$$u = \alpha + \varepsilon\varphi + O(\varepsilon^2)$$

$$\varepsilon\varphi_t + (\alpha + \varepsilon\varphi)(\varepsilon\varphi_x) + \int_{-\infty}^{\infty} K(x-y)\,\varepsilon\varphi_y(y,t)\,dy = 0$$

$$\varphi_t + \alpha\varphi_x + \int_{-\infty}^{\infty} K(x-y)\,\varphi_y(y,t)\,dy = 0 \qquad \varphi(x,t) = e^{i(kx-\omega t)}$$

$$-\omega e^{i(kx-\omega t)} + \alpha k e^{i(kx-\omega t)} + k\int_{-\infty}^{\infty} K(x-y)\,e^{i(ky-\omega t)}\,dy = 0 \qquad (e^{i\omega t})$$

$$\xi = x - y \Rightarrow d\xi = -dy$$

$$(-\omega + \alpha k)e^{ikx} + k\int_{-\infty}^{\infty} K(\xi)\,e^{ikx - ik\xi}\,d\xi = 0 \qquad (e^{-ikx})$$

$$(-\omega + \alpha k) + k\underbrace{\int_{-\infty}^{\infty} K(\xi)\,e^{-ik\xi}\,d\xi}_{\mathcal{F}\{K\}(k)} = 0 \qquad K(\xi) = \mathcal{F}^{-1}\{c\}(\xi)$$

$$(-\omega + \alpha k) + k\,\mathcal{F}\{\mathcal{F}^{-1}\{c\}\}(k) = 0$$

$$(-\omega+\alpha K)+ \quad K C(K)=0 \Rightarrow \omega = \alpha K + K C(K)$$

$$\alpha=0 \Rightarrow \frac{\omega(K)}{K}=C(K)$$

$$u_t + \vartheta u_x + u u_x + \gamma u_{xxx} = 0$$

$$u = \alpha + \varepsilon\varphi + O(\varepsilon^2)$$

$$O(\varepsilon): \varphi_t + (\vartheta+\alpha)\varphi_x + \gamma\varphi_{xxx} = 0$$

$$\varphi = e^{i[kx - \omega(k)t]}$$

$$\omega(k) = \alpha k + \vartheta k + (-\gamma)k^3$$

$$\tanh x = x - \frac{x^3}{3} + O(x^5)$$

$$c(k) = \sqrt{\frac{g}{k}\tanh(kh)}$$

$$\sqrt{\tanh(kh)} = \left(kh - \frac{k^3h^3}{3}\right)^{1/2}$$

$$= \sqrt{kh}\left(1 - \frac{k^2h^2}{3}\right)^{1/2}$$

$$= \sqrt{kh}\left(1 - \frac{k^2h^2}{6}\right)$$

$$C(k) = \sqrt{gh}\left(1 - \frac{k^2 h^2}{6}\right)$$

$$\omega(k) = \alpha k + \sqrt{gh}\, k + \left(-\sqrt{gh}\,\frac{h^2}{6}\right) k^3$$

$$v = \sqrt{gh}$$

$$\gamma = \frac{v h^2}{6}$$

$u = u_0 + \delta \bar{u}$

$w = w_0 + \delta \bar{w}$ $\qquad (u_0, w_0) = (0,0)$

$$\delta \bar{u}_t = \delta \bar{u}_{xx} + \cancel{2(1+a)\,\delta u_0 \bar{u}} - 3 u_0^2 \delta \bar{u} - a \delta \bar{u} + \delta \bar{w}$$

$$\delta \bar{w}_t = \epsilon (\cancel{u_0} + \delta \bar{u})$$

$$\bar{u}_t = \bar{u}_{xx} - a\bar{u} + \bar{w}$$

$$\bar{w}_t = \epsilon \bar{u}$$

$$\bar{u} = \bar{w} = e^{i(kx - \omega t)}$$

$$-i\omega \bar{u} = -k^2 \bar{u} - a\bar{u} + \bar{w}$$

$$-i\omega \bar{w} = \epsilon \bar{u}$$

$$(-i\omega + k^2 + a)\bar{u} - \bar{w} = 0$$

$$\epsilon \bar{u} + i\omega \bar{w} = 0$$

$$\underbrace{\begin{bmatrix} -i\omega + k^2 + a & -1 \\ \epsilon & i\omega \end{bmatrix}}_{} \begin{bmatrix} \bar{u} \\ \bar{w} \end{bmatrix} = \vec{0}$$

$$\det m = 0$$

$$(-i\omega + k^2 + a) i\omega + \epsilon = 0$$

$a, \epsilon > 0$

$$+\omega^2 + (k^2 + a) i\omega + \epsilon = 0$$

$$\omega_{1,2} = \frac{-(k^2+a)i \pm \sqrt{-(k^2+a)^2 - 4\epsilon}}{2}$$

$$= \frac{-(k^2+a)i \pm i\sqrt{(k^2+a)^2 + 4\epsilon}}{2}$$

$$\omega_{1,2} = \frac{-(k^2+a) \pm \sqrt{(k^2+a)^2 + 4\epsilon}}{2} i$$

$O(\delta^0)$:

$$u_0(u_0 - a)(1 - u_0) + w_0 = 0$$

$$\varepsilon u_0 = 0$$

$$(u_0, w_0) = (0, 0)$$

Nonlinearity

$$u_t = -uu_x$$

(Inviscid Burgers eqn.)

Similar to advection:

$$u_t + cu_x = 0$$

with $c = u$.

Characteristics

$$X(t) = x_0 + u(X(t), t)t$$

$$\frac{d}{dt}u(X(t), t) = u_x X'(t) + u_t$$

$$X'(t) = u$$

$$\frac{du}{dt} = uu_x + u_t = 0$$

So u is constant along each characteristic, and char's are straight lines.

Solution via characteristics exists up to some finite time

After that?

In order to have single-valued solutions at later times, we allow solutions with discontinuities

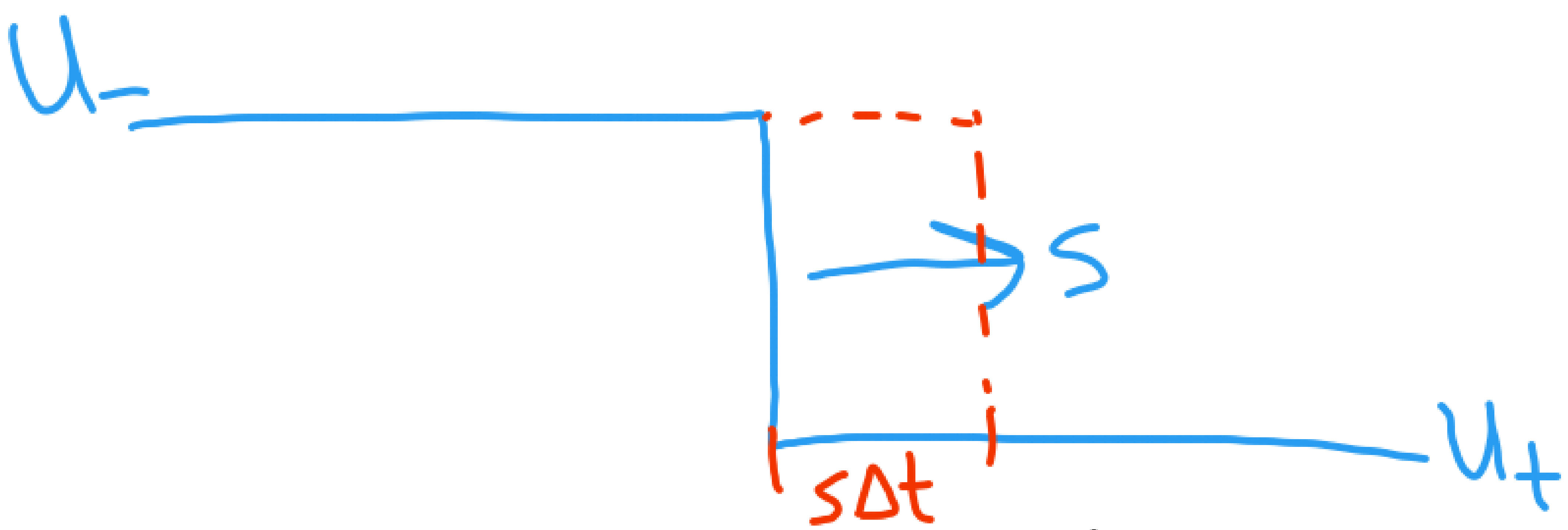
What condition(s) should such a discontinuity satisfy?

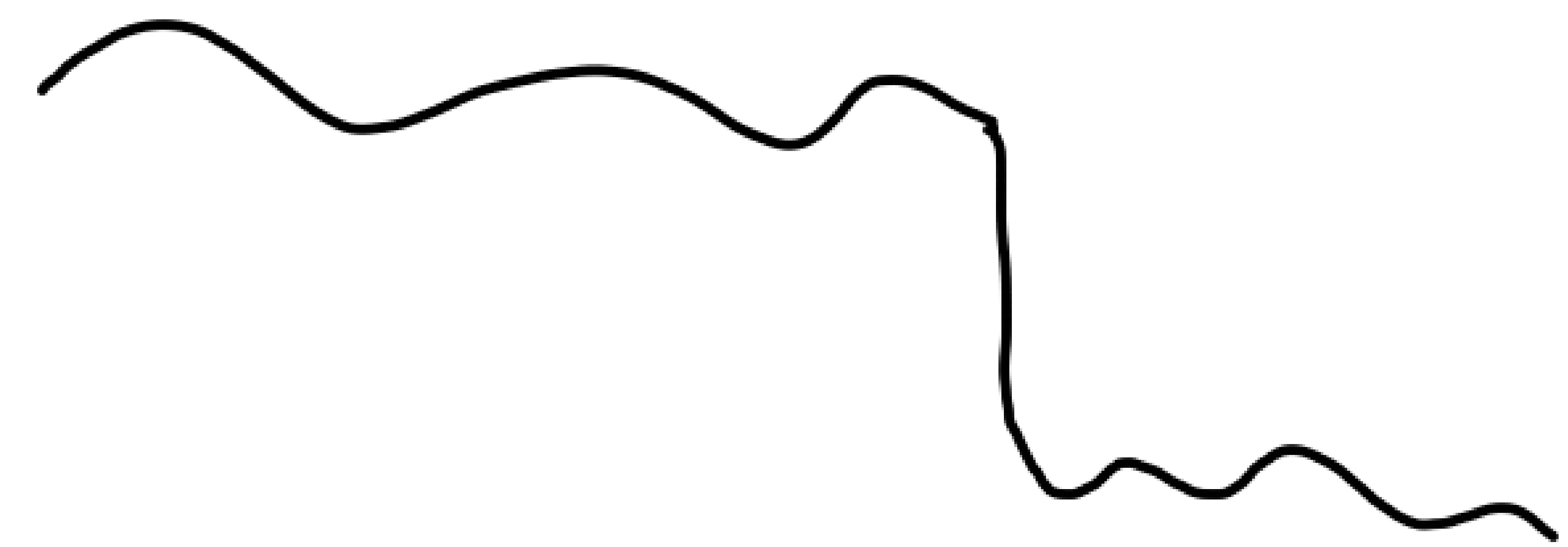
$$\frac{d}{dt}\int_{-\infty}^{\infty} u(x,t)\,dx = \int_{-\infty}^{\infty} u_t(x,t)\,dx$$

$$= -\int_{-\infty}^{\infty} u u_x\,dx = -\int_{-\infty}^{\infty} \frac{1}{2}(u^2)_x\,dx$$

$$= -\frac{1}{2}u^2\Big|_{-\infty}^{\infty} = \frac{1}{2}\left((u(-\infty))^2 - (u(+\infty))^2\right)$$

Consider a single d.c. moving at speed s, separating states u_+, u_-:





In a time interval Δt,

$\int_{-\infty}^{\infty} u\,dx$ changes by $s\Delta t(u_- - u_+)$

So $$\frac{d}{dt}\int_{-\infty}^{\infty} u\,dx = s(u_- - u_+) = \frac{1}{2}\left((u_-)^2 - (u_+)^2\right)$$

Rankine Hugoniot conditions

$$s = \frac{1}{2}\frac{u_-^2 - u_+^2}{u_- - u_+} = \frac{1}{2}(u_- + u_+)$$

$u_t + u u_x = -u_{xxx}$ Dispersive shock

$u_t + \underbrace{u u_x}_{\text{generates high } k} = \underbrace{u_{xx}}_{\text{dissipates high } k}$ Viscous shock