

The influence of surface tension on the circular hydraulic jump

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We present the results of a combined theoretical and experimental investigation of the influence of surface tension σ on the laminar circular hydraulic jump. An expression is deduced for the magnitude of the radial curvature force per unit length along a circular jump, $F_c = -\sigma(s - \Delta R)/R_j$, where R_j is the jump radius, and s and ΔR are, respectively, the arclength along the jump surface and radial distance between the nearest points at the nose and tail of the jump at which the surface is horizontal. This curvature force is dynamically significant when $2\sigma/(\rho g R_j \Delta H)$ becomes appreciable, where ΔH is the jump height, ρ the fluid density and g the acceleration due to gravity. The theory of viscous hydraulic jumps (Watson 1964) is extended through inclusion of the curvature force, and yields a new prediction for the radius of circular hydraulic jumps. Our experimental investigation demonstrates that the surface tension correction is generally small in laboratory settings, but appreciable for jumps of small radius and height.

1. Introduction

Circular hydraulic jumps may arise when a descending vertical fluid jet impacts a rigid horizontal boundary; two such jumps are presented in figure 1. The first theoretical investigation of hydraulic jumps was undertaken by Rayleigh (1914), who developed the theory appropriate for inviscid jumps by application of continuity and momentum principles. The inviscid theory for the circular hydraulic jump resulting from the impact of a vertical jet on a horizontal plate is detailed in Birkhoff & Zarantonello (1957). The inviscid theory is known to be inadequate in predicting the radius of laboratory-scale circular jumps: the layer thickness preceeding the jump is typically sufficiently thin that diffusion of vorticity from the lower boundary is dynamically significant.

Watson (1964) considered the influence of fluid viscosity on the circular hydraulic jump, and described the flow in terms of a Blasius sublayer developing near the point of impact and a far-field similarity solution. Application of the momentum theorem at the jump yielded predictions for the jump radius for both laminar and turbulent flows. Watson's theory has been tested in a number of experimental investigations, including those of Watson himself, Olson & Turkdogan (1966), Ishigai *et al.* (1977), Nakoryakov *et al.* (1978), Bouhadepf (1978), Craik *et al.* (1981), Errico (1986) and Vasista (1989). The agreement has ranged from good to poor, being generally good when the jump radius is more than ten times the depth beyond the jump and poor in the opposite limit of small jump radii. In their experimental study of water jumps, Liu & Lienhard (1993) observe that Watson's predictions are least satisfactory in the

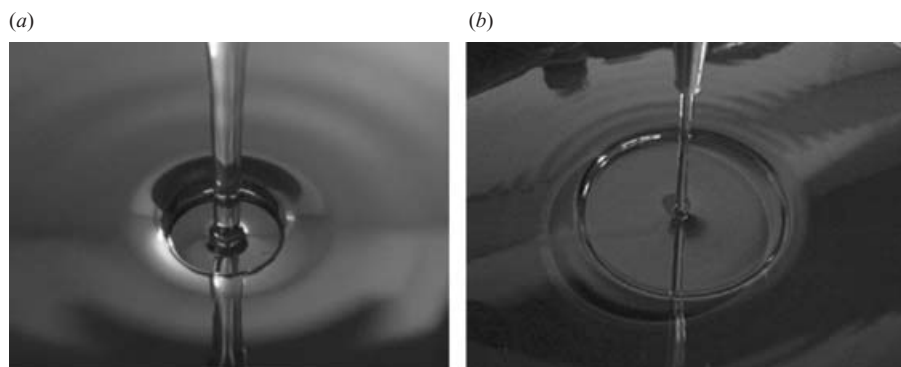


FIGURE 1. Circular hydraulic jumps arising from the impact of a vertical fluid jet on a horizontal boundary. (a) Nozzle radius 0.5 cm. (b) Nozzle radius 0.2 cm.

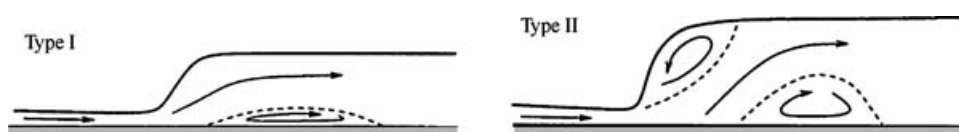


FIGURE 2. A schematic illustration of the two principal types of steady laminar hydraulic jumps. The progression from Type I to II arises as the outer layer depth is increased.

limit of relatively weak jumps, specifically when the ratio of the layer depths after and before the jump is small. The general consensus is thus that Watson's (1964) theoretical predictions are least satisfactory for jumps of small radius and height. Craik *et al.* (1981) focused on this small jump regime; their data for the jump radius thus underscored the shortcomings of Watson's theory.

In his theoretical developments, Watson (1964) neglected the influence of surface tension, and assumed that the flow beyond the jump was unidirectional. Subsequent theoretical studies have focused on the shortcomings of the latter assumption. The separation of the flow and the associated recirculation eddy beyond the jump was identified by Tani (1949), and has subsequently been highlighted in the experimental studies of Ishigai *et al.* (1977), Nakoryakov, Pokusaev & Troyan (1978), Craik *et al.* (1981), Liu & Lienhard (1993) and Ellegaard *et al.* (1996). Figure 2 is a schematic illustration of the two distinct types of laminar circular jumps that arise in water: Type I, marked by unidirectional surface flow, but boundary layer separation beyond the jump; Type II, marked by reversed surface flow adjoining the jump. The transition from Type I to II arises as the outer depth is increased. As a result of the pronounced boundary layer separation prevalent in the Type II jumps, the flow need not decelerate significantly as it passes through the jump: the discontinuity in radial speed assumed in the theoretical developments of Watson (1964) need not arise. Nevertheless, the numerous experimental investigations have considered Type I, II, and unsteady jumps, and generally indicate that Watson's (1964) prediction for the jump radius provides an adequate leading-order description. Craik *et al.* (1981) point out that in Type I jumps, Watson's assumption is expected to be adequate. In order to highlight the dynamic influence of the previously neglected curvature force, we focus our experimental investigation on the steady Type I jumps.

The modelling of the boundary layer separation beyond the jump has been the subject of a number of theoretical and numerical investigations (Bohr, Dimon &

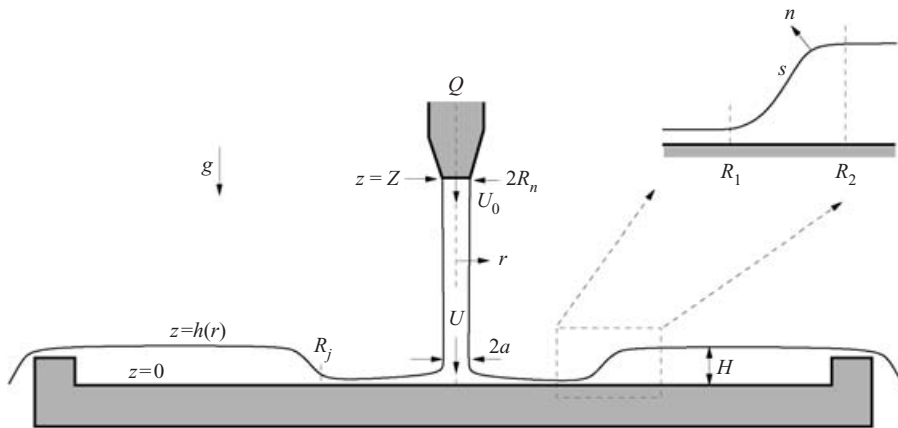


FIGURE 3. A schematic illustration of the hydraulic jump. A fluid jet of radius a and speed U impacts a horizontal fluid reservoir of outer depth H .

Putkaradze 1993; Bohr, Putkaradze & Watanabe 1997; Higuera 1997; Yokoi & Xiao 2000, 2002). Bowles & Smith (1992) and Higuera (1994) examine the flow in the neighbourhood of a planar jump via a boundary layer analysis. Both studies highlight the dominant influence of dynamic pressure and surface tension in the neighbourhood of the jump; moreover, they yield predictions for the jump profiles that are in good agreement with the observations of Craik *et al.* (1981). Yokoi & Xiao (2000, 2002) present numerical simulations of the flow in the circular hydraulic jump that similarly indicate the importance of surface tension and dynamic pressure in the vicinity of the radial hydraulic jump. While these studies have highlighted the subtle dynamical balance that exists in the neighbourhood of the jump, and so the deficiencies of Watson's (1964) assumption concerning the uniformity of the downstream flow, they have not led to a revised estimate of the jump radius. We thus focus our experimental investigation on the parameter regime where the influence of boundary layer separation is least pronounced and that of surface tension most pronounced, namely on Type I jumps of small radius and height. We emphasize that this small jump regime is the most poorly described by Watson's (1964) theory.

While both Craik *et al.* (1981) and Liu & Lienhard (1993) suggested the importance of surface tension on the size and frontal stability of the circular hydraulic jump, a theoretical investigation of the influence of surface tension on the form of a circular hydraulic jump has yet to be undertaken. In a companion paper (Bush, Hosoi & Aristoff 2003), we elucidate the role of surface tension in breaking the axial symmetry of the jump, while in the present study we examine its influence on the jump radius. In §2, a simple expression is deduced for the radial force per unit length associated with the curvature of a circular hydraulic jump, and the predictions of Watson (1964) for the jump radius are appropriately revised. The results of an accompanying experimental study are presented in §3.

2. Theory

We consider a descending vertical fluid jet of radius a and speed U transporting a flux $Q = \pi U a^2$ and impacting a horizontal plane covered by a fluid of depth H (figure 3). The depth of the fluid just upstream of the jump is denoted by h . The

momentum theorem requires that the difference in pressure force across the jump balance the difference in radial momentum flux, and is typically expressed as

$$\int_0^H \rho g z \, dz - \int_0^h \rho g z \, dz = \int_0^h \rho u^2(z) \, dz - \int_0^H \rho u^2(z) \, dz. \quad (2.1)$$

While this statement is sufficient for the case of a planar jump, it is incomplete for a circular jump in that it neglects the radial component of the surface tension force associated with the azimuthal curvature of the jump. The resulting radial curvature force may be expressed as

$$F_c = \int_S \sigma (\nabla \cdot \mathbf{n}) (\mathbf{n} \cdot \hat{\mathbf{r}}) \, dS. \quad (2.2)$$

Defining the jump surface $J(r, z) = z - h(r) = 0$ indicates that its unit outward normal

$$\mathbf{n} = \frac{\nabla J(r, z)}{|\nabla J(r, z)|} = \frac{\hat{\mathbf{z}} - h_r \hat{\mathbf{r}}}{(1 + h_r^2)^{1/2}}. \quad (2.3)$$

Substituting

$$\nabla \cdot \mathbf{n} = -\frac{1}{r} \frac{\partial}{\partial r} \frac{r h_r}{(1 + h_r^2)^{1/2}}, \quad \mathbf{n} \cdot \hat{\mathbf{r}} = -\frac{h_r}{(1 + h_r^2)^{1/2}}, \quad (2.4)$$

and $dS = r \, d\theta \, dr (1 + h_r^2)^{1/2}$ into (2.2) thus yields

$$F_c = 2\pi\sigma \int_{R_1}^{R_2} h_r \frac{\partial}{\partial r} \frac{r h_r}{(1 + h_r^2)^{1/2}} \, dr, \quad (2.5)$$

where R_1 and R_2 represent the inner and outer bounds on the jump, defined as the nearest points up- and downstream of the jump at which the slope h_r vanishes (see figure 3). Integrating twice by parts and noting that $h_r = 0$ at radii R_1 and R_2 yields

$$F_c = -2\pi\sigma \left[r(1 + h_r^2)^{1/2} \Big|_{R_1}^{R_2} - \int_{R_1}^{R_2} (1 + h_r^2)^{1/2} \, dr \right]. \quad (2.6)$$

Recognizing the last term as the arclength, s , of the jump surface between R_1 and $R_2 = R_1 + \Delta R$ yields the simple result

$$F_c = -2\pi\sigma (s - \Delta R). \quad (2.7)$$

We note that in the limit of no jump, $s = \Delta R$, the surface tension force vanishes identically. In the limit of an abrupt jump, $\Delta R \rightarrow 0$, the arclength s approaches the jump height ΔH , so that one obtains the curvature force on a cylindrical section of radius R_j and height ΔH :

$$F_c = -2\pi\sigma \Delta H. \quad (2.8)$$

Inclusion of the surface tension force per unit length (2.8) along the jump thus converts (2.1) to the form

$$\frac{1}{2}g(H^2 - h^2) + \sigma \frac{\Delta H}{R_j} = \int_0^h \rho u^2(z) \, dz - \int_0^H \rho u^2(z) \, dz, \quad (2.9)$$

where we adopt the abrupt jump result (2.8) for the sake of simplicity.

Figure 4 indicates the relevant physical picture for the case of laminar viscous flow, as presented originally by Watson (1964). Viscosity results in the diffusion of vorticity across the radially spreading fluid layer on a timescale $\tau \sim h^2/\nu$. In that time,

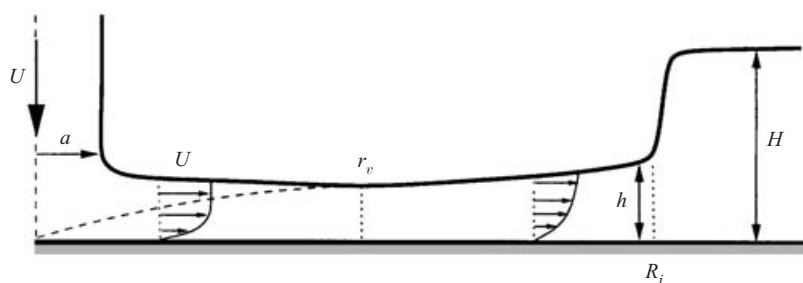


FIGURE 4. A schematic illustration of the boundary layer structure established within the circular hydraulic jump (Watson 1964). The viscous boundary layer grows until reaching the surface at a radial distance $r_v = 0.315aRe^{1/3}$ from the point of impact.

the flow travels a radial distance $r_v \sim U\tau \sim Ua^4/(4\nu r_v^2)$. This scaling anticipates that the boundary layer will span the entire layer depth at a radius $r_v \sim aRe^{1/3}$, where $Re = Qa/\nu$ is the jet Reynolds number. Watson employed the Kármán–Polhausen integral method and matched the near-source Blasius sublayer onto a far-field similarity solution in order to obtain the appropriate coefficient, $r_v = 0.315aRe^{1/3}$; Bowles & Smith (1992) deduced a similar result by way of an exact numerical solution. The flow necessarily changes form at $r = r_v$. For $r < r_v$, the surface speed is that of the incoming jet U , a Blasius sublayer develops from the lower boundary (figure 4), and the jump radius is defined by

$$\frac{R_j g H^2 a^2}{Q^2} \left(1 + \frac{2}{B_o}\right) + \frac{a^2}{2\pi^2 R_j H} = 0.10132 - 0.1297 \left(\frac{R_j}{a}\right)^{3/2} Re^{-1/2}, \quad (2.10)$$

where $B_o = \rho g R_j \Delta H / \sigma$ is the jump Bond number. For $r > r_v$, the surface speed is diminished relative to the incoming jet speed, and the jump radius is given by

$$\frac{R_j g H^2 a^2}{Q^2} \left(1 + \frac{2}{B_o}\right) + \frac{a^2}{2\pi^2 R_j H} = 0.01676 \left[\left(\frac{R_j}{a}\right)^3 Re^{-1} + 0.1826 \right]^{-1}. \quad (2.11)$$

We emphasize that the curvature force is dynamically significant when it is comparable to the hydrostatic pressure force, specifically when $2/B_o$ is appreciable. Equations (2.10) and (2.11) differ from those of Watson (1964) only through inclusion of the $O(B_o^{-1})$ surface tension correction on the left-hand side. They rest on the same assumptions concerning the flow profiles, specifically that $H/h \gg 1$, the radial flow speed is constant beyond the jump, and that radial gradients in the hydrostatic pressure prior to the jump are negligible relative to viscous stresses. The latter assumption is valid in the parameter regime examined in our experimental study, which is marked by jumps of small radial extent and relatively viscous fluids. We note that the inviscid prediction for the jump radius is obtained from (2.10) in the limit of $Re \rightarrow \infty$. We proceed by testing the range of validity and utility of the relations (2.10) and (2.11) for the jump radius.

3. Experimental study

Glycerol–water solutions were pumped through a flow meter and source nozzle, then impacted the centre of a circular glass target plate of diameter 36 cm that formed the base of a reservoir. The fluid then proceeded through the jump, over the edges

of the reservoir, and into the surrounding tank, from which it was recycled through the pump. The nozzle height was variable, and the reservoir depth controlled by the outer wall, which was threaded so that its vertical position could be adjusted by a twisting motion. The system was levelled to 1 part in 24 000 by adjusting its three support legs and measuring the deflection from horizontal of the impact plate and reservoir rim with a Sterret level. The depth H of the reservoir was measured with a micrometer point gauge accurate to $100\ \mu\text{m}$, and the radial position of the jump from radial gradations on the target plate surface. Five source nozzles (with exit radii 0.15, 0.2, 0.25, 0.38 and 0.5 cm) were fashioned from stainless steel pipe. The nozzles were smoothed and tapered near their exits in order to suppress turbulence and ensure laminar outflow in the parameter regime of interest (McCarthy & Malloy 1974).

The variable flow pump (Cole Parmer, Model 75225-00) was capable of fluxes in the range of $0\text{--}100\text{ ml s}^{-1}$ for the fluids examined in our study. The flow rate was measured with an AW Company Model JFC-01 digital flowmeter, that gave an accuracy of 0.1% over the range considered. Viscosity measurements accurate to 0.14% were made with Cannon-Fenske Routine tube viscometers. Fluid density was measured with an Anton-Parr 35N densitometer accurate to 0.01%. Surface tension measurements accurate to 0.1 dyn cm^{-1} were made with a Kruss K10 surface tensiometer. The glycerol–water solutions examined had viscosities in the range 1–30 cS, densities $1.0\text{--}1.2\text{ g cm}^{-3}$ and surface tensions between 65 and 70 cS. Outer layer depths were varied from 2 mm to 1 cm.

The jet speed U and radius a at the point of impact, $z = 0$, may be related to those at the source ($z = Z$), U_0 and R_N (see figure 3). Application of Bernoulli's equation and continuity, $Q = \pi R_N^2 U_0 = \pi a^2 U$, yields

$$\frac{U}{U_0} = \left[1 + \frac{2gZ}{U_0^2} + \frac{2\sigma}{\rho U_0^2} \left(\frac{1}{R_N} - \frac{1}{a} \right) \right]^{1/2}, \quad \frac{a}{R_N} = \left[1 + \frac{2gZ}{U_0^2} + \frac{2\sigma}{\rho U_0^2} \left(\frac{1}{R_N} - \frac{1}{a} \right) \right]^{-1/4}. \quad (3.1)$$

The jet radii near the point of impact were measured with calipers, and found to agree with (3.1) to within measurement error.

A preliminary experiment was performed in order to clearly demonstrate the significance of surface tension on the circular hydraulic jump. Flow conditions were set to establish a laminar circular jump of radius 3 cm; subsequently, a small volume (1–2 drops) of surfactant (either a commercial detergent or superwetting agent) was added to the test fluid. The jump responded to the resulting decrease in surface tension by expanding smoothly to a new radius that was approximately 20% larger. The shape of the jump changed substantially, becoming less abrupt once the surfactant was added. The surfactant thus diminishes the radial curvature force (2.7) through simultaneously reducing the surface tension σ and the geometric factor ($s - \Delta R$). This qualitative experiment makes clear the significance of the previously neglected curvature force, and motivates the more comprehensive experimental investigation that follows.

For each of the five nozzles used and each test fluid, the dependence of jump radius on Q and H was determined. Figure 5 indicates the observed dependence of the jump radius. In order to facilitate comparison with our theoretical predictions (2.10) and (2.11), we plot the dependence of their left-hand sides on $(R_j/a)^3 Re^{-1}$. In figure 5(a), the data are plotted with Watson's (1964) original theoretical curve, specifically (2.10) and (2.11) without the surface tension corrections ($B_o \rightarrow \infty$). The data are seen to be adequately described by Watson's (1964) predictions for large

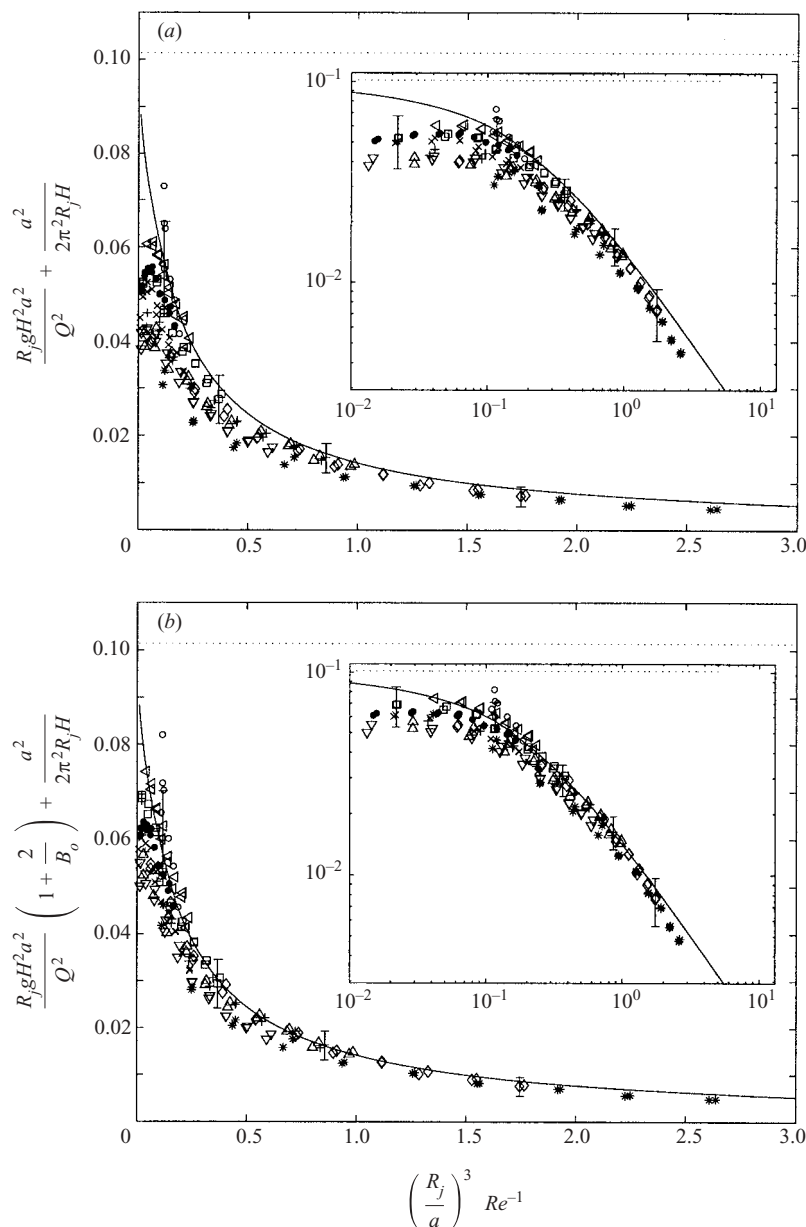


FIGURE 5. The observed dependence of the jump radius R_j on the system parameters. (a) The solid line represents the theoretical prediction of Watson (1964). (b) The solid line represents the theoretical predictions (2.10) and (2.11). The insets represent the equivalent log-log representations, and the dotted lines the predictions of inviscid theory. *, $R_N = 0.15$ cm, $\nu = 21$ cS; \diamond , $R_N = 0.2$ cm, $\nu = 21$ cS; +, $R_N = 0.25$ cm, $\nu = 21$ cS; \square , $R_N = 0.38$ cm, $\nu = 21$ cS; \triangleleft , $R_N = 0.5$ cm, $\nu = 21$ cS; \triangle , $R_N = 0.2$ cm, $\nu = 10$ cS; ∇ , $R_N = 0.25$ cm, $\nu = 10$ cS; \times , $R_N = 0.38$ cm, $\nu = 10$ cS; \bullet , $R_N = 0.5$ cm, $\nu = 10$ cS; \circ , $R_N = 0.5$ cm, $\nu = 1$ cS. Characteristic error bars are shown.

jumps ($(R_j/a)^3 Re^{-1} > 1$), but less so for smaller jumps. The surface tension correction is included in figure 5(b), and improves the agreement between theory and experiment. Specifically, the standard error is reduced from 15% to 10% over the entire data set.

The influence of surface tension is most significant for jumps of small radius and small height, and serves to decrease the jump radius by as much as 30% for the smallest jumps examined.

While a wealth of data has been presented in the literature on the radius of the circular hydraulic jump, it is not generally presented in such a way as to be readily incorporated into our study. Moreover, many of the experiments were performed on a flat impact plate with no outer wall; thus the outer depth H was neither controlled nor reported (Tani 1949; Olson & Turkdogan 1966; Bohr *et al.* 1993). We note however that the data of both Watson (1964) and Errico (1986) are similar to ours, yielding good agreement with Watson's (1964) theory for large jumps (large $(R_j/a)^3 Re^{-1}$), but overpredicting the jump radii of the smallest jumps by as much as 50%. We thus expect that their data would be similarly improved by our surface tension correction.

Craik *et al.* (1981) examined circular water jumps, and focused on the small jump regime in order to highlight the shortcomings of Watson's (1964) theory. Their raw data were examined, and are also better described by our theoretical curve than that of Watson (1964); however, their parameter regime was marked by unsteady irregular flows. This variance from the steady laminar flow assumed in the theory is presumably responsible for the considerable scatter in their data. All of the data presented in our experimental study correspond to steady laminar Type I jumps, where Watson's (1964) model for the flow is expected to be most relevant.

The principal measurement errors in our experimental study were those in the jet radius (100 μm), outer fluid depth (100 μm) and jump radius (1 mm). We note that the outer fluid depth was not strictly constant, but rather varied by approximately 200 μm over its radial extent. Relatively small (approximately 5%) variations in surface tension were observed to arise from day to day owing to some combination of evaporation of water from the solution, uptake of atmospheric water, and diffusion of soluble surfactant from the bulk to the surface. The resulting errors were obviated by frequent surface tension measurements.

4. Discussion

The dynamic significance of surface tension on the circular hydraulic jump was clearly demonstrated by the substantial (20%) jump expansion prompted by the addition of surfactant. However, these experiments were not pursued owing to the complications that accompany the introduction of surface tension gradients. For example, one expects Marangoni stresses to influence both the incident jet (Hancock & Bush 2002) and flow regimes marked by pronounced surface divergence, such as the Type II jumps (figure 2). The influence of surface tension gradients on the circular hydraulic jump is left as a subject for future consideration.

While previous experimental studies have indicated the potential significance of surface tension on circular hydraulic jumps of limited radial extent, its influence had not previously been characterized. We have deduced in (2.7) a new expression for the radial curvature force per unit length along a circular hydraulic jump, and in (2.10) and (2.11) the associated correction required to the theoretical expression of Watson (1964) for the jump radius. Our study demonstrates that surface tension becomes dynamically significant when the radial curvature force becomes comparable with the hydrostatic pressure forces, that is, when $2/B_o$ becomes appreciable. While the influence of surface tension is generally weak in terrestrial experiments, it becomes appreciable for jumps of small radius and height. Moreover, its influence will be

heightened dramatically in a microgravity setting, or when internal jumps arise between immiscible fluids of comparable density.

The surface-tension correction to the predicted jump radius in equations (2.10) and (2.11) has improved the agreement between experiment and theory, particularly in the weak-jump regime. The persistent discrepancy between experiment and theory is presumably due to shortcomings in the approximations inherent in Watson's (1964) theoretical description of the flow. One potential source of error is the neglect of the details of the flow in the vicinity of the jet impact ($r < a$) (Schlichting 1960), a region whose influence is expected to be greatest in the small-jump regime examined here. The trend apparent in figure 5, that the jumps produced by the nozzles with largest exit radii produce the most satisfactory agreement between theory and experiment, suggests another potential source of error: the influence of nozzle source conditions on the impinging jet. While an attempt was made to reduce this influence by smoothing and tapering the exit nozzle (McCarthy & Malloy 1974), such an influence cannot be eliminated entirely without meticulous attention to detail in nozzle design (Bergthorsson *et al.* (2003)).

Another potential source of discrepancy between theory and experiments is the dynamic influence of the flow beyond the jump (Craik *et al.* 1981; Liu & Lienhard 1993; Higuera 1994). Bowles & Smith (1992) point out that the influence of the curvature force will be partially offset by the dynamic pressure associated with streamline curvature in the wake of the jump. Specifically, one expects a relatively low dynamic pressure to arise beyond the jump, and so increase its radius. While we focused our attention on the laminar Type I regime in order to minimize such an influence, it may not have been entirely negligible. Finally, in the absence of measured jump profiles, we applied the surface tension correction (2.8) appropriate for sudden jumps. For a jump shape corresponding to a quarter-circle with radius ΔH , the curvature force should be reduced by a factor of $(\pi/2 - 1) \approx 0.57$; for a semicircular jump with diameter ΔH , it should be increased by a factor of $\pi/2$. We note that, as in the experiments with water reported by Craik *et al.* (1981), the abruptness of the jump generally increased with increasing outer depth and decreasing jump radius.

The great majority of previous experimental studies were undertaken with water; consequently, the jumps were relatively unstable. In particular, the small-jump regime of interest here, where the curvature force is most pronounced, is typically marked by unsteady irregular water jumps (Craik *et al.* 1981; Liu & Lienhard 1993). Through increasing the viscosity of the test fluid, we were able to focus on the parameter regime characterized by small steady laminar jumps where Watson's dynamical description is most likely to be relevant, and where the surface tension correction (2.7) is most significant. We note that the curvature force (2.7) is still relevant for the case of unsteady or turbulent jumps (where time-averaged values will apply for unsteady jumps) and so may be incorporated in a similar fashion.

Finally, while our study indicates that the influence of surface tension on the radius of the hydraulic jump is generally small in laboratory settings, we demonstrate in a companion paper (Bush *et al.* 2003) that it plays a critical role in the jump stability, specifically in prompting the axisymmetry-breaking instabilities responsible for the polygonal jumps reported by Ellegaard *et al.* (1998).

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REFERENCES

- BERGTHORSSON, J., SONE, K., MATTNER, T., DIMOTAKIS, P., GOODWIN, D. & MEIRON, D. 2003 Impinging laminar jets. *Phys. Fluids* (submitted).
- BIRKHOFF, G. & ZARANTONELLO, E. 1957 *Jets, Wakes and Cavities*. Academic.
- BOHR, T., DIMON, P. & PUTKARADZE, V. 1993 Shallow-water approach to the circular hydraulic jump. *J. Fluid Mech.* **254**, 635–648.
- BOHR, T., PUTKARADZE, V. & WATANABE, S. 1997 Averaging theory for the structure of hydraulic jumps and separation in laminar free-surface flows. *Phys. Rev. Lett.* **79**, 1038–1042.
- BOUHADEFF, P. 1978 Etalement en couche mince d'un jet liquide cylindrique vertical sur un plan horizontal. *Z. Angew. Math. Phys.* **29**, 157–167.
- BOWLES, R. & SMITH, F. 1992 The standing hydraulic jump: Theory, computations and comparisons with experiments. *J. Fluid Mech.* **242**, 145–168.
- BUSH, J., HOSOI, A. & ARISTOFF, J. 2003 The stability of the circular hydraulic jump. *J. Fluid Mech.* (submitted).
- CRAIK, A., LATHAM, R., FAWKES, M. & GIBBON, P. 1981 The circular hydraulic jump. *J. Fluid Mech.* **112**, 347–362.
- ELLEGAARD, C., HANSEN, A., HAANING, A., HANSEN, K. & BOHR, T. 1996 Experimental results on flow separation and transitions in the circular hydraulic jump. *Physica Scripta* **T67**, 105–110.
- ELLEGAARD, C., HANSEN, A., HAANING, A., MARCUSSEN, A., BOHR, T., HANSEN, T. & WATANABE, S. 1998 Creating corners in kitchen sink flows. *Nature* **392**, 767–768.
- ERRICO, M. 1986 A study of the interaction of liquid jets with solid surfaces. PhD thesis, University of California, San Diego.
- HANCOCK, M. & BUSH, J. 2002 Fluid pipes. *J. Fluid Mech.* **466**, 285–304.
- HIGUERA, F. 1994 The hydraulic jump in a viscous laminar flow. *J. Fluid Mech.* **274**, 69–92.
- HIGUERA, F. 1997 The circular hydraulic jump. *Phys. Fluids* **9**, 1476–1478.
- ISHIGAI, S., NAKANISHI, S., MIZUNO, M. & IMAMURA, T. 1977 Heat transfer of the impinging round water jet in the interference zone of film flow along the wall. *Bull. JSME* **20**, 85–92.
- LIU, X. & LIENHARD, J. 1993 The hydraulic jump in circular jet impingement and in other thin liquid films. *Exps. Fluids* **15**, 108–116.
- MCCARTHY, M. & MALLOY, N. 1974 Review of stability of liquid jets and the influence of nozzle design. *Chem. Engng J.* **7**, 1–20.
- NAKORYAKOV, V., POKUSAEV, B. & TROYAN, E. 1978 Impingement of an axisymmetric liquid jet on a barrier. *Intl J. Heat Mass Transfer* **21**, 1175–1184.
- OLSON, R. & TURKDOGAN, E. 1966 Radial spread of a liquid stream on a horizontal plate. *Nature* **211**, 813–816.
- RAYLEIGH, LORD 1914 On the theory of long waves and bores. *Proc. R. Soc. Lond. A* **90**, 324.
- SCHLICHTING, H. 1960 *Boundary Layer Flow*. McGraw-Hill.
- TANI, I. 1949 Water jump in the boundary layer. *J. Phys. Soc. Japan* **4**, 212–215.
- VASISTA, V. 1989 Experimental study of the hydrodynamics of an impinging liquid jet. B. Eng. thesis, MIT.
- WATSON, E. 1964 The spread of a liquid jet over a horizontal plane. *J. Fluid Mech.* **20**, 481–499.
- YOKOI, K. & XIAO, F. 2000 Relationships between a roller and a dynamic pressure distribution in circular hydraulic jumps. *Phys. Rev. D* **61**, 1016–1019.
- YOKOI, K. & XIAO, F. 2002 Mechanism of structure formation in circular hydraulic jumps: Numerical studies of strongly deformed free-surface shallow flows. *Physica D* **161**, 202–219.