Convergence and stability for IBVP discretizations

Last time: $u_t = u_{xx}$

Discretize in space: $U'(t) = A_h U(t)$

$$A_h = \frac{1}{h^2}\begin{bmatrix} -2 & 1 & & & \\ 1 & -2 & \ddots & & \\ & \ddots & \ddots & \ddots & \\ & & \ddots & \ddots & 1 \\ & & & 1 & -2 \end{bmatrix}$$

The Lipschitz Constant blows up as $h \to 0$.

Then discretize in time: (one-step method)

$$U^{n+1} = R(kA_h)U^n$$

e.g. with FE method:

$$U^{n+1} = U^n + kA_h U^n$$
$$= (I + kA_h)U^n$$

Trapezoidal method:

$$U^{n+1} = U^n + \frac{k}{2}\left(A_h U^n + A_h U^{n+1}\right)$$

$$\left(I-\frac{k}{2}A_h\right)U^{n+1}=\left(I+\frac{k}{2}A_h\right)U^n$$

$$U^{n+1}=\underbrace{\left(I-\frac{k}{2}A_h\right)^{-1}\left(I+\frac{k}{2}A_h\right)}_{R(kA_h)}U^n$$

$R(kA_h)$ is a matrix depending on k and h. We denote it by $B_{k,h}$. So we have

$$U^{n+1}=B_{k,h}U^n.$$

So $U^n=B_{k,h}^n U^0.$

Let $u^n=\begin{bmatrix} u(x_1,t_n) \\ u(x_2,t_n) \\ \vdots \\ u(x_m,t_n) \end{bmatrix}$

Then

$$u^{n+1}=B_{k,h}u^n+k\tau^n$$

So $U^{n+1}-u^{n+1}=B_{k,h}(\underbrace{U^n-u^n}_{E^n})-k\tau^n$

$$E^{n+1}=B_{k,h}E^n-k\tau^n$$

Let $T = NK$ be fixed.

i.e. $N = \frac{T}{K}$.

We want to prove that

$$\lim_{K,h\to 0} \|E^N\| = 0.$$

We assume:

① Consistency: $\|\tau\| \to 0$ as $K,h \to 0$
$\|E^0\| \to 0$ as $h \to 0$

② Lax-Richtmeyer stability: $\|B^n_{K,h}\| \leq C(T)$

$C(T)$ is indep. of K, h, and n.

$$E^N = B^N_{K,h} E^0 - K \sum_{j=0}^{N-1} B^{N-1-j}_{K,h} \tau^j$$

$$\|E^N\| \leq \|B^N_{K,h}\| \cdot \|E^0\| + K \sum_{j=0}^{N-1} \|B^{N-1-j}_{K,h}\| \cdot \|\tau^j\|$$

$$\|E^N\| \leq C(T) \|E^0\| + K C(T) N \max_j \|\tau^j\|$$

$$\|E^N\| \leq C(T) \underbrace{\left[\|E^0\| + T \max_j \|\tau^j\|\right]}_{\text{Vanishes as } K,h\to 0}$$

so $\lim_{K,h\to 0} \|E^N\| = 0$.

Proof of stability:

With Euler's method

$$B_{k,h} = I + kA_h$$

Take $\|\cdot\| = \|\cdot\|_2$:

$$\|B_{k,h}\|_2 = \max_{\mu \in \sigma(B)} |\mu|$$

$\mu = 1 + k\lambda$ where $\lambda \in \sigma(A_h)$

$$\lambda = \frac{2}{h^2}(\cos(p\pi h) - 1)$$

So $\|B_{k,h}\| = \max_{1 \le p \le m} \left| 1 + \frac{2k}{h^2}(\cos(p\pi h) - 1) \right|$

We need $\|B^n_{k,h}\| \le C(T)$ $\forall\, n, k, h$

A sufficient condition is $\|B\| \le 1$.

i.e.

$$-1 \le 1 + \frac{2k}{h^2}(\cos(p\pi h) - 1) \le 1$$

$$-1 \le \frac{k}{h^2}(\cos(p\pi h) - 1) \le 0 \qquad -1 \le \cos(p\pi h) \le 1$$

$$-1 \le -2\frac{k}{h^2} \le 0$$

$$\frac{k}{h^2} \le \frac{1}{2} \iff k \le \frac{h^2}{2}.$$

This is the condition for absolute stability!

Recall: For ODEs, we did NOT need absolute stability to prove convergence.

Why do we need it now?

We need KL to remain bounded.

Von Neumann analysis

A straightforward way to study stability of linear PDE discretizations with periodic bdy. conditions.

$$u_t + u_{xx} = 0 \quad 0 < x < 1$$

$$u(x,0) = \eta(x)$$

$$u(0,t) = u(1,t)$$

for $2 \le j \le m-1$

Discretization: $U_j^{n+1} = U_j^n + \frac{k}{h^2}\left(U_{j+1}^n - 2U_j^n + U_{j-1}^n\right)$

Boundaries: $U_1^{n+1} = U_1^n + \frac{k}{h^2}\left(U_2^n - 2U_1^n + U_m^n\right)$

$$U_m^{n+1} = U_m^n + \frac{k}{h^2}\left(U_1^n - 2U_m^n + U_{m-1}^n\right)$$

Ansatz: $U_j^n = g_\xi^n e^{ijh\xi}$

$$g^{n+1} e^{ijh\xi} = g^n e^{ijh\xi} + \frac{k}{h^2} g^n \left(e^{ih\xi(j+1)} - 2e^{ih\xi j} + e^{ih\xi(j-1)}\right)$$

$$e^{i\theta} + e^{-i\theta} = 2\cos(\theta)$$

$$g = 1 + \frac{k}{h^2}\left(e^{ih\xi} - 2 + e^{-ih\xi}\right)$$

$$g = 1 + \frac{k}{h^2}\left(2\cos(h\xi) - 2\right)$$

$$g = 1 + \frac{2k}{h^2}\left(\cos(h\xi) - 1\right) \Rightarrow \text{stable if } |g| \le 1$$

What have we done?

We have $U^{n+1} = BU^n$

Where $B = I + k\hat{A}_h$

with $\hat{A}_h = \frac{1}{h^2}\underbrace{\begin{bmatrix} -2 & 1 & & & 1 \\ 1 & -2 & \ddots & & \\ & \ddots & \ddots & \ddots & \\ & & \ddots & \ddots & 1 \\ 1 & & & 1 & -2 \end{bmatrix}}_{\text{circulant matrix}}$

Every linear PDE FD discretization with periodic BCs is circulant.

Every circulant matrix has the same eigenvectors.

Typically, vN analysis provides conditions that are necessary for stability with other BCs. (but maybe not sufficient)

A note about L-R stability

If we can show that

$$\|B\| < 1 + \alpha k$$

then $\|B^n\| \le \|B\|^n < (1+\alpha k)^n = 1 + n\alpha k + \cdots$

$$\le e^{\alpha T}$$

So this is enough.