Integrable Systems

Integrable PDEs:

- Have a countably infinite number of conserved quantities

$$\int F(u)\,dx$$

- Possess a Lax Pair (two DEs such that the PDE is their compatibility)
- Can be solved using inverse scattering

Scaling Symmetry

$$u_t + uu_x + u_{xxx} = 0$$

$$\hat{x} = \varepsilon x \qquad \hat{t} = \varepsilon^3 t \qquad \hat{u} = \varepsilon^{-2} u$$

$$\partial_x = \varepsilon \partial_{\hat{x}} \qquad \partial_t = \varepsilon^3 \partial_{\hat{t}} \qquad u = \varepsilon^2 \hat{u}$$

$$\rightarrow \cancel{\varepsilon^5} \hat{u}_{\hat{t}} + \cancel{\varepsilon^5} \hat{u}\hat{u}_{\hat{x}} + \cancel{\varepsilon^5} \hat{u}_{\hat{x}\hat{x}\hat{x}} = 0$$

How to derive it:

Assume $\hat{x} = \varepsilon^a x \quad \hat{t} = \varepsilon^b t \quad u = \varepsilon^c \hat{u}$

$$\partial_x \to \varepsilon^a \partial_{\hat{x}} \qquad \partial_t \to \varepsilon^b \partial_{\hat{t}}$$

$$\varepsilon^{b+c} \hat{u}_{\hat{t}} + \varepsilon^{2c+a} \hat{u}\hat{u}_{\hat{x}} + \varepsilon^{3a+c} \hat{u}_{\hat{x}\hat{x}\hat{x}} = 0$$

$$b + c = 2c + a = 3a + c$$

$$b - 3a = 0$$

$$\boxed{b = 3a}$$

$$c - 2a = 0$$

$$\boxed{c = 2a}$$

$a = 1$

$b = 3$

$c = 2$

We call a, b, c the "weights" of $\partial_x, \partial_t,$ and u

We write:

$$[\partial_x] = 1 \quad [\partial_t] = 3 \quad [u] = 2$$

Conserved densities of KdV

$\int q(u)\,dx$ is conserved if there exist $f(u)$ such that

$$q_t = -f_x \quad \text{i.e.} \quad q_t + f_x = 0$$

Since $\int q_t\,dx = -\int f_x\,dx$

$$\frac{d}{dt}\int q\,dx = \lim_{x\to\infty}\left(f(u(x,t)) - f(u(-x,t))\right)$$

(We assume $f \to 0$ as $|x| \to \infty$)

The weights should be equal:

$$[q_t] = [q] + \underset{3}{[\partial_t]} = [f] + \underset{1}{[\partial_x]} = [f_x]$$

$$[f] = [q] + 2.$$

Smallest weight for q?

$[q] = 2$: $q = u$

$[f] = 4$: $f = c_1 u^2 + c_2 u_{xx}$

$$q_t + f_x = u_t + (c_1 u^2 + c_2 u_{xx})_x$$

$$= u_t + 2c_1 u u_x + c_2 u_{xxx} \equiv 0$$

$$c_1 = \frac{1}{2} \quad c_2 = 1$$

So $\int u\,dx$ is conserved

The conservation law is just

$$u_t + uu_x + u_{xxx} = 0.$$

$[q] = 3$: $q = u_x$

Trivially: $q_t = u_{xt} = \underbrace{(u_t)_x}_{-f_x}$

$$f(u) = u_t$$

$[u] = 2$ $[\partial_x] = 1$ $[\partial_t] = 3$

$$q_t + f_x = 0$$

$$u_t = -uu_x - u_{xxx}$$

$[q] = 4$: $q = \cancel{d_2 u_{xx}} + d_1 u^2$ $d_1 = 1$

$[f] = 6$: $f(u) = c_1 u^3 + c_2 u_{xxxx} + c_3 u_x^2 + c_4 u u_{xx}$

Since u_{xx} is an x-derivative: $(u_x)_x$ it's automatically conserved.

$$(u^2)_t + (c_1 u^3 + c_2 u_{xxxx} + c_3 u_x^2 + c_4 u u_{xx})_x = 0$$

$$2uu_t + 3c_1 u^2 u_x + c_2 u_{5x} + 2c_3 u_x u_{xx} + c_4 (u_x u_{xx} + u u_{xxx}) = 0$$

$$\underline{2}u(-uu_x \underline{-u_{xxx}}) + \underline{3c_1} u^2 u_x + \underline{c_2 u_{5x}} + (2c_3 + c_4) u_x u_{xx} \underline{+ c_4} u u_{xxx} = 0$$

$$(3c_1-2)u^2u_x+(c_4-2)uu_{xxx}+c_2u_{5x}+(2c_3+c_4)u_xu_{xx}=0$$

$3c_1-2=0$ $\quad c_4-2=0$ $\quad c_2=0$ $\quad 2c_3+c_4=0$

$c_1=\frac{2}{3}$ $\quad c_4=2$ $\quad 2c_3=-2$

$c_3=-1$

So

$$(u^2)_t+\left(\frac{2}{3}u^3-u_x^2+2uu_{xx}\right)_x=0$$

$$\int w\,dx=\text{constant}$$

$$u_t+uu_x=\varepsilon u_{xx}\rightarrow\frac{d}{dt}\int u^2dx<0$$

$$u_t+uu_x=-\varepsilon u_{xxx}\rightarrow\frac{d}{dt}\int u^2dx=0$$

Lax Pairs

Pair of linear operators $X(u,\lambda)$, $T(u,\lambda)$ such that

if

$$\psi_x = X(u,\lambda)\psi \quad (1)$$

$$\psi_t = T(u,\lambda)\psi \quad (2)$$

then the condition

$$\psi_{xt} = \psi_{tx}$$

gives the PDE.

For KdV, (1) is the time-independent Schrodinger equation with u as the potential.

$$\omega\psi = -\psi_{xx} + \underset{\substack{\uparrow \\ u}}{V(x)}\psi$$

$\int_a^b |\psi|^2 dx$ = probability of observing particle in $[a,b]$

The inverse scattering method

$$
\begin{tikzcd}
u(x,0) \arrow[d, "\text{Forward scattering (1)}"'] & & u(x,t) \\
S(0) \arrow[rr, "\text{time}", "\text{evolution (2)}"'] & & S(t) \arrow[u, "\text{Inverse scattering (3)}"']
\end{tikzcd}
$$

eigenvalue problem

$$(u-\partial_x^2)\psi = \omega\psi$$

eigenvalue

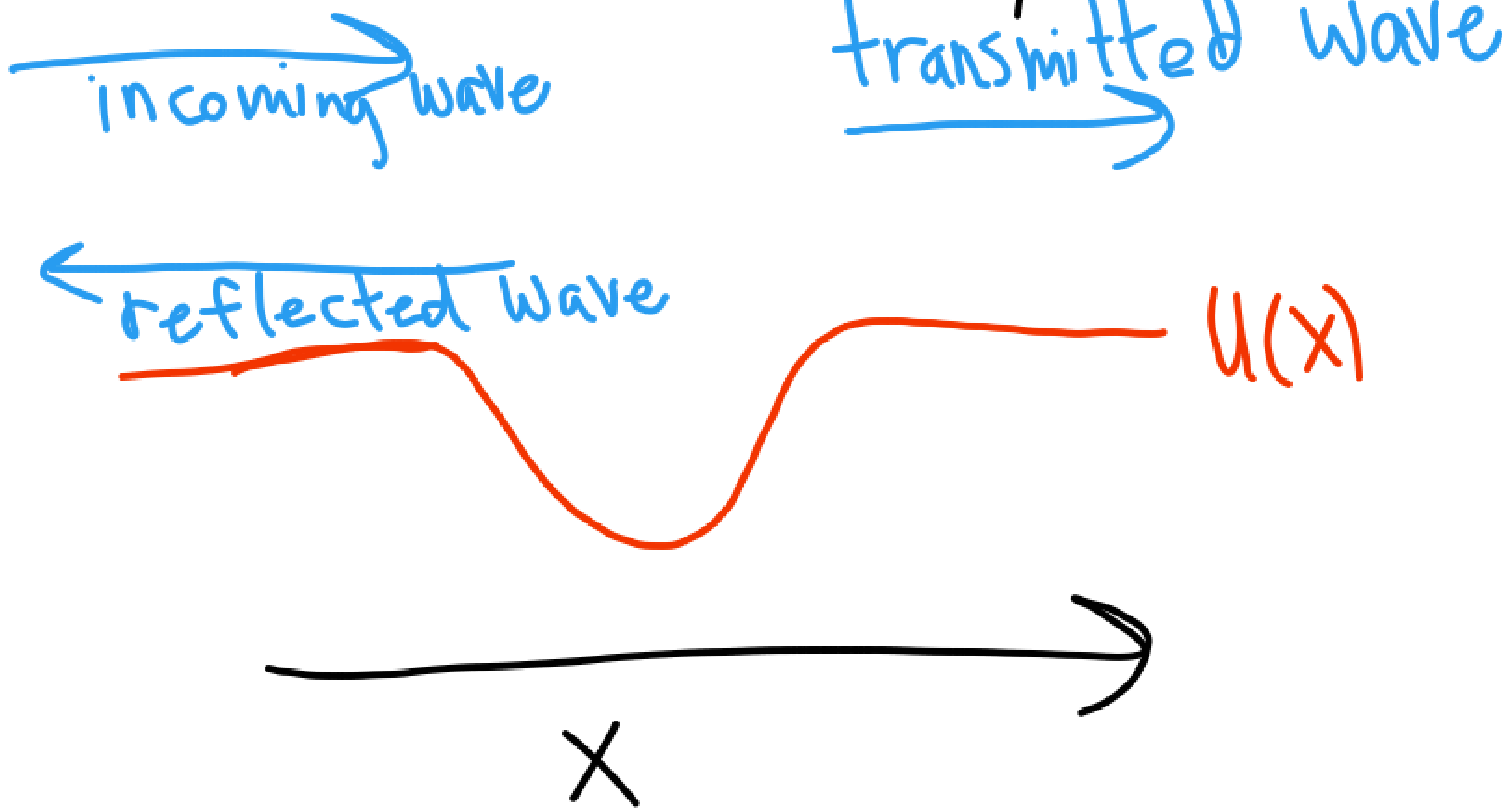
An analogy

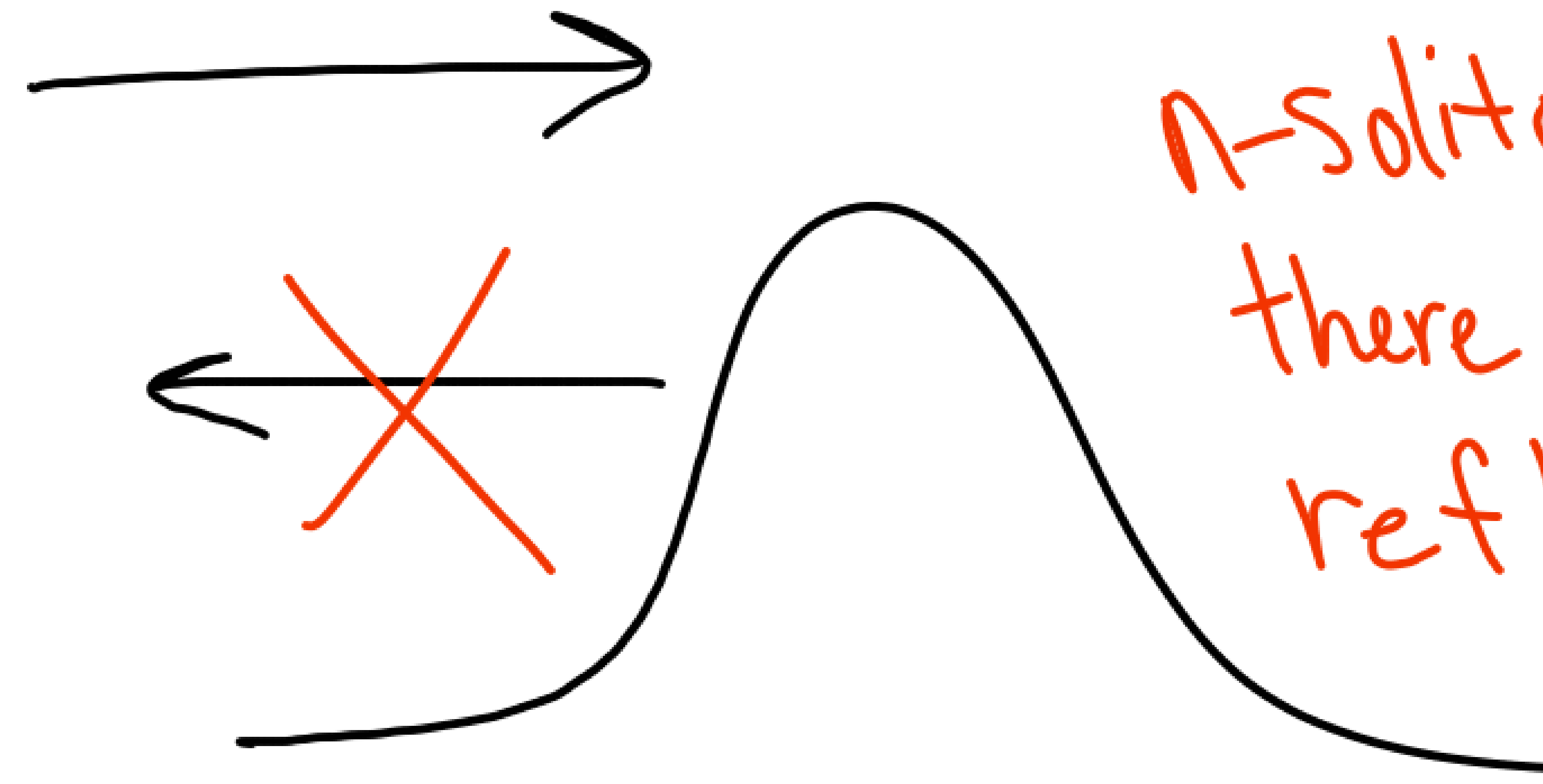
Solution of linear PDE:

$$
\begin{tikzcd}
u(x,0) \arrow[rr, "\text{PDE}", "\text{solution}"'] \arrow[d, "\text{F.T.}"'] & & u(x,t) \\
\hat{u}(k,0) \arrow[rr, "e^{-i\omega t}"] & & \hat{u}(k,t) \arrow[u, "\text{IFT}"']
\end{tikzcd}
$$

Forward scattering

$$\omega\psi = -\psi_{xx} + u\psi$$





if u is a n-soliton solution, there is no reflection

S(t) relates the incoming and outgoing.