Derive 3rd order splitting Method for

$$\begin{cases} \partial_t q(t) = (A+B)q \\ q(0) = q_0 \end{cases}, \quad t \in [0,T]$$

$\Delta t = T/n$, $n \in \mathbb{N}$

Step 1: Apply Lie splitting with order $A \to B$

$$\hat{q}(0) = q_0 \quad \begin{cases} \partial_t u_k^{(1)}(t) = A u_k^{(1)}(t) \\ u_k^{(1)}((k-1)\Delta t) = \hat{q}((k-1)\Delta t) \end{cases}$$

$$\begin{cases} \partial_t u_k^{(2)}(t) = B u_k^{(2)}(t) \\ u^{(2)}((k-1)\Delta t) = u_k^{(1)}(k\Delta t) \end{cases}$$

Step2 : Apply Lie splitting at kth step with $B \to A$

$$\begin{cases} \partial_t v_k^{(1)}(t) = B\, v_k^{(1)}(t) \\ v_k^{(1)}((k-1)\Delta t) = \hat{q}((k-1)\Delta t) \end{cases}$$

$$\begin{cases} \partial_t v_k^{(2)}(t) = A\, v_k^{(2)}(t) \\ v_k^{(2)}((k-1)\Delta t) = v_k^{(1)}(k\Delta t) \end{cases}$$

$$v_k^{(2)}(t)$$

Step3: After the kth time step,

Take a weighted sum

$$\hat{q}(x,\Delta t) = \theta u_k^{(2)}(\Delta t) + (1-\theta)\, v_k^{(2)}(\Delta t) \qquad \theta \in [0,1]$$

if $\theta = 0$, $\quad \hat{q}(x,\Delta t) = v_k^{(2)}(\Delta t)$

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splitting Error:

$$E(\Delta t) := \hat{q}(x,\Delta t) - q(x,\Delta t)$$

$$= \theta\left(e^{\Delta t B} e^{\Delta t A}\right) q_0 + (1-\theta)\left(e^{\Delta t A} e^{\Delta t B}\right) q_0$$

$$- e^{\Delta t (A+B)} q_0$$

$$E_{\Delta t^2} = \Delta t^2 \left[\frac{1}{2}(AB - BA) + \theta(BA - AB) \right] q_0$$

$$= \Delta t^2 \left[\left(\frac{1}{2} - \theta \right) AB + \left(\theta - \frac{1}{2} \right) BA \right] q_0$$

$$\theta = \frac{1}{2}$$

$$e^{\Delta t (A+B)} = I + \Delta t (A+B) + \frac{\Delta t^2}{2}(A+B)^2 + \frac{\Delta t^3}{6}(A+B)^3 + \mathcal{O}(\Delta t^4)$$

$$(A+B)^3 \downarrow (A+B)(A+B)(A+B)$$

$$ABA \quad BAB$$

$$E = \overset{1-\theta}{\frac{1}{2}}\left(e^{\Delta t B} e^{\Delta t A}\right) q_0 + \frac{1}{2}\left(e^{\Delta t A} e^{\Delta t B}\right) - e^{\Delta t (A+B)} q_0$$

$$e^{\Delta t(B+A)} = e^{\Delta t B} e^{\Delta t A}$$

$$E_{\Delta t^3} = \frac{\Delta t^3}{2}\left(\frac{1}{6} A^3 + \frac{1}{2} BA^2 + \frac{1}{2} B^2 A + \frac{1}{6} B^3\right) q_0$$

$$+ \frac{\Delta t^3}{2}\left(\frac{1}{6} B^3 + \frac{1}{2} AB^2 + \frac{1}{2} A^2 B + \frac{1}{6} A^3\right) q_0$$

$$- \frac{\Delta t^3}{6}\left(A^3 + A^2 B + ABA + AB^2 + BA^2 + BAB + B^2 A + B^3\right) q_0$$

$$= \frac{\Delta t^3}{12}\left(BA^2 + B^2 A + AB^2 + A^2 B - 2ABA - 2BAB\right) q_0$$

$$= \frac{\Delta t^3}{12}\left(\left[A - B, [A, B]\right]\right)$$

Csomos 2005
Weighted Sequential Splitt

$$[A, B] = AB - BA$$

$$e^{A}e^{B} = e^{C}$$

$$C = (A+B) + \underbrace{\frac{1}{2}[A,B] + \frac{1}{2}}_{\Delta t^3}\left([\cdots\cdots]\right)$$

$$e^{\Delta t B}e^{\Delta t A} = \left(I + \Delta t B + \frac{\Delta t^2}{2}B^2 + \frac{\Delta t^3}{6}B^3 + \cdots\right)\left(I + \Delta t A + \frac{\Delta t^2}{2}A^2 + \frac{\Delta t^3}{6}A^3 + \cdots\cdots\right)$$

$$= I + \Delta t(B+A) + \Delta t^2\left(BA + \frac{1}{2}B^2 + \frac{1}{2}A^2\right)$$

$$+ \Delta t^3\left(\frac{1}{6}B^3 + \frac{1}{6}A^3 + \frac{1}{2}BA^2 + \frac{1}{2}B^2A\right)$$

$$q^{n+1} = \cdots e^{b_2 \Delta t B} e^{a_2 \Delta t A} e^{b_1 \Delta t B} e^{a_1 \Delta t A} q^n$$

$$a_1 = 1 \qquad b_1 = \frac{-1}{24}$$

$$a_2 = \frac{-2}{3} \qquad b_2 = \frac{3}{4}$$

$$a_3 = \frac{2}{3} \qquad b_3 = \frac{7}{24}$$

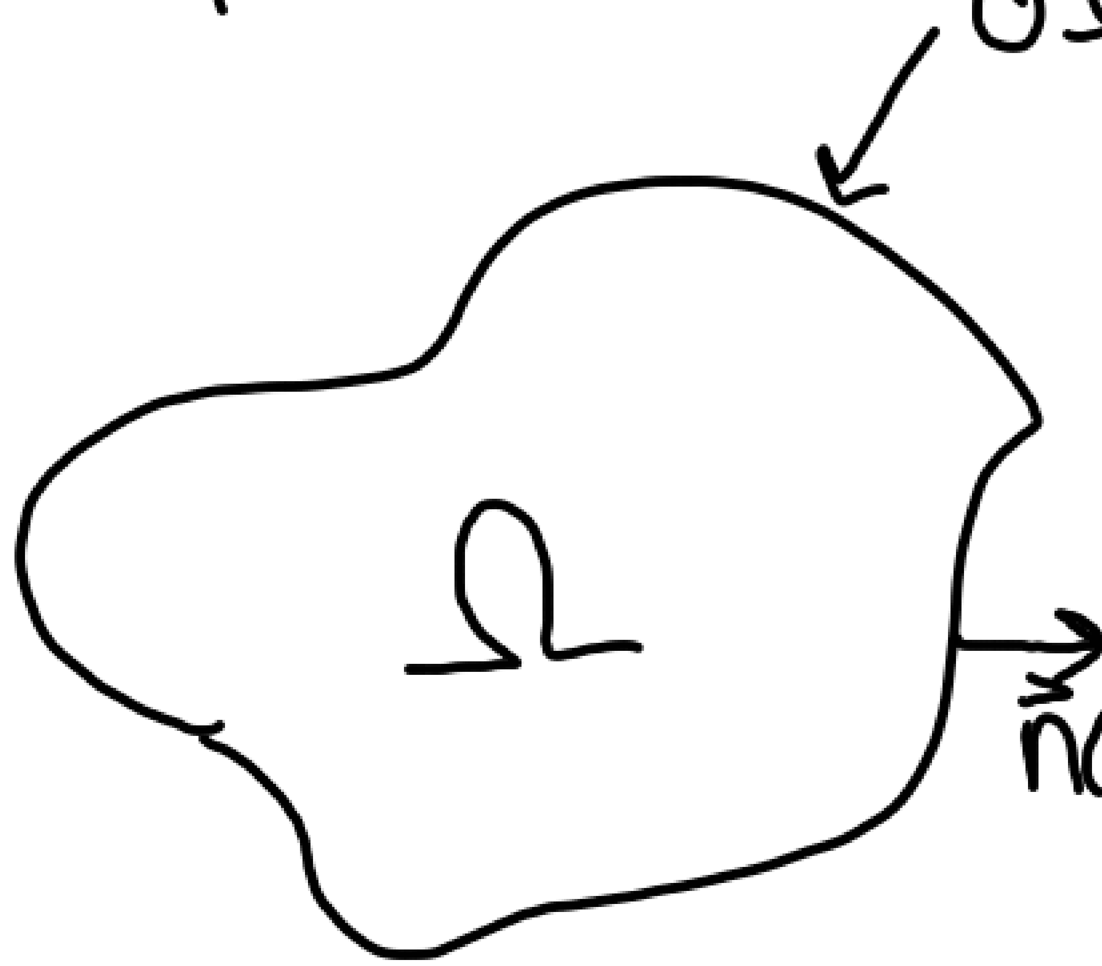
Baker-Campbell-Hausdorff
(BCH)

Hairer, Lubich, Wanner
Geometric Integration

Multidimensional Conservation Laws

$q(x,y,t)$

$\partial\Omega$



$$\vec{n}(s) = \begin{bmatrix} n^x \\ n^y \end{bmatrix}$$

Outward unit normal

Total mass

$$\frac{d}{dt}\overbrace{\iint_\Omega q(x,y,t)\,dx\,dy}^{\text{Total mass}}$$

$$\underbrace{\frac{d}{dt}\iint_\Omega q(x,y,t)\,dx\,dy}_{\text{rate of change of mass}}$$

Total flux through boundary:

$$-\int_{\partial\Omega} \vec{n}(s)\cdot\vec{f}(q(s,t))\,ds$$

$$\vec{f} = \begin{bmatrix} f(q) \\ g(q) \end{bmatrix}$$

$$\frac{d}{dt}\iint_{\Omega} q(x,y,t)\,dx\,dy = -\int_{\partial\Omega} \vec{n}(s)\cdot\vec{f}(q(x(s),y(s),t))\,ds$$

$$\iint_{\Omega} q_t(x,y,t)\,dx\,dy = -\iint_{\Omega} \nabla\cdot\vec{f}(q(x,y,t))\,dx\,dy$$

$$\iint_{\Omega} \underbrace{\left(q_t + \nabla\cdot\vec{f}(q)\right)}_{\text{must vanish pointwise}}\,dx\,dy = 0$$

$$q_t + \nabla\cdot\vec{f}(q) = 0 \quad \text{or} \quad q_t + f(q)_x + g(q)_y = 0$$

Under what conditions is this hyperbolic?

Linear 2D cons. law:

$$q_t + Aq_x + Bq_y = 0$$

$$A = R_A \Lambda_A R_A^{-1}$$

$$B = R_B \Lambda_B R_B^{-1}$$

$$q_t + R_A \Lambda_A R_A^{-1} q_x + R_B \Lambda_B R_B^{-1} q_y = 0$$

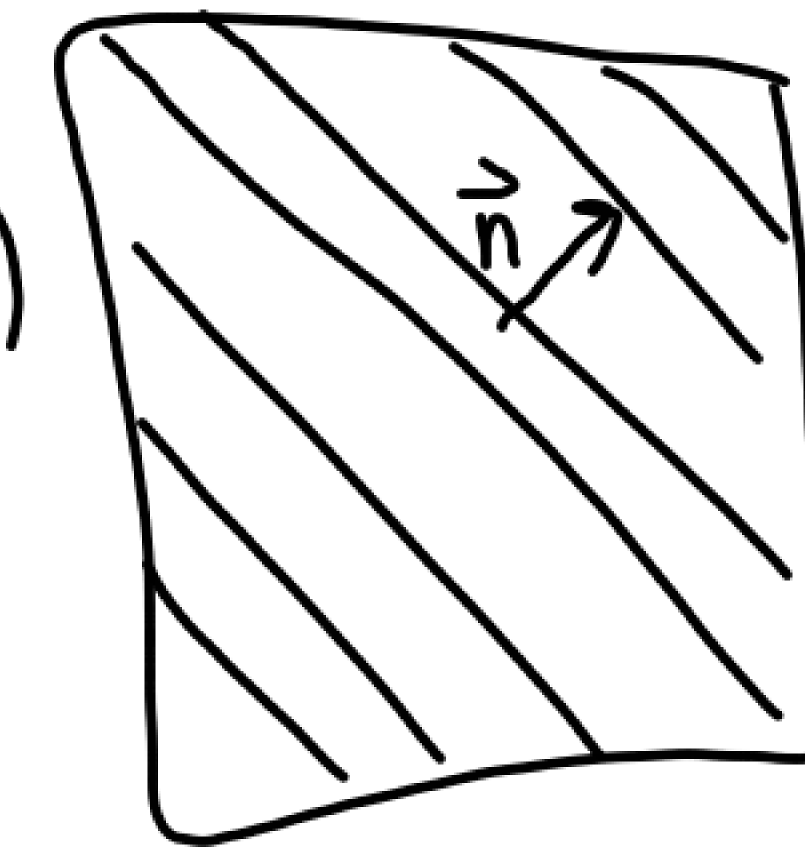
Left multiplication by R_A^{-1} or R_B^{-1} doesn't diagonalize this, unless $R_A = R_B$.

Suppose we choose plane wave initial data: $\vec{x} = \begin{bmatrix} x \\ y \end{bmatrix}$

$$q(x,y,0) = \mathring{q}(\vec{n} \cdot \vec{x}) = \mathring{q}(n^x x + n^y y)$$

We want

$$q(x,y,t) = \mathring{q}(\vec{n} \cdot \vec{x} - st)$$



Then

$$q_t = -s\mathring{q}'(\xi)$$

$$q_x = n^x \mathring{q}'(\xi)$$

$$q_y = n^y \mathring{q}'(\xi)$$

So

$$-s\mathring{q}'(\xi) + n^x A \mathring{q}'(\xi) + n^y B \mathring{q}'(\xi) = 0$$

$$(n^x A + n^y B)\mathring{q}'(\xi) = s\mathring{q}'(\xi)$$

s must be an eigenvalue of $n^x A + n^y B$.

$\mathring{q}'(\xi)$ must be an eigenvector of $n^x A + n^y B$ (for each ξ).

Definition: $q_t + Aq_x + Bq_y = 0$ is hyperbolic if $\forall\, n^x, n^y$

$$n^x A + n^y B$$

is diagonalizable with real eigenvalues.

$$q_t + f(q) + g(q) = 0$$

is hyperbolic if $\forall\, n^x, n^y, q$

$$n^x f'(q) + n^y g'(q)$$

is diagonalizable w/real eigenvalues.

Example: 2D Acoustics

$$p_t + K\nabla\cdot\vec{u} = 0 \qquad \vec{u} = \begin{bmatrix} u \\ v \end{bmatrix}$$

$$\vec{u}_t + \frac{1}{\rho}\nabla p = 0$$

$$p_t + K(u_x + v_y) = 0$$

$$u_t + \frac{1}{\rho} p_x = 0$$

$$v_t + \frac{1}{\rho} p_y = 0$$

$$q_t + A q_x + B q_y = 0$$

$$q = \begin{bmatrix} p \\ u \\ v \end{bmatrix} \quad A = \begin{bmatrix} 0 & K & 0 \\ 1/\rho & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$B = \begin{bmatrix} 0 & 0 & K \\ 0 & 0 & 0 \\ 1/\rho & 0 & 0 \end{bmatrix}$$

$$R_A = \begin{bmatrix} -Z & 0 & Z \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \quad \lambda_A = -c, 0, +c$$

$$R_B = \begin{bmatrix} -Z & 0 & Z \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \quad \lambda_B = -c, 0, +c$$

$$Z = \sqrt{K\rho}$$

$$c = \sqrt{\frac{K}{\rho}}$$

Homework:
FVMHP 18.2

Planar Riemann Problem