Reminder: Starting Thursday, Ramadan schedule (10:00 - 10:55)

Spectrum of $\frac{d^2}{dx^2}$ on $[0,1]$ with homog. Dirichlet B.C.s?

Neumann Boundary Conditions

$u''(x) = f(x) \quad 0 < x < 1$

$\cancel{u(0) = \alpha} \quad u'(0) = \sigma_0$

$u(1) = \beta$

e.g. if $\sigma_0 = 0$: insulated

How to discretize this?

① One-sided FD

$$u'(0) \approx \underbrace{\frac{U_1 - U_0}{h}}_{D_+u(x_0)} = \sigma_0$$

Only 1st order

$$\frac{1}{h^2}\begin{bmatrix} -h & h & & & \\ 1 & -2 & 1 & & \\ & \ddots & \ddots & \ddots & \\ & & \ddots & \ddots & \ddots \\ & & 1 & -2 & 1 \\ & & & & h^2 \end{bmatrix}\begin{bmatrix} U_0 \\ \vdots \\ \vdots \\ U_{m+1} \end{bmatrix} = \begin{bmatrix} \sigma_0 \\ f(x_1) \\ \vdots \\ f(x_m) \\ \beta \end{bmatrix}$$

To get 2nd-order accuracy: also use U_2:

$$u'(x_0) \approx \frac{1}{h}\left(-\frac{3}{2}U_0 + 2U_1 - \frac{1}{2}U_2\right) = \sigma_0$$

② Defect correction

$$D_+ u(x_0) = u'(x_0) + \frac{h}{2} u''(x_0) + O(h^2)$$

We know $u''(x_0) = f(x_0)$

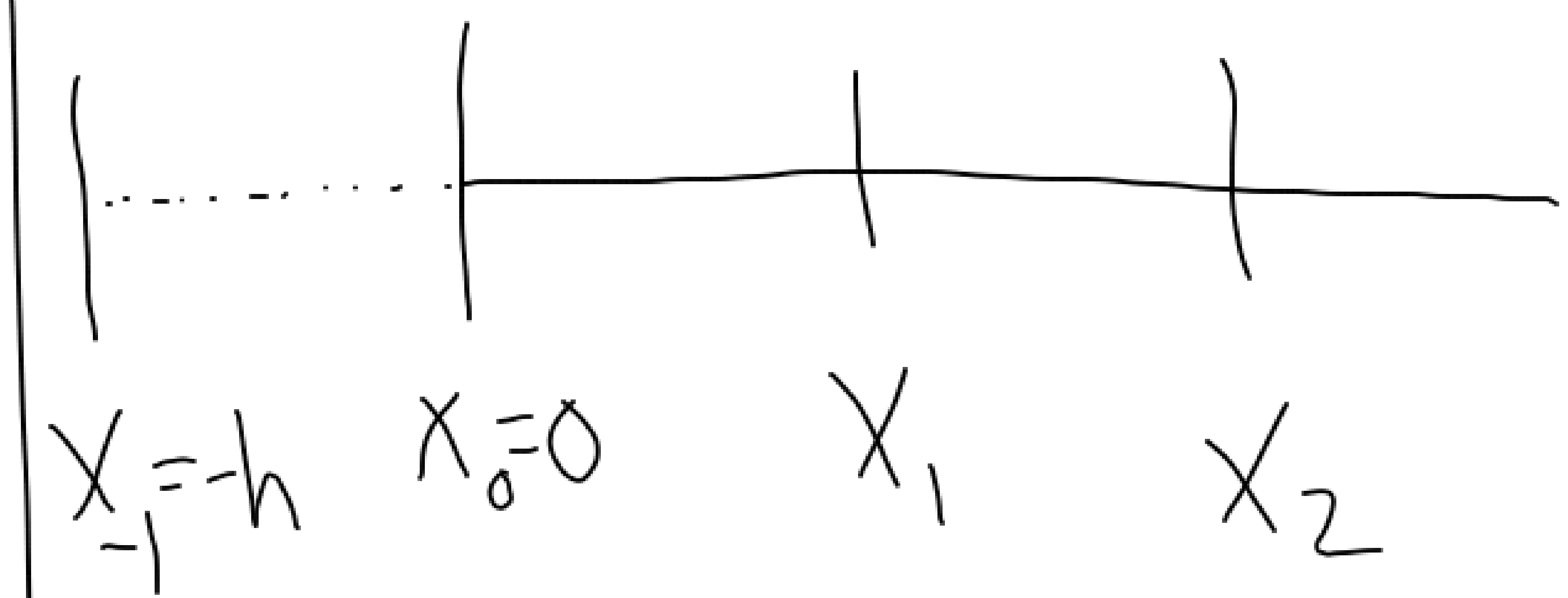
So $$\frac{u(x_1) - u(x_0)}{h} = u'(x_0) + \frac{h}{2} f(x_0) + O(h^2)$$

so we set

$$\frac{U_1 - U_0}{h} = \sigma_0 + \frac{h}{2} f(x_0)$$

2nd order accurate

③ Ghost point method



(*) $$u'(x_0) \approx \frac{U_1 - U_{-1}}{2h} = \sigma_0$$

We will approximately impose the ODE at x_0: $$\frac{U_1 - 2U_0 + U_{-1}}{h^2} = f(x_0)$$

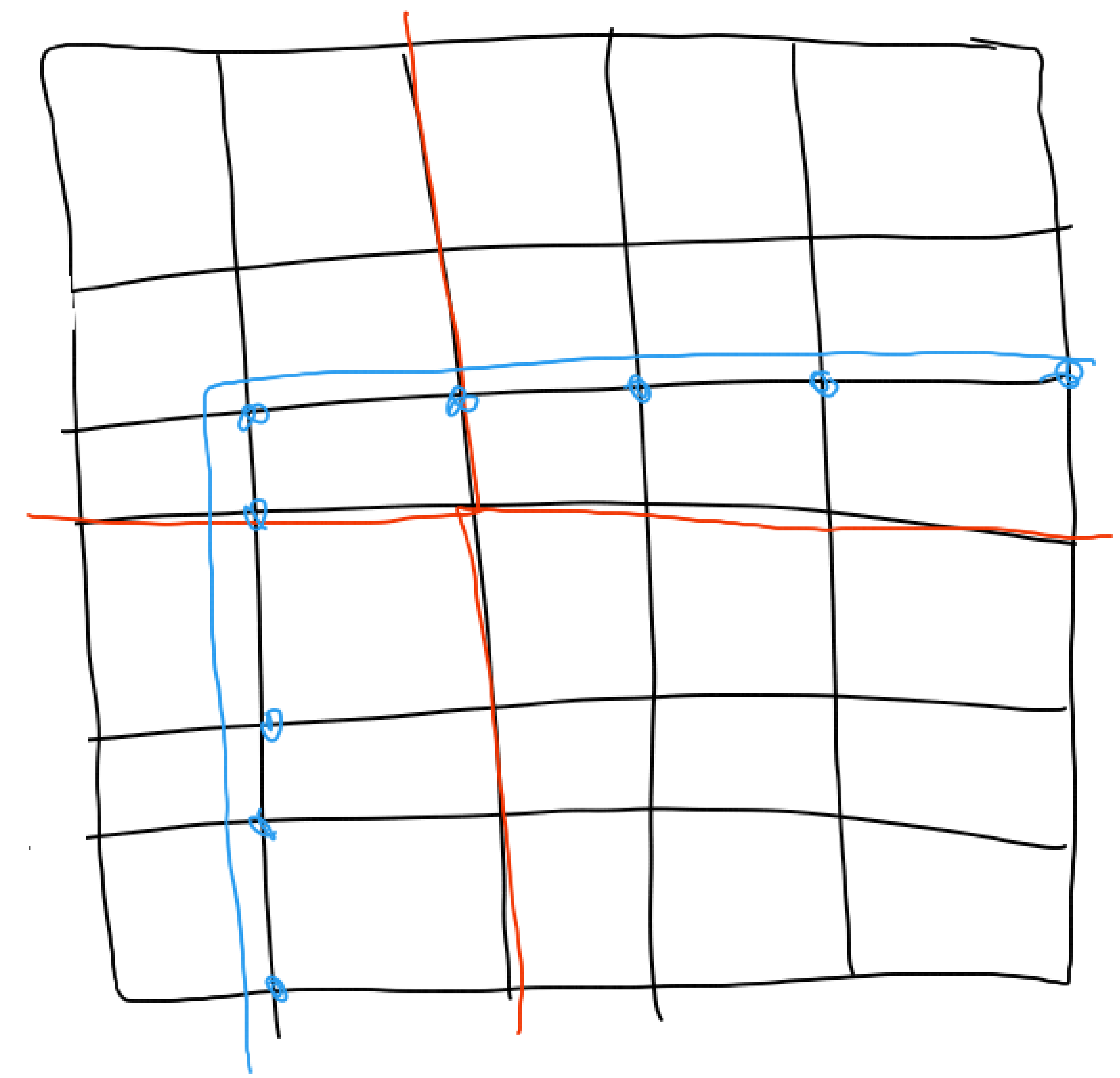
$$U_{-1} = h^2 f(x_0) + 2U_0 - U_1$$

substitute in (*):

$$\rightarrow \frac{U_1 - 2U_0 - h^2 f(x_0) + U_1}{2h} = \sigma_0$$

$$\frac{U_1 - U_0}{h} - \frac{h}{2} f(x_0) = \sigma_0$$

Ghost points are also used for domain decomposition methods



What if we have Neumann conditions at both ends?

e.g.

$$u'(0) = u'(1) = 0$$
$$u''(x) = f(x)$$

If $u(x)$ is a solution, then so is $u(x) + C$.

If $f(x) = 0$: $u(x) = C$

$$\int_0^1 u''(x)\,dx = \int_0^1 f(x)\,dx$$

$$u'(1) - u'(0) = \int_0^1 f(x)\,dx$$

In general with $u'(0) = \sigma_0$, $u'(1) = \sigma_1$

We require $\sigma_1 - \sigma_0 = \int_0^1 f(x)\,dx$

Otherwise, no solution.

Discretization:

$$\underbrace{\frac{1}{h^2}\begin{bmatrix} -h & h & & & \\ 1 & -2 & 1 & & \\ & \ddots & \ddots & \ddots & \\ & & 1 & -2 & 1 \\ & & & -h & h \end{bmatrix}}_{} \begin{bmatrix} U_0 \\ \vdots \\ \vdots \\ U_{m+1} \end{bmatrix} = \begin{bmatrix} \sigma_0 \\ f(x_1) \\ \vdots \\ f(x_m) \\ \sigma_1 \end{bmatrix}$$

Does this have a unique solution?

Find x so that $Ax=0$ $\quad x = \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix}$

So A is singular, and this problem has zero or ∞-many solutions.