Stability and Convergence of one-step methods

$$u'(t) = f(u)$$
$$u(t_0) = \eta \qquad t \in [t_0, T]$$

$$E^N = U^N - u(t_N)$$

where $N = \dfrac{T - t_0}{K}$

We want to prove that

$$\lim_{K \to 0} \|E^N\| = 0$$

3 steps:

1. Euler's method applied Dahlquist's problem $u' = \lambda u + g(t)$
2. Euler's method applied to $u' = f(u)$ (f Lipschitz)
3. Any RK method, any Lipschitz f

Dahlquist's problem:

$$u'(t) = \lambda u(t) + g(t) \qquad u(t_0) = \eta$$

Solution: $u(t) = e^{\lambda(t-t_0)}\eta + \int_{t_0}^{t} e^{\lambda(t-\tau)} g(\tau)\,d\tau$

Apply Euler's method:

$$\frac{U^{n+1} - U^n}{k} = \lambda U^n + g(t_n)$$

Local Truncation Error

$$\frac{u(t_{n+1}) - u(t_n)}{k} = \lambda u(t_n) + g(t_n) + \tau^n$$

$$\tau^n = O(k)$$

$$\frac{U^{n+1} - u(t_{n+1}) - (U^n - u(t_n))}{k}$$

$$= \lambda(U^n - u(t_n)) - \tau^n$$

$$\frac{E^{n+1} - E^n}{k} = \lambda E^n - \tau^n$$

$$\boxed{E^{n+1} = (1 + k\lambda)E^n - k\tau^n}$$

Euler's method: $U^{n+1} = (1 + k\lambda)U^n + kg(t_n)$

$$E^N = (1+k\lambda)E^{N-1} - k\tau^{N-1}$$

$$E^N = (1+k\lambda)\left((1+k\lambda)E^{N-2} - k\tau^{N-2}\right) - k\tau^{N-1}$$

$$E^N = (1+k\lambda)^2 E^{N-2} - (1+k\lambda)k\tau^{N-2} - k\tau^{N-1}$$

$$\vdots$$

$$E^N = \underbrace{(1+k\lambda)^N E^0 - k\sum_{m=1}^{N} (1+k\lambda)^{N-m} \tau^{m-1}}$$

usually vanishes

Discrete analog Duhamel's principle.

$$|E^N| = k\left|\sum_{m=1}^{N} (1+k\lambda)^{N-m} \tau^{m-1}\right|$$

$$|E^N| \le k \sum_{m=1}^{N} |1+k\lambda|^{N-m} |\tau^{m-1}|$$

triangle inequality

Lemma: $|1+k\lambda| \le e^{k|\lambda|}$

$$e^{k|\lambda|} \ge 1 + k|\lambda|$$

$$|1+k\lambda| \le 1 + k|\lambda|.$$

$$|E^N| \le k \sum_{m=1}^{N} e^{k|\lambda|(N-m)} |\tau^{m-1}|$$

$$\le k \sum_{m=1}^{N} e^{k|\lambda|N} |\tau^{m-1}|$$

Note: take $t_0 = 0$. Then $kN = T$.

$$|E^N| \leq k \sum_{m=1}^{N} e^{T|\lambda|} |\tau^{m-1}|$$

$$= k e^{|\lambda| T} \sum_{m=1}^{N} |\tau^{m-1}|$$

$$\leq N k e^{|\lambda| T} \max_{1 \leq m \leq N} |\tau^{m-1}|$$

$$|E^N| \leq \underbrace{T e^{|\lambda| T}}_{\text{Indep. of } k} \underbrace{\|\tau\|_\infty}_{= O(k)}$$

$$\Rightarrow \lim_{k \to 0} |E^N| = 0$$

Consider $T = 10$

$|\lambda| = 10$

$$|E^N| \leq 10 e^{100} \|\tau\|_\infty$$

Gehrmund Dahlquist:

"Such constants don't belong in numerical analysis!"

② Converge of Euler's method for Lipschitz IVPs

$$u'(t) = f(u)$$
$$u(0) = \eta$$

Assume: $\|f(v) - f(w)\| \le L\|v - w\|$

$$\frac{U^{n+1} - U^n}{k} = f(U^n) \quad (1)$$

$$\frac{u(t_{n+1}) - u(t_n)}{k} = f(u(t_n)) + \tau^n \quad (2)$$

Subtract (2) from (1)

$$\frac{E^{n+1} - E^n}{k} = f(U^n) - f(u(t_n)) + \tau^n$$

$$E^{n+1} = E^n + k\left(f(U^n) - f(u(t_n))\right) + k\tau^n$$

$$\|E^{n+1}\| \le \|E^n\| + k\|f(U^n) - f(u(t_n))\| + k\|\tau^n\|$$

$$\|E^{n+1}\| \le \|E^n\| + kL\|E^n\| + k\|\tau^n\|$$

$$\|E^{N+1}\| \le (1 + kL)\|E^N\| + k\|\tau^N\| \quad (3)$$

We substitute (3) into itself repeatedly to obtain

$$\|E^N\| \le (1+kL)\underbrace{\|E^0\|}_{\text{vanishes}} + k\sum_{m=1}^{N}(1+kL)^{N-m}\|\tau^{m-1}\|$$

$$\|E^N\| \le k\sum_{m=1}^{N}(1+kL)^{N}\|\tau^{m-1}\|$$

$$\|E^N\| \le (1+kL)^N \sum_{m=1}^{N}\|\tau^{m-1}\|$$

Recall: $1+kL \le e^{kL}$

$$\|E^N\| \le e^{NkL}\, kN \max_m \|\tau^{m-1}\|$$

$$\|E^N\| \le T e^{TL} \max_m \|\tau^{m-1}\|$$

$$\Rightarrow \lim_{k\to 0}\|E^N\| = 0$$

③ Convergence of RK methods

$$U^* = U^n + \tfrac{1}{2}k f(U^n)$$

$$U^{n+1} = U^n + k f(U^*)$$

$$\frac{U^{n+1}-U^n}{k} = \underbrace{f\left(U^n + \tfrac{1}{2}k f(U^n)\right)}_{\Psi(U^n)}$$

Claim: if f is L.C. then Ψ is also L.C.

Proof: Assume $\|f(v)-f(w)\| \le L\|v-w\|$.

Then

$$\begin{aligned}\|\Psi(v)-\Psi(w)\| &= \|f(v+\tfrac{1}{2}kf(v))-f(w+\tfrac{1}{2}kf(w))\| \\ &\le L\|v+\tfrac{1}{2}kf(v)-(w+\tfrac{1}{2}kf(w))\| \\ &= L\|v-w+\tfrac{1}{2}k(f(v)-f(w))\| \\ &\le L(\|v-w\|+\tfrac{1}{2}k\|f(v)-f(w)\|) \\ &\le L\|v-w\|+\tfrac{1}{2}kL^2\|v-w\| \\ &= (L+\tfrac{1}{2}kL^2)\|v-w\|\end{aligned}$$

So $L+\frac{1}{2}kL^2$ is a Lipschitz constant for Ψ.

Similarly, for any RK method we can show that Ψ is L.C...

The rest of the proof is identical to ② if we replace $f \to \Psi$, $L \to L+\frac{1}{2}kL^2$.

We end up with:

$$\|E^N\| \le \underbrace{\left(1+kL+\tfrac{1}{2}k^2L^2\right)}_{\le e^{kL}}\|E^{N-1}\| + k\|\tau^{N-1}\|$$

So eventually we find

$$\|E^N\| \le T e^{LT} \underbrace{\max_m \|\tau^{m-1}\|}_{\mathcal{O}(k^p) \text{ where } p = \text{local order of accuracy}}$$