Jacobi's Method and Multigrid

$$u''(x) = f(x) \qquad 0<x<1$$
$$u(0) = \alpha$$
$$u(1) = \beta$$
$$\Rightarrow AU = F$$

Jacobi's Method

Let $G = \begin{bmatrix} 0 & 1 & & \\ 1 & \ddots & \ddots & \\ & \ddots & \ddots & 1 \\ & & 1 & 0 \end{bmatrix}$

and $A = \frac{1}{h^2}\begin{bmatrix} -2 & 1 & & \\ 1 & \ddots & \ddots & \\ & \ddots & \ddots & 1 \\ & & 1 & -2 \end{bmatrix}$

$$A = \frac{1}{h^2}(G - 2I)$$

$$AU = F$$

$$(GU - 2U)\frac{1}{h^2} = F$$

$$U = \frac{1}{2}(GU - h^2 F) \quad (*)$$

Jacobi iteration:

$$U^{[k+1]} = \frac{1}{2}(GU^{[k]} - h^2 F) \quad (**)$$

U is a fixed point of this iteration.

If we start with $U^{[0]} \neq U$, will $\lim_{k \to \infty} U^{[k]} = U$?

Let $e^{[k]} = U^{[k]} - U$

Take $(**) - (*)$

$$U^{[k+1]} - U = \frac{1}{2}(G(U^{[k]} - U))$$

$$e^{[k+1]} = \frac{1}{2} G e^{[k]}$$

Let $\tilde{G} = \frac{1}{2} G$.

$\tilde{G}$ is symmetric.

What are the eigenvalues and eigenvectors of G?

Let $Av = \lambda v$ $\qquad A = \frac{1}{h^2}(G - 2I)$

$\frac{1}{h^2}(G - 2I)v = \lambda v$ $\qquad \lambda_p = \frac{2}{h^2}(\cos(p\pi h) - 1)$

$$\left.\begin{aligned} Gv - 2v &= h^2 \lambda v \\ Gv &= (h^2\lambda + 2)v \end{aligned}\right]$$

A and G have the same eigenvectors

E.V.'s of G: $h^2 \lambda_p + 2 = \cancel{h^2} \frac{2}{\cancel{h^2}}(\cos(p\pi h) - 1) + 2$

$= 2\cos(p\pi h)$

E.V.'s of $\tilde{G}$: $\cos(p\pi h) = \tilde{\gamma}_p$ $\quad p = 1, 2, \ldots, m$

$h = \frac{1}{m+1}$

Note that $|\tilde{\gamma}_p| < 1$.

So

$\|e^{[k]}\|_2 = \|\tilde{G}^k e^{[0]}\|_2 \leq \|\tilde{G}^k\|_2 \cdot \|e^{[0]}\|_2$

$\leq \|\tilde{G}\|_2^k \cdot \|e^{[0]}\|_2$ Because G is symmetric

$= \max_p |\tilde{\gamma}_p|^k \cdot \|e^{[0]}\|$

So $\lim_{k \to \infty} \|e^{[k]}\| = 0$.

How fast will it converge? slowly

$\max_p |\tilde{\gamma}_p| = 1 - \Theta(h^2)$

Under-relaxed Jacobi

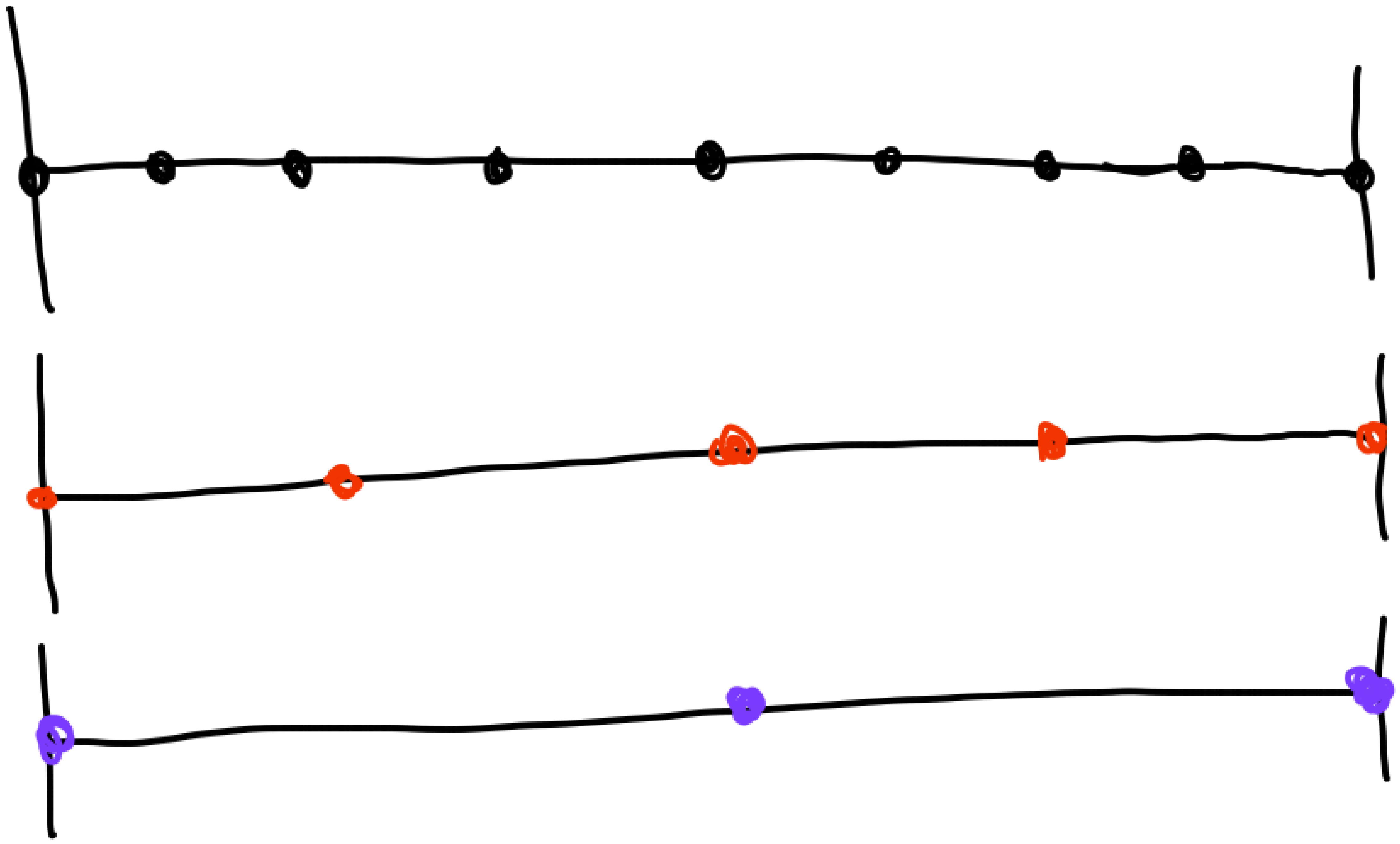
$$\hat{U}^{[k+1]} = \frac{1}{2}\left(GU^{[k]} - h^2 F\right)$$

$$U^{[k+1]} = U^{[k]} + \omega\left(\hat{U}^{[k+1]} - U^{[k]}\right)$$

$$= \underbrace{\left((1-\omega)I + \frac{\omega}{2}G\right)}_{\hat{G}} U^{[k]} - \omega\frac{h^2}{2}F$$

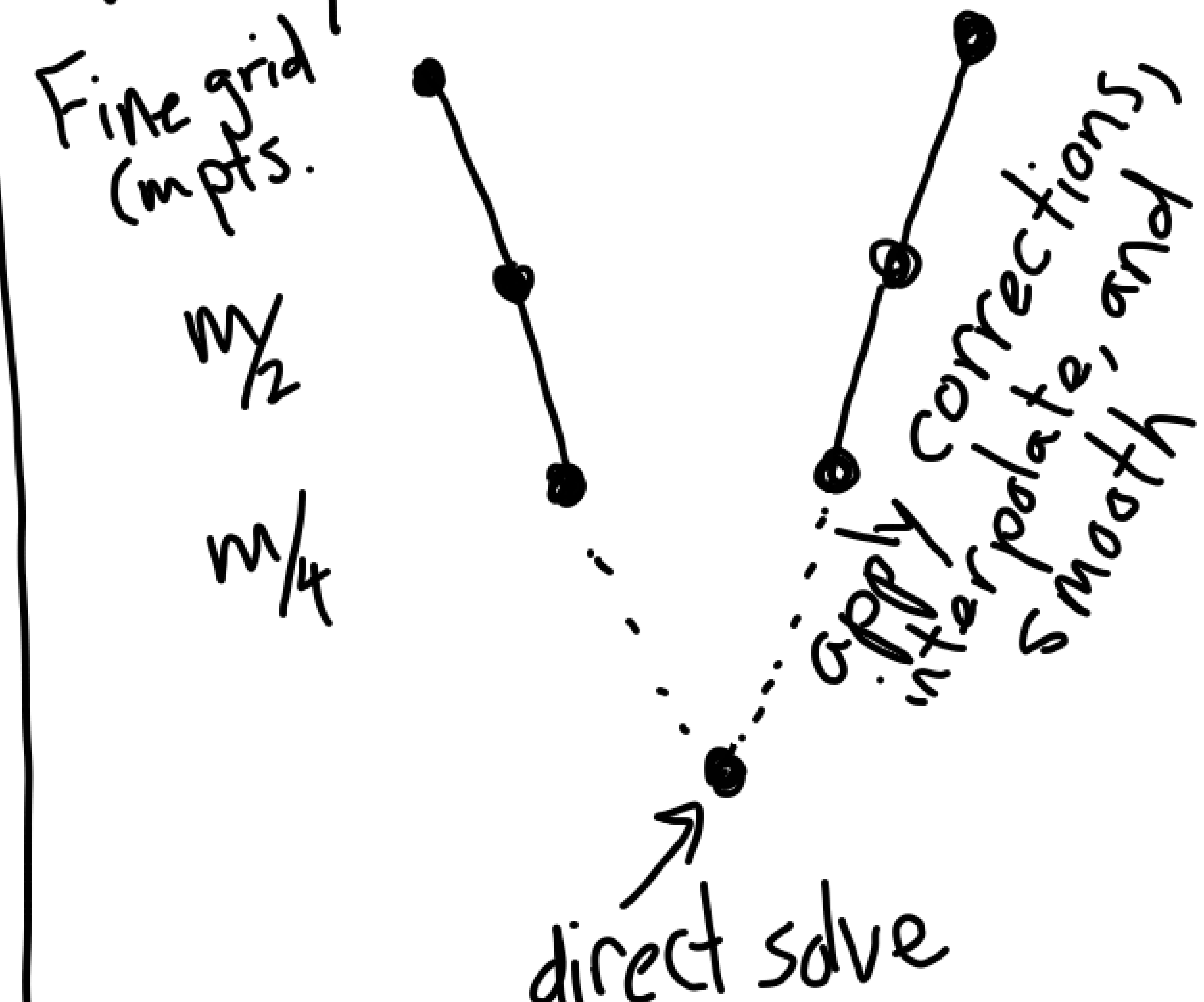
Multigrid

Start on a (fine) grid with m points. Apply under-relaxed Jacobi to remove the high-frequency error. (solve $AU=F$) After ν iterations, restrict the solution to a coarser grid.



In practice we coarsen until we can apply a direct solver.

V-cycle



Define $e_\nu = U_\nu - U$

soln. after ν iterations on fine grid.

$AU_\nu - F = -r_\nu$ residual

$AU - F = 0$

$Ae_\nu = -r_\nu$ solve this on coarse grid then subtract e_ν from U_ν.

Complexity:

$$\mathcal{O}(m \log m)$$

"fast algorithm"