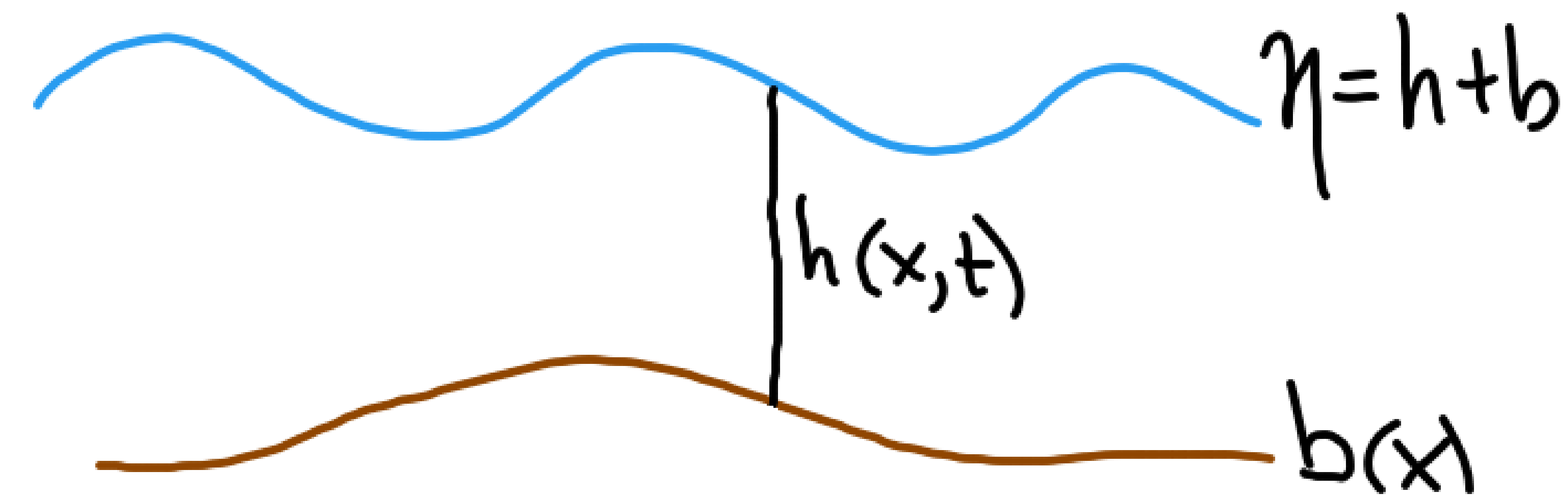
Balance Laws

$$\underbrace{q_t + f(q)_x}_{\text{hyperbolic}} = \underbrace{\psi(q)}_{\text{source term(s)}}$$

Examples:

1. Fluid dynamics with
   - chemical reactions
   - viscosity/diffusion/heat transfer (parabolic)
   - gravity/self-gravity

2. Shallow water equations with varying bathymetry



$$h_t + (hu)_x = 0$$

$$(hu)_t + \left(hu^2 + \tfrac{1}{2}gh^2\right)_x = -ghb_x$$

If $u(x,t) = 0$ and $\eta(x,t) = \bar{\eta}$ then h, u are constant in time.

③ Dispersive wave equations. e.g. Korteweg-de Vries (KdV):

$$u_t + \left(\tfrac{1}{2}u^2\right)_x = -u_{xxx}$$

Model for long-wavelength water waves

Consider

$$q_t + a q_x = -\beta q \qquad a>0$$

1st-order method:

$$Q_i^{n+1} = Q_i^n - a\frac{\Delta t}{\Delta x}\left(Q_i^n - Q_{i-1}^n\right) - \Delta t \beta Q_i^n$$

Lax-Wendroff 2nd-order

$$q_t = \underbrace{(-a\partial_x - \beta)}_{\mathcal{L}} q = \mathcal{L} q$$

$$q(x, t+\Delta t) = e^{\Delta t \mathcal{L}} q(x,t)$$

$$e^{\Delta t \mathcal{L}} = 1 + \Delta t \mathcal{L} + \frac{\Delta t^2}{2}\mathcal{L}^2 + \cdots$$

$$= 1 - (a\partial_x + \beta)\Delta t + \frac{\Delta t^2}{2}(a\partial_x + \beta)^2 + \mathcal{O}(\Delta t^3)$$

$$= 1 - \Delta t(a\partial_x + \beta) + \frac{\Delta t^2}{2}\left(a^2\partial_x^2 + 2a\beta\partial_x + \beta^2\right) + \mathcal{O}(\Delta t^3)$$

So a 2nd-order method is

$$Q_i^{n+1} = Q_i^n - a\frac{\Delta t}{2\Delta x}\left(Q_{i+1}^n - Q_{i-1}^n\right) - \Delta t \beta Q_i^n + a^2\frac{\Delta t^2}{2\Delta x^2}\left(Q_{i+1}^n - 2Q_i^n + Q_{i-1}^n\right)$$

$$+ \underbrace{\frac{\Delta t^2}{2\Delta x} a\beta\left(Q_{i+1}^n - Q_{i-1}^n\right)}_{\text{interaction of hyp. and source term}} + \frac{\Delta t^2}{2}\beta^2 Q_i^n$$

Operator splitting methods
("fractional step")

Idea: alternate between solving

$$q_t + f(q)_x = 0$$

and $q_t = \psi(q)$

For example:

$$Q_i^* = Q_i^n - a\frac{\Delta t}{\Delta x}\left(Q_i^n - Q_{i-1}^n\right)$$

$$Q_i^{n+1} = Q_i^* - \Delta t B Q_i^*$$

$$Q_i^{n+1} = Q_i^n - a\frac{\Delta t}{\Delta x}\left(Q_i^n - Q_{i-1}^n\right) - \Delta t B Q_i^n$$

$$- \frac{\Delta t^2}{\Delta x} a B \left(Q_i^n - Q_{i-1}^n\right)$$

1st-order in time.

Consider

$$q_t = (A+B)q$$

$$q(x,t+\Delta t) = e^{\Delta t(A+B)} q(x,t)$$

$$= \left(1 + \Delta t(A+B) + \frac{\Delta t^2}{2}(A+B)^2\right) q(x,t) + O(\Delta t^3)$$

$$= 1 + \Delta t(A+B) + \Delta t^2\left(\frac{1}{2}A^2 + \frac{1}{2}B^2 + \frac{1}{2}BA + \frac{1}{2}AB\right)$$

Suppose we alternately solve

$$q_t = Aq$$

and $q_t = Bq$

exactly.

$(A+B)(A+B)$

This method is known as Lie-Trotter splitting or Godunov splitting.

Then $Q^{n+1} = e^{\Delta t B} e^{\Delta t A} Q^n$

$$Q^{n+1} = \left(I + \Delta t B + \frac{\Delta t^2}{2} B^2\right)\left(I + \Delta t A + \frac{\Delta t^2}{2} A^2\right) Q^n + \mathcal{O}(\Delta t^3)$$

$$= \left[I + \Delta t(A+B) + \Delta t^2\left(\tfrac{1}{2}A^2 + \tfrac{1}{2}B^2 + BA\right)\right] Q^n + \mathcal{O}(\Delta t^3)$$

This is 1st-order accurate unless A, B commute

Strang splitting

① Solve $q_t = Aq$ with step size $\frac{\Delta t}{2}$

② Solve $q_t = Bq$ " " " Δt

③ Solve $q_t = Aq$ " " " $\frac{\Delta t}{2}$

$$Q^{n+1} = e^{\frac{\Delta t}{2}A} e^{\Delta t B} e^{\frac{\Delta t}{2}A} Q^n$$

2nd-order accurate

Two steps: $Q^{n+2} = \left(e^{\frac{\Delta t}{2}A} e^{\Delta t B} e^{\frac{\Delta t}{2}A} e^{\frac{\Delta t}{2}A} e^{\Delta t B} e^{\frac{\Delta t}{2}A}\right) Q^n$

$$= e^{\frac{\Delta t}{2}A} e^{\Delta t B} e^{\Delta t A} e^{\Delta t B} e^{\frac{\Delta t}{2}A} Q^n$$

Homework exercise:
Design a 3rd-order operator splitting method for

$$q_t = Aq + Bq$$

Method-of-lines discretization

Given $q_t = Aq + Bq$

first discretize $A \to A_{\Delta x}$

$B \to B_{\Delta x}$

$$Q'(t) = (A_{\Delta x} + B_{\Delta x})Q(t)$$

Then apply a Runge-Kutta or linear multistep method.

Implicit-Explicit (ImEx) methods

If A is nonlinear, nonstiff

B is linear, stiff

we would like to integrate A with an explicit method and B with an implicit method.

ImEx Runge-Kutta methods

$$q_t = \underbrace{f(q)}_{\text{non-stiff}} + \underbrace{g(q)}_{\text{stiff}}$$

$$y^i = Q^n + \Delta t \sum_{j=1}^{i-1} a_{ij} f(y^j) + \Delta t \sum_{j=1}^{s} \hat{a}_{ij} g(y^j) \qquad 1 \le i \le s$$

$$Q^{n+1} = Q^n + \Delta t \sum_{j=1}^{s} b_j f(y^j) + \Delta t \sum_{j=1}^{s} \hat{b}_j g(y^j)$$

$$\begin{array}{c|c|c|c} c & A & \hat{A} & \hat{c} \\ \hline & b^T & \hat{b}^T & \end{array}$$