AMCS 252 Homework 3

Exercise 1

Let $f(u) = \log(u)$.

(a) Determine the best possible Lipschitz constant for this function over $2 \leq u \leq \infty$.

(b) Is $f(u)$ Lipschitz continuous over $0 < u < \infty$?

(c) Consider the initial value problem

$$\begin{aligned} u'(t) &= \log(u(t)) \\ u(0) &= 2. \end{aligned}$$

Explain why we know that this problem has a unique solution for all $t \geq 0$ based on the existence and uniqueness theory described in Section 5.2.1. (Hint: argue that f is Lipschitz continuous in a domain that the solution never leaves, though the domain is not symmetric about $\eta = 2$ as assumed in the theorem quoted in the book.)

Exercise 2

Consider the system of ODEs

$$\begin{aligned} u_1'(t) &= 3u_1 + 4u_2 \\ u_2'(t) &= 5u_1 - 6u_2. \end{aligned}$$

Determine the Lipschitz constant for this system in the maximum norm and the 1-norm.

Exercise 3

The initial value problem

$$v''(t) = -4v, \quad v(0) = v_0, \quad v'(0) = v_0',$$

has the solution $v(t) = v_0 \cos(2t) + \frac{1}{2} v_0' \sin(2t)$. Determine this solution by rewriting the ODE as a first-order system $u' = Au$ so that $u(t) = e^{tA} u(0)$ and then computing the matrix exponential using (D.30) in Appendix D.

Exercise 4 Consider the SIR model

$$\begin{aligned} x'(t) &= -\beta x y \\ y'(t) &= \beta x y - \gamma y \\ z'(T) &= \gamma y, \end{aligned}$$

where x, y, z represent susceptible, infected, and removed proportions of the population. Let initial conditions $x(0), y(0), z(0)$ be given such that $x(0) + y(0) + z(0) = 1$. Since $x + y + z = 1$ for all time, we can study the system by considering just the first two differential equations and setting $z(t) = 1 - x(t) - y(t)$.

Consder the domain

$$D = \{(x, y) : x \geq 0, y \geq 0, x + y \leq 1\}.$$

Show that the SIR model has a unique solution for all $t > 0$ whenever $(x(0), y(0)) \in D$, as follows:

1. Show that if $(x(0), y(0)) \in D$, then $(x(t), y(t)) \in D$ for all $t > 0$. Hint: to show that x, y remain non-negative, consider the behavior of the SIR system when $x = 0$ or $y = 0$. Be sure to state your reasoning clearly and carefully.

2. Show that the function

$$f : \begin{bmatrix} x \\ y \end{bmatrix} \to \begin{bmatrix} -\beta xy \\ \beta xy - \gamma y \end{bmatrix}$$

is Lipschitz continuous for $(x, y) \in D$.