NLS

$$i a_t = -a_{xx} + |a|^2 a$$

$$a(x,t) = e^{i\int v\,dx} \rho^{\frac{1}{2}}$$

a) $$a_t = e^{i\int v\,dx}\left[\rho^{\frac{1}{2}} i\int V_t\,dx + \frac{\rho_t}{2\rho^{\frac{1}{2}}}\right]$$

b) $$a_x = e^{i\int v\,dx}\left[\rho^{\frac{1}{2}}. iV + \frac{\rho_x}{2\rho^{\frac{1}{2}}}\right] \Rightarrow \rho\, a_x = \rho i V a(x,t) + \frac{\rho_x}{2} a$$

$$= a(x,t)\left[iV + \frac{\rho_x}{2\rho}\right]$$

c) $$a_{xx} = a(x,t)\left[-V^2 + \frac{iV\rho_x}{\rho} + \frac{\rho_x}{4\rho^2}\right]$$

$$+ a\left[iV_x + \frac{\rho\rho_{xx} - \rho_x^2}{2\rho^2}\right]$$

$$z = x + iy$$

$$|z|^2 = \mathrm{Re}\, z^2 + \mathrm{Im}\, z^2$$

$$|a|^2 = \left| e^{i\int v\,dx} \rho^{\frac{1}{2}} \right|^2$$

$$= \underbrace{\left| e^{i\int v\,dx} \right|^2}_{1} \left| \rho^{\frac{1}{2}} \right|^2 =$$

$$= |\rho|$$

NLS $\quad i a_t = -a_{xx} + |a|^2 a$

$$\underbrace{|\rho| \qquad \mathrm{Re}\{w\}}$$

$$\underbrace{\int V_t\, dx}_{w} - \frac{i}{2}\frac{\rho_t}{\rho} = -V^2 + \frac{\rho_x^2}{4\rho^2} + \frac{\rho\rho_{xx} - \rho_x^2}{2\rho^2} - |\rho|$$

$$+\, i\underbrace{\left[\frac{V\rho_x}{\rho} + V_x \right]}_{\mathrm{Im}\{z\}}$$

$$\boxed{\rho_t = -2V\rho_x - 2\rho V_x}$$

$$\boxed{V_t = \frac{\rho_x^3 + \rho^2 \rho_{xxx} - 2\rho\rho_x\rho_{xx}}{2\rho^3} - 2VV_x - \frac{\rho}{|\rho|}\rho_x}$$

b) Find linear dispersion Rel.

$V=0,\ \rho=1$

$$\tilde{V} = \varepsilon V \qquad \tilde{\rho} = 1+\varepsilon\rho \quad \xRightarrow{O(\varepsilon)} \begin{cases} \varepsilon\rho_t = -2\varepsilon V_x + O(\varepsilon^2) \\ \varepsilon V_t = -\varepsilon\rho_x + \frac{\varepsilon}{2}\rho_{xxx} + O(\varepsilon^2) \end{cases}$$

$\varepsilon=1$

$$\begin{cases} \rho_t = -2V_x \\ V_t = -\rho_x + \frac{1}{2}\rho_{xxx} \end{cases} \Longrightarrow V_{tx} = \frac{1}{2}\rho_{xxxx} - \rho_{xx} = -\frac{1}{2}\rho_{tt}$$

$$= V_{xt}$$

$$\Longrightarrow \rho_{tt} = 2\rho_{xx} - \rho_{xxxx}$$

$$\rho(x,t) = e^{ikx - i\omega t}$$

$$\boxed{\omega = \pm\sqrt{2k^2 + k^4}}$$

c) Re-write in terms of

$$\xi = \varepsilon(x - \beta t) \quad , \quad \tau = \varepsilon^3 t$$

$$\partial_x = \partial_\xi \xi_x + \partial_\tau \tau_x = \varepsilon \partial_\xi$$

$$\partial_t = \partial_\xi \xi_t + \partial_\tau \tau_t = -\beta\varepsilon \partial_\xi + \varepsilon^3 \partial_\tau$$

Re-write the ρ_t, V_t system

in terms of ξ, τ

$$\varepsilon^2 \rho_\tau = \beta \rho_\xi - 2(V\rho_\xi - \rho V_\xi)$$

$$\varepsilon^2 V_\tau = \beta V_\xi - 2VV_\xi - \rho_\xi$$

$$+ \varepsilon^2 \left(\frac{\rho_\xi^3 - 2\rho\rho_\xi\rho_{\xi\xi} + \rho^2 \rho_{\xi\xi\xi}}{2\rho^3} \right)$$

$$O(\varepsilon^0): \quad 0 = 0 + 0$$

$$O(\varepsilon^2): \quad 0 = 0 + 0 + 2V_\xi^1$$

$$0 = \beta V_\xi^1 + \rho_\xi^1$$

$$V^1 = V^1(\tau)$$

$$\rho^1 = \rho^1(\tau)$$

KdV: $u_t = u u_x + u_{xxx}$

$$V^1_\tau = V^1 V^1_\xi + V^1_{\xi\xi\xi}$$

$O(\varepsilon^4)$: $$\rho^1_\tau = \beta \rho^2_\xi + 2V^2_\xi$$

$$V^1_\tau = \beta V^2_\xi - \rho^2_\xi$$

$$V = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_m \end{bmatrix}$$

$$D = \begin{bmatrix} v_1 & & & \\ & v_2 & & \\ & & \ddots & \\ & & & v_m \end{bmatrix}$$