Poisson's Equation

$$\nabla^2 u = f(x,y)$$

More generally:

$$\nabla \cdot \left(K(x,y) \nabla u(x,y) \right) = f(x,y)$$

Applications:

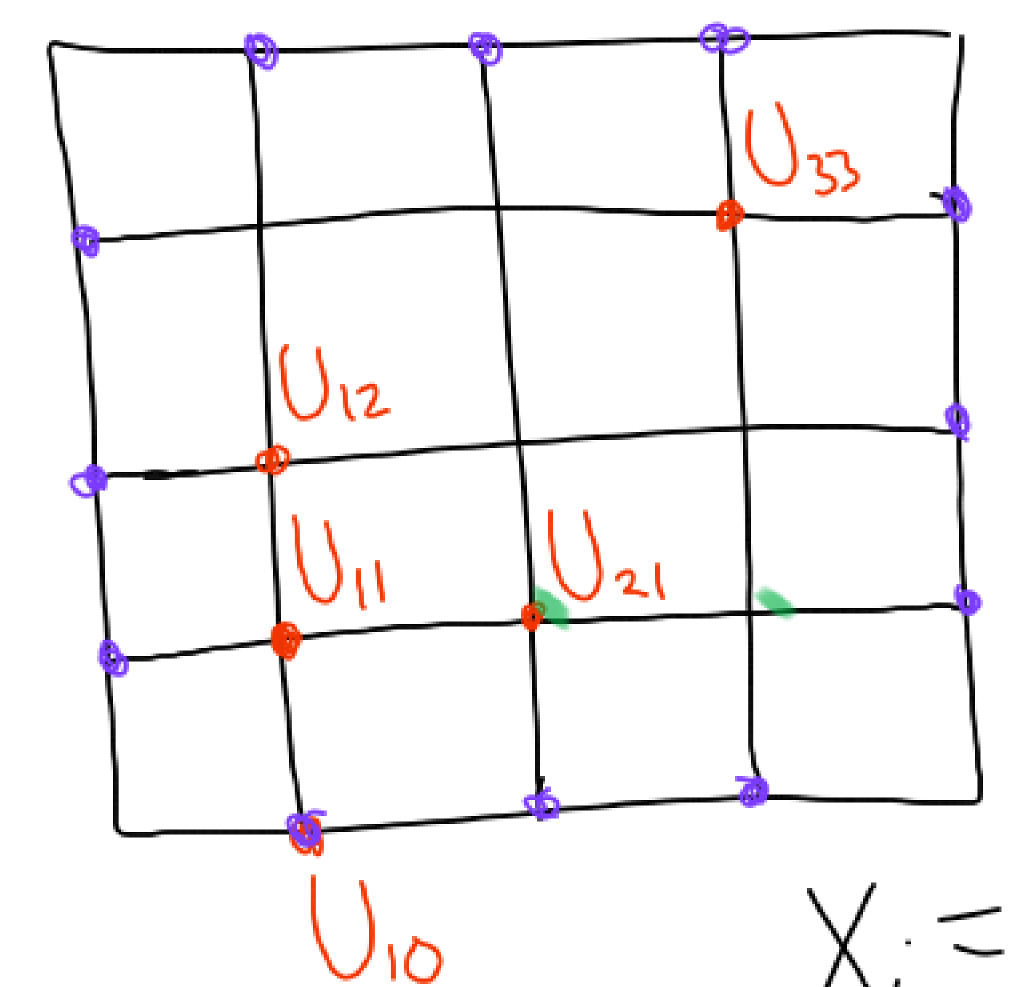
u	f	K
Temperature	Heat source	Heat conductivity
Gravitational potential	Mass	—
Electrical potential	Electric charge	Permitivity
Chemical concentration	Source	

Discretization in 2D

$$u_{xx} + u_{yy} = f(x,y) \qquad 0<x<1, \quad 0<y<1$$

$u(x,0) = \alpha(x)$ $\qquad u(0,y) = \gamma(y)$

$u(x,1) = \beta(x)$ $\qquad u(1,y) = \delta(y)$



$x_i = ih$

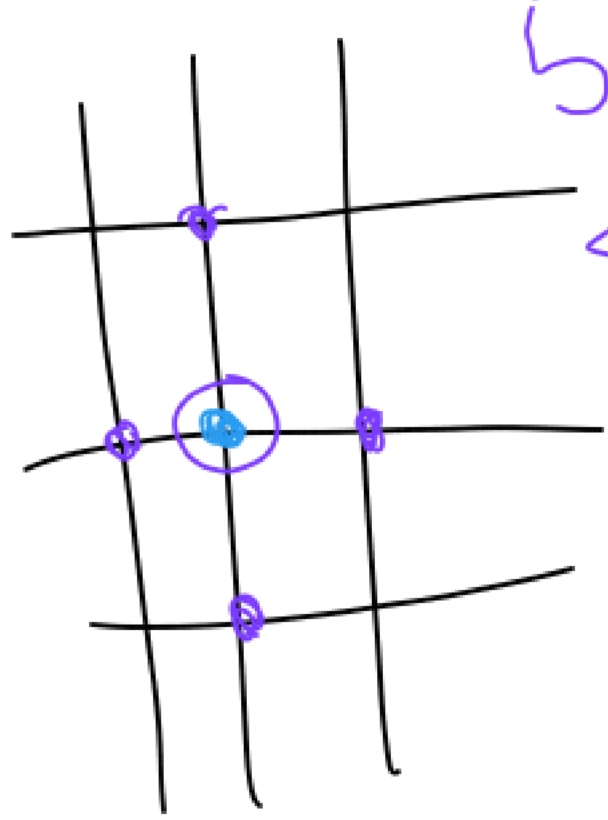
$y_j = jh$

$$\Delta x = \Delta y = h^2$$

$$u_{xx}\Big|_{(x_i,y_j)} \approx \frac{U_{i+1,j} - 2U_{i} + U_{i-1,j}}{h^2}$$

$1 \le i,j \le m$

$$u_{yy}\Big|_{(x_i,y_j)} \approx \frac{U_{i,j+1} - 2U_{ij} + U_{i,j-1}}{h^2}$$



5-point stencil

Row-wise ordering:

$$U = [U_{11}, U_{21}, U_{31}, \ldots, U_{12}, U_{22}, \ldots, \ldots, U_{m1}, \ldots, U_{mm}]^T$$

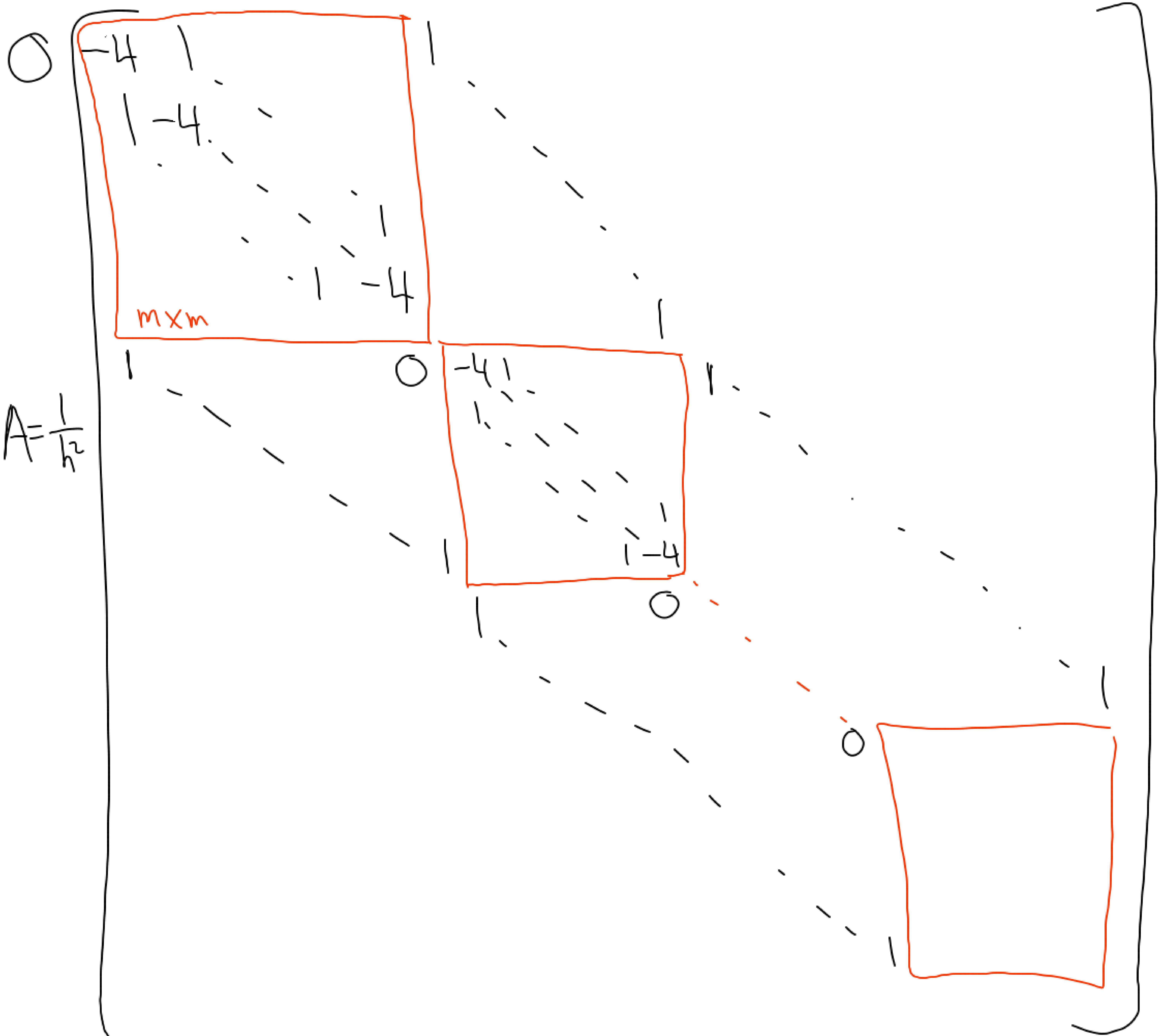
$$\frac{U_{i+1,j} - 2U_{ij} + U_{i-1,j}}{h^2} + \frac{U_{i,j+1} - 2U_{ij} + U_{i,j-1}}{h^2} = f(x_i,y_j)$$

$1 \le i,j \le m$

m^2 equations

$AU = F$

$A = \frac{1}{h^2}$



Sparse matrix

$m^2 \times m^2$

but only

$\sim 5m^2$ non-zero entries

Consistency

$$\frac{u(x_i+h,y_j)-2u(x_i,y_j)+u(x_i-h,y_j)}{h^2}+\frac{u(x_i,y_j+h)-2u(x_i,y_j)+u(x_i,y_j-h)}{h^2}=f(x_i,y_j)+\tau_{ij}$$

$$\cancel{u_{xx}}+\frac{h^2}{12}u_{xxxx}+\mathcal{O}(h^4)+\cancel{u_{yy}}+\frac{h^2}{12}u_{yyyy}+\mathcal{O}(h^4)=\cancel{f(x_i,y_j)}+\tau_{ij}$$

$$\tau_{ij}=\frac{h^2}{12}\left(u_{xxxx}+u_{yyyy}\right)+\mathcal{O}(h^4)$$ 2nd-order

$\|\tau\|=\mathcal{O}(h^2)$ as $h\to 0$

Global error:

$$\hat{U}_{ij}=u(x_i,y_j)$$

$$E=U-\hat{U}$$

$$AU=F$$

$$A\hat{U}=F+\tau$$

$$AE=-\tau$$

$$E=-A^{-1}\tau$$

$$\|E\|\leq\|A^{-1}\|\cdot\|\tau\|$$

We need $\|A^{-1}\|<C$ (stability)

in order to conclude

$\lim_{h\to 0}\|E\|=0$ (convergence)

Take $\|\cdot\|_2$.

What are the eigenvalues of A?

Let

$$AV = \lambda V$$

Assume V is separable:

$$V_{ij} = R_i S_j \qquad 1 \le i,j \le m$$

$$\frac{V_{i+1,j} - 2V_{ij} + V_{i-1,j}}{h^2} + \frac{V_{i,j+1} - 2V_{ij} + V_{i,j-1}}{h^2} = \lambda V_{ij}$$

$$\frac{S_j}{h^2}\left(R_{i+1} - 2R_i + R_{i-1}\right) + \frac{R_i}{h^2}\left(S_{j+1} - 2S_j + S_{j-1}\right) = \lambda R_i S_j$$

$$\underbrace{\frac{R_{i+1} - 2R_i + R_{i-1}}{R_i h^2}}_{\text{indep. of } j} + \underbrace{\frac{S_{j+1} - 2S_j + S_{j-1}}{S_j h^2}}_{\text{indep. of } i} = \lambda$$

indep. of i, j

So

$$\frac{R_{i+1} - 2R_i + R_{i-1}}{R_i h^2} = C_1 \qquad 1 \le i \le m$$

$$\frac{S_{j+1} - 2S_j + S_{j-1}}{S_j h^2} = C_2 \qquad 1 \le j \le m$$

Let $R_i = \xi^i$ for $\xi \in \mathbb{C}$

$$\xi^{i+1} - 2\xi^i + \xi^{i-1} = C_1 h^2 \xi^i$$

$$\xi^2 - (2 + C_1 h^2)\xi + 1 = 0$$

$$C_1 = \frac{2}{h^2}\left(\cos(p\pi h) - 1\right)$$

Similarly: $\quad 1 \le p, q \le m$

$$C_2 = \frac{2}{h^2}\left(\cos(q\pi h) - 1\right) \qquad h = \frac{1}{m+1}$$

$$\lambda_{pq} = C_1 + C_2 = \frac{2}{h^2}\left(\cos(p\pi h) + \cos(q\pi h) - 2\right)$$

$$\lambda_{pq} \approx \frac{\cancel{2}}{\cancel{h^2}}\left(\cancel{1} - \frac{p^2\pi^2\cancel{h^2}}{\cancel{2}} + \cancel{1} - \frac{q^2\pi^2\cancel{h^2}}{\cancel{2}} + O(h^4) - \cancel{2}\right)$$

(for small h)

$$\approx -(p^2+q^2)\pi^2 \qquad \text{Smallest: } \approx -2\pi^2$$

(in magnitude)

So $\|A^{-1}\| < \frac{1}{2\pi^2}$ as $h \to 0$.