Jacobi's Method and Multigrid

$$u''(x) = f(x) \quad 0<x<1$$
$$u(0) = \alpha$$
$$u(1) = \beta$$
$$\Downarrow$$
$$AU = F$$

Jacobi's Method

$$G = \begin{bmatrix} 0 & 1/2 & & & \\ 1/2 & \ddots & \ddots & & \\ & \ddots & \ddots & \ddots & \\ & & \ddots & \ddots & 1/2 \\ & & & 1/2 & 0 \end{bmatrix} \qquad A = \frac{1}{h^2}\begin{bmatrix} -2 & 1 & & & \\ 1 & \ddots & \ddots & & \\ & \ddots & \ddots & \ddots & \\ & & \ddots & \ddots & 1 \\ & & & 1 & -2 \end{bmatrix}$$

$$A = \frac{2}{h^2}(G - I)$$
$$AU = F \Rightarrow \frac{2}{h^2}(G-I)U = F$$
$$GU - U = \frac{h^2}{2}F$$
$$U = GU - \frac{h^2}{2}F$$

1. Pick an initial guess $U^{[0]}$
2. Compute $U^{[k+1]} = GU^{[k]} - \frac{h^2}{2}F$
3. Repeat 2 until $\underbrace{\|F-AU\|}_{\text{residual}} < \varepsilon$

U is a fixed point.

Will $U^{[k]} \to U$ as $k \to \infty$? Yes

If so, how quickly?

$$e^{[k]} = U^{[k]} - U$$

$$U^{[k+1]} - U = GU^{[k]} - \frac{h^2}{2}F - U$$

$$e^{[k+1]} = GU^{[k]} - \cancel{\frac{h^2}{2}F} - (GU - \cancel{\frac{h^2}{2}F})$$

$$e^{[k+1]} = Ge^{[k]} \Rightarrow e^{[k]} = G^k e^{[0]}$$

G is symmetric $\Rightarrow$ G has a complete set of orthogonal eigenvectors V_p

$(GV_p = \gamma_p V_p)$

Let $e^{[0]} = \sum_{p=1}^{m} c_p V_p$

Then $G^k e^{[0]} = \sum_{p=1}^{m} c_p G^k V_p$

$$e^{[k]} = \sum_{p=1}^{m} c_p \gamma_p^k V_p$$

How large is $|\gamma_p|$?

$$A = \frac{2}{h^2}(G - I) \qquad A w_p = \lambda_p w_p$$

$$\lambda_p = \frac{2}{h^2}(\cos(p\pi h) - 1)$$

$$A w_p = \frac{2}{h^2}(G - I) w_p = \lambda_p w_p \qquad p = 1, 2, \dots, m$$

$$h = \frac{1}{m+1}$$

$$(G - I) w_p = \frac{h^2}{2} \lambda_p w_p$$

$$G w_p = \underbrace{\left(\frac{h^2}{2}\lambda_p + 1\right)}_{\gamma_p} w_p$$

$$\gamma_p = \cos(p\pi h)$$

$|\gamma_p| < 1$ for all $p = 1, 2, \dots, m$

so $\lim_{k\to\infty} \|e^{[k]}\| = 0$

For small h:

$$\cos(p\pi h) = \underbrace{1 - \frac{p^2\pi^2 h^2}{2}}_{\text{Very close to } 1} + O(h^4)$$

Also: $p = m$

$$\cos\left(\frac{m\pi}{m+1}\right) = -1 + O(h^2)$$

Under-relaxed Jacobi

$$\hat{U}^{[k+1]} = G U^{[k]} - \frac{h^2}{2} F$$

$$U^{[k+1]} = U^{[k]} + \omega\left(\hat{U}^{[k+1]} - U^{[k]}\right)$$

$$0 < \omega < 1$$

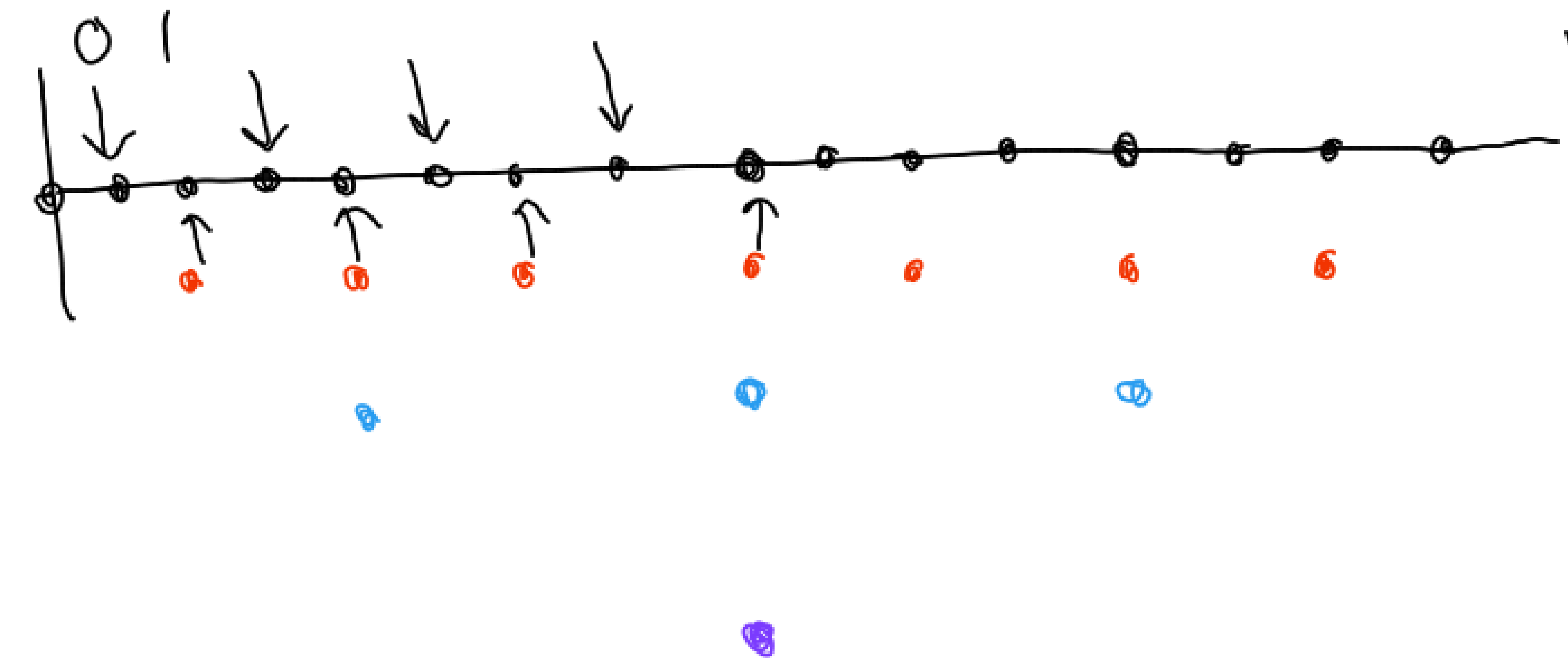
$$U^{[k+1]} = (1-\omega) U^{[k]} + \omega \hat{U}^{[k+1]}$$

$$U^{[k+1]} = (1-\omega) I U^{[k]} + \omega G U^{[k]} - \omega \frac{h^2}{2} F$$

$$= \underbrace{\left((1-\omega) I + \omega G\right)}_{\hat{G}} U^{[k]} - \omega \frac{h^2}{2} F$$

$$\hat{\gamma}_p = \omega \gamma_p + (1-\omega)$$

Multigrid



Solve $AU=F$ on a grid with m points using underrelaxed Jacobi (ν iterations) with $\omega = \frac{2}{3}$.

Then take only every 2nd grid point and repeat

Let U_ν denote solution on fine grid

$$F - AU_\nu = r_\nu$$

$$F - AU = 0$$

$$-A(\underbrace{U_\nu - U}_{e_\nu}) = r_\nu \qquad e_\nu = U_\nu - U$$

$$\boxed{Ae_\nu = -r_\nu}$$

solve this eqn. on coarse grids.

Then correct: $U = U_\nu - e_\nu$

Then interpolate → then smooth again (UR Jacobi)

V-cycle

$m = 2^k - 1$ pts. $AU = F$

$2^{k-1} - 1$ $Ae_1 = -r_1$

$2^{k-2} - 1$ $Ae_2 = r_2$

Direct solve

At each step:

- correct e
- Interpolate
- Smooth

$\mathcal{O}(m \log(m))$ work total

"Fast"