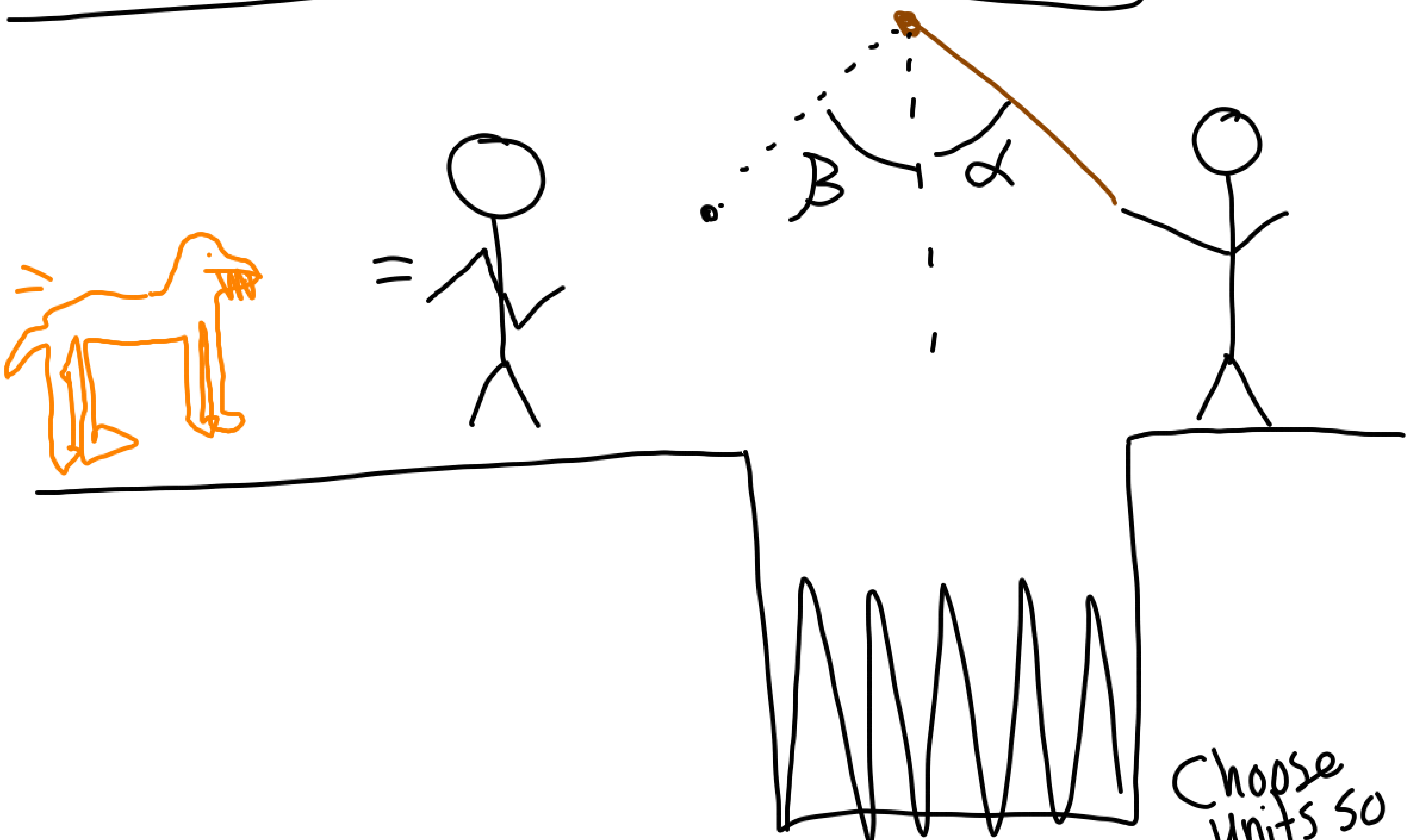
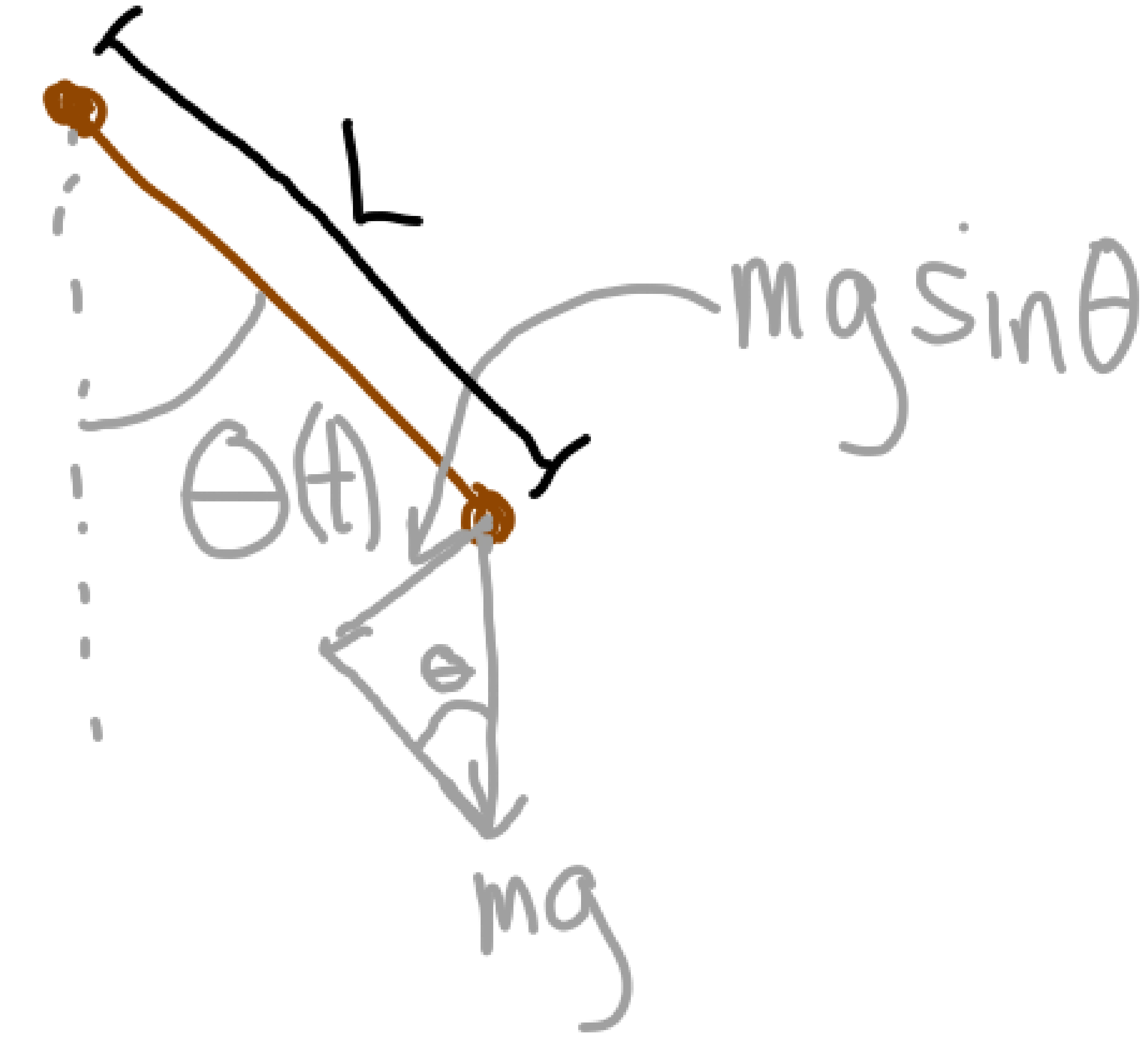
Nonlinear BVPs





$$F = ma$$

$$-mg\sin(\theta(t)) = mL\theta''(t)$$

$$\theta''(t) = -\frac{L}{g}\sin(\theta(t))$$

Choose units so $\frac{L}{g} = 1$

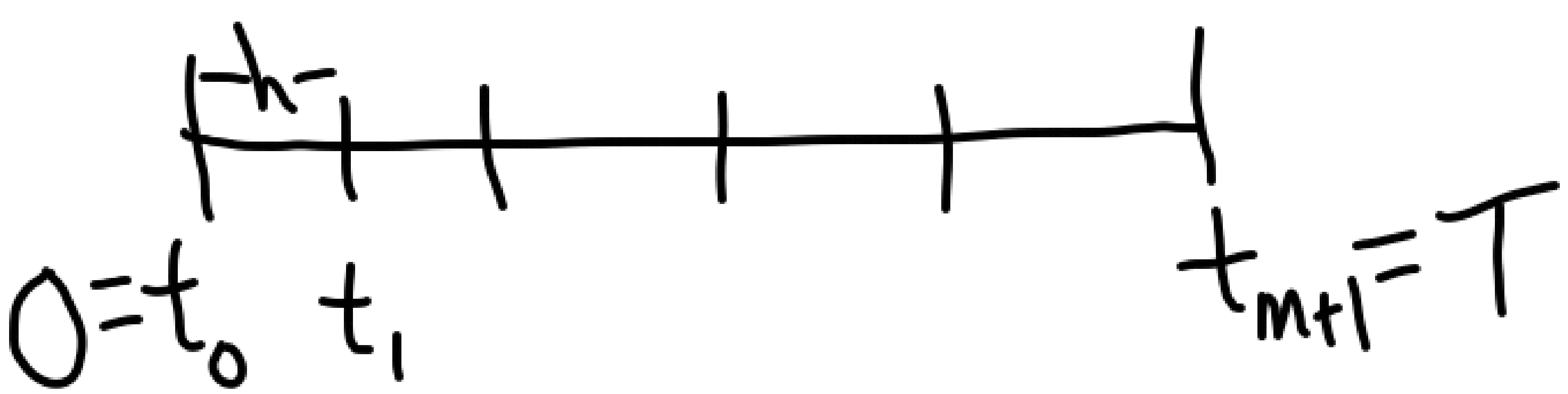
$$\theta''(t) = -\sin(\theta(t))$$

BVP: $\Theta''(t) = -\sin(\Theta(t))$

$\Theta(0) = \alpha$

$\Theta(T) = \beta$

Discretize:



$h = \frac{1}{m+1}$

$$\Theta''(t) = \frac{\Theta_{i+1} - 2\Theta_i + \Theta_{i-1}}{h^2}$$

$i = 1, 2, \ldots, m$

$\Theta_0 = \alpha \quad \Theta_{m+1} = \beta$

We have

$$\overbrace{\frac{\Theta_{i+1} - 2\Theta_i + \Theta_{i-1}}{h^2} + \sin(\Theta_i)}^{G(\Theta)} = 0 \qquad (*)$$

for $i = 1, 2, \ldots, m$.

Let Θ_* denote the exact solution of $(*)$: $G(\Theta_*) = 0$.

Let $\Theta^{[0]}$ denote an initial guess.

Then

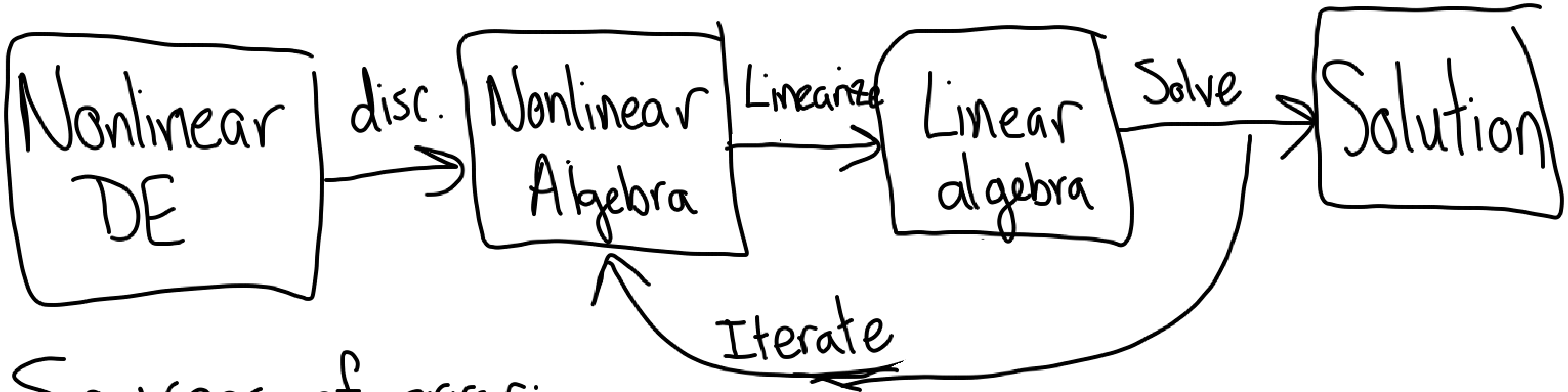
$$G(\theta_*) = G(\theta^{[0]}) + G'(\theta^{[0]})\overbrace{(\theta_* - \theta^{[0]})}^{\delta^{[0]}} + O(\|\theta_* - \theta^{[0]}\|^2)$$

Here

$$J(\theta) = G'(\theta) = \begin{bmatrix} \frac{\partial G_1}{\partial \theta_1} & \frac{\partial G_1}{\partial \theta_2} & \cdots & \frac{\partial G_1}{\partial \theta_m} \\ \frac{\partial G_2}{\partial \theta_1} & & & \vdots \\ \vdots & & & \vdots \\ \frac{\partial G_m}{\partial \theta_1} & \cdots & & \cdots \end{bmatrix}$$

Newton's method

1. Choose initial guess $\theta^{[0]}$
2. For $k = 1, 2, \ldots$ solve
   $$J(\theta^{[k-1]})\delta^{[k-1]} = -G(\theta^{[k-1]})$$
   $$\theta^{[k]} = \theta^{[k-1]} + \delta^{[k-1]}$$
3. Stop when
   $$\|G(\theta^{[k]})\| < \varepsilon$$



Sources of error:

1. Truncation error
2. Linearization (iteration)
3. Rounding errors

Ideally, we balance the different sources of error (for efficiency)

Consistency

Local truncation error
(subst. $\theta(t_i)$ for θ_i)

$$\frac{\theta(t_{i+1})-2\theta(t_i)+\theta(t_{i-1})}{h^2}+\sin(\theta(t_i)) = \tau_i$$

$$\cancel{\theta''(t_i)}+\frac{1}{12}h^2\theta^{(4)}(t_i)+O(h^4)+\cancel{\sin(\theta(t_i))}=\tau_i$$

$$\tau_i=\frac{1}{12}h^2\theta^{(4)}(t_i)+O(h^4)$$

So this method is 2nd-order accurate.

Stability

$$\hat{\theta}=\begin{bmatrix}\theta(t_1)\\ \vdots \\ \theta(t_m)\end{bmatrix}$$

$$E=\theta_*-\hat{\theta}$$

$$G(\theta_*)=0$$

$$G(\hat{\theta})=\tau$$

$$G(\hat{\theta})-G(\theta_*)=\tau$$

$$\tau=\begin{bmatrix}\tau_1\\ \vdots \\ \tau_m\end{bmatrix}$$

$$G(\theta_*) = G(\hat{\theta}) + J(\hat{\theta})E + \mathcal{O}(\|E\|^2)$$

If $\|E\|$ is small, we can discard higher-order terms and have (approximately).

$$\tau \approx -J(\hat{\theta})E$$

$$E \approx (-J(\hat{\theta}))^{-1}\tau$$

$$\|E\| \leq \|(J(\hat{\theta}))^{-1}\| \cdot \|\tau\|$$

Stability requires that $\|J^{-1}\| < C$ for some C.

Sketch of Why $\|J^{-1}\|$ is bounded:

$$J = \frac{1}{h^2}\hat{A} + D$$

$$\hat{A} = \text{tridiag}(1,-2,1) \quad D = \begin{bmatrix} \sin(t_1) & & \\ & \ddots & \\ & & \sin(t_m) \end{bmatrix}$$

We know $\left\|\left(\frac{1}{h^2}\hat{A}\right)^{-1}\right\| < C$.

$$J = \frac{1}{h^2}\left(\hat{A} + h^2 D\right)$$

$$J^{-1} = h^2\left(\hat{A} + h^2 D\right)^{-1} \to h^2\hat{A}^{-1} \text{ as } h \to 0$$

$$h^2\hat{A}^{-1} = \left(\frac{1}{h^2}\hat{A}\right)^{-1}$$ so $\|J^{-1}\| < C$ as $h \to 0$.