



$m^2$  equations

Need to choose  
an ordering

For example, row-wise  
Ordering:

$$U = \begin{bmatrix} U_{11} \\ U_{21} \\ U_{31} \\ U_{12} \\ U_{22} \\ \vdots \\ U_{mm} \end{bmatrix} \quad F = \begin{bmatrix} f(x_1, y_1) \\ f(x_2, y_1) \\ \vdots \\ f(x_m, y_m) \end{bmatrix}$$

$$AU = F$$

$$A = \frac{1}{h^2}$$

$m^2 \times m^2$

Sparse  
 $\sim 5m^2$  non-zero  
entries

$m \times m$

# Consistency

Substitute:  $U_{ij} \rightarrow u(x_i, y_j)$

$$\frac{1}{h^2} [-4u(x_i, y_j) + u(x_i + h, y_j) + u(x_i - h, y_j) + u(x_i, y_j + h) + u(x_i, y_j - h)] = f(x_i, y_j) + \tau_{ij}$$

Expand about  $(x_i, y_j)$ :

$$\cancel{u_{xx}} + \frac{1}{12} h^2 u_{xxxx} + O(h^4) + \cancel{u_{yy}} + \frac{1}{12} h^2 u_{yyyy} + O(h^4) = \cancel{f(x_i, y_j)} + \tau_{ij}$$

$$\tau_{ij} = \frac{1}{12} h^2 (u_{xxxx} + u_{yyyy}) + O(h^4)$$

2nd-order accurate  
(locally)

# Global error

Exact solution

Vector:  $\hat{U}_{ij} = u(x_i, y_j)$

Error:  $U - \hat{U}$

$$AU = F$$

$$A\hat{U} = F + \tau$$

$$AE = -\tau$$

$$E = -A^{-1}\tau$$

$$\|E\| = \|A^{-1}\tau\| \leq \|A^{-1}\| \cdot \|\tau\|$$

So we need to show that

$$\|A^{-1}\| < C \text{ as } h \rightarrow 0. \text{ (stability)}$$

(Since we know  $\|\tau\| = O(h^2)$ )

i.e. We need to show that the e.v.'s of  $A$  are bounded away from zero as  $h \rightarrow 0$ .

$$Av = \lambda v: \frac{V_{i+1,j} - 2V_{i,j} + V_{i-1,j}}{h^2} + \frac{V_{i,j+1} - 2V_{i,j} + V_{i,j-1}}{h^2} = \lambda V_{i,j} \quad (1 \leq i, j \leq m)$$

Note:

$V_{0,j}, V_{m+1,j}, V_{i,0}, V_{i,m+1}$  defined by

B.C.s.

Assume  $V_j = R_i S_j$  (separability)

$$S_j \cdot \frac{R_{i+1} - 2R_i + R_{i-1}}{h^2} + R_i \frac{S_{j+1} - 2S_j + S_{j-1}}{h^2} = \lambda R_i S_j$$

Divide  $R_i S_j$ :  $C_1$   $C_2$

$$\frac{R_{i+1} - 2R_i + R_{i-1}}{h^2 R_i} + \frac{S_{j+1} - 2S_j + S_{j-1}}{h^2 S_j} = \lambda$$

Each must be equal to a constant

$$R_{i+1} - 2R_i + R_{i-1} = h^2 R_i C_1$$

$$R_{i+1} - (2 + h^2 C_1) R_i + R_{i-1} = 0 \quad 1 \leq i \leq m$$

$R_0, R_{m+1}$  given by B.C.s

$R_i = \sum^i$  for some  $\sum^i \in \mathbb{C}$   
(problem we solved before)

$$C_1 = \frac{2}{h^2} (\cos(p\pi h) - 1) \quad p = 1, 2, \dots, m$$

$$C_2 = \frac{2}{h^2} (\cos(q\pi h) - 1) \quad q = 1, 2, \dots, m$$

$$\lambda = C_1 + C_2 = \frac{2}{h^2} (\cos(p\pi h) + \cos(q\pi h) - 2)$$

Smallest:  $p=q=1$

$$\lambda_1 = -2\pi^2 + \mathcal{O}(h^2)$$

$$\Rightarrow \|A^{-1}\| < \frac{1}{2\pi^2} + \varepsilon$$

for small  $h$ .

$$\Rightarrow \|E\| = \mathcal{O}(h^2).$$