Hyperbolic systems with periodic coefficients

Outline:

- Motivation
- Linear wave equation
  - Homogenization
  - Dispersion relation
- Nonlinear equations

Homogenization of linear Wave equ. with periodic coefficients

Acoustics:

$$p_t + K(x)u_x = 0$$

$$\rho(x)u_t + p_x = 0$$

Char. speed:

$$c = \pm\sqrt{\frac{K}{\rho}}$$

Let

$$K(x+\delta) = K(x)$$

$$\rho(x+\delta) = \rho(x)$$

Introduce:

$$y = \frac{x}{\delta}$$

So

$$\partial_x \rightarrow \partial_x + \partial_y \frac{dy}{dx} = \partial_x + \delta^{-1}\partial_y$$

We can write ρ, K as functions of y with period 1.

We have

$$p_t + K(y)(u_x + \delta^{-1} u_y) = 0$$

$$\rho(y) u_t + p_x + \delta^{-1} p_y = 0$$

Let

$$p(x,y,t) = p^0(x,y,t) + \delta p^1(x,y,t) + \cdots$$

$$u(x,y,t) = u^0(x,y,t) + \delta u^1(x,y,t) + \cdots$$

We assume all functions are periodic w.r.t. y with period 1.

$\mathcal{O}(\delta^{-1})$: $K u^0_y = 0 \qquad p^0_y = 0$

$$u^0 = u^0(x,t) \qquad p^0 = p^0(x,t)$$

$\mathcal{O}(\delta^0)$: $p^0_t + K(y) u^0_x = -K(y) u^1_y$

$$\rho(y) u^0_t + p^0_x = -p^1_y \qquad (1)$$

$$\Rightarrow -u'_y = \frac{p^0_t}{K(y)} + u^0_x \qquad (2)$$

Integrate $^{(1)-(2)}$ over one period in y:

$$-\int_0^1 u'_y\,dy = \int_0^1 \left(\frac{p^0_t}{K(y)} + u^0_x\right)dy$$

$$(3) \quad 0 = \overline{K^{-1}}\, p^0_t + u^0_x \qquad \text{where } \overline{K^{-1}} = \int_0^1 \frac{1}{K(y)}\,dy$$

$$p^0_t + \left(\overline{K^{-1}}\right)^{-1} u^0_x = 0$$

From (1) we get

$$\bar{\rho}\, u^0_t + p^0_x = 0. \quad (4)$$

System (3)-(4) is just our original system but with the variable coefficients replaced by averages.

Before going to the next order, we want equations for u', p'. We take (4) minus (1) and integrate w.r.t. y:

$$\int_0^y p'_y\,dy = -\int_0^y (\rho - \bar{\rho})\,u^0_t\,dy = -u^0_t \int_0^y (\rho - \bar{\rho})\,dy$$

(5) $\Rightarrow \quad p^1(x,y,t) = -u^0_t [\![\rho]\!](y) + \overline{p}^1(x,t)$ ← "constant" of integration

We define here

$$[\![f]\!](y) = \int_0^y (f(\xi) - \overline{f})\,d\xi - \underbrace{\int_0^1\!\!\int_0^y (f(\xi)-\overline{f})\,d\xi}_{\text{This is chosen so } \overline{[\![f]\!]}=0.}$$

Similarly, from (3)-(2) we get

(6) $\quad u^1(x,y,t) = -[\![1/k(y)]\!]\,p^0_t + \overline{u}^1(x,t)$

$\mathcal{O}(\delta^1)$: $$p^1_t + K(y)\,u^1_x + K(y)\,u^2_y = 0$$

$$\rho(y)\,u^1_t + p^1_x + p^2_y = 0$$

Solve for u^2_y, p^2_y and substitute (5) and (6):

$$-u^2_y = \frac{p^1_t}{K(y)} + u^1_x = \frac{-[\![\rho]\!](y)}{K(y)}\,u^0_{tt} + \frac{1}{K(y)}\,\overline{p^1_t} - [\![K^{-1}]\!]\,p^0_{tx} + \overline{u^1_x}$$

$$-p^2_y = \rho(y)\,u^1_t + p^1_x = -\rho(y)[\![K^{-1}]\!](y)\,p^0_{tt} + \rho(y)\,\overline{u^1_t} - u^0_{tx}[\![\rho]\!] + \overline{p^1_x}$$

Integrate these over one period in y:

$$-\overline{K^{-1}[\![\rho]\!]}\,u^0_{tt} + \overline{K^{-1}}\,\overline{p^1_t} + \overline{u^1_x} = 0 \qquad (7)$$

$$-\overline{\rho[\![K^{-1}]\!]}\,p^0_{tt} + \overline{\rho}\,\overline{u^1_t} + \overline{p^1_x} = 0 \qquad (8)$$

(3) + δ(7): $$\overline{K^{-1}}\,p^0_t + u^0_x + \delta\left(\overline{K^{-1}}\,\overline{p^1_t} + \overline{u^1_x} - \overline{K^{-1}[\![\rho]\!]}\,u^0_{tt}\right) = 0$$

(4) + δ(8): $$\overline{\rho}\,u^0_t + p^0_x + \delta\left(\overline{\rho}\,u^1_t + \overline{p^1_x} - \overline{\rho[\![K^{-1}]\!]}\,p^0_{tt}\right) = 0$$

Now $$\overline{u} = u^0 + \delta \overline{u^1} + O(\delta^2)$$

$$\overline{p} = p^0 + \delta \overline{p^1} + O(\delta^2)$$

$C_2 = -C_1 =: -\alpha$

So
$$\overline{K^{-1}}\,\overline{p}_t + \overline{u}_x = \delta \overbrace{\overline{K^{-1}[[\rho]]}}^{C_1}\, \overline{u}_{tt} + O(\delta^2)$$

$$\overline{\rho}\,\overline{u}_t + \overline{p}_x = \delta \underbrace{\overline{\rho[[K^{-1}]]}}_{C_2}\, \overline{p}_{tt} + O(\delta^2)$$

What is the dispersion relation for this equation?

We will use that

$$\overline{f[[g]]} = -\overline{g[[f]]}$$

and in particular $\overline{f[[f]]} = 0.$

$$\overline{K^{-1}}\,\overline{p}_t + \overline{u}_x = \delta\,\overline{K^{-1}[\![\rho]\!]}\,\overline{u}_{tt} + \mathcal{O}(\delta^2)$$

$$\overline{\rho}\,\overline{u}_t + \overline{p}_x = \delta\,\overline{\rho[\![K^{-1}]\!]}\,\overline{p}_{tt} + \mathcal{O}(\delta^2)$$

$$\overline{K^{-1}}\,\overline{p}_t + \overline{u}_x = \mathcal{O}(\delta)$$

$$\overline{\rho}\,\overline{u}_t + \overline{p}_x = \mathcal{O}(\delta)$$

$$\Rightarrow \overline{\rho}\,\overline{u}_{tt} = -\overline{p}_{xt} = \left(\overline{K^{-1}}\right)^{-1} u_{xx} \Rightarrow \overline{u}_{tt} = \frac{1}{\overline{K^{-1}}\,\overline{\rho}}\,\overline{u}_{xx} + \mathcal{O}(\delta)$$

$$\overline{p}_{tt} = \frac{1}{\overline{K^{-1}}\,\overline{\rho}}\,\overline{p}_{xx} + \mathcal{O}(\delta) \qquad \hat{c}^2 = \frac{1}{\overline{K^{-1}}\,\overline{\rho}}$$

$$\overline{K^{-1}}\,\overline{p}_t + \overline{u}_x = \delta\alpha\,\hat{c}^2\,\overline{u}_{xx} + \mathcal{O}(\delta^2)$$

$$\overline{\rho}\,\overline{u}_t + \overline{p}_x = -\delta\alpha\,\hat{c}^2\,\overline{p}_{xx} + \mathcal{O}(\delta^2)$$

$$\begin{bmatrix} \overline{K^{-1}}\partial_t & \partial_x - \delta\alpha\hat{c}^2\partial_x^2 \\ \partial_x + \delta\alpha\hat{c}^2\partial_x^2 & \overline{\rho}\partial_t \end{bmatrix} \rightarrow \begin{bmatrix} -i\omega\overline{K^{-1}} & i\xi + \xi^2\delta\alpha\hat{c}^2 \\ i\xi - \xi^2\delta\alpha\hat{c}^2 & -\overline{\rho}i\omega \end{bmatrix}$$

$$\det M = 0 \Rightarrow -\frac{\omega^2}{\hat{c}^2} - \left(-\xi^2 - \xi^4\delta^2\alpha^2\hat{c}^4\right) = 0$$

$$\omega^2 = \hat{c}^2\xi^2 + \hat{c}^4\xi^4\delta^2\alpha^2$$

$$\omega = \pm\hat{c}\xi\sqrt{1+\hat{c}^2\xi^2\delta^2\alpha^2} \approx \pm\hat{c}\xi\left(1+\tfrac{1}{2}\hat{c}^2\xi^2\delta^2\alpha^2\right)$$

$$\approx \pm\left(\hat{c}\xi + \tfrac{1}{2}\hat{c}^3\xi^3\delta^2\alpha^2\right) \quad \text{Dispersion!}$$

Note:

- The leading dispersive term is $\mathcal{O}(\xi^3)$, like KdV dispersion
- $\hat{c}$ is just an averaged version of the characteristic speed in a homogeneous medium: $c = \sqrt{\frac{K}{\rho}}$
- The dispersion depends on

$$\alpha = \overline{K^{-1}[\![\rho]\!]}$$

Notice that if $K(x) = \beta \frac{1}{\rho(x)}$ then

$$\alpha = \beta\, \overline{\rho[\![\varphi]\!]} = 0.$$

The impedance is defined as

$Z(x) = \sqrt{K(x)\rho(x)}$. If Z is constant, then there is no dispersion!

$$\overline{K^{-1}}\,\overline{p}_t + \overline{u}_x = \delta \overbrace{\overline{K^{-1}[\![\rho]\!]}}^{c_1}\,\overline{u}_{tt} + O(\delta^2)$$

$$\overline{\rho}\,\overline{u}_t + \overline{p}_x = \delta \overbrace{\overline{\rho[\![K^{-1}]\!]}}^{c_2}\,\overline{p}_{tt} + O(\delta^2)$$

Instead of converting t-derivatives to x-derivatives, we could rewrite this as a system of 4 first-order equations:

$$\boxed{\begin{aligned} &\overline{u}_t = v \qquad \overline{p}_t = q \\ &q_t = \frac{1}{\delta c_1}\left(\overline{K^{-1}}\,q + \overline{u}_x\right) \\ &v_t = \frac{1}{\delta c_2}\left(\overline{\rho}\,v + \overline{p}_x\right) \end{aligned}}$$

This is a 1st-order "hyperbolic" system but it's actually dispersive!

$$\begin{bmatrix} \overline{p} \\ \overline{u} \\ q \\ v \end{bmatrix}_t + \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1/\delta c_1 & 0 & 0 \\ 1/\delta c_2 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \overline{p} \\ \overline{u} \\ q \\ v \end{bmatrix}_x = \begin{bmatrix} q \\ v \\ \frac{\overline{K^{-1}}}{\delta c_1} q \\ \frac{\overline{\rho}}{\delta c_2} v \end{bmatrix}$$

All eigenvalues are zero.