$$u_t = Lu + f(u)$$

$$u_t = Lu$$

$$u_t = f(u)$$

$$\left(F^{-1}F - a_{ii}hF^{-1}DF\right)^{-1}$$

$$F^{-1}\underbrace{\left(I - a_{ii}hD\right)}_{L\text{ factor}}^{-1}F$$

$$\sum_j b_j = 1$$

$$\sum_j b_j c_j = \frac{1}{2}$$

$$\sum_j b_j c_j^2 = \frac{1}{3}$$

$$\sum_{i,j} b_i a_{ij} c_j = \frac{1}{6}$$

$$\sum_j \tilde{b}_j = 1$$

$$\sum_j \hat{b}_j \tilde{c}_j = \frac{1}{2}$$

$$\sum_j \hat{b}_j c_j = \sum_j b_j \tilde{c}_j = 1/2$$

Exponential Integrators

The linear scalar IVP

$$u'(t) = \lambda u + g(t) \quad u(0) = u_0$$

has solution:

$$u(t) = e^{\lambda t} u_0 + \int_0^t e^{\lambda(t-\tau)} g(\tau)\, d\tau$$

Similarly: $u'(t) = Lu + f(u) \quad u(0) = u_0$

has solution: $u(t) = e^{tL} u_0 + \int_0^t e^{(t-\tau)L} f(u(\tau))\, d\tau$ $\qquad \Delta t = t_{n+1} - t_n$

We can write: $u(t_{n+1}) = e^{\Delta t L} u(t_n) + \int_0^{\Delta t} e^{(\Delta t-\tau)L} f(u(t_n+\tau))\, d\tau$

We approximate

$$U^n \approx u(t_n)$$

and

$$f(u(t_n+\tau)) = f(U^n)$$

$$L = F^{-1} D F$$

$$L^{-1} = F^{-1} D^{-1} F$$

This gives

$$U^{n+1} = e^{\Delta t L} U^n + \int_0^{\Delta t} e^{(\Delta t-\tau)L} f(U^n)\, d\tau$$

$$U^{n+1} = e^{\Delta t L} U^n - \left[e^{(\Delta t-\tau)L} L^{-1} f(U^n) \right|_{\tau=0}^{\Delta t}$$

$$U^{n+1} = e^{\Delta t L} U^n - (I - e^{\Delta t L}) L^{-1} f(U^n)$$

Exponential Time Differencing Euler Method

$$U^{n+1} = e^{\Delta t L} U^n + \Delta t\, \phi_1(\Delta t L)\, f(U^n)$$

where $\phi_1(z) = \dfrac{e^z - 1}{z}$

Lawson Methods

$$u'(t) = Lu + f(u)$$

$$e^{-tL}u'(t) = e^{-tL}Lu + e^{-tL}f(u)$$

$$e^{-tL}(u'(t) - Lu) = e^{-tL}f(u)$$

Introduce "twisted" variable

$$w(t) = e^{-tL}u(t)$$

Note that:

$$w'(t) = e^{-tL}u'(t) - Le^{-tL}u(t)$$

$$= e^{-tL}(u'(t) - Lu)$$

$$u(t) = e^{tL}w(t)$$

So we have

$$w'(t) = e^{-tL}f(e^{tL}w(t)) \quad (*)$$

Lawson methods are obtained by discretizing this equation.

Lawson-Euler:

$$w^{n+1} = w^n + \Delta t\, e^{-t_nL}f(e^{t_nL}w^n)$$

$$e^{-t_{n+1}L}u^{n+1} = e^{-t_nL}u^n + \Delta t\, e^{-t_nL}f(u^n)$$

$$u^{n+1} = e^{\Delta tL}u^n + \Delta t\, e^{\Delta tL}f(u^n)$$

$$u^{n+1} = e^{\Delta tL}(u^n + \Delta t f(u^n))$$

Equivalent to first-order operator splitting

One can develop high order Lawson methods just by applying standard RK or multistep methods to (*).