Linear Multistep Methods

$$u'(t) = f(u)$$

$$u(t_0) = \eta$$

A LMM takes the form:

$$\sum_{j=0}^{r} \alpha_j U^{n+j} = k \sum_{j=0}^{r} \beta_j f(U^{n+j})$$

We already know $U^1, U^2, \ldots, U^{n+r-1}$

We will find U^{n+r}

If $\beta_r = 0$: explicit

If $\beta_r \neq 0$: implicit

Accuracy

$$\frac{1}{k}\sum_{j=0}^{r}\alpha_j u(t_n)+\sum_{j=0}^{r}\sum_{i=1}^{\infty}\frac{j^{i-1}k^{i-1}}{(i-1)!}\left(\alpha_j\frac{j}{i}-\beta_j\right)u^{(i)}(t_n)=\tau^n$$

Local truncation error

$$\sum_{j=0}^{r}\alpha_j u(t_n+jk)=k\sum_{j=0}^{r}\beta_j u'(t_n+jk)+k\tau^n$$

$$u(t_n+jk)=\sum_{i=0}^{\infty}\frac{(jk)^i}{i!}u^{(i)}(t_n)=u(t_n)+\sum_{i=1}^{\infty}\frac{(jk)^i}{i!}u^{(i)}(t_n)$$

$$u'(t_n+jk)=\sum_{i=0}^{\infty}\frac{(jk)^i}{i!}u^{(i+1)}(t_n)=\sum_{i=1}^{\infty}\frac{(jk)^{i-1}}{(i-1)!}u^{(i)}(t_n)$$

$$\frac{1}{k}\left[\sum_{j=0}^{r}\alpha_j u(t_n)+\sum_{j=0}^{r}\sum_{i=1}^{\infty}\left(\alpha_j\frac{(jk)^i}{i!}-k\beta_j\frac{(jk)^{i-1}}{(i-1)!}\right)u^{(i)}(t_n)\right]=\tau^n$$

We want $\tau^n=\mathcal{O}(k^p)$ with p as large as possible.

$$\left.\begin{aligned}\mathcal{O}\left(\tfrac{1}{k}\right):&\ \sum_{j=0}^{r}\alpha_j=0\\ \mathcal{O}(k^0):&\ \sum_{j=0}^{r}(j\alpha_j-\beta_j)=0\end{aligned}\right\}\ \text{Consistency (1st order)}$$

$$\mathcal{O}(k^1):\ \sum_{j=0}^{r}\left(\alpha_j\frac{j}{2}-\beta_j\right)j=0$$

etc.

What is the maximum order achievable for a given r?

Examples:

2-step Adams-Bashforth:

$$U^{n+2} = U^{n+1} + \frac{k}{2}\left(3f(U^{n+1}) - f(U^n)\right)$$

Leapfrog: $U^{n+2} = U^n + 2kf(U^{n+1})$

Backward Diff Formula (2-step):

$$U^{n+2} = \frac{4}{3}U^{n+1} - \frac{1}{3}U^n + 2kf(U^{n+2})$$

A 2-step 1st-order method: Unstable!

$$U^{n+3} = 3U^{n+2} - 2U^{n+1} + kf(U^n)$$

$$U^{n+2} = 3U^{n+1} - 2U^n - kf(U^n)$$

Test problem:

$$u'(t) = 0 \quad (*)$$

$$u(0) = 0$$

$$U^0 = 0 \qquad U^1 = k$$

$$U^{n+3} - 3U^{n+2} + 2U^{n+1} = kfU^n$$

$\alpha_1 = 2$, $\alpha_2 = -3$ $\alpha_3 = 1$ $\beta_0 = 1$

$$\sum (j\alpha_j - \beta_j) = 2 - 6 + 3 - 1 = -2 \neq 0$$

$\alpha_0 = 2$, $\alpha_1 = -3$, $\alpha_2 = 1$ $\beta_0 = 1$

$$-3 + 2 - 1 = -2$$

Zero-stability

If we apply a LMM to (*) we get

$$\underbrace{\sum_{j=0}^{r} \alpha_j U^{n+j} = 0}_{\text{Linear difference equation}} \quad (**)$$

Ansatz: $U^j = \zeta^j$

$$\sum_{j=0}^{r} \alpha_j \zeta^j = \rho(\zeta) = 0$$

We call $\rho(\zeta)$ the 1st characteristic polynomial of the LMM.

It has r roots:

$$\zeta_1, \dots, \zeta_r$$

If they are distinct, the general solution of (**) is

$$U^n = \sum_{i=1}^{r} c_i \zeta_i^n$$

What about repeated roots?

Example: $U^{n+2} - 2U^{n+1} + U^n = 0$

$$\rho(\zeta) = \zeta^2 - 2\zeta + 1 = 0$$

$$\zeta_1 = 1 \quad \zeta_2 = 1$$

One fundamental soln. is $U^n = c_1 1^n = c_1$.

The other is

$$U^n = c_2 n 1^n = c_2 n$$

Check: $n+2 - 2(n+1) + n$
$= 0$

In general a root of multiplicity m leads to the fundamental solutions:

$$\zeta_i^n,\ n\zeta_i^n,\ n^2\zeta_i^n,\ \ldots,\ n^{m-1}\zeta_i^n$$

Under what conditions do the solutions of (**) remain bounded as $n \to \infty$?

We need

$$|\zeta_i| \le 1 \quad \forall\, 1 \le i \le r$$

and if $|\zeta_i| = 1$ then ζ_i has multiplicity 1.

"the root condition"

If this holds, we say the method is <u>zero-stable</u>.

AB2: $U^{n+2} = U^{n+1} \dots$

$$U^{n+2} - U^{n+1} = \dots$$

$\rho(\zeta) = \zeta^2 - \zeta$ roots: $\zeta = 0, 1$

Leapfrog: $U^{n+2} = U^n + 2kf(U^n)$

$\rho(\zeta) = \zeta^2 - 1$ roots: $\zeta = \pm 1$

Unstable: $U^{n+2} = 3U^{n+1} - 2U^n + kf(U^n)$

$\rho(\zeta) = \zeta^2 - 3\zeta + 2$

roots: $\zeta = 1, 2$ Not zero stable

Any zero-stable and consistent LMM is convergent for the IVP

$$u'(t) = f(u)$$
$$u(t_0) = \eta$$

if f is Lipschitz continuous.

Proof: Write the LMM as one-step method:

$$V^n = \begin{bmatrix} U^n \\ U^{n+1} \\ \vdots \\ U^{n+r-1} \end{bmatrix} \qquad V^{n+1} = \begin{bmatrix} U^{n+1} \\ U^{n+2} \\ \vdots \\ U^{n+r} \end{bmatrix}$$

$$\sum_{j=0}^{r} \alpha_j U^{n+j} = k \sum_{j=0}^{r} \beta_j f(U^{n+j})$$

$$U^{n+r} = \frac{1}{\alpha_r}\left(\sum_{j=0}^{r-1} -\alpha_j U^{n+j} + \right)$$

Then the LMM is

$$V^{n+1} = \underbrace{\begin{bmatrix} 0 & 1 & 0 & \cdots & \cdots \\ 0 & 0 & 1 & 0 & \cdots \\ & 0 & & \ddots & 0 \\ & & & & 1 \\ \frac{-\alpha_0}{\alpha_r} & \frac{-\alpha_1}{\alpha_r} & \cdots & \cdots & \frac{-\alpha_{r-1}}{\alpha_r} \end{bmatrix}}_{\text{Companion matrix}} V^n + \frac{k}{\alpha_r} \begin{bmatrix} 0 \\ \vdots \\ 0 \\ \sum_{j=0}^{r} \beta_j f(U^{n+j}) \end{bmatrix}$$

$$V^{n+1} = CV^n + kw^n$$

This sequence is bounded iff $\|C^n\|$ is bounded. (as $n \to \infty$)

The eigenvalues of C are the roots of $\rho(\xi)$. So $\|C^n\|_2$ is bounded iff the root condition is satisfied, i.e. if the LMM is zero-stable.

For any one-step method, $\rho(\zeta) = \zeta - 1$.

So all one-step methods are zero-stable.