Time Stepping for PDEs of mixed type

Semilinear PDE:

$$u_t = \underbrace{f(u)}_{\text{non-stiff}} + \underbrace{Lu}_{\text{stiff}}$$

Examples:

KdV: $u_t + uu_x + u_{xxx} = 0$

(surface water waves)

Navier-Stokes: $q_t + \underbrace{\nabla \cdot f(q)}_{\text{hyperbolic, nonlinear}} = \underbrace{P(q)}_{\text{parabolic, linear}}$

Burgers: $u_t + uu_x = u_{xx}$

NLS: $i\psi_t = \psi_{xx} + |\psi|^2\psi + V(x)\psi$

Reactive fluid dynamics*

① Operator splitting (fractional step)

② Implicit-Explicit (ImEx)

③ Exponential

Lie-Trotter/Godunov splitting

ⓐ Solve $u_t = f(u)$ over one step

ⓑ Solve $u_t = Lu$ over one step

Repeat

For part ⓐ: use an explicit method

For part ⓑ: use an implicit

We can:

- Take large time steps
- Solve only linear equations

Disadvantage: Splitting error

Consider the linear eqn.

$$u_t = Au + Bu = (A+B)u$$

Exact solution: $u(t+k) = e^{k(A+B)} u(t)$

$$e^{k(A+B)} = I + k(A+B) + \frac{k^2}{2}(A+B)^2 + O(k^3)$$

$$= I + k(A+B) + \frac{k^2}{2}(\underline{A^2 + B^2 + AB + BA}) + O(k^3)$$

Assume ⓐ and ⓑ are each solved exactly. Then

$$e^{KB} e^{KA} = e^{K(B+A)}$$

$$U^{n+1} = e^{KB} e^{KA} U^n$$

$$= \left(I + KB + \frac{K^2}{2}B^2 + O(K^3)\right)\left(I + KA + \frac{K^2}{2}A^2 + O(K^3)\right)U^n$$

$$= \left(I + KA + KB + K^2\left(\frac{1}{2}A^2 + \frac{1}{2}B^2 + BA\right) + O(K^3)\right)U^n$$

$$= \left(I + K(A+B) + \frac{K^2}{2}\underline{\left(A^2 + B^2 + 2BA\right)} + O(K^3)\right)U^n$$

These would match only if A and B commute.

In general, this method is 1st-order accurate.

Strang Splitting

ⓐ Take a half step: $u_t = f(u)$

ⓑ Take a full step: $u_t = Lu$

ⓒ Take a half step: $u_t = f(u)$

2nd-order accurate

It's possible to design methods of any order of accuracy.

$$u'(t) = Lu + f(u)$$