A guide to the numerical zoo

AMCS 252, Spring 2025

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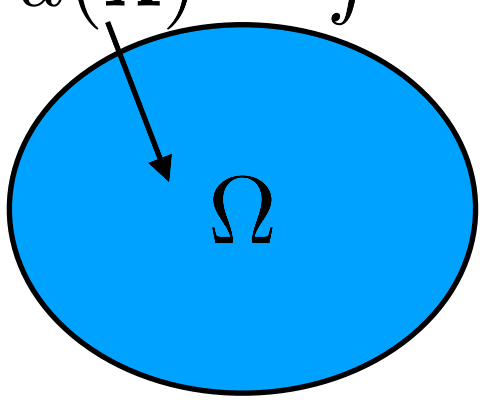
The purpose of this lecture

- Introduce some general classifications of numerical methods
- Give you a basic idea of how each kind works and what their advantages are
- We won't go into details

Three types of problems

- Steady state
    - boundary -value problem
    - Solution doesn't change in time

$$\nabla^2 u(\mathbf{x}) = f$$



$\partial\Omega$

$$u(\mathbf{x}) = g(\mathbf{x}) \quad (\mathbf{x} \in \partial\Omega)$$

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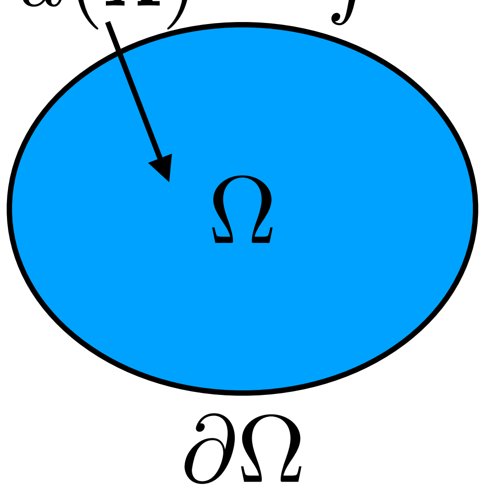

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- Time-dependent
  - Initial-value problem
  - Solution changes in time

$$u'(t) = f(t, u)$$

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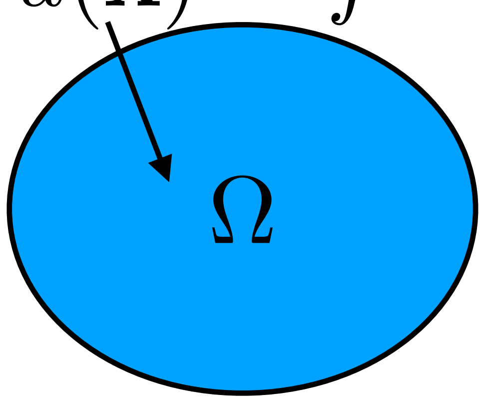

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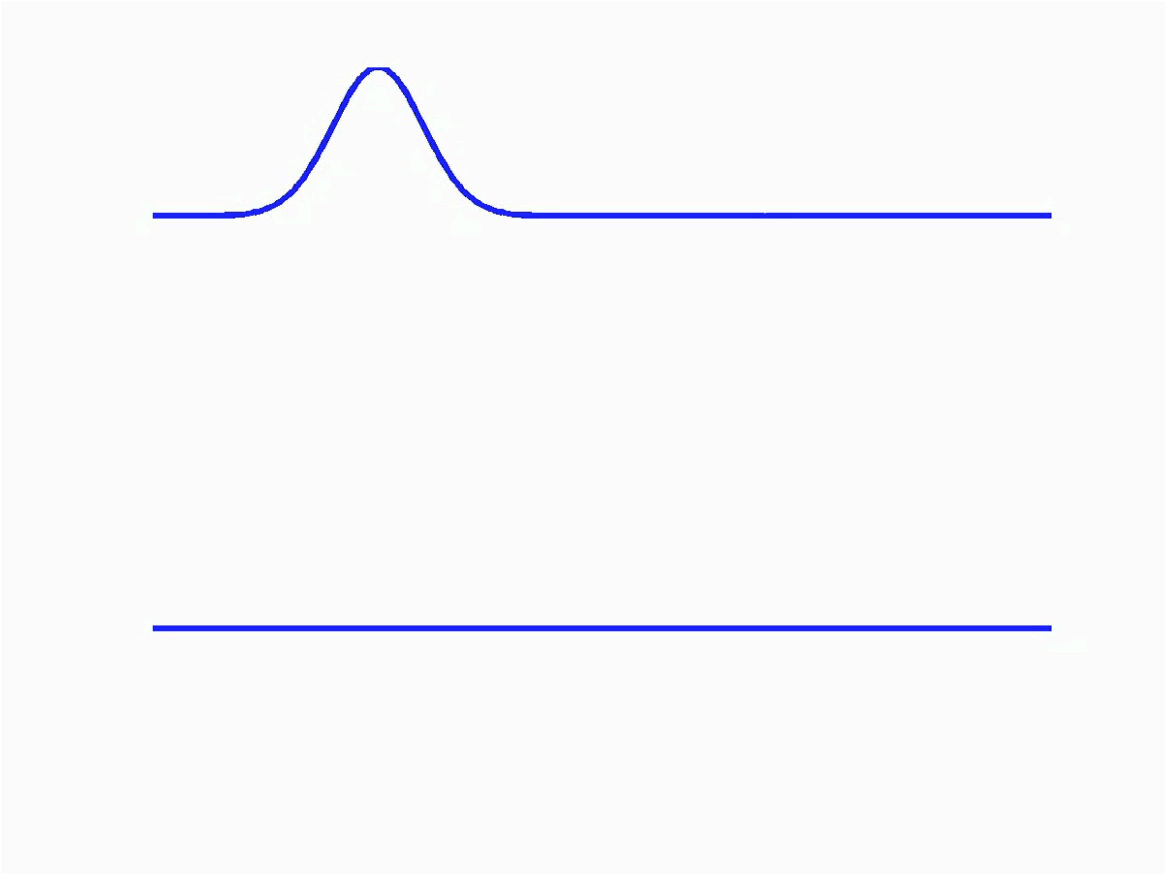
$$u'(t) = f(t, u)$$
$$u(0) = u_0$$

- Initial boundary value problem

$$u_t = \nabla^2 u(\mathbf{x}) - f$$
$$u(t, \mathbf{x} \in \partial\Omega) = g(\mathbf{x})$$
$$u(t = 0, \mathbf{x}) = u_0(\mathbf{x})$$

Linear vs. Nonlinear problems

- Linear problems
  - May be exactly solvable
  - Can use techniques like superposition
  - Discretizations lead to linear algebra
  - Examples:
    - diffusion of heat
    - electromagnetic waves
    - acoustic waves
    - gravitational potential



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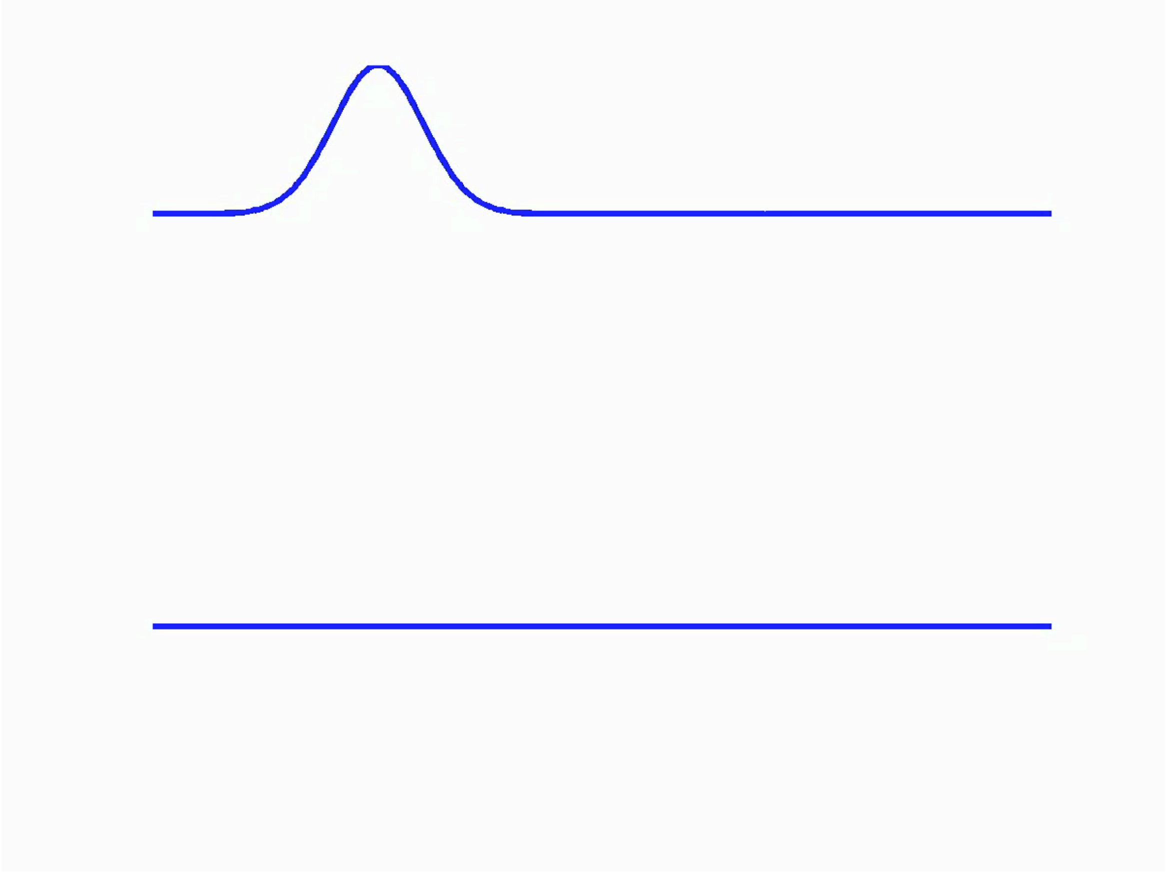

Linear vs. Nonlinear problems

- Nonlinear problems
  - Rarely have exact solutions
  - No superposition
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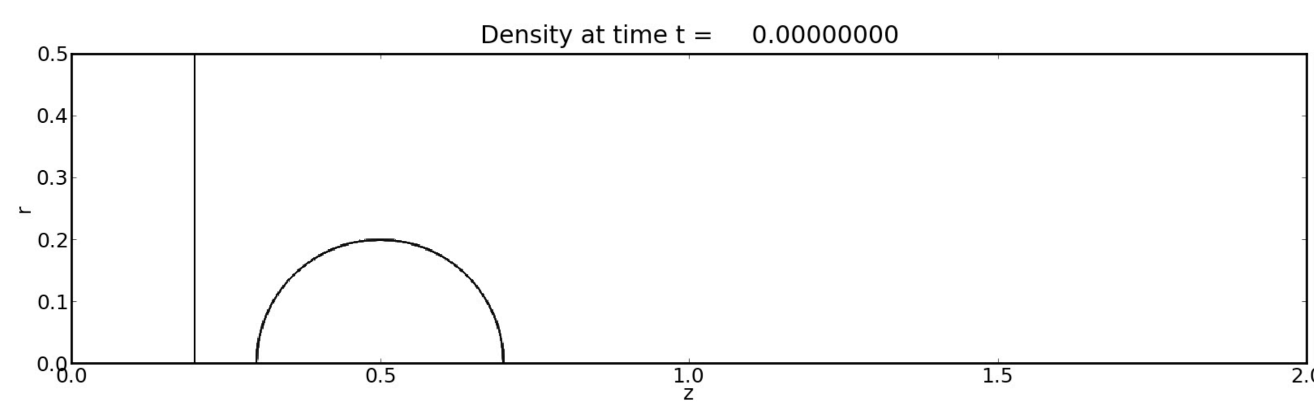
  - Solution may not be unique
  - Examples:
    - Pendulum
    - Spread of disease
    - Water waves
    - Fluid dynamics



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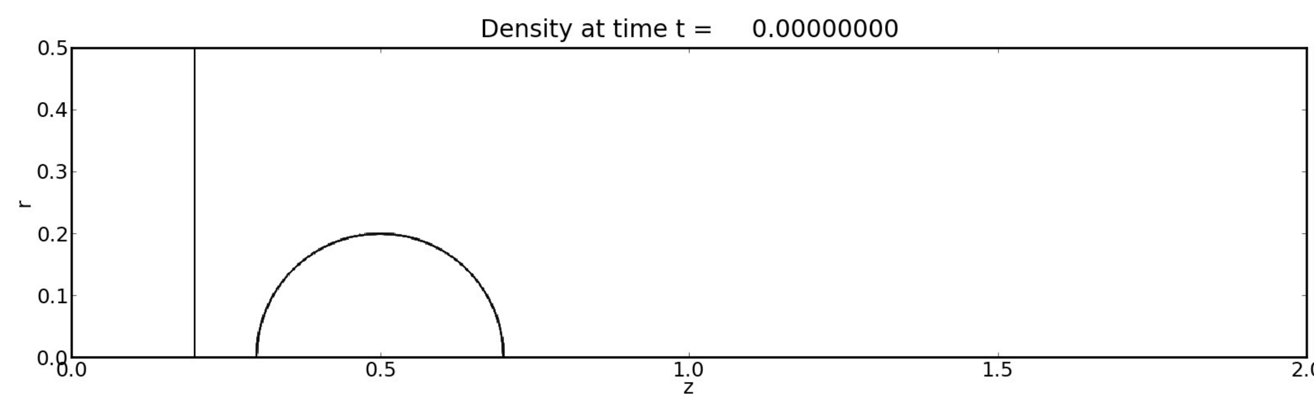
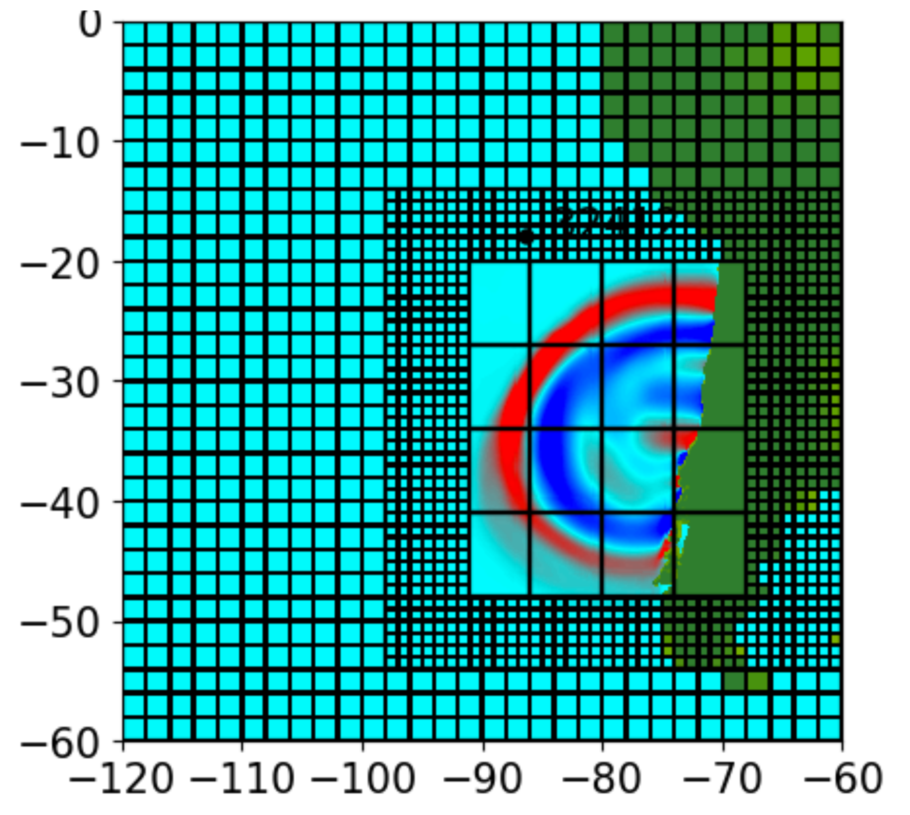


Discretization

- Solutions of ODEs and PDEs live in continuous, infinite-dimensional spaces!

- To compute with them, we must replace those with discrete spaces that are finite-dimensional

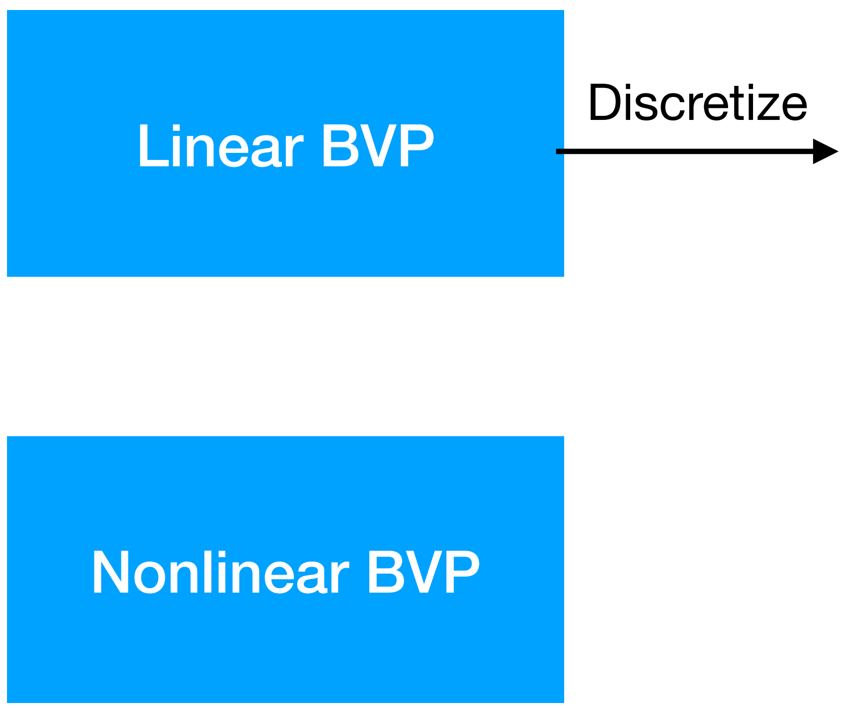


Discretization of boundary value problems

Linear BVP

Nonlinear BVP

Discretization of boundary value problems



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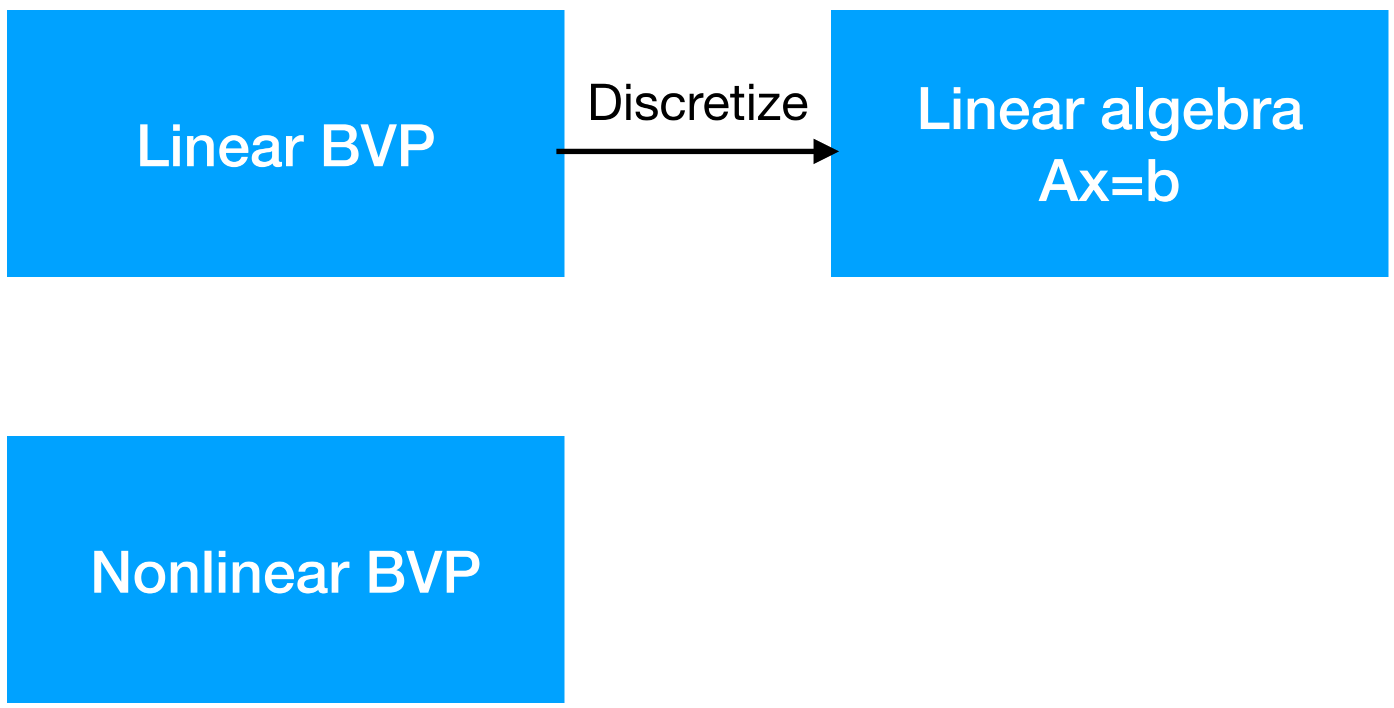

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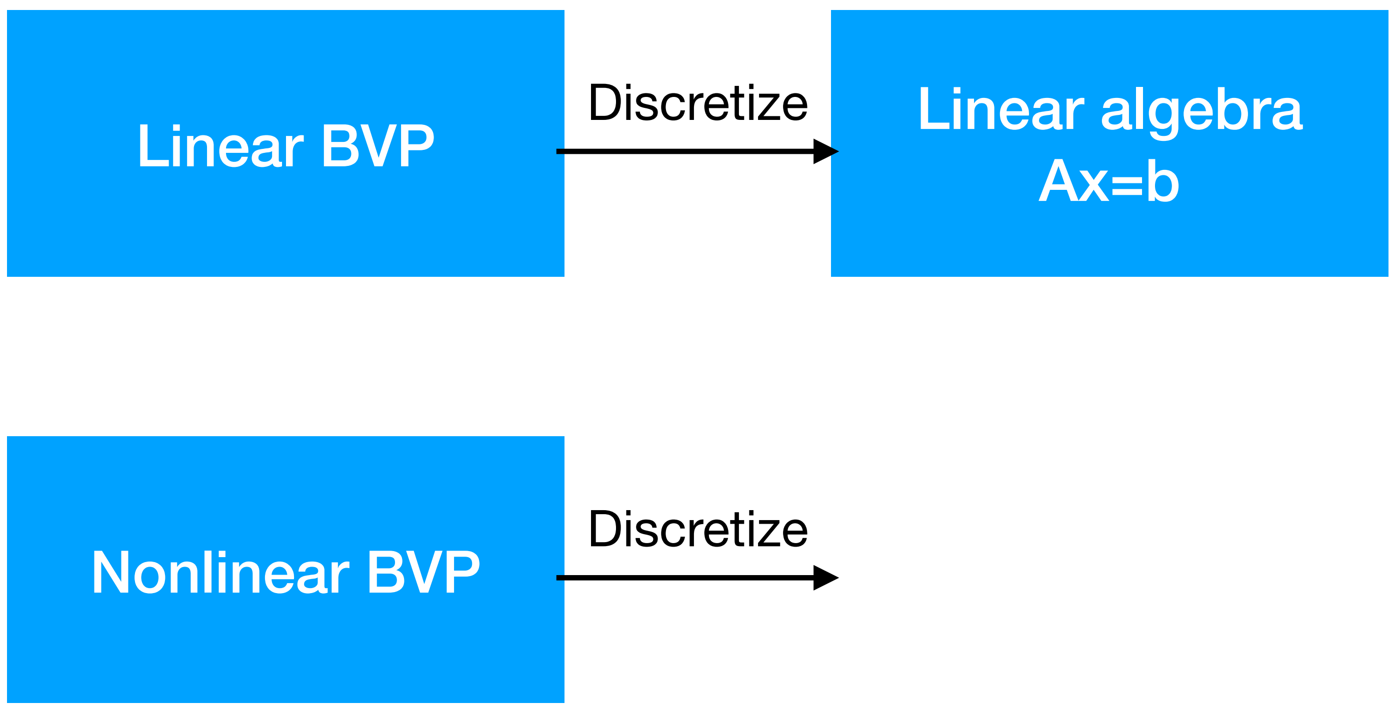

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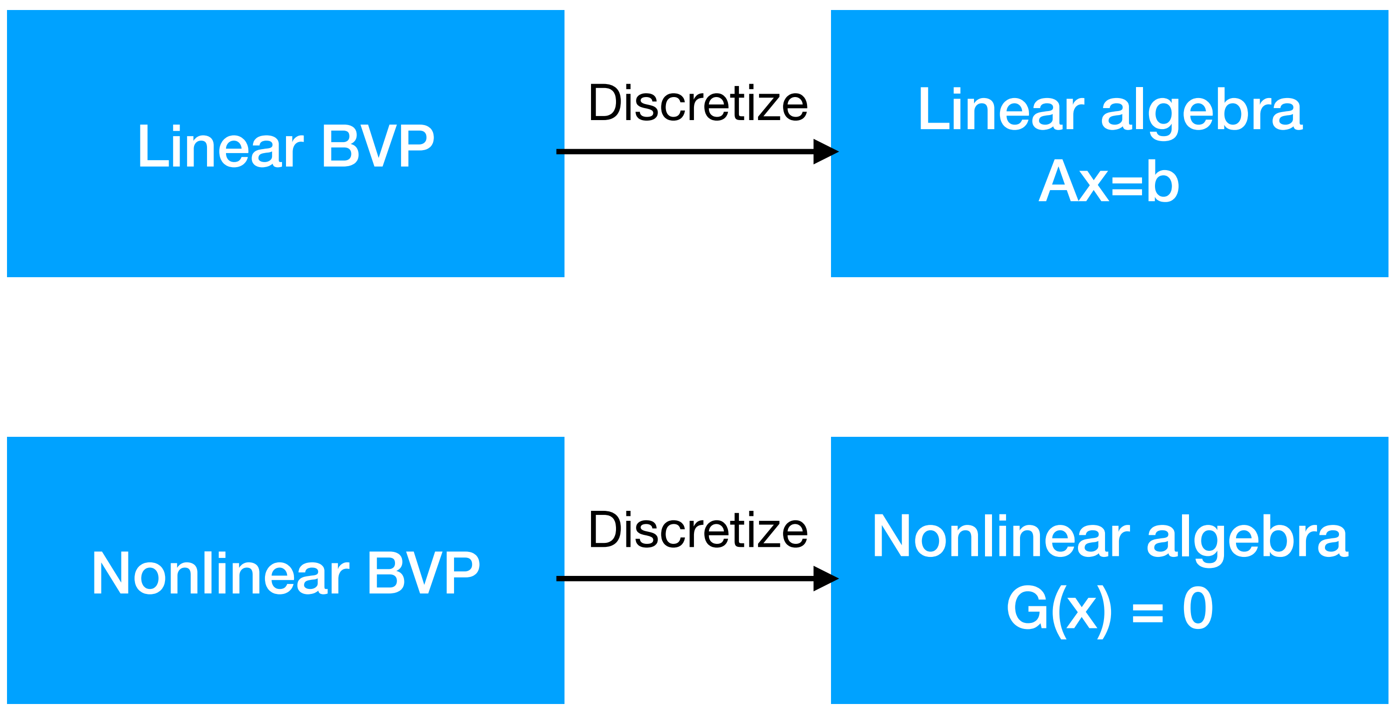

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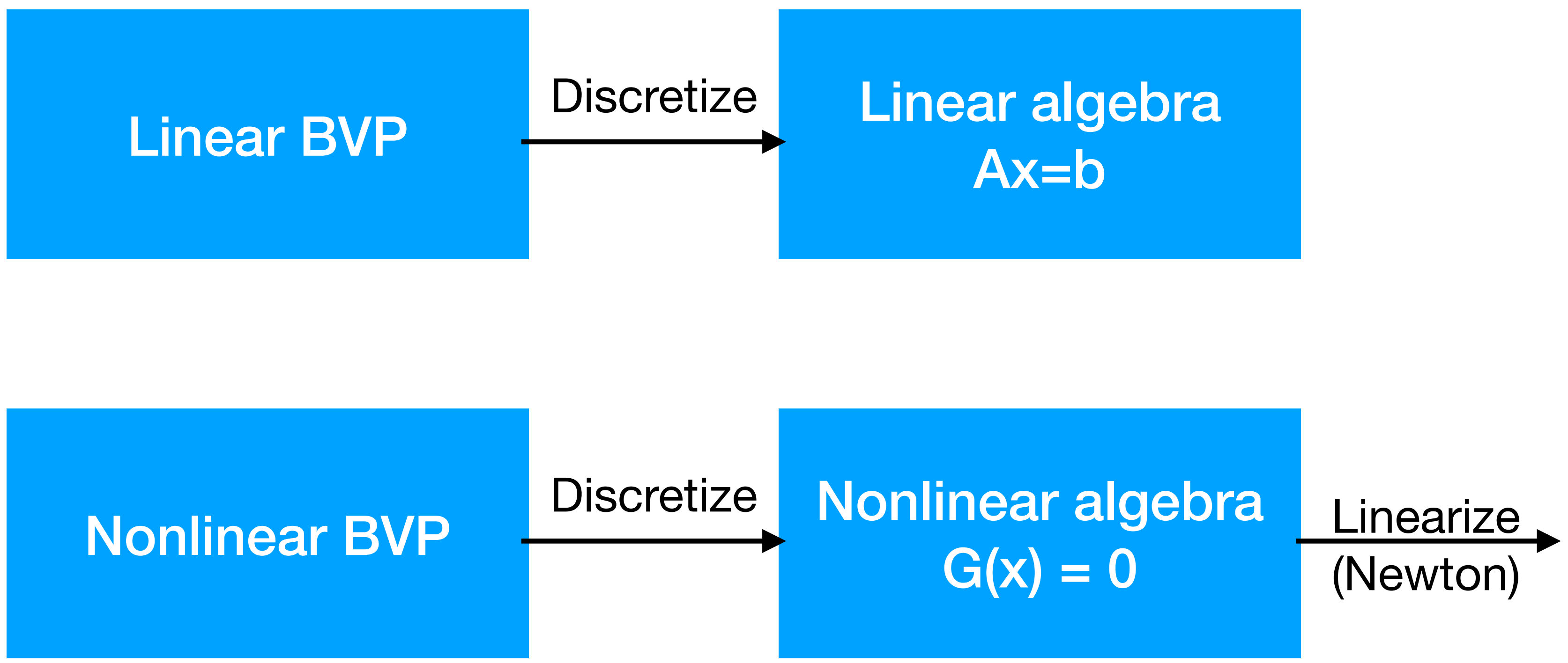

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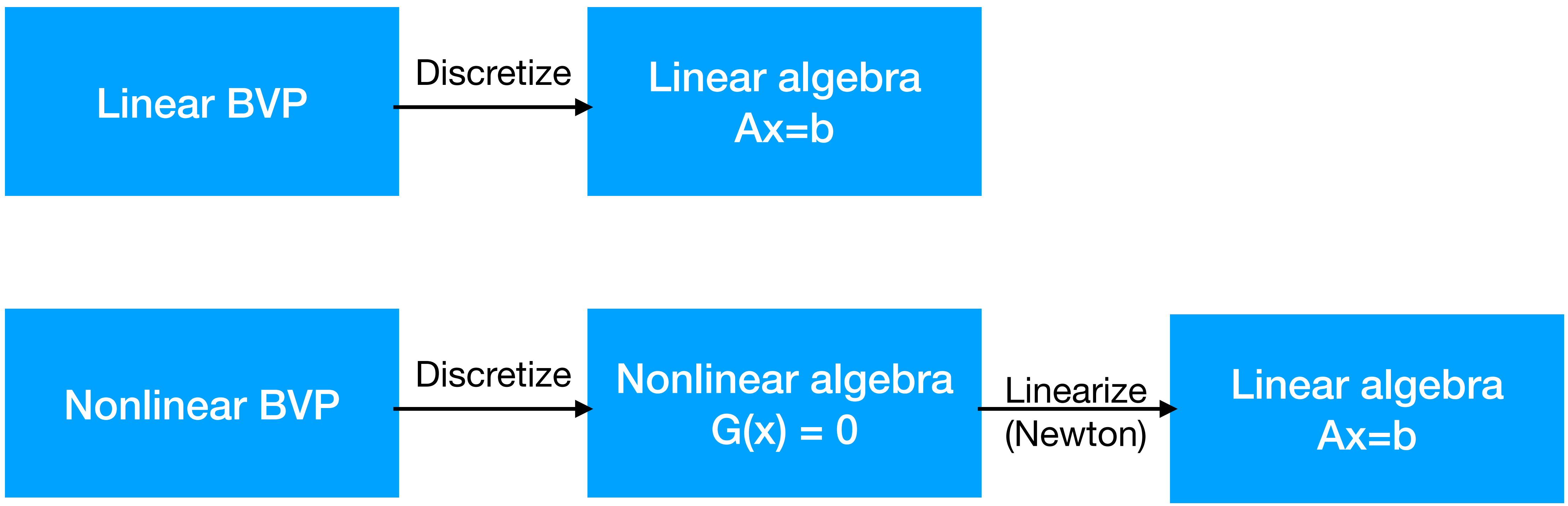

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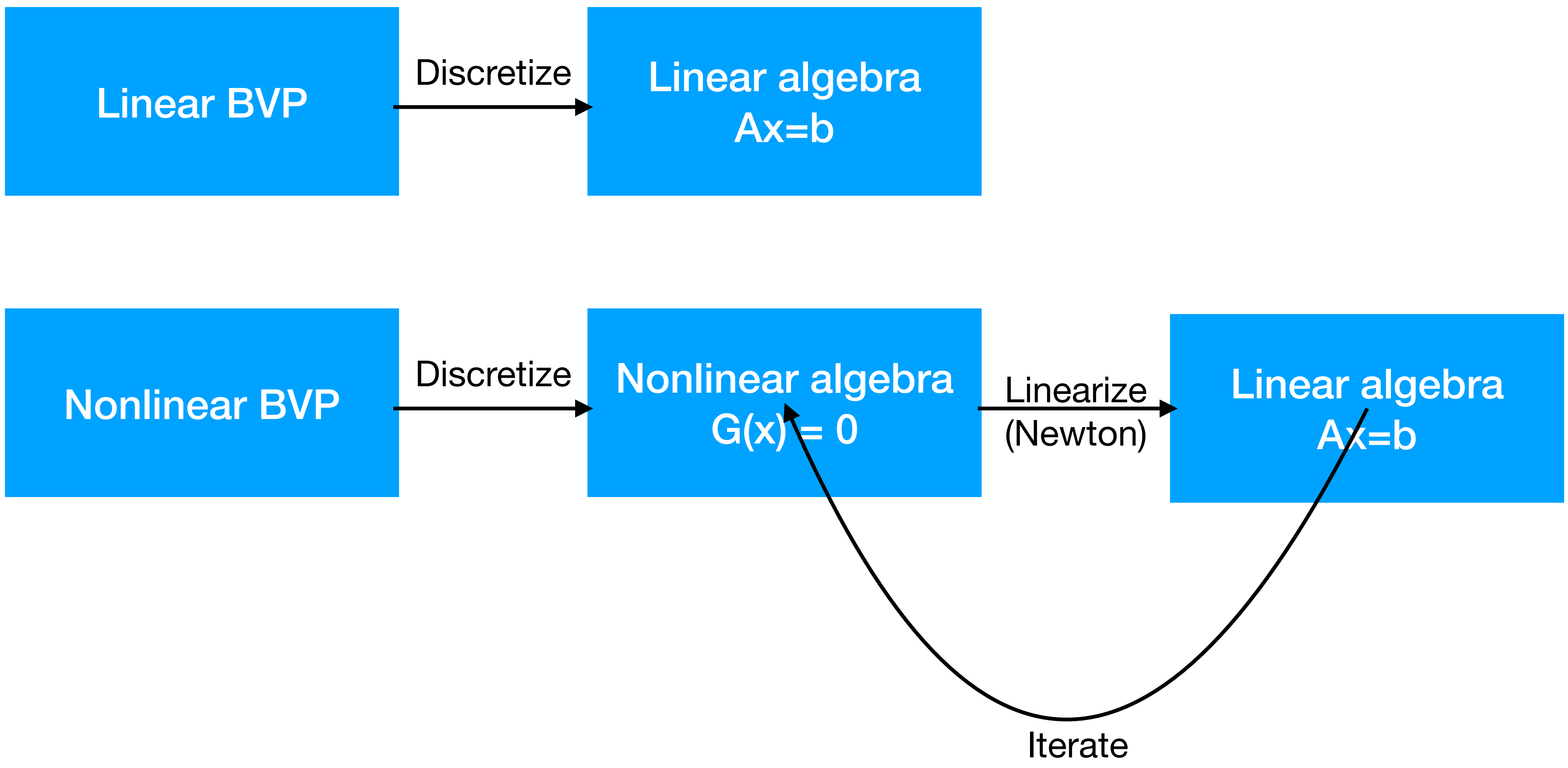

3 “finite”s

- Finite differences

- Finite volumes

- Finite elements

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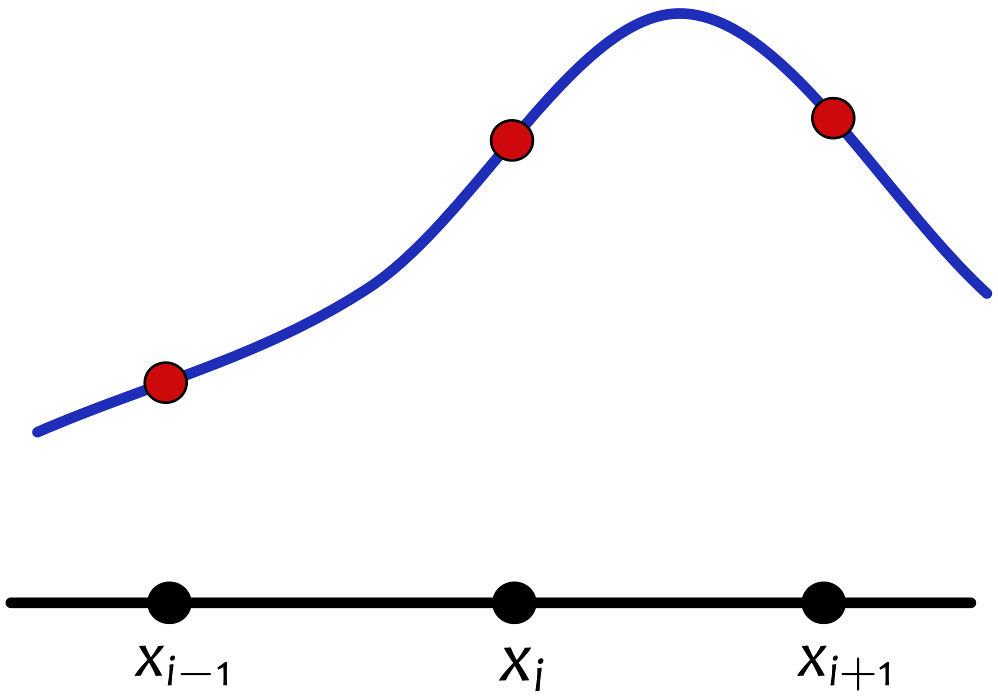
- Finite differences

  $u(x_i, t)$



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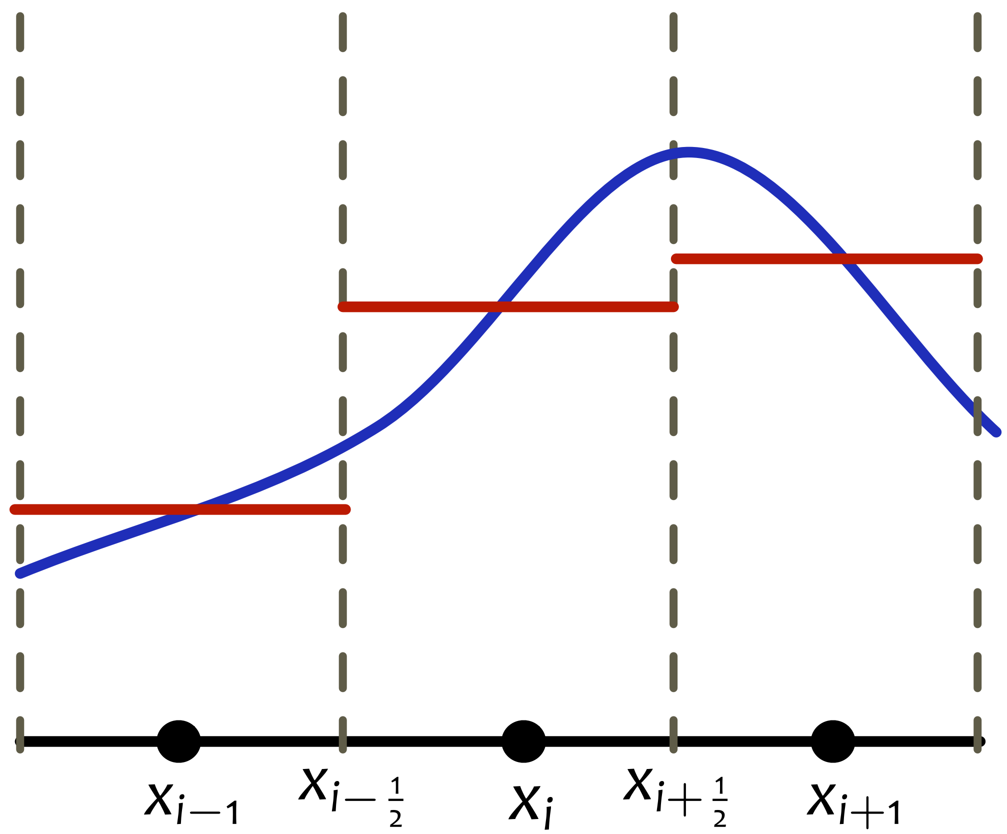
$$u(x_i, t)$$

- Finite volumes

$$\int_{x_{i-1/2}}^{x_{i+1/2}} u(x,t)dx$$

- Finite elements



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$$u(x_i, t)$$

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$$\int_{x_{i-1/2}}^{x_{i+1/2}} u(x,t)dx$$

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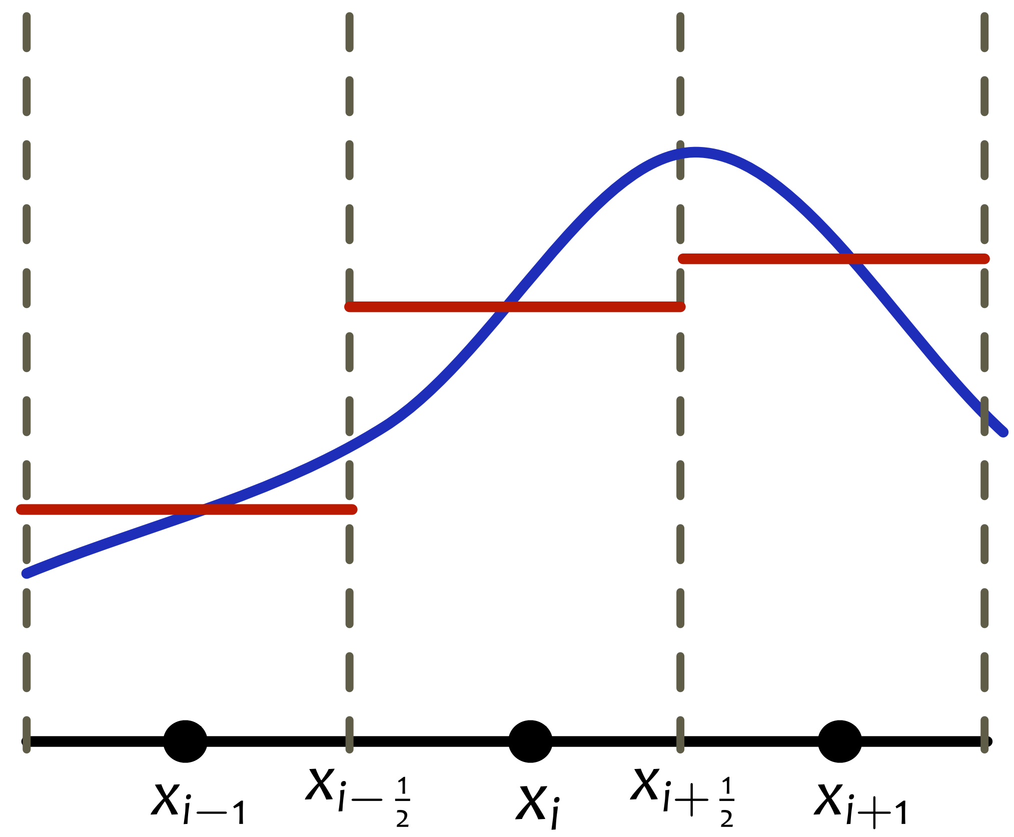
$$\int_{x_{i-1/2}}^{x_{i+1/2}} u(x,t)\phi_j(x)dx$$



3 “finite”s

FDM

- Easier to code
- Lower computational cost
- Works best with simple geometries
- Simplest to understand

FVM

Falls somewhere in-between

FEM

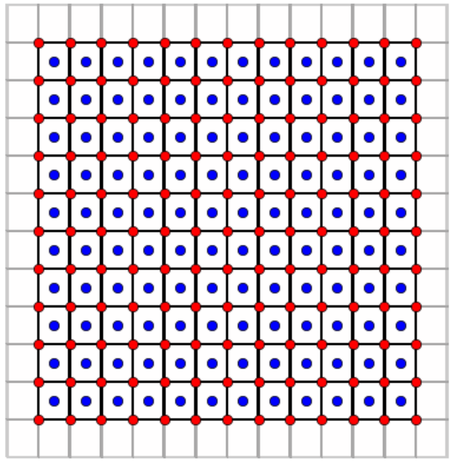
- Harder to code
- Higher computational cost
- Works easily with complex geometries
- More sophisticated mathematical foundation

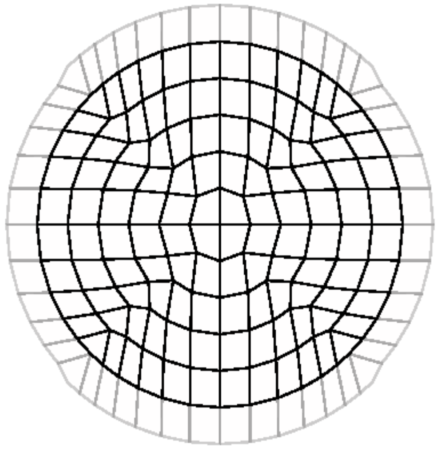
Basis functions

- Spectral methods
  - Basis functions have global support
  - More accurate for smooth solutions
  - Work best in simple geometries
  - Dense linear algebra

- “Local” methods
  - Basis functions have compact support
  - More suited to non-smooth solutions
  - More suited to complex geometries
  - Sparse linear algebra

Grids





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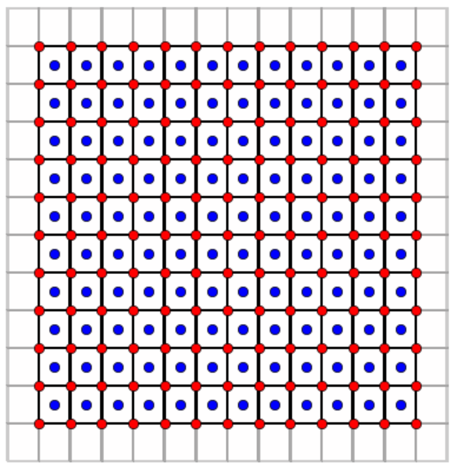
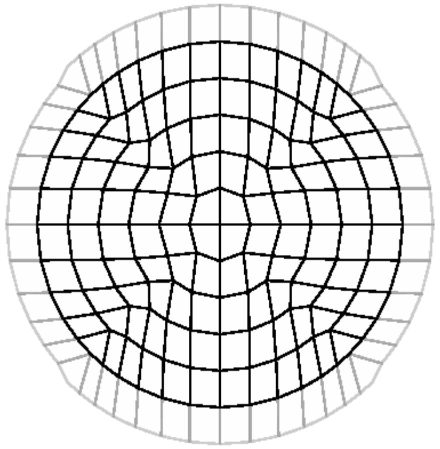
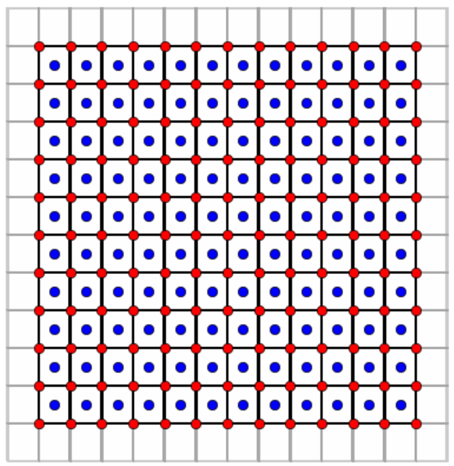
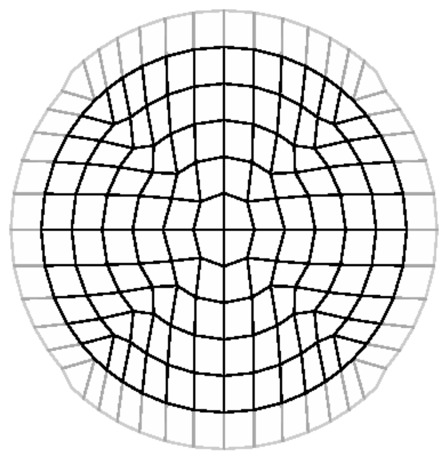




- Structured grid

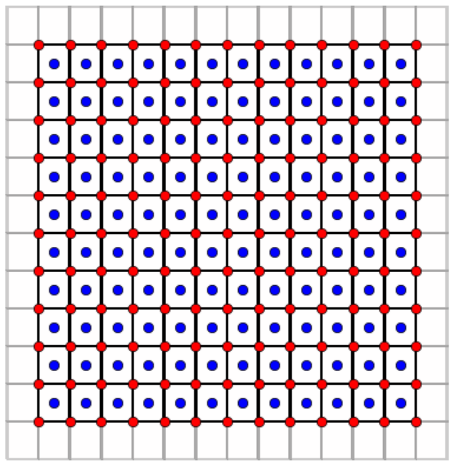
Grids

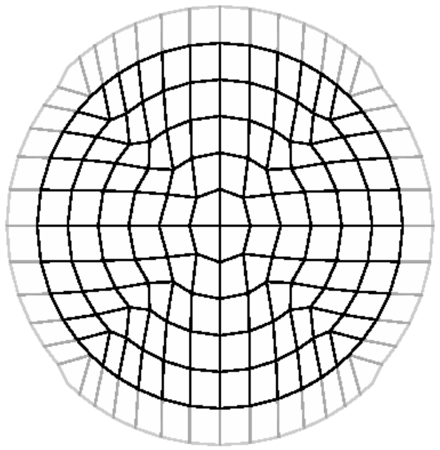




- Structured grid
  - Natural ordering

Grids





- Structured grid
  - Natural ordering
  - Neighbors are obvious

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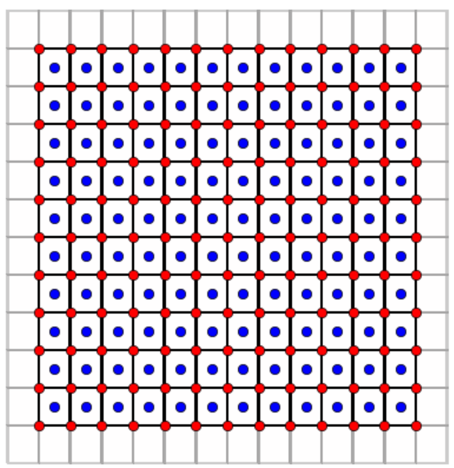

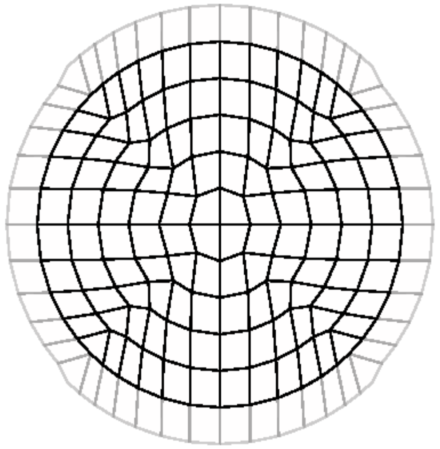

- Structured grid
  - Natural ordering
  - Neighbors are obvious
  - Computationally efficient

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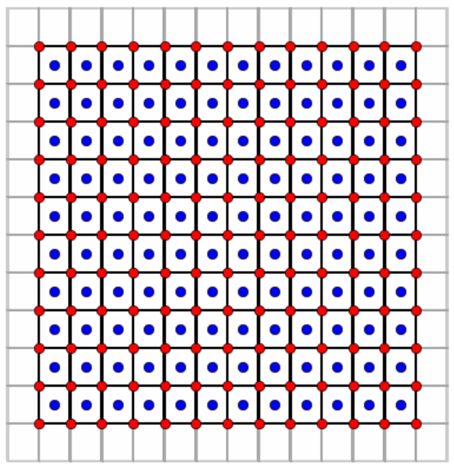

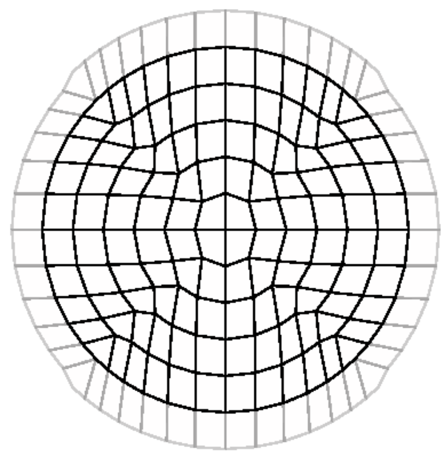

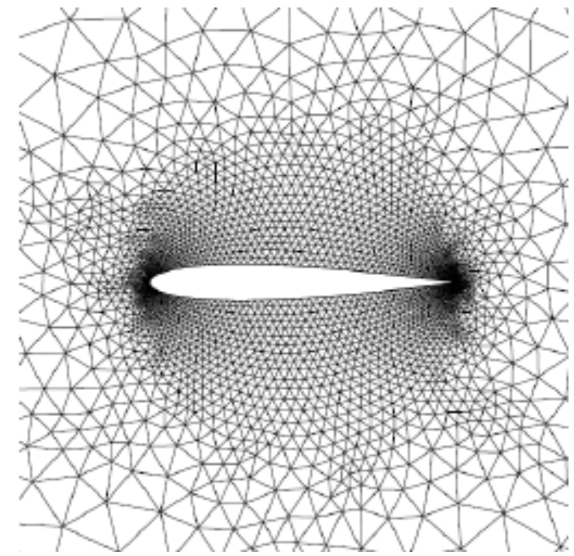

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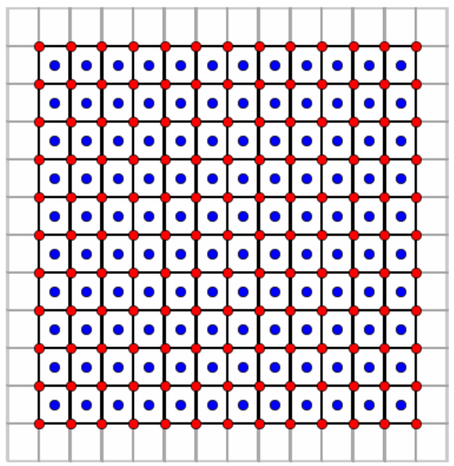

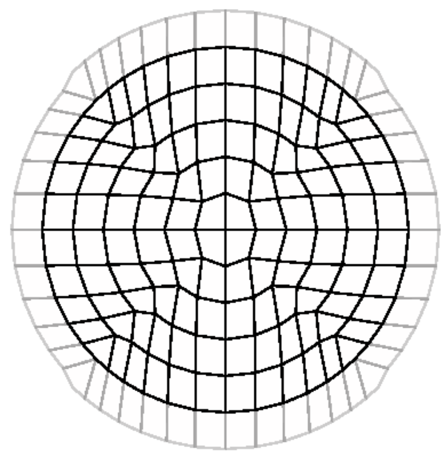

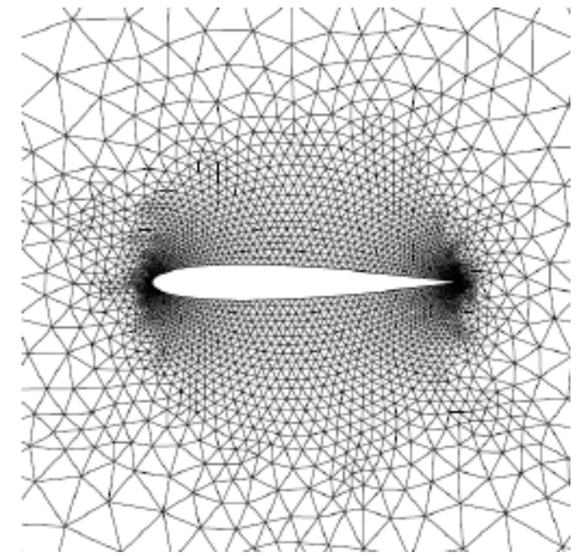

- Structured grid
  - Natural ordering
  - Neighbors are obvious
  - Computationally efficient

- Unstructured grids
  - Need lookup table for neighbors
  - Arbitrary geometries
  - Local refinement

Explicit vs. Implicit

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- Explicit
  - Easier to program
  - Low cost per step
  - Efficient for wave equations (hyperbolic PDEs)
- Implicit
  - Harder to program
  - Higher cost per step
  - Efficient for stiff probems (elliptic, parabolic PDEs)

Accuracy and cost

- Goal 1: Achieve a desired level of accuracy for the lowest cost
- Goal 2: Achieve the greatest possible accuracy for a given cost
- “Method x is expensive” or “method y is fast” isn’t meaningful