Exercise 2

The Riemann problem for acoustic waves at a material interface

Consider the Riemann problem given by the variable-coefficient acoustics equations

$$p_t + K(x)u_x = 0 \tag{1}$$

$$u_t + \frac{1}{\rho(x)}p_x = 0 \tag{2}$$

with

$$(\rho(x), K(x)) = \begin{cases} (\rho_\ell, K_\ell) & x < 0 \\ (\rho_r, K_r) & x > 0 \end{cases} \tag{3}$$

$$(p(x,0), u(x,0)) = \begin{cases} (p_\ell, u_\ell) & x < 0 \\ (p_r, u_r) & x > 0. \end{cases} \tag{4}$$

In this exercise we will find the solution to this problem and in the process rediscover the classical transmission and reflection coefficients.

(a) The above system can be written as $q_t + A(x)q_x = 0$. Show that for any real, positive values $K(x), \rho(x)$, the matrix $A(x)$ has one positive and one negative eigenvalue, so that the system is hyperbolic with one family of waves going in each direction.

Notice that we cannot define a single set of characteristic variables, since they depend on the coefficients that vary in space.

(b) The Riemann solution will consist of a wave (or discontinuity) going to the left in the medium with coefficients (ρ_ℓ, K_ℓ) and a wave (or discontinuity) going to the right in the medium with coefficients (ρ_r, K_r). To find this solution we must decompose the jump $q_r - q_\ell$ into a linear combination of the corresponding eigenvectors of the matrices $A(x < 0)$ and $A(x > 0)$. That is, we must use the eigenvector corresponding to left-going waves in the left medium and the eigenvector corresponding to right-going waves in the right medium. Write down and solve this linear system of equations.

(c) Now suppose that the initial data corresponds to a jump that is incident from the left; i.e., that $q_r - q_\ell$ is a multiple of the left-going wave family eigenvector of the left medium. For instance, you could take $q_r = [0, 0]^T$ and $q_\ell = [-Z_\ell, 1]^T$. Compute the solution of the Riemann problem, and then find:

- The ratio of the left-going wave to the initial jump; this is the reflection coefficient.
- The ratio of the right-going wave to the initial jump; this is the transmission coefficient.