Reminder:
Homework 1
due Thursday

$u''(x) = f(x) \quad 0 < x < 1$

$\cancel{u(0) = \alpha}$ $\quad u'(0) = 0$

Insulation

$u(1) = \beta$

More generally we could impose a desired flux:

$u'(0) = \sigma_0$

How to discretize?

① One-sided FD

(1a) Defect correction

② Ghost point method

One-sided FD

$$D_+ u(x=0) = \frac{U_1 - U_0}{h} = \sigma_0$$

$$\frac{1}{h^2}\begin{bmatrix} -h & h & & & \\ 1 & -2 & 1 & & \\ & \ddots & \ddots & \ddots & \\ & & 1 & -2 & 1 \\ & & & & h^2 \end{bmatrix} \begin{bmatrix} U_0 \\ \vdots \\ \vdots \\ U_{m+1} \end{bmatrix} = \begin{bmatrix} \sigma_0 \\ f(x_1) \\ \vdots \\ f(x_m) \\ \beta \end{bmatrix}$$

For 2nd-order accuracy:

$$U'(x_0) \approx aU_2 + bU_1 + cU_0$$

$$\Rightarrow U'(x_0) \approx \frac{1}{h}\left(-\frac{3}{2}U_0 + 2U_1 - \frac{1}{2}U_2\right)$$

Alternatively:

Recall that

$$D_+ u(\bar{x}) = u'(\bar{x}) + \underbrace{\frac{h}{2}u''(\bar{x})}_{\text{We know this}} + O(h^2)$$

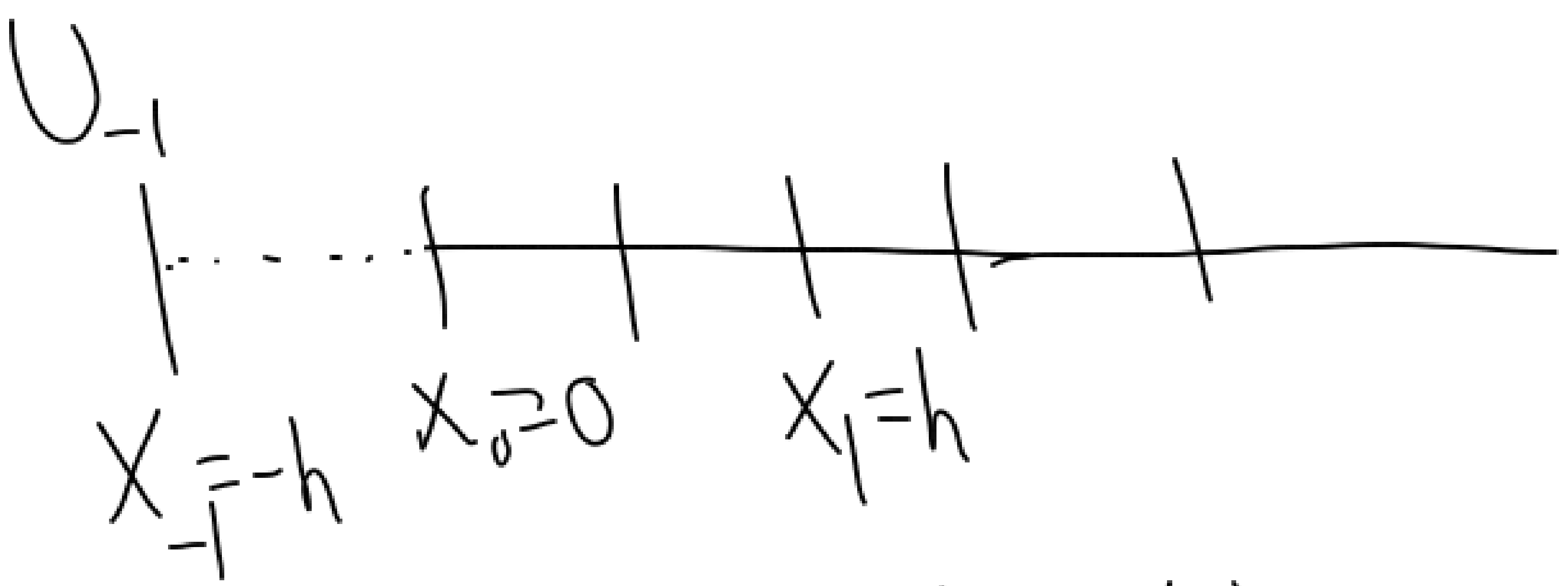
So we use

$$\frac{U_1 - U_0}{h} = \sigma_0 + \frac{h}{2} f(x_0)$$

Defect Correction

2nd-order accurate

② Ghost-point method



$$U'(x_0) \approx \frac{U_1 - U_{-1}}{2h} = \sigma_0$$

We also impose $U''(x) = f(x)$ at x_0:

$$\frac{U_1 - 2U_0 + U_{-1}}{h^2} = f(x_0)$$

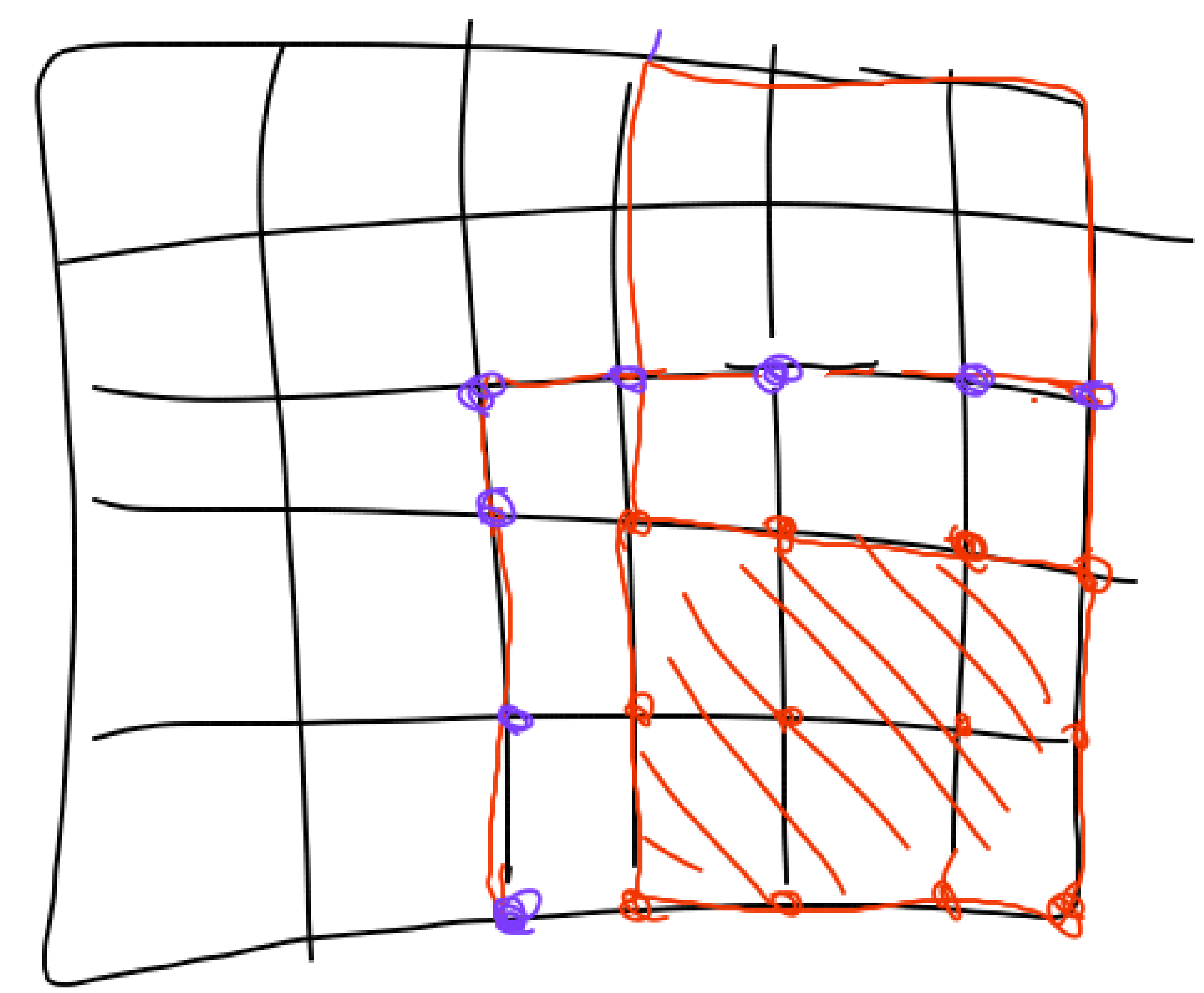
Solve: $U_{-1} = h^2 f(x_0) + 2U_0 - U_1$

Substitute:

$$\frac{U_1 - h^2 f(x_0) - 2U_0 + U_1}{2h} = \sigma_0$$

$$\frac{U_1 - U_0}{h} = \sigma_0 + \frac{h}{2} f(x_0)$$

Domain Decomposition



h-refinement: use a finer grid

p-refinement: use a higher-order discretization

Which is more is efficient?

Depends on smoothness

What if both ends are insulated?

$u'(0)=0 \quad u''(x)=f(x)$

$u'(1)=1$

If $u^*(x)$ is a solution,

$u^*(x)+C$ is also for all $C\in\mathbb{R}$

We have either 0 or ∞ solutions.

$$U''(x) = f(x)$$
$$U'(0) = \sigma_0$$
$$U'(1) = \sigma_1$$

$$\int_0^1 U''(x)\,dx = \int_0^1 f(x)\,dx$$
$$U'(1) - U'(0) = \int_0^1 f(x)\,dx$$
$$\sigma_1 - \sigma_0 = \int_0^1 f(x)\,dx$$

Solution exist iff this holds.

$$\underbrace{\frac{1}{h^2}\begin{bmatrix} -h & h & & & \\ 1 & -2 & 1 & & \\ & \ddots & \ddots & \ddots & \\ & & 1 & -2 & 1 \\ & & & -h & h \end{bmatrix}}_{A} \begin{bmatrix} \\ U \\ \\ \end{bmatrix} = \begin{bmatrix} 0 \\ f \\ 0 \end{bmatrix}$$

$$A \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix}$$

A is singular