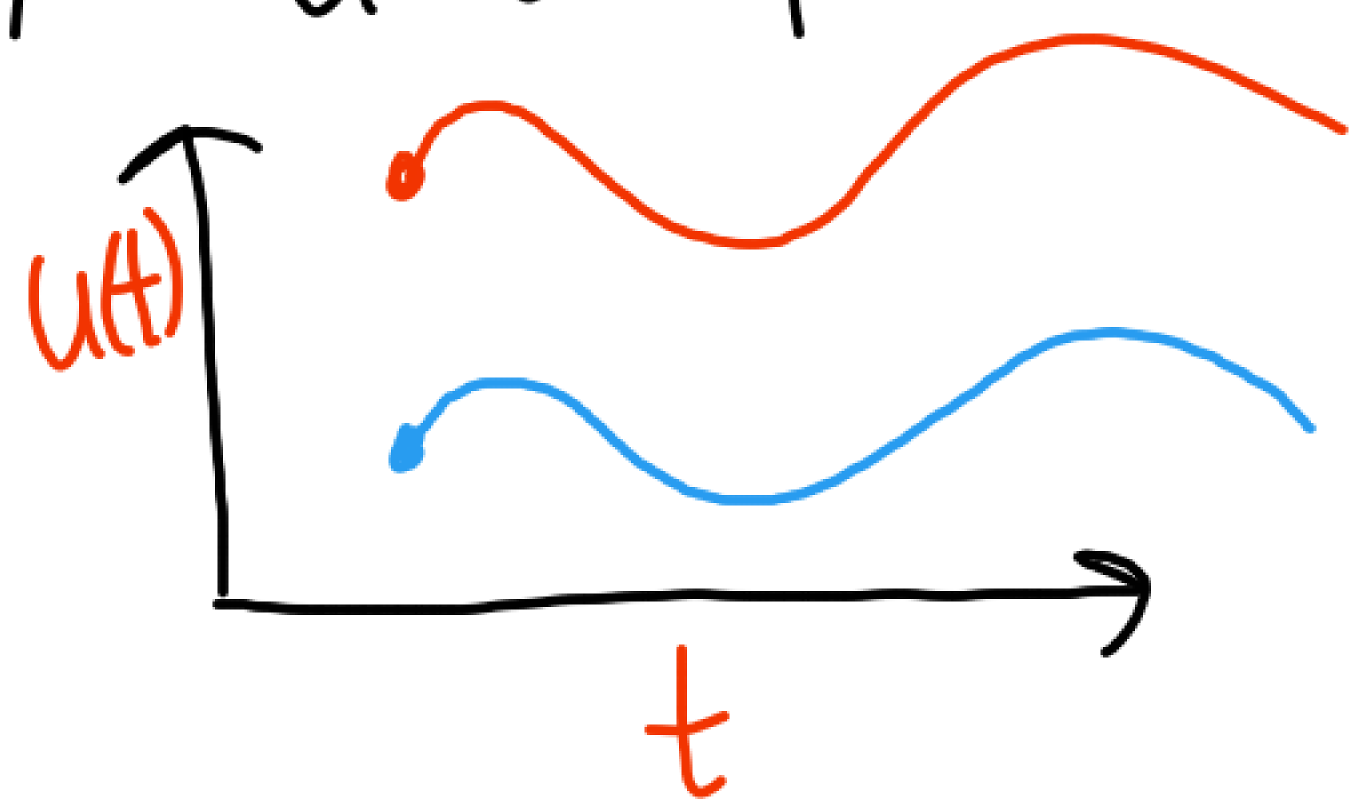
Meaning of the Lipschitz constant

$$u(t) = \eta + \int_{t_0}^{t} g(\tau)\,d\tau$$

Examples:

① $u'(t) = g(t) \qquad u(t_0) = \eta$

$L = 0$

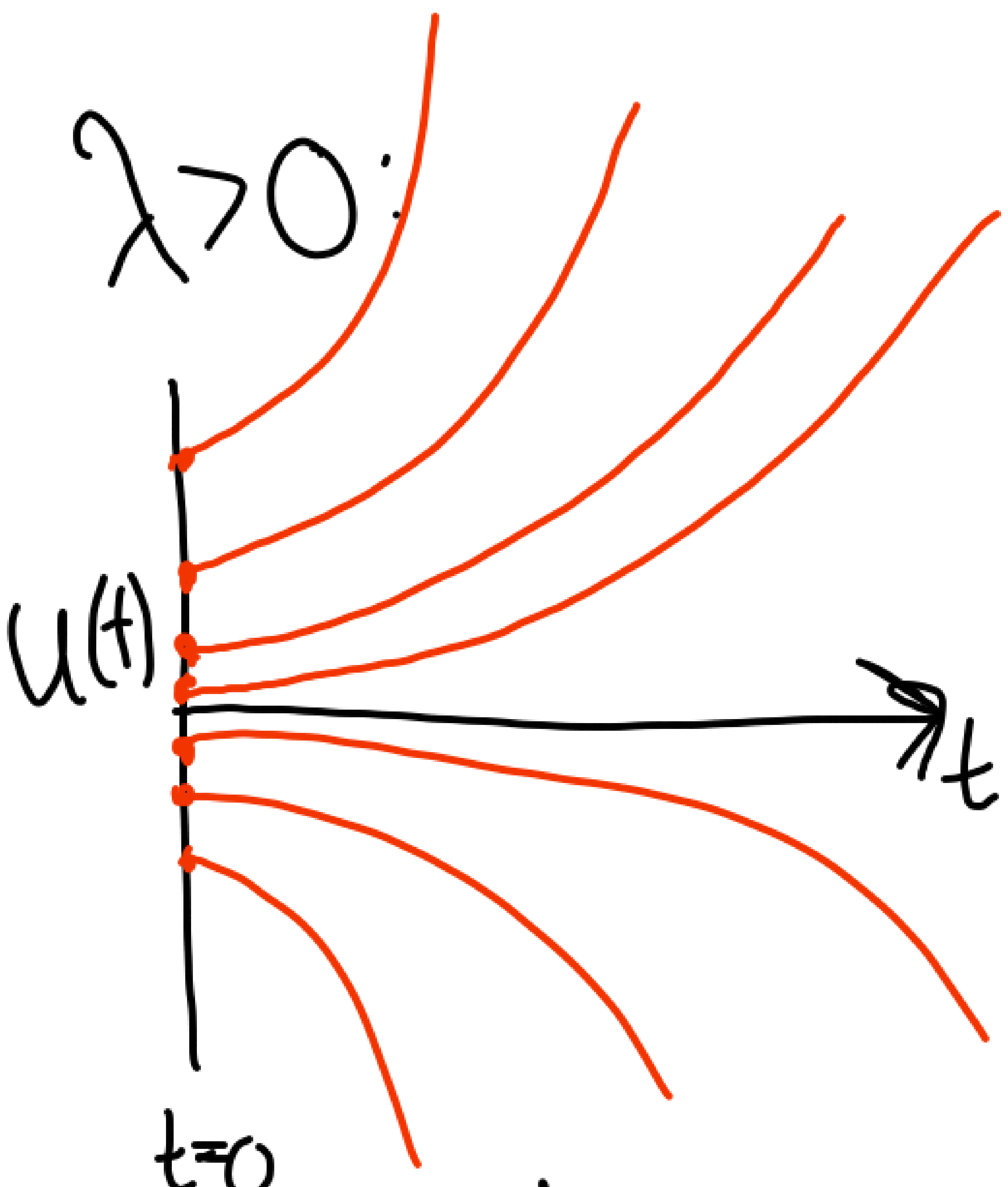


② $u'(t) = \lambda u(t)$
$u(0) = \eta$

$u(t) = e^{\lambda t}\eta$

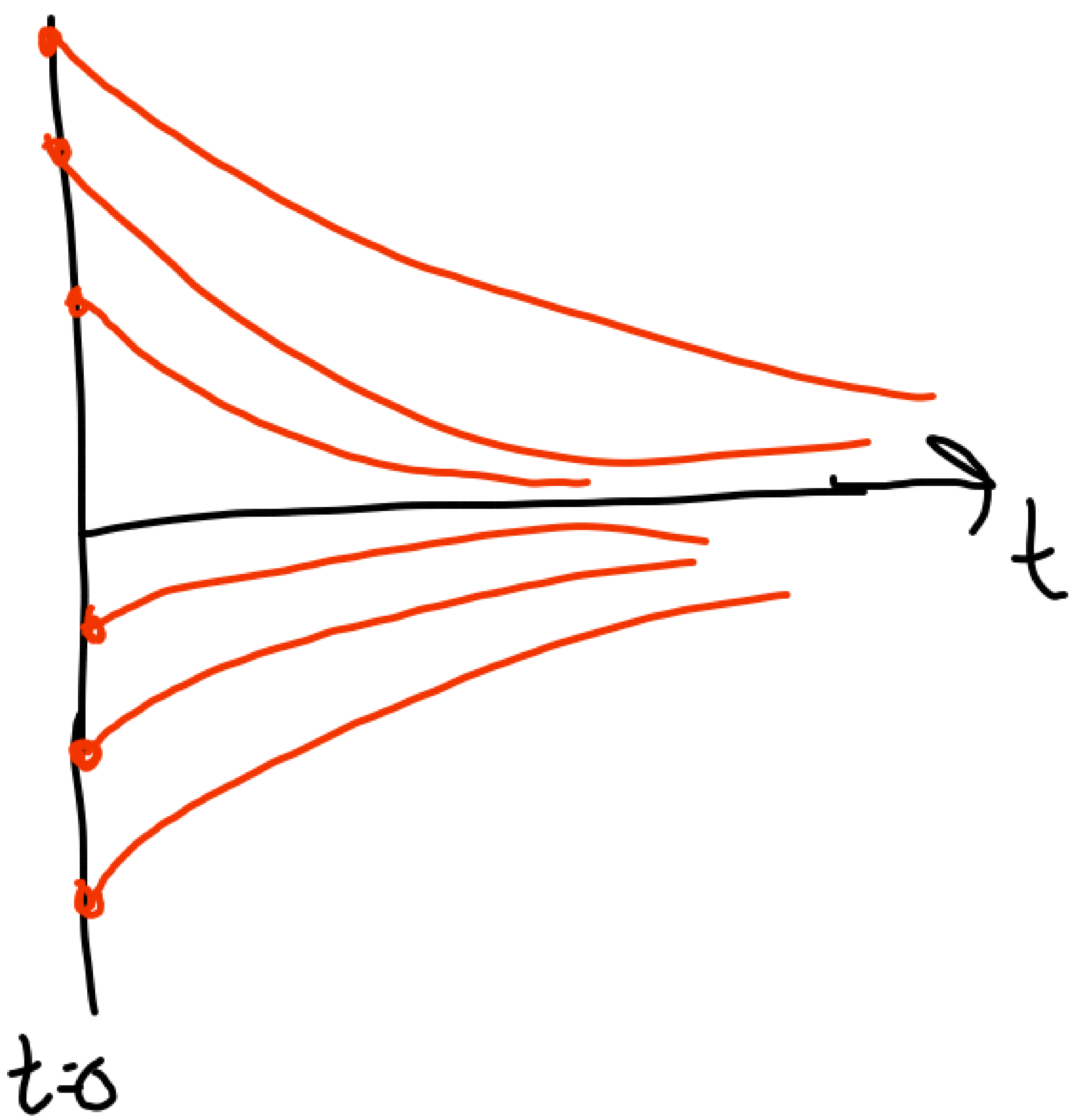
$L = |\lambda|$

$\lambda > 0$:



Solutions diverge

$\lambda < 0$



Trajectories can either diverge or converge at a rate bounded by L.