Traveling fronts and waves

$$u_t + u u_x = \begin{cases} \varepsilon u_{xx} \\ \varepsilon u_{xxx} \end{cases}$$

① Viscous traveling fronts

We assume $u(x,t) = \tilde{u}(x-ct)$
$= \tilde{u}(\xi)$
$\xi = x - ct$

$u_t \to -c\tilde{u}'$

$u_x \to \tilde{u}'$

$u_{xx} \to \tilde{u}''$

$u_x = \tilde{u}'(\xi)\,\xi_x = \tilde{u}'(\xi)$

Drop tildes

$\to -cu' + uu' = \varepsilon u''$

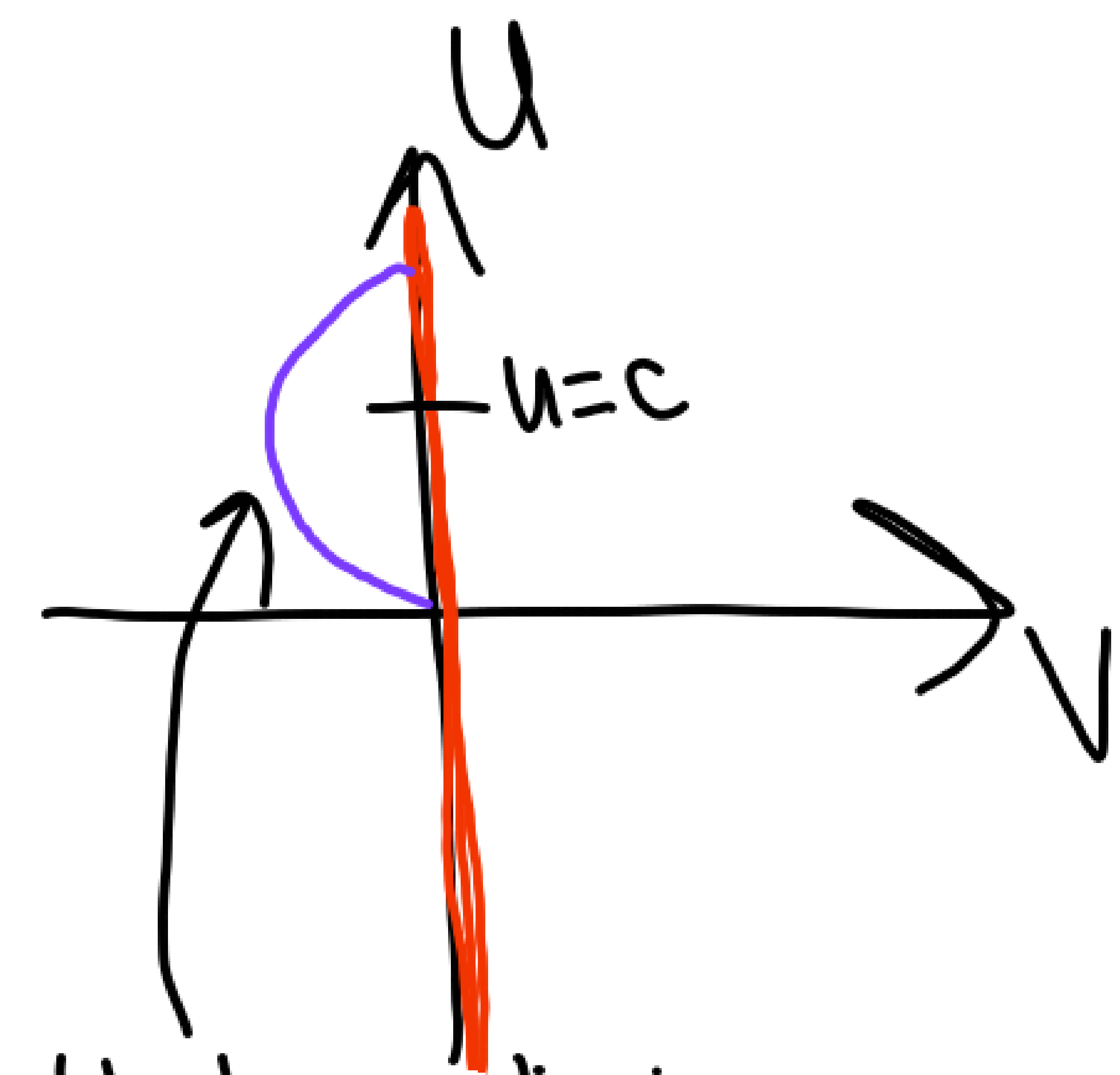
$u'(\xi) = v$

$v'(\xi) = \frac{1}{\varepsilon}(u-c)v$

Equilibria: $u' = v' = 0$

$\iff v = 0$

$$J=\begin{bmatrix}\frac{\partial u'}{\partial u} & \frac{\partial u'}{\partial v}\\ \frac{\partial v'}{\partial u} & \frac{\partial v'}{\partial v}\end{bmatrix}=\begin{bmatrix}0 & 1\\ \cancel{\frac{v}{\varepsilon}}\ 0 & \frac{u-c}{\varepsilon}\end{bmatrix}$$



Heteroclinic Connection

Eigenvalues of J: $\lambda_1=0,\ \lambda_2=\frac{u-c}{\varepsilon}$

If $u>c$: $\lambda_2>0 \Rightarrow$ unstable

$u<c$: $\lambda_2<0 \Rightarrow$ stable

Eigenvectors of J: $w_1=\begin{bmatrix}1\\0\end{bmatrix}$

$w_2=\begin{bmatrix}0\\1\end{bmatrix}$

Dispersive case:

$$u_t + uu_x + \varepsilon u_{xxx} = 0 \qquad \varepsilon > 0$$

$$u(x,t) = \tilde{u}(x-ct) = \tilde{u}(\xi)$$

$$\tilde{u}'(\xi)\xi_t + \tilde{u}\tilde{u}'(\xi)\xi_x + \varepsilon\tilde{u}'''(\xi)\xi_x = 0$$

$$-c\tilde{u}' + \tilde{u}\tilde{u}' + \varepsilon\tilde{u}''' = 0$$

$$-cu' + \left(\tfrac{1}{2}u^2\right)' + \varepsilon u''' = 0$$

$$-cu + \tfrac{1}{2}u^2 + \varepsilon u'' = \alpha$$

$$v' = u'' = \frac{\alpha + cu - \frac{1}{2}u^2}{\varepsilon}$$

$$u' = v$$

$$v' = \frac{\alpha + cu - \frac{1}{2}u^2}{\varepsilon}$$

Equilibria:

$$u' = v = 0$$

$$\frac{\alpha + cu - \frac{1}{2}u^2}{\varepsilon} = 0$$

$$u^2 - 2cu - 2\alpha = 0$$

$$u = c \pm \underbrace{\sqrt{c^2 + 2\alpha}}_{>0}$$

$$J = \begin{bmatrix} 0 & 1 \\ \frac{c-u}{\varepsilon} & 0 \end{bmatrix}$$

At equilibrium:

$$J = \begin{bmatrix} 0 & 1 \\ \frac{\mp\sqrt{c^2+2\alpha}}{\varepsilon} & 0 \end{bmatrix}$$

$$\lambda = \pm\sqrt{\mp\frac{\sqrt{c^2+2\alpha}}{\varepsilon}}$$

$+$: Saddle (hyperbolic)

$-$: Center

Homoclinic Connection

Separatrix