Homework 1 and Course Project requirements are now on the course website.

Finite Difference Approximations

How can we approximate $u'(\bar{x})$?

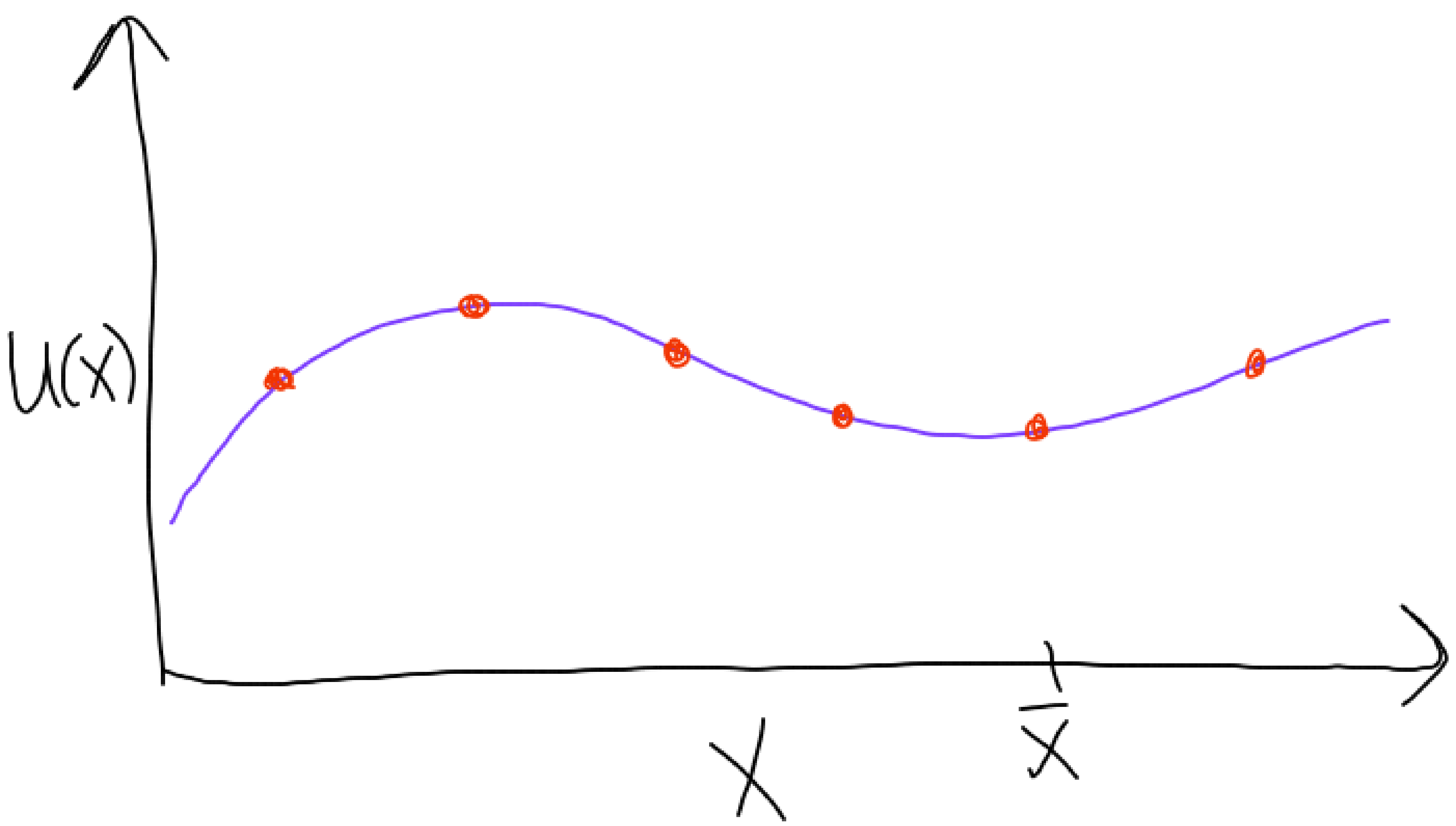
$$u'(x) = \lim_{h \to 0} \frac{u(x+h) - u(x)}{h}$$

This suggests

$$u'(\bar{x}) \approx D_+ u(\bar{x}) = \frac{u(\bar{x}+h) - u(\bar{x})}{h}$$

$$D_- u(\bar{x}) = \frac{\bar{u}(\bar{x}) - u(\bar{x}-h)}{h}$$

or $$D_0 u(\bar{x}) = \frac{u(\bar{x}+h) - u(\bar{x}-h)}{2h}$$



Taylor Series

$$u(x) = \sum_{j=0}^{\infty} \frac{u^{(j)}(\bar{x})}{j!}(x-\bar{x})^j$$

$$u(x) = \sum_{j=0}^{p} \frac{u^{(j)}(\bar{x})}{j!}(x-\bar{x})^j + \mathcal{O}\left((x-\bar{x})^{p+1}\right)$$

Note:

$f(h) = \mathcal{O}(h^p)$ means

there exist $C>0$, $h_0>0$

such that $|f(h)| < Ch^p$ for all $h<h_0$.

$$u(\bar{x}+h) = \sum_{j=0}^{\infty} \frac{h^j}{j!} u^{(j)}(\bar{x})$$

$$= u(\bar{x}) + h u'(\bar{x}) + \frac{h^2}{2} u''(\bar{x}) + \frac{h^3}{6} u'''(\bar{x}) + \mathcal{O}(h^4)$$

$$D_+ u(\bar{x}) = \frac{u(\bar{x}+h) - u(\bar{x})}{h} = u'(\bar{x}) + \frac{h}{2} u''(\bar{x}) + \frac{h^2}{6} u'''(\bar{x}) + \mathcal{O}(h^3)$$

Leading truncation error

$$u(\bar{x}-h) = u(\bar{x}) - h u'(\bar{x}) + \frac{h^2}{2} u''(\bar{x}) - \frac{h^3}{6} u'''(\bar{x}) + \mathcal{O}(h^4)$$

$$D_- u(\bar{x}) = \frac{u(\bar{x}) - u(\bar{x}-h)}{h} = u'(\bar{x}) - \frac{h}{2} u''(\bar{x}) + \frac{h^2}{6} u'''(\bar{x}) + \mathcal{O}(h^3)$$

These methods are 1st-order accurate

$$D_0 u(\bar{x}) = \frac{u(\bar{x}+h) - u(\bar{x}-h)}{2h}$$

$$= u'(\bar{x}) + \frac{2h^3}{6} \cdot \frac{1}{2h} u'''(\bar{x}) + O(h^4)$$

$$= u'(\bar{x}) + \frac{h^2}{6} u'''(\bar{x}) + O(h^4)$$

LTE

2nd-order accurate

Suppose we want to approximate $u''(\bar{x})$ using values $u(\bar{x}), u(\bar{x}+h), u(\bar{x}+2h)$

We want

$$u''(\bar{x}) \approx a u(\bar{x}) + b u(\bar{x}+h) + c u(\bar{x}+2h)$$

$$u(\bar{x}+2h) = u + 2hu' + 2h^2 u'' + \frac{4}{3} h^3 u''' + O(h^4)$$

$$u''(\bar{x}) \approx au + b\left(u + hu' + \frac{h^2}{2} u'' + \frac{h^3}{6} u'''\right)$$
$$+ c\left(u + 2hu' + 2h^2 u'' + \frac{4}{3} h^3 u'''\right) + O(h^4)$$

$(a+b+c)u = 0$ $\qquad$ $(b+2c)hu' = 0$

$a+b+c=0$ $\qquad$ $b+2c=0$

$$\left(\frac{b}{2} + 2c\right) h^2 u'' = u'' \iff \frac{b}{2} + 2c = \frac{1}{h^2}$$

We could also try to satisfy

$$\left(\frac{b}{6} + \frac{4c}{3}\right)h^3 u''' = 0$$

$b = -2c$

$$u''(\bar{x}) = \frac{u(\bar{x}) - 2u(\bar{x}+h) + u(\bar{x}+2h)}{h^2} + O(h)$$

$-c + 2c = \frac{1}{h^2}$

$c = \frac{1}{h^2}$

$b = \frac{-2}{h^2}$

$a = -b - c = \frac{1}{h^2}$

A general approach to FD formulas

Given $u(x_1), u(x_2), \dots, u(x_n)$ find a FD approximation for $u^{(k)}(\bar{x})$ (as accurate as possible)

$$u^{(k)}(\bar{x}) \approx \sum_{i=1}^{n} c_i u(x_i)$$

$$u^{(k)}(\bar{x}) \approx \sum_{i=1}^{n} c_i \sum_{j=0}^{\infty} (x_i - \bar{x})^j \frac{u^{(j)}(\bar{x})}{j!}$$

$$u(\bar{x}): \sum_{i=1}^{n} c_i u(\bar{x}) = 0 \Rightarrow \sum_{i=1}^{n} c_i = 0$$

$$u'(\bar{x}): \sum_{i=1}^{n} c_i (x_i - \bar{x}) u'(\bar{x}) = 0 \Rightarrow \sum_{i} c_i (x_i - \bar{x}) = 0$$

$$\begin{bmatrix} 1 & 1 & \cdots & 1 \\ x_1 - \bar{x} & x_2 - \bar{x} & \cdots & x_n - \bar{x} \\ \frac{(x_1-\bar{x})^2}{2!} & \cdots & \cdots & \frac{(x_n-\bar{x})^2}{2!} \\ \vdots & & & \vdots \\ \frac{(x_1-\bar{x})^{n-1}}{(n-1)!} & \cdots & \cdots & \frac{(x_n-\bar{x})^{n-1}}{(n-1)!} \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ \vdots \\ c_n \end{bmatrix} = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \leftarrow k+1 \text{ entry}$$

Vandermonde Matrix

As long as the x_i are distinct, this system has a unique solution.

Obviously we should have $n > k$.

How large is the error?

$$O(h^{n-k})$$