$$\omega^2 = \hat{c}^2 \xi^2 + \hat{c}^4 \xi^4 \delta^2 \alpha^2$$

$$\omega = \pm \hat{c} \xi \sqrt{1 + \hat{c}^2 \xi^2 \delta^2 \alpha^2}$$

$$\bar{\rho}$$

$$\begin{bmatrix} \hat{p} \\ \hat{u} \end{bmatrix}_t = \begin{bmatrix} 0 & \frac{-d\xi^2 - i\xi}{d_1} \\ \frac{d\xi^2 - i\xi}{d_2} & 0 \end{bmatrix} \begin{bmatrix} \hat{p} \\ \hat{u} \end{bmatrix}$$

Dispersion relations for hyperbolic systems

Linear hyperbolic system:

$$q_t + A q_x = 0$$

$$(I\partial_t + A\partial_x)q = 0$$

$$\underbrace{(-i\omega I + ikA)}_{ikM}\hat{q} = 0$$

$$\det(M) = 0$$

$$M = A - \frac{\omega}{k}I$$

$$\det\left(A - \frac{\omega}{k}I\right) = 0$$

$\Rightarrow \frac{\omega}{k} = \lambda$ where λ is an eigenvalue

$$\left.\begin{aligned} c_p &= \frac{\omega}{k} = \lambda \\ c_g &= \omega'(k) = \lambda \end{aligned}\right\}$$ c_p and c_g are equal to the characteristic speeds

Waves are characterized by the fact that information moves at finite speed.

What about dispersive equations?

KdV: $u_t + uu_x - u_{xxx} = 0$

$$\omega(k) = ku_0 + k^3$$

$$\left.\begin{aligned} c_p &= \frac{\omega}{k} = u_0 + k^2 \\ c_g &= \omega'(k) = u_0 + 3k^2 \end{aligned}\right\} \text{both are unbounded as } |k| \to \infty$$

BBM: $u_t + uu_x - u_{txx} = 0$

$$\omega(k) = \frac{ku_0}{1+k^2}$$

$$\left.\begin{aligned} c_p &= \frac{\omega}{k} = \frac{u_0}{1+k^2} \\ c_g &= \omega'(k) = u_0 \frac{1-k^2}{1+k^2} \end{aligned}\right\} \text{Bounded as } |k| \to \infty$$

Lagrangian Mechanics

Many dynamical systems correspond to "stationary points" of some "Lagrangian":

$$L(u, u_x, u_{xx}, \ldots, u_t, u_{tt}, \ldots, u_{xt} \ldots)$$

The Fundamental Thm. of the calculus of variations says that

$$S = \int L \, dx\, dt$$

is extremized when the variational derivative vanishes:

$$\frac{\delta S}{\delta u} = 0 \leftarrow \text{Euler-Lagrange Equation}$$

The variational derivative is:

$$\frac{\delta S}{\delta u} = \frac{\partial L}{\partial u} - \frac{\partial}{\partial x}\frac{\partial L}{\partial u_x} + \frac{\partial^2}{\partial x^2}\frac{\partial L}{\partial u_{xx}} + \cdots$$

$$- \frac{\partial}{\partial t}\frac{\partial L}{\partial u_t} + \frac{\partial^2}{\partial t^2}\frac{\partial L}{\partial u_{tt}} + \cdots$$

$$+ \frac{\partial^2}{\partial x \partial t}\frac{\partial L}{\partial u_{xt}} + \cdots$$

See Deconinck 6.3.

Example 1

N particles in 1D moving under force of a potential.

Position of particle i: $x_i(t)$

Potential: $V(x_1, x_2, \dots, x_N)$

Newton says:

$$m_i x_i''(t) = -\frac{\partial V}{\partial x_i}$$

This eqn. can be obtained by defining the Lagrangian

$$L(x, x'(t)) = \underbrace{\sum_{i=1}^{N} \frac{1}{2} m_i (x_i'(t))^2}_{\text{Kinetic Energy}} - \underbrace{V(x_1, \dots, x_N)}_{\text{Potential Energy}}$$

Euler-Lagrange eqns. are

$$0 = \frac{\partial L}{\partial x_i} - \frac{d}{dt}\frac{\partial L}{\partial x_i'} = -\frac{\partial V}{\partial x_i} - \frac{d}{dt}(m_i x_i'(t))$$

$$= -\frac{\partial V}{\partial x_i} - m_i x_i''(t)$$

The Lagrangian for BBM:

$$u = \phi_x$$

$$L(\phi_t, \phi_x, \phi_{xxx}) = -\frac{\phi_t \phi_x}{2} + \frac{\phi_t \phi_{xxx}}{2} - \frac{\phi_x^3}{6}$$

The E.L. eqn. for this is

$$0 = -\frac{\partial}{\partial t}\frac{\partial L}{\partial \phi_t} - \frac{\partial}{\partial x}\frac{\partial L}{\partial \phi_x} - \frac{\partial^3}{\partial x^3}\frac{\partial L}{\partial \phi_{xxx}}$$

$$= -\frac{\partial}{\partial t}\left(-\frac{\phi_x}{2} + \frac{\phi_{xxx}}{2}\right) - \frac{\partial}{\partial x}\left(-\frac{\phi_t}{2} - \frac{(\phi_x)^2}{2}\right) - \frac{\partial^3}{\partial x^3}\frac{\phi_t}{2}$$

$$0 = \frac{\phi_{xt}}{2} - \frac{\phi_{xxxt}}{2} + \frac{\phi_{xt}}{2} + \phi_x \phi_{xx} - \frac{\phi_{xxxt}}{2}$$

$$\phi_{xt} - \phi_{xxxt} + \phi_x \phi_{xx} = 0$$

$$u_t + u u_x - u_{xxt} = 0$$