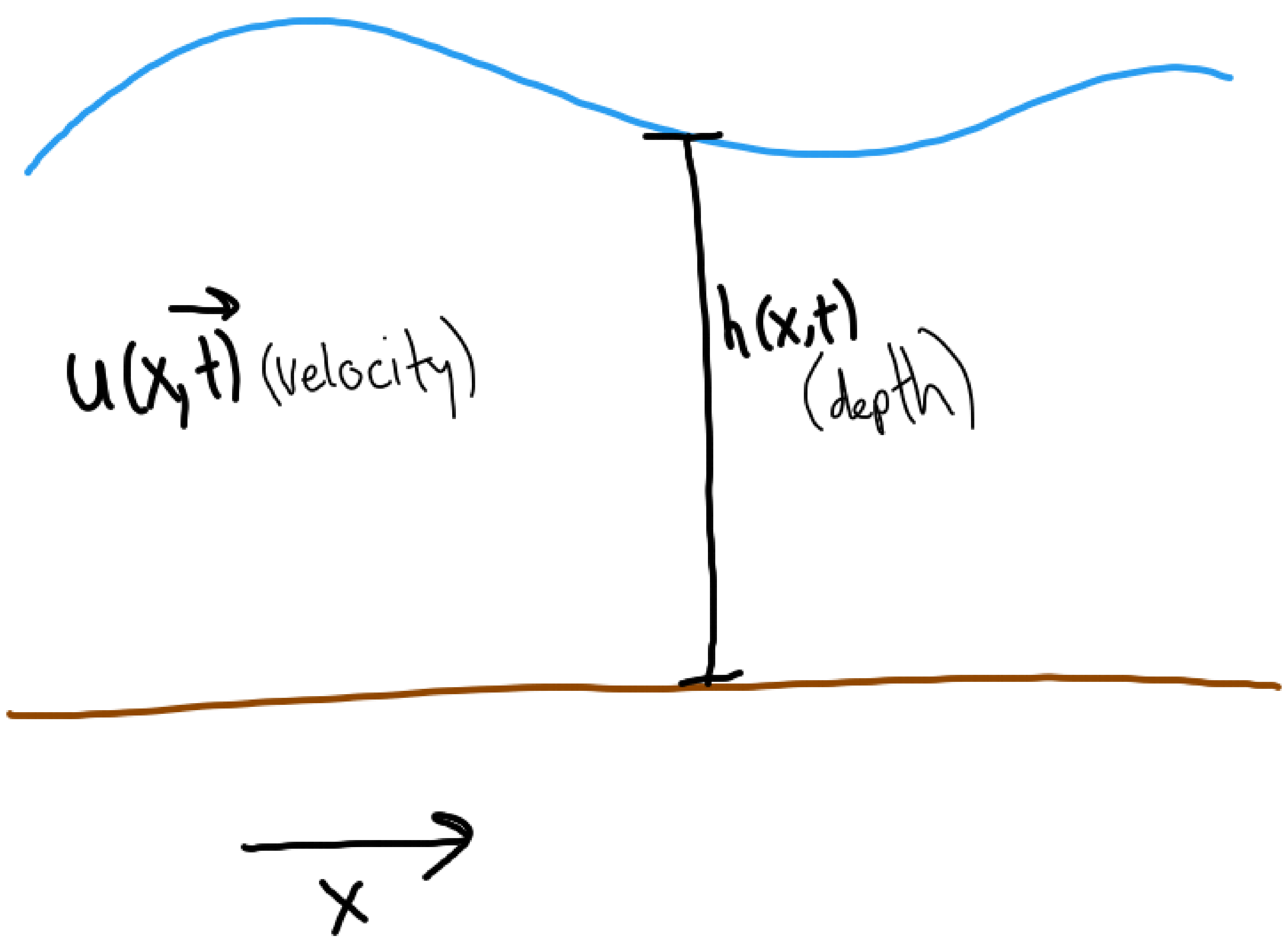
The shallow water (Saint-Venant) equations



Uniform density: $\bar{\rho}$

Mass in interval (x_1, x_2): $\int_{x_1}^{x_2} \bar{\rho} h \, dx$

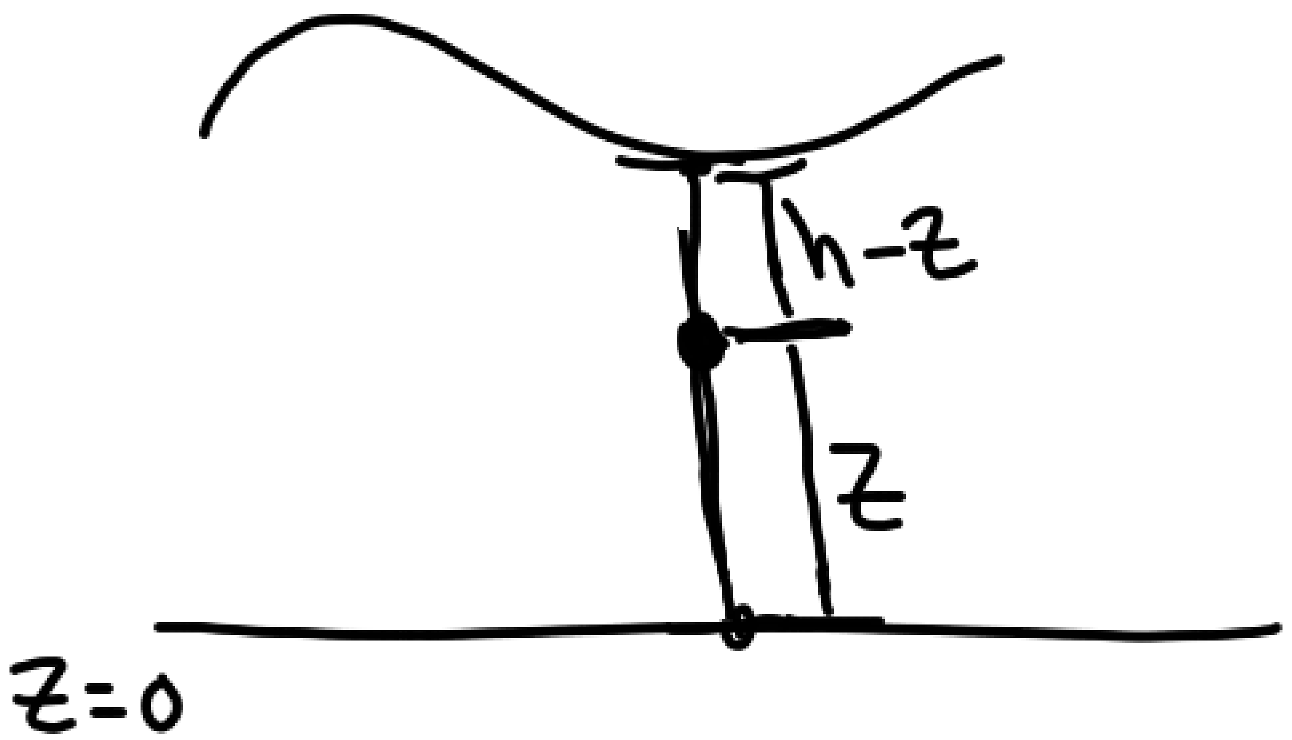
Flux: $\bar{\rho} h u$

Continuity eqn.: $(\bar{\rho} h)_t + (\bar{\rho} h u)_x = 0$

$$\Rightarrow h_t + (hu)_x = 0$$ (conservatio of mass)

Conservation of momentum:

$$(hu)_t + \left(hu^2 + \tfrac{1}{2} g h^2\right)_x = 0$$



Hydrostatic equilibrium

Weight of water above z: $\bar{\rho}(h-z)g$

Pressure at each point must counteract this.

Total pressure in a column of water:

$$\int_0^h \bar{\rho}(h-z)g\,dz = \bar{\rho}g\left[hz - \frac{z^2}{2}\right]_0^h$$

$$= \bar{\rho}g\left(\tfrac{1}{2}h^2 - 0\right) = \tfrac{1}{2}\bar{\rho}gh^2$$

$$\underbrace{\begin{bmatrix} h \\ hu \end{bmatrix}_t}_{q} + \underbrace{\begin{bmatrix} hu \\ hu^2 + \frac{1}{2}gh^2 \end{bmatrix}_x}_{f(q)} = 0$$

$$q_t + f(q)_x = 0$$

Quasilinear form:

$$q_t + \underbrace{f'(q)}_{\text{Flux Jacobian}} q_x = 0$$

Compare to advection: $q_t + a q_x = 0$

$$f(q)=\begin{bmatrix} q^2 \\ \frac{(q^2)^2}{q^1}+\frac{1}{2}g(q^1)^2 \end{bmatrix}$$

$$f'(q)=\begin{bmatrix} 0 & 1 \\ -\left(\frac{q^2}{q^1}\right)^2+gq^1 & 2\frac{q^2}{q^1} \end{bmatrix}$$

$$f'(q)=\begin{bmatrix} 0 & 1 \\ gh-u^2 & 2u \end{bmatrix}$$

Characteristic speeds (eigenvalues of $f'(q)$):

$$\det(\lambda I-f'(q))=0$$

$$\lambda(\lambda-2u)-(gh-u^2)=0$$

$$\lambda^2-2u\lambda+u^2-gh=0$$

$$\lambda_{\pm}=u\pm\frac{\sqrt{\cancel{4u^2}-\cancel{4u^2}+\cancel{4}gh}}{\cancel{2}}=u\pm\sqrt{gh}$$

If $q(x,t)=q_0+\varepsilon q_1(x,t)$

Then $(q_0+\varepsilon q_1)_t+f'(q_0+\varepsilon q_1)(q_0+\varepsilon q_1)_x=0$

$$\varepsilon q_{1,t}+\left(f'(q_0)+\mathcal{O}(\varepsilon)\right)\varepsilon q_{1,x}=0$$

$$q_{1,t}+f'(q_0)q_{1,x}=\mathcal{O}(\varepsilon)$$

Linear wave equation

$c=u_0\pm\sqrt{gh_0}$

$$r_- = \begin{bmatrix} 1 \\ u-\sqrt{gh} \end{bmatrix} \quad r_+ = \begin{bmatrix} 1 \\ u+\sqrt{gh} \end{bmatrix}$$

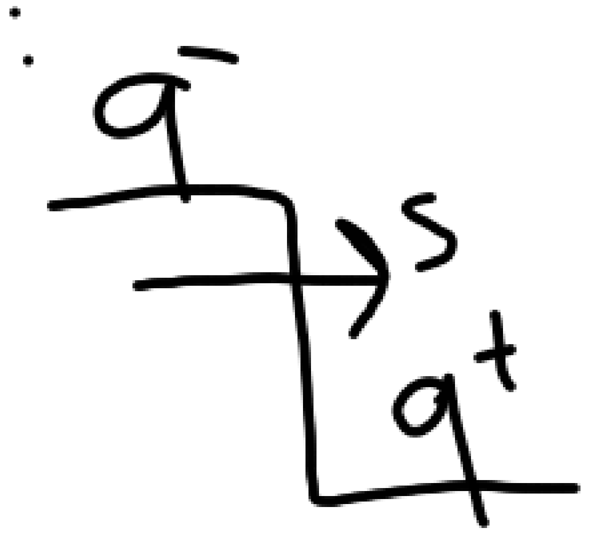
The Riemann problem

$$q_t + f(q)_x = 0$$

Initial data

$$q(x, t=0) = \begin{cases} q_\ell & x<0 \\ q_r & x>0 \end{cases}$$

Rankine-Hugoniot conditions:



$$s[q] = [f(q)]$$

where $[q] = q^+ - q^-$

The Lax entropy condition

$$\lambda(q^-) > s > \lambda(q^+)$$

Simple Waves

A simple wave solution has the property

$$q(x,t)_x = \alpha(x,t)\, r_\pm(q(x,t))$$

Then $$q_t + f'(q) q_x = q_t + \alpha(x,t)\, \lambda_\pm(q(x,t))\, r_\pm(q(x,t)) = 0$$

It behaves like the solution of a scalar conservation law.

By choosing an appriate parameterization ξ

$$q(x,t)_x = \tilde{q}'(\xi) = r_{\pm}(q(\xi)) \qquad r_{\pm} = \begin{bmatrix} 1 \\ u \mp \sqrt{gh} \end{bmatrix}$$

$$\left.\begin{aligned} q^1 &= h'(\xi) = 1 \\ q^2 &= (hu)'(\xi) = u \mp \sqrt{gh} = \frac{q^2}{q^1} \mp \sqrt{g q^1} \end{aligned}\right\}$$

System of ODEs.
Specify an "initial condition" $(h_*, (hu)_*)$ to find a solution

Solution: $\underbrace{u \pm 2\sqrt{gh}}_{\text{Riemann invariant}} = u_* \pm 2\sqrt{gh_*}$