Nonconvex flux

(FVMHP 16.1)

Burgers': $f(q) = \frac{1}{2}q^2$

$f'(q) = q$

$f''(q) = 1$

LWR Traffic flow:

$f(q) = q(1-q)$

$f'(q) = 1-2q$

$f''(q) = -2$

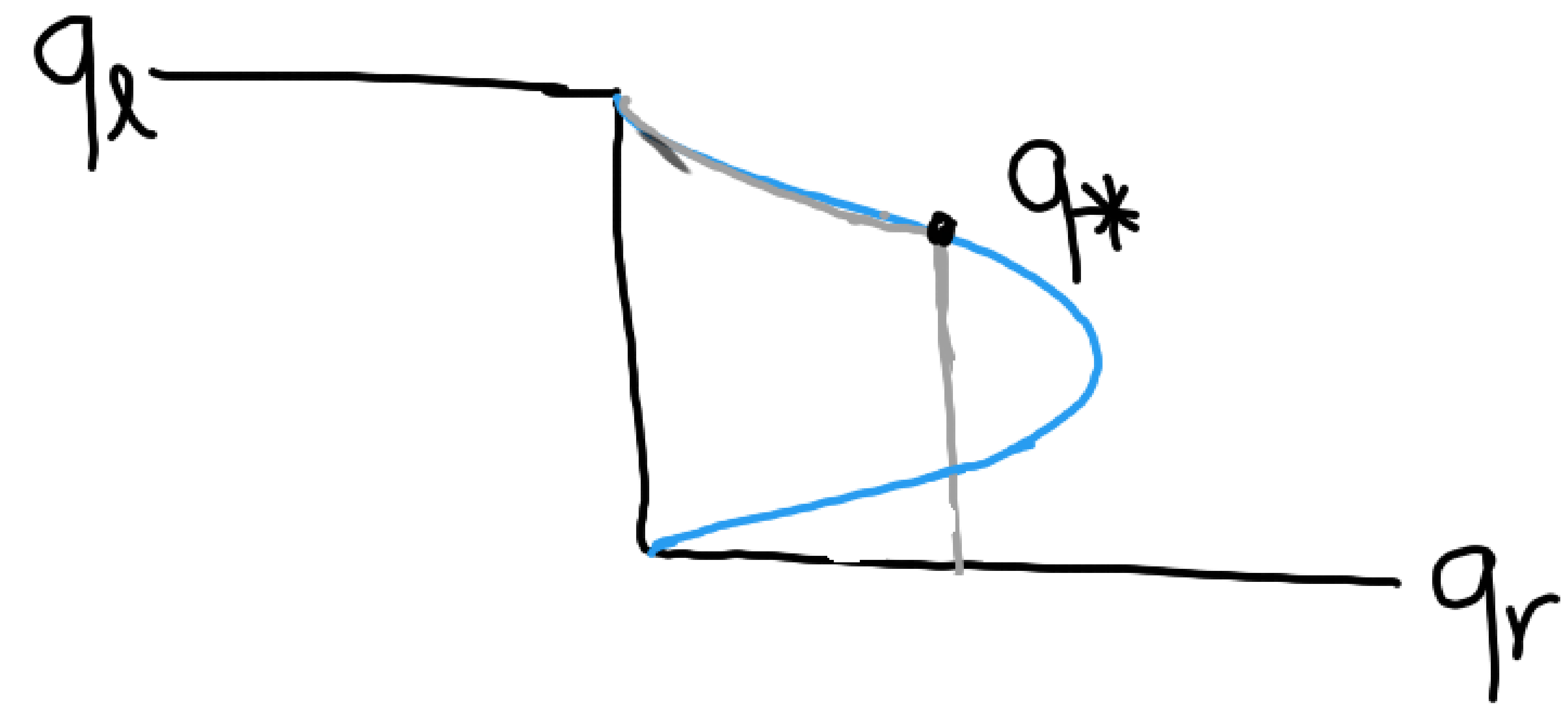
So the Riemann solution has a single shock or rarefaction.

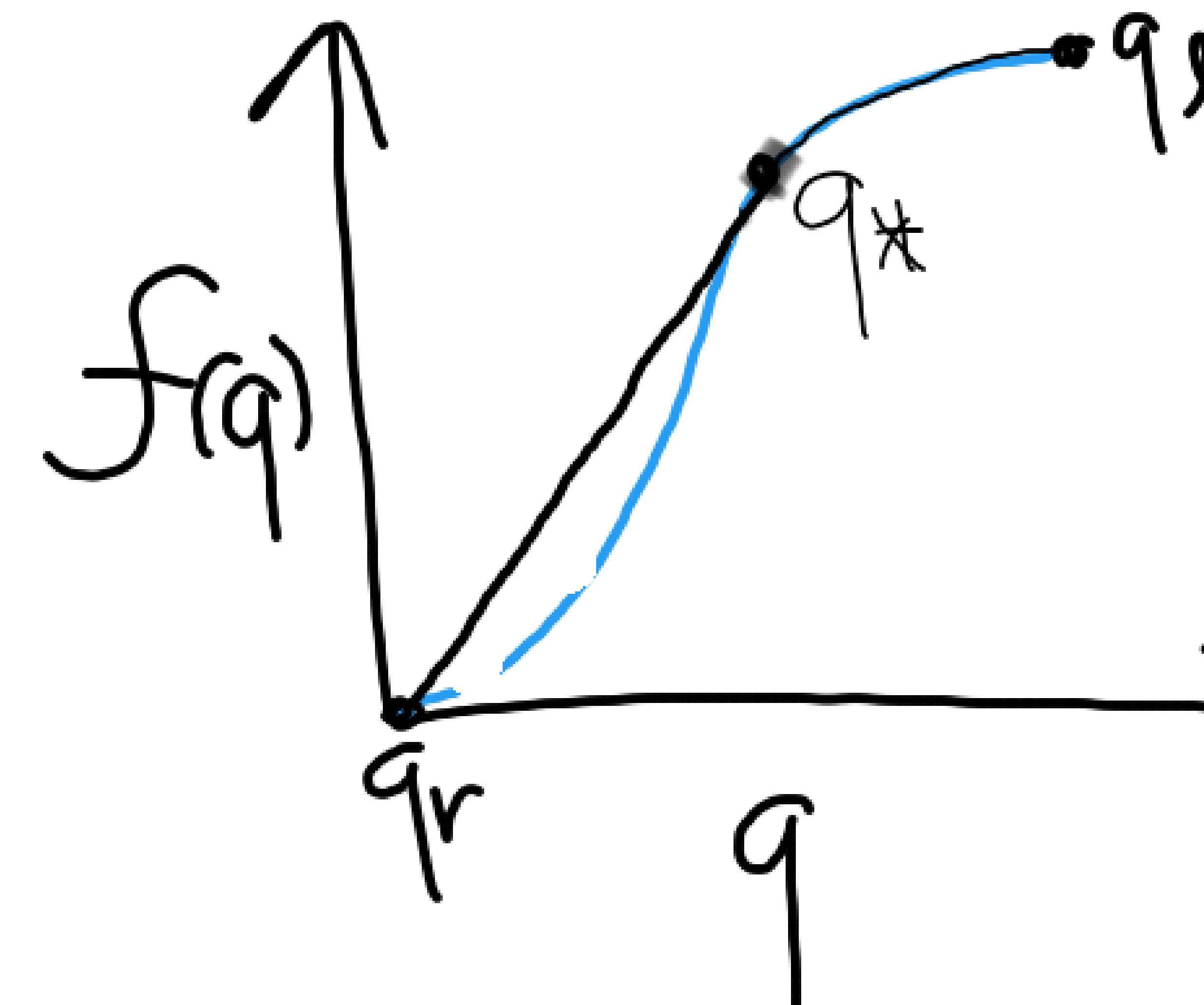
So the char. speed $f'(q)$ varies monotonically.

Buckley-Leverett

Models flow of oil and water.

$$q_t + \Big(\underbrace{\frac{q^2}{q^2 + a(1-q^2)}}_{f(q)}\Big)_x = 0$$





$$f'(q_*) = \frac{f(q_*) - f(q_r)}{q_* - q_r}$$

Oleinik's entropy condition

$$\frac{f(q)-f(q_\ell)}{q-q} \geq s \geq \frac{f(q_r)-f(q)}{q_r-q}$$

must hold for all $q \in (\min(q_\ell,q_r), \max(q_\ell,q_r))$ for a shock connecting states q_ℓ, q_r.

Osher's solution

$q(x,t) = \tilde{q}\left(\frac{x}{t}\right)$ where

$$\tilde{q}(\xi) = \begin{cases} \underset{q\in(q_\ell,q_r)}{\operatorname{argmin}} \left(f(q)-\xi q\right) & \text{if } q_\ell \leq q_r \\ \underset{q\in(q_r,q_\ell)}{\operatorname{argmax}} \left(f(q)-\xi q\right) & \text{if } q_r \leq q_\ell \end{cases}$$

Numerical Methods

We want $f(q(\xi=0))$

We have:

$$\tilde{q}(\xi=0) = \begin{cases} \operatorname{argmin} f(q) & q_\ell \le q_r \\ \operatorname{argmax} f(q) & q_r \le q_\ell \end{cases}$$

For the 2nd-order corrections in LWL

We can use $W = q_r - q_\ell \quad s = \dfrac{f(q_r) - f(q_\ell)}{q_r - q_\ell}$

Note that we can have

$$|f'(q)| > \max(|f'(q_\ell)|, |f'(q_r)|)$$

for some q between q_ℓ and q_r.

So if we choose Δt based on the speed

$$\max_i |f'(Q_i)|$$

it might not satisfy the CFL condition.