Feb. 12: HW 1 due

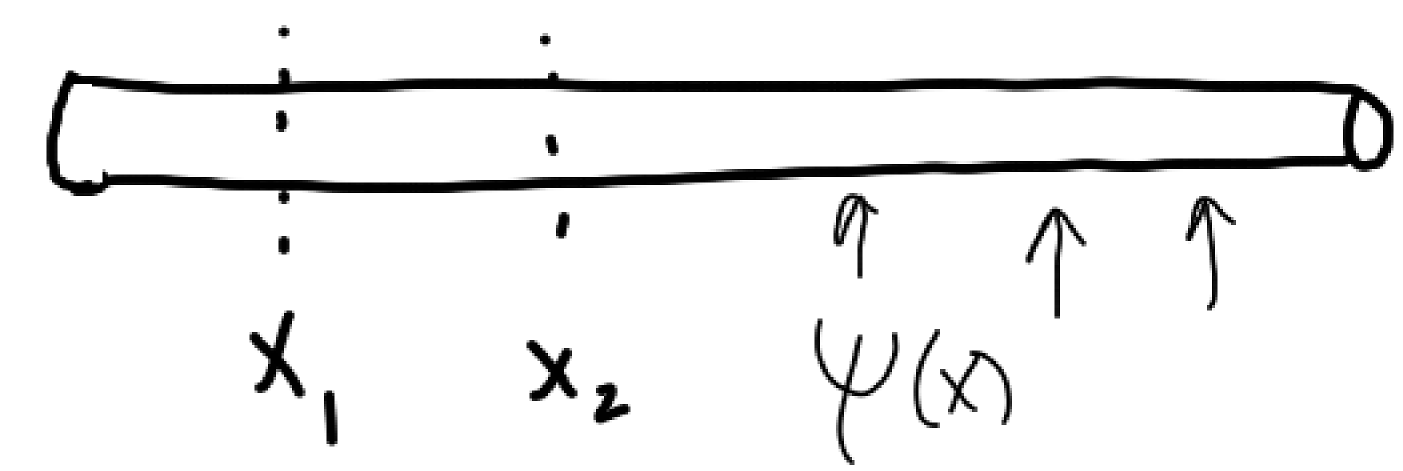
No class

Feb. 18 - March 18:

Ramadan schedule

10:00 - 11:00

Boundary Value Problems



$U(x,t)$: heat

$\psi(x)$: heat source or sink

Heat in interval $[x_1, x_2]$:

$$\frac{d}{dt}\int_{x_1}^{x_2} U(x,t)\,dx = \int_{x_1}^{x_2} \psi(x)\,dx + F(x_1,t) - F(x_2,t)$$

$$\int_{x_1}^{x_2} \frac{\partial}{\partial t} u(x,t)\,dx = \int_{x_1}^{x_2} \psi(x)\,dx - \int_{x_1}^{x_2} \frac{\partial}{\partial x} F(x,t)\,dx$$

Fick's law of diffusion: $F(u(x,t)) = -\underset{\substack{\uparrow \\ \text{Heat conductivity}}}{K} u_x$

$$\int_{x_1}^{x_2} \left(u_t - \psi - K u_{xx}\right) dx = 0$$

Conservation law

Must hold pointwise:

$$u_t = \psi + K u_{xx}$$

Heat equation

We suppose that a steady state is reached:

as $t \to \infty$, $u_t \to 0$.

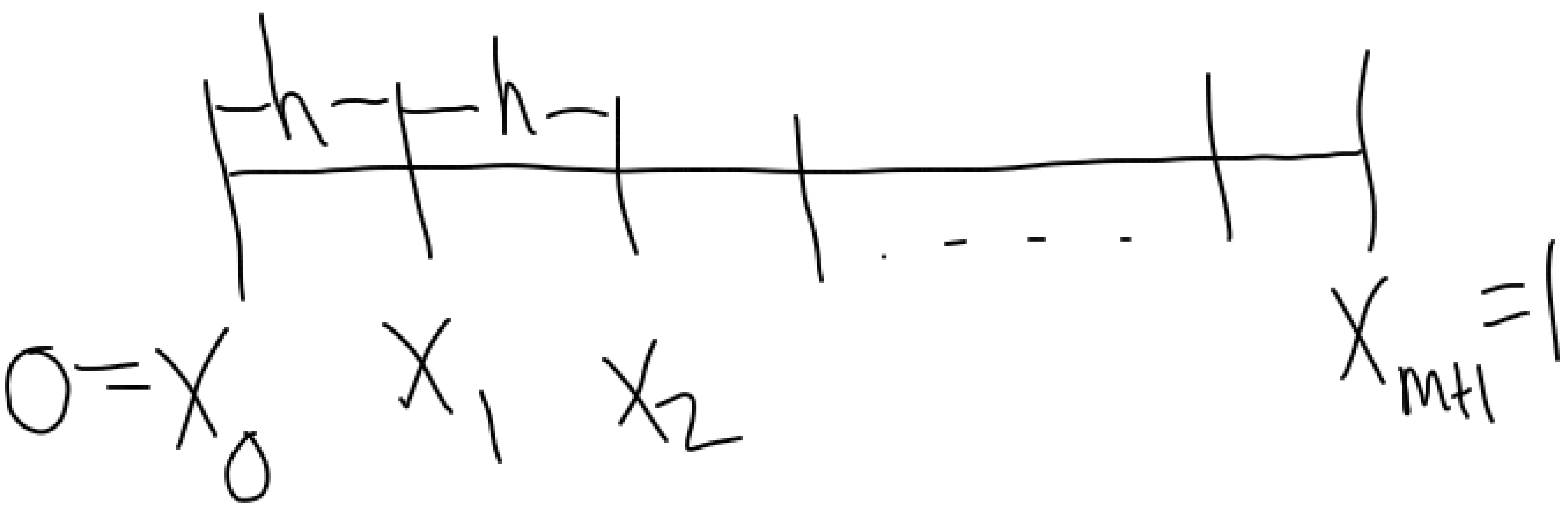
$$\psi(x) + K u_{xx} = 0$$

$$u_{xx} = -\psi/K = f(x)$$

$$\boxed{\begin{aligned} &u''(x) = f(x) \quad 0 < x < 1 \\ &u(0) = \alpha \\ &u(1) = \beta \end{aligned}}$$

Poisson's equation

Discretization



$$x_j = jh \qquad U_j \approx u(x_j)$$

$$h = \frac{1}{m+1}$$

$$U_0 = \alpha \qquad U_{m+1} = \beta$$

$$u''(x_j) \approx \frac{U_{j+1} - 2U_j + U_{j-1}}{h^2} = f(x_j)$$

$$j = 1, 2, \dots, m$$

$$\underbrace{\frac{1}{h^2}\begin{bmatrix} -2 & 1 & & & & \\ 1 & -2 & 1 & & 0 & \\ & 1 & -2 & \ddots & & \\ & & \ddots & \ddots & \ddots & \\ & 0 & & 1 & -2 & 1 \\ & & & & 1 & -2 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} U_1 \\ \vdots \\ \vdots \\ U_m \end{bmatrix}}_{U} = \underbrace{\begin{bmatrix} f(x_1) - \frac{\alpha}{h^2} \\ f(x_2) \\ \vdots \\ f(x_{m-1}) \\ f(x_m) - \frac{\beta}{h^2} \end{bmatrix}}_{F}$$

$j=1$: $$\frac{U_2 - 2U_1 + \alpha}{h^2} = f(x_1)$$

$$\frac{U_2 - 2U_1}{h^2} = f(x_1) - \frac{\alpha}{h^2}$$

$$\frac{U_{m+1} - 2U_m + U_{m-1}}{h^2} = f(x_m)$$

$AU = F$

A is: tridiagonal

$m \times m$

How accurate is our solution?

Define:

$$\hat{U} = \begin{bmatrix} u(x_1) \\ u(x_2) \\ \vdots \\ \vdots \\ u(x_m) \end{bmatrix}$$

Global error:

$$E = U - \hat{U}$$

We want

$\|E\| \to 0$

as $h \to 0$.

(i.e. $m \to \infty$)

(Convergence)

Local truncation error

Substitute: $U_j \to u(x_j)$

$$\frac{u(x_j+h) - 2u(x_j) + u(x_j-h)}{h^2} = f(x_j) + \tau_j$$

We find

$$\frac{u(x_j+h) - 2u(x_j) + u(x_j-h)}{h^2} = u''(x_j) + \frac{1}{12}h^2 u^{(4)}(x_j) + O(h^4)$$

so

$$\cancel{u''(x_j)} + \frac{1}{12}h^2 u^{(4)}(x_j) + O(h^4) = \cancel{f(x_j)} + \tau_j$$

$$\tau_j = \frac{1}{12}h^2 u^{(4)}(x_j) + O(h^4)$$

Define $\tau = \begin{bmatrix} \tau_1 \\ \vdots \\ \vdots \\ \tau_m \end{bmatrix}$.

Then we have

$$A\hat{U} = F + \tau$$
$$AU = F$$
$$A(U - \hat{U}) = -\tau$$
$$AE = -\tau$$
$$E = -A^{-1}\tau$$

$$\|E\| = \|A^{-1}\tau\| \le \|A^{-1}\| \cdot \|\tau\|$$

Here we mean the induced matrix norm:

$$\|M\| = \sup_{\|x\|=1} \frac{\|Mx\|}{\|x\|}$$

$$\|E\| \le \|A^{-1}\| \cdot \|\tau\|$$

Consistency

We say a discretization is consistent if $\|\tau\| \to 0$ as $h \to 0$.

Stability

We say a discretization is stable if the global error is related to the local error by a bounded function.

Grid-function norms

Vector norms:

$$\|v\|_1 = \sum_{j=1}^{m} |v_j|$$

$$\|v\|_2 = \left(\sum_{j=1}^{m} v_j^2\right)^{1/2}$$

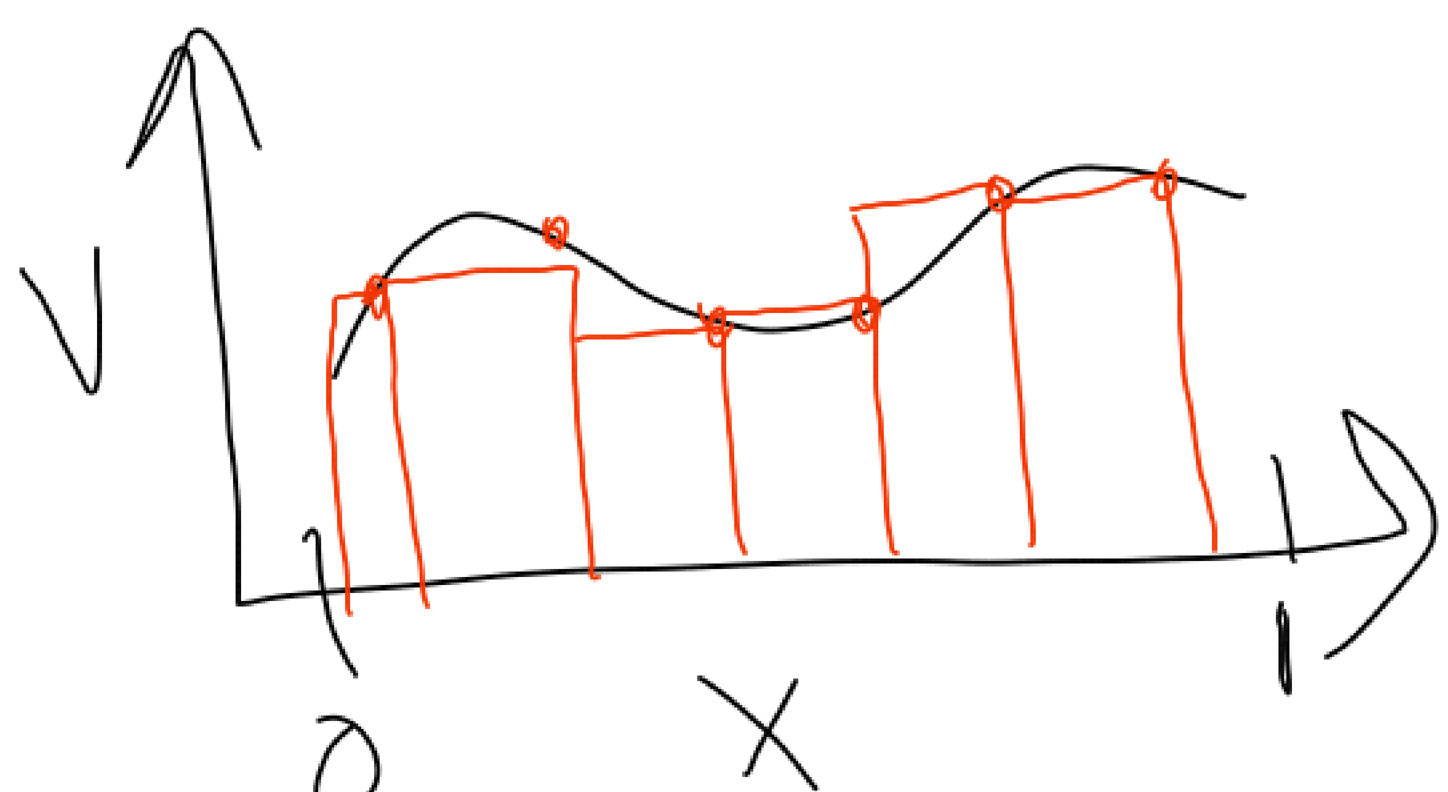
$$\|v\|_\infty = \max_j |v_j|$$

Function norms

$$\|v(x)\|_1 = \int_0^1 |v(x)|\,dx$$

$$\|v(x)\|_2 = \left(\int_0^1 |v(x)|^2\,dx\right)^{1/2}$$

$$\|v(x)\|_\infty = \max_x |v(x)|$$



Grid-function norms

$$\|v\|_1 = h\sum_{j=1}^{m} |v_j|$$

$$\|v\|_2 = \left(h\sum_{j=1}^{m} |v_j|^2\right)^{1/2}$$

$$\|v\|_\infty = \max_j |v_j|$$

$$\tau = \begin{bmatrix} \frac{1}{12}h^2 u^{(4)}(x_1) \\ \frac{1}{12}h^2 u^{(4)}(x_2) \\ \vdots \\ \vdots \end{bmatrix} + O(h^4)$$

$$\|\tau\| = O(h^2)$$

So our method is consistent. ✓

$$\|E\| \leq \|A^{-1}\| \, O(h^2)$$

We will show that $\|A^{-1}\| < C$ as $h \to 0$.

2-norm convergence

We need to show that

$$\|A^{-1}\|_2 < C \text{ as } h \to 0.$$

Recall: $\|M\|_2 = \rho(M) = \max_{1 \leq j \leq m} |\lambda_j|$

What are the eigenvalues of A^{-1}?

$$Av = \lambda v \Rightarrow v = \lambda A^{-1} v \Rightarrow A^{-1} v = \frac{1}{\lambda} v$$

What are the e.v.s of A?

$$\lambda_p = \frac{2}{h^2}(\cos(p\pi h) - 1)$$

$p = 1, 2, \dots, m$

$$\cos(x) = 1 - \frac{1}{2}x^2 + \mathcal{O}(x^4)$$

$$\cos(p\pi h) = 1 - \frac{1}{2}p^2\pi^2h^2 + \mathcal{O}(h^4)$$

$$\lambda_p = -p^2\pi^2 + \mathcal{O}(h^2)$$

Smallest in modulus: $\lambda_1 = -\pi^2$

As $h \to 0$, $\|A^{-1}\| \to \frac{1}{\pi^2}$

So take $C = \frac{1}{\pi^2} + \varepsilon$.

So $\|E\| = \mathcal{O}(h^2)$.