Godunov's Theorem

Definition: A scheme is said to monotonicity preserving (MP) if

$$Q_i^n \geq Q_{i-1}^n \quad \forall i$$

implies

$$Q_i^{n+1} \geq Q_{i-1}^{n+1} \quad \forall i.$$

Lemma. Consider a scheme

$$Q_i^{n+1} = \sum_{j=-r}^{j=r} \alpha_j Q_{i+j}^n. \quad (1)$$

If this scheme is MP, then

$$\alpha_j \geq 0 \quad \forall j.$$

Proof: Let $Q_i^n = \begin{cases} 0 & i < K \\ 1 & i \geq K \end{cases}$

Then $Q_i^{n+1} - Q_{i-1}^{n+1} = \sum_{j=-r}^{r} \alpha_j \left(Q_{i+j}^n - Q_{i+j-1}^n\right)$

$$= \alpha_{K-i}$$

Assume, by way of contradiction, that $\exists\, p$ s.t. $\alpha_p < 0$. Take $K-i = p$

$i = K-p$

Then $Q_{K-p}^{n+1} - Q_{K-p-1}^{n+1} = \alpha_p < 0.$

So $Q_{K-p}^{n+1} < Q_{K-p-1}^{n+1}.$

Theorem (Godunov)

Let a MP scheme (1) be given for the solution of

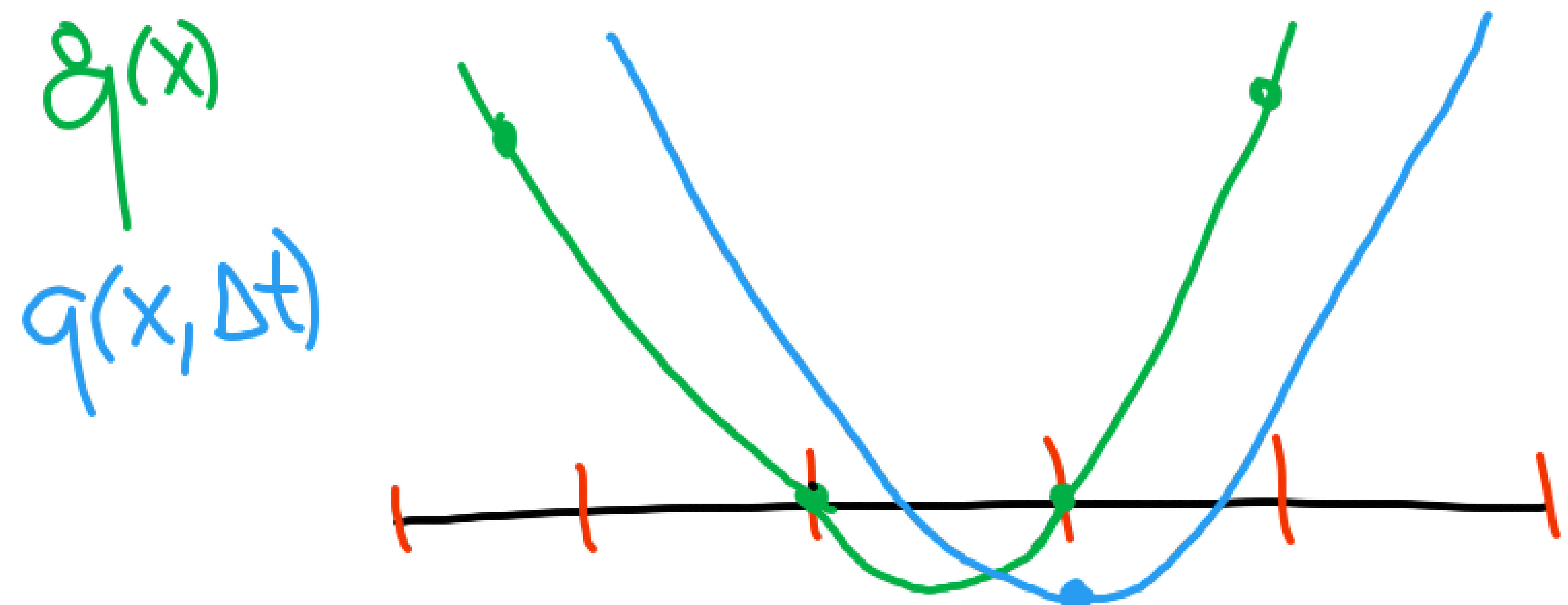
$$q_t + a q_x = 0.$$

Then either:

(i) $\frac{a\Delta t}{\Delta x} \in \mathbb{Z}$

(ii) The scheme is at most 1st-order accurate.

Proof. Suppose the scheme is 2nd-order accurate. Then it must be exact for quadratic initial data.



So some solution value should be negative after the first step. But if the scheme is MP, then according to (1) the new solution values must be positive. (Since $Q_i^0 \geq 0 \;\forall i$, $\alpha_j \geq 0 \;\forall j$)

For example: $\mathring{q}(x) = \left(\frac{x}{\Delta x} - \frac{1}{2}\right)^2 - \frac{1}{4}$

Total Variation

$$TV(q(x)) = \sup \sum_{j=1}^{N} |q(\xi_j) - q(\xi_{j-1})|$$

The supremum is taken over all sequences $-\infty = \xi_0 < \xi_1 < \cdots < \xi_N = \infty$

For a grid function:

$$TV(Q) = \sum_{-\infty}^{\infty} |Q_i - Q_{i-1}|$$

For a differentiable function

$$TV(q) = \int_{-\infty}^{\infty} |q'(x)|\,dx$$

For advection:

$$q_t + a q_x = 0$$
$$q(x,0) = \mathring{q}(x)$$

$$TV(q(x,t)) = TV(\mathring{q}(x))$$

For scalar conservation laws

$$q_t + f(q)_x = 0$$

$$q(x,t) = \mathring{q}(x)$$

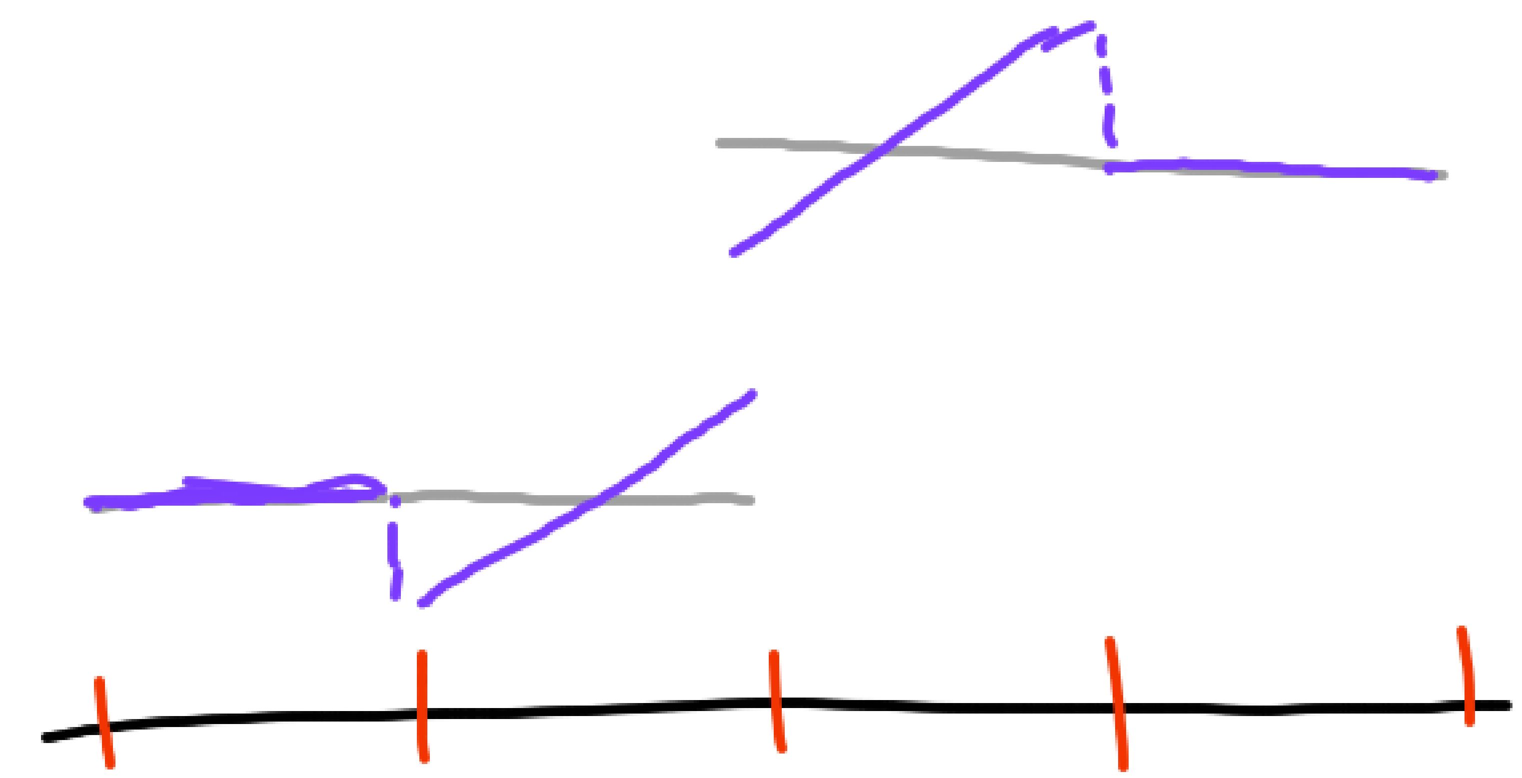
$$TV(q(x,t)) \le TV(\mathring{q}(x))$$

We say a numerical scheme is Total Variation Diminishing (TVD) if

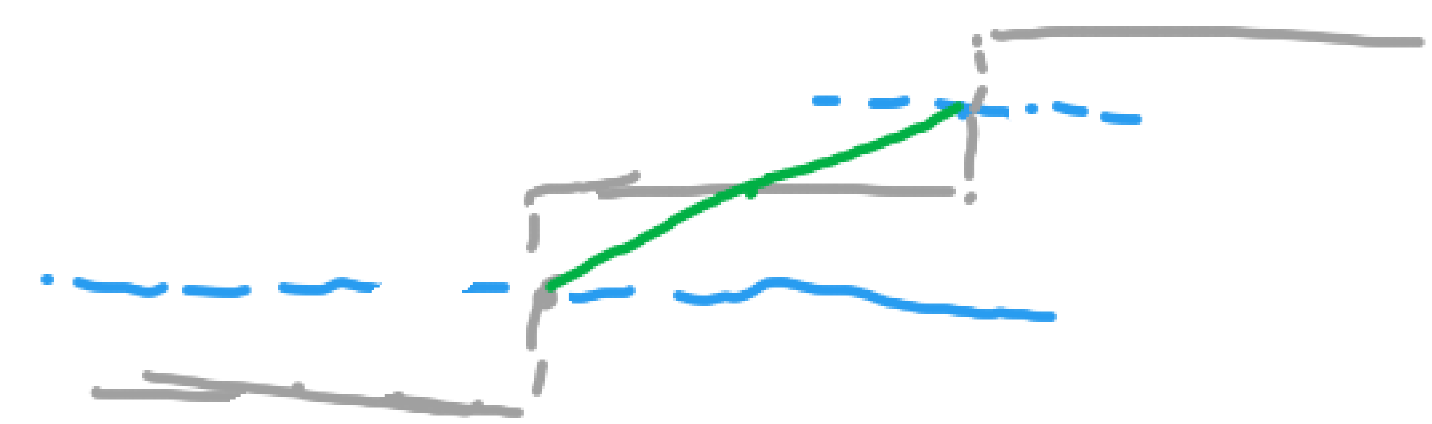
$$TV(Q^{n+1}) \le TV(Q^n)$$

Last time: piecewise linear reconstion

$$\tilde{q}(x) = Q_i^n + \sigma(x - x_i) \qquad x \in (x_{i-1/2}, x_{i+1/2})$$



The linear reconstruction increases the TV.



for $x \in (x_{i-1/2}, x$

We want $\tilde{q}(x) \in \left[\frac{Q_{i-1}+Q_i}{2}, \frac{Q_i+Q_{i+1}}{2}\right]$

We can achieve this if we choose

$$\sigma_{mm} = \text{minmod}\left(\frac{Q_{i+1}-Q_i}{\Delta x}, \frac{Q_i - Q_{i-1}}{\Delta x}\right)$$

where

$$\text{minmod}(a,b) = \frac{\text{sign}(a)+\text{sign}(b)}{2} \cdot \min(|a|,|b|)$$

$$\text{sign}(z) = \begin{cases} 1 & \text{if } z>0 \\ -1 & \text{if } z<0 \\ 0 & \text{if } z=0 \end{cases}$$

$_{i+1/2})$

This yields a TVD scheme.

① Reconstruct $\tilde{q}(x) = Q_i^n + (x-x_i)\sigma_{mm}$

② Solve Riemann problem at each interface using states

$$\lim_{x \to x_{i-1/2}^{\pm}} \tilde{q}(x)$$

③ Evolve in time (integrate with RK method)

The scheme is 2nd order (for smooth solutions) in L_1, L_2.
It is 1st-order accurate in L_∞.

Thm.

Any TVD scheme is at most 2nd-order accurate.

Homework: 6.3

In general minmod is more dissipative than is necessary to enforce TVD.

There are many other 2nd-order accurate TVD slope limiters

For example:

$$\sigma_i = \frac{(Q_{i+1}-Q_i)(Q_i-Q_{i-1})}{Q_{i+1}-Q_{i-1}}\left(\text{sign}(Q_{i+1}-Q_i)+\text{sign}(Q_i-Q_{i-1})\right)$$

(Van Leer)