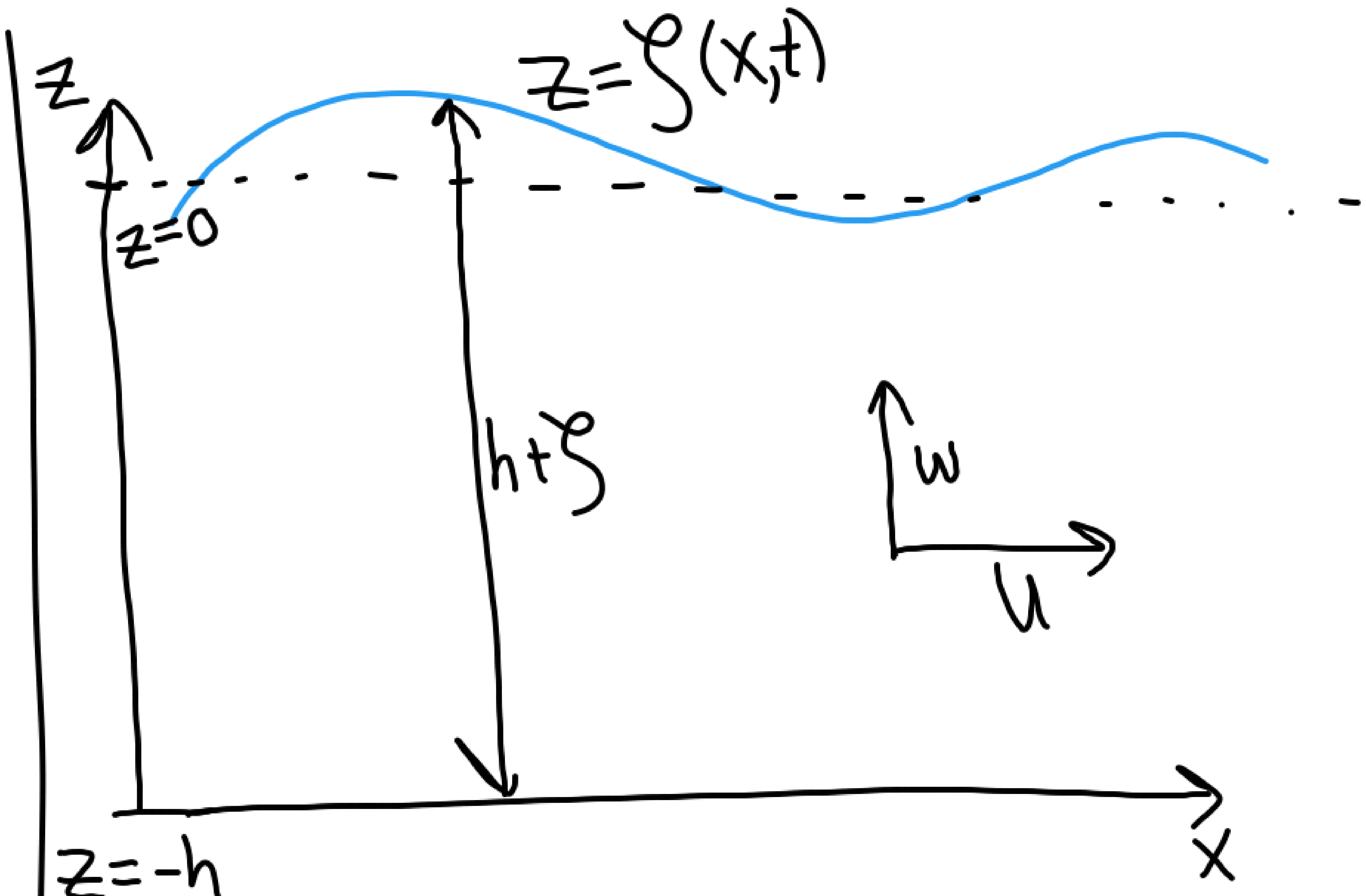
Surface waves in Shallow water

1. Governing Equations
2. Linear dispersion
3. Multiple scale perturbation theory



Incompressible, inviscid, irrotational flow

$\Rightarrow$ potential flow $\nabla\phi = \begin{bmatrix} u \\ w \end{bmatrix} = \begin{bmatrix} \phi_x \\ \phi_z \end{bmatrix}$

① $\nabla^2\phi = 0$ $\quad -h < z < \zeta$, $\; -\infty < x < \infty$

② $\phi_z = 0$ at $z=-h$

(no flow through bottom)

③ $\zeta_t + \phi_x \zeta_x = \phi_z$ at $z=\zeta(x,t)$

"Kinematic condition"

④ $\underbrace{\phi_t + g\zeta + \frac{1}{2}|\nabla\phi|^2 = 0}_{\text{Bernoulli}}$ at $z=\zeta(x,t)$

"Dynamic condition"

Difficult to solve either analytically or computationally.

Linear dispersion relation

Assume: $\zeta, \phi_x, \phi_z \ll 1$

So any products can be neglected

$\phi_{xx} + \phi_{zz} = 0$ $\quad -h<z<0\approx\zeta$

$\phi_z = 0$ $\quad z=-h$

$\zeta_t = \phi_z$ $\quad z=0$

$\phi_t = -g\zeta$ $\quad z=0$

$\phi_{tt} = -g\phi_z$ $\quad z=0$

Ansatz: $\phi(x,z,t) = e^{i(kx+\omega t)} f(z)$

$$-k^2\phi + e^{ik}e^{i\omega t}f''(z)=0$$

$$f''(z)=k^2 f(z)$$

$$f(z)=ae^{kz}+be^{-kz}$$

$$\phi_z = e^{ikx}e^{i\omega t}k\left(ae^{kz}-be^{-kz}\right)$$

$$\phi_z\Big|_{z-h} = e^{ikx}e^{i\omega t}k\left(ae^{-kh}-be^{kh}\right)=0$$

$$ae^{-kh}-be^{kh}=0$$

$$b=ae^{-2kh} \qquad f(z)=a\left(e^{kz}+e^{-k(2h+z)}\right)$$

$$-\omega^2\cancel{e^{ikx}}\cancel{e^{i\omega t}}\cancel{a}\left(e^{kz}+e^{-k(z+2h)}\right)=-g\cancel{e^{ikx}}\cancel{e^{i\omega t}}\cancel{a}k\left(e^{kz}-e^{-k(2h+z)}\right)$$

$$\omega^2 = gk\frac{1-e^{-2hk}}{1+e^{-2hk}}$$

$$=gk\frac{e^{hk}+e^{-hk}}{e^{hk}-e^{-hk}}$$

$$\omega^2 = gk\tanh(kh)$$

$$\omega = \pm\sqrt{gk\tanh(kh)}$$

Long waves in shallow water

$|kh| \ll 1$: $\tanh(kh) \simeq kh$

$$\omega \simeq \pm |k| \sqrt{gh}$$

$$c_p = \frac{\omega}{k} \simeq \pm \sqrt{gh}$$

To leading order, these are not dispersive.

Short waves in deep water

$|kh| \gg 1$: $\tanh(kh) \simeq \operatorname{sgn}(k)$

$$\omega \simeq \pm \sqrt{gk\operatorname{sgn}(k)} = \pm \sqrt{g|k|}$$

$$c_p = \frac{\omega}{k} \simeq \pm \sqrt{\frac{g}{|k|}} \leftarrow \text{dispersion}$$

Multiple scale perturbation theory

1. Choose small parameter ε
2. Choose scales for all variables; nondimensionalize
3. Assume power series expansion in ε for each dependent variable
4. Equate terms with like powers of ε

4b) Require that secular terms vanish

Step 1: small parameter

$$\varepsilon = (kh)^2 \ll 1$$

long wave
shallow water

$$\varepsilon = \frac{|\zeta|}{h} \ll 1$$

small amplitude
wave

Step 2: rescaling

Typical depth scale: h

$$z^* = \frac{z}{h}$$

Length scale: $k^{-1} = \frac{h}{\sqrt{\varepsilon}}$

$$x^* = \frac{x}{k^{-1}} = \frac{\sqrt{\varepsilon}}{h} x$$

Time scale: $\omega^{-1} \approx \frac{1}{k\sqrt{gh}} = \sqrt{\frac{h}{g\varepsilon}}$

$$t^* = \frac{t}{\omega^{-1}}$$

$$t^* = \sqrt{\frac{g\varepsilon}{h}}\, t$$

$$\zeta^* = \frac{\zeta}{h\varepsilon}$$

$$\phi^* = \frac{\phi}{h\sqrt{\varepsilon g h}}$$

$$\nabla^2 \phi = \phi_{xx} + \phi_{zz}$$

$$= \left(\frac{\sqrt{\varepsilon}}{h}\right)^2 \phi_{x^*x^*} + \left(\frac{1}{h}\right)^2 \phi_{z^*z^*}$$

$$= \cancel{h\sqrt{\varepsilon g h}}\, \frac{\varepsilon}{\cancel{h^2}} \phi^*_{x^*x^*} + \frac{\cancel{h\sqrt{\varepsilon g h}}}{\cancel{h^2}} \phi^*_{z^*z^*} = 0$$

$$\varepsilon \phi^*_{x^*x^*} + \phi^*_{z^*z^*} = 0$$

Now we drop the *'s

(1) $\varepsilon\phi_{xx} + \phi_{zz} = 0$

(2) $\phi_z = 0$ at $z = -1$

$$\left.\begin{array}{l} (3)\ \varepsilon\zeta_t + \varepsilon^2\phi_x\zeta_x = \phi_z \\ (4)\ \phi_t + \zeta + \frac{1}{2}(\phi_z)^2 + \frac{\varepsilon}{2}(\phi_x)^2 = 0 \end{array}\right\} \text{at } z = \varepsilon\zeta$$