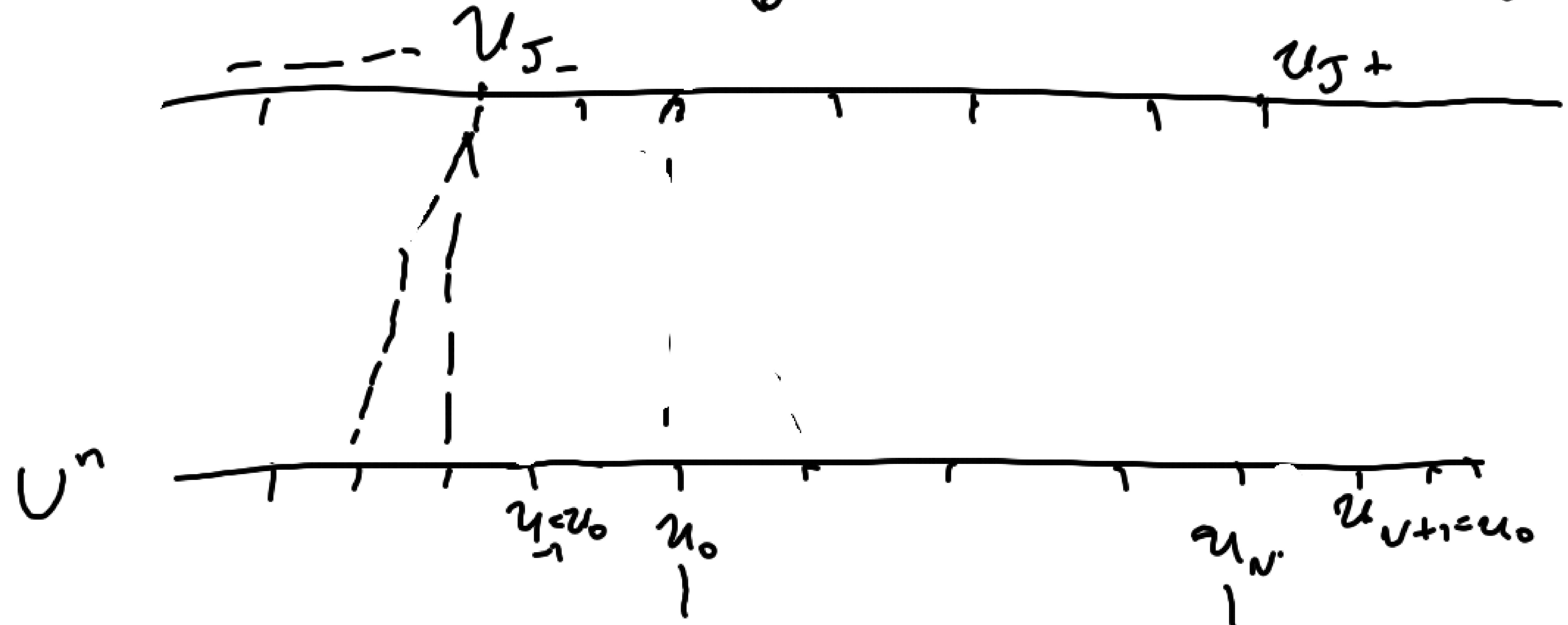
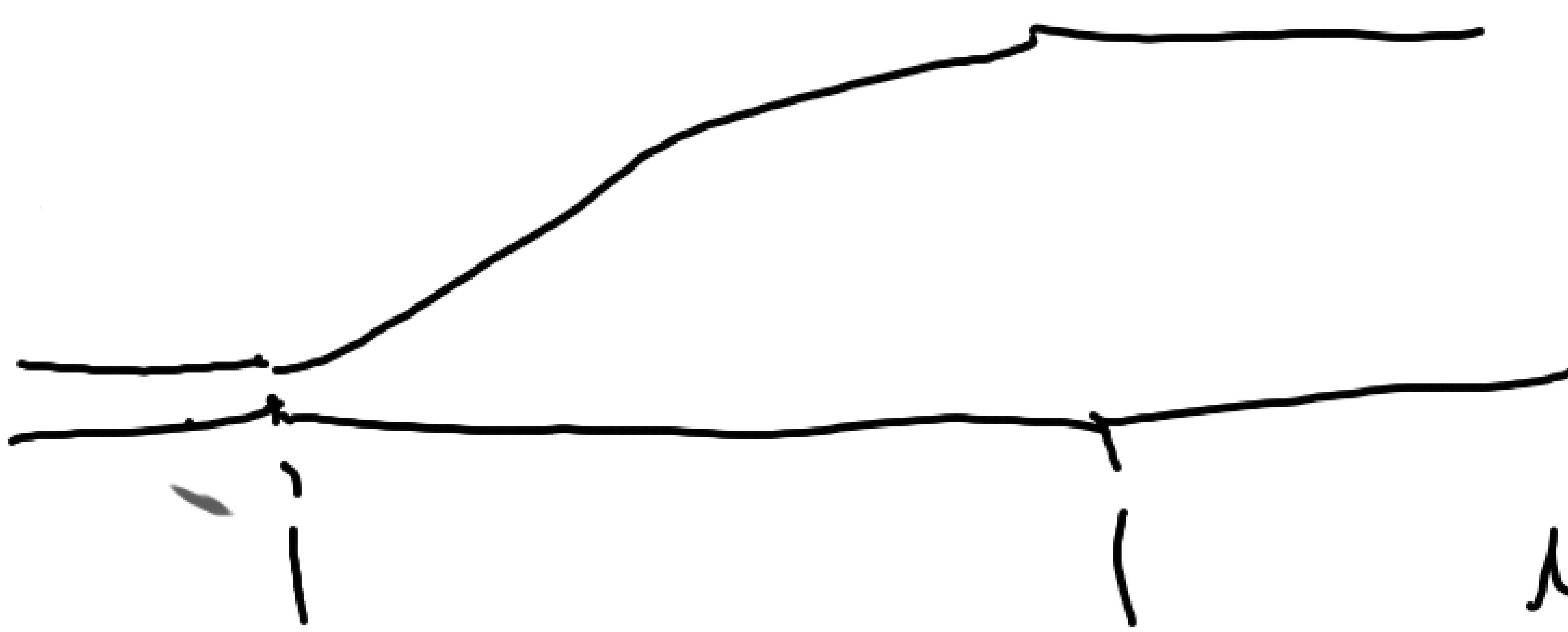
(1) $u_i^{n+1} = u_i^n + \lambda\left(\tilde{f}(u^n_{i+1-k}, \dots, u_{i+1+k}) - f(u^n_{i-k}, \dots, u^n_{i+k})\right)$



(1) TVD $\Longrightarrow$ MP $\forall U^n$

Assume (1) TVD $\wedge$ $\exists \tilde{U}^n$ non-decreasing s.t. $\tilde{u}_i^{n+1} < \tilde{u}_{i-1}^{n+1}$

$$\tilde{u}_i^n = \begin{cases} m, & \text{if } i \leq 0 \\ M, & \text{if } i \geq N \end{cases}$$



$M \geq m$

$$\tilde{u}_i^{n+1} = \begin{cases} m, & \text{if } i \leq J_- \\ M, & \text{if } i \geq J_+ \end{cases}$$

$$D = \{ J_- + 1 \le i \le J_+ : \tilde{u}_i^{n+1} < \tilde{u}_{i-1}^{n+1} \}$$

$$I = \{ J_- + 1 \le i \le J_+ : \tilde{u}_{i-1}^{n+1} \le \tilde{u}_i^{n+1} \}$$

$$TV(\tilde{u}^{n+1}) = \sum_{i=-\infty}^{\infty} |\tilde{u}_i^{n+1} - \tilde{u}_{i-1}^{n+1}|$$

$$= \sum_{i=J_-+1}^{J_+} |\tilde{u}_i^{n+1} - \tilde{u}_{i-1}^{n+1}|$$

$$= \sum_{i\in I} |\tilde{u}_i^{n+1} - \tilde{u}_{i-1}^{n+1}| + \underbrace{\sum_{i\in D} |\tilde{u}_i^{n+1} - \tilde{u}_{i-1}^{n+1}|}_{= C > 0}$$

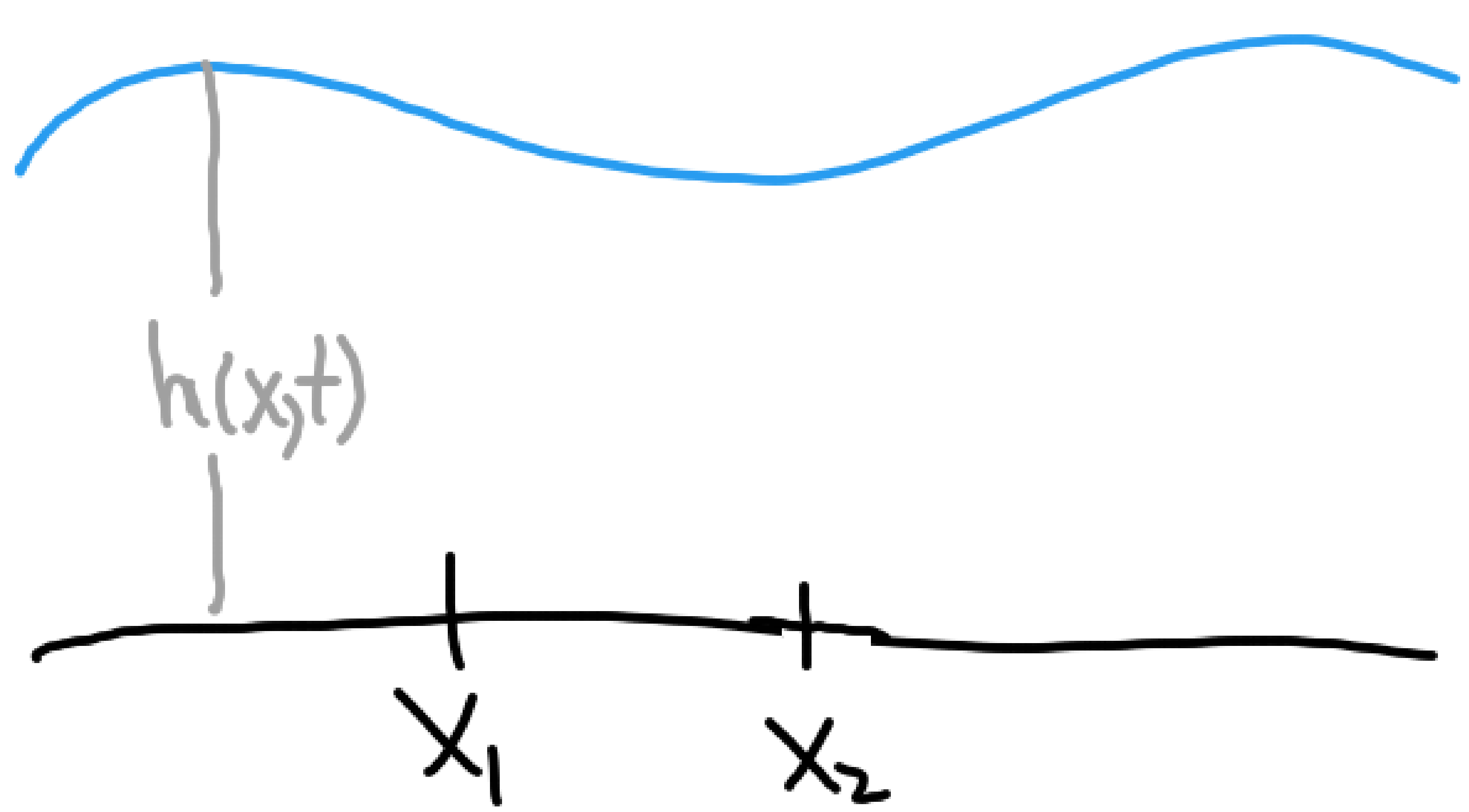
$$\ge |\tilde{u}_{J_+}^{n+1} - \tilde{u}_{J_-}^{n+1}| + C$$

$$= |M - m| + C = TV(\tilde{u}^n) + C$$

The shallow water equations

Incompressible fluid with density $\bar{\rho}$ and depth: $h(x,t)$

Velocity: $u(x,t)$



Mass in $[x_1, x_2]$: $\int_{x_1}^{x_2} \bar{\rho} h(x,t)\,dx$

Flux of mass: $\bar{\rho} h u$

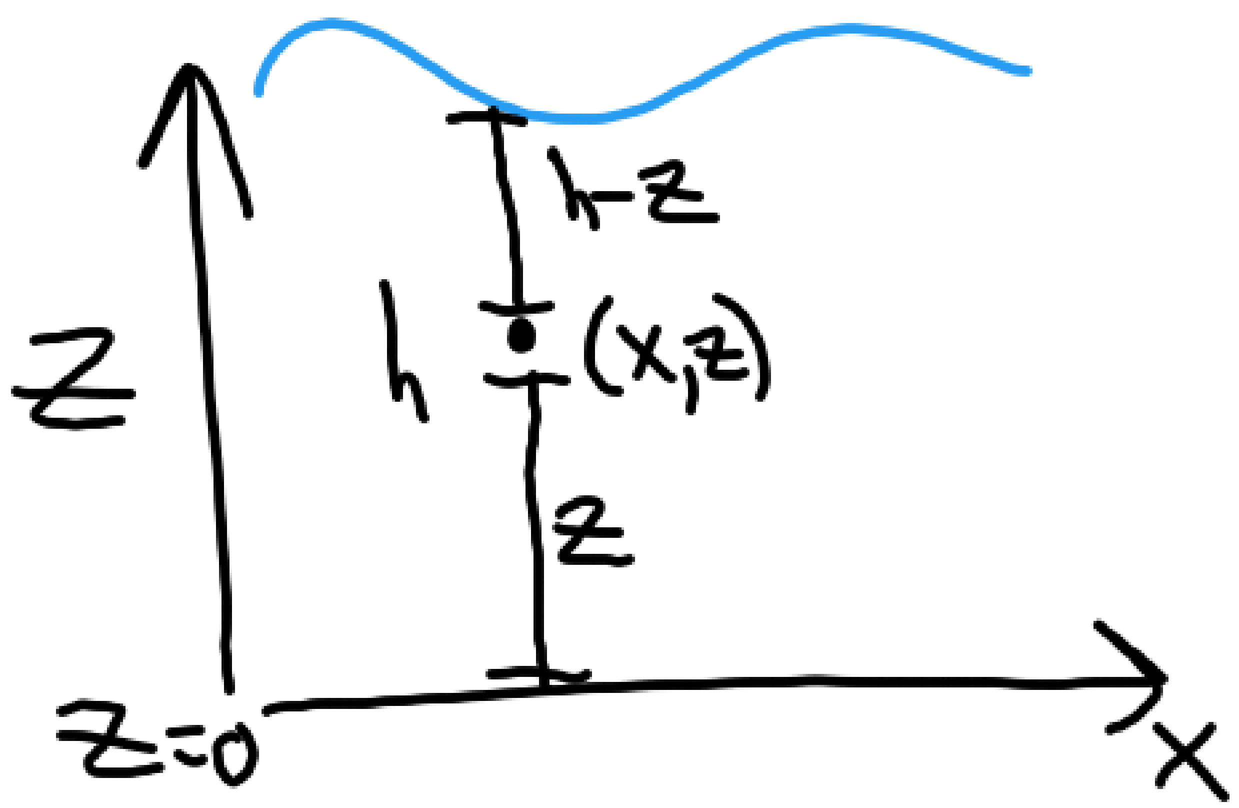
Conservation of mass:

$$(\bar{\rho} h)_t + (\bar{\rho} h u)_x = 0$$

or $h_t + (hu)_x = 0$

Conservation of momentum:

$$(\bar{\rho} h u)_t + (\bar{\rho} h u^2 + P)_x = 0$$



Weight of water above (x,z):

$$p(x,z) = g\bar{\rho}(h-z)$$

The total pressure of the column of water is

$$P(x) = \int_0^h p(x,z)\,dz$$

$$= \int_0^h g\bar{\rho}(h-z)\,dz$$

$$= g\bar{\rho}\left(h^2 - \frac{h^2}{2}\right)$$

$$= \frac{1}{2}g\bar{\rho}h^2$$

So:
$$\boxed{\begin{aligned} &(hu)_t + \left(hu^2 + \tfrac{1}{2}gh^2\right)_x = 0 \\ &h_t + (hu)_x = 0 \end{aligned}}$$

To get this from the Euler equations, we would need to assume that $h \ll \lambda$ = wavelength of a typical wave.

$$q_t + f(q)_x = 0 \implies q_t + f'(q)\, q_x = 0$$

$$q = \begin{bmatrix} h \\ hu \end{bmatrix} \quad f(q) = \begin{bmatrix} hu \\ hu^2 + \frac{1}{2} g h^2 \end{bmatrix}$$

$$q = \begin{bmatrix} q_1 \\ q_2 \end{bmatrix} \quad f(q) = \begin{bmatrix} q_2 \\ \frac{1}{2} g q_1^2 + \frac{q_2^2}{q_1} \end{bmatrix}$$

$$f'(q) = \begin{bmatrix} 0 & 1 \\ g q_1 - \frac{q_2^2}{q_1^2} & 2\frac{q_2}{q_1} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ gh - u^2 & 2u \end{bmatrix}$$

If $q(x,t) = q_0 + \varepsilon \hat{q}(x,t)$:

$$(q_0 + \varepsilon \hat{q}(x,t))_t + f'(q_0 + \varepsilon \hat{q}(x,t))(q_0 + \varepsilon \hat{q}(x,t))_x = 0$$

$$\varepsilon \hat{q}_t + (f'(q_0) + \mathcal{O}(\varepsilon))\, \varepsilon \hat{q}_x = 0$$

$$\hat{q}_t + f'(q_0)\, \hat{q}_x = \mathcal{O}(\varepsilon)$$

So small perturbations propagate with speeds equal to the eigenvalues of $f'(q_0)$.

$$\det(f'(q) - \lambda I) = \det \begin{bmatrix} -\lambda & 1 \\ gh - u^2 & 2u - \lambda \end{bmatrix}$$

$$\lambda(\lambda-2u)-(gh-u^2)=0$$

$$\lambda^2-2u\lambda+u^2-gh=0$$

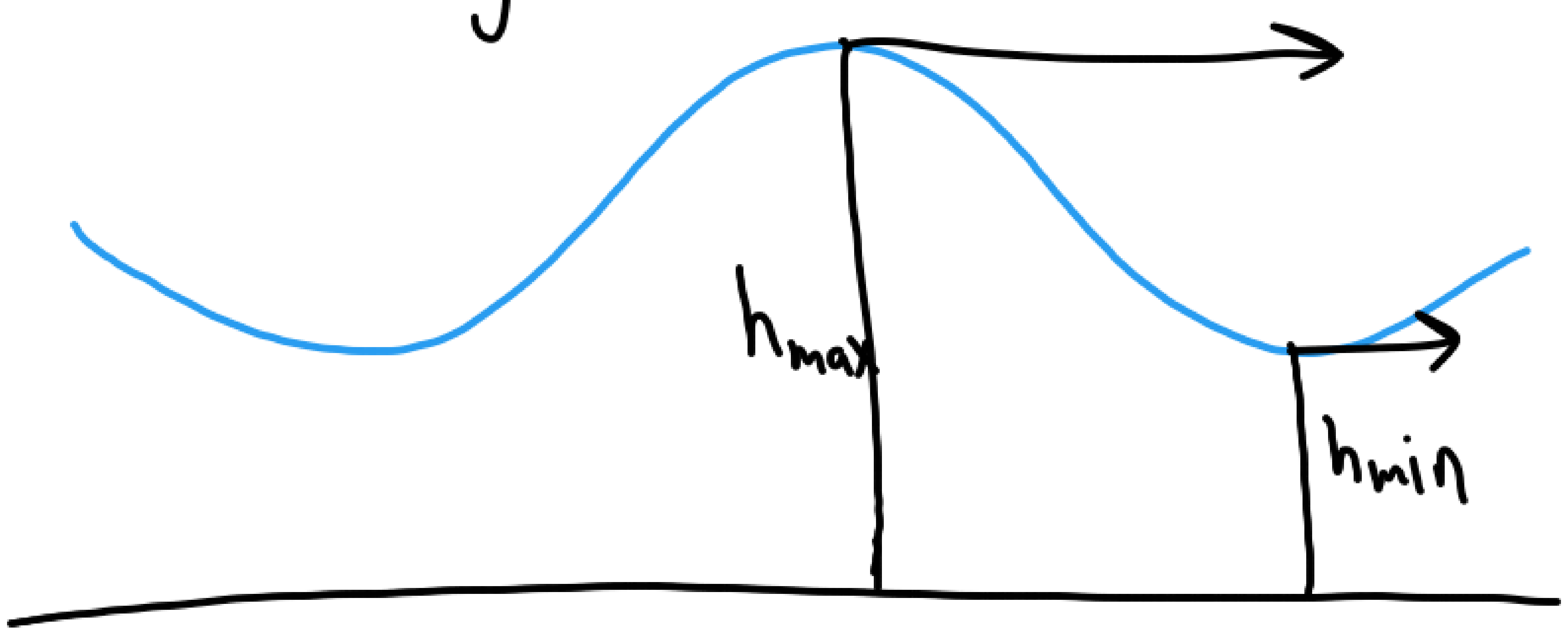
$$\lambda=\frac{\cancel{2}u\pm\sqrt{\cancel{4u^2}-\cancel{4}(\cancel{u^2}-gh)}}{\cancel{2}}$$

$$=u\pm\sqrt{gh}$$

$$\lambda_1=u-\sqrt{gh}$$

$$\lambda_2=u+\sqrt{gh}$$

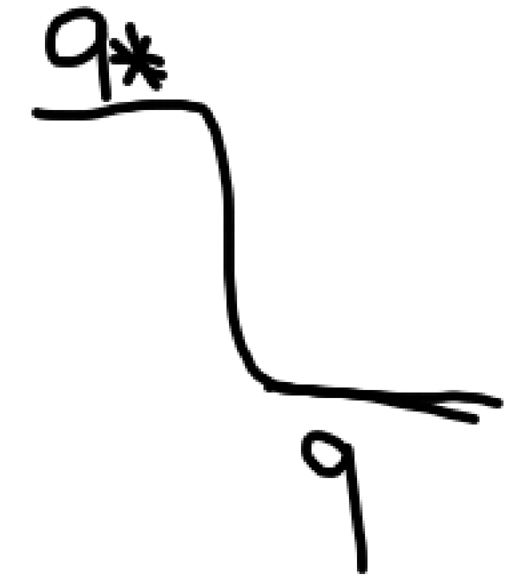
$$r_1=\begin{bmatrix}1\\ u-\sqrt{gh}\end{bmatrix}\qquad r_2=\begin{bmatrix}1\\ u+\sqrt{gh}\end{bmatrix}$$



$$\sqrt{gh_{max}}>\sqrt{gh_{min}}$$

The Rankine-Hugoniot Condition

$$s[q]=[f(q)]$$



$$s(h_*-h)=h_*u_*-hu$$

$$s(h_*u_*-hu)=h_*u_*^2+hu^2+\frac{g}{2}(h_*^2-h^2)$$

$$\Rightarrow s_{1,2}=u_*\mp\sqrt{gh\frac{h_*+h}{2h_*}}$$

Note that $s\to\lambda_{1,2}$ as $(h,u)\to(h_*,u_*)$

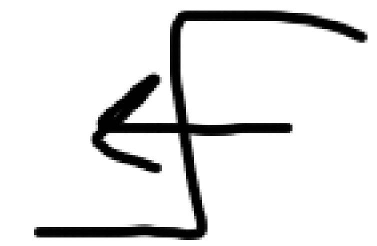
The Lax entropy condition says that for a 1-shock

$$\lambda_1(q_\ell)>s_1>\lambda_1(q_r)$$

$$u_\ell-\sqrt{gh_\ell}>u_\ell-\sqrt{gh_r\frac{h_\ell+h_r}{2h_\ell}}$$

$$\sqrt{h_r\frac{h_\ell+h_r}{2h_\ell}}>\sqrt{h_\ell}$$

$$\sqrt{h_r\frac{h_\ell+h_r}{2}}>h_\ell \iff h_r>h_\ell$$

For a 1-shock: $h_r > h_\ell$ 

2-shock: $h_\ell > h_r$ 

Homework: 13.7 :

13.8