Convergence for IBVP Discretizations

Last time: $u_t = u_{xx}$

Semi-Discretize: $\underbrace{U'(t) = A_h U(t)}_{\text{IVP}}$

IVP
system of m ODEs

$$A_h = \frac{1}{h^2}\begin{bmatrix} -2 & 1 & & & \\ 1 & \ddots & \ddots & & \\ & \ddots & \ddots & \ddots & \\ & & \ddots & \ddots & 1 \\ & & & 1 & -2 \end{bmatrix}$$

$m \to \infty$ as $h \to 0$

$\|A_h\| = O\left(\frac{1}{h^2}\right) \to \infty$ as $h \to 0$

Next, discretize in time

Forward Euler:

$$U^{n+1} = U^n + kA_hU^n$$

$$U^{n+1} = (I + kA_h)U^n$$

Trapezoidal Method:

$$U^{n+1} = U^n + \frac{k}{2}\left[A_hU^n + A_hU^{n+1}\right]$$

$$U^{n+1} - \frac{k}{2}A_hU^{n+1} = U^n + \frac{k}{2}A_hU^n$$

$$U^{n+1} = \underbrace{\left(I - \frac{k}{2}A_h\right)^{-1}\left(I + \frac{k}{2}A_h\right)}_{R(kA_h)}U^n$$

Abs. stability function

With any one-step time discretization, we get

$$U^{n+1} = \underbrace{R(kA_h)}_{B_{k,h}} U^n$$

(matrix depending on k and h)

$$U^{n+1} = BU^n$$

$$U^N = B^N U^0$$

Fix the final time T

Let $N = \frac{T}{k}$

Let $\hat{U}^n = \begin{bmatrix} u(x_1, t_n) \\ \vdots \\ \vdots \\ u(x_m, t_n) \end{bmatrix}$ $\quad E^n = U^n - \hat{U}^n$

$\hat{U}$ satisfies $\hat{U}^{n+1} = B\hat{U}^n + k\tau^n$

local truncation error

Thus $U^{n+1} - \hat{U}^{n+1} = B(U^n - \hat{U}^n) - k\tau^n$

$$E^{n+1} = BE^n - k\tau^n$$

$$\Rightarrow E^N = B_{k,h}^N E^0 - k\sum_{j=0}^{N-1} B_{k,h}^{N-j-1}\tau^j \quad (1)$$

We want to prove that

$$\lim_{k,h\to 0} \|E^N\| = 0$$

We need

(1) Consistency: $\|\tau^j\| \to 0$ as $k,h \to 0$
and $\|E^0\| \to 0$ as $h \to 0$

(2) Lax-Richtmeyer stability $\|B^n_{k,h}\| < C_T$ Need to prove this

C_T is independent of k, h, n

Take norms of (1):

$$\|E^N\| = \left\|B^N_{k,h} E^0 - k \sum_{j=0}^{N-1} B^{N-1-j}_{k,h} \tau^j\right\|$$

$$\leq \|B^N_{k,h}\| \cdot \|E^0\| + k \sum_{j=0}^{N-1} \|B^{N-1-j}_{k,h}\| \cdot \|\tau^j\|$$

$$\leq C_T \|E^0\| + kN C_T \max_j \|\tau^j\|$$

$$\leq C_T \left(\|E^0\| + T \max_j \|\tau^j\|\right) \to 0 \text{ as } k,h \to 0.$$

Proof of stability:

With Euler's method

$$B_{k,h} = I + kA_h$$

Take $\|\cdot\| = \|\cdot\|_2$

$$\|B_{k,h}\|_2 = \max_{\mu \in \sigma(B)} |\mu|$$

Eigenvalues of A: $\lambda_p = \frac{2}{h^2}(\cos(p\pi h) - 1)$

Eigenvalues of B: $\mu_p = 1 + k\lambda_p$

So $\|B_{k,h}\| = \max_{1 \leq p \leq m} \left|1 + \frac{2k}{h^2}(\cos(p\pi h) - 1)\right|$

A sufficient condition for $\|B^n_{k,h}\|_2$ to be bounded is that

$$|\mu_p| \leq 1 \quad \forall p, k, h.$$

i.e.

$$-1 \le 1+\frac{2k}{h^2}(\cos(p\pi h)-1) \le 1$$

$$-1 \le \frac{k}{h^2}(\cos(p\pi h)-1) \le 0$$

always true

$$\frac{k}{h^2}(\underbrace{1-\cos(p\pi h)}_{(0,2)}) \le 1$$

$$\frac{k}{h^2}\cdot 2 \le 1 \iff k \le \frac{h^2}{2}$$

This is just the condition for abs. stability.

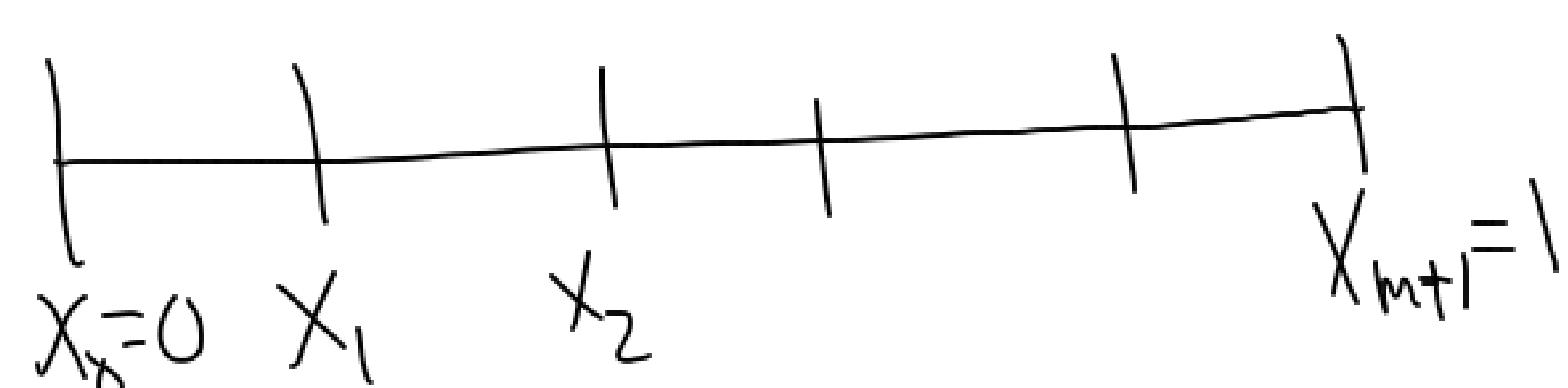
Von Neumann analysis

A simple way to derive stability conditions for linear PDE discretizations with periodic boundary conditions.
(give necessary and often sufficient conditions in the case of other B.C.s.)

$$u_t = u_{xx} \qquad 0 \le x \le 1$$

$$u(x,t=0)=\eta(x)$$

$$\left.\begin{aligned} u(0,t)&=u(1,t)\\ u_x(0,t)&=u_x(1,t)\end{aligned}\right] \text{Periodic BCs.}$$



Discretization:

$$U_j^{n+1} = U_j^n + \frac{k}{h^2}\left(U_{j+1}^n - 2U_j^n + U_{j-1}^n\right)$$
$$(1 \le j \le m)$$

$$U_{m+1}^n = U_0^n$$

Ansatz: $U_j^n = g^n e^{ijh\xi}$ We require $|g| \le 1$

Substitute:

$$g^{n+1} e^{ih\xi j} = g^n e^{ijh\xi} + \frac{k}{h^2} g^n \left(e^{ih\xi(j+1)} - 2e^{ih\xi j} + e^{ih\xi(j-1)}\right)$$

$$g = 1 + \frac{k}{h^2}\left(e^{ih\xi} + e^{-ih\xi} - 2\right)$$

$$g = 1 + \frac{k}{h^2}\left(2\cos(h\xi) - 2\right) = 1 + \frac{2k}{h^2}\left(\cos(h\xi) - 1\right)$$

$$|g| \le 1 \iff k \le \frac{h^2}{2}$$

Why this works:

We have $U^{n+1} = BU^n$

Where $B = R(kA_h) = I + kA_h$

with
$$A_h = \frac{1}{h^2}\begin{bmatrix} -2 & 1 & & & 1 \\ 1 & \ddots & \ddots & & \\ & \ddots & \ddots & \ddots & \\ & & \ddots & \ddots & 1 \\ 1 & & & 1 & -2 \end{bmatrix}$$

This is a circulant matrix.

Every FD discretization of a linear PDE with periodic BCs gives a circulant matrix.

All circulant matrices have (essentially) the same eigenvectors: $V_\xi = \begin{bmatrix} e^{ih\xi} \\ e^{2ih\xi} \\ \vdots \\ \vdots \end{bmatrix}$.

$$\begin{bmatrix} a & b & c & d & e \\ e & a & b & c & d \\ d & e & a & b & c \\ c & d & e & a & b \\ b & c & d & e & a \end{bmatrix}$$

A note about Lax-Richtmeyer Stability:

If we can prove

$$(*) \quad \|B_{k,h}\| \leq 1 + \alpha k \qquad \alpha > 0$$

then

$$\|B^N\| \leq \|B\|^N \leq 1 + \alpha N k + \cdots$$
$$\leq e^{\alpha T} = C_T$$

So $(*)$ is sufficient for L-R stability.