Linear Schrodinger equation

$$iu_t = u_{xx} + Vu$$

$$\omega(k) = -k^2 + V$$

$$e^{i(kx+(k^2-V)t)}$$

$$e^{i(kx-\omega t)}$$

$$e^{ik(x-\frac{\omega}{k}t)}$$

$$e^{ik(x-ct)}$$

Higher-order time derivatives

$$u_{tt} = u_{xx}$$

$$-\omega^2 = -k^2$$

$$\omega(k) = \pm k$$

Phase velocity:

$$c = \frac{\omega(k)}{k}$$

Systems

$$u_t = v_x$$

$$v_t = u_x$$

$$\begin{bmatrix} u \\ v \end{bmatrix}_t = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix}_x$$

$$\begin{bmatrix} \partial_t & -\partial_x \\ -\partial_x & \partial_t \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix} = 0$$

$$\partial_t \to -i\omega$$

$$\partial_x \to ik$$

$$\begin{bmatrix} -i\omega & -ik \\ -ik & -i\omega \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix} = 0$$

$$\begin{bmatrix} \omega & k \\ k & \omega \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix} = 0$$

$$\det \begin{bmatrix} \omega & k \\ k & \omega \end{bmatrix} = 0$$

$$\omega^2 - k^2 = 0$$

$$\omega = \pm k$$

$$u_t = \alpha v_{xx}$$

$$v_t = \beta u_{xx}$$

$$-i\omega u + \alpha k^2 v = 0$$

$$-i\omega v + \beta k^2 u = 0$$

$$\underbrace{\begin{bmatrix} -i\omega & \alpha k^2 \\ \beta k^2 & -i\omega \end{bmatrix}} \begin{bmatrix} u \\ v \end{bmatrix} = 0$$

$$\det(\uparrow) = 0$$

$$-\omega^2 - \alpha\beta k^4 = 0$$

$$\omega^2 = -\alpha\beta k^4$$

$$\omega = \pm\sqrt{-\alpha\beta}\, k^2$$

Wave-like (dispersive)

if $\cancel{\alpha} R < 0 \iff \omega(k) \in \mathbb{R}$

so $e^{-i\omega t}$ is oscillatory.

$$\omega^2 = 10 - k^2$$

$$\omega = \pm\sqrt{10 - k^2}$$

$k \le \sqrt{10}$: $\omega(k) \in \mathbb{R}$

$k > \sqrt{10}$: $\omega(k) = ir$, $r \in \mathbb{R}$

$$u_t + u_{xxx} = 0$$

The method of stationary phase

$$u(x,t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \hat{u}(k,0) e^{i(kx-\omega t)} \, dk$$

$$\hat{u}(k,0) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} u(x,0) e^{-ikx} \, dx$$

Each mode propagates with velocity

$$c(k) = \frac{\omega(k)}{k}.$$

What happens to the solution as $t \to \infty$?

We assume

$$\omega''(k) \neq 0$$

So that modes propagate at different speeds.

$$u(x,t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \hat{u}(k,0) e^{i\phi(k)t} \, dk$$

$$\phi(k) = k\frac{x}{t} - \omega(k)$$

For $t \gg 1$, modes k, $k+\delta$ have very different phases and interfere destructively.

But if $\phi'(K)=0$, then modes K, $K+\delta$ will have similar phase.

$$\phi'(K) = \frac{x}{t} - W'(K)$$

$$\phi'(K)=0 \Leftrightarrow W'(K) = \frac{x}{t}$$

So we define

$$c_g = W'(K)$$

$W = -K^3$

$c_g = -3K^2$

Let $\phi'(K_0) = 0$

$$\phi(K) = \phi(K_0) + \cancel{(K-K_0)\phi'(K_0)}$$
$$+ \frac{1}{2}(K-K_0)^2 \phi''(K_0) + \dots$$

$$\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\hat{u}(k,0)e^{i\phi(k)t}\,dk \approx \frac{1}{\sqrt{2\pi}}\int_{k_0-\delta}^{k_0+\delta}\hat{u}(k,0)e^{i\left(\phi(k_0)+\frac{(k-k_0)^2}{2}\phi''\right)t}\,dk$$

$$\approx \frac{2}{\sqrt{2\pi}}\hat{u}(k_0,0)e^{i\phi(k_0)t}\int_{k_0}^{k_0+\delta} e^{\frac{i(k-k_0)^2}{2}\phi''(k_0)t}\,dk$$

Substitute $K^2 = \frac{1}{2}(k-k_0)^2|\phi''(k_0)|t$

Use $\int_0^\infty e^{\pm ix^2}dx = \sqrt{\frac{\pi}{2}}e^{\pm i\pi/4}$ (Fresnel)

Eventually we get: $u(x,t) \approx \hat{u}(k_0,0)\sqrt{\frac{2}{|\phi''(k_0)|t}}\; e^{i\phi(k_0)t}\; e^{i\frac{\pi}{4}\operatorname{sgn}(\phi''(k_0))}$

This approximates the solution along the ray $\frac{x}{t} = \omega'(k_0)$.

Decays like $t^{-1/2}$.

$$u_t + u_{xxx} = 0$$

$$\omega = -k^3$$

$$\phi(k) = k\frac{x}{t} + k^3$$

$$\phi'(k) = \frac{x}{t} + 3k^2$$

$$\frac{x}{t} = -3k^2$$

$$k = \pm\sqrt{\frac{-x}{3t}}$$