Convergence for scalar conservation laws

We say $q(x,t)$ is a weak solution of

$$q_t + f(q)_x = 0 \qquad (1)$$

if for all $\phi(x,t) \in C_0^1$

$$\int_0^\infty \int_{-\infty}^{\infty} \left(\phi_t q + \phi_x f(q)\right) dx\,dt = -\int_{-\infty}^{\infty} q(x,0)\phi(x,0)\,dx$$

Lax-Wendroff Theorem

Consider a sequence of grids $j=1,2,\ldots$ such that $\Delta x_j \to 0$, $\Delta t_j \to 0$ as $j \to \infty$. Let $Q^{(j)}$ denote a piecewise-constant approximation computed on grid j using a method consistent with (1) and conservative:

$$Q_i^{n+1} = Q_i^n - \frac{\Delta t}{\Delta x}\left(F_{i+1/2}^n - F_{i-1/2}^n\right)$$

Assume there exist $R(t) > 0$ s.t.

$$TV(Q^{(j)}(x,t)) < R(t)$$

for all j.

Assume $\exists\ \tilde{q}(x,t)$ s.t.

$$\lim_{j\to\infty} \|Q^{(j)} - \tilde{q}\|_{1,\Omega} = 0 \quad \forall\, \Omega = [a,b]\times[0,T]$$

Where

$$\|g(x)\|_{1,\Omega} = \int_0^T \int_a^b |g(x)|\,dx\,dt.$$

Then $\tilde{q}(x,t)$ is a weak solution of (1).

Proof

Let $\phi \in C_0^1$ be given and let $\Phi_i^n = \phi(x_i, t_n)$.

$$\sum_{n=0}^{\infty} \sum_{i=-\infty}^{\infty} \Phi_i^n (Q_i^{n+1} - Q_i^n) = -\frac{\Delta t}{\Delta x} \sum_n \sum_i \Phi_i^n \left(F_{i+1/2}^n - F_{i-1/2}^n\right)$$

(Sums are effectively finite since ϕ has compact support)

Now do summation by parts:

$$\sum_{n=0}^{N} \Phi_i^n (Q_i^{n+1} - Q_i^n) = \Phi_i^0 (Q_i^1 - Q_i^0)$$

$$+ \Phi_i^1 (Q_i^2 - Q_i^1) + \dots + \Phi_i^N (Q_i^{N+1} - Q_i^N)$$

$$= -\Phi_i^0 Q_i^0 + Q_i^1(\Phi_i^0 - \Phi_i^1) + Q_i^2(\Phi_i^1 - \Phi_i^2) + \cdots + Q_i^N(\Phi_i^{N-1} - \Phi_i^N) + Q_i^{N+1}\Phi_i^N$$

$$= -\Phi_i^0 Q_i^0 - \sum_{n=1}^{N} Q_i^n(\Phi_i^n - \Phi_i^{n-1}) + \cancelto{0}{Q_i^{N+1}\Phi_i^N}$$

These expressions converge to those in the dfn. of weak solution as $\Delta x, \Delta t \to 0$.

Similarly,

$$-\sum_{i=-\infty}^{\infty} \Phi_i^n \left(F_{i+1/2}^n - F_{i-1/2}^n\right) = \sum_{i=-\infty}^{\infty} F_{i+1/2}^n \left(\Phi_{i+1}^n - \Phi_i^n\right)$$

So we have

$$-\sum_n \sum_i Q_i^n(\Phi_i^n - \Phi_i^{n-1}) - \sum_i \Phi_i^0 Q_i^0 = \sum_n \sum_i F_{i+1/2}(\Phi_{i+1}^n - \Phi_i^n)\frac{\Delta t}{\Delta x}$$

$$\Delta x \Delta t \left[\underbrace{\sum_n \sum_i \frac{\Phi_i^n - \Phi_i^{n-1}}{\Delta t} Q_i^n}_{\approx \iint \phi_t q \, dx\, dt} + \underbrace{\sum_n \sum_i \frac{\Phi_{i+1}^n - \Phi_i^n}{\Delta x} F_{i+1/2}^n}_{\iint \phi_x f(q)\, dx\, dt} \right] = \underbrace{-\Delta x \sum_i \Phi_i^0 Q_i^0}_{-\int \phi(x,0) q(x,0)\, dx}$$

To prove convergence to the vanishing-viscosity weak solution, we need:

1. Convergence (to something)
2. "something" is a weak soln. (Lax-Wendroff)
3. "something" is entropy-satisfying

How to show part 1?
(for scalar problems)

For linear PDEs, we prove convergence of linear discretizations by showing consistency + stability (Lax-Richtmeyer 1956).

For nonlinear PDEs with strong solutions, the same approach can be used (Strang 1964).

Compactness

A space K is compact if every sequence

$$\{K_1, K_2, \ldots\} \subset K$$

has a subsequence

$$\{K_{i_1}, K_{i_2}, \ldots\}$$

that converges to an element of K

Examples:

An closed, bounded set in $\mathbb{R}^n$ is compact.

Consider $L_1 = \{V(x) : \underbrace{\int_{-\infty}^{\infty} |V(x)| dx}_{\|V\|_1} < \infty\}$

Let

$$V_j(x) = \begin{cases} 1 & \text{if } j < x < j+1 \\ 0 & \text{else} \end{cases}$$

Then $V_j(x) \in B_1 = \{V(x) : \|V(x)\|_1 \leq 1\}$

But there is no convergent subsequence.

Consider: $L_{1,\pi} = \{V(x) : \|V\|_1 < \infty \text{ and } V(x) = 0 \ \forall |x| > \pi\}$

Let

$$V_j(x) = \begin{cases} \sin(jx) & x \in [-\pi, \pi] \\ 0 & \text{else} \end{cases}$$

Consider $L_{1,M,R} = \{ v(x) : \|v\|_1 < \infty,\ \text{supp}(v) \subset [-M,M],\ TV(v) \le R \}$

This space is compact.

Theorem: Let a consistent and conservative discretization $Q_i^{n+1} = Q_i^n - \frac{\Delta t}{\Delta x}\left(F_{i+1/2}^n - F_{i-1/2}^n\right)$

of the scalar conservation law $q_t + f(q)_x = 0$ be given. Assume the numerical flux is Lipschitz continuous and that $TV(Q^n) < R$ indep. of the mesh.

Then $$\lim_{\Delta t, \Delta x \to 0} \min_{q \in W} \|Q - q\|_1 = 0.$$

Where W is the set of all weak solutions.

Proof. By way of contradiction, suppose that $\exists\, \varepsilon > 0$ s.t.

$$\min_{q \in W} \|Q^{(j)} - q\|_1 > \varepsilon$$

for all small enough $\Delta t, \Delta x$.

Consider a sequence of $Q^{(j)}$ with $j \to \infty$. This lies in a compact set, so there is a convergent subsequence:

$$Q^{(j_1)}, Q^{(j_2)}, \ldots\ldots,$$

Then $\lim_{i \to \infty} \|Q^{(j_i)} - v\| = 0$

By the Lax-Wendroff Thm., v must be a weak solution (contradiction).