$$u_t + uu_x + u_{xx} + u_{xxxx} = 0$$

steepening

Antidiffusion

Hyperviscosity

Hyperbolic PDEs with Periodic coefficients

Acoustics

$$p_t + K(x) u_x = 0$$

$$\rho(x) u_t + p_x = 0$$

If $K(x)$, $\rho(x)$ are constant, this is equivalent to

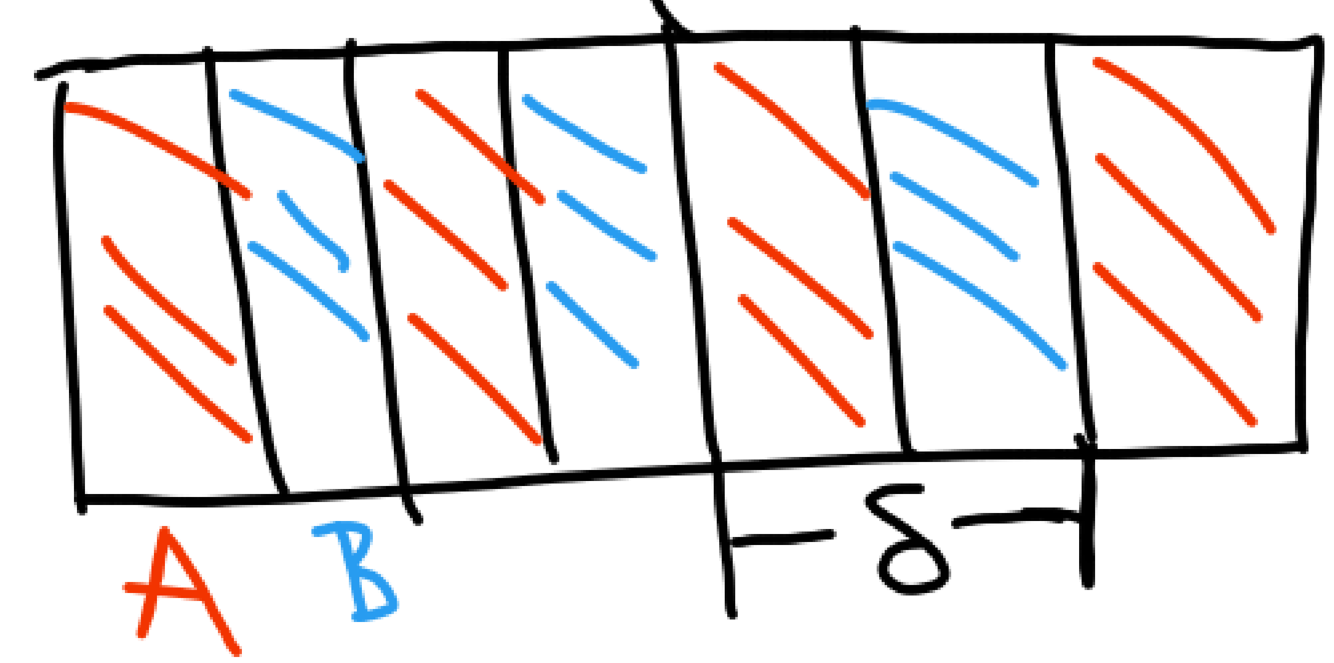
$$u_{tt} = c^2 u_{xx}$$

where $c = \pm\sqrt{\frac{K}{\rho}}$.

Suppose that

$$K(x+\delta) = K(x)$$

$$\rho(x+\delta) = \rho(x)$$



Goal: find a constant-coefficient PDE that describes the behavior of long-wavelength waves.

We assume $\delta \xi \ll 1$.

We introduce the fast scale

$$y = \frac{x}{\delta}$$

So $\partial_x \to \partial_x + \delta^{-1}\partial_y$

We have

(a) $\delta p_t + K(y)(u_x\delta + u_y) = 0$

(b) $\delta\rho(y)u_t + \delta p_x + p_y = 0$

Let $p(x,y,t) = \sum_{j=0}^{\infty} \delta^j p^j(x,y,t)$

$$u(x,y,t) = \sum_{j=0}^{\infty} \delta^j u^j(x,y,t)$$

We assume that all functions are periodic in y.

$\mathcal{O}(\delta^0)$:

$$u^0_y = 0$$

$$p^0_y = 0$$

$p^0 = p^0(x,t) \quad u^0 = u^0(x,t)$

We will find eqns. that describe

$$\bar{p}(x,t) = \int_0^1 p(x,y,t)\,dy$$

$$\bar{u}(x,t) = \int_0^1 u(x,y,t)\,dy$$

$\mathcal{O}(\delta^1)$: $p^0_t + K(y)u^0_x + K(y)u^1_y = 0$ (2)

$\rho(y)u^0_t + p^0_x + p^1_y = 0$ (1)

Integrate (1) w.r.t. y:

$$\int_0^1 \left(\rho(y)u^0_t + p^0_x + p^1_y\right) dy = 0$$

(4) $\boxed{\bar{\rho}\, u^0_t + p^0_x = 0}$

Integrate (2):

$$\int_0^1 \left(K^{-1} p^0_t + u^0_x + u^1_y\right) dy = 0$$

(3) $\boxed{\overline{K^{-1}}\, p^0_t + u^0_x = 0}$

Averages of original equations

Take (4) minus (1) and integrate w.r.t. y, from 0 to y:

$$\int_0^y -\left(\rho(z) - \bar{\rho}\right) u^0_t \, dz = \int_0^y p^1_z(x,z,t)\, dz$$

Define $[\![f]\!](y) = \int_0^y \left(f(z) - \bar{f}\right) dz$

(5)
$$p^1(x,y,t) = -u^0_t [\![\rho]\!](y) + \overline{p^1}(x,t)$$

Similarly from (3)−(2) we get

(6) $$u^1(x,y,t) = -p^0_t [\![K^{-1}]\!](y) + \overline{u^1}(x,t)$$

$$-\int_0^1\int_0^y (f(z)-\bar{f})\,dz\,dy$$

$\downarrow$

So $\overline{[\![f]\!]}=0$

$\mathcal{O}(\delta^2)$: $p^1_t + K(y)u^1_x + K(y)u^2_y = 0$

$$\rho(y)u^1_t + p^1_x + p^2_y = 0$$

solving for u^2_y:

$$-u^2_y = \frac{1}{K(y)}p^1_t + u^1_x$$

Substitute (5):

$$-u^2_y = \frac{1}{K(y)}\left(-u^0_{tt}[\![\rho]\!] + \overline{p^1_t}\right) - p^0_{tx}[\![K^{-1}]\!] + \overline{u^1_x}$$

Integrate:

$$0 = -\overline{K^{-1}[\![\rho]\!]}u^0_{tt} + \overline{K^{-1}}\,\overline{p^1_t} + \overline{u^1_x}$$

Similarly we get

$$-\overline{\rho[\![K^{-1}]\!]}p^{\circ}_{tt} + \bar{\rho}\overline{u}'_t + \overline{p}'_x = 0$$

$$u_t + u_x + u_{xxx} = 0$$

$$q_t + A(x)q_x = 0$$

$$A = \begin{bmatrix} 0 & K(x) \\ 1/\rho(x) & 0 \end{bmatrix} \quad q = \begin{bmatrix} p \\ u \end{bmatrix}$$

Eigenvalues of A:

$$\lambda_{\pm} = \pm\sqrt{\frac{K(x)}{\rho(x)}}$$

Eigenvectors of A:

$$\begin{bmatrix} 1 \\ \sqrt{K\rho} \end{bmatrix} \quad \begin{bmatrix} -1 \\ \sqrt{K\rho} \end{bmatrix}$$

$$R = \begin{bmatrix} 1 & -1 \\ Z(x) & Z(x) \end{bmatrix}$$

$$q_t + R\Lambda R^{-1} q_x = 0$$

If $Z(x) = Z_0$, then $R(x) = R_0$

and we can write

$$R^{-1} q_t + \Lambda R^{-1} q_x = 0$$

$w = R^{-1} q$

$$w_t + \Lambda w_x = 0$$

If $Z(x)$ is not constant:

$$\left(R^{-1}(x) q\right)_x = \left(R^{-1}\right)' q + w_x$$

Impedance variation causes reflection.