Stiffness (Ch. 8)

Two reasons to restrict k:

1. To ensure sufficient accuracy
   $k \le k_{acc}$

2. To ensure stability
   $k \le k_{stab}$

For explicit methods
we have $k_{stab} \approx \frac{1}{L}$

L: Lipschitz const.

If $L \gg 1$, this means
we must take $k \ll 1$.

Prothero-Robinson Problem

$$u'(t) = \lambda(u(t) - \cos(t)) - \sin(t).$$

$$L = |\lambda|$$

With I.V. $u(0) = 1$, we have $u = \cos(t)$

With I.V. $u(t_0) = \eta$ we have

$$u(t) = \cos(t) + e^{\lambda(t-t_0)}(\eta - \cos(t_0))$$

We say this problem is stiff:

1. There is a slowly-varying solution, but nearby solutions are rapidly-varying.

2. Implicit methods give an accurate solution with much less work (compared to explicit methods)

i.e. $K_{stab} \ll K_{acc}$ (for explicit methods)

Consider: $u'(t) = \lambda u(t)$ $\quad |\lambda| \gg 1$, $\lambda < 0$

Chemical Reaction Problem

$$A \xrightarrow{K_1} B \xrightarrow{K_2} C$$

(A) $u_1'(t) = -K_1 u_1(t)$

(B) $u_2'(t) = K_1 u_1(t) - K_2 u_2(t)$

(C) $u_3'(t) = K_2 u_2(t)$

$$\begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix}' = \begin{bmatrix} -K_1 & 0 & 0 \\ K_1 & -K_2 & 0 \\ 0 & K_2 & 0 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix}$$

$\lambda = 0, -K_1, -K_2$

What type of method Should we use for a Stiff problem?

- A-stable (or $A(\alpha)$-stable)
  $\rightarrow$ then $K_{stab} = \infty$
  So we can take $K \approx K_{acc}$.

Backward Euler: $U^{n+1} = U^n + Kf(U^{n+1})$

(Implicit)

$$R(z) = \frac{1}{1-z} \quad \lim_{z \to -\infty} R(z) = 0$$

Trapezoidal Method: $U^{n+1} = U^n + \frac{K}{2}\left(f(U^n) + f(U^{n+1})\right)$

$$R(z) = \frac{1 + z/2}{1 - z/2} \quad \lim_{z \to -\infty} R(z) = -1$$

$|\lambda K| \gg 1 \quad \lambda < 0$

For very stiff problems, we want

$$\lim_{z \to -\infty} R(z) = 0 \quad \text{(L-stability)}$$

Recall $R(z) \approx e^z$ and $\lim_{z \to -\infty} e^z = 0$

Good methods for stiff problems:

1. Diagonally implicit RK methods that are A-stable, L-stable.

2. Backward Differentiation Methods ($A(\alpha)$-stable and L-stable)