Approximate Riemann Solvers

Recall the Lax-Wendroff-LeVeque method

$$Q_i^{n+1} = Q_i^n - \frac{\Delta t}{\Delta x}\left[A^+\Delta Q_{i-1/2} + A^-\Delta Q_{i+1/2}\right] - \frac{\Delta t}{\Delta x}\left[\tilde{F}_{i+1/2} - \tilde{F}_{i-1/2}\right]$$

where $A^\pm \Delta Q_{i-1/2} = \sum_{p=1}^{m} (\lambda^p)^\pm \alpha^p_{i-1/2} r^p$

$A = R\Lambda R^{-1}$

$R\alpha_{i-1/2} = \Delta Q_{i-1/2}$

and $\tilde{F}_{i-1/2} = \frac{1}{2}\sum_{p=1}^{m} |\lambda^p|\left(1 - \frac{\Delta t}{\Delta x}|\lambda^p|\right)\tilde{W}^p_{i-1/2}$

$W^p_{i-1/2} = \alpha^p_{i-1/2} r^p$

① Linearized solvers

$$q_t + f(q)_x = 0$$

$$\downarrow$$

$$q_t + f'(q)q_x = 0$$

$$\downarrow$$

Locally: $\underbrace{q_t + \hat{A}(q)q_x = 0}$

We solve this linear system with

$$\hat{A}_{i-1/2} = \hat{A}(q^{\ell}_{i-1/2}, q^{r}_{i-1/2})$$

Roe's method

What properties should we require for $\hat{A}$?

① Consistency: $\hat{A}(q_\ell, q_r) \to f'(q)$ as $q_\ell, q_r \to q$

② Diagonalizable with real eigenvalues

③ $\hat{A}(q_\ell, q_r)(q_r - q_\ell) = f(q_r) - f(q_\ell)$

Prop. ③ implies conservation, and also that our solution will be exact if the exact solution consists only of a single shock.

If we choose

$$\hat{A}(q^l,q^r)=f'(\hat{q}(q^l,q^r))$$

we get ① and ②.

From ③ we have:

$$f'(\hat{q})=\begin{bmatrix}0 & 1\\ g\hat{h}-\hat{u}^2 & 2\hat{u}\end{bmatrix} \quad q^r-q^l=\begin{bmatrix}h_r-h_l\\ h_ru_r-h_lu_l\end{bmatrix}$$

$$f(q_r)-f(q_l)=\begin{bmatrix}h_ru_r-h_lu_l\\ h_ru_r^2-h_lu_l^2+\frac{1}{2}g(h_r^2-h_l^2)\end{bmatrix}$$

$$h_ru_r-h_lu_l=h_ru_r-h_lu_l$$

$$(g\hat{h}-\hat{u}^2)(h_r-h_l)+2\hat{u}(h_ru_r-h_lu_l)=h_ru_r^2-h_lu_l^2+\frac{1}{2}g(h_r^2-h_l^2)$$

$$g\hat{h}(h_r-h_l)=\frac{1}{2}g(h_r^2-h_l^2)\Rightarrow\hat{h}=\frac{h_r+h_l}{2}$$

$$\hat{u}^2(h_l-h_r)+2\hat{u}(h_ru_r-h_lu_l)-(h_ru_r^2-h_lu_l^2)=0$$

$$\hat{u}=\frac{\cancel{2}(h_ru_r-h_lu_l)}{\cancel{2}(h_l-h_r)}\pm\frac{\sqrt{\cancel{4}(h_ru_r-h_lu_l)^2+\cancel{4}(h_l-h_r)(h_ru_r^2-h_lu_l^2)}}{\cancel{2}(h_l-h_r)}$$

The radical simplifies:

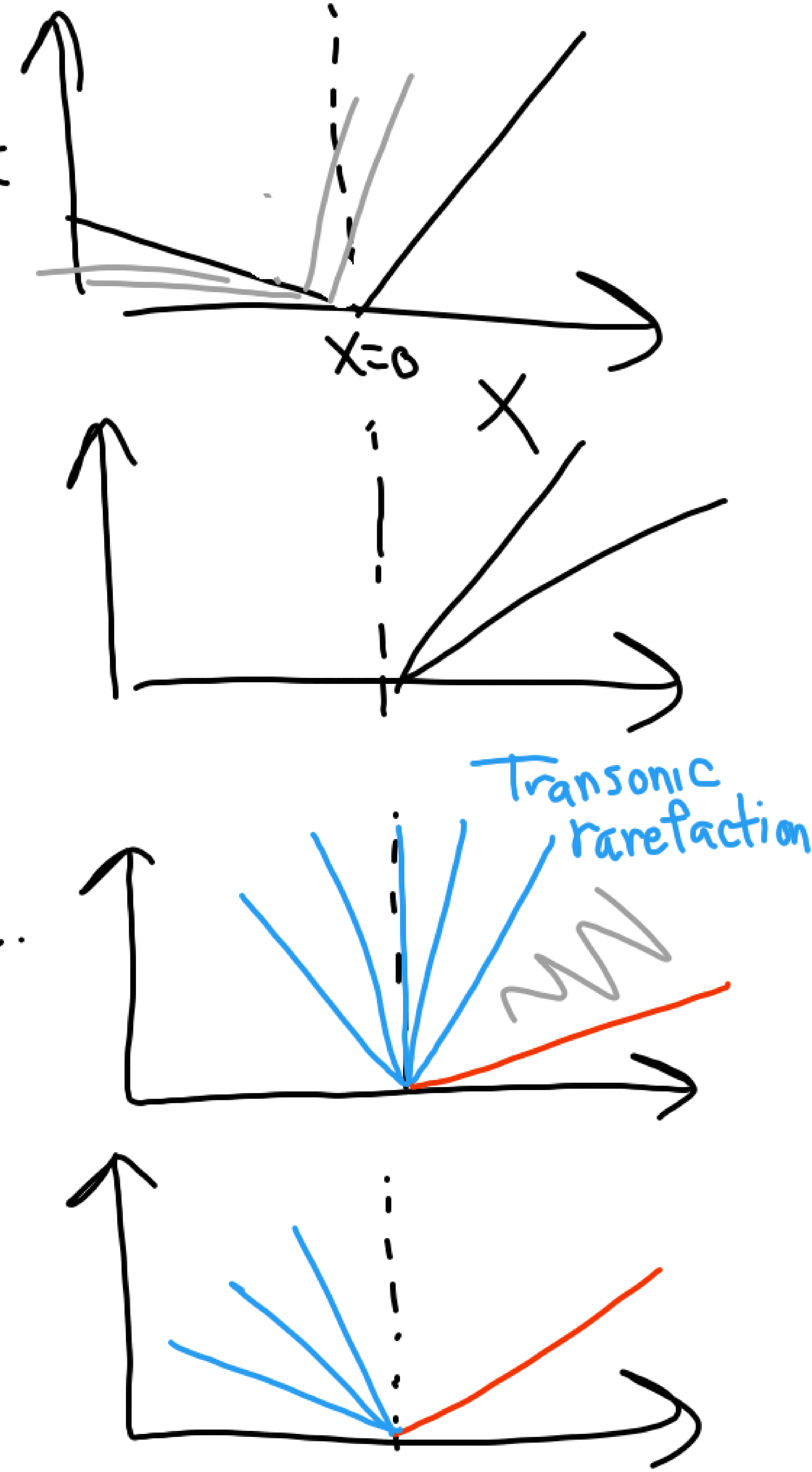
$$\cancel{h_r^2u_r^2}+\cancel{h_l^2u_l^2}-2h_rh_lu_ru_l-\cancel{h_r^2u_r^2}-\cancel{h_l^2u_l^2}+h_rh_l(u_r^2+u_l^2)$$

$$=h_rh_l(u_r-u_l)^2$$

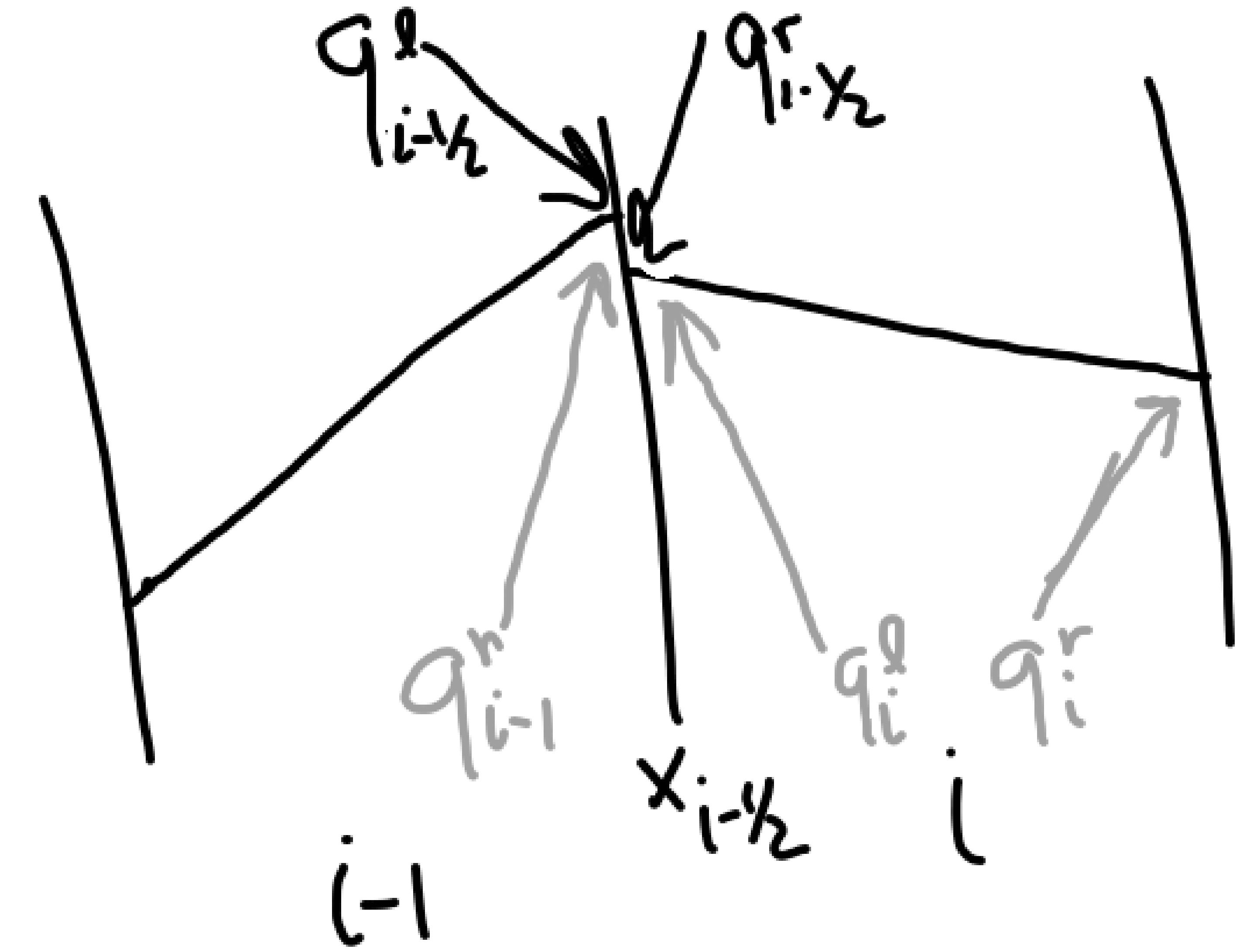
So we get:
$$\hat{u}_\pm=\frac{h_ru_r-h_lu_l\pm\sqrt{h_rh_l}\,(u_r-u_l)}{h_r-h_l}$$

$$\hat{u}=\frac{u_r\sqrt{h_r}+u_l\sqrt{h_l}}{\sqrt{h_r}+\sqrt{h_l}}$$

This is the Roe average. (taking "+").



For problems with transonic rarefactions, Roe's method can lead to entropy-violating solutions.



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