Modified Equation Analysis

Idea: Given a PDE and a discretization, find a different PDE such that the numerical solution of the original PDE exactly satisfies the modified PDE.

Example: $u_t + au_x = 0$

Forward time, centered space:

$$\frac{U_j^{n+1} - U_j^n}{k} + \frac{a}{2h}\left(U_{j+1}^n - U_{j-1}^n\right) = 0 \quad (1)$$

Suppose there exists $v(x,t)$ such that if we replace U_j^n by $v(x_j, t_n)$ then (1) is satisfied.

Theorem: Let $Av = \lambda v$.

If $A = A^*$, $\lambda \in \mathbb{R}$.

If $A = -A^*$, $\mathrm{Re}(\lambda) = 0$.

$$Av = \lambda v$$

$$v^* A v = \lambda v^* v$$

$$\lambda = \frac{v^* A v}{v^* v}$$

$$\lambda^* = \frac{v^* A^* v}{v^* v} = \begin{cases} \dfrac{v^* A v}{v^* v} = \lambda & \text{if } A^* = A \\ -\dfrac{v^* A v}{v^* v} = -\lambda & \text{if } A^* = -A \end{cases}$$

$$\frac{v(x_j, t_n + k) - v(x_j, t_n)}{k} + \frac{a}{2h}\left(v(x_j + h, t_n) - v(x_j - h, t_n)\right) = 0$$

$$v(x_j, t_n \pm k) = v(x_j, t_n) \pm k v_t(x_j, t_n) + \frac{k^2}{2} v_{tt}(x_j, t_n) + \mathcal{O}(k^3)$$

$$v(x_j \pm h, t_n) = v \pm h v_x + \frac{h^2}{2} v_{xx} \pm \frac{h^3}{6} v_{xxx} + \mathcal{O}(k^4)$$

$$v_t + \frac{k}{2} v_{tt} + \mathcal{O}(k^2) + a v_x + \mathcal{O}(h^2) = 0$$

$$v_t + a v_x = -\frac{k}{2} v_{tt} + \mathcal{O}(k^2, h^2)$$

$$v_t + a v_x = \underbrace{-\frac{k a^2}{2} v_{xx}}_{\text{anti-diffusive}} + \mathcal{O}(k^2, h^2)$$

This problem is ill-posed.

$$V_t = -aV_x + \mathcal{O}(k, h^2) \qquad V_{tx} = V_{xt}$$

$$V_{tx} = -aV_{xx} + \mathcal{O}(k, h^2)$$

$$V_{tt} = -aV_{xt} + \mathcal{O}(k, h^2)$$

$$V_{tt} = a^2 V_{xx} + \mathcal{O}(k, h^2)$$

Lax-Friedrichs

$$U_j^{n+1} = \frac{1}{2}\left(U_{j+1}^n + U_{j-1}^n\right) - \frac{ka}{2h}\left(U_{j+1}^n - U_{j-1}^n\right)$$

$$U_j^n \to v(x_j, t_n)$$

$$v(x_j, t_n + k) = \frac{1}{2}\left(v(x_j + h, t_n) + v(x_j - h, t_n)\right) - \frac{ka}{2h}\left(v(x_j + h, t_n) - v(x_j - h, t_n)\right)$$

$$v + kv_t + \frac{k^2}{2} v_{tt} + \mathcal{O}(k^3) = \frac{1}{2}\left(2v + h^2 v_{xx} + \mathcal{O}(h^3)\right) - \frac{ka}{2h}\left(2hv_x + \frac{h^3}{3} v_{xxx} + \mathcal{O}(h^5)\right)$$

$$\cancel{v} + \cancel{k}v_t + \frac{k^{\cancel{2}}}{2} v_{tt} = \cancel{v} + \frac{h^2}{2k} v_{xx} - \cancel{k}av_x + \mathcal{O}(k^3, h^3, kh^2)$$

$$v_t + av_x = \frac{h^2}{2k} v_{xx} - \frac{k}{2} v_{tt} + \mathcal{O}\left(k^2, \frac{h^3}{k}, h^2\right)$$

$$v_t + av_x = v_{xx}\left(\frac{h^2}{2k} - a^2\frac{k}{2}\right) + \mathcal{O}\left(k^2, \frac{h^3}{k}, h^2\right)$$

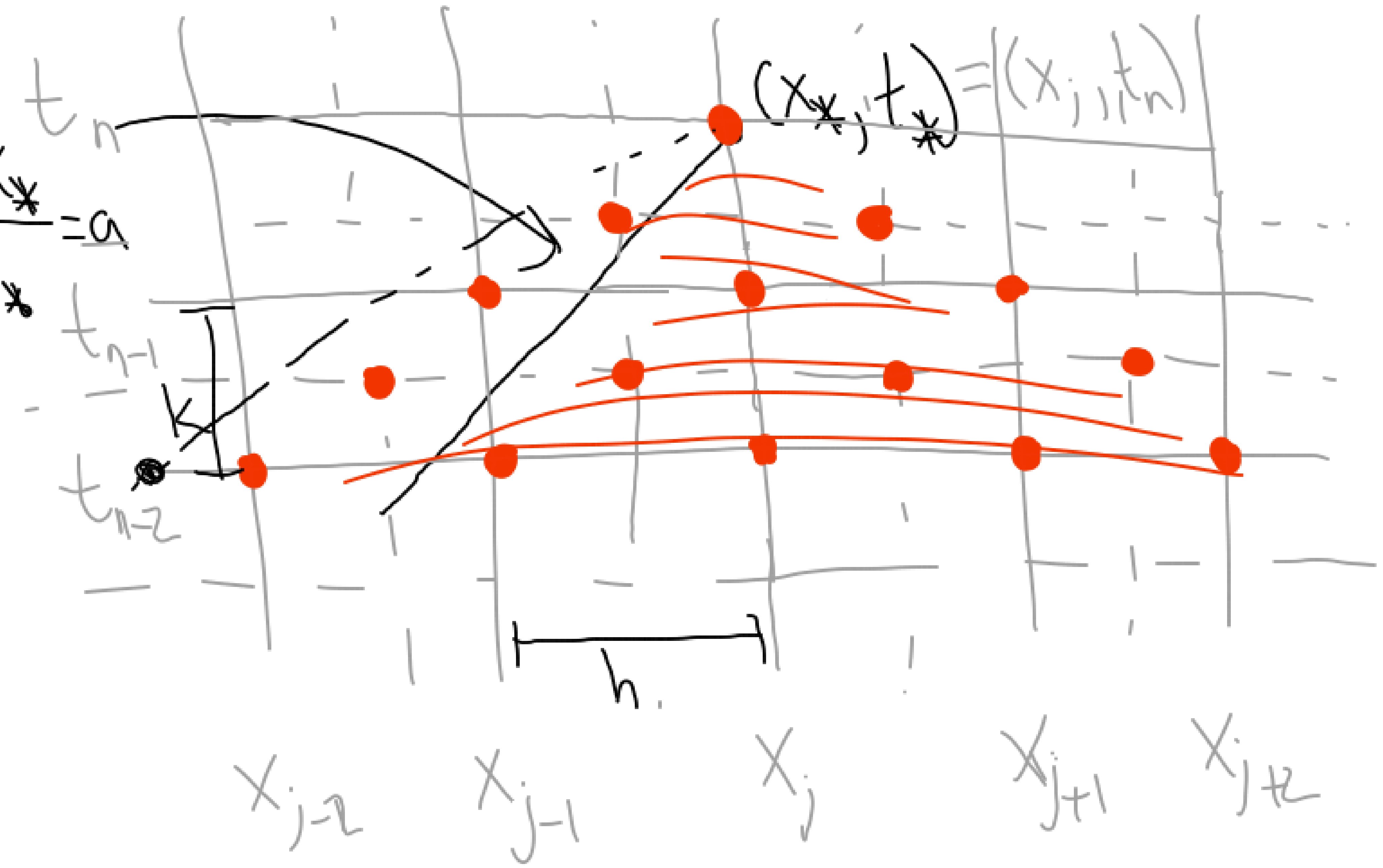
We need $\frac{h^2}{2k} \geq \frac{ka^2}{2} \iff \frac{k^2a^2}{h^2} \leq 1 \iff \underbrace{\frac{ka}{h}}_{\text{CFL number } \nu} \leq 1$

$$ka \leq h$$

CFL Condition

(Courant, Friedrichs, Lewy, 1927)

$$\frac{x-x_*}{t-t_*}=a$$



A consistent numerical method cannot be convergent unless the numerical domain of dependence contains the true domain of dependence as $k, h \to 0$.

Domain of dependence

Given (x_*, t_*), the DoD is the set of pts. in the (x,t) plane whose value influences $u(x_*, t_*)$.

Centered in time (Leapfrog)
Centered in space

$$\frac{U_j^{n+1}-U_j^{n-1}}{2k}+a\,\frac{U_{j+1}^{n}-U_{j-1}^{n}}{2h}=0$$

$$\frac{v(x_j,t_n+k)-v(x_j,t_n-k)}{2k}+a\,\frac{v(x_j+h,t_n)-v(x_j-h,t_n)}{2h}=0$$

$$v_t+\frac{k^2}{6}v_{ttt}+\mathcal{O}(k^4)+a\left(v_x+\frac{h^2}{6}v_{xxx}+\mathcal{O}(h^4)\right)=0$$

$$v_t+av_x=-\frac{k^2}{6}v_{ttt}-\frac{ah^2}{6}v_{xxx}+\mathcal{O}(h^4,k^4)$$

$$v_t+av_x=av_{xxx}\left(\frac{k^2a^2}{6}-\frac{h^2}{6}\right)+\mathcal{O}(h^4,k^4,k^2h^2)$$

≤ 0 if $\frac{k^2a^2}{6}\leq\frac{h^2}{6}\Leftrightarrow\frac{ka}{h}\leq 1$

$$v_t+av_x=\alpha v_{xxx}+\mathcal{O}(\quad)$$

$$v_{tt}=a^2v_{xx}+\mathcal{O}(k^2,h^2)$$

$$v_{ttt}=a^2v_{xxt}+\mathcal{O}(\quad)$$

$$v_t=-av_x+\mathcal{O}(\quad)$$

$$v_{txx}=-av_{xxx}+\mathcal{O}(\quad)$$

$$v_{ttt}=-a^3v_{xxx}+\mathcal{O}(k^2,h^2)$$

Ansatz: $v(x,t)=e^{i(\xi x-\omega t)}=e^{i\xi(x-\frac{\omega}{\xi}t)}$

Phase velocity c

$$v_t=-i\omega v$$

$$v_x=i\xi v$$

$$v_{xxx}=-i\xi^3 v$$

Dispersion relation

$$-i\omega v+i\xi a v=-i\xi^3\alpha v$$

$$\omega=\xi a+\xi^3\alpha\qquad\frac{\omega}{\xi}=a+\xi^2\alpha$$

$$V(x,t) = e^{i\xi(x-(a+\alpha\xi^2)t)}$$

For small ξ, $c \approx a$

For large ξ, c and a differ greatly.