NLS from KdV

$$u_t = u u_x + u_{xxx}$$

Linearize about $u=0$:

$$u = 0 + \varepsilon \tilde{u}(x,t)$$

$$\varepsilon \tilde{u}_t = \varepsilon^2 \tilde{u}\tilde{u}_x + \varepsilon \tilde{u}_{xxx}$$

$$\tilde{u}_t = \tilde{u}_{xxx} + \mathcal{O}(\varepsilon^2)$$

$$\omega(k) = k^3$$

$\omega'(k) = 3k^2$ Group velocity

$\frac{\omega(k)}{k} = k^2$ Phase velocity

So we can write the solution as

$$u(x,t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \hat{u}(k) e^{i(kx - \omega(k)t)}\, dk$$

where $\hat{u}(k)$ is the Fourier transform of the initial data $u(x,t=0)$.

Now suppose we have a wavepacket solution with dominant wavenumber k_0.

Then we write

$$\omega(k) = \underbrace{\omega(k_0)}_{\omega_0} + (k-k_0)\underbrace{\omega'(k_0)}_{\omega_0'} + \tilde{\Omega}(k)$$

Here $\tilde{\Omega}(k) = \mathcal{O}((k-k_0)^2)$.

We want to express the solution as a product of rapid oscillations and a slowly-varying envelope.

We have

$$u(x,t)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\hat{u}(k)e^{ikx-it(\omega_0+(k-k_0)\omega_0'+\tilde{\Omega})}dk$$

We assume that A varies slowly compared to $e^{i(k_0x-\omega_0t)}$

$$ikx-it(\omega_0+(k-k_0)\omega_0'+\tilde{\Omega})$$

$$=ik_0x-i\omega_0t+i(k-k_0)(x-\omega_0't)-i\tilde{\Omega}t$$

$$=i(k_0x-\omega_0t)+i(k-k_0)\xi-i\tilde{\Omega}t$$

So $$u(x,t)=\frac{1}{\sqrt{2\pi}}e^{i(k_0x-\omega_0t)}\int_{-\infty}^{\infty}\hat{u}(k)e^{i(k-k_0)\xi-i\tilde{\Omega}(k)t}dk$$

$$=\frac{1}{\sqrt{2\pi}}e^{i(k_0x-\omega_0t)}\int_{-\infty}^{\infty}\hat{u}(k+k_0)e^{ik\xi-i\Omega(k)t}dk=e^{i(k_0x-\omega_0t)}A(\xi,t)$$

Where $\Omega(k)=\tilde{\Omega}(k-k_0)$

We introduce the expansions

$$u = \varepsilon u^1 + \varepsilon^2 u^2 + \varepsilon^3 u^3 + \mathcal{O}(\varepsilon^4)$$

and $\xi = \varepsilon(x - \omega_0' t)$ $\quad \tau_2 = \varepsilon^2 t$

Why not $\tau_1 = \varepsilon t$? (no benefit)

with $\varepsilon \ll 1$.

So we transform the partial derivatives:

$$\partial_x \to \partial_x + \varepsilon \partial_\xi$$

$$\partial_t \to \partial_t + \varepsilon^2 \partial_{\tau_2} - \varepsilon \omega_0' \partial_\xi$$

We assume $u^1 = e^{i(k_0 x - \omega_0 t)} A(\xi, \tau_2) + \text{c.c.}$

$$u_t = u u_x + u_{xxx}$$

$$u_t + \varepsilon^2 u_{\tau_2} - \varepsilon \omega_0' u_\xi = u(u_x + \varepsilon u_\xi)$$

$$+ u_{xxx} + 3\varepsilon u_{xx\xi} + 3\varepsilon^2 u_{x\xi\xi} + \varepsilon^3 u_{\xi\xi\xi}$$

$$\varepsilon u_t^1 + \varepsilon^2 u_t^2 + \varepsilon^3 u_t^3 + \varepsilon^3 u_{\tau_2}^1 - \varepsilon^2 \omega_0' u_\xi^1$$

$$- \varepsilon^3 \omega_0' u_\xi^2 = \varepsilon^2 u^1 u_x^1 + \varepsilon^3 (u^1 u_x^2 + u^2 u_x^1)$$

$$+ \varepsilon^3 u^1 u_\xi^1 + \varepsilon u_{xxx}^1 + \varepsilon^2 u_{xxx}^2 + \varepsilon^3 u_{xxx}^3 + 3\varepsilon^2 u_{xx\xi}^1$$

$$+ 3\varepsilon^3 u_{xx\xi}^2 + 3\varepsilon^3 u_{x\xi\xi}^1 + \mathcal{O}(\varepsilon^4)$$

$\mathcal{O}(\varepsilon^1)$: $u^1_t = u^1_{xxx}$

$\mathcal{O}(\varepsilon^2)$: $u^2_t - \omega_0' u^1_\xi = u^1 u^1_x + u^2_{xxx} + 3u^1_{xx\xi}$

$$u^2_t - u^2_{xxx} = u^1 u^1_x + \omega_0' u^1_\xi + 3u^1_{xx\xi}$$

Substitute: $u^1 = e^{i(k_0 x - \omega_0 t)} A(\xi, \tau_2) + c.c.$

$$u^2_t - u^2_{xxx} = \left(e^{i(k_0x-\omega_0 t)} A + c.c.\right)\left(ik_0 e^{i(k_0x-\omega_0 t)} A + c.c.\right)$$
$$+\left(\omega_0' e^{i(k_0x-\omega_0 t)} A_\xi + c.c.\right) - \left(3k_0^2 e^{i(k_0x-\omega_0 t)} A_\xi + c.c.\right)$$

$$u^2_t - u^2_{xxx} = \underbrace{(\omega_0' - 3k_0^2)}_{=0} e^{i(k_0x-\omega_0 t)} A_\xi + ik_0 e^{2i(k_0x-\omega_0 t)} A^2 + c.c.$$

So $u^2_t - u^2_{xxx} = ik_0 e^{2i(k_0x-\omega_0 t)} A^2 + c.c.$

Now we assume a wavepacket ansatz for u^2:

$$u^2 = e^{2i(k_0x-\omega_0 t)} B(\xi, \tau_2) + c.c.$$
$$+ C(\xi, \tau_2)$$

We get

$$(-2i\omega_0 + 8ik_0^3)\cancel{e^{2i(k_0x-\omega_0 t)}} B$$
$$= ik_0 \cancel{e^{2i(k_0x-\omega_0 t)}} A^2$$

$\omega_0 = k_0^3$

$$6ik_0^3 B = ik_0 A^2$$

$$B = \frac{1}{6k_0^2} A^2$$

Next steps:

1. Write $\mathcal{O}(\varepsilon^3)$ eqn.
2. Substitute u^1, u^2
3. We get

Each must $=0$

$$u^3_t - u^3_{xxx} = \sum_{j=1}^{3} e^{ij(k_0 x - \omega_0 t)} f_j(\xi, \tau_2)$$

$$\underbrace{+ 3k_0^2 C_\xi + A\bar{A}_\xi + \bar{A}A_\xi}_{\text{must } =0}$$

$$3k_0^2 C_\xi + (|A|^2)_\xi = 0 \Rightarrow 3k_0^2 C + |A|^2 = D(\tau_2)$$

$$C(\xi, \tau_2) = \frac{-1}{3k_0^2}|A|^2 + D(\tau_2)$$

$$f_1 = -A_{\tau_2} + \frac{i}{6k_0} + |A|^2 A + 3ik_0 A_{\xi\xi} + ik_0 AC = 0$$

$$(i\partial_{\tau_2} + k_0 D)A = -3k_0 A_{\xi\xi} + \frac{1}{6k_0}|A|^2 A$$

Now substitute

$$A(\xi, \tau_2) = a(\xi, \tau_2) e^{ik_0 \int_0^{\tau_2} D(\hat{\tau}_2)\, d\hat{\tau}_2}$$

So $A_{\tau_2} = (a_{\tau_2} + ik_0 D a) e^{ik_0 \int D\, d\tau_2}$

$$ia_{\tau_2} = -3k_0 a_{\xi\xi} + \frac{1}{6k_0}|a|^2 a$$

Nonlinear Schrodinger (defocusing)

$$V = \frac{1}{6K_0}|a|^2$$

$$\left. i a_{\tau_2} = -3K_0 a_{\xi\xi} + V a \right\}$$ looks like linear Schr.