Nonlinear BVPs

$$\theta(0) = \alpha$$
$$\theta(T) = \beta$$

$$F = ma$$
$$a = \theta''(t)L$$
$$-mg\sin(\theta(t)) = m\theta''(t)L$$
$$\theta''(t) = -\frac{g}{L}\sin(\theta(t))$$

Choose units so g/L

$$\theta''(t) = -\sin(\theta(t))$$

For small θ:

$$\sin(\theta(t)) = \theta(t) + O(\theta^3)$$

$$\Rightarrow \theta''(t) = -\theta(t) \quad \times$$

$\theta(0) = \alpha$

$\theta(T) = \beta$

$\theta''(t) = -\sin(\theta)$

Shooting:

Ignore $\theta(T)=\beta$

Pick $\theta'(0)$

Try to "hit" the endpoint condition $\theta(T)=\beta$

(Solving an IVP)

Discretize:

$$h = \frac{T}{m+1}$$

$1 \le i \le m$

$$\theta''(t_i) \approx \underbrace{\frac{\theta_{i+1} - 2\theta_i + \theta_{i-1}}{h^2} + \sin(\theta_i)}_{G(\theta)} = 0$$

$\theta_0 = \alpha$

$\theta_{m+1} = \beta$

We need to solve this nonlinear algebraic system.

$$G(\theta) = 0$$

Let θ_* denote the exact solution of $G(\theta)=0$.

Let $\theta^{[0]}$ denote an initial guess.

$$\delta = \theta_* - \theta^{[0]}.$$

$$0 = G(\theta_*) = G(\theta^{[0]} + \delta)$$

$$G(\theta_*) = G(\theta^{[0]}) + G'(\theta^{[0]})\delta + O(\|\delta\|^2)$$

$$G'(\theta) = J(\theta) = \begin{bmatrix} \frac{\partial G_1}{\partial \theta_1} & \frac{\partial G_1}{\partial \theta_2} & \cdots & \\ \frac{\partial G_2}{\partial \theta_1} & & & \\ \vdots & & & \frac{\partial G_m}{\partial \theta_n} \end{bmatrix}$$

We approximate:

$$G(\theta^{[0]}) + J(\theta^{[0]})\delta = 0$$

$$J(\theta^{[0]})\delta = -G(\theta^{[0]})$$

$$\downarrow$$

$$(1)\quad J(\theta^{[k]})\delta^{[k]} = G(\theta^{[k]})$$

Newton's method:

1. Solve (1) for $\delta^{[k]}$
2. Set $\theta^{[k+1]} = \theta^{[k]} + \delta^{[k]}$
3. If $\|G(\theta^{[k+1]})\| < \varepsilon$ we're done. Otherwise go to 1.

$$G_i = \frac{1}{h^2}\left(\theta_{i+1} - 2\theta_i + \theta_{i-1}\right) + \sin\theta_i$$

$$J = \frac{1}{h^2}\begin{bmatrix} -2 & 1 & & \\ 1 & \ddots & \ddots & \\ & \ddots & \ddots & 1 \\ & & 1 & -2 \end{bmatrix} + \begin{bmatrix} \cos\theta_1 & & & \\ & \cos\theta_2 & & \\ & & \ddots & \\ & & & \cos\theta_m \end{bmatrix}$$

Consistency

$$\frac{1}{h^2}\left(\theta(t_{i+1}) - 2\theta(t_i) + \theta(t_{i-1})\right) + \sin(\theta(t_i)) = \tau_i$$

$$\cancel{\theta''(t_i)} + \frac{1}{12}h^2\theta''''(t_i) + O(h^4) \cancel{+ \sin(\theta(t_i))} = \tau_i$$

$$\tau_i = \frac{1}{12}h^2\theta''''(t_i) + O(h^4)$$

Stability

$$\text{Let } \hat{\theta} = \begin{bmatrix} \theta(t_1) \\ \vdots \\ \theta(t_m) \end{bmatrix}$$

$$\tau = G(\hat{\theta}) \qquad G(\theta_*) = 0$$

$$\tau = G(\hat{\theta}) - G(\theta_*) \qquad E = \theta_* - \hat{\theta}$$

$$G(\theta_*) = G(\hat{\theta}) + J(\hat{\theta})(\theta_* - \hat{\theta}) + O(\|E\|^2)$$

$$\tau = -J(\hat{\theta})E + O(\|E\|^2)$$

If the $\mathcal{O}(\|E\|^2)$ can be neglected (it can)

$$-J^{-1}(\hat{\theta})\tau = E$$

$$\|E\| = \|J^{-1}\tau\| \le \|J^{-1}\|\,\|\tau\|$$

We need that $\|J^{-1}\| < C$ as $h \to 0$.

This holds because

$$J_h = A_h + D$$ where D is independent of h

and we already proved

$$\|A_h^{-1}\| < C \text{ as } h \to 0.$$

$$(A_h + D)^{-1} = \left(\frac{1}{h^2}\tilde{A} + D\right)^{-1}$$
$$= h^2\left(\tilde{A} + h^2 D\right)^{-1}$$

$$(A_h + D)^{-1} = A_h^{-1}\left(I + h^2\hat{A}^{-1}D\right)^{-1}$$
$$= A_h^{-1}\left(I - h^2\hat{A}^{-1}D + \mathcal{O}(h^4)\right)$$
$$= A_h^{-1} - D + \mathcal{O}(h^2)$$

As $h \to 0$,

$$J^{-1} = (A_h + D)^{-1} \to A_h^{-1}$$