Reminders:

- Homework 4 due today
- Proposal revisions due Thursday (if requested)

$$\begin{array}{c|c} c & A \\ \hline & b^T \end{array}$$

Method with m stages:

$$\frac{m(m-1)}{2} + m = \frac{m(m+1)}{2}$$

coefficients

9th order: 486 eqns.

10th order: 1205 eqns.

1970: 10th order, 17 stages

2019: 10th order, 16 stages*

Initial-Boundary Value Problems: The Heat Equation

$$\frac{\partial u}{\partial t} = K\frac{\partial^2 u}{\partial x^2} + f(x,t) \qquad \begin{array}{l} 0 \le x \le 1 \\ 0 \le t \le T \end{array}$$

$u(x,t)$: distribution of heat

K: Heat conductance

f: source/sink of heat

$$u_t = K u_{xx} + f(x,t)$$

Initial data: $u(x,0) = \eta(x)$

Boundary data:

$$u(0,t) = \alpha(t)$$

$$u(1,t) = \beta(t)$$

Simple case: $K=1 \quad f=0$

$\alpha(t)=0 \quad \beta(t)=0$

$$u_t = u_{xx}$$

Exact solution

Fourier series of I.C.:

$$u(x,0)=\sum_{p=0}^{\infty}\hat{u}_p(0)\sin(p\pi x)$$

Ansatz:

$$u(x,t)=\sum_{p=0}^{\infty}\hat{u}_p(t)\sin(p\pi x)$$

Substitute into PDE:

$$\sum_{p=0}^{\infty}\hat{u}_p'(t)\sin(p\pi x)=\sum_{p=0}^{\infty}\hat{u}_p(t)\left(-p^2\pi^2\right)\sin(p\pi x)$$

Decouple: $\hat{u}_p'(t)=-p^2\pi^2\,\hat{u}_p(t)$

Solution:

$$\hat{u}_p(t)=e^{-p^2\pi^2 t}\,\hat{u}_p(0)$$

$$\Rightarrow u(x,t)=\sum_{p=0}^{\infty}\hat{u}_p(0)e^{-p^2\pi^2 t}\sin(p\pi x)$$

Discretization

We will use the method of lines:

1. Discretize in space → IV ODE
2. Apply a RKM or LMM in time.

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1. $x_j = jh \qquad j = 0, 1, \ldots, m+1$

$h = \frac{1}{m+1}$

$U_j(t) \approx u(x_j, t)$

$u_{xx}(x_j, t) = \dfrac{U_{j+1} - 2U_j + U_{j-1}}{h^2}$

We get an ODE system

$$\underbrace{\begin{bmatrix} U_1 \\ \vdots \\ \vdots \\ U_m \end{bmatrix}'}_{U} = \underbrace{\frac{1}{h^2} \begin{bmatrix} -2 & 1 & & \\ 1 & \ddots & \ddots & \\ & \ddots & \ddots & 1 \\ & & 1 & -2 \end{bmatrix}}_{A} \begin{bmatrix} U_1 \\ \vdots \\ \vdots \\ U_m \end{bmatrix}$$

$U'(t) = AU \Rightarrow U(t) = e^{tA} U(0)$

Semi-discrete system

$A = R\Lambda R^{-1}$ $p = 1, 2, \ldots, m$

Here: $\lambda_p = \frac{2}{h^2}(\cos(p\pi h) - 1)$

$r_{jp} = \sin(p\pi x_j) = \sin(p\pi j h)$

$U'(t) = R\Lambda R^{-1} U(t)$

$R^{-1} U'(t) = \Lambda R^{-1} U(t)$ $\qquad \hat{U}(t) = R^{-1} U(t)$

$\hat{U}'(t) = \Lambda \hat{U}(t)$

$\hat{U}(t) = e^{t\Lambda} \hat{U}(0)$

$\cos(p\pi h) \approx 1 - \frac{1}{2} p^2 \pi^2 h^2$

So $\lambda_p = -p^2 \pi^2 + O(h^2)$

for small ph.

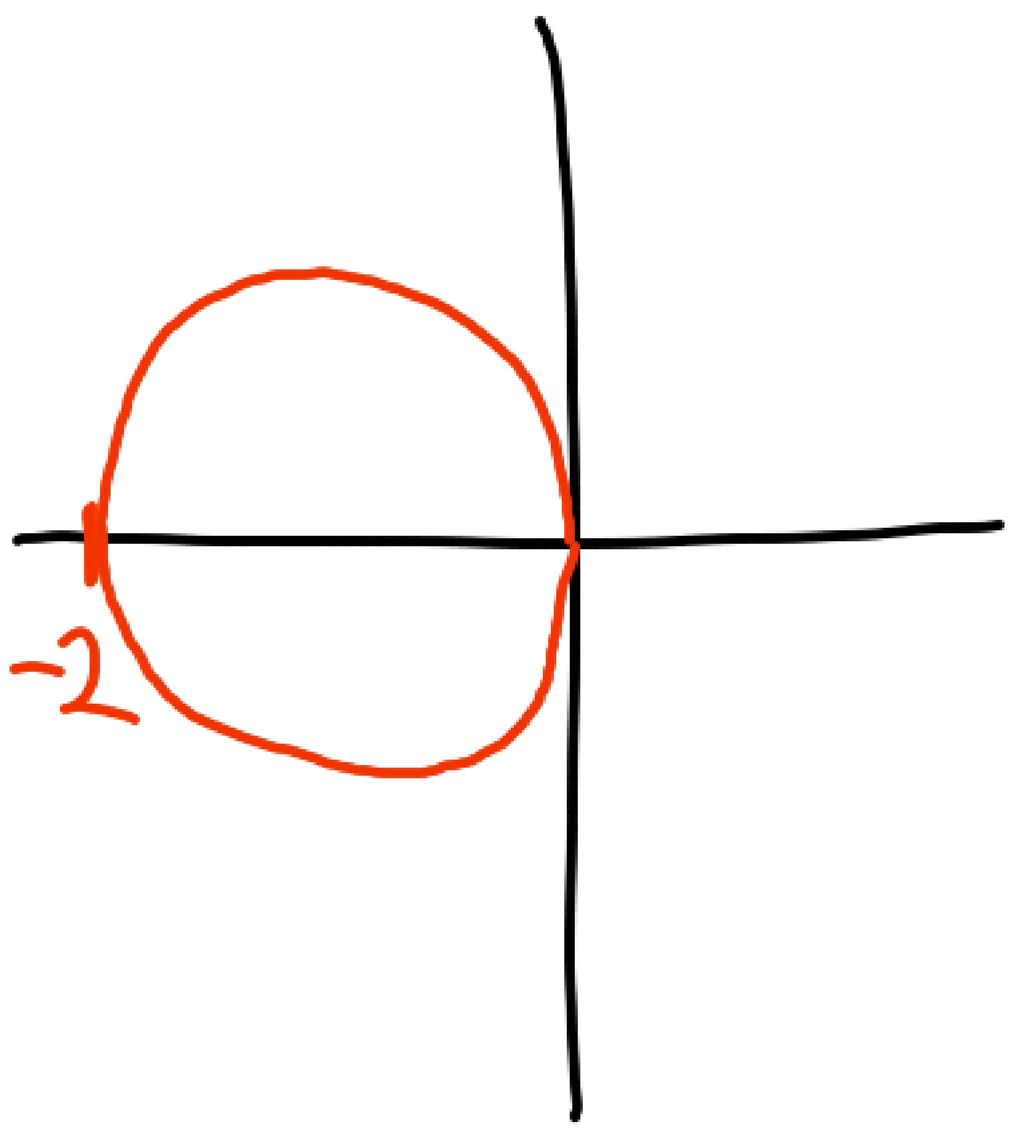
Very accurate for small ph.

How large is the largest eigenvalue?

$-\frac{4}{h^2}$

We need $-\frac{4}{h^2} k \in S$ (the region of absolute stability)

For instance, with Explicit Euler:



$$-\frac{4K}{h^2} \geq -2$$

$$K \leq \frac{h^2}{2}$$

Requires a very small step size.

We should use an A-stable or $A(\alpha)$-stable method. Then there is no stability restriction on K.

One nice method is TR-BDF2 (DIRK)

$$\begin{array}{c|ccc} 0 & 0 & 0 & 0 \\ 1/2 & 1/4 & 1/4 & 0 \\ 1 & 1/3 & 1/3 & 1/3 \\ \hline & 1/3 & 1/3 & 1/3 \end{array}$$

Need to solve a tridiagonal system of size m for each stage.

This method is:

A-stable
L-stable
2nd-order accurate

Local truncation error

(Explicit Euler)

$$U_j^{n+1} = U_j^n + \frac{k}{h^2}\left(U_{j+1}^n - 2U_j^n + U_{j-1}^n\right)$$

To find LTE:

$$u(x_j, t_{n+1}) = u(x_j, t_n) + \frac{k}{h^2}\left(u(x_{j+1}, t_n) - 2u(x_j, t_n) + u(x_{j-1}, t_n)\right) + k\tau_j^n$$

$$\Rightarrow \tau_j^n = \frac{k}{2}u_{tt} - \frac{h^2}{12}u_{xxxx} + \mathcal{O}(k^2) + \mathcal{O}(h^4)$$

1st-order in t

2nd-order in x

In general if we use a time stepping of order p we get

$$\tau_j^n = \mathcal{O}(k^p) + \mathcal{O}(h^2)$$