Review

$0 < x < 1$

$u''(x) = f(x)$

$u(0) = \alpha$

$u(1) = \beta$

$$\downarrow$$

$$\frac{U_{j+1} - 2U_j + U_{j-1}}{h^2} = f(x_j)$$

$$\downarrow$$

$$AU = F$$

Exact solution satisfies:

$$A\hat{U} = F + \tau$$

E: global error $U - \hat{U}$

$$AE = -\tau$$

$$\|E\| \leq \|A^{-1}\| \cdot \|\tau\|$$

We showed that

$$\|A^{-1}\| < C \quad \text{as } h \to 0$$

so $\|E\| \to 0$ as $h \to 0$

We claimed the eigenvalues of

A are $\lambda_p = \frac{2}{h^2}(\cos(p\pi h) - 1)$ $\star$

Let's derive this

Toeplitz matrix

$$\hat{A} = h^2 A = \begin{bmatrix} -2 & 1 & & \\ 1 & \ddots & \ddots & \\ & \ddots & \ddots & 1 \\ & & 1 & -2 \end{bmatrix}$$

$$\hat{A} v = \hat{\lambda} v$$

$$\boxed{v_{j+1} - 2v_j + v_{j-1} = \hat{\lambda} v_j \quad j = 1, 2, \ldots, m}$$

$$\boxed{v_0 = 0 \quad v_{m+1} = 0}$$

Linear difference equations

Ansatz: $v_j = \xi^j \quad \xi \in \mathbb{C}$

$$\xi^{j+1} - 2\xi^j + \xi^{j-1} = \hat{\lambda} \xi^j$$

$$\xi^2 - (2 + \hat{\lambda})\xi + 1 = 0$$

$$\xi_\pm = 1 + \frac{\hat{\lambda}}{2} \pm \frac{\sqrt{\hat{\lambda}^2 + 4\lambda}}{2}$$

General solution:

$$v_j = a\xi_+^j + b\xi_-^j$$

Now use BCs

$$V_0 = a + b = 0$$

$$b = -a$$

$$V_j = a(\xi_+^j - \xi_-^j)$$

$$V_{m+1} = a(\xi_+^{m+1} - \xi_-^{m+1}) = 0$$

So $\xi_+^{m+1} = \xi_-^{m+1}$

$$\xi_+^{2m+2} = (\xi_+ \xi_-)^{m+1} = 1$$

$\xi_+ \xi_- = 1$ (Vieta)

$$\xi_+^{2m+2} = e^{2\pi i p} \quad h = \frac{1}{m+1}$$

$$\xi_+ = e^{\frac{\pi i}{m+1} p} = e^{p\pi h i}$$

$$\xi_- = e^{-p\pi h i} \qquad p = 1, 2, \dots, m$$

$$\xi_+ + \xi_- = 2 + \hat{\lambda} = e^{p\pi h i} + e^{-p\pi h i}$$

$$2 + \hat{\lambda} = 2\cos(p\pi h)$$

$$\hat{\lambda} = 2(\cos(p\pi h) - 1)$$

$$\lambda = \frac{\hat{\lambda}}{h^2} = \frac{2}{h^2}(\cos(p\pi h) - 1)$$

Max-norm Stability

We want to show that

$$\|A^{-1}\|_\infty < C$$

$$\|M\|_\infty = \sup_{\|v\|_\infty = 1} \frac{\|Mv\|_\infty}{\|v\|_\infty}$$

$$= \max_{1 \le i \le m} \sum_{j=1}^{m} |m_{ij}|$$

$$AU = F \qquad U = BF = \sum_{j=0}^{m} F_j B_j$$

$$\frac{1}{h^2}\begin{bmatrix} h^2 & & & & \\ 1 & -2 & 1 & & \\ & \ddots & \ddots & \ddots & \\ & & 1 & -2 & 1 \\ & & & & h^2 \end{bmatrix}\begin{bmatrix} U_0 \\ U_1 \\ \vdots \\ U_m \\ U_{m+1} \end{bmatrix} = \begin{bmatrix} \alpha \\ f(x_1) \\ \vdots \\ f(x_m) \\ \beta \end{bmatrix}$$

Define $B = A^{-1}$.

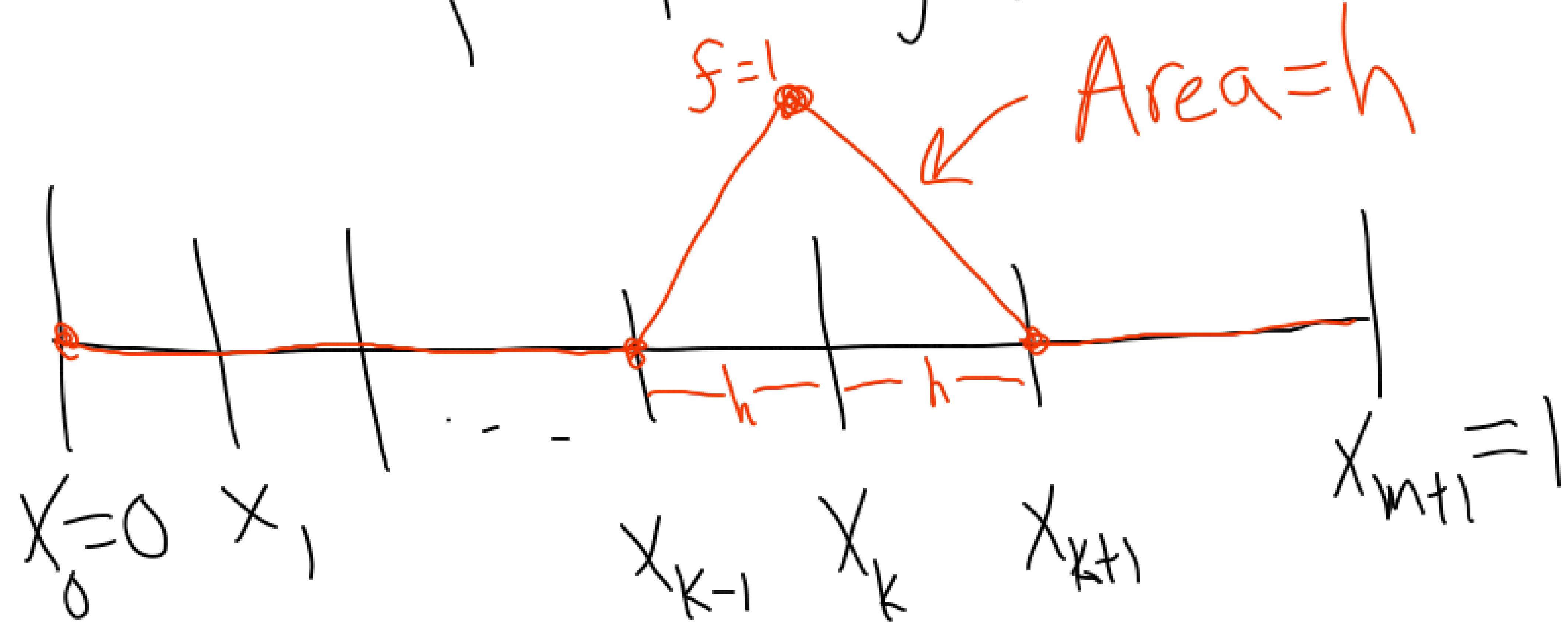
$$B = \begin{bmatrix} B_0 \,\Big|\, B_1 \,\Big|\, \cdots \,\Big|\, B_m \,\Big|\, B_{m+1} \end{bmatrix}$$

If $F_j = 0 \ \forall j \neq K$ and $F_K = 1$,

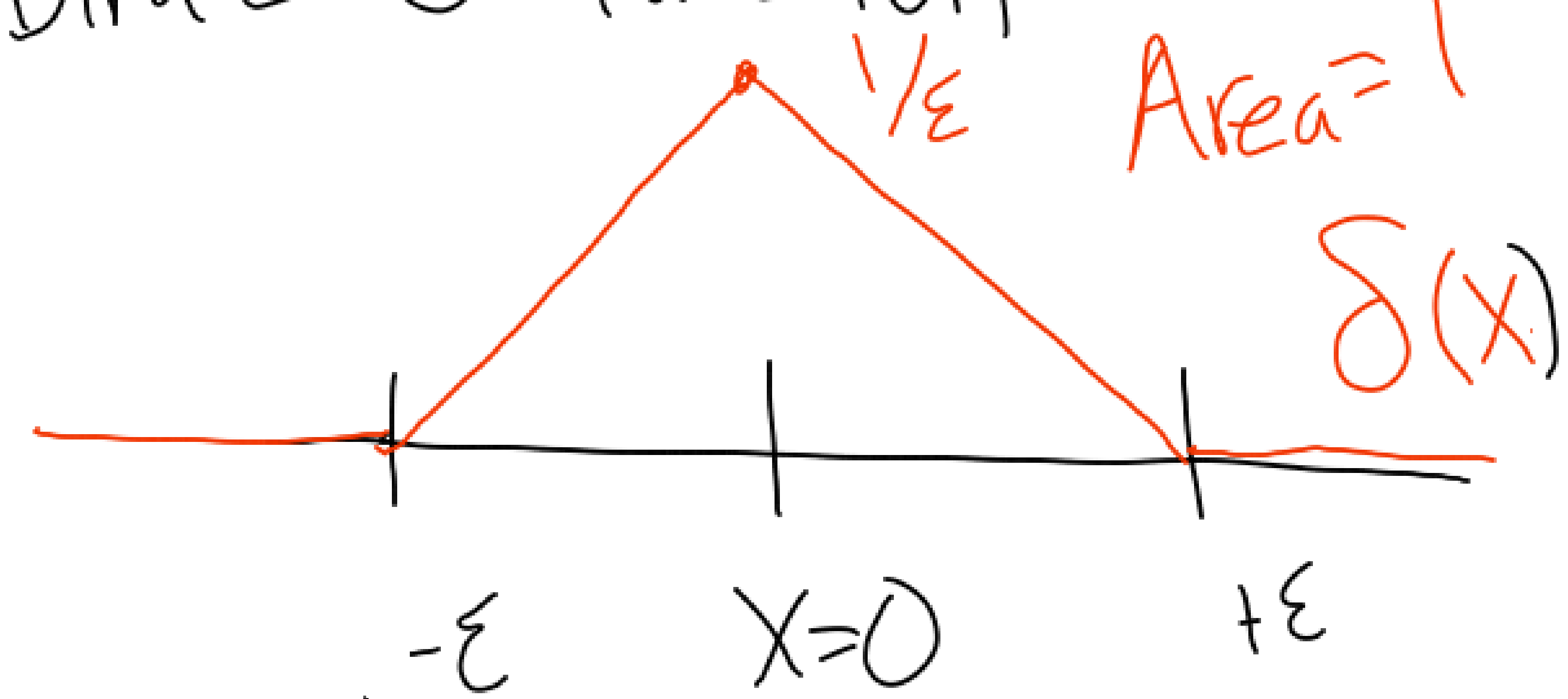
$$U = B_K$$

Suppose $\alpha = \beta = 0$

$$f(x_j) = \begin{cases} 0 & j \neq k \\ 1 & j = k \end{cases}$$



Dirac δ-function



$$\delta_\varepsilon(x) = \begin{cases} \dfrac{\varepsilon + x}{\varepsilon^2} & -\varepsilon \le x \le 0 \\ \dfrac{\varepsilon - x}{\varepsilon^2} & 0 \le x \le \varepsilon \\ 0 & |x| \ge \varepsilon \end{cases}$$

As $\varepsilon \to 0$, $\delta_\varepsilon(x) = 0$ everywhere except $x = 0$.

We have $\int_{-\infty}^{\infty} \delta_\varepsilon(x) = 1$

we call the limit as $\varepsilon \to 0$ the Dirac δ-function.

We have

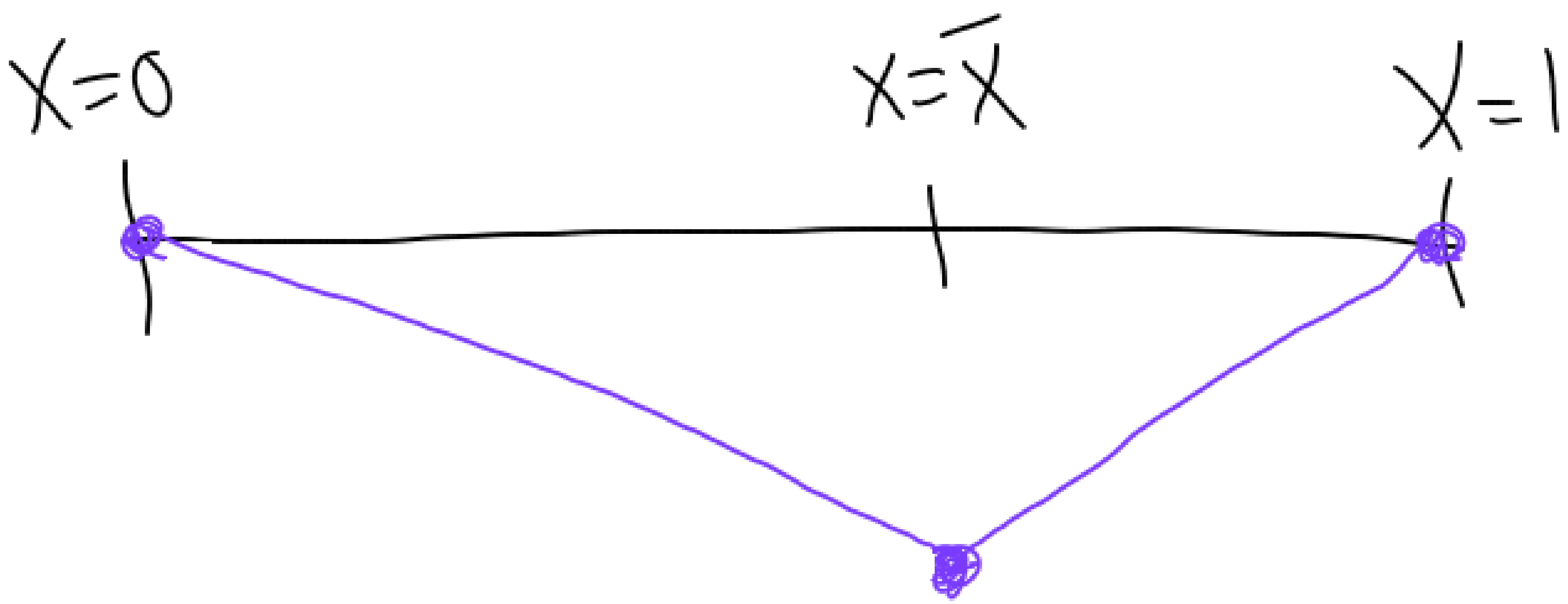
$$\int_{-\infty}^{\infty} \delta(x - \bar{x}) f(x)\, dx = f(\bar{x}).$$

Consider the BVP

$$u''(x) = \delta(x-\bar{x})$$

$$u(1) = u(0) = 0$$

$u(x)$ is linear except at $x=\bar{x}$



$$u'(\bar{x}+\varepsilon) - u'(\bar{x}-\varepsilon) = \int_{\bar{x}-\varepsilon}^{\bar{x}+\varepsilon} u''(x)\,dx = 1$$

Let $u(x) = \begin{cases} c_1 x & 0 \le x \le \bar{x} \\ c_2(x-1) & \bar{x} \le x \le 1 \end{cases}$

$c_1\bar{x} = c_2(\bar{x}-1)$ (continuity)

$c_2 - c_1 = 1$

$c_2 = 1 + c_1$

$c_1\bar{x} = (1+c_1)(\bar{x}-1)$

$\cancel{c_1\bar{x}} = \bar{x} + \cancel{c_1\bar{x}} - c_1 - 1$

$c_1 = \bar{x} - 1$

$c_2 = \bar{x}$

$$u(x) = G(x;\bar{x}) = \begin{cases} (\bar{x}-1)x & 0 \le x \le \bar{x} \\ \bar{x}(x-1) & \bar{x} \le x \le 1 \end{cases}$$

"Green's function"

Any function can be written

$$f(x) = \int_{-\infty}^{\infty} f(\bar{x})\,\delta(x-\bar{x})\,d\bar{x}$$

So the general solution of

$$u''(x) = f(x) \qquad u(0)=u(1)=0$$

is

$$u(x) = \int_0^1 f(\bar{x})\,G(x;\bar{x})\,dx$$

The solution of

$$u''(x) = h\delta(x-x_k)$$
$$u(0)=u(1)=1$$

is

$$u(x) = hG(x;x_k).$$

In fact

$$B_{ij} = hG(x_i;x_j) \qquad \begin{matrix} 1 \le j \le m \\ 0 \le i \le m+1 \end{matrix}$$

What about B_0, B_{m+1}?

$$u''(x)=0$$

$$u(0)=1 \quad u(1)=0$$

$$u(x)=1-x$$

$$B_{i0}=1-x_i$$

$$u''(x)=0$$

$$u(0)=0 \quad u(1)=1$$

$$u(x)=x$$

$$B_{i,m+1}=x_i$$

What is $\|B\|_\infty$? $\quad h=\frac{1}{m+1}$

$$\|B\|_\infty \leq 1+1+hm=2+\frac{m}{m+1}$$

$$<3$$