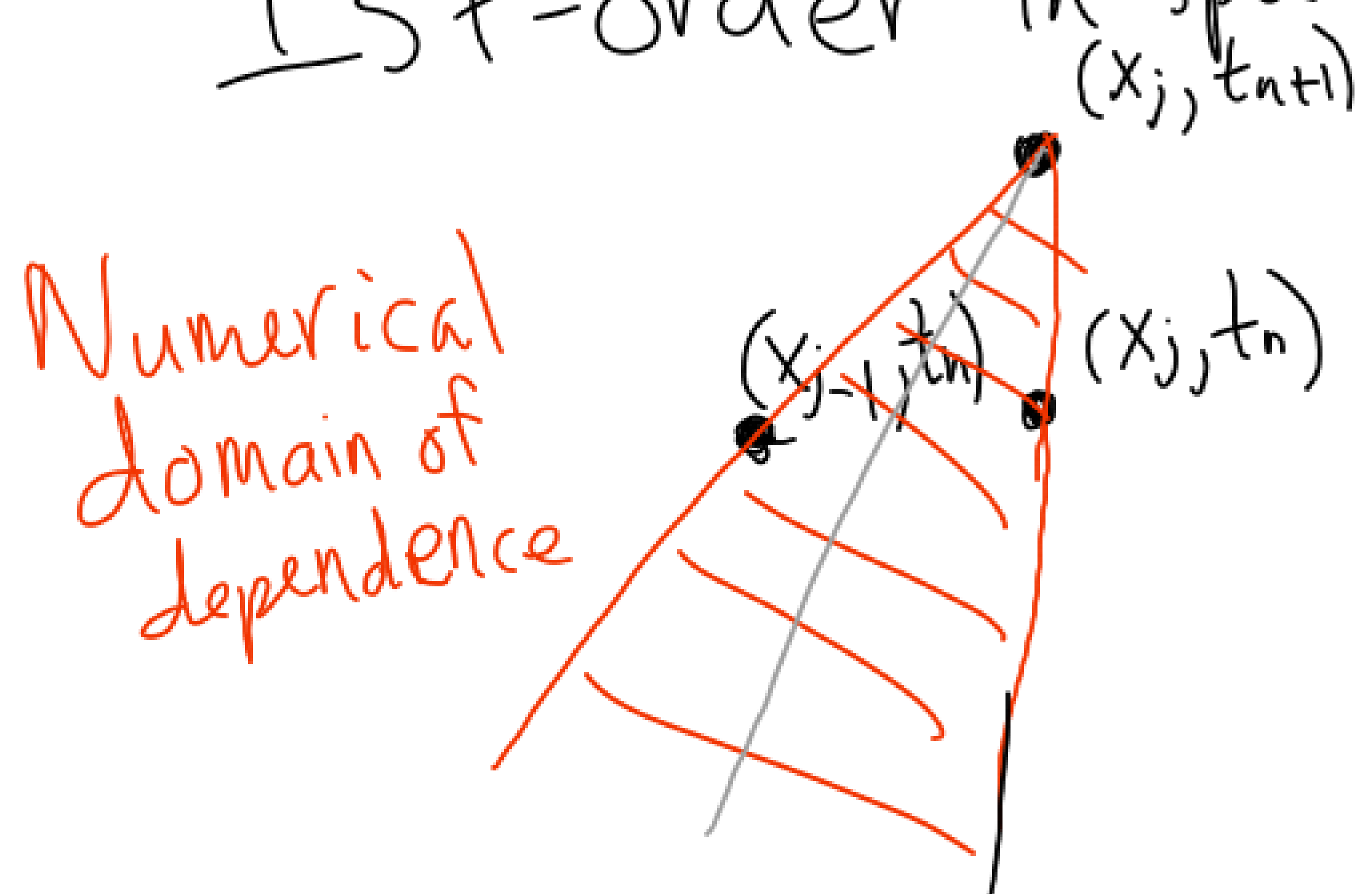
# The Upwind Method

$$u_t + a u_x = 0 \qquad a > 0$$

$$\frac{U_j^{n+1} - U_j^n}{k} + a\,\frac{U_j^n - U_{j-1}^n}{h} = 0$$

1st-order in space and time




CFL condition:

$$0 \le \frac{ka}{h} \le 1$$

## Von Neumann analysis

$$U_j^n \to g^n e^{ijh\xi}$$

We want $|g| \le 1$ $\forall \xi$

$$\frac{g^{n+1} - g^n}{k} e^{ijh\xi} + a g^n \frac{e^{ijh\xi} - e^{i(j-1)h\xi}}{h} = 0$$

$$\frac{g-1}{k} + a\,\frac{1 - e^{-ih\xi}}{h} = 0$$

$$\nu = \frac{ka}{h}$$

$$g = 1 - \frac{ka}{h}\left(1 - e^{-ih\xi}\right)$$

$$|g|^2 = \left(1 - \nu - \nu e^{-ih\xi}\right)\left(1 - \nu - \nu e^{ih\xi}\right)$$

$$|g|^2 = (1-\nu)^2 + \nu^2 - (1-\nu)\nu\left(e^{-ih\xi} + e^{ih\xi}\right)$$

$$|g|^2 = 1 - 2\nu + 2\nu^2 - 2\nu(1-\nu)\cos(h\xi)$$

$\cos(h\xi) = +1$: $|g|^2 = 1 - 2\nu + 2\nu^2 - 2\nu + 2\nu^2$

$= 1 - 4\nu + 4\nu^2 = (1-2\nu)^2 \Rightarrow 0 \le \nu \le 1$

$\cos(h\xi) = -1$: $|g|^2 = 1 - 2\nu + 2\nu^2 + 2\nu - 2\nu^2$

$= 1$

So we obtain the same condition as the CFL.

# Method of lines stability analysis

Semi-discrete scheme:

$$U_j'(t) = -a\frac{U_j(t) - U_{j-1}(t)}{h}$$

$$U'(t) = -\frac{a}{h}\underbrace{\begin{bmatrix} 1 & & & & -1 \\ -1 & \ddots & & & \\ & \ddots & \ddots & & \\ & & \ddots & \ddots & \\ & & & -1 & 1 \end{bmatrix}}_{L} U(t)$$

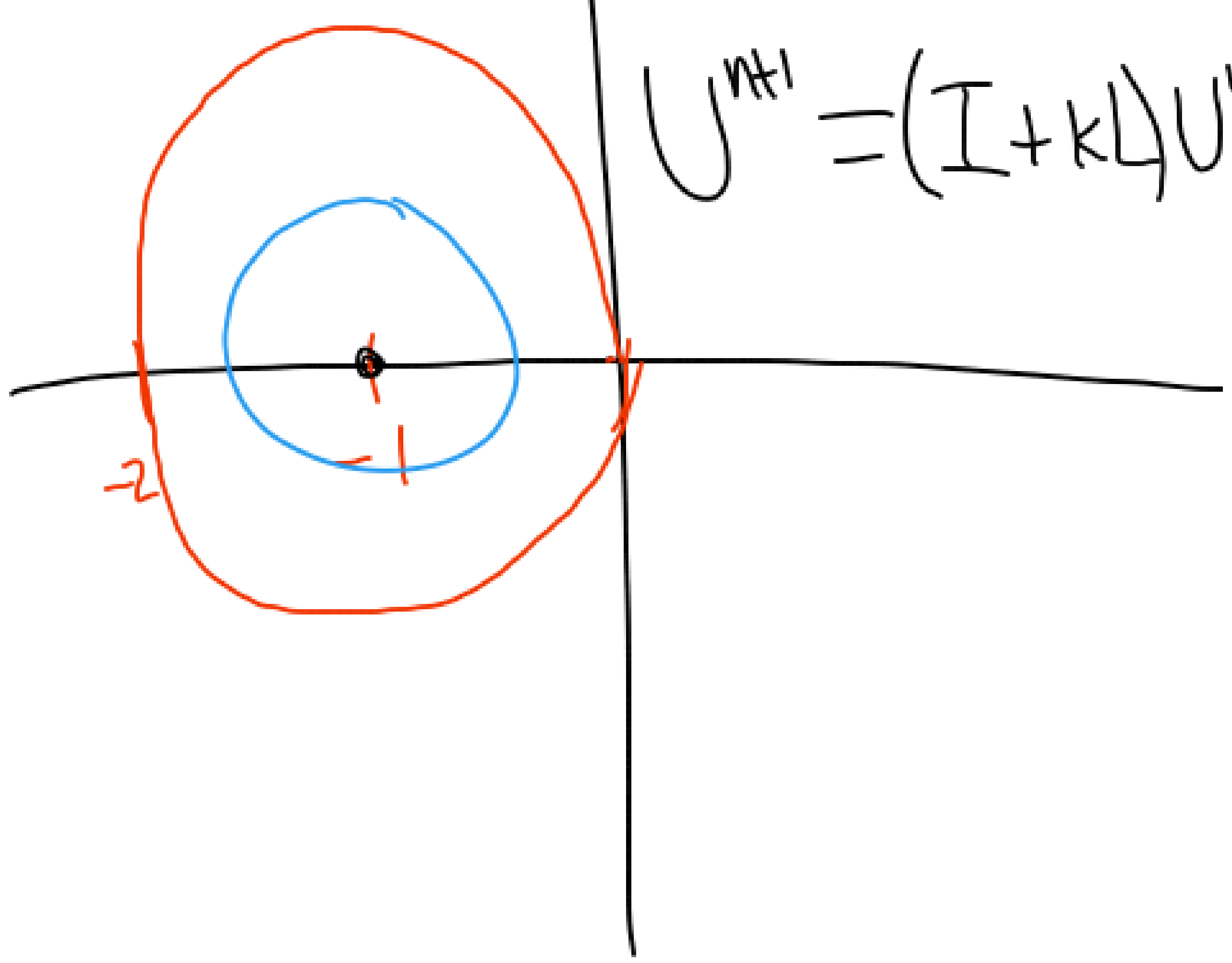
Eigenvalues of $L$: $\lambda = \frac{-a}{h}$

So we need $-\frac{ka}{h} \in S$ ← region of abs. stability

Using Euler's method:

$$U^{n+1} = U^n + kLU^n$$

$$U^{n+1} = (I+kL)U^n$$




$$-2 \le \frac{-ka}{h} \le 0$$

$$0 \le \frac{ka}{h} \le 2$$

What's wrong?

For stability, we really need $\|I+kL\|_2 \le 1$.

This is not the same as $|\lambda| \le 1$ for all $\lambda \in \sigma(I+kL)$.

(i.e. $\rho(I+kL) \le 1$)
(spectral radius)

In general
$\rho(A) \neq \|I+kL\|_2$.

Suppose $A$ is diagonalizable.

Then $Ar_j = \lambda_j r_j$

Let $\left[\begin{array}{c|c|c|c} r_1 & r_2 & \cdots & r_m \end{array}\right]$

and $\Lambda = \begin{bmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_m \end{bmatrix}$.

$$AR = R\Lambda \qquad \kappa(R) = \|R\|_2 \|R^{-1}\|_2$$

$$A = R\Lambda R^{-1}$$

$$\|A\|_2 \le \|R\|_2 \|\Lambda\|_2 \|R^{-1}\|_2$$

$$\|A\|_2 \le \rho(\Lambda)\,\kappa(R)$$

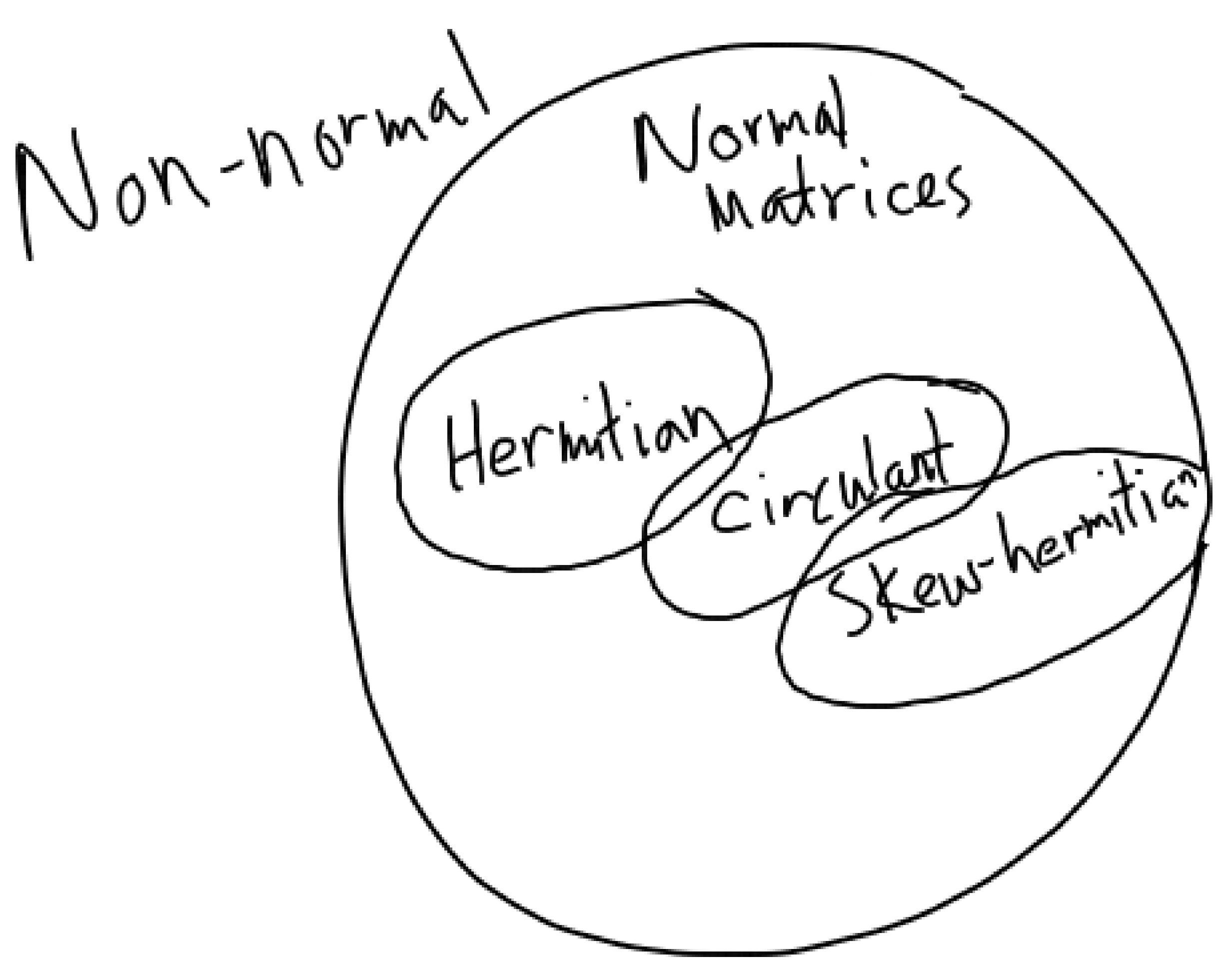
We have $\|A\|_2 = \rho(A)$ if $\kappa(R) = 1$

i.e. if $R$ is unitary.

Otherwise we will have $\|A\|_2 > \rho(A)$.

If $R$ is unitary, we say

$A$ is normal.




An equivalent defintion of normal matrix

$$A^*A = AA^*$$

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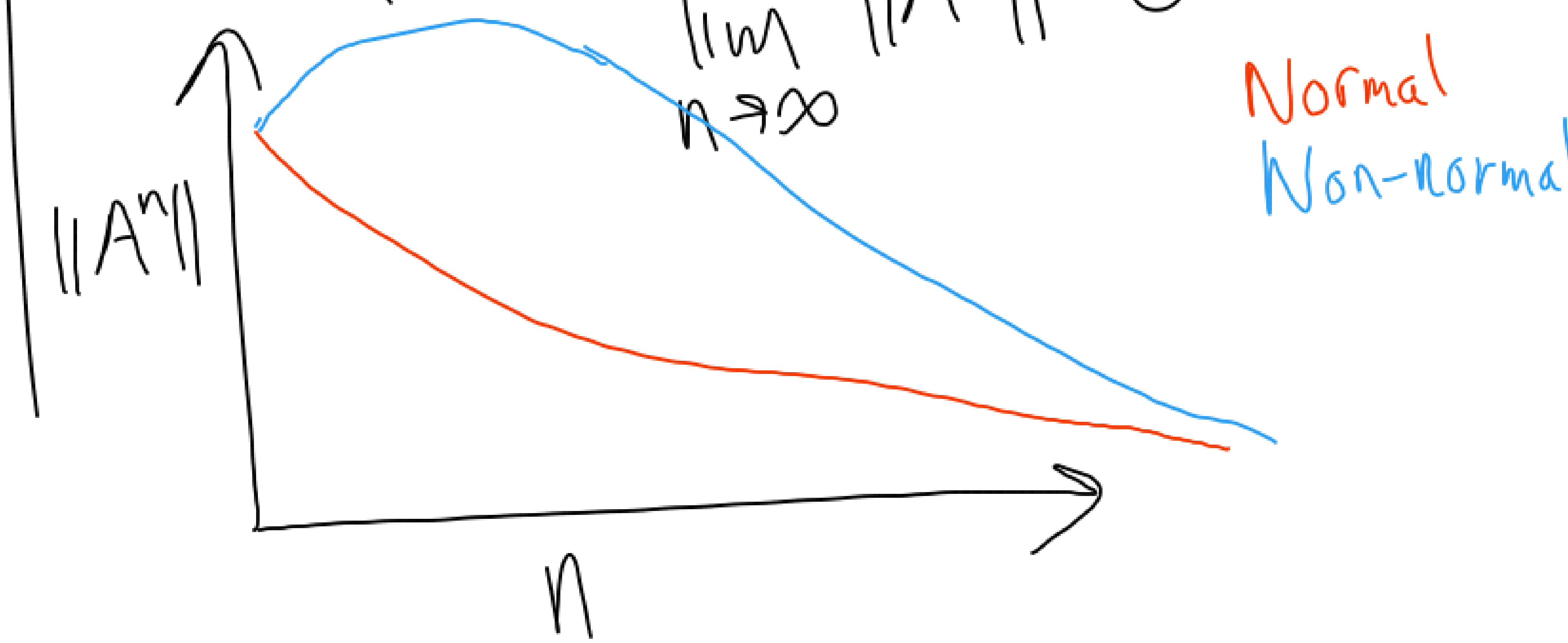
For a normal matrix

if $\rho(A) < 1$ then $\|A^n\|_2 < 1$ and $\lim_{n\to\infty} \|A^n\| = 0$

For non-normal matrices, if $\rho(A) < 1$,

then $\lim_{n\to\infty} \|A^n\| = 0$



For non-normal matrices, it can be useful to look at the $\varepsilon$-pseudospectrum:

$$\{\lambda \in \sigma(A+M_\varepsilon) : \|M_\varepsilon\| \le \varepsilon\}$$

Spectra and Pseudospectra
Embree, Trefethen