Initial Boundary Value Problems

$$u_t = K u_{xx} + f(x,t)$$

Temp. — Heat conductance — Heat source/sink

$$0 \le x \le 1$$

$$u(x,0) = \eta(x)$$

$$u(0,t) = \alpha(t)$$

$$u(1,t) = \beta(t)$$

Consider: $K = 1$

$\alpha(t) = \beta(t) = 0 \qquad f(x,t) = 0$

Exact Solution:

Initial cond.: $\underbrace{u(x,0)}_{\eta(x)} = \sum_{p=0}^{\infty} \hat{u}_p(0) \sin(p\pi x)$

Ansatz:

$$u(x,t) = \sum_{p=0}^{\infty} \hat{u}_p(t) \sin(p\pi x)$$

$$\sum_{p=0}^{\infty} \hat{u}_p'(t) \sin(p\pi x) = -\pi^2 \sum_{p=0}^{\infty} p^2 \hat{u}_p(t) \sin(p\pi x)$$

$$\Rightarrow \quad \hat{u}_p'(t) = -\pi^2 p^2 \hat{u}_p(t)$$

$$\hat{u}_p(t) = e^{-p^2\pi^2 t} \hat{u}_p(0)$$

$$u(x,t) = \sum_{p=0}^{\infty} e^{-p^2\pi^2 t}\, \hat{u}_p(0) \sin(p\pi x)$$

Discretization

Method of lines:

① Discretize in space
$\Rightarrow$ system of ODEs (IVP)

② Apply a RK or LMM to integrate in time.

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① $x_j = jh \qquad j = 0, 1, \ldots, m+1$

$h = \frac{1}{m+1}$

$U_j(t) \approx u(x_j, t)$

$$u_{xx}\Big|_{x=x_j} \approx \frac{U_{j+1}(t) - 2U_j(t) + U_{j-1}(t)}{h^2}$$

ODE system:

$$\underbrace{\begin{bmatrix} U_1'(t) \\ U_2'(t) \\ \vdots \\ U_m'(t) \end{bmatrix}}_{U} = \frac{K}{h^2} \underbrace{\begin{bmatrix} -2 & 1 & & \\ 1 & -2 & \ddots & \\ & \ddots & \ddots & 1 \\ & & 1 & -2 \end{bmatrix}}_{A} \begin{bmatrix} U_1 \\ \vdots \\ U_m \end{bmatrix} + \underbrace{\begin{bmatrix} f(x_1,t) \\ f(x_2,t) \\ \vdots \\ f(x_m,t) \end{bmatrix}}_{F}$$

$$U'(t) = AU + F(t)$$

Take $F = 0$: $\quad U(t) = e^{tA} U(0)$

$A = R \Lambda R^{-1}$

$\lambda_p = \frac{2}{h^2}(\cos(p\pi h) - 1) \qquad r_{jp} = \sin(p\pi j h)$

Largest eigenvalue: $\lambda_{max} \approx \frac{-4}{h^2}$

$$U' = AU$$

$$U' = R\Lambda R^{-1}U$$

$$R^{-1}U' = \Lambda R^{-1}U \qquad \hat{U} = R^{-1}U$$

$$\hat{U}' = \Lambda \hat{U}$$

$$\hat{U}(t) = e^{t\Lambda}\hat{U}(0)$$

Comparing with the exact solution, we should have

$$\lambda_p \approx -p^2\pi^2.$$

$$\cos x \approx 1 - \frac{1}{2}x^2 + O(x^4) \quad \Big| \text{ for } x \sim 0$$

$$\lambda_p \approx \frac{2}{h^2}\left(1 - \frac{1}{2}p^2\pi^2h^2 + O(p^4h^4) - 1\right)$$

$$\lambda_p \approx -p^2\pi^2 + O(h^2p^4)$$

We need

$$k\lambda_{max} \in S$$

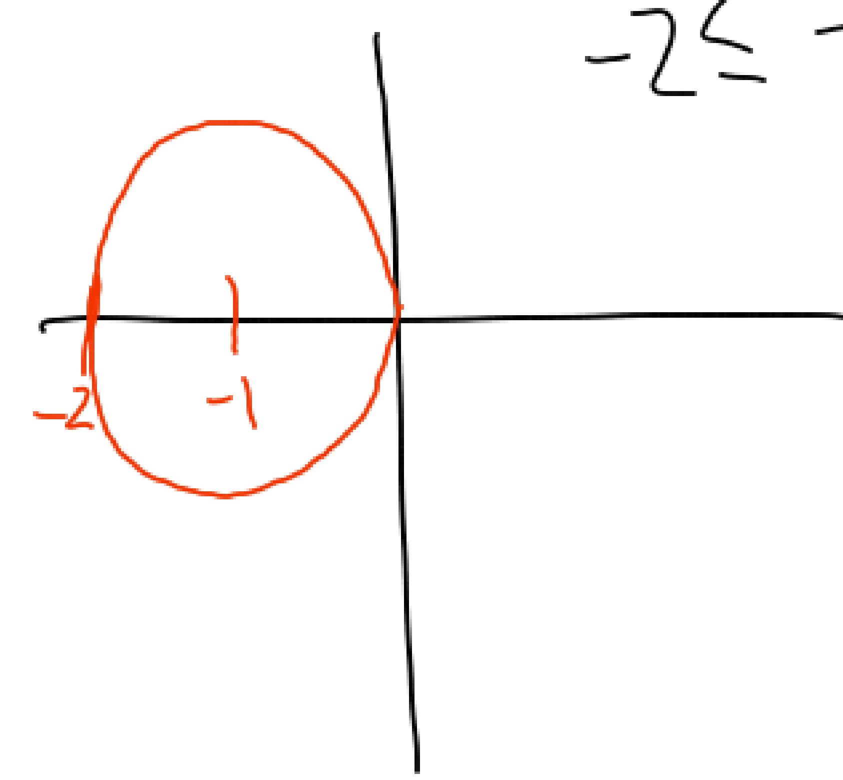
i.e. for any explicit method we have

$$|k\lambda_{max}| \leq C$$

$$\frac{4k}{h^2} \leq C$$

$$k \leq O(h^2)$$

Explicit Euler



$$-2 \leq \frac{-4k}{h^2} \leq 0$$

$$\Rightarrow k \leq \frac{h^2}{2}$$

We should use an A-stable or $A(\alpha)$-stable method.

One nice method is TR-BDF2

0	0	0	0
1/2	1/4	1/4	0
1	1/3	1/3	1/3
	1/3	1/3	1/3

This is:

A-stable
L-stable
2nd-order accurate

$$u_t = K(x,t)\, u_{xx}$$

$$u_t = f(u)$$

For all $\lambda \in \sigma(f'(u))$ $\forall\, u \in D$

$$K\lambda \in S$$

Local truncation error

Explicit Euler + 3-pt. CD

$$U_j^{n+1} = U_j^n + \frac{K}{h^2}\left(U_{j+1}^n - 2U_j^n + U_{j-1}^n\right)$$

To find LTE:

$$u(x_j, t_n + K) = u(x_j, t_n) + \frac{K}{h^2}\left(u(x_j + h, t_n) - 2u(x_j, t_n) + u(x_j - h, t_n)\right) + \tau_j^n$$

$$\tau_j^n = \frac{K}{2} u_{tt} - \frac{h^2}{12} u_{xxxx} + \mathcal{O}(K^2) + \mathcal{O}(h^4)$$

In general if we use a method of order p in time and q in space, then

$$\tau_j^n = \mathcal{O}(h^q) + \mathcal{O}(K^p)$$