(1) $\varepsilon\phi_{xx} + \phi_{zz} = 0$

(2) $\phi_z = 0$ at $z = -1$

(3) $\varepsilon\zeta_t + \varepsilon^2\phi_x\zeta_x = \phi_z$

(4) $\phi_t + \zeta + \frac{1}{2}(\phi_z)^2 + \frac{\varepsilon}{2}(\phi_x)^2 = 0$

$\left.\begin{matrix}(3)\\(4)\end{matrix}\right\}$ at $z = \varepsilon\zeta$

$$\phi(x,z,t) = \sum_{n=0}^{\infty}(z+1)^n\phi^n(x,t)$$

Substitute into (1):

$$\varepsilon\sum_{n=0}^{\infty}(z+1)^n\phi^n_{xx} + \sum_{n=2}^{\infty}n(n-1)(z+1)^{n-2}\phi^n = 0$$

$$\sum_{n=0}^{\infty}(z+1)^n\left(\varepsilon\phi^n_{xx} + (n+2)(n+1)\phi^{n+2}\right) = 0$$

Each term must vanish:

$$\varepsilon\phi^n_{xx} + (n+2)(n+1)\phi^{n+2} = 0 \quad n = 0, 1, \dots$$

$$\phi^{n+2} = -\frac{\varepsilon\phi^n_{xx}}{(n+2)(n+1)} \quad (5)$$

From (2): $\phi_z\big|_{z=-1} = \phi^1 = 0$

Since $\phi_z = \sum_{n=1} n(z+1)^{n-1}\phi^n(x,t)$

So $\phi^3 = \phi^5 = \dots = 0$

From (5) we have no

(6) $\phi(x,z,t) = \phi^0 - \frac{\varepsilon}{2!}(z+1)^2\phi^0_{xx} + \frac{\varepsilon^2}{4!}(z+1)^4\phi^0_{xxxx}$
$+ O(\varepsilon^3)$

$$\phi_z(x,z,t) = -\varepsilon(z+1)\phi^0_{xx} + \frac{\varepsilon^2}{3!}(z+1)^3\phi^0_{xxxx} + O(\varepsilon^3)$$

$$\phi_z\Big|_{z=\varepsilon\zeta} = -\varepsilon(\varepsilon\zeta+1)\phi^0_{xx} + O(\varepsilon^2)$$

$$= -\varepsilon\phi^0_{xx} + O(\varepsilon^2)$$

$$\zeta_t = -\phi^0_{xx} + O(\varepsilon) \quad (\text{from } (3))$$

$$\phi^0_t + \zeta = O(\varepsilon) \quad (\text{from } (4)) \qquad u^0 = \phi^0_x$$

Diff. w.r.t. x:

$$\phi^0_{tx} = -\zeta_x + O(\varepsilon) \qquad \phi_x = u$$

$$\boxed{u^0_t + \zeta_x = O(\varepsilon)}$$

$$\boxed{\zeta_t + u^0_x = O(\varepsilon)}$$

To leading order we get the (linear) wave equation.

Now we introduce

$$\phi^0_x = u^0 + \varepsilon u^1 + \varepsilon^2 u^2 + \cdots$$

$$\zeta = \zeta^0 + \varepsilon\zeta^1 + \varepsilon^2\zeta^2 + \cdots$$

$$\phi_z(x,z,t)\Big|_{z=\varepsilon\zeta} = -\varepsilon(1+\varepsilon\zeta)\phi^0_{xx} + \frac{\varepsilon^2}{3!}(1+\varepsilon\zeta)^3\phi^0_{xxxx} + O(\varepsilon^3)$$

$$= -\varepsilon\phi^0_{xx} - \varepsilon^2\zeta\phi^0_{xx} + \frac{\varepsilon^2}{6}\phi^0_{xxxx} + O(\varepsilon^3)$$

$$= -\varepsilon\left(u^0_x + \varepsilon u^1_x + O(\varepsilon^2)\right) - \varepsilon^2\zeta u^0_x + \frac{\varepsilon^2}{6}u^0_{xxx} + O(\varepsilon^3)$$

$$\phi_z\Big|_{z=\varepsilon\zeta} = -\varepsilon u^0_x - \varepsilon^2\left(u^1_x + \zeta^0 u^0_x - \frac{1}{6}u^0_{xxx}\right) + O(\varepsilon^3)$$

Similarly:

$$\phi_x\Big|_{z=\varepsilon\zeta} = u^0 + \varepsilon\left(u^1 - \frac{1}{2}u^0_{xxx}\right) + O(\varepsilon^2)$$

We also introduce

$\tau_0 = t \quad \tau_1 = \varepsilon t$

We treat these as indep. variables

$$\frac{\partial f}{\partial t} = f_{\tau_0}\frac{d\tau_0}{dt} + f_{\tau_1}\frac{d\tau_1}{dt}$$

so

$$\frac{\partial}{\partial t} \to \frac{\partial}{\partial \tau_0} + \varepsilon\frac{\partial}{\partial \tau_1}$$

$$\phi_{xt}\Big|_{z=\varepsilon\rho} = u^0_{\tau_0} + \varepsilon\left(u^0_{\tau_1} + u^1_{\tau_0} - \tfrac{1}{2}u^0_{xx\tau_0}\right) + O(\varepsilon^2)$$

$$\rho_t\Big|_{z=\varepsilon\rho} = \rho^0_{\tau_0} + \varepsilon\left(\rho^0_{\tau_1} + \rho^1_{\tau_0}\right) + O(\varepsilon^2)$$

Substitute these 4 equations into (3) and the x-derivative of (4).

$$\left.\begin{aligned} &(3) \quad \varepsilon\rho_t + \varepsilon^2\phi_x\rho_x = \phi_z \\ &(4) \quad \phi_{tx} + \rho_x + \tfrac{1}{2}\partial_x\left(\phi_z^2 + \varepsilon\phi_x^2\right) = 0 \end{aligned}\right\} \text{at } z = \varepsilon\rho$$

From (3), $O(\varepsilon^1)$ terms:

$$\rho^0_{\tau_0} = -u^0_x$$

From (4), $O(\varepsilon^0)$ terms:

$$u^0_{\tau_0} = -\rho^0_x$$

$$(3)\ O(\varepsilon^2):\ u^1_x + \rho^1_{\tau_0} = -\rho^0_{\tau_1} - u^0\rho^0_x - u^0_x\rho^0 + \tfrac{1}{6}u^0_{xxx} \qquad (A)$$

$$(4)\ O(\varepsilon^1):\ u^1_{\tau_0} + \rho^1_x = -\left(u^0_{\tau_1} - \tfrac{1}{2}u^0_{xx\tau_0} + u^0u^0_x\right) \qquad (B)$$

LHS: linear wave eqn.

RHS: dispersion

In (3) there are no terms of $O(\varepsilon^0)$

Characteristic variables:

$$\ell = x + \tau_0$$

$$r = x - \tau_0$$

$$\partial_x = \partial_\ell \ell_x + \partial_r r_x = \partial_\ell + \partial_r$$

$$\partial_{\tau_0} = \partial_\ell \ell_{\tau_0} + \partial_r r_{\tau_0} = \partial_\ell - \partial_r$$

We can write

$$\rho^0 = f(r) + g(\ell)$$

$$u^0 = f(r) - g(\ell)$$

Replace x, τ_0 by ℓ, r derivatives

Replace ρ^0, u^0 by f, g

In (A) and (B)

(A) $u'_\ell + u'_r + \rho'_\ell - \rho'_r = -(f+g)_{\tau_1} - (f-g)(f+g)_x - (f-g)_x(f+g) + \frac{1}{6}(f-g)_{xxx}$

$$u'_\ell + u'_r + \rho'_\ell - \rho'_r = -f_{\tau_1} - g_{\tau_1} - (f-g)(f_\ell + f_r + g_\ell + g_r) - (f_\ell + f_r - g_\ell - g_r) + \frac{1}{6}(f_{rrr} - g_{\ell\ell\ell})$$

Simplify this.

Do the same for (B).

Then add (A)+(B):

$$2(\rho'_\ell + u'_\ell) = -2f_{\tau_1} - 3ff_r - \frac{1}{3}f_{rrr} + f_r g + g g_\ell + f g_\ell - \frac{2}{3} g_{\ell\ell\ell}$$

Integrate w.r.t. ℓ:

$$2(\rho' + u') = \underbrace{-\ell\left(2f_{\tau_1} + 3ff_r + \frac{1}{3}f_{rrr}\right)}_{\text{This would blow up (secular)}} + \int f_r g \, d\ell + \frac{1}{2}g^2 + \int f g_\ell \, d\ell - \frac{2}{3} g_{\ell\ell}$$

If the initial data goes to zero as $|x| \to \infty$

These will vanish after long enough time.

We require that the secular terms vanish:

$$2f_{\tau_1} + 3ff_r + \frac{1}{3}f_{rrr} = 0 \qquad \text{(KdV)}$$

If take (A)−(B), we get a KdV equation for the right-going part of the solution.

These must vanish for large ℓ.