$$u_t + a u_x = 0$$

$$-\infty < x < \infty$$

$$u(x,t=0) = u_0(x)$$

Fourier ansatz: Frequency

$$u(x,t) = e^{i(kx - \omega(k)t)}$$

wavenumber

$$u_t = -i\omega(k)u$$

$$u_x = iku$$

$$-i\omega(k)u + ikau = 0 \qquad u \neq 0$$

$$\omega(k) = ak$$

$$\Downarrow$$

$$u(x,t) = e^{i(kx - kat)} = e^{ik(x-at)}$$

$$\mathcal{F}(u(x,0)) = \hat{u}(k,t=0) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} u(x,t=0)\,e^{ikx}\,dx$$

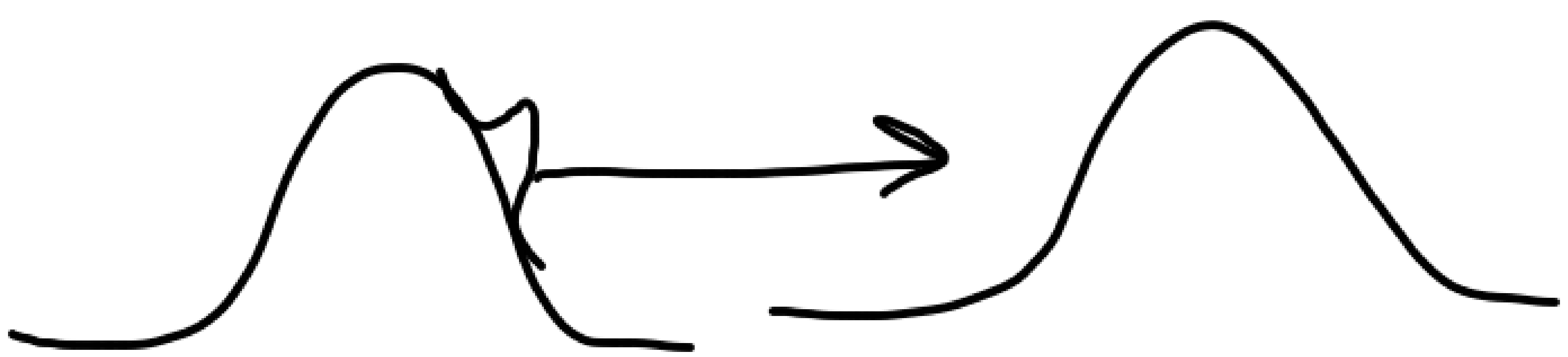
$$u_0(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \hat{u}(k,0)\,e^{ikx}\,dk$$

$$\hat{u}(k,t) = \hat{u}(k,0)\,e^{-ikat}$$

$$u(x,t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \hat{u}(k,0) e^{ikx} e^{-ikat} \, dk$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \hat{u}(k,0) e^{ik(x-at)} \, dk$$

$$= u(x-at, 0).$$



Waves:

- Finite speed of propagation
- Conserved quantities

For advection: $\frac{d}{dt}\int_{-\infty}^{\infty} u \, dx = 0$

$$\frac{d}{dt}\int_{-\infty}^{\infty} f(u) \, dx$$

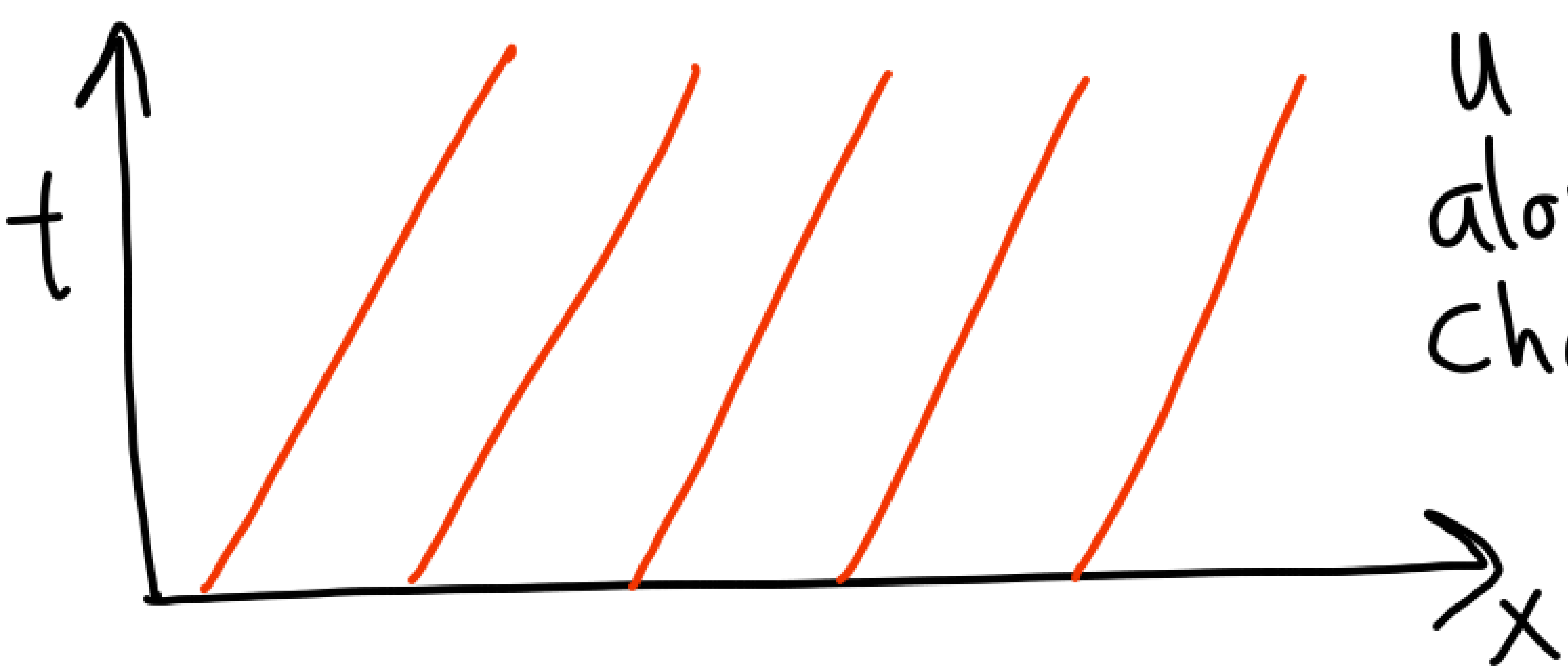
$\vdots$

Characteristics

$$X(t) = x_0 + at$$

$$\frac{d}{dt} u(X(t), t) = u_x X'(t) + u_t$$

$$= a u_x + u_t = 0$$



u is constant along each characteristic.

Heat equation

$$u_t = K u_{xx}$$

$$u(x,t) = e^{i(kx - \omega t)}$$

$$-i\omega u = K(ik)^2 u \qquad u \neq 0$$

$$-i\omega = -Kk^2$$

$$\omega = -iKk^2$$

diffusion

$$u = e^{ikx} \underline{e^{-Kk^2 t}}$$

Evolution equation

$$u_t = \sum_{j=0}^{m} a_j \partial_x^j u$$

Odd derivatives lead to wave-like behavior (translation)

Even derivatives lead to dissipation (or growth)

$$u = e^{i(kx - wt)}$$

$$-iw = \sum_{j=0}^{m} a_j (ik)^j$$

$$w = i \sum_{l=0}^{m/2} a_{2l} (ik)^{2l} + i \sum_{n=0}^{\frac{m}{2}-1} a_{2n+1} (ik)^{2n+1}$$

$$w = \underbrace{i \sum_{l} (-1)^l a_{2l} k^{2l}}_{\substack{\text{imag} \\ \text{(even-order} \\ \text{deriv.)}}} - \underbrace{\sum_{n=0}^{\frac{m}{2}-1} (-1)^n a_{2n+1} k^{2n}}_{\substack{\text{real} \\ \text{(odd-order} \\ \text{deriv.)}}}$$

$$u_t + u_x + u_{xxx} = 0$$

$$-i\omega + ik + (ik)^3 = 0$$

$$\omega - k + k^3 = 0$$

$$\omega(k) = k - k^3$$

$$u(x,t) = e^{ik(x - \underbrace{(1-k^2)}_{\text{speed of mode } k \, = c} t)}$$

Different wavenumber modes move at different speeds.

$$\lim_{|k| \to \infty} c = -\infty$$

Information travels infinitely fast.

Since the wavenumber k mode travels with speed c.

$$u_t + \varepsilon u_{xxx} = 0$$

BBM: $u_t + uu_x - u_{xxt} = 0$

$$c = \frac{\omega(k)}{k} = \bar{u}\,\frac{1}{1+k^2}$$

Finite speed of propagation

Linearize: $u = \bar{u} + \varepsilon v(x,t)$ $\quad \varepsilon \ll 1$

$$\varepsilon v_t + (\bar{u} + \varepsilon v)\varepsilon v_x - \varepsilon v_{xxt} = 0$$

$$v_t + \bar{u} v_x + \cancel{\varepsilon v v_x} - v_{xxt} = 0$$

KdV: $u_t + uu_x + u_{xxx} = 0$

$$v_t + \bar{u} v_x - v_{xxt} = 0$$

$$e^{i(kx-\omega t)} \longrightarrow -i\omega + \bar{u}ik - (ik)^2(-i\omega) = 0$$

$$e^{ik(x-\frac{\omega}{k}t)} \qquad \omega - \bar{u}k + k^2\omega = 0$$

$$e^{ik(x-ct)} \qquad \omega(1+k^2) = \bar{u}k$$

$$\omega(k) = \bar{u}\,\frac{k}{1+k^2}$$

Homework: Deconinck

Either: 2.8 or 2.11

(at end of chapter 2)

Reading: Whitham Ch. 1

Deconinck Ch. 2 up to end of 2.2