Lax-Wendroff method

$$u_t + a u_x = 0$$

$$u_t = -a u_x$$

$$u(x,t+k) = u(x,t) + k u_t(x,t) + \frac{k^2}{2} u_{tt}(x,t) + \mathcal{O}(k^3)$$

$$u_{tt} = -a u_{xt} \qquad u_{xt} = -a u_{xx}$$

$$u_{tt} = a^2 u_{xx}$$

$$u(x,t+k) = u(x,t) - k a u_x(x,t) + \frac{k^2 a^2}{2} u_{xx}(x,t) + \mathcal{O}(k^3)$$

$$U_j^{n+1} = U_j^n - \frac{ka}{2h}\left(U_{j+1}^n - U_{j-1}^n\right) + \frac{k^2 a^2}{2h^2}\left(U_{j+1}^n - 2U_j^n + U_{j-1}^n\right)$$

2nd order in space time

Hyperbolic Systems of PDEs

Example: Acoustics

$p(x,t)$: pressure

$u(x,t)$: velocity

$$p_t = -K u_x$$

Bulk modulus

$F = ma$

density

$$p_x = \rho u_t$$

$$q = \begin{bmatrix} p \\ u \end{bmatrix}$$

$$p_t + K u_x = 0$$

$$u_t + \frac{1}{\rho} p_x = 0$$

$$A = \begin{bmatrix} 0 & K \\ 1/\rho & 0 \end{bmatrix}$$

$$(1) \quad q_t + A q_x = 0$$

Linear, 1st-order hyperbolic system

We say (1) is hyperbolic if A is diagonalizable with real eigenvalues

$$AR = R\Lambda \qquad R = \left[\, r^+ \,\middle|\, r^- \,\right]$$

$$A = R\Lambda R^{-1} \qquad \Lambda = \begin{bmatrix} \lambda^+ & \\ & \lambda^- \end{bmatrix}$$

$$q_t + R\Lambda R^{-1} q_x = 0$$

$$w = R^{-1} q$$

$$R^{-1} q_t + \Lambda R^{-1} q_x = 0$$

$$w = \begin{bmatrix} w^1 \\ w^2 \end{bmatrix}$$

$$w_t + \Lambda w_x = 0$$

$$\left.\begin{aligned} w^1_t + \lambda_1 w^1_x &= 0 \\ w^2_t + \lambda_2 w^2_x &= 0 \end{aligned}\right\}$$ Advection eqns. if $\lambda_1, \lambda_2 \in \mathbb{R}$.

$$\lambda^2 - \frac{K}{\rho} = 0 \qquad \lambda^\pm = \pm\sqrt{\frac{K}{\rho}}$$

Speed of sound

Eigenvectors:

$$r^\pm = \begin{bmatrix} r_1^\pm \\ r_2^\pm \end{bmatrix}$$

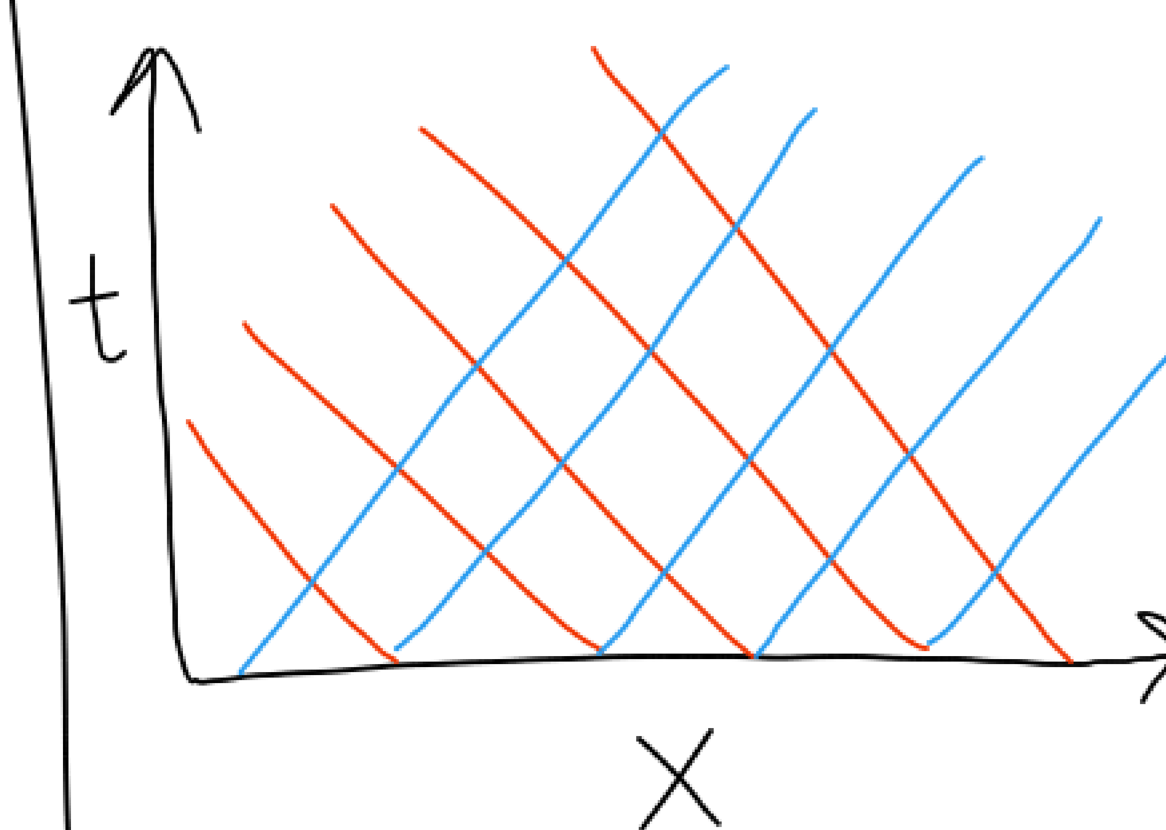
$$A r^\pm = \lambda^\pm r^\pm$$

$$K r_2^\pm = \lambda^\pm r_1^\pm$$

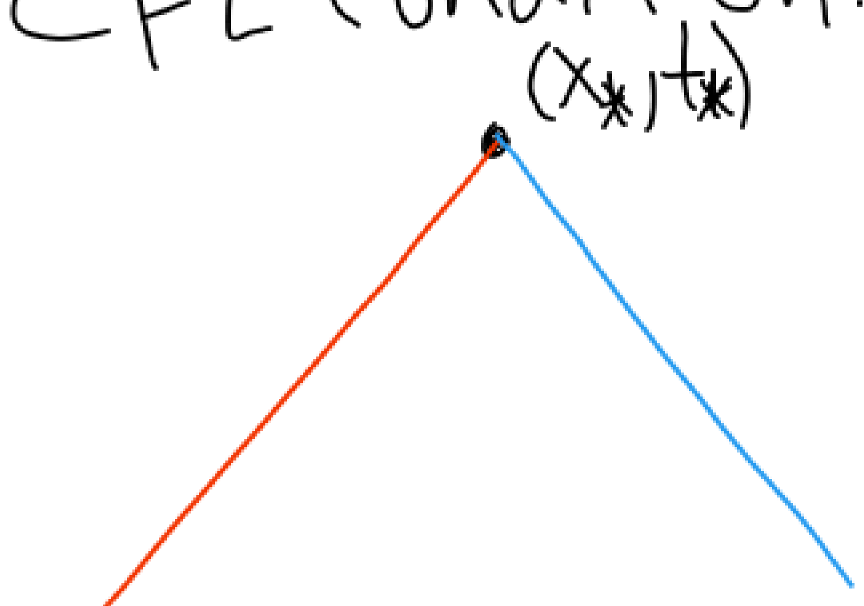
$$r_1^\pm = \frac{K}{\lambda^\pm} r_2^\pm = \pm\overbrace{\sqrt{K\rho}}^{Z\text{: impedance}}\, r_2^\pm$$

$$R = \begin{bmatrix} -Z & +Z \\ 1 & 1 \end{bmatrix}$$

We have 2 families of characteristics



CFL condition:



Need info from both sides

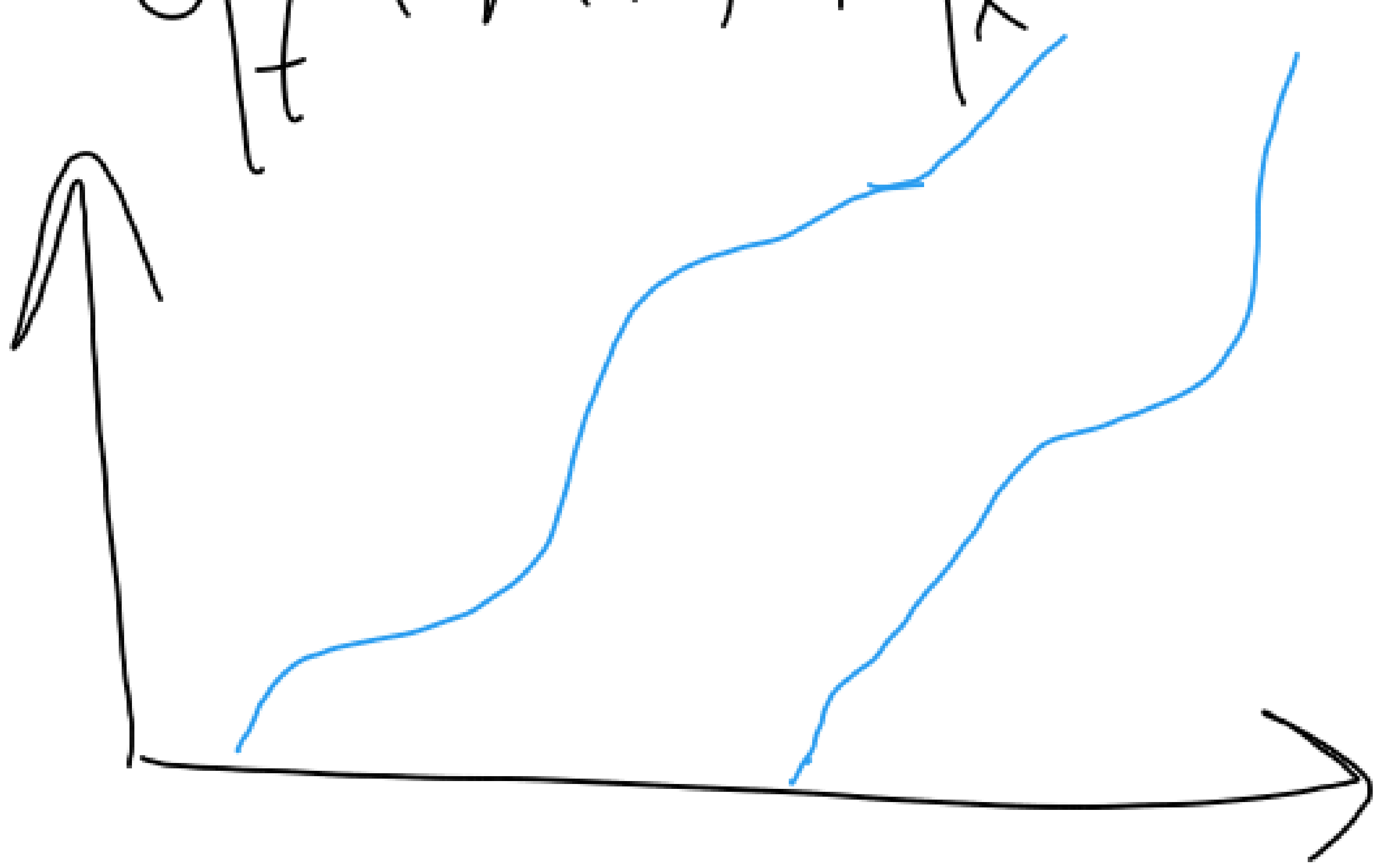
Lax Wendroff:

$$U_j^{n+1} = U_j^n - \frac{ka}{2h}\left(U_{j+1}^n - U_{j-1}^n\right) + \frac{k^2a^2}{2h^2}\left(U_{j+1}^n - 2U_j^n + U_{j-1}^n\right)$$

For Systems:

$$Q_j^{n+1} = Q_j^n - \frac{k}{2h}A\left(Q_{j+1}^n - Q_{j-1}^n\right) + \frac{k^2}{2h^2}A^2\left(Q_{j+1}^n - 2Q_j^n + Q_{j-1}^n\right)$$

$$q_t + A(x,t)q_x = 0$$



$$q_t + f(q)_x = 0$$

Quasilinear form:

(2) $q_t + f'(q)q_x = 0$

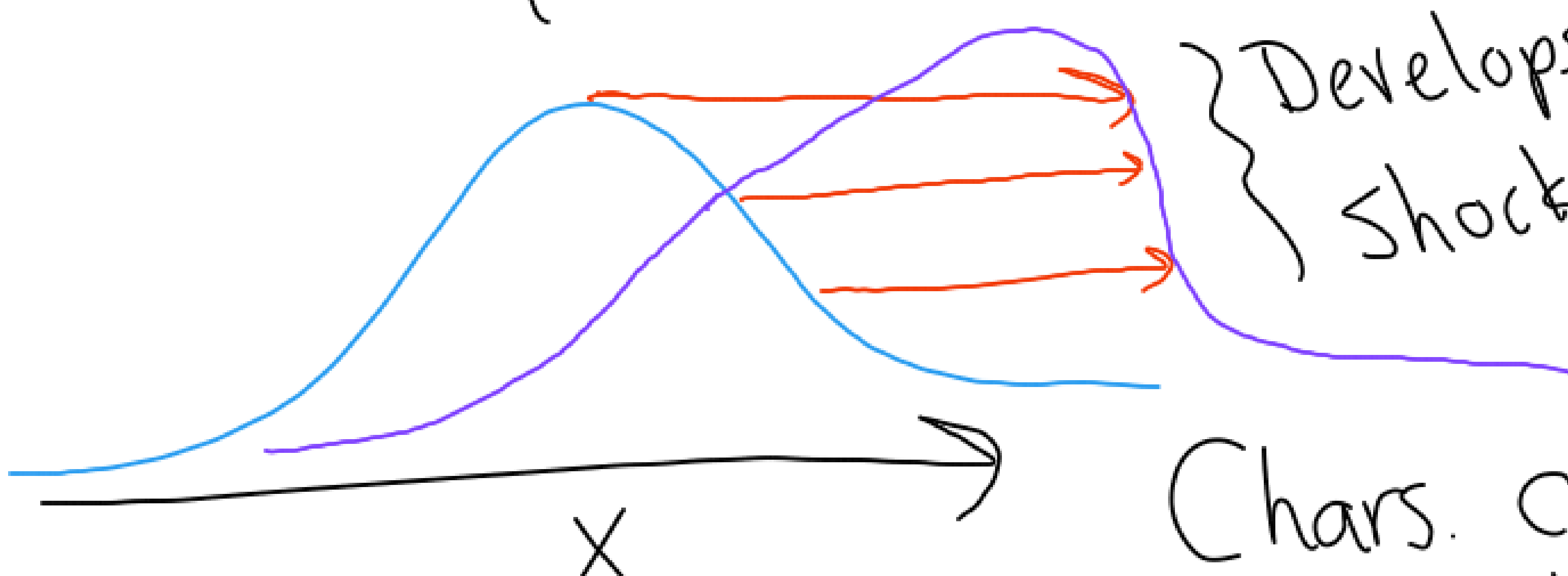
We say (2) is hyperbolic if $f'(q)$ is diagonalizable w/real eigenvalues.

Example:

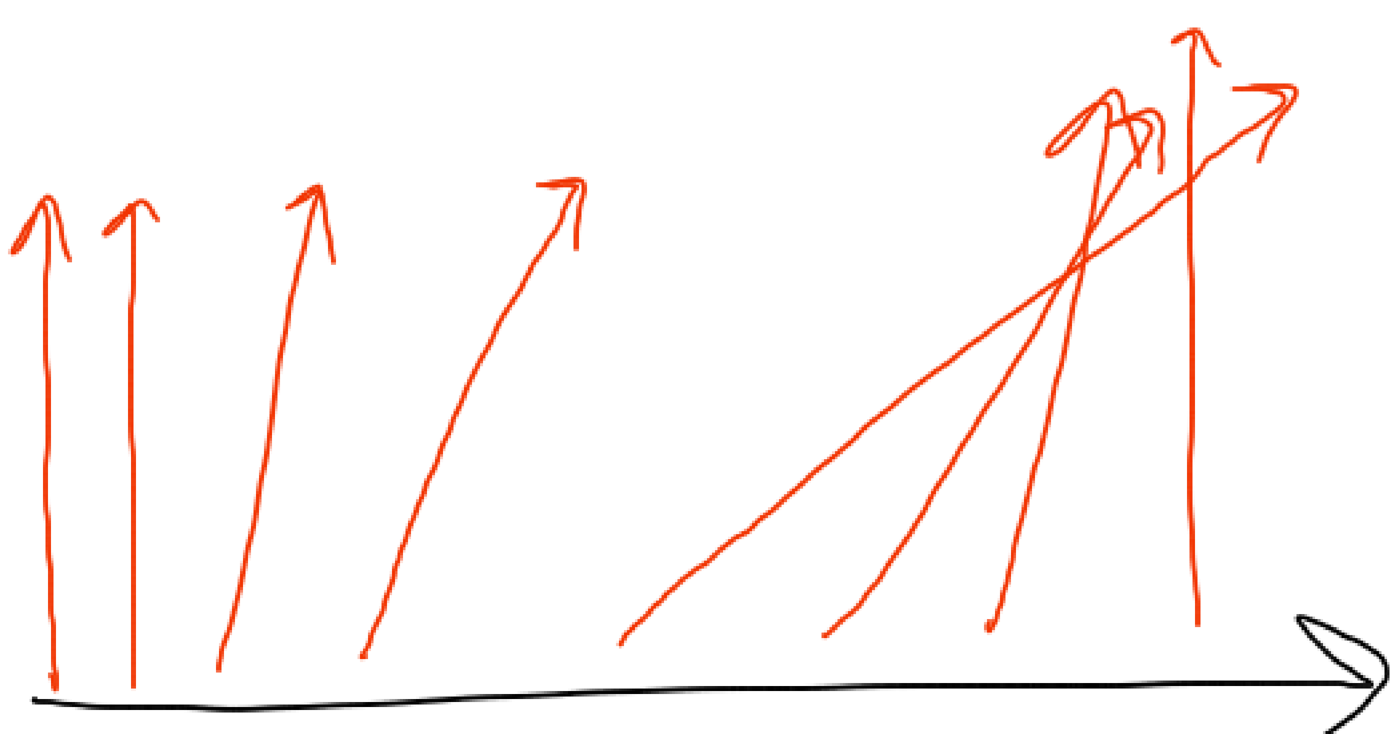
Burgers' eqn.

$$u_t + \left(\tfrac{1}{2}u^2\right)_x = 0$$

$$u_t + u u_x = 0$$



Develops into a shock (discontinuity)



Chars. cross after finite time.

At this time, $u_x \to -\infty$.

No strong solution exists after this time.

Need numerical methods that can approximate discontinuous functions.

Standard (FD, FV, FE) methods fail here.

Special numerical methods have been developed for this using "slope limiters" and "Riemann solvers".

See AMCS 333