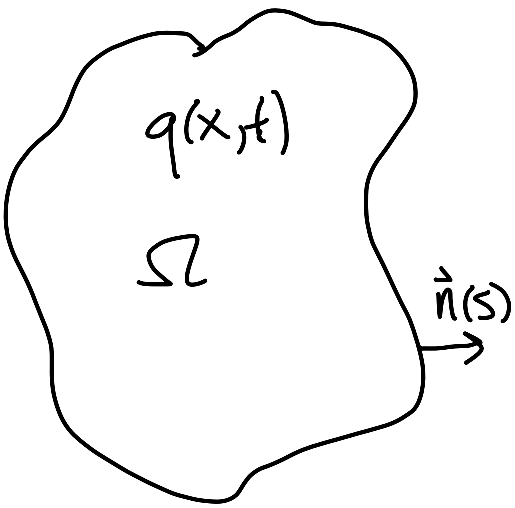
github.com/ketch/AMCS-333-2025




$$\frac{d}{dt}\iint_\Omega q(\vec{x},t)\,d\vec{x} = -\oint_{\partial\Omega} \vec{n}\cdot\vec{F}(s)\,ds$$

$$= -\iint_\Omega \nabla\cdot\vec{F}(q,\vec{x},t)\,d\vec{x}$$

$$\iint_\Omega \underbrace{\left(\frac{\partial q}{\partial t} + \nabla\cdot\vec{F}\right)}_{\text{must vanish pointwise}} d\vec{x} = 0 \Rightarrow q_t + \nabla\cdot\vec{F} = 0$$

Differential form of conservation

More generally, we could have a "balance law"

$$q_t + \nabla\cdot\vec{F} = S(q,x,t)$$

convective term(s)

source term(s)

Often a hyperbolic model is obtained from

$$q_t + \nabla\cdot\vec{F} = \underbrace{\varepsilon\nabla^2 q}_{\text{parabolic term(s)}} \qquad q_t + f(q)_x = \varepsilon q_{xx}$$

taking $\varepsilon \to 0$. ("vanishing viscosity")

We might also consider a "vanishing dispersion" limit:

$$q_t + f(q)_x = \varepsilon q_{xxx} \qquad (\varepsilon \to 0)$$

e.g. the Korteweg-de Vries equation

$$q_t + (q^2)_x + q_{xxx} = 0$$

# 1D Advection

$q \in \mathbb{R}$

$$q_t + a q_x = 0 \qquad f(q) = aq$$

$x \in (-\infty, \infty)$

$$q(x,0) = q_0(x)$$

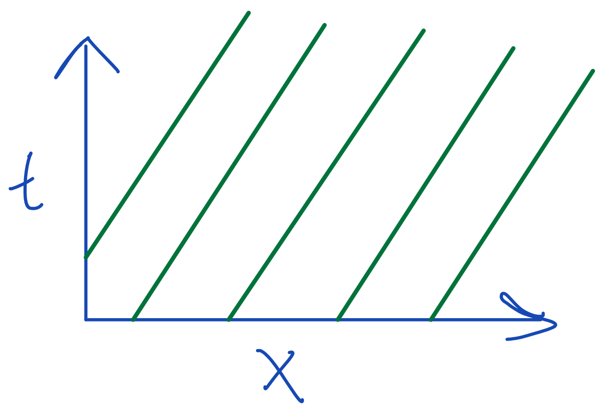
$$q(x,t) = q_0(x - at)$$

$$X(t) = x_0 + at \qquad \frac{d}{dt} q(X(t),t) = \frac{\partial}{\partial x} q(x,t) X'(t) + \frac{\partial}{\partial t} q(x,t)$$

$$= a q_x + q_t = 0$$

$$\begin{bmatrix} 1 \\ a \end{bmatrix} \cdot \nabla q = 0$$

Solution is constant along $\underbrace{x - at = \text{const}}_{\text{characteristics}}$.



# The Riemann Problem

$$q_t + f(q)_x = 0$$

$$q(x,0) = \begin{cases} q_\ell & x<0 \\ q_r & x>0 \end{cases}$$

Solution usually depends only on $x/t$

## Advection

$$q_t + a q_x = 0$$

$$q(x,0) = \begin{cases} q_\ell & x<0 \\ q_r & x>0 \end{cases}$$

We can't use $q_x$ when $q$ is discontinuous.

$$\int_{t_1}^{t_2}\int_{x_1}^{x_2} \left(q_t + f(q)_x\right) dx\,dt = 0$$

Integral form of the conservation law:

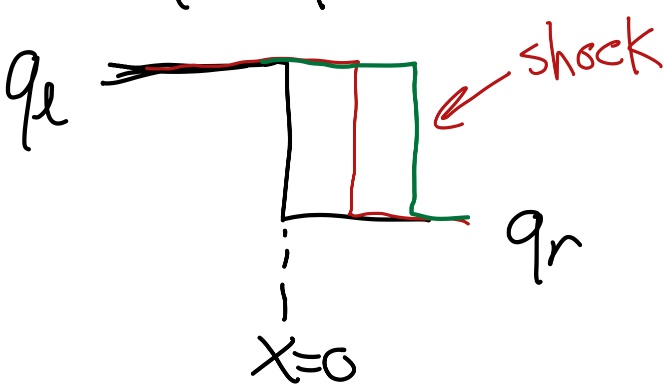
$$\int_{x_1}^{x_2} \left(q(x,t_2) - q(x,t_1)\right) dx + \int_{t_1}^{t_2} f(q(x_2,t)) - f(q(x_1,t))\,dt$$

In fact $q(x,t) = q_0(x-at)$ satisfies this even if $q_0$ is not continuous.

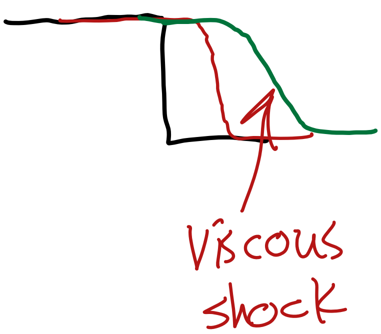
Solution of RP for advection:

$$q(x,t) = \begin{cases} q_\ell & x < at \\ q_r & x > at \end{cases}$$

$$q_t + f(q)_x = 0$$




$$q_t + f(q)_x = \varepsilon q_{xx}$$




$$q_t + f(q)_x = \varepsilon q_{xxx}$$