So far we've studied

- A elliptic PDE:

$$\nabla^2 u = f$$

- A parabolic PDE:

$$u_t = \nabla^2 u + f$$

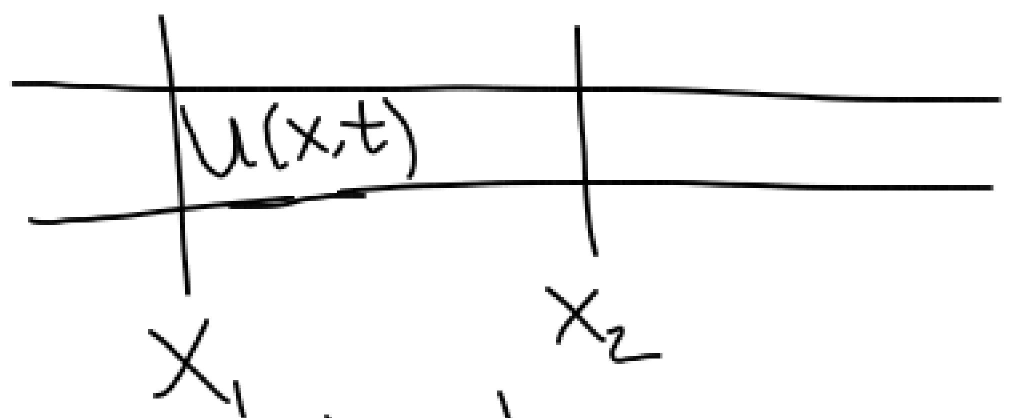
Today we'll study a hyperbolic PDE.

Hyperbolic PDEs model waves:

- Surface water waves
- Pressure waves (sound)
- EM waves
- Fluid dynamics

$\vdots$

Flow of fluid in a narrow channel



u: concentration (per unit length)

f: rate of flow (flux)

Total amount in (x_1, x_2):

$$\int_{x_1}^{x_2} u(x,t)\,dx$$

Change in time:

$$\frac{d}{dt}\int_{x_1}^{x_2} u(x,t)\,dx = f(u(x_1,t)) - f(u(x_2,t))$$

If u is smooth enough

$$\int_{x_1}^{x_2} \frac{\partial}{\partial t} u(x,t)\,dx = -\int_{x_1}^{x_2} \frac{\partial}{\partial x} f(u(x,t))\,dx$$

$$\int_{x_1}^{x_2} (u_t + f(u)_x)\,dx = 0$$

Conservation Law

→ Integrand must vanish pointwise

$$u_t + f(u)_x = 0$$

Simplest case: $f(u) = au$

$$u_t + a u_x = 0$$ Advection equation

IVP: $u(x,0) = \eta(x) \qquad x \in \mathbb{R}$

Solution: $u(x,t) = \eta(x - at)$

Check: $-a\eta' + a\eta' = 0$

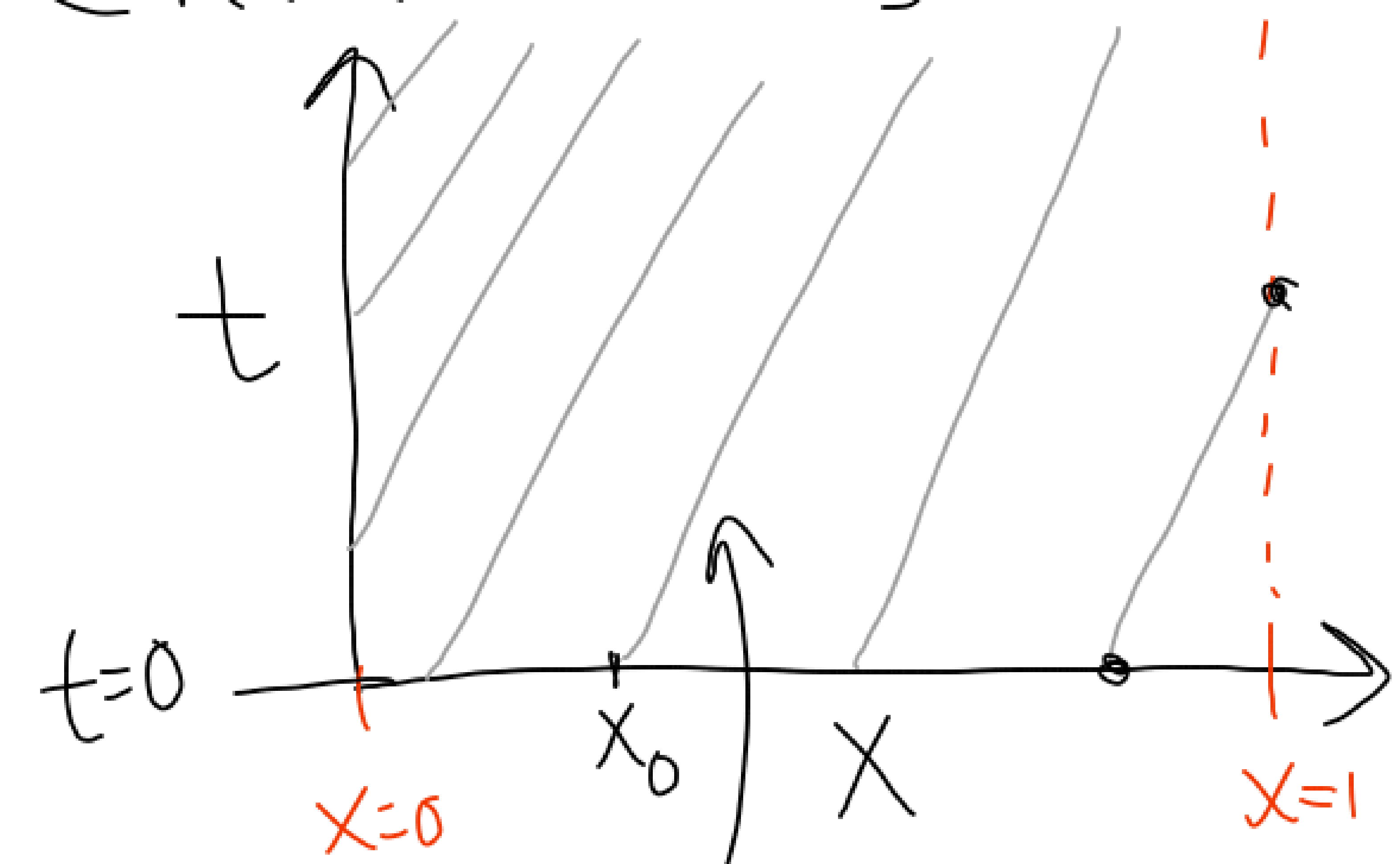
IBVP $\quad u_t + a u_x = 0 \qquad 0 \le x \le 1$

IC: $\quad u(x, t=0) = \eta(x)$

BCs: $\quad u(x=0,t) = g(t) \quad (\text{if } a>0)$

or $u(x=1,t) = g(t) \quad (\text{if } a<0)$

Characteristics



$x = x_0 + at$

u is constant along each characteristic.

Discretization

Centered in space, Euler in time

$$\frac{U_j^{n+1}-U_j^n}{K} + a\,\frac{U_{j+1}^n - U_{j-1}^n}{2h} = 0$$

$$U_j^{n+1} = U_j^n - \frac{Ka}{2h}\left(U_{j+1}^n - U_{j-1}^n\right)$$

Is this stable?

Von Neumann analysis: $U_j^n = g^n e^{ijh\xi}$

$$g^{\cancel{n}+1}\cancel{e^{ijh\xi}} = \cancel{g^n}\cancel{e^{ijh\xi}} - \frac{Ka}{2h}\cancel{g^n}\cancel{e^{ijh\xi}}\left(e^{ih\xi} - e^{-ih\xi}\right)$$

$$e^{\pm i\theta} = \cos\theta \pm i\sin\theta$$

$$g = 1 - \frac{Ka}{2h}\left(2i\sin(h\xi)\right)$$

$\nu = \frac{Ka}{h}$: CFL number

$$g = 1 - \frac{Ka}{h} i\sin(h\xi) = 1 - \nu i\sin(h\xi)$$

$$|g| = \sqrt{1 + \frac{K^2a^2}{h^2}\sin^2(h\xi)} \le 1 + O(K)$$

$$|g| \le \sqrt{1 + \frac{K^2a^2}{h^2}} \Rightarrow \frac{K^2}{h^2} = O(K)$$

If we take $K = O(h)$, this method is unstable.

Method of lines analysis

Discretize only in space:

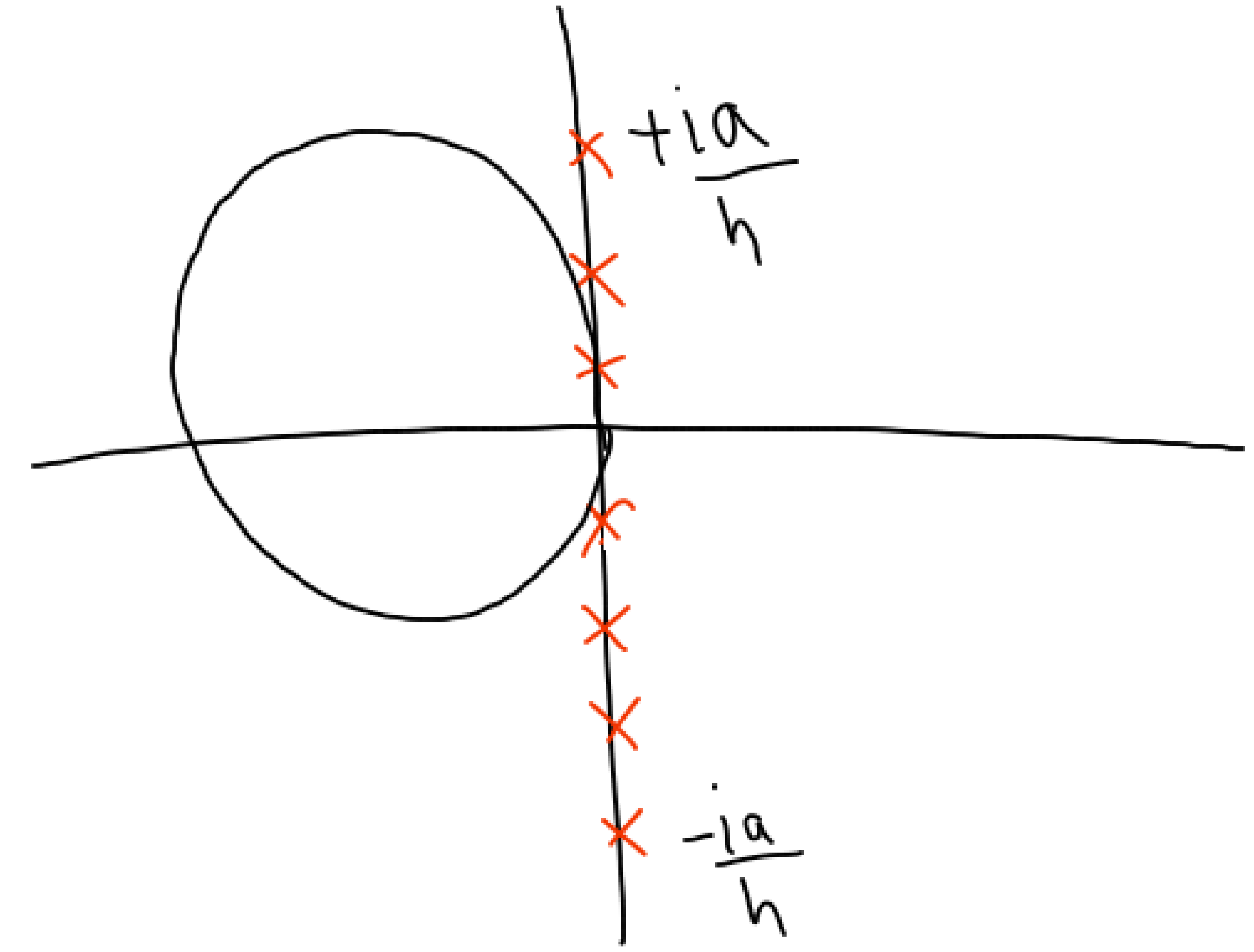
$$U_j'(t) = \frac{a}{2h}\left(U_{j+1}(t) - U_{j-1}(t)\right)$$

$$U'(t) = \frac{a}{2h}\begin{bmatrix} 0 & 1 & & & \\ -1 & \ddots & \ddots & & \\ & \ddots & \ddots & \ddots & \\ & & \ddots & \ddots & 1 \\ & & & -1 & 0 \end{bmatrix} U(t)$$

$$U'(t) = AU(t)$$

What are the eigenvalues?

$A^T = -A$



Thm. Symmetric matrices have real eigenvalues. ①

Skew-symmetric matrices have imaginary eigenvalues. ②

Proof: ① $A^* = A \qquad Av = \lambda v \qquad v^*A = \lambda^* v^*$

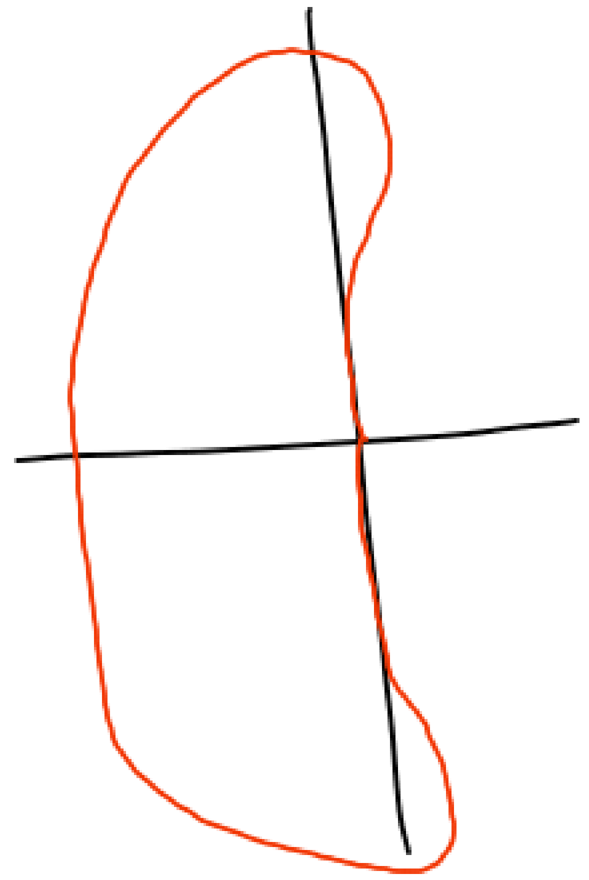
$v^TA^T = \lambda v^T$

② $A^T = -A \qquad Av = \lambda v \qquad v^TA = -\lambda v^T \Rightarrow A^T\bar{v} = -\lambda^*\bar{v}$

$A^Tv = -\lambda v$

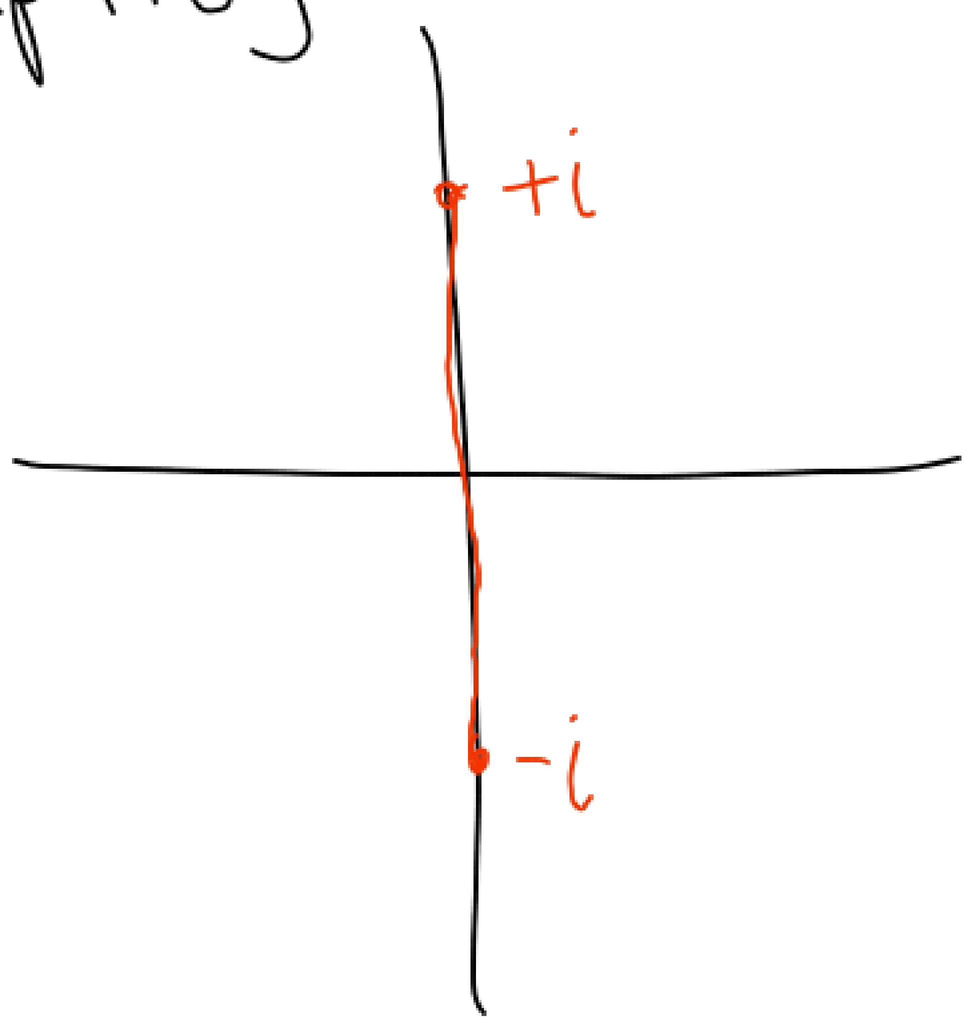
What time discretization should we use?

RK4



ITR

Leapfrog



Lax-Friedrichs

$$U_j^{n+1} = \frac{U_{j+1}^n + U_{j-1}^n}{2} - \frac{ka}{2h}\left(U_{j+1}^n - U_{j-1}^n\right)$$

Observe:

$$\frac{U_{j+1}^n + U_{j-1}^n}{2} = U_j^n + \frac{1}{2}\left(U_{j+1}^n - 2U_j^n + U_{j-1}^n\right)$$

So the LF method can be written:

$$\frac{U_{j+1}^n - U_j^n}{K} + \frac{a}{2h}\left(U_{j+1}^n - U_{j-1}^n\right) = \frac{h^2}{2K}\,\frac{U_{j+1}^n - 2U_j^n + U_{j-1}^n}{h^2}$$

This looks like a discretization of:

$$u_t + au_x = \frac{h^2}{2K}u_{xx}$$

Advection-diffusion