Review

$$U''(x) = f(x) \qquad 0 < x < 1$$
$$u(0) = \alpha$$
$$u(1) = \beta$$

$$\downarrow$$

$$\frac{U_{j+1} - 2U_j + U_{j-1}}{h^2} = f(x_j)$$

$$\downarrow$$

$$AU = F$$
$$A\hat{U} = F + \tau$$
$$AE = -\tau$$
$$\|E\| \le \|A^{-1}\| \, \|\tau\|$$

We showed that

$$\|A^{-1}\|_2 < C$$

as $h \to 0$

So $\|E\|_2 \to 0$ as $h \to 0$

We claimed that the eigenvalues of A are

$$\lambda_p = \frac{2}{h^2}\left(\cos(p\pi h) - 1\right)$$
$$p = 1, 2, \dots, m.$$

Let's prove this.

$$\hat{A} = h^2 A = \begin{bmatrix} -2 & 1 & & & \\ 1 & \ddots & \ddots & & \\ & \ddots & \ddots & \ddots & \\ & & \ddots & \ddots & 1 \\ & & & 1 & -2 \end{bmatrix}$$

Toeplitz
Tridiagonal

$$\hat{A} v = \hat{\lambda} v$$

$$v_{j+1} - 2v_j + v_{j-1} = \hat{\lambda} v_j$$
$$j = 1, 2, \dots, m$$

$$v_0 = v_{m+1} = 0$$

$$v_j = \xi^j \qquad \xi \in \mathbb{C}$$

$$\xi^{j+1} - 2\xi^j + \xi^{j-1} = \hat{\lambda} \xi^j$$

$$\xi^2-(2+\hat{\lambda})\xi+1=0$$

$$\xi_\pm=\frac{2+\hat{\lambda}}{2}\pm\frac{\sqrt{(2+\hat{\lambda})^2-4}}{2}$$

$$=1+\frac{\hat{\lambda}}{2}\pm\frac{\sqrt{\hat{\lambda}^2+4\hat{\lambda}}}{2}$$

General solution:

$$V_j=a\xi_+^j+b\xi_-^j$$

$$V_0=a+b=0$$

$$b=-a$$

$$V_j=a(\xi_+^j-\xi_-^j)$$

$$V_{m+1}=a(\xi_+^{m+1}-\xi_-^{m+1})=0$$

So $\xi_+^{m+1}=\xi_-^{m+1}$

$$\xi_+^{2m+2}=(\xi_-\xi_+)^{m+1}$$

$\xi_+\xi_-=1$ (Vieta's Thm.)

$\xi_+^{2m+2}=1$ $\quad \xi_+=e^{\frac{2\pi i}{2m+2}p}$

$p=1,2,\ldots,m$

$$\xi_\pm=e^{\pm\frac{\pi i}{m+1}p}$$

$$\xi_++\xi_-=2+\hat{\lambda}$$

$h=\frac{1}{m+1}$

$$2\cos\left(\frac{p\pi}{m+1}\right)=2+\hat{\lambda}$$

$$\hat{\lambda}=2(\cos(p\pi h)-1)$$

$$\lambda=\frac{2}{h^2}(\cos(p\pi h)-1)$$

Today we will prove

$$\|A^{-1}\|_\infty < C$$

$$\|A\|_\infty = \sup_{v \neq 0} \frac{\|Av\|_\infty}{\|v\|_\infty}$$

$$= \max_{1 \le i \le m} \sum_{j=1}^{m} |a_{ij}|$$

$$\frac{1}{h^2}\begin{bmatrix} h^2 & & & & \\ 1 & -2 & 1 & & \\ & \ddots & \ddots & \ddots & \\ & & 1 & -2 & 1 \\ & & & & h^2 \end{bmatrix}\begin{bmatrix} U_0 \\ U_1 \\ \vdots \\ U_{m+1} \end{bmatrix} = \begin{bmatrix} \alpha \\ f(x_1) \\ \vdots \\ f(x_m) \\ \beta \end{bmatrix}$$

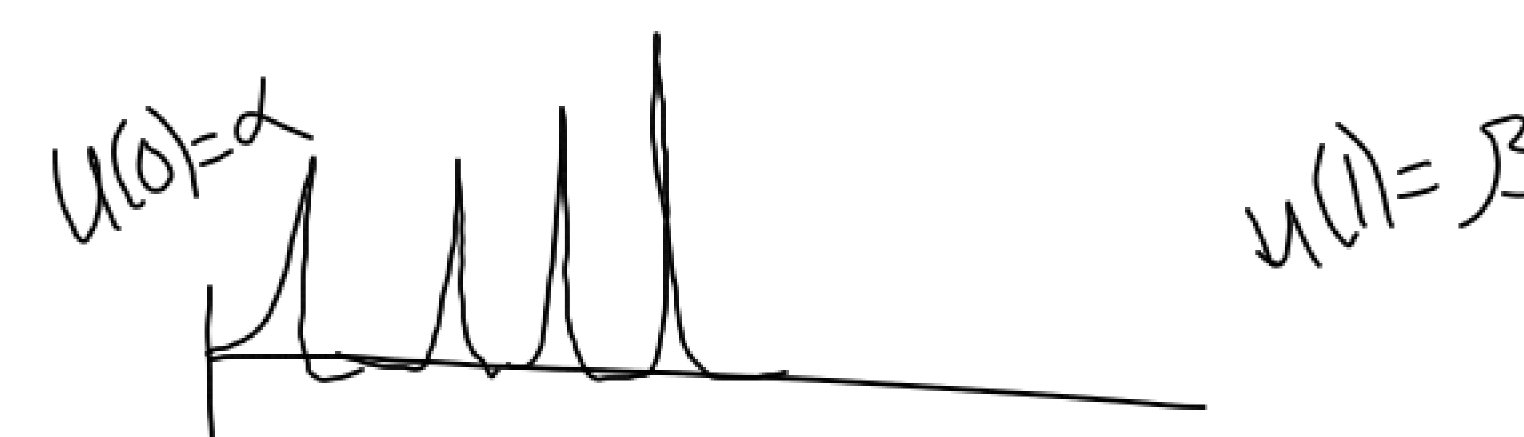
$$B = A^{-1} \quad B = \begin{bmatrix} B_0 & B_1 & \cdots & B_{m+1} \end{bmatrix}$$

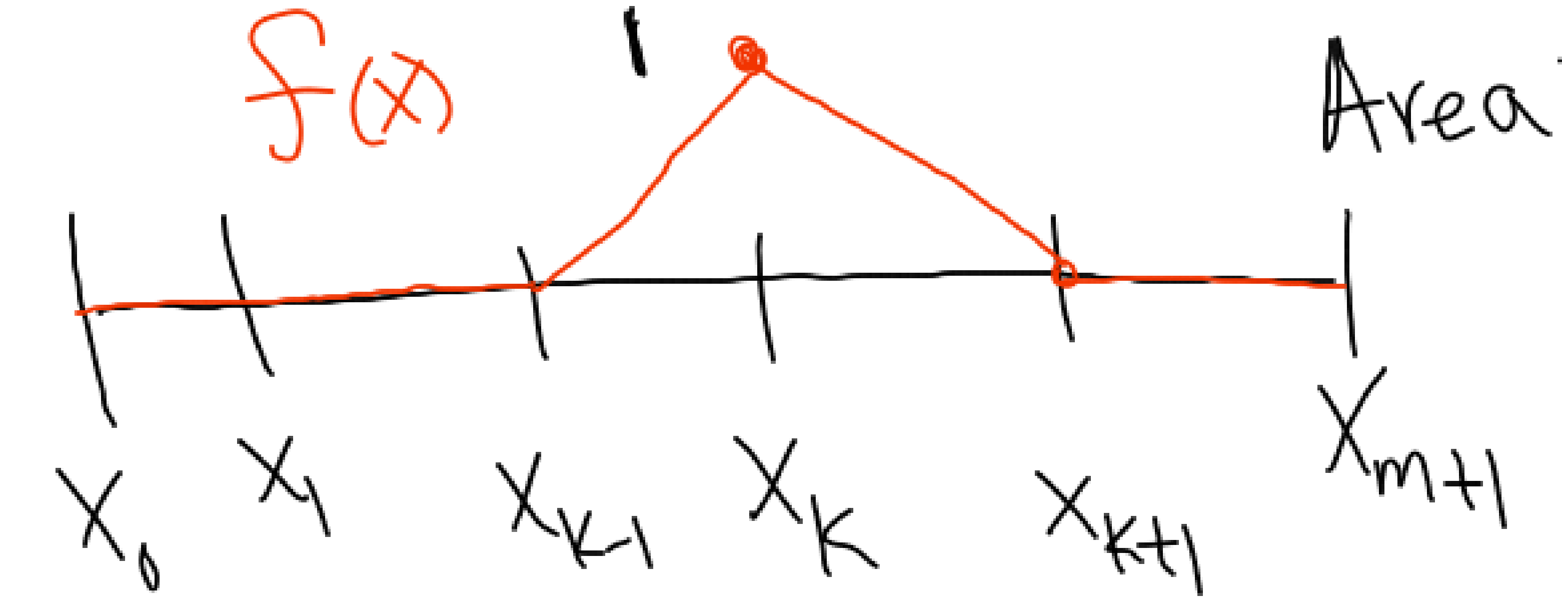
Suppose $\alpha = \beta = 0$

and $f(x_j) = \begin{cases} 0 & j \neq K \\ 1 & j = K \end{cases}$

$$AU = F \Rightarrow U = BF = \sum_{j=0}^{m+1} f_j B_j$$

In this case $U = B_K$

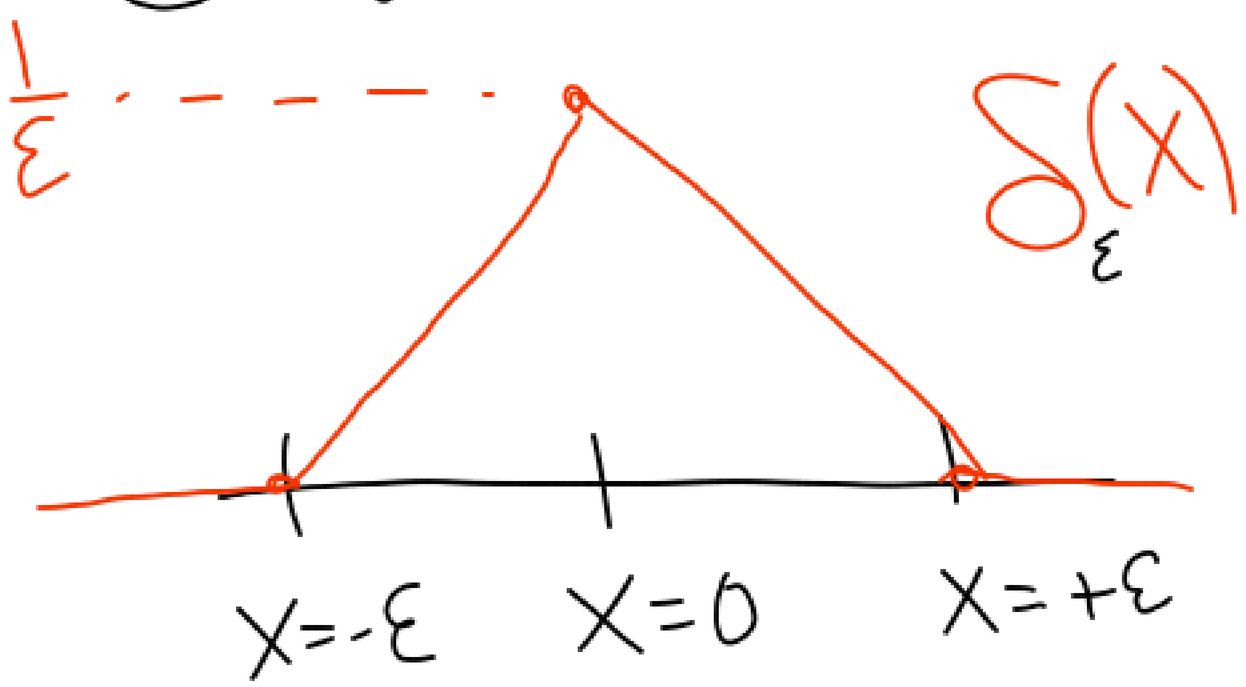




Area: h

$f(x) \simeq h\,\delta(x-x_k)$

δ-function



$$\delta_\varepsilon(x) = \begin{cases} \dfrac{\varepsilon+x}{\varepsilon^2} & -\varepsilon \le x \le 0 \\ \dfrac{\varepsilon-x}{\varepsilon^2} & 0 \le x \le \varepsilon \\ 0 & |x| > \varepsilon \end{cases}$$

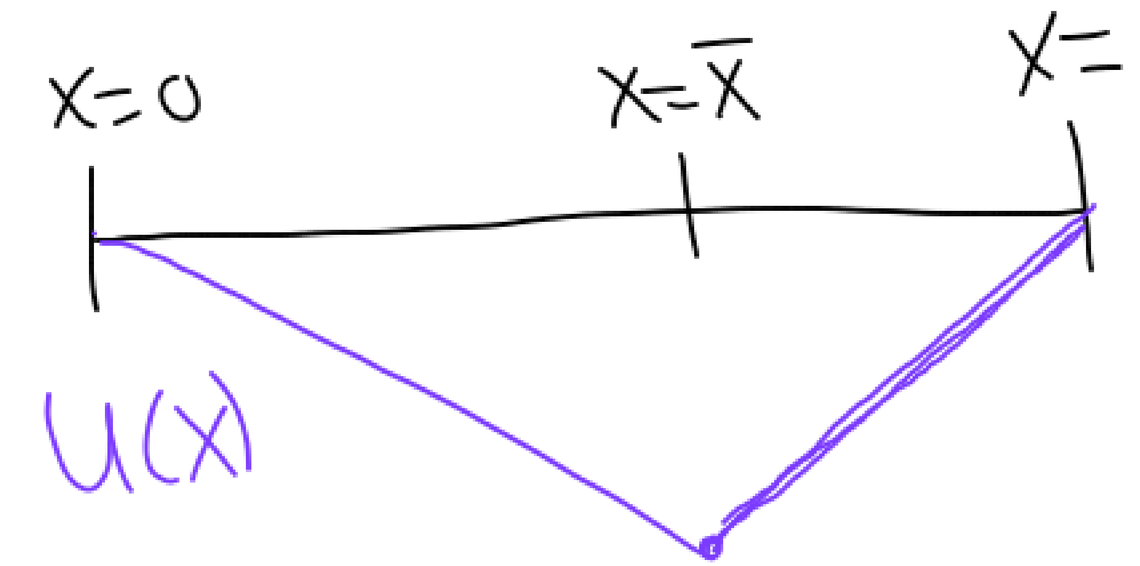
As $\varepsilon \to 0$, $\delta_\varepsilon(x) = 0$ everywhere except $x=0$

and $\displaystyle\int_{-\infty}^{\infty} \delta_\varepsilon(x)\,dx = 1$

Consider the BVP

$$U''(x) = \delta(x-\bar{x})$$
$$u(1) = u(0) = 0$$

Linear except at $x=\bar{x}$:



$$U'(\bar{x}+\varepsilon) - U'(\bar{x}-\varepsilon) = \int_{\bar{x}-\varepsilon}^{\bar{x}+\varepsilon} U''(x)\,dx$$

$$= 1$$

Let $u(x) = \begin{cases} c_1 x & x < \bar{x} \\ c_2(x-1) & x > \bar{x} \end{cases}$

at $x = \bar{x}$ $\quad c_1\bar{x} = c_2(\bar{x}-1)$

$$c_2 - c_1 = 1$$

$$c_2 = 1 + c_1$$

$$c_1\bar{x} = (1+c_1)(\bar{x}-1)$$

$$\cancel{c_1\bar{x}} = \bar{x} + \cancel{c_1\bar{x}} - c_1 - 1$$

$$c_1 = \cancel{\bar{x}} - 1$$

$$c_2 = \bar{x}$$

$$AB = I$$

$$BA = I$$

So $u(x) = \begin{cases} x(\bar{x}-1) & x < \bar{x} \\ \bar{x}(x-1) & x > \bar{x} \end{cases} = G(x;\bar{x})$

Green's function

Any function $f(x)$ can be written

$$f(x) = \int_{-\infty}^{\infty} f(\bar{x})\,\delta(x-\bar{x})\,d\bar{x}$$

So the general solution of

$$u''(x) = f(x) \qquad u(0) = u(1) = 0$$

is

$$u(x) = \int_0^1 f(\bar{x})\,G(x;\bar{x})\,d\bar{x}$$

The solution of $u(0)=u(1)=0$

$$u''(x) = h\,\delta(x - x_k)$$

is then

$$u(x) = h\,G(x; x_k)$$

In fact

$$B_{ij} = h\,G(x_i; x_j) \qquad 1 \le i, j \le m$$

What about B_0, B_{m+1}?

Consider

$$u''(x) = 0 \quad u(0) = 1 \quad u(1) = 0$$

$$u(x) = 1 - x \qquad B_{i0} = 1 - x_i$$

$$u''(x) = 0 \quad u(0) = 0 \quad u(1) = 1$$

$$u(x) = x$$

$$B_{i,m+1} = x_i$$

$$\|B\|_\infty \le 1 + 1 + mh \qquad h = 1/(m+1)$$

$$< 3$$