Absolute Stability

$$u'(t) = \lambda u(t) \qquad (*)$$
$$u(0) = \eta$$
$$u(t) = e^{\lambda t}\eta$$

We found that the error satisfies:

$$\|E^N\| \leq T e^{|\lambda| T} \|\tau\|_\infty$$

$T = Nk$

Will the error really grow exponentially?

For Euler's method applied to $(*)$, we found

$$E^{n+1} = (1+k\lambda)E^n - k\tau^n$$

The error will grow if $\underline{\underline{|1+k\lambda| > 1}}$.

Another example

$$u'(t) = \lambda(u(t) - \cos(t)) - \sin(t)$$

$$u(0) = 1$$

$$\Rightarrow u(t) = \cos(t)$$

Lipschitz const.: $|\lambda|$

We see error growth if $|1+k\lambda| > 1$

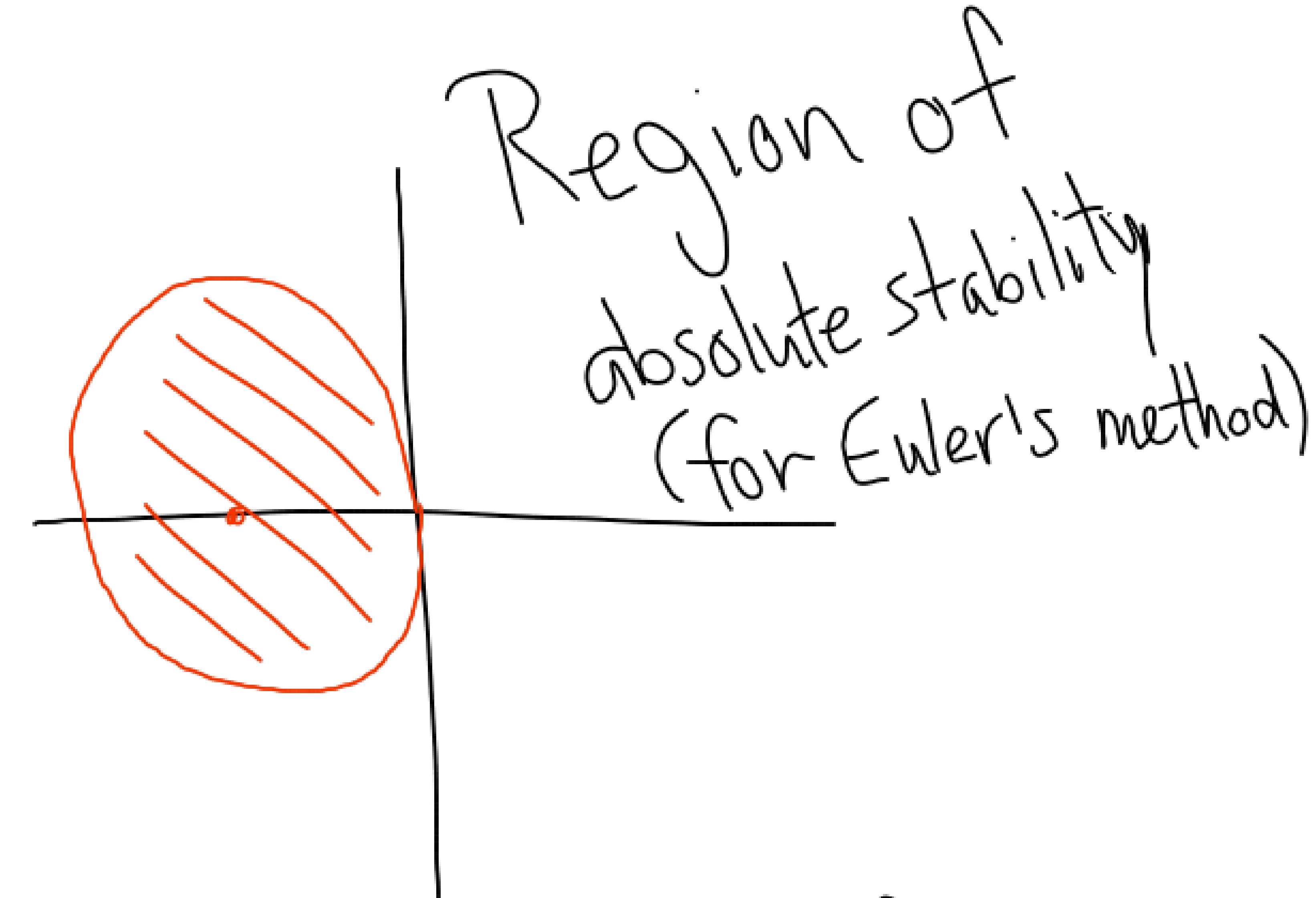
$$|1+k| < 1 \iff -1 < 1 + k\lambda < 1$$

$$-2 < k\lambda < 0$$

$\lambda < 0$: $\quad 0 < k < \frac{2}{\lambda}$

$\lambda = -250$: $\quad 0 < k < \frac{1}{125}$

Let $S = \{ z \in \mathbb{C} : |1+z| \leq 1 \}$



We have small errors if $k\lambda \in S$.

Implicit Euler method

$$U^{n+1} = U^n + kf(U^{n+1})$$

$$f(u) = \lambda u$$

$$\Rightarrow U^{n+1} = U^n + k\lambda U^{n+1} \qquad z = k\lambda$$

$$(1-z)U^{n+1} = U^n$$

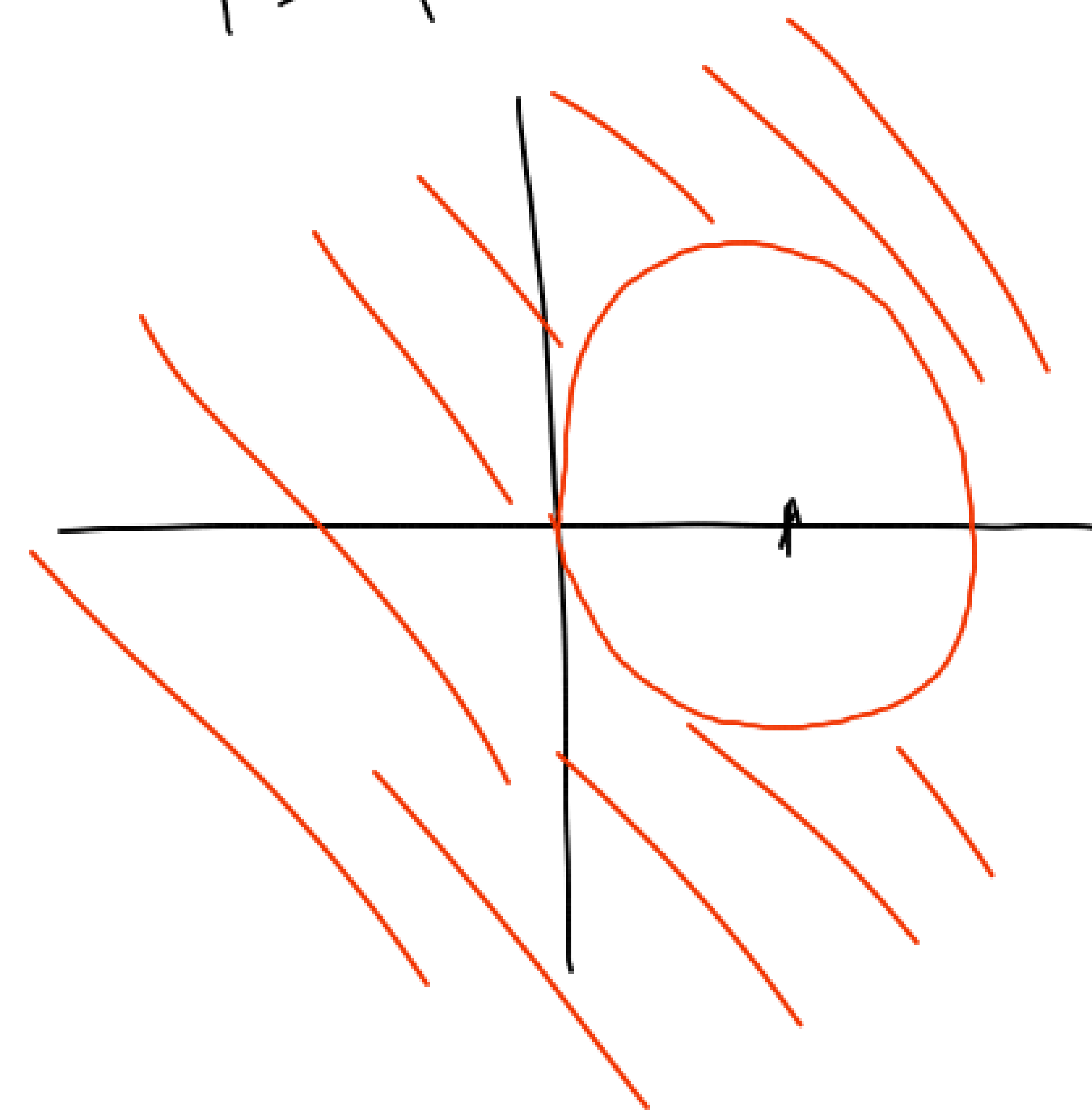
$$U^{n+1} = \frac{1}{1-z}U^n \qquad U^{n+1} = (I-kA)^{-1}U^n$$

$$E^{n+1} = \frac{1}{1-z}E^n - k\tau^n$$

We need $\left|\frac{1}{1-z}\right| \le 1$

$$S = \left\{z \in \mathbb{C} : \left|\frac{1}{1-z}\right| \le 1\right\}$$

$$|z-1| \ge 1$$



Trapezoidal Method

$$U^{n+1} = U^n + \frac{k}{2}\left(f(U^n) + f(U^{n+1})\right)$$

$$f(u) = \lambda u$$

$$\Rightarrow U^{n+1} = U^n + \frac{z}{2}\left(U^n + U^{n+1}\right)$$

$$\left(1 - \frac{z}{2}\right)U^{n+1} = \left(1 + \frac{z}{2}\right)U^n$$

$$U^{n+1} = \frac{1+\frac{z}{2}}{1-\frac{z}{2}}U^n \qquad R(z) = \frac{1+\frac{z}{2}}{1-\frac{z}{2}}$$

For any one-step method, we'll find

$$U^{n+1} = \underbrace{R(z)}_{\text{Absolute stability fcn.}} U^n$$

In general the abs. stab. region is

$$S=\{z\in\mathbb{C} : |R(z)|\le 1\}$$

Trapezoidal:

$$\left|\frac{2+z}{2-z}\right|\le 1$$

$$|2+z|\le|2-z|$$

$$z=x+iy$$

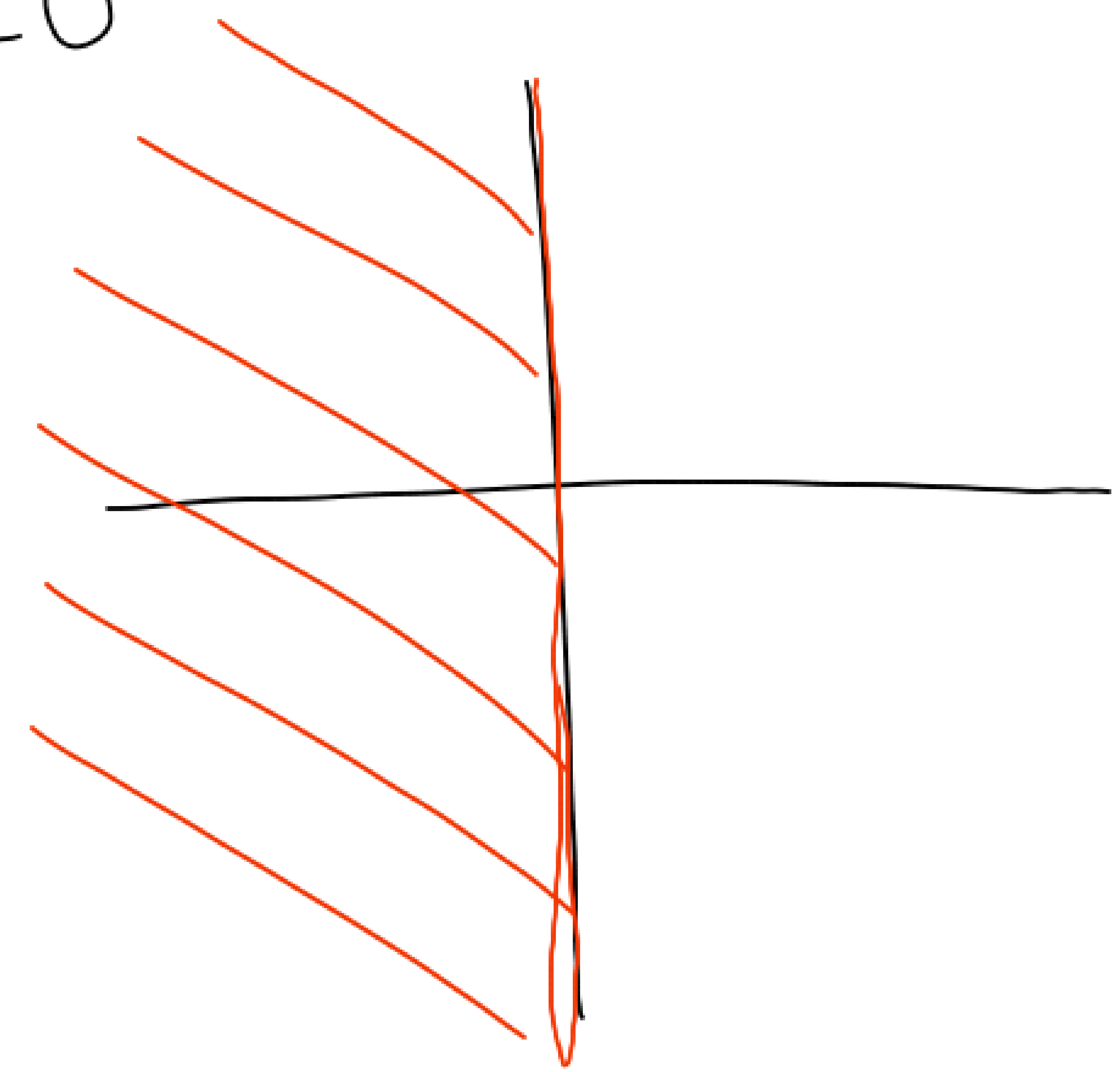
$$|2+x+iy|^2\le|2-x-iy|^2$$

$$(2+x)^2+\cancel{y^2}\le(2-x)^2+\cancel{y^2}$$

$$4+4x+x^2\le 4-4x+x^2$$

$$x\le -x$$

$$x\le 0$$



Consider $u'(t)=Au(t)+g(t)$

$A\in\mathbb{R}^{m\times m}$

For Euler's method, we have

$$E^{n+1}=(I+kA)E^n-k\tau^n$$

The errors will not grow

$$\|I+kA\|\le 1.$$

Using the 2-norm, this is equivalent to

$$|1+k\lambda|\le 1 \quad \text{for all } \lambda\in\sigma(A).$$

In general, we need all eigenvalues of kA to lie within S.

Linearized Pendulum

$$u''(t) = -\sin(u(t)) \approx -u(t)$$

$$\boxed{\begin{aligned} u'(t) &= v(t) \\ v'(t) &= -u(t) \end{aligned}} \qquad \begin{bmatrix} u(0) \\ v(0) \end{bmatrix} = \eta$$

$$\begin{bmatrix} u' \\ v' \end{bmatrix} = \underbrace{\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}}_{A} \begin{bmatrix} u \\ v \end{bmatrix} \Rightarrow u(t) = e^{tA}\eta$$

Eigenvalues of A:

$$\lambda = \pm i$$

Energy: $u^2 + v^2$

Conserved by trapezoidal

Absolute Stability for LMMs

$$\sum_{j=0}^{r} \alpha_j U^{n+j} = k \sum_{j=0}^{r} \beta_j f(U^{n+j})$$

$f(u) = \lambda u$:

$$\sum_{j=0}^{r} \alpha_j U^{n+j} = k\lambda \sum_{j=0}^{r} \beta_j U^{n+j}$$

$$\sum_{j=0}^{r} (\alpha_j - z\beta_j) U^{n+j} = 0$$

Solutions: $U^n = \zeta^n$

$$\sum_{j=0}^{r} (\alpha_j - z\beta_j)\zeta^j = 0$$

2nd char. polynomial $\pi(\zeta; z)$

The global error satisfies:

$$\sum_{j=0}^{r}\left[(\alpha_j - z\beta_j)E^{n+j} + k\tau^{n+j}\right] = 0$$

We have abs. stab. if the roots of $\pi(\zeta;z)$ satisfy the root condition:

$|\zeta| \le 1$ and if ζ is a multiple root then $|\zeta| < 1$.

$$S = \left\{ z \in \mathbb{C} : \begin{array}{l}\text{the roots of } \pi(\zeta;z) \\ \text{satisfy the root condition}\end{array} \right\}$$

How to find S?

The boundary locus method

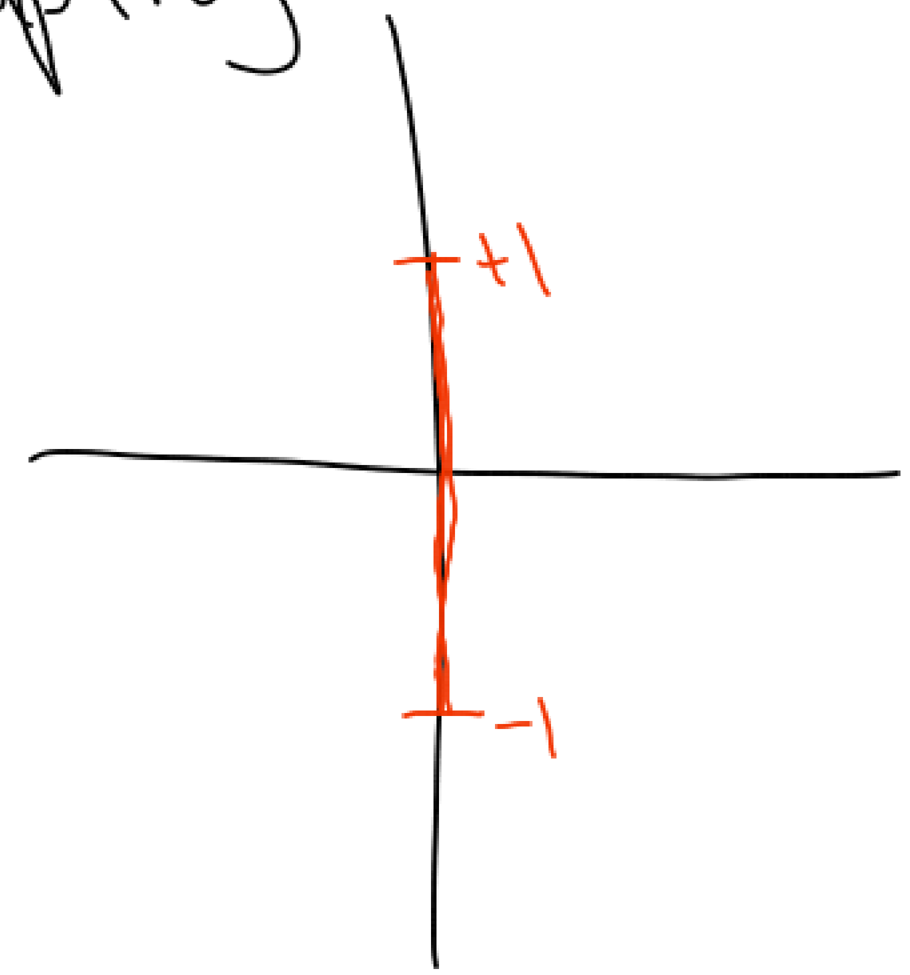
On the boundary of S, some root satisfies $|\zeta| = 1 = e^{i\theta}$.

$$\pi = \sum_{j=0}^{r}(\alpha_j - z\beta_j)e^{i\theta j} = 0$$

$$\sum_{j=0}^{r}\alpha_j e^{i\theta j} = z\sum_{l=0}^{r}\beta_l e^{i\theta l}$$

$$z = \frac{\sum_{j=0}^{r}\alpha_j e^{i\theta j}}{\sum_{l=0}^{r}\beta_l e^{i\theta l}} \Bigg\} \text{evaluate for } 0 \le \theta \le 2\pi$$

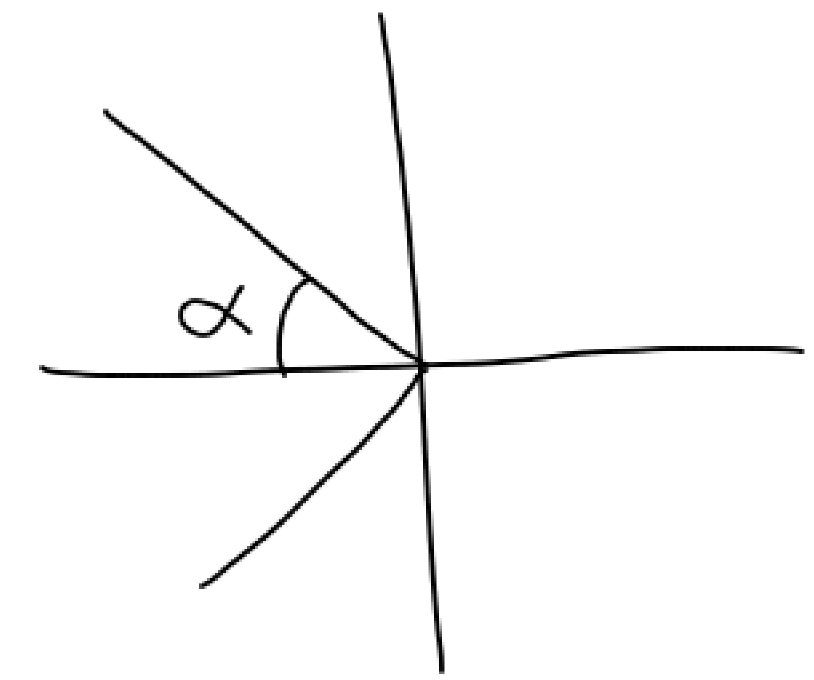
Leapfrog



We say a method is A-stable if $\mathbb{C}^- \subset S$.

$\{z \in \mathbb{C} : \mathrm{Re}(z) \leq 0\}$

We say a method is $A(\alpha)$ if the wedge (sector)



is in S.