

$$U_{xx} + U_{yy} = f(x, y)$$

$$0 < x < 1, \quad 0 < y < 1$$

$$U(x, 0) = \alpha(x) \quad U(0, y) = \gamma(y)$$

$$U(x, 1) = \beta(x) \quad U(1, y) = \delta(y)$$

$$U_{xx} \Big|_{(x,y)=(x_i,y_j)} \approx \frac{U_{(i+1,j)} - 2U_{(i,j)} + U_{(i-1,j)}}{(\Delta x)^2}$$

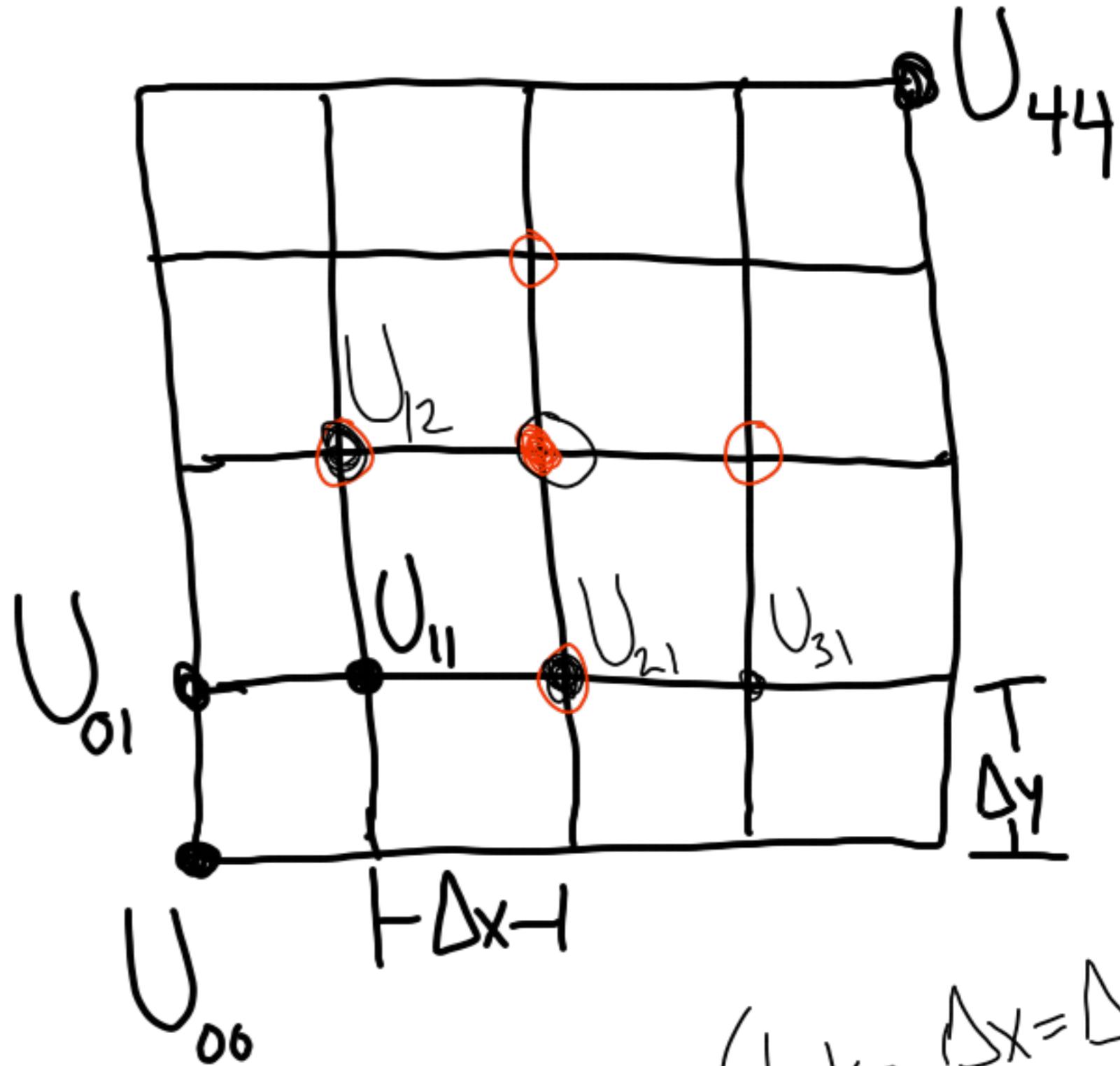
$$U_{yy} \Big|_{(x,y)=(x_i,y_j)} \approx \frac{U_{(i,j+1)} - 2U_{(i,j)} + U_{(i,j-1)}}{(\Delta y)^2}$$

$$\begin{aligned} 1 &\leq i \leq M \\ 1 &\leq j \leq M \end{aligned}$$

5-point Laplacian: (take  $\Delta x = \Delta y = h$ )

$$\nabla^2 u \approx \frac{1}{h^2} \left[ -4U_{(i,j)} + U_{(i+1,j)} + U_{(i-1,j)} + U_{(i,j+1)} + U_{(i,j-1)} \right]$$

$$= f(x_i, y_j)$$



$m^2$  equations

Need to choose  
an ordering

For example, row-wise  
Ordering:

$$U = \begin{bmatrix} U_{11} \\ U_{21} \\ U_{31} \\ U_{12} \\ U_{22} \\ \vdots \\ U_{mm} \end{bmatrix} \quad F = \begin{bmatrix} f(x_1, y_1) \\ f(x_2, y_1) \\ \vdots \\ f(x_1, y_m) \\ f(x_2, y_m) \\ \vdots \\ f(x_m, y_m) \end{bmatrix}$$

$$AU = F$$

$$A = \frac{1}{h^2}$$

$m \times m$

SPARSE  
 $\sim 5m^2$  non-zero entries

A  $5 \times 5$  matrix  $A$  is shown. The matrix has a central  $3 \times 3$  block of  $-4$ 's. In the top-left corner, there is a  $2 \times 2$  block of  $1$ 's. In the bottom-right corner, there is another  $2 \times 2$  block of  $1$ 's. The matrix is labeled  $A = \frac{1}{h^2}$ . A red arrow points to the  $m \times m$  dimension of the matrix. Another red arrow points to the  $1 \times 1$  dimension of the matrix. A third red arrow points to the  $5 \times 5$  dimension of the matrix.

## Consistency

Substitute:  $U_{ij} \rightarrow U(x_i, y_j)$

$$\frac{1}{h^2} \left[ -4U(x_i, y_j) + U(x_i + h, y_j) + U(x_i - h, y_j) + U(x_i, y_j + h) + U(x_i, y_j - h) \right] = f(x_i, y_j) + \tau_{ij}$$

Expand about  $(x_i, y_j)$ :

$$\cancel{U_{xx}} + \frac{1}{12} h^2 U_{xxxx} + \mathcal{O}(h^4) + \cancel{U_{yy}} + \frac{1}{12} h^2 U_{yyyy} + \mathcal{O}(h^4) = \cancel{f(x_i, y_j)} + \tau_{ij}$$

$$\tau_{ij} = \frac{1}{12} h^2 (U_{xxxx} + U_{yyyy}) + \mathcal{O}(h^4)$$

2nd-order accurate  
(locally)

# Global error

Exact solution

Vector:  $\hat{U}_{ij} = u(x_i, y_j)$

Error:  $U - \hat{U}$

$$AU = F$$

$$A\hat{U} = F + \tau$$

$$AE = -\tau$$

$$E = -A^{-1} \tau$$

$$\|E\| = \|A^{-1} \tau\| \leq \|A^{-1}\| \cdot \|\tau\|$$

So we need to show that  $\|A^{-1}\| < C$  as  $h \rightarrow 0$ . (stability)

(since we know  $\|\tau\| = \mathcal{O}(h^2)$ )

i.e. We need to show that the e.v's of  $A$  are bounded away from zero as  $h \rightarrow 0$ .

$$Av = \lambda v: \frac{V_{(i+1,j)} - 2V_{ij} + V_{(i-1,j)}}{h^2} + \frac{V_{(i,j+1)} - 2V_{ij} + V_{(i,j-1)}}{h^2} = \lambda V_{ij}$$

Note:

$V_{0j}, V_{M+1,j}, V_{i,0}, V_{i,M+1}$  defined by

B.C.S.

Assume  $V_{ij} = R_i S_j$  (separability)

$$S_j \cdot \frac{R_{i+1} - 2R_i + R_{i-1}}{h^2} + R_i \frac{S_{j+1} - 2S_j + S_{j-1}}{h^2} = \lambda R_i S_j$$

Divide  $R_i S_j$  by  $C_1$

$$\frac{R_{i+1} - 2R_i + R_{i-1}}{h^2 R_i} + \frac{S_{j+1} - 2S_j + S_{j-1}}{h^2 S_j} = \lambda$$

Each must be equal to a constant

$$R_{i+1} - 2R_i + R_{i-1} = h R_i C_1$$

$$R_{i+1} - (2 + h^2 C_1) R_i + R_{i-1} = 0 \quad 1 \leq i \leq m$$

$R_0, R_{m+1}$  given  
by B.C.s

$$R_i = S^i \quad \text{for some } S^i \in \mathbb{C}$$

(problem we solved before)

$$C_1 = \frac{2}{h^2} (\cos(p\pi h) - 1) \quad p = 1, 2, \dots, m$$

$$C_2 = \frac{2}{h^2} (\cos(q\pi h) - 1) \quad q = 1, 2, \dots, m$$

$$\lambda = C_1 + C_2 = \frac{2}{h^2} (\cos(p\pi h) + \cos(q\pi h) - 2)$$

Smallest:  $p=q=1$

$$\lambda_{\parallel} = -2\pi^2 + \mathcal{O}(h^2)$$

$$\Rightarrow \|A^{-1}\| < \frac{1}{2\pi^2} + \varepsilon$$

for small  $h$ .

$$\Rightarrow \|E\| = \mathcal{O}(h^2).$$