Finite difference Formulas
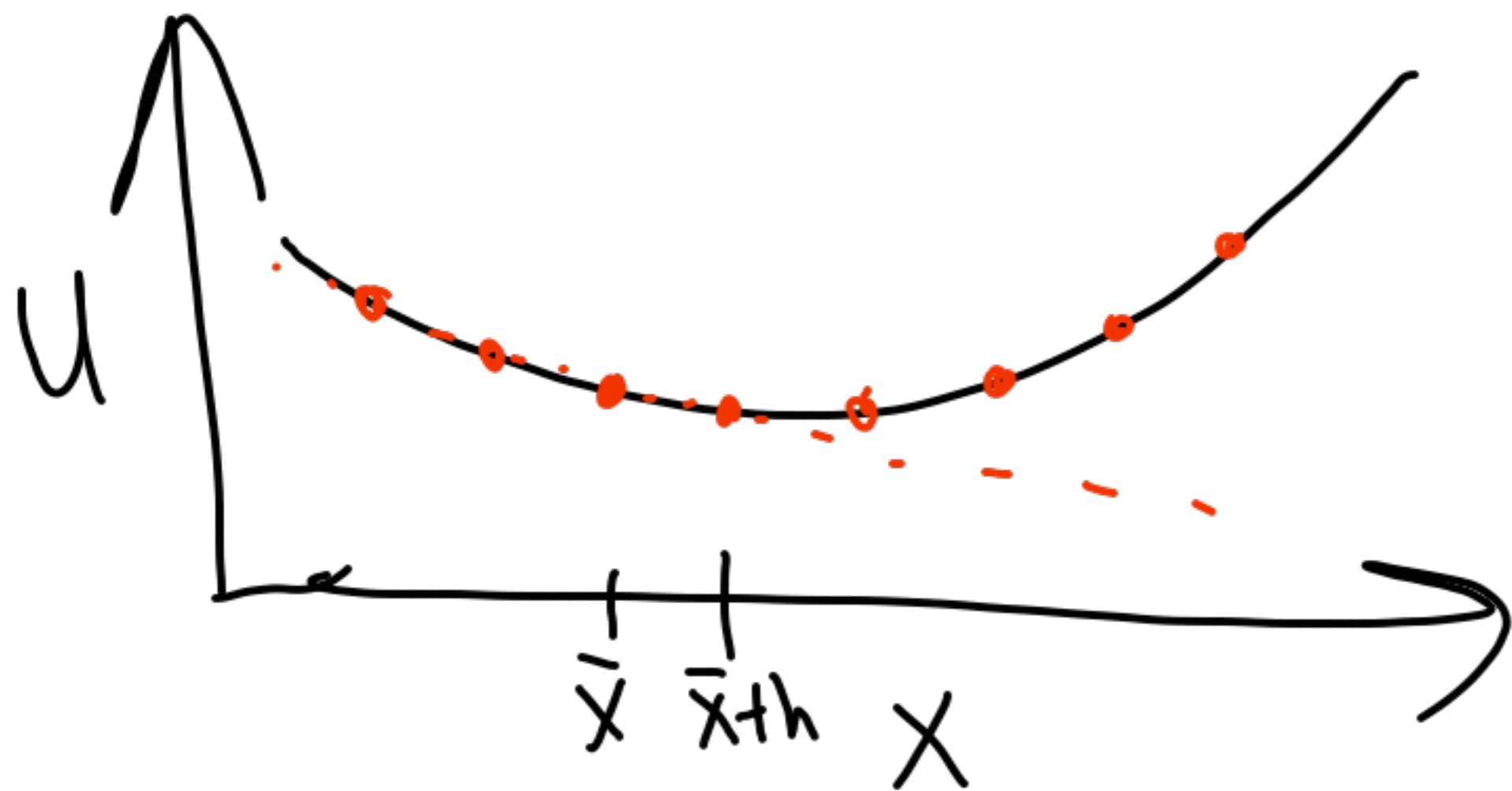
$$U'(\bar{x}) = \lim_{h \to 0} \frac{U(\bar{x}+h) - U(\bar{x})}{h}$$

This suggests the approximations:

$$U'(\bar{x}) \approx \frac{U(\bar{x}+h) - U(\bar{x})}{h} = D_+ U(\bar{x})$$

$$U'(\bar{x}) \approx \frac{U(\bar{x}) - U(\bar{x}-h)}{h} = D_- U(\bar{x})$$

$$U'(\bar{x}) \approx \frac{U(\bar{x}+h) - U(\bar{x}-h)}{2h} = D_0 U(\bar{x})$$

Taylor says:

$$U(\bar{x}) = \sum_{j=0}^{\infty} U^{(j)}(x) \frac{(\bar{x}-x)^j}{j!}$$

Truncation error

$$= \sum_{j=0}^{P} U^{(j)}(x) \frac{(\bar{x}-x)^j}{j!} + \mathcal{O}\left(|\bar{x}-x|^{P+1}\right)$$

$$D_- u(\bar{x}) = U'(\bar{x}) - \frac{h}{2}U''(\bar{x}) + \mathcal{O}(h^2)$$

$$D_0 u(\bar{x}) = U'(\bar{x}) + \frac{h^2}{6}U''(\bar{x}) + \mathcal{O}(h^4)$$

Second-order accurate

$$U(\bar{x}+h) = U(\bar{x}) + hu'(\bar{x}) + \frac{h^2}{2}U''(\bar{x}) + \frac{h^3}{6}U'''(\bar{x}) + \mathcal{O}(h^4)$$

$$U(\bar{x}-h) = U(\bar{x}) - hu'(\bar{x}) + \frac{h^2}{2}U''(\bar{x}) - \frac{h^3}{6}U'''(\bar{x}) + \mathcal{O}(h^4)$$

$$D_+ u(\bar{x}) = U'(\bar{x}) + \frac{h}{2}U''(\bar{x}) + \frac{h^2}{6}U'''(\bar{x}) + \mathcal{O}(h^3)$$

leading trunc. error

First-order approximation.

Deriving FD formulas

Suppose we want to approximate

$u''(x)$ using $u(x), u(x+h), u(x+2h)$.

$$u(x+2h) = u(x) + 2hu'(x) + 2h^2 u''(x) + \frac{4}{3}h^3 u'''(x) + \mathcal{O}(h^4)$$

We want

$$u''(x) \approx a\, u(x) + b\, u(x+h) + c\, u(x+2h)$$

Expanding: $a u(x) + b\left(u(x) + hu'(x) + \frac{h^2}{2}u''(x) + \frac{h^3}{6}u'''(x)\right) + c\left(\quad\right) + \mathcal{O}(h^4) = u''(x) + \mathcal{O}(h^?)$

$u(x):\quad a + b + c = 0$

$u'(x)\quad bh + 2ch = 0$

$u''(x):\quad b\frac{h^2}{2} + 2ch^2 = 1$

$u'''(x):\quad b\frac{h^3}{6} + c\frac{4}{3}h^3 = 0$ Not possible

$b = -2c$

$a = -b - c = 2c - c = c$

$\frac{1}{2}b + 2c = \frac{1}{h^2}$

$-c + 2c = \frac{1}{h^2}$

$c = \frac{1}{h^2} = a$

$b = \frac{-2}{h^2}$

$$u''(x) \approx \frac{u(x) - 2u(x+h) + u(x+2h)}{h^2}$$

1st-order formula

A General Approach

We want: $\sum_{i=1}^{n} c_i u(x_i) \approx u^{(k)}(\bar{x})$

Given values
$u(x_1), u(x_2), \ldots, u(x_n)$
find a FD formula to
approximate $u^{(k)}(\bar{x})$
as accurately as possible.

We have: $\sum_{i=1}^{n} c_i \sum_{j=0}^{\infty} \frac{(x_i-\bar{x})^j}{j!} u^{(j)}(\bar{x}) \approx u^{(k)}(\bar{x})$

Taylor series:
$$u(x_i) = \sum_{j=0}^{\infty} \frac{(x_i-\bar{x})^j}{j!} u^{(j)}(\bar{x})$$

We have (vandermonde)

$$\begin{bmatrix} 1 & - & - & - & - & - & 1 \\ (x_1-\bar{x}) & (x_2-\bar{x}) & - & - & - & & x_n-\bar{x} \\ \vdots & & & & & & \vdots \\ \frac{(x_1-\bar{x})^{n-1}}{(n-1)!} & - & - & - & - & & \frac{(x_n-\bar{x})^{n-1}}{(n-1)!} \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ \vdots \\ \vdots \\ c_n \end{bmatrix} = \begin{bmatrix} 0 \\ \vdots \\ \vdots \\ 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

$\leftarrow$ entry $k+1$

$X_j \neq X_i$ for $j \neq i$

$\Rightarrow$ non-singular

• We always have
  a unique solution.

What is the order
  of accuracy?

$\mathcal{O}(h^{n-k})$

$Sin(x) = 0 + x + 0 - \dfrac{x^3}{6} \cdots$