The stability of the circular hydraulic jump

By **JOHN W. M. BUSH,**
A. E. HOSOI† AND **JEFFREY M. ARISTOFF**
Department of Mathematics
†Department of Mechanical Engineering
Massachusetts Institute of Technology, 77 Massachusetts Avenue, Cambridge, MA 02139, USA

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We present the results of an experimental investigation of the striking flow structures that may arise when a vertical jet of fluid impinges on a thin fluid layer overlying a horizontal boundary. Ellegaard *et al.* (1998, 1999) demonstrated that the axial symmetry of the circular hydraulic jump may be broken, resulting in steady polygonal jumps. In addition to these polygonal forms, our experiments reveal a new class of steady asymmetric jump forms that include structures resembling cat's eyes, three- and four-leaf clovers, bowties and butterflies. A parameter study reveals the dependence of the jump structure on the governing parameters. The symmetry-breaking responsible for the asymmetric jumps is demonstrated to result from a capillary instability of the circular jump. For all steady nonaxisymmetric forms observed, the wavelength of instability of the jump is related to the surface tension, σ, fluid density ρ and speed U of the radial outflow at the jump through $\lambda = (68 \pm 7)\ \sigma/(\rho U^2)$.

1. Introduction

The circular hydraulic jump may arise when a fluid jet falling vertically at high Reynolds number strikes a horizontal plate. Fluid is expelled radially, and the layer generally thins until reaching a critical radius at which the layer depth increases abruptly (Figure 1). Predictions for the jump radius based on inviscid theory were presented by Rayleigh (1914). The dominant influence of fluid viscosity on the jump radius was elucidated by Watson (1964), who developed an appropriate description of the boundary layer that develops from the impact plate. Subsequent theoretical studies of the circular jump have focused principally on describing the boundary layer separation and the associated dynamic pressure distribution behind the jump (Bohr *et al.* 1993, 1997; Yokoi & Xiao 2000, 2002; Bowles & Smith 1992; Ellegaard *et al.* 1996; Higuera 1994, 1997).

An extensive review of the circular hydraulic jump is presented by Bush & Aristoff (2003), who elucidate the influence of surface tension σ on the circular hydraulic jump. They calculate the magnitude of the curvature force acting on the jump, and demonstrate that it becomes appreciable relative to the hydrostatic pressure force when $2\sigma/(\rho g R_j \Delta H)$ becomes significant, where R_j is the jump radius and ΔH the jump height. The viscous theory of Watson (1964) was suitably modified, and the surface tension correction found to substantially decrease the discrepancy between experimental observations and theoretical predictions of the jump radii for jumps of small radius and height. The curvature force decreases the radius of a circular water jump of characteristic dimensions $R_j \sim 2$cm and $\Delta H \sim 3$mm by approximately 10%.

Both Craik *et al.* (1981) and Liu & Lienhard (1993) noted asymmetric instabilities in circular hydraulic jumps, and suggested the importance of surface tension in the frontal instability; however, no mechanism for instability was proposed. Moreover, as their working fluid was water, the resulting asymmetric jumps were irregular and unsteady. Liu & Lienhard (1993) characterized the dependence of the resulting unsteady jump forms on the governing dimensionless groups. The observed dependence on the jump Weber number clearly indicated the significance of surface tension on the jump stability.

Ellegaard *et al.* (1998) identified that a striking instability may transform the circular hydraulic jump into steady regular polygons. Experiments were conducted with ethylene glycol, which has a viscosity of approximately 10cS, and a source nozzle of radius 0.5 cm at elevations of 1-5 cm above the lower boundary ejecting fluxes between 30 and 50 ml/s. The dependence of the jump planform on the nozzle height and flux rate was reported by Ellegaard *et al.* (1999); however, the dependence of the flow structure on the governing dimensionless parameters was not elucidated. While a suggestion was made that the jump forms could be understood if an effective line tension was ascribed to the jump, no clear mechanistic explanation of the instability was given. We here extend the experimental study of these authors in order to gain further insight into the problem.

In §2, we describe the experimental technique employed in our study, and describe the variety of jump shapes observed in our exploration of parameter space. In §3, we identify the dimensionless groups that govern the system, and detail the dependence of the jump form on these parameters. The dependence of the mean jump radius of the asymmetric forms is investigated in §4. A scaling argument that indicates the dependence of the wavelength of instability of the steady asymmetric jumps on the governing parameters is presented in §5.

2. Experimental technique

Figure 1 is a schematic illustration of our experimental apparatus. Glycerol-water solutions were pumped through the flowmeter and source nozzle, resulting in a falling jet that impacted the centre of a circular glass target plate of diameter 36cm. The nozzle height was varied between 1 and 5 cm above the target plate. Beyond the hydraulic jump, the fluid ultimately spilled over the edges of the reservoir, and was recycled through the pump. The reservoir depth H was controlled by an adjustable outer wall and measured with a micrometer point gauge. The asymmetric jump structures were extremely sensitive to any variations from horizontal; consequently, great care was taken in leveling the system in order to ensure that the reservoir spilled uniformly over the bounding outer wall. The system was leveled to 1 part in 24,000 by adjusting its three support legs, and measuring the deflection from horizontal of the impact plate and reservoir rim with a Sterret level. The position of the jump was measured from radial gradations on the target plate surface.

The variable flow pump (Cole Parmer, Model 75225-00) was capable of fluxes in the range of 0-100 ml/s for the fluids examined in our study. The flow rate was measured with an AW Company Model JFC-01 digital flowmeter accurate to 0.1% over the range considered. Viscosity measurements accurate to 0.14% were made with Cannon-Fenske Routine tube viscometers. Fluid density was measured with an Anton-Parr 35N densitometer, accurate to 0.01%. Surface tension measurements accurate to 0.1 dynes/cm were made with a Kruss K10 surface tensiometer. The glycerol-water solutions examined had viscosities in the range 1-35 cS, densities 1.0-1.2 g/cc and surface tensions between 60 and 70 cS. Outer layer depths were varied from 0.2 to 1.5 cm. Three nozzles were used, of radii 0.2, 0.38 and 0.5 cm; their inner surfaces were smoothed and tapered near their exits in order to suppress turbulence and encourage laminar outflow in the parameter regime considered (McCarthy & Malloy (1974); however, see also Bergthorsson *et al.* (2003)). Flow speeds could be measured by tracking microbubbles introduced in the source fluid using a Redlake Motionscope Model PCI 8000S high-speed video camera. Adequate resolution of the bubbles typically required that we record at 500 frames per second with 0.001 second exposure times. The video footage was analyzed using Midas Version 2.08 particle tracking software.

The variety of strictly circular jumps that may arise has been documented by Craik *et al.* (1981) and Liu & Lienhard (1993). Owing to its relevance to the jump stability, we recall the transitions of the circular jump that arise as the outer depth is increased, from Type I to IIa and IIb jumps (see Figure 2). The former is the standard circular hydraulic jump in which the surface flow is everywhere radially outward; the interior flow is radial outwards everywhere except within a recirculating region just downstream of the jump (Tani 1949). The Type IIa jump is similarly marked by a subsurface 'separation bubble', but also by a region of reversed surface flow adjoining the jump. As the outer depth increases further, the jump transforms into a Type IIb jump marked by a tiered or 'double-jump' structure. The axial symmetry-breaking instabilities reported by Ellegaard *et al.* (1998, 1999) and observed in our study occur exclusively for Type II jumps. At higher flow rates, the jumps become irregular and time-dependent, and may ultimately be marked by air entrainment at their base. At the highest flow rates examined, the flow within the thin film becomes turbulent (Watson 1964). We here restrict our attention to the case of laminar upstream flow.

The 0.5 cm nozzle radius corresponds to that used in the study of Ellegaard *et al.* (1998, 1999) and yielded the most regular polygonal jumps. Examples of the polygonal jump structures observed in our experiments are presented in Figure 3. With this large nozzle, we reproduced the data of Ellegaard *et al.* (1999) that indicates the dependence of the jump structure on jet height and flow rate. Varying the nozzle size and fluid viscosity allowed us to explore a new regime marked by steady stable structures that included oval-, cat's eye-, bowtie-, butterfly- and clover-shaped jumps, henceforth referred to as the 'clover regime'. Shapes arising in the clover regime are presented in Figure 4. We note that the jumps arising in the clover regime are marked by the tiered structure characteristic of the Type IIb jumps. Some polygonal and clover forms were subject to weak time-dependent fluctuations, typically characterized by a net rotational motion of the entire jump structure, or the propagation of wave disturbances towards a single point on the jump.

The flow that accompanies the polygonal jumps is described by Ellegaard *et al.* (1999), and was visualized here through tracking microbubbles suspended in the flow. The axial symmetry within the radially expanding film is broken when the flow reaches the jump. Fluid is partially redirected along the jump front and so funneled towards the corners of the jump, resulting in radial jets emerging from the corners. In the clover regime, the vortices adjoining the jump have a limited vertical extent, thus giving jumps in the clover regime their two-tiered structure.

The significance of surface tension on the frontal instability was clearly and simply demonstrated. The flow parameters were set in order to obtain a steady pentagonal jump of mean radius 2cm. A small volume (1-2 drops) of surfactant (either a superwetting agent or a commercial detergent) was added to the working fluid, and resulted in three qualitative changes in the jump. First, the jump expanded, with its mean radius increasing by approximately 20%. Second, the polygonal instability was suppressed, and the jump assumed a circular form. Third, the jump became significantly less abrupt. The increase in mean radius is consistent with the theoretical predictions of Bush & Aristoff (2003) for the influence of the curvature force . The suppression of the polygonal form clearly indicates the importance of surface tension on the jump stability, the nature of which will be elucidated in §5.

3. Parameter study

We consider a nozzle of radius R_N ejecting fluid downward at uniform speed U_0; the associated flux is $Q = \pi R_N^2 U_0$. The jet descends a distance Z in a gravitational field $-g\hat{z}$ before striking a rigid horizontal boundary covered by fluid of outer depth H (Figure 1). The source conditions may be eliminated from consideration by straightforward application of Bernoulli's equation (e.g. Bush & Aristoff 2003). However, nozzle source conditions generally lead to variance from the predictions of inviscid theory (e.g. Bergthorsson *et al.* 2003); consequently, the jet radius at impact was measured with calipers.

We thus consider a vertical jet of radius a and flux Q of fluid with viscosity ν and density ρ impacting a horizontal boundary covered by fluid of depth H at a speed $U = Q/(\pi a^2)$. The free surface is characterized by a constant surface tension σ. There are thus 7 physical variables, a, H, Q, ν, ρ, g and σ expressible in terms of three fundamental units. Dimensional analysis indicates that the system may be uniquely prescribed by four dimensionless groups. We choose the jet Reynolds number $Re = Q/(\nu a)$, the Weber number $We = \rho Q^2/(\sigma H^3)$, the Bond number $B = \rho g a^2/\sigma$, and the relative magnitudes of the impacting jet radius and the outer layer depth, a/H. We proceed by elucidating the dependence of the form of the jump structure on these four governing dimensionless groups. We note that the commonly used Froude number is expressible in terms of the Bond and Weber numbers.

Two-dimensional projections of the four-dimensional parameter space are presented in Figures 5 and 6. Each corresponds to a series of experiments with a fixed nozzle radius and fluid. Figure 5 illustrates the dependence of jump structure on Re and We for a source nozzle of radius 0.5cm located a distance 2.5 cm above the impact plane ejecting a glycerol-water solution with viscosity ν = 10 cS and density ρ =1.12 g/cm^3. Each diagonal trace indicates the influence of increasing flux Q at a fixed outer depth H. Critical H values for the transitions between the various flow forms are indicated. At the lowest H value examined, 0.45 cm, the jumps assume the circular Type I planform. When the layer depth is increased to $H = 0.57$ cm, the jumps assume the Type IIa planform, and so are marked by a recirculating vortex and reversed surface flow adjoining the jump. Further increasing H prompts the symmetry-breaking instability responsible for polygonal jumps (Figure 3). Polygons with number of sides between 3 and 10 occupy well-defined regions of parameter space, with the larger number of sides being observed at higher flow rates and mean jump radii. For $H > 0.83$ cm, no jumps arise.

Figure 6 illustrates an equivalent regime diagram describing a series of experiments conducted with a nozzle of radius 0.2 cm located a height 2.5 cm above the plane and ejecting fluid of viscosity 32 cS. As the outer layer depth increases progressively, the jump evolves from circular Type I to Type II forms, and then to an asymmetric form. In this parameter regime, the asymmetric forms are those arising in the clover regime (Figure 4): cat's eyes, bowties, butterflies or clovers. At the largest H values examined, the jump is turbulent: air is entrained at the base of the jump, giving rise to irregular, unsteady motion such as that observed by Liu & Lienhard (1993) in water.

A series of three-dimensional regime diagrams illustrating the dependence of the jump structure on (Re, We, a/H) are presented in Figures 7-9. Each regime diagram corresponds to a particular nozzle size and so to a particular Bond number. The data falls out in distinct planes corresponding to two-dimensional projections such as those presented in Figures 5 and 6, each corresponding to a different value of fluid viscosity. For the sake of clarity, we classify jumps as being either Type I, II, polygons, clovers or turbulent, and do not distinguish between the various types of polygonal and clover-shaped jumps. This distinction is made in Figures 5 and 6 for the data planes indicated, respectively, in Figures 7 and 9. We note that the polygonal and clover jump regimes are both confined to limited regions of parameter space.

We note that the jump structures were subject to strong hysteretic effects: the point of transition from one jump shape to another depended on whether the flow rate was increasing or decreasing. For each experiment, once the source parameters were established, we disrupted the jump structure by blowing on it: this standard initialization eliminated hysteresis from consideration.

4. Jump radius

Watson (1964) elucidated the dominant influence of viscosity on the circular hydraulic jump, and Bush & Aristoff (2003) calculated the correction to Watson's (1964) theoretical prediction for the jump radius required by consideration of surface tension. Viscosity results in vorticity diffusing from the lower boundary until spanning the fluid layer at a radial distance $r_v = 0.315aRe^{1/3}$ from the point of impact. For $r < r_v$, the surface speed is that of the incoming jet U, and the jump radius is defined by

$$\frac{R_j g H^2 a^2}{Q^2}\left(1 + \frac{2}{B_j}\right) + \frac{a^2}{2\pi^2 R_j H} = 0.10132 - 0.1297\left(\frac{R_j}{a}\right)^{3/2} Re^{-1/2} \quad , \tag{4.1}$$

where $B_j = \rho g R_j \Delta H/\sigma$ is the jump Bond number. For $r > r_v$, the surface speed is diminished relative to the incoming jet speed,

$$U(r) = \frac{27c^2}{8\pi^4}\frac{Q^2}{\nu(r^3 + l^3)} \tag{4.2}$$

where $c = 1.402$ and $l = 0.567aRe^{1/3}$, and the jump radius is given by

$$\frac{R_j g H^2 a^2}{Q^2}\left(1 + \frac{2}{B_j}\right) + \frac{a^2}{2\pi^2 R_j H} = 0.01676\left[\left(\frac{R_j}{a}\right)^3 Re^{-1} + 0.1826\right]^{-1} \quad . \tag{4.3}$$

Finally, the layer depth in this outer regime ($r > r_v$) is given by

$$h(r) = \frac{2\pi^2}{3\sqrt{3}}\frac{\nu(r^3 + l^3)}{Qr} \quad . \tag{4.4}$$

We note that, owing to the influence of viscosity, the layer depth has a minimum at a critical radius that may be computed from (4.4).

Equations (4.1) and (4.3) were originally presented in Bush & Aristoff (2003) and differ from those of Watson (1964) only through inclusion of the $O(B_j^{-1})$ surface tension correction on the left-hand side. They rest on the same assumptions concerning the flow structure, specifically, that the ratio of layer depths directly down- and upstream of the jump is large ($H/h \gg 1$), the flow is unidirectional in the thin film, and radial gradients in the hydrostatic pressure upstream of the jump are negligible relative to viscous stresses. Finally, Watson's prediction for jump radius rests on the assumption that the radial flow speed is constant beyond the jump, an assumption generally expected to be violated owing to separation beyond the jump, and to be least adequate for the strongly nonlinear Type II jumps. Watson's predictions for the circular jump radius were found by a number of investigators to be adequate for laminar jumps of large radius and depth, but to yield poor agreement in the opposite small-jump limit (Olson & Turkdogan 1966; Ishigai *et al.*

1977; Nakoryakov *et al.* 1978; Bouhadepf 1978; Craik *et al.* 1981; Errico 1986; Vasista 1989; Liu & Lienhard 1993), where the surface tension correction becomes significant (Bush & Aristoff 2003). This small-jump regime is precisely that examined by Ellegaard *et al.* (1998, 1999) in which the polygonal jumps arise; consequently, we anticipate the relevance of this surface tension correction in our experimental study.

An experimental investigation of the dependence of the radii of strictly circular jumps on the governing parameters is presented in Bush & Aristoff (2003); we here focus our attention on the asymmetric jumps. The mean radius $\overline{R}_j$ of the asymmetric jumps was calculated from images by computing a suitable approximation to

$$\overline{R}_j = \left(\frac{1}{2\pi} \int_0^{2\pi} r^2 d\theta \right)^{1/2} \quad . \tag{4.5}$$

For example, for polygons with number of sides $n > 6$, $\overline{R}_j$ was simply taken as the mean of the minimum and maximum jump radii. Figure 10 indicates the observed dependence of the mean jump radius $\overline{R}_j$. In order to facilitate comparison with equations (4.1) and (4.3), we plot the dependence of $\frac{\overline{R}_j g H^2 a^2}{Q^2} \left(1 + \frac{2}{B_j}\right) + \frac{a^2}{2\pi^2 \overline{R}_j H}$ on $(\overline{R}_j/a)^3 Re^{-1}$. The dotted horizontal line indicates the inviscid theory, obtained from (4.1) in the limit of $Re \to \infty$, which is obviously inadequate in describing our data. The solid curves represent the jump radii predicted by (4.1) and (4.3). The mean radii of all steady structures observed, polygons and clovers, are reasonably well-described by the theoretical predictions.

We note that the discrepancy between theory and experiments for the mean jump radius is substantially larger than for the Type I circular jumps examined by Bush & Aristoff (2003). We expect this discrepancy to be a consequence of shortcomings in Watson's description of the flow in the strongly nonlinear regime examined here. While variance of the flow near the point of impact from purely radial violates the assumptions made in the development of equations (4.2) and (4.4), this did not introduce substantial errors in Bush & Aristoff's (2003) study of the Type I jumps. Moreover, the assumption that radial gradients in the hydrostatic pressure prior to the jump are negligible relative to viscous stresses is valid in the small-jump parameter regime examined in our study. We thus expect that the principal source of discrepancy between predicted and observed jump radii is the neglect of the influence of dynamic pressure downstream of the jump, an influence likely to be most pronounced for these strongly nonlinear structures (Bowles & Smith 1992; Higuera 1994; Yokoi & Xiao 2000, 2002). The principal source of measurement error arose from approximations made in calculating the mean jump radii of markedly asymmetric forms.

5. Jump stability

We propose a physical picture in which the instability of the initially circular jump is analogous to the Rayleigh-Plateau capillary pinch-off of a fluid thread. Rayleigh (1879) demonstrated that a fluid thread bound by surface tension σ will become unstable to a varicose instability in order to minimize surface energy. The Ohnesorge number, $Oh = \sigma R/(\mu\nu)$, a Reynolds number based on the capillary wavespeed σ/μ, prescribes the form of pinch-off of a fluid thread of radius R and dynamic viscosity $\mu = \rho\nu$ (Weber 1931; Chandrasekhar 1961). At high Oh, the pinch-off is resisted by fluid inertia, the timescale of instability is $(R^3\rho/\sigma)^{1/2}$, and the most unstable wavelength is $2\sqrt{2}\pi R$. At low Oh, the pinch-off is resisted by fluid viscosity, the timescale of instability is $\mu R/\sigma$, and the most unstable wavelength increases with Oh. The jumps in our problem have characteristic height of $\Delta H \sim 0.2 - 1$ cm and an Ohnesorge number of $O(100)$; consequently, the observed wavelengths of 0.7 - 2.0 cm are roughly consistent with the anticipated result of $2\sqrt{2}\pi\Delta H$.

According to our physical picture, the jump is viewed as the inner portion of a torus whose axisymmetry is broken as the system acts to minimize surface area (Figure 11). In order to deduce the most unstable wavelength of instability, we must deduce the dominant curvature of the unperturbed circular jump. Since $R_j \gg \Delta H$, the curvature of the free surface in a vertical plane aligned with the mean flow necessarily dominates that associated with its azimuthal curvature: we thus need consider only the curvature in a vertical radial plane, $1/R$ (Figure 11). The shape of the jump is determined by a balance between some combination of inertia, viscosity, gravity and surface tension. As in the study of Ellegaard *et al.* (1998, 1999), asymmetric jumps only arose as an instability of the Type II circular jumps, that are adjoined by a toroidal vortex. In all experiments considered in our study, the characteristic vortex Reynolds number is large; for example, for a jump vortex with radius 2mm and flow speed 30cm/s in a fluid of viscosity 30cS, the vortex Reynolds number is 200. One thus expects that the dominant curvature at the jump, $1/R$, will be set by a balance between surface tension and inertia:

$$\rho U_v^2 \sim \sigma/R \tag{5.1}$$

U_v is the speed in the vortex adjoining the jump, taken to be the surface speed of the fluid layer at the base of the jump. For $\overline{R}_j < r_v$, this speed corresponds to that of the incoming jet $U_v = U$, while for $\overline{R}_j > r_v$, it is defined by (4.2) evaluated at the mean jump radius $\overline{R}_j$. The radius of curvature of the jump is thus prescribed by $R \sim \sigma/(\rho U_v^2)$. The theoretical description of the Rayleigh-Plateau instability indicates that the radius of curvature of the most unstable azimuthal mode has a wavelength proportional to R; therefore, we anticipate that

$$\lambda \;=\; C_1 \,\frac{\sigma}{\rho U_v^2} \;\;, \tag{5.2}$$

where C_1 is a coefficient to be determined.

Figure 12 indicates the observed dependence of the wavelength of instability of the jump on the governing flow parameters. All jump shapes observed in the polygon and clover regimes are included. The wavelength is computed as $\lambda = 2\pi\overline{R}_j/n$, where n is the number of sides of the jump structure. Four-leaf clovers were taken as having eight sides, three-leaf clovers six, and bowties and butterflies four. For each data point, r_v was computed; either the incident jet speed U or (4.2) was assigned to U_v according to the relative magnitudes of r_v and $\overline{R}_j$. The data provides satisfactory agreement with (5.2). The constant of proportionality in (5.2) is thus deduced to be $C_1 = 68 \pm 7$.

The precise dynamical balance existing at the jump is complex, involving gravity, curvature, inertia and pressure (Bowles & Smith (1992), Higuera (1994), Yokoi & Xiao (2000, 2002)). The principal shortcoming of our simple scaling argument is the relatively crude approximation made in deducing the dominant curvature of the jump, specifically, that it may be obtained by balancing exclusively inertial and curvature forces. Another source of error in our prediction for the wavelength of instability results from the discrete number of sides; specifically, the mean circumference need not correspond to an integer multiple of the most unstable wavelength. The associated errors necessarily decrease as the number of sides increases.

6. Discussion

We have presented the results of an experimental investigation of the viscous hydraulic jump. In addition to the Type I, IIa and IIb circular jumps and the steady polygonal jump forms identified by previous investigators, we have identified a new class of steady asymmetric jumps that arise as an instability of the tiered Type IIb jumps, that we refer to as clovers (Figure 4). Our exploration of parameter space (Figures 5-9) has underscored the limited parameter regime in which both polygonal and clover jumps arise; it is presumably thus that none of these striking flow structures had been observed prior to the experiments of Ellegaard *et al.* (1998).

We note that the influence of fluid viscosity on the circular hydraulic jump is two-fold. First, viscosity acts to hasten the diffusion of vorticity across the fluid layer and so decelerate the flow. The concomitant decrease in the jump radius necessarily heightens the influence of surface tension on the jump. Second, viscosity acts to regularize the asymmetric frontal structure; it is thus, presumably, that the stable, steady polygonal or clover forms do not arise in water, where unsteady irregular frontal instabilities are the norm (Craik *et al.* (1981), Liu & Lienhard (1993)).

Our study has clearly identified the importance of surface tension in prompting the axisymmetry-breaking instabilities observed in the circular hydraulic jump. While the significance of surface tension on jump stability was suggested by Craik *et al.* (1981) and Liu & Lienhard (1993), the precise mechanism for instability had not previously been considered. Ellegaard *et al.* (1998, 1999) suggest that the symmetry-breaking instability responsible for the polygonal jumps may be anticipated if one ascribes an effective line tension to the jump; however, they did not identify the origins of such a line tension. We here identify the instability as a manifestation of the Rayleigh-Plateau pinch-off of the initially circular jump, taken to be the inner section of a torus.

The addition of surfactant was observed to convert the polygonal jumps to circular jumps of large mean radius. The associated reduction in surface tension results in expansion of the jump according to the theoretical developments of Bush & Aristoff (2003), and may result in the suppression of capillary pinch-off according to the present study. The influence of surfactant on the jump structure is complex, however, and it is thus that a more quantitative study of the influence of surfactants was not undertaken. In particular, the surfactant acts to suppress motions marked by non-zero surface divergence such as those arising in the Type II jumps. The surfactant may thus impact the jump structure not only through decreasing the surface, but also through suppressing the Type II planform that is a prerequisite for symmetry-breaking instability. The influence of surfactant on the stability of hydraulic jumps is left as a subject for future consideration.

Our experiments indicate that the mean jump radius is adequately described by the theory of Bush & Aristoff (2003), and that the wavelength of instability is given by

$$\lambda \;=\; (68 \pm 7)\, \frac{\sigma}{\rho U_v^2} \;\; , \tag{6.1}$$

where U_v is the speed of the surface flow at the mean jump radius as predicted by the theory of Watson (1964). We note that these two results do not uniquely prescribe the jump shape in the nonlinear regime; for example, a six-sided jump could correspond to either a heptagon or a three-leaf clover. The nonlinear jump shape may only be inferred from the source conditions by reference to our regime diagrams (Figures 5-9). We note that the coefficient appearing in (6.1) is larger than that appropriate for a cylindrical thread, $2\sqrt{2}\pi$. This may indicate the influence of the downstream flow, specifically the dynamic pressure associated with the jump vortex, on the pinch-off at the jump. The influence of rotational motion on the classic Rayleigh-Plateau instability is currently under investigation.

While the observed number of sides could be adequately understood in terms of the scaling result (6.1) anticipated on the basis of linear theory, many of the observed jump shapes (Figures 3 and 4) are strongly nonlinear. Some were marked by sharp cusps in the corners as frequently arise in convergent flows dominated by viscosity and surface tension, for example, in the four-roll mill (Joseph *et al.* (1991)). Surface tension typically serves to regularize the corner flows unless the flows are sufficiently vigorous to prompt air entrainment at the cusp (Eggers (2000)). Our observations indicate that entrainment of air may arise at both the base of the jump and at the corners of the polygonal jumps. While the present study has identified capillary pinch-off as the source of the symmetry-breaking instabilities in the hydraulic jump, the calculation of fully developed nonlinear shapes is left as a subject for future investigation.

While the capillary instability is best known as the source of pinch-off of a cylindrical fluid thread (Rayleigh (1879)), it prompts analogous instabilities in a variety of geometries; for example, in liquid menisci bound at solid edges (Langbein (1990)). Figure 13 illustrates a number of free surface flows in which symmetry-breaking is prompted by a capillary instability. The capillary pinch-off of the toroidal rim on a radially expanding circular fluid sheet (Savart 1833) leads to the release of discrete droplets at the sheet edge. The hydraulic jump instability bears a resemblance to that arising when a partially filled cylinder is rotated about its horizontal axis of symmetry (Hosoi & Mahadevan 1999, Figure 13). The frontal instability so observed is likewise caused by a capillary instability of a curved fluid front; however, in their flow, the characteristic Reynolds number is small. The dominant force balance at the front is thus between viscous stresses and capillary forces, consideration of which yields a most unstable wavelength that scales as $H(\mu U/\sigma)^{-1/3}$ where U is the speed of the cylinder wall and H is the fluid depth. By way of contrast, the hydraulic jump considered here is characterized by a high Re: the curvature of the jump and the concomitant wavelength of azimuthal instability are prescribed by a balance between curvature and inertial forces.

While the majority of the observed shapes could be classified as belonging in either circular, polygon, clover or turbulent regimes, there are a number of exceptions and oddities. First, there is an unsteady flow regime between the polygon and fully turbulent regime in which the roller vortex adjoining the jump lifted off, resulting in an apparent crown on the jump (Figure 14). Second, situations arose where the outer layer was too deep to support a hydraulic jump, but where a roller vortex was observed to form. In certain parameter regimes, this vortex became unstable and assumed a roughly polygonal shape. Such an instability is clearly not a manifestation of capillary pinch-off, but may be related to the instability mechanism responsible for the break-up of a circular smoke ring into polygons (Widnall & Tsai (1977)). Such oddities are left as problems for future consideration.

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