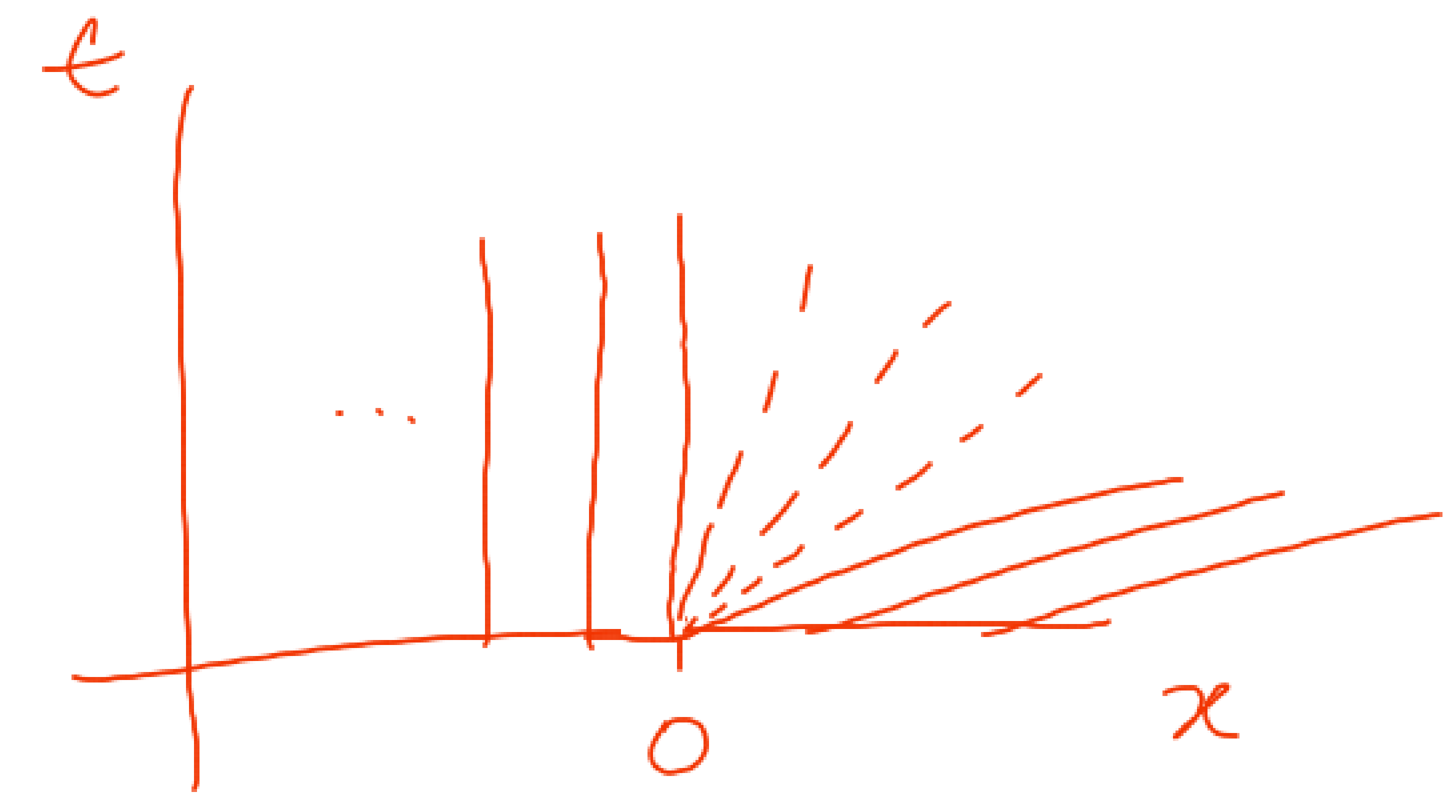
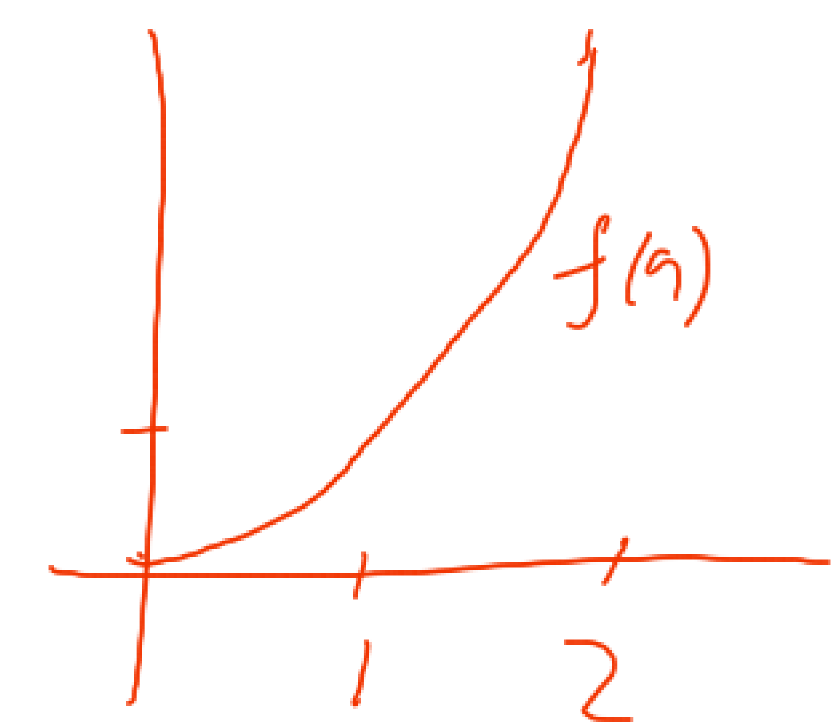
a) $f(q)=q^3$ with $q_\ell = 0$, $q_r = 2$

$f'(q) = 3q^2$, $f'(q_\ell) = 0$, $f'(q_r) = 3(2)^2 = 12$

$q_\ell = 0 < 2 = q_r$



$H = \{(q,y) : 0 \le q \le 2 \text{ and } y \ge f(q)\}$



$q = \tilde{q}\left(\frac{x}{t}\right)$ $\quad f'\left(\tilde{q}\left(\underbrace{\frac{x}{t}}_{\xi}\right)\right) = \frac{x}{t}$

$f'(\tilde{q}(\xi)) = 3[\tilde{q}(\xi)]^2 = \frac{x}{t} \implies \tilde{q} = +\sqrt{\frac{x}{3t}}$ in $0 \le \frac{x}{t} \le 12$

$f(q) = q^3 \quad f'(q) = 3q^2 \qquad q_\ell = 2 \qquad q_r = -1$

$$\tilde{q}(\xi) = \underset{\substack{q_r \le q \le q_\ell \\ -1 \le q \le 2}}{\operatorname{argmax}} \left[\underbrace{q^3 - \xi q}_{h(q)} \right]$$

$h'(q) = 3q^2 - \xi$

$h'(\bar{q}) = 0 \Rightarrow \bar{q} = -\sqrt{\frac{\xi}{3}}$

$h''(\bar{q}) = 6\bar{q}$

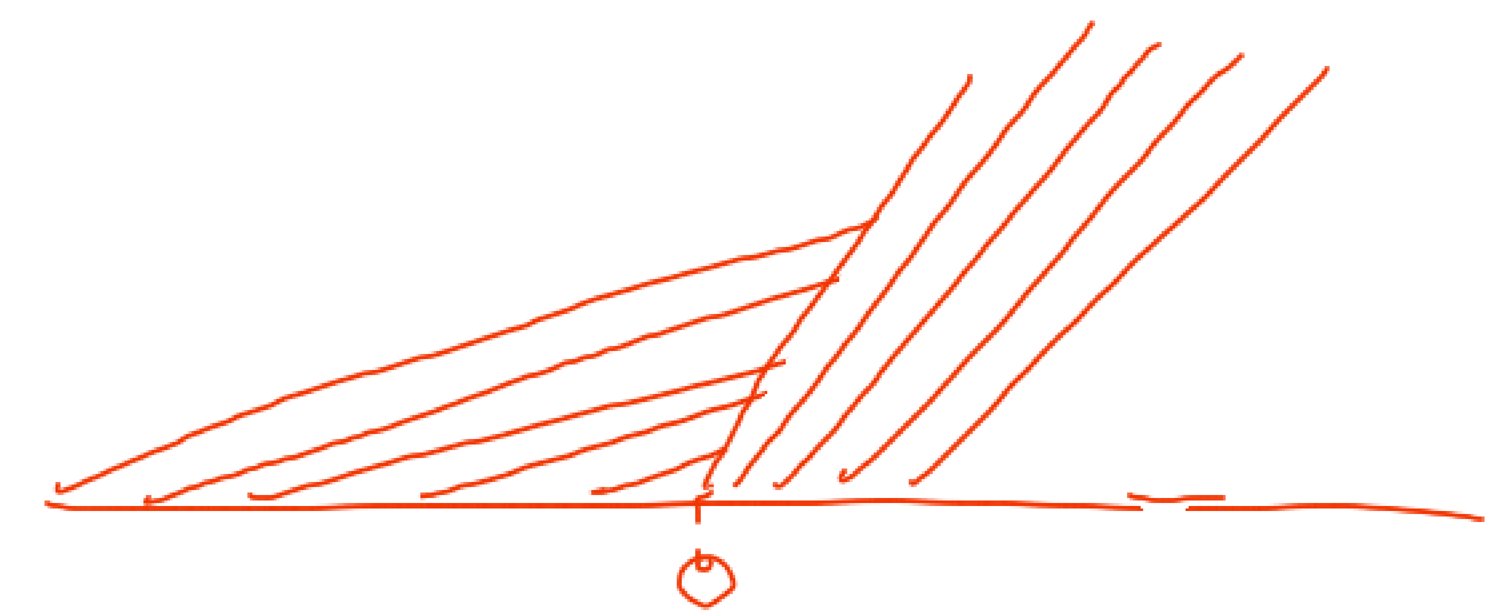
If $\xi < 0$: $\tilde{q}(\xi) = q_\ell = 2$

If $0 \le \xi < 3$: $h(q_\ell) = 8 - 2\xi > 2$

$h(\bar{q}) = \frac{2}{3} \frac{\xi\sqrt{\xi}}{\sqrt{3}} \le \frac{2}{3}$

$h(q_r) = -1 + \xi < 2$

$\tilde{q}(\xi) = q_\ell = 2$

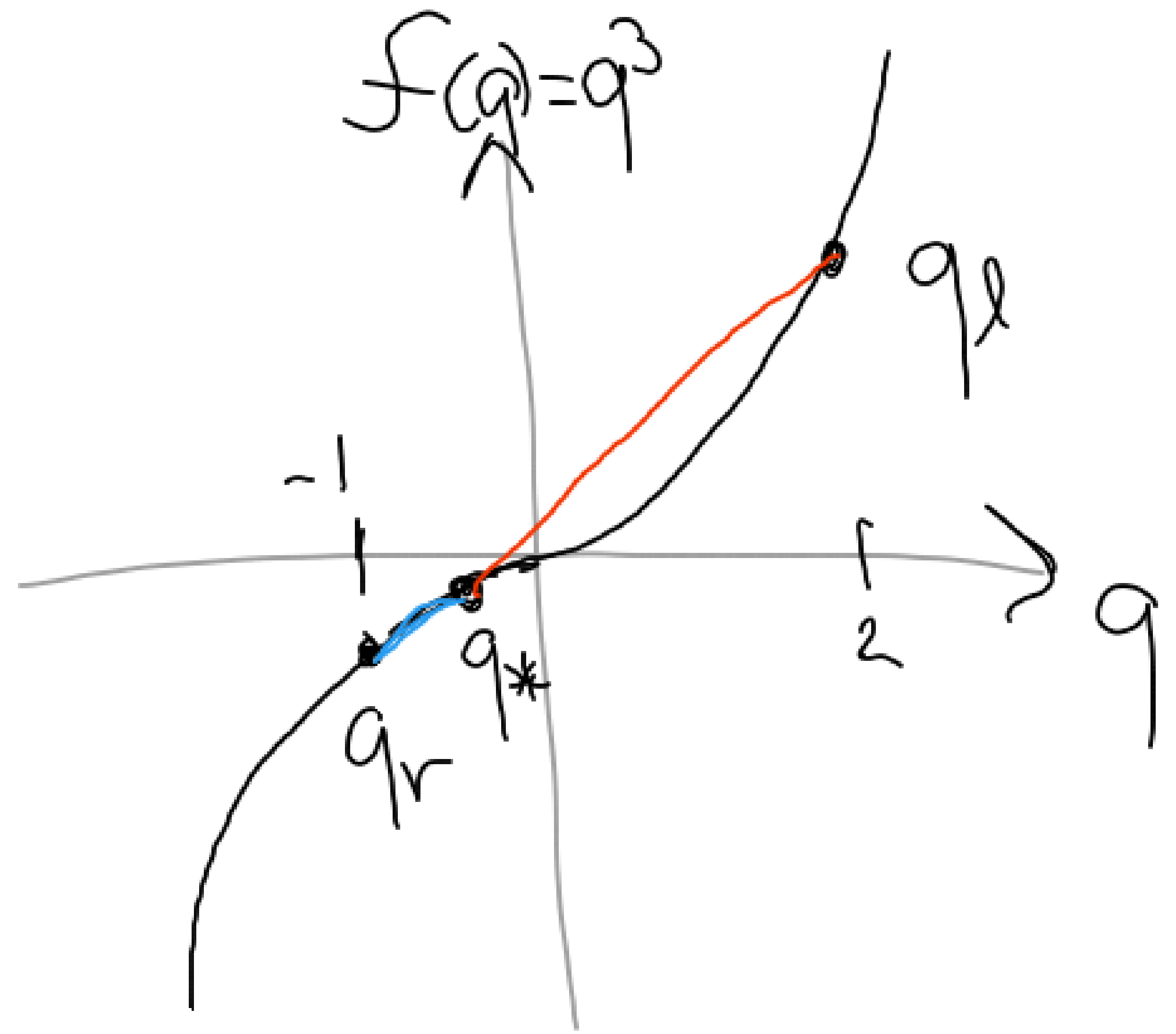


If $\xi > 3$:

$h(q_\ell) = 8 - 2\xi < 2$

$h(q_r) = -1 + \xi > 2$

$\tilde{q}(\xi) = q_r = -1$

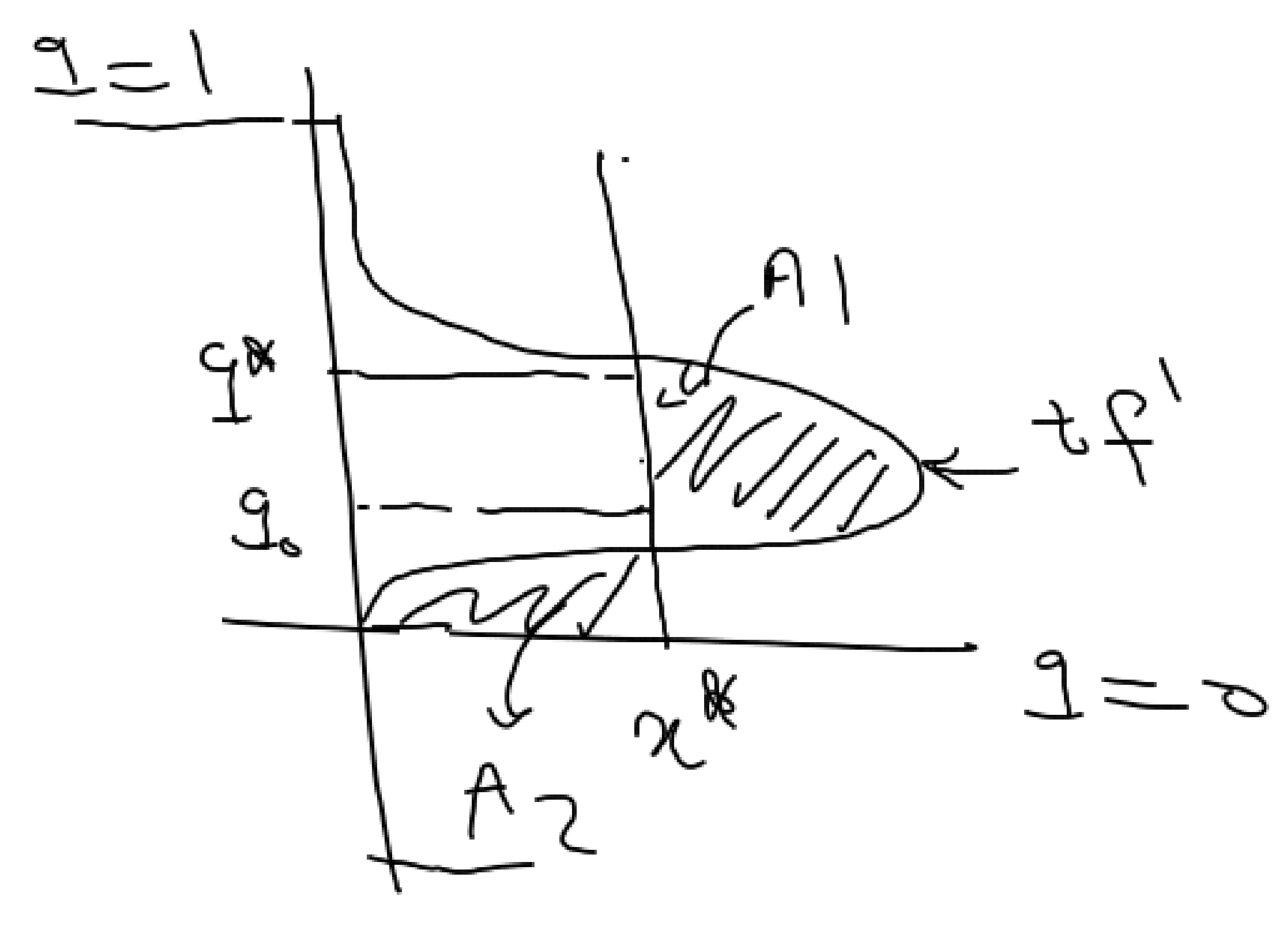


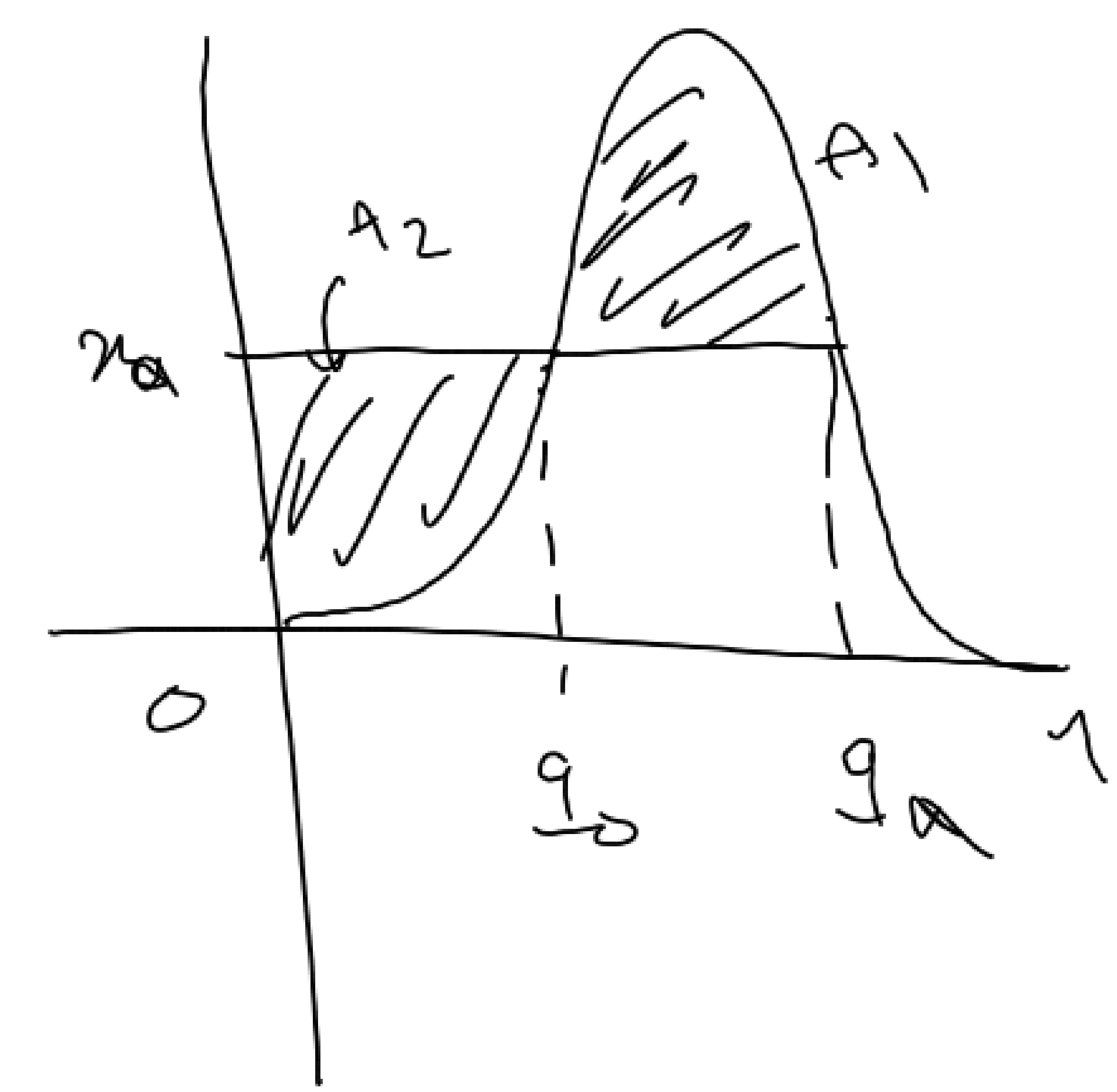
$$\frac{f(q_l)-f(q_*)}{q_l-q_*}=f'(q_*)$$

16.2

$$f(q)=\frac{q^2}{q^2+a(1-q)^2}$$

$$f'(q)=\frac{2a\,q(1-q)}{[q^2+a(1-q)^2]^2}$$





$$A_1 = \int_{q_0}^{q_\alpha} (t f' - x_\alpha)\, dq$$

$$= t\left(f(q_\alpha) - f(q_0)\right) - x_\alpha (q_\alpha - q_0)$$

$$A_2 = \int_{0}^{q_0} (x_\alpha - t f')\, dq$$

$$= x_\alpha q_0 - t\left(f(q_0) - \underbrace{\cancel{f(0)}}_{0}\right)$$

$A_1 = A_2$ if

$$t\left(f(q_\alpha) - \cancel{f(q_0)}\right) - x_\alpha (q_\alpha - \cancel{q_0}) =$$

$$= \cancel{x_\alpha q_0} - \cancel{t f(q_0)}$$

$$t f(q_\alpha) - x_\alpha q_\alpha = 0$$

$$\Rightarrow x_\alpha = \frac{t f(q_\alpha)}{q_\alpha}$$ Differentiate w.r.t q^α

$$f'(q^*) = \frac{x_*}{t}$$

$$x_* = f'(q_*)\, t$$

$$\frac{t f(q_*)}{q_*} = \frac{2a q^*(1-q^*)}{\left[q^2 + a(1-q^*)^2\right]^2}\, t$$

$$\frac{t}{\cancel{q_*}} \cdot \frac{q_*^2}{q_*^2 + a(1-q)^2} = \frac{2a q^*(1-q^*)}{\left[q^{*2} + a(1-q^*)^2\right]^2}\, t$$

$$\cancel{q_*}\, \cancel{t} = \frac{2a \cancel{q^*}(1-q^*)}{\left(q^{*2} + a(1-q^*)^2\right)^2}\, \cancel{t}$$

$$q^{*2} + a(1-q^*)^2 = 2a(1-q^*)$$

$$q_*^2 + a(1 - 2q^* + q^{*2}) = 2a - 2aq^*$$

$$q_*^2 + a - 2aq^* + aq^{*2} = 2a - 2aq^*$$

$$q_* = \pm\sqrt{\frac{a}{1+a}}$$

Substitute back in x_* to find the shock location:

$$x_*(t) = \frac{(1+a)}{2a\sqrt{\frac{a}{1+a}}}\, t$$

from the Rankine – Hugoniot cond:

$$S = \frac{f(\underline{q}_*) - f(0)}{\underline{q}_* - 0} = \frac{f(\underline{q}_*)}{\underline{q}_*}.$$

Non-strict hyperbolicity

$$q_t + f(q)_x = 0$$

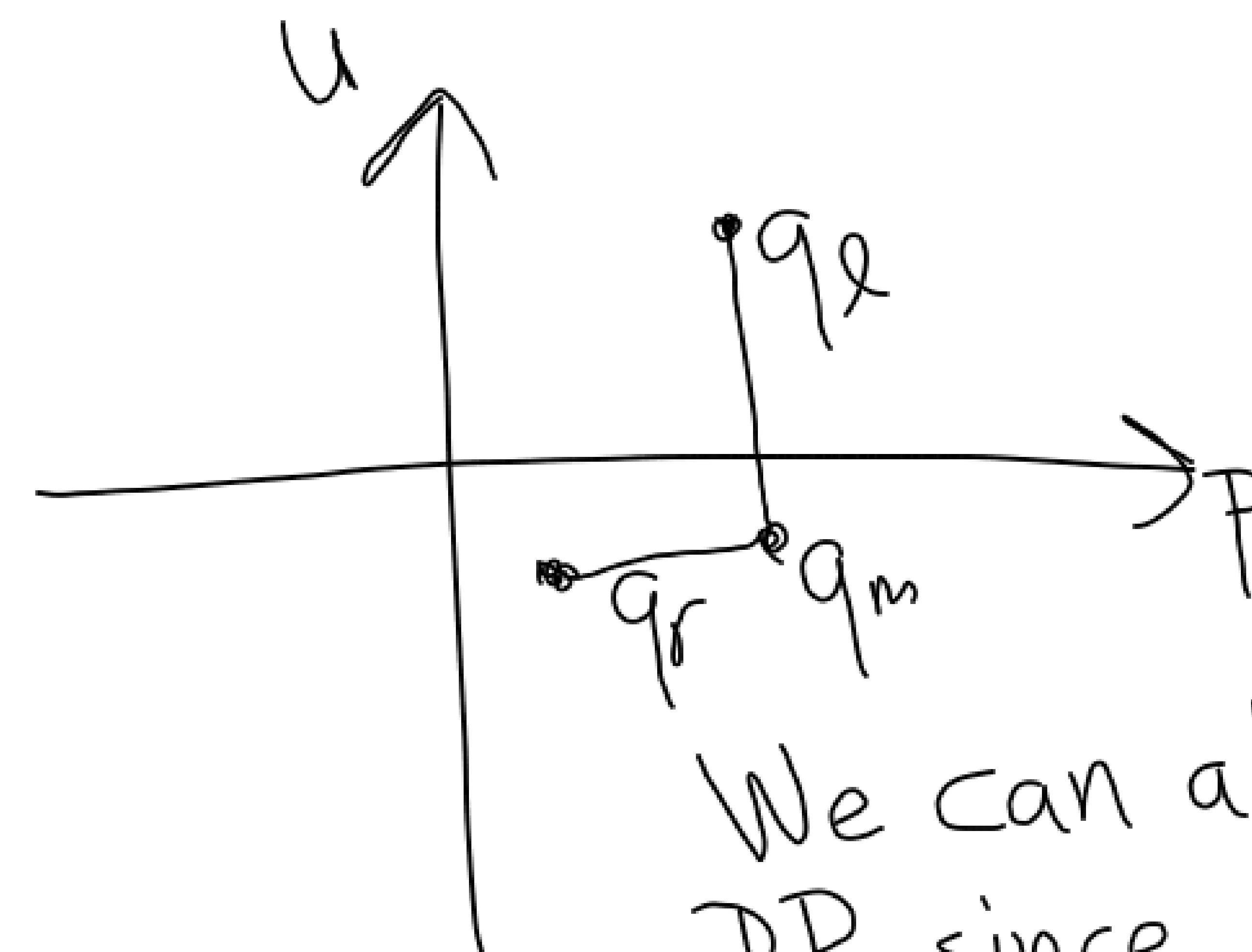
is strictly hyperbolic if the eigenvalues of $f'(q)$ are distinct.

In this case there must be a complete set of lin. indep. eigenvectors.

Acoustics: $\lambda^1 = -\sqrt{K/\rho}$

$\lambda^2 = +\sqrt{K/\rho}$

$r^1 = \begin{bmatrix} 1 \\ \sqrt{K\rho} \end{bmatrix}$ $r^2 = \begin{bmatrix} -1 \\ \sqrt{K\rho} \end{bmatrix}$



We can always solve the RP since $\{r^1, r^2\}$ form a basis for $\mathbb{R}^2$.

Shallow water

$$\lambda^1 = u - \sqrt{gh}$$
$$\lambda^2 = u + \sqrt{gh}$$

Euler (1D)

$$\lambda^1 = u - c$$
$$\lambda^2 = u$$
$$\lambda^3 = u + c$$

In 2D, there is another eigenvalue $\lambda = u$.

Spatially varying flux

$$q_t + f(q,x) = 0$$

LWR Traffic flow

$$\rho_t + (V(x)\rho(1-\rho))_x = 0$$

$V(x)$: speed limit

$0 \le \rho \le 1$

We can write this as $q_t + f(q)_x = 0$ if we set

$$q = \begin{bmatrix} \rho \\ V \end{bmatrix} \quad f(q) = \begin{bmatrix} q^2 q^1 (1-q^1) \\ 0 \end{bmatrix}$$

$$f'(q) = \begin{bmatrix} q^2(1-2q^1) & q^1(1-q^1) \\ 0 & 0 \end{bmatrix}$$

$$f'(q) = \begin{bmatrix} V(1-2\rho) & \rho(1-\rho) \\ 0 & 0 \end{bmatrix}$$

$$\lambda^1 = V(1-2\rho)$$

$$\underbrace{\lambda^2 = 0}_{\text{Zero-Velocity Wave}}$$

The flux must be continuous.

Consider the Riemann problem:

$$(\rho, V) = \begin{cases} (\rho_l, V_l) & x < 0 \\ (\rho_r, V_r) & x > 0 \end{cases}$$

$$\lim_{x \to 0^-} V_l\, \rho(x,t)(1-\rho(x,t)) = \lim_{x \to 0^+} V_r\, \rho(x,t)(1-\rho(x,t))$$

If $V_l \neq V_r$, then ρ must be discontinuous at $x=0$.

Lax Entropy condition

$f'(\rho) = 1 - 2\rho$

Shock adjacent to ρ_l: $f'(\rho_l) > s > f'(\rho_*)$

$$\Rightarrow \rho_* > \rho_l$$

Shock adjacent to ρ_r: $f'(\rho_*) > s > f'(\rho_r)$

$$\rho_r > \rho_*$$