$$\lambda_1 = u - \sqrt{gh}$$

$$\lambda_2 = u + \sqrt{gh}$$

$$r_1 = \begin{bmatrix} 1 \\ u - \sqrt{gh} \end{bmatrix}$$

$$r_2 = \begin{bmatrix} 1 \\ u + \sqrt{gh} \end{bmatrix}$$

Integral Curves

$$q(x,t)_x = \alpha(x)\, r_p(q(x,t))$$

$$\begin{aligned} q_t + f'(q)q_x &= q_t + f'(q)\alpha(x)\, r_p(q(x,t)) \\ &= q_t + \alpha(x)\lambda_p(q(x,t))\, r_p(q(x,t)) \end{aligned}$$

Simple wave solution

Behaves like solution of scalar conservation law.

We can eliminate α by using an appropriate parameterization:

$$q(x,t) = \tilde{q}(\xi)$$

$$q_x = \tilde{q}'(\xi) = r_p(\tilde{q}(\xi))$$

$$\tilde{h}'(\xi) = 1 \Rightarrow \xi = h$$

$$(\widetilde{hu})'(\xi) = \tilde{u} \pm \sqrt{g\tilde{h}} = \tilde{u} \pm \sqrt{g\xi}$$

$$\tilde{q}_2'(\xi) = \frac{\tilde{q}_2}{\tilde{q}_1} \pm \sqrt{g\tilde{q}_1}$$

The solution is

$$\underbrace{u \mp 2\sqrt{gh}}_{\text{Riemann Invariants}} = u_* \pm \sqrt{2gh_*}$$

$$w_1 = u + 2\sqrt{gh}$$

$$w_2 = u - 2\sqrt{gh}$$

Centered Rarefaction Waves

$$q(x,t=0)=\begin{cases} q_l & x<0 \\ q_r & x>0 \end{cases}$$

The solution is a similarity solution:

$$q(x,t)=\tilde{q}\left(\frac{x}{t}\right)=\tilde{q}(\xi)$$

For a rarefaction in char. family p, we must have

$$\lambda_p(\tilde{q}(\xi))=\xi$$

i.e. $u(\xi)\pm\sqrt{gh(\xi)}=\xi$

Using this with the corresponding R.I., we can solve for h, u in the rarefaction.

For $\xi=0$ we get

$$h=\frac{u^2}{g}$$

To find the Riemann solution:

1. Determine nature of each wave
2. Solve for middle state
3. Solve for structure inside rarefaction.