# Quiz 2

**Problem 1** (10 points)**.** Use the following three facts about determinants to compute the determinant of a matrix using row operations.

a. If $B$ is diagonal, then $\det(B) = b_{11} \cdot b_{22} \cdots b_{nn}$.

b. If $B$ arises from $A$ by a type I row operation, i.e., interchanging two rows, then $\det(B) = -\det(A)$.

c. If $B$ arises from $A$ by a type III row operation, i.e., $r_i + ar_j \to r_i$, that is, row $i$ is replaced by row $i$ plus a scalar multiple of row $j$, where $i \neq j$. Then $\det(A) = \det(B)$.

Compute $\det(A)$ by:

1. Reducing $A$ to a triangular matrix $B$ using only type I and III operations. (I would say echelon form, except for the issue with pivots being 1).
2. Keep track of how many row swaps were made.
3. Compute $\det(B)$ by multiplying the diagonal elements of $B$.

$$A = \begin{bmatrix} 2 & 6 & 3 & 2 \\ 4 & 2 & 3 & 2 \\ 2 & 2 & 2 & 1 \\ 4 & 2 & 1 & 5 \end{bmatrix}$$

Show the work for the above computation here.

On your own, don't include this in the quiz, try computing this determinant by expanding on a row or column.

Discuss which method, "expansion along a row or column" or "using elementary row operations" is, in general, a faster method of computing a determinant.

I will only write this out using row reduction, the other way ... expansion along a row/column would be too painful.

$$\begin{bmatrix}2&6&3&2\\4&2&3&2\\2&2&2&1\\4&2&1&5\end{bmatrix} \xrightarrow[\substack{r_2-2r_1\to r_2\\ r_3-r_1\to r_3\\ r_4-2r_1\to r_4}]{} \begin{bmatrix}2&6&3&2\\0&-10&-3&-2\\0&-4&-1&-1\\0&-10&-5&1\end{bmatrix}$$

$$\xrightarrow[\substack{r_3-4/10r_2\to r_3\\ r_4-10r_2\to r_4}]{} \begin{bmatrix}2&6&3&2\\0&-10&-3&-2\\0&0&1/5&-1/5\\0&0&-2&3\end{bmatrix}$$

$$\xrightarrow[r_4+10r_3\longrightarrow r_3]{} \begin{bmatrix}2&6&3&2\\0&-10&-3&-2\\0&0&1/5&-1/5\\0&0&0&-1\end{bmatrix}$$

We had two row swaps so $\det(A) = (2)(-10)(1/5)(1) = -4$.

I leave the expansion along a column to the reader. You should note that it is a much longer and more involved process. For finding $\det(A)$ it is much quicker to use the method of applying elimination and keeping track of how the determinant changes.

Formally for an $n \times n$ matrix, reducing to a triangular matrix requires at most $(n-1)+(n-2)+...+1 = (n)(n-1)/2 \approx n^2$ many operations. Expending a determinant is a recursive procedure so it is a little harder to compute. For $2 \times 2$, let's say 2 operations are used. For $3 \times 3$, we have $3 \times 2$, since you must do 3 $2 \times 2$'s. For $4 \times 4$, you need $4 \times 3 \times 2$, recognize this! It requires $n!$ operations, that is HUGE! Worse than exponential.

**Problem 2** (5 points)**.** Let $A$ be as above, consider $A\boldsymbol{x} = \boldsymbol{b}$ where $\boldsymbol{b} = (-3,-3,-2,1)$. Find $x_1$ using Cramer's rule. (You may use MATLAB/Octave to compute the determinants, but write out what you are computing.)

$$x_1 = \frac{\det\begin{bmatrix}-3&6&3&2\\-3&2&3&2\\-2&2&2&1\\1&2&1&5\end{bmatrix}}{\det\begin{bmatrix}2&6&3&2\\4&2&3&2\\2&2&2&1\\4&2&1&5\end{bmatrix}} = -2$$

**Problem 3** (10 points; 2 points each)**.** Decide if each of the following are true or false and provide a small proof or counterexample in each case.

(a) ______ If $A$ is an $n \times n$ matrix all of whose entries are integers and $\det(A) = \pm 1$, then $A^{-1}$ also has only integer entries.

This is true as can be seen by considering the adjoint.

(b) ______ If $A$ and $B$ are similar, then $\det(A) = \det(B)$.

Here two $n \times n$ matrices $A$ and $B$ are called *similar* iff $A = SBS^{-1}$ for some invertible $S$.

This is clearly true since

$$\det(A) = \det(SBS^{-1}) = \det(S)\det(B)\det(S^{-1}) = \det(S)\det(S)^{-1}\det(B) = \det(B)$$

(c) ______ Three vectors in $\mathbb{R}^3$, $\boldsymbol{v}_1$, $\boldsymbol{v}_2$, and $\boldsymbol{v}_3$ are co-planar iff $\det(\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3) = 0$.

This is true. The three vectors are co-planar iff they don't span $\mathbb{R}^3$ iff they are linearly dependent iff the determinant is 0.

(d) ______ $\det(A^2 + B^2) \geq 0$ for all $n \times n$ matrices $A$ and $B$ with real entries that commute.

This is also true. Here there is a nice trick, consider the complex matrix $A + Bi$. Since $A^2 + B^2 = (A+Bi)(A-Bi)$ we have $\det(A^2+B^2) = \det(A+Bi)\det(A-Bi) = \det(A+Bi)\overline{\det(A+Bi)} = |\det(A+Bi)|^2$.

(e) ______ The determinant can be viewed as a multilinear function $\det : (\mathbb{R}^n)^n \to \mathbb{R}$ with the properties that $\det(\boldsymbol{e}_1, \ldots, \boldsymbol{e}_n) = 1$ and $\det(\boldsymbol{v}_1, \ldots, \boldsymbol{v}_n) = \det(\boldsymbol{v}'_1, \ldots, \boldsymbol{v}'_n)$, where $(\boldsymbol{v}'_1, \ldots, \boldsymbol{v}'_n)$ is the result of swapping two of the vectors in $(\boldsymbol{v}_1, \ldots, \boldsymbol{v}_n)$.

False. There was a typo here, I should have had $\det(v_1, \ldots, v_n) = -\det(v'_1, \ldots, v'_n)$, and I intended this to be true. But as it is, false is correct.

This was just a check to see that you are looking at the notes that I post. (See here.)

**Problem 4** (10 points)**.** Submit the completion certificate for the OnRamp tutorial from MATLAB in the MATLAB shared drive.