Stability and convergence of one-step methods

$$u'(t) = f(u) \qquad t \in [0,T]$$
$$u(0) = \eta \qquad f \text{ is Lipschitz continuous}$$

$$N = \frac{T}{k} \qquad U^n \approx u(t_n)$$

$$E^N = U^N - u(T)$$

We want to prove that

$$\lim_{k\to 0} \|E^N\| = 0$$

Proof in steps:

1. Euler's method applied to linear ODE
2. Euler's method applied to a nonlinear ODE system
3. Any explicit Runge-Kutta method applied to a nonlinear ODE system

Dahlquist's Problem

$$u'(t) = \lambda u(t) + g(t)$$
$$u(0) = \eta$$

Solution:
$$\underbrace{u(t) = e^{\lambda t}\eta + \int_0^t e^{(t-\tau)\lambda} g(\tau)\,d\tau}_{\text{Duhamel's principle}}$$

Apply Euler's method:

$$U^{n+1} = U^n + k(\lambda U^n + g(t_n))$$

$$u(t_{n+1}) = u(t_n) + k(\lambda u(t_n) + g(t_n)) + k\tau^n$$

$$\tau^n = \mathcal{O}(k)$$

Subtract:

$$E^{n+1} = E^n + k\lambda E^n - k\tau^n$$

$$\boxed{E^{n+1} = (1+k\lambda)E^n - k\tau^n}$$

$$E^n = (1+k\lambda)E^{n-1} - k\tau^{n-1}$$

Substitute:

$$E^{n+1} = (1+k\lambda)\left((1+k\lambda)E^{n-1} - k\tau^{n-1}\right) - k\tau^n$$

$$E^{n+1} = (1+k\lambda)^2 E^{n-1} - k(1+k\lambda)\tau^{n-1} - k\tau^n$$

$$E^N = \cancel{(1+k\lambda)^N E^0} - k\sum_{m=1}^{N}(1+k\lambda)^{N-m}\tau^{m-1}$$

$$|E^N| = k\left|\sum_{m=1}^{N}(1+k\lambda)^{N-m}\tau^{m-1}\right|$$

$$|E^N| \le k\sum_{m=1}^{N}|1+k\lambda|^{N-m}|\tau^{m-1}|$$

Lemma: $|1+k\lambda| \le e^{k|\lambda|}$ $(k\ge 0)$

Proof: $e^{k|\lambda|} = 1 + k|\lambda| + \cdots$

$$e^{k|\lambda|} \ge 1 + k|\lambda| \ge |1+k\lambda|$$

$$|E^N| \le k\sum_{m=1}^{N} e^{k|\lambda|(N-m)}|\tau^{m-1}|$$

$kN = T$

$$\le k\sum_{m=1}^{N} e^{k|\lambda|N}|\tau^{m-1}|$$

$$\le kNe^{kN|\lambda|}\max_{0\le m\le N-1}|\tau^m|$$

$$|E^N| = \underbrace{Te^{T|\lambda|}}_{\text{Indep. of } k}\underbrace{\|\tau\|_\infty}_{\mathcal{O}(k)}$$

So this vanishes as $k \to 0$.

What if $T=10$ and $|\lambda|=10$?

Then we have

$$|E^N| \le 10e^{100}\|\tau\|_\infty \quad !!!$$

Step 2: Euler applied to Nonlinear ODE system

$$u'(t) = f(u) \qquad u(t) \in \mathbb{R}^m$$

$$u(0) = \eta$$

$$\|f(v) - f(w)\| \le L\|v - w\|$$

$$\forall \; v, w \in D$$

We have:

$$U^{n+1} = U^n + kf(U^n)$$

$$u(t_{n+1}) = u(t_n) + kf(u(t_n)) + k\tau^n$$

Subtract:

$$E^{n+1} = E^n + k(f(U^n) - f(u(t_n))) - k\tau^n$$

$$\|E^{n+1}\| \le \|E^n\| + k\|f(U^n) - f(u(t_n))\| + k\|\tau^n\|$$

$$\|E^{n+1}\| \le \|E^n\| + kL\|U^n - u(t_n)\| + k\|\tau^n\|$$

$$\|E^{n+1}\| \le (1+kL)\|E^n\| + k\|\tau^n\|$$

Substitute this into itself repeatedly

$$\|E^N\| \le \cancel{(1+kL)\|E^0\|} + k\sum_{m=1}^{N}(1+kL)^{N-m}\|\tau^{m-1}\|$$

Following the same steps, we obtain

$$\|E^N\| \le Te^{TL} \max_{0 \le m \le N-1} \|\tau^m\|$$

So $\|E^N\| = \mathcal{O}(k)$

③ Convergence of RK methods

Example:

$$U^* = U^n + \tfrac{1}{2}k f(U^n)$$

$$U^{n+1} = U^n + k f(U^*)$$

Rewrite:

$$\frac{U^{n+1} - U^n}{k} = \underbrace{f\left(U^n + \tfrac{1}{2}k f(U^n)\right)}_{\Psi(U^n)}$$

Claim: if f is L.C., then Ψ is L.C..

Proof

Assume $\|f(v) - f(w)\| \le L\|v - w\|$

Then

$\|\Psi(v) - \Psi(w)\| =$

$$\|f(v + \tfrac{1}{2}k f(v)) - f(w + \tfrac{1}{2}k f(w))\|$$

$$\le L\|v + \tfrac{1}{2}k f(v) - (w + \tfrac{1}{2}k f(w))\|$$

$$\le L\|v - w + \tfrac{1}{2}k(f(v) - f(w))\|$$

$$\le L\|v - w\| + \tfrac{1}{2}kL\|f(v) - f(w)\|$$

$$\le L\|v - w\| + \tfrac{1}{2}kL^2\|v - w\|$$

$$\le (L + \tfrac{1}{2}kL^2)\|v - w\|$$

So Ψ is L.C. with constant $L + \tfrac{1}{2}kL^2$

So we have

$$U^{n+1} = U^n + k\Psi(U^n)$$

We follow the previous proof but with

$$f \to \Psi$$

$$L \to L + \tfrac{1}{2}kL^2$$

We get

$$\|E^n\| \le T e^{T(L + \frac{1}{2}kL^2)} \max_m \|\tau^m\|$$

which is the same bound when $k \to 0$.