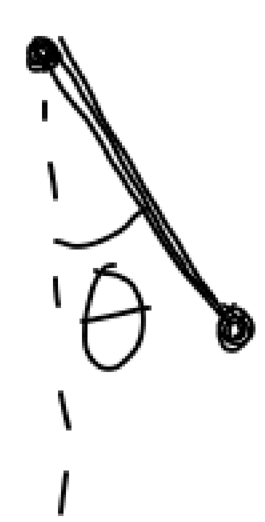
Initial Value Problems
(Chapter 5)

Examples:

① Rigid Pendulum



$$\theta''(t) = -\sin(\theta(t))$$

2nd-order ODE

$$\theta(t_0) = \alpha$$
$$\theta'(t_0) = \beta$$

We can rewrite this as a system of 2 first-order ODEs:

$$\Omega(t) = \theta'(t) \qquad \theta(t_0) = \alpha$$
$$\Omega'(t) = -\sin(\theta(t)) \qquad \Omega(t_0) = \beta$$

So we can focus only on numerical methods for first-order ODEs.

② SIR (Infectious Disease Transmission)

β: contact/transmission rate

$1/\gamma$: typical time of infectiousness

$$\left.\begin{aligned} S(t)&: \text{Susceptible} \\ I(t)&: \text{Infectious} \\ R(t)&: \text{Removed} \end{aligned}\right\} \text{fractions}$$

$$\frac{d}{dt}(S+I+R)=0$$

$$S+I+R=1$$

$$S'(t)=-\beta S I$$

$$I'(t)=\beta S I-\gamma I$$

$$R'(t)=\gamma I$$

$$0\le S\le 1$$
$$0\le I\le 1$$
$$0\le R\le 1$$

Need initial data:

$$S(t_0)=S_0$$
$$I(t_0)=I_0$$
$$R(t_0)=1-S_0-I_0$$

Solutions of Linear IVPs

① $u'(t) = \lambda u(t)$ $\quad \lambda \in \mathbb{C}$

$u(t_0) = \eta$ $\quad u : [t_0, \infty) \to \mathbb{C}$

$$u(t) = e^{\lambda(t-t_0)}\eta$$

② $u'(t) = A u(t)$ $\quad u(t) : [t_0, \infty) \to \mathbb{C}^m$

$u(t_0) = \eta$ $\quad A \in \mathbb{C}^{m \times m}$

$$u(t) = e^{(t-t_0)A}\eta$$

Matrix exponential:

$$e^M = I + M + \frac{1}{2}M^2 + \cdots$$

$$= \sum_{j=0}^{\infty} \frac{1}{j!} M^j$$

③ $u'(t) = Au(t) + g(t)$

$u(t_0) = \eta$

$$\underbrace{u(t) = e^{(t-t_0)A}\eta + \int_{t_0}^{t} e^{(t-\tau)A} g(\tau)\,d\tau}_{\text{Duhamel's principle}}$$

Existence and Uniqueness

Does the IVP

$$u'(t) = f(u)$$
$$u(t_0) = \eta$$

have a unique solution?

If $f(u) = Au$ (linear) then $\exists$ a unique solution for $t \in [t_0, \infty)$.

In general, no.

$$u'(t) = (u(t))^2$$
$$u(0) = \eta > 0$$

$$u(t) = \frac{1}{\eta^{-1} - t}$$

Solution exists only for $t \in (0, \eta^{-1})$.

Is $f(u) = u^2$ L.C?

Yes, for $[\eta, M)$

No, for $[\eta, \infty)$

$$u'(t) = \sqrt{u(t)}$$
$$u(0) = 0$$

$$u(t) = 0$$
$$u(t) = \frac{1}{4}t^2$$

Not unique!

Is $f(u) = \sqrt{u}$ L.C.?

$D = [0, M]$ No

$D = [\varepsilon, \infty)$ Yes

Lipschitz Continuity

Given a function f and a domain D, we say f is L.C. on D if there exists $0 < L < \infty$ s.t.

$$\|f(v) - f(w)\| \leq L \|v - w\|$$

for all $v, w \in D$.

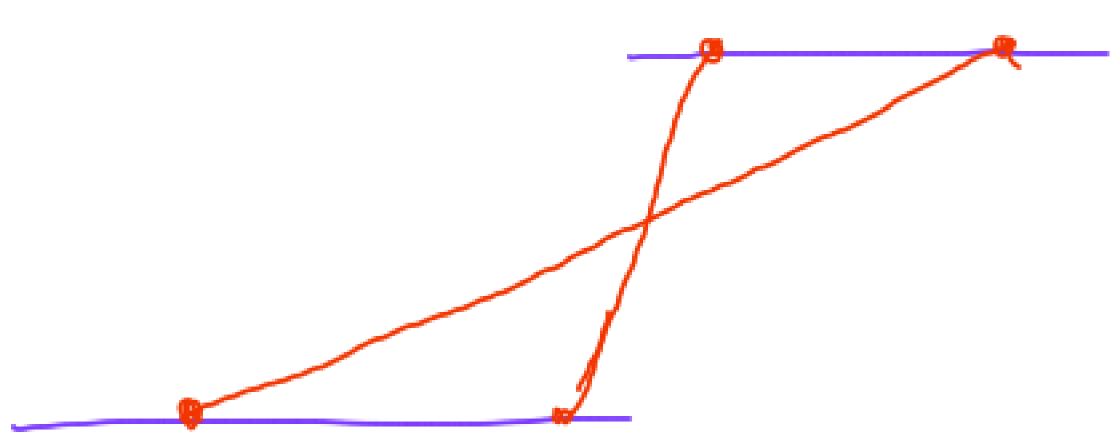
If such an L exists, we call it a Lipschitz constant for f on D.

If f is differentiable, we can take

$$L = \sup_{v \in D} \|f'(v)\|.$$

Heaviside function:

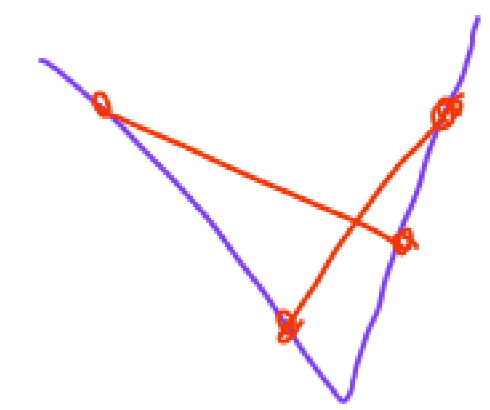
$$f(u) = \begin{cases} 0 & x<0 \\ 1 & x>0 \end{cases}$$



LC $\Rightarrow$ continuity

$f(u) = |u|$ $\quad D=[-1,1]$ ✓ LC

$D=(-\infty,\infty)$ ✓ LC



Theorem: Given the IVP

$$u'(t) = f(u(t))$$
$$u(t_0) = \eta$$

$a \in (0,\infty)$

let $D = [\eta - a, \eta + a]$

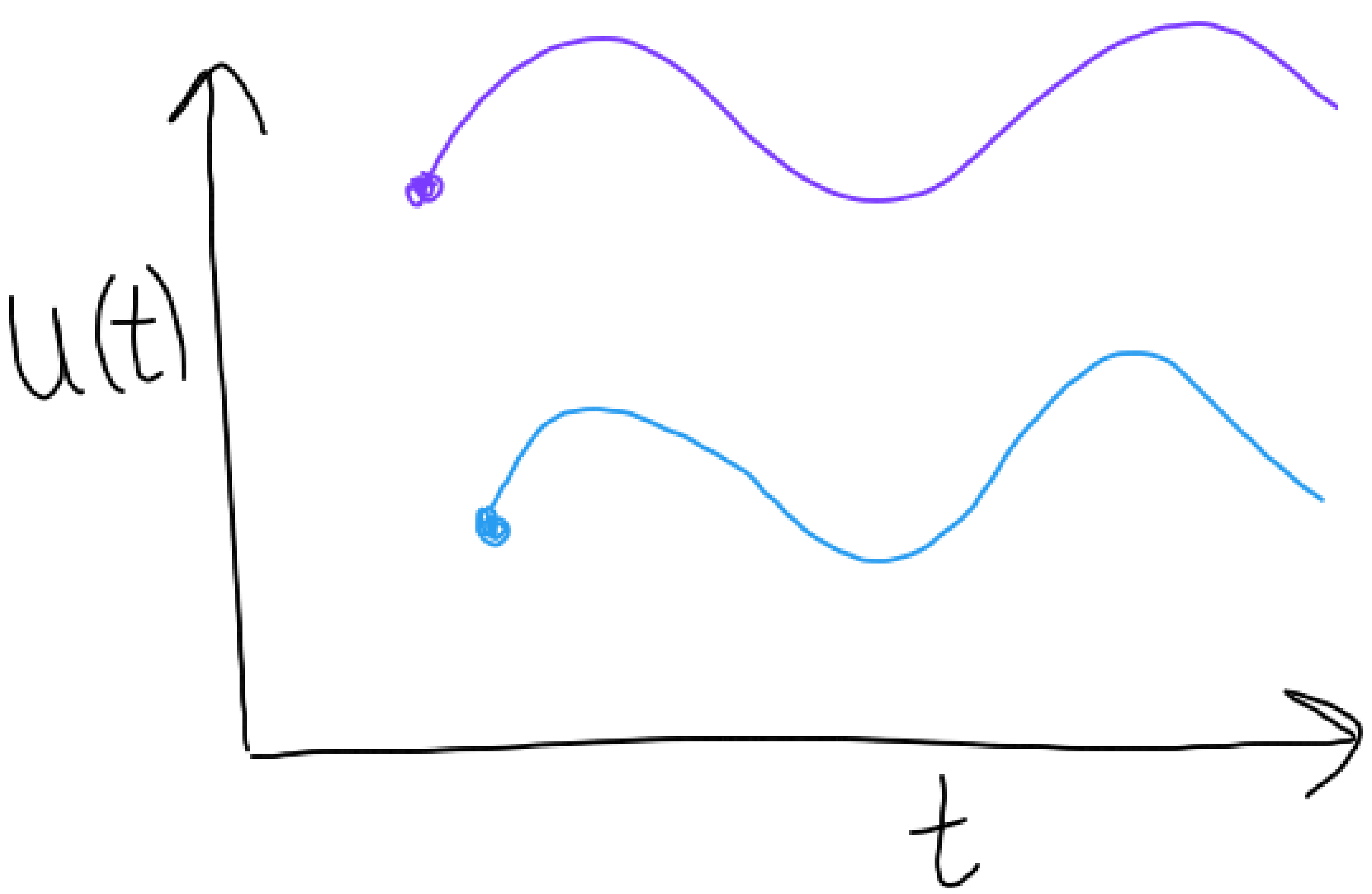
Suppose f is L.C. for $\eta - a \le u \le \eta + a$.

Then a unique solution exists for (at least) $t \le t_0 + \dfrac{a}{\sup_{u \in D} |f(u)|}$.

Meaning of the Lipschitz Constant

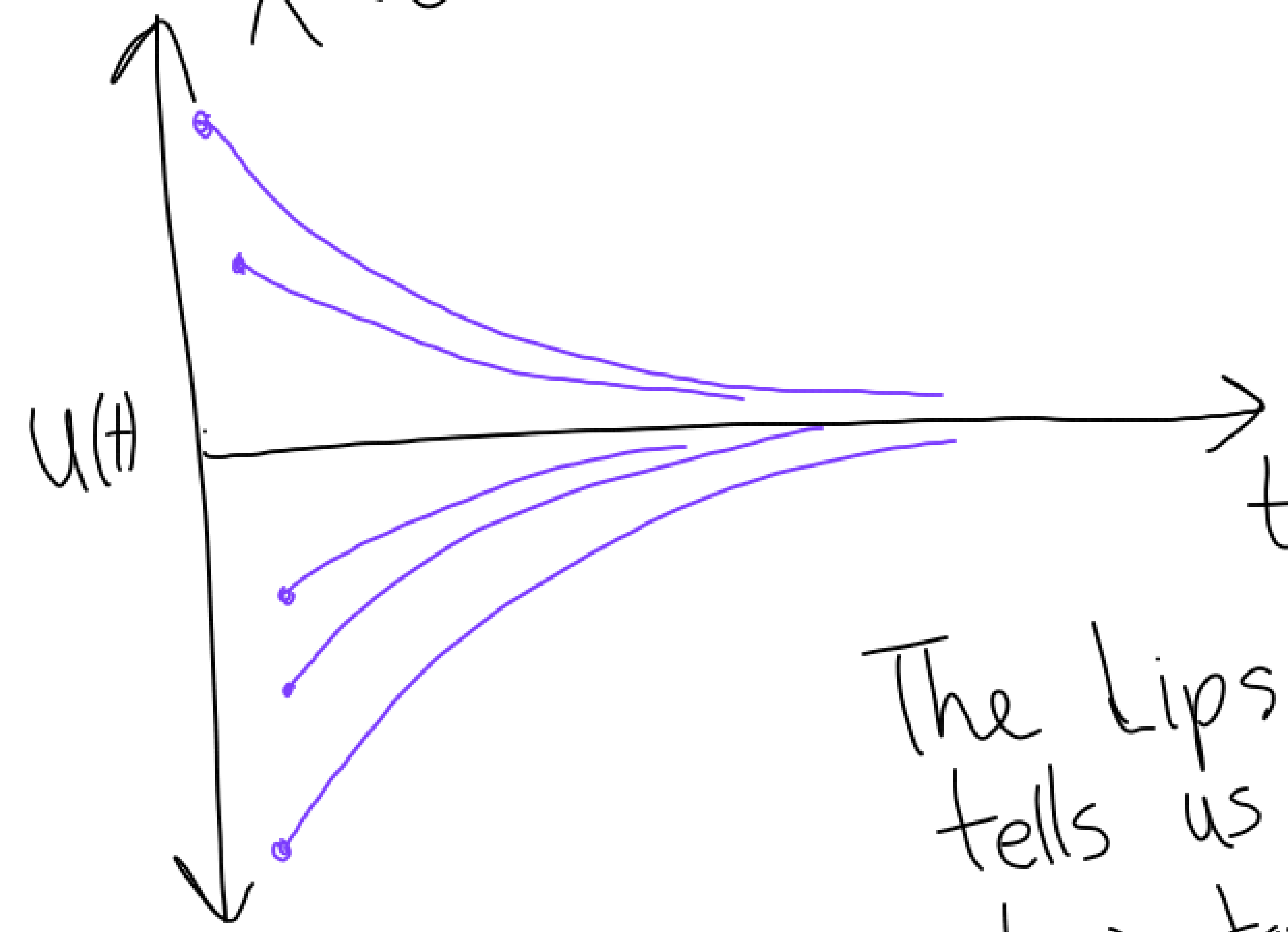
Examples:

1. $u'(t) = g(t)$ $\quad L = 0$
   $u(t_0) = \eta$



2. $u'(t) = \lambda u(t)$ $\quad L = |\lambda|$
   $u(t_0) = \eta$

$\lambda < 0$:



The Lipschitz constant tells us about how rapidly trajectories diverge (or converge).

$\lambda > 0$: