Spectral Methods

Example: Advection-diffusion

$$u_t + a u_x = \varepsilon u_{xx}$$

$$u(x, t=0) = u_0(x)$$

Boundary conditions: Periodic

Ansatz: $u(x,t) = \hat{u}(t) e^{i\xi x}$

$$\hat{u}'(t) e^{i\xi x} + a i \xi \hat{u} e^{i\xi x} = -\varepsilon \xi^2 e^{i\xi x} \hat{u}$$

$$\hat{u}'(t) = -(a i \xi + \varepsilon \xi^2)\hat{u}$$

Solution:

$$u(t) = \exp\left(-t(a i \xi + \varepsilon \xi^2)\right)\hat{u} \qquad (1)$$

Superposition:
Any linear combination of solutions of a linear PDE is also a solution.

IFT: $u_0(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \hat{u}_0(\xi) e^{i\xi x} \, d\xi$

FT: $\hat{u}(t,\xi) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} u(x,t) e^{i\xi x} \, dx$

Exact solution of any linear PDE:

$$
\begin{tikzcd}
u_0(x) \arrow[d, "FT"'] & u(x,t) \\
\hat{u}_0(\xi) \arrow[r, "\text{Exact time evolution (1)}"'] & \hat{u}(x,t) \arrow[u, "IFT"']
\end{tikzcd}
$$

Discrete Fourier Analysis

(Spectral Methods)

$u(x,t) \rightarrow U(t)$

F, F^{-1}: DFT, IDFT

$$\xi = \begin{bmatrix} \xi_1 \\ \xi_2 \\ \vdots \\ \xi_m \end{bmatrix}$$

Wavenumber vector

$$D = \begin{bmatrix} i\xi_1 & & & \\ & i\xi_2 & & \\ & & \ddots & \\ & & & i\xi_m \end{bmatrix}$$

Semi-discrete spectral method

$$U'(t) = F^{-1}(-aD - \varepsilon D^2) F U(0)$$

Solution

$$U(t) = \exp\left(tF^{-1}(-aD - \varepsilon D^2)F\right)U(0)$$

$$\exp(tF^{-1}\Lambda F) = I + tF^{-1}\Lambda F + \frac{t^2}{2}F^{-1}\Lambda F F^{-1}\Lambda F + \cdots$$

$$= I + tF^{-1}\Lambda F + \frac{t^2}{2}F^{-1}\Lambda^2 F + \cdots$$

$$= F^{-1}\exp(t\Lambda)F$$

So

$$U(t) = \underbrace{F^{-1}\exp(-t(aD + \varepsilon D^2))}_{m \times m \text{ dense matrix-vector product}}\boxed{FU(0)}$$

$FU(0)$ can be computed with $\mathcal{O}(m \log m)$ flops via FFT

Pseudospectral Methods

$$u_t + \underbrace{a(x)u_x}_{\text{FT gives convolution (expensive)}} = 0$$

Idea: Compute spatial derivative(s) in wavenumber (ξ) space but compute products in physical (x) space.

$$U_x \approx F^{-1}DFU$$

Semi-discrete scheme:

$$U'(t) = AF^{-1}DFU$$

$$A = \begin{bmatrix} a(x_1) & & & \\ & a(x_2) & & \\ & & \ddots & \\ & & & a(x_m) \end{bmatrix}$$

Now we need to discretize in time (e.g. via RK).

Or leapfrog:

$$U^{n+1} = U^{n-1} + 2k f(U^n)$$

Nonlinear PDEs

$$u_t + u u_x + u_{xxx} = 0$$

Korteweg-de Vries eqn.

Surface water waves

What is the effect of the term

u_{xxx}?

$$u_t + u_{xxx} = 0$$

Ansatz:

$$u(x,t) = e^{i(\xi x - \omega t)}$$

$$-i\omega \cancel{u} - i\xi^3 \cancel{u} = 0$$

$$\omega = -\xi^3$$

So

$$u(x,t) = e^{i\xi(x+\xi^2 t)}$$

Solution moves at speed $-\xi^2$.

$$\int u\,dx \qquad \int u^2\,dx$$