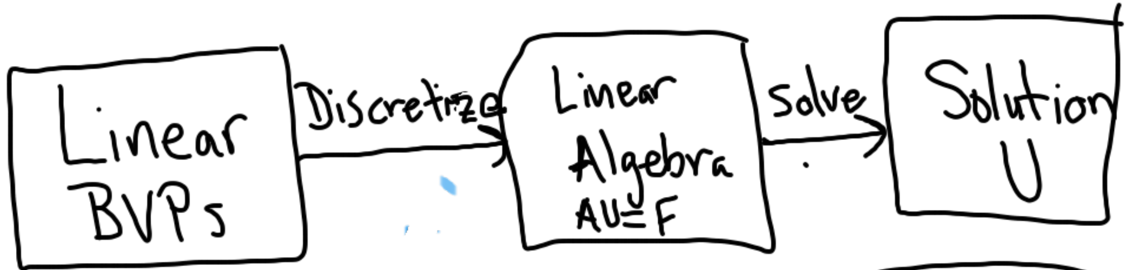
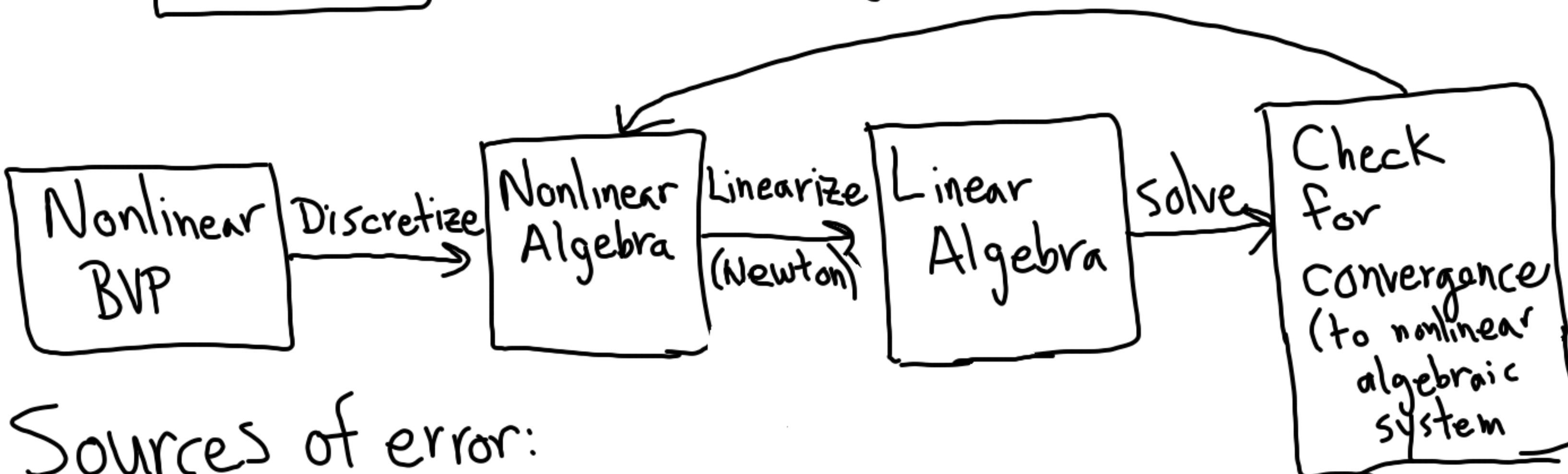
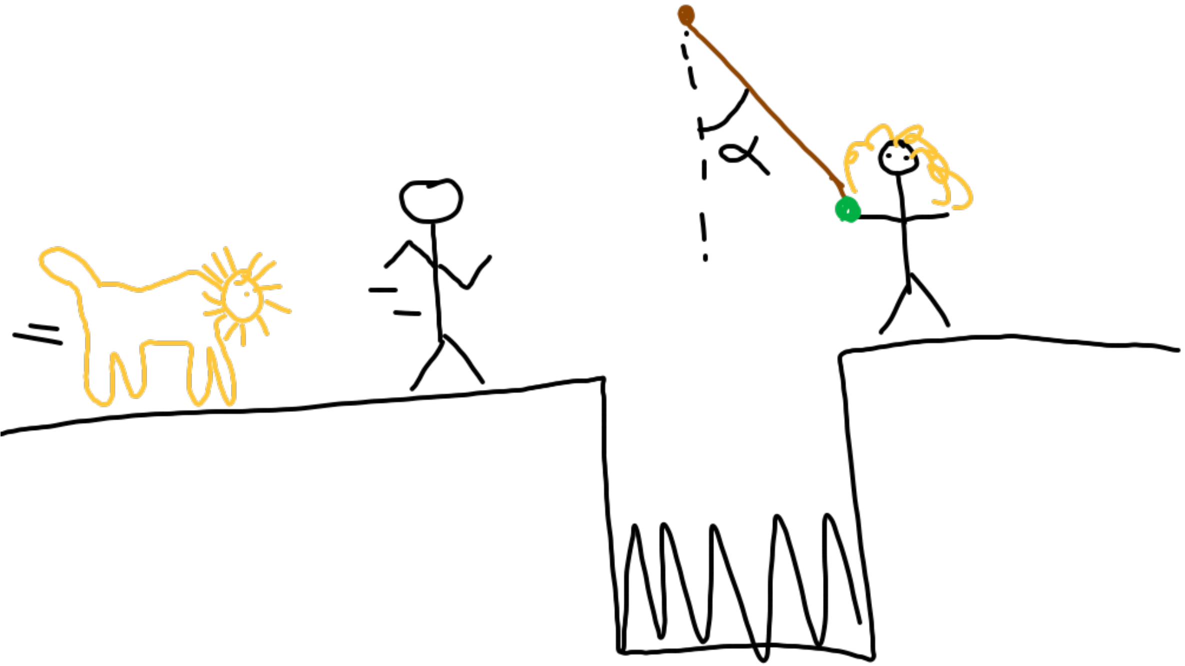




Sources of error:

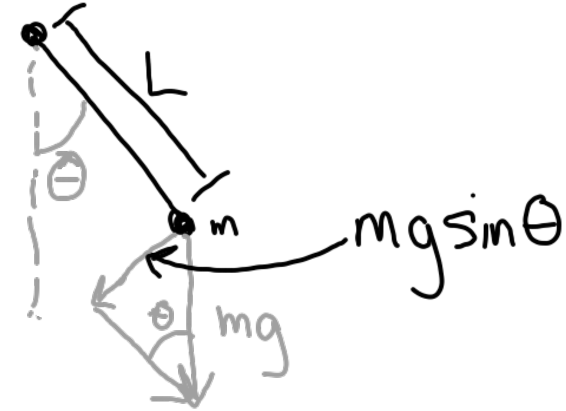
1. Discretization
2. Rounding
3. Linearization



BVP: $\theta(0) = \alpha$

$$\theta(T) = \beta$$

$$\theta''(t) = -\sin(\theta(t))$$



$$F = ma \qquad a = \theta''(t)L$$

$$-mg\sin\theta(t) = mL\theta''(t)$$

$$\theta''(t) = -\frac{g}{L}\sin(\theta(t))$$

Choose units so $\frac{g}{L} = 1$:

$$\theta''(t) = -\sin(\theta(t))$$

$$h = \frac{1}{m+1}$$

$$\theta''(t_i) \approx \frac{\theta_{i+1} - 2\theta_i + \theta_{i-1}}{h^2}$$

$$\theta_0 = \alpha$$

$$\theta_{m+1} = \beta$$

$$\boxed{\frac{\theta_{i+1} - 2\theta_i + \theta_{i-1}}{h^2} + \sin\theta_i = 0} \quad G(\theta) = 0$$

for $i = 1, 2, \ldots, m$.

Let θ_* denote the exact solution:

$$G(\theta_*) = 0$$

and $\theta^{[0]}$ an initial guess. δ

$$0 = G(\theta_*) = G(\theta^{[0]}) + G'(\theta^{[0]})(\theta_* - \theta^{[0]}) + \mathcal{O}(\|\theta_* - \theta^{[0]}\|^2)$$

$G'(\theta) = J(\theta)$ is the Jacobian:

$$J(\theta) = \begin{bmatrix} \frac{\partial G_1}{\partial \theta_1} & \frac{\partial G_1}{\partial \theta_2} & \cdots & - \\ \frac{\partial G_2}{\partial \theta_1} & & & \\ \vdots & & & \\ \vdots & & & \frac{\partial G_m}{\partial \theta_m} \end{bmatrix}$$

$$J(\theta^{[0]})\delta = -G(\theta^{[0]})$$

$$\theta^{[1]} = \theta^{[0]} + \delta$$

Newton's method:

1. Start with initial guess $\theta^{[0]}$, $K=0$
2. Solve $J(\theta^{[K]})\delta = -G(\theta^{[K]})$
3. $\theta^{[K+1]} = \theta^{[K]} + \delta$
4. If not converged, increment K and go back to 2.

$$G_i = \frac{1}{h^2}(\theta_{i+1} - 2\theta_i + \theta_{i-1}) + \sin(\theta_i)$$

$$J_{ij} = \begin{cases} \frac{1}{h^2} & j = i \pm 1 \\ \frac{-2}{h^2} + \cos(\theta_i) & j = i \\ 0 & |j-i| > 1 \end{cases}$$

$$J = \frac{1}{h^2}\begin{bmatrix} -2 & 1 & & \\ 1 & \ddots & \ddots & \\ & \ddots & \ddots & 1 \\ & & 1 & -2 \end{bmatrix} + \begin{bmatrix} \cos\theta_1 & & & \\ & \cos\theta_2 & & \\ & & \ddots & \\ & & & \cos\theta_N \end{bmatrix}$$

Consistency

Local truncation error:

$$\tau_i = \frac{1}{h^2}\left(\theta(t_{i+1}) - 2\theta(t_i) + \theta(t_{i-1})\right) + \sin(\theta(t_i))$$

$$\tau_i = \underline{\theta''(t_i)} + \frac{1}{12}h^2\theta^{(4)}(t_i) + \mathcal{O}(h^4) + \underline{\sin(\theta(t_i))}$$

$$\tau_i = \frac{1}{12}h^2\theta^{(4)}(t_i) + \mathcal{O}(h^4)$$

So the method is consistent and locally 2nd-order

Stability

Let $\hat{\theta} = \begin{bmatrix} \theta(t_1) \\ \vdots \\ \vdots \\ \theta(t_m) \end{bmatrix}$, $\tau = \begin{bmatrix} \tau_1 \\ \vdots \\ \vdots \\ \tau_m \end{bmatrix} = G(\hat{\theta})$

$$G(\theta) = 0$$

$$\tau = G(\hat{\theta}) - G(\theta)$$

$$E = \theta - \hat{\theta}$$

$$G(\theta) = G(\hat{\theta}) + J(\hat{\theta})E + \mathcal{O}(\|E\|^2)$$

$$-\tau = J(\hat{\theta})E + \mathcal{O}(\|E\|^2)$$

It's not clear that we can ignore $O(\|E\|^2)$ terms, since our goal is to show that $\|E\|$ is small.

But if we do, we have

$$E \approx -J(\hat{\theta})^{-1}\tau$$

$$\|E\| \leq \|J(\hat{\theta})^{-1}\| \cdot \|\tau\|$$

Stability requires that

$$\|J(\hat{\theta})^{-1}\| < C$$

for small enough h.

In fact this holds because $J(\theta)$ is close to A as $h \to 0$.