Name: ____________________ Exam 3 - MAT345

There is a possible 210 points, so that is a possible bonus of 10 points. there will be no take-home part. For take-home, if you want to earn points back on the T/F, you can do the T/F justifications. I will not post the correct T/F answers until after class on Thursday.

1 True/False

Each problem is points for a total of 80 points. (10 problems worth 8 points each.)

Problem 1.1. For the exam, you need only indicate True or False. No justification is required. If you want to earn back some points, you can supply full justifications for **all** of the problems. You may earn back 50% of lost points.

a) False There is a 3×3 matrix with eigenvalues $\lambda_1 = 1/2$ and $\lambda_2 = -1/3$ with

$$E_{\lambda_1} = \operatorname{span}\left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix} \right\} \text{ and } E_{\lambda_2} = \operatorname{span}\left\{ \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix} \right\}$$

If this were true, then $E_{\lambda_1} \cap E_{\lambda_2} = \{\mathbf{0}\}$ and as

$$\begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}$$

we see that this is false.

b) True There is a 3×3 matrix with eigenvalues $\lambda_1 = 1/2$ and $\lambda_2 = -1/3$ with

$$E_{\lambda_1} = \operatorname{span}\left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix} \right\} \text{ and } E_{\lambda_2} = \operatorname{span}\left\{ \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}$$

This is true, and we can write down the matrix quite easily

$$A = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 0 \\ 1 & 2 & 1 \end{bmatrix} \begin{bmatrix} 1/2 & 0 & 0 \\ 0 & 1/2 & 0 \\ 0 & 0 & -1/3 \end{bmatrix} \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 0 \\ 1 & 2 & 1 \end{bmatrix}^{-1}$$

c) True If a 3×3 matrix A has eigenvalues $\lambda_1 = 1/2$ and $\lambda_2 = -1/3$, and no others, and

$$E_{\lambda_1} = \operatorname{span}\left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \right\} \text{ and } E_{\lambda_2} = \operatorname{span}\left\{ \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix} \right\}$$

Then there is no diagonal matrix D similar to A.

In this case one of the eigenvalues has algebraic multiplicity 2, but geometric multiplicity 1, so A is deficient and hence not diagonalizable.

d) True If a 3×3 matrix has eigenvalues 1, $1/2$, and $-1/4$, then A is diagonalizable.

If an $n \times n$ matrix has n distinct eigenvalues, then it is diagonalizable.

e) True There is a 3×3 symmetric matrix with eigenvalues $\lambda_1 = 1/2$ and $\lambda_2 = -1/3$ with

$$E_{\lambda_1} = \operatorname{span}\left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix} \right\} \text{ and } E_{\lambda_2} = \operatorname{span}\left\{ \begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix} \right\}$$

This is true since $E_{\lambda_1} \perp E_{\lambda_2}$, the eigenvalues are real, and there is a basis of eigenvectors for $\mathbb{R}^3$.

f) False For A a 2×3 matrix, there is always a unique $\hat{\boldsymbol{x}} \in \mathbb{R}^3$ so that $\|\boldsymbol{b} - A\hat{\boldsymbol{x}}\| \leq \|\boldsymbol{b} - A\boldsymbol{x}\|$ for all $\boldsymbol{x} \in \mathbb{R}^3$.

There is always a unique $\hat{\boldsymbol{b}}$ so that $\hat{\boldsymbol{b}} = A\hat{\boldsymbol{x}}$ for some $\hat{\boldsymbol{x}}$ that satisfies the condition, but the $\hat{\boldsymbol{x}}$ is not unique.

g) True For A a 2×3 matrix, there is always a unique $\hat{\boldsymbol{b}} \in \mathbb{R}^2$ so that $\hat{\boldsymbol{b}} = A\hat{\boldsymbol{x}}$ for some $\hat{\boldsymbol{x}} \in \mathbb{R}^3$, and $\|\boldsymbol{b} - A\hat{\boldsymbol{x}}\| \leq \|\boldsymbol{b} - A\boldsymbol{x}\|$ for all $\boldsymbol{x} \in \mathbb{R}^3$.

This is true and $\hat{\boldsymbol{b}} = \operatorname{proj}_{\mathrm{CS}(A)}(\boldsymbol{b})$.

h) True The sheer matrix $A = \left[\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right]$ can be written as

$$A = UDV$$

where $U^{-1} = U^T$ and $V^{-1} = V^T$ and D is diagonal. (All matrices are real.)

This is just the SVD theorem applied to A.

i) False The sheer matrix $A = \left[\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right]$ can be written

$$A = UDU^{-1}$$

where D is diagonal. (All matrices are real.)

This matrix is the standard example of a non-diagonalizable matrix.

j) True Every finite dimensional vector space has an orthonormal basis.

This is what the Gram-Schmidt procedure always produces.

2 Multiple Choice (30 points; 10 points each)

Problem 2.1 (10 points)**.** Which of the following are equivalent to U being unitary for a real $n \times n$ matrix U? Mark "Y" if the property is equivalent and "N" otherwise.

$\boxed{\text{Y}}$ U preserves lengths, i.e., $\|U\boldsymbol{x}\| = \|x\|$.

$\boxed{\text{Y}}$ $U^H = U^{-1}$.

$\boxed{\text{N}}$ $\det(U) = \pm 1$

$\boxed{\text{Y}}$ The columns of U form an orthonormal basis for $\mathbb{R}^n$.

$\boxed{\text{N}}$ U preserves signed volume, i.e., $\det(U\boldsymbol{x}_1, \ldots, U\boldsymbol{x}_n) = \det(\boldsymbol{x}_1, \ldots, \boldsymbol{x}_n)$.

Problem 2.2 (10 points)**.** Which of the following is equivalent to $\hat{\boldsymbol{x}}$ is a least-squares solution to $A\boldsymbol{x} = \boldsymbol{b}$? Write "Y" if the property is equivalent for all matrices A and "N" otherwise.

$\boxed{\text{N}}$ $A\hat{\boldsymbol{x}} = \boldsymbol{b}$.

$\boxed{\text{Y}}$ $A\hat{\boldsymbol{x}} = \mathrm{proj}_{\mathrm{CS}(A)}(b)$.

$\boxed{\text{Y}}$ $A\hat{\boldsymbol{x}} = UU^T\boldsymbol{b}$ where the columns of U form an orthonormal basis for $\mathrm{CS}(A)$.

$\boxed{\text{Y}}$ $A^TA\hat{\boldsymbol{x}} = A^T\boldsymbol{b}$.

$\boxed{\text{Y}}$ $\boldsymbol{b} - A\hat{\boldsymbol{x}} \perp \mathrm{CS}(A)$.

Problem 2.3 (10 points)**.** Which of the following are theorems that we have learned in Topics 5, 6, or 7?

$\boxed{\text{Y}}$ Every real $m \times n$ matrix A of rank k can be written as $A = U\Sigma V^T$ where U is $m \times k$, V is $n \times k$ both with orthonormal columns, and Σ is a $k \times k$ diagonal matrix with $\Sigma_{i,i} \geq \Sigma_{j,j} > 0$ for $i \leq j$, that is, with descending positive diagonal entries.

$\boxed{\text{Y}}$ Every symmetric $n \times n$ matrix A can be written as $A = UDU^T$ where D is diagonal and U is unitary.

$\boxed{\text{N}}$ Every $n \times n$ matrix can be written as SDS^{-1} where D is diagonal.

$\boxed{\text{Y}}$ Let V be an inner-product space and $S \subseteq V$ a subspace. Then for every $\boldsymbol{v} \in V$, there is a unique $\hat{\boldsymbol{v}} \in S$ so that $\|\boldsymbol{v} - \hat{\boldsymbol{v}}\| \leq \|\boldsymbol{v} - \boldsymbol{s}\|$ for all $\boldsymbol{s} \in S$.

$\boxed{\text{Y}}$ Let V be an inner-product space and $S \subseteq V$ a subspace. Then for every $\boldsymbol{v} \in V$, there is a unique $\hat{\boldsymbol{v}} \in S$ so that $\boldsymbol{v} - \hat{\boldsymbol{v}} \perp S$.

3 Computational (100 points; 5 problems, each worth 20 points.)

Show all computations so that you make clear what your thought processes are.

Problem 3.1 (20 pts)**.** Diagonalize A if possible. If not diagonalizable, explain why.

$$A = \begin{bmatrix} \frac{1}{3} & \frac{1}{3} \\ -\frac{1}{3} & \frac{7}{6} \end{bmatrix}$$

Find A^{100} exactly. (This needs to be done by hand.)

Show your work even if you use tech. Here, you really should just use tech to check your work. Show how you find the eigenvalues and the eigenvectors.

$P_A(t) = (1/3-t)(7/6-t)+1/9 = t^2-9/6t+7/18+1/9 = t^2-3/2t+1/2 = (t-1/2)(t-1)$

so the eigenvalues are $\lambda_1 = 1$ and $\lambda_2 = 1/2$.

For $\lambda_1 = 1$:

$$\mathrm{NS}\begin{bmatrix} -2/3 & 1/3 \\ -1/3 & 1/6 \end{bmatrix} = \mathrm{NS}\begin{bmatrix} -2 & 1 \\ 0 & 0 \end{bmatrix} = \mathrm{span}\left\{\begin{bmatrix} 1 \\ 2 \end{bmatrix}\right\}$$

For $\lambda_2 = 1/2$:

$$\mathrm{NS}\begin{bmatrix} -1/6 & 1/3 \\ -1/3 & 2/3 \end{bmatrix} = \mathrm{NS}\begin{bmatrix} -1 & 2 \\ 0 & 0 \end{bmatrix} = \mathrm{span}\left\{\begin{bmatrix} 2 \\ 1 \end{bmatrix}\right\}$$

So letting $S = \left[\begin{smallmatrix} 1 & 2 \\ 2 & 1 \end{smallmatrix}\right]$ and hence $S^{-1} = (-1/3)\left[\begin{smallmatrix} 1 & -2 \\ -2 & 1 \end{smallmatrix}\right]$ we have

$$A = -\frac{1}{3}\begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 1/2 \end{bmatrix}\begin{bmatrix} 1 & -2 \\ -2 & 1 \end{bmatrix}$$

so

$$A^{100} = -\frac{1}{3}\begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 2^{-100} \end{bmatrix}\begin{bmatrix} 1 & -2 \\ -2 & 1 \end{bmatrix}$$

Problem 3.2 (20 pts)**.** Let P be the plane through the origin and the points $(1, 1, 0)$ and $(1, -1, 0)$. Let $p : \mathbb{R}^3 \to P$ be the orthogonal projection onto P. Let L be the line through the origin perpendicular to P. p is a linear map and hence given by a matrix A.

a) What are the eigenvalues of A? Explain without calculation.

All points on the plane P are fixed, so 1 is an eigenvalue with $E_1 = P$. All points on L are mapped to 0, so 0 is an eigenvalue with $E_0 = \ker(p) = L$.

b) For each eigenvalue, what is the associated eigenspace in terms of P and L?

already answered in the answer to (a)

c) Given the answer to the first two questions, write $A = SDS^{-1}$. (Read the next item first, you might kill two birds with one stone here.)

This could just be answered in (d), but given the answer to (a) and (b) we have the following. $P = \text{span}\{(1, 1, 0), (1, -1, 0)\}$ and $L = \text{span}\{(0, 0, 1)\}$. So

$$A = \begin{bmatrix} 1 & 1 & 0 \\ 1 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 1 & 0 \\ 1 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}^{-1}$$

d) If possible, write $A = UDU^T$ for some unitary U.

Given the preceding, we can just normalize the two vectors we chose for the basis to P to get

$$A = \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} & 0 \\ 1/\sqrt{2} & -1/\sqrt{2} & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} & 0 \\ 1/\sqrt{2} & -1/\sqrt{2} & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Note here that $U^T = U$.

Note: There is almost nothing that you need to calculate here. This is checking that you understand what eigenvalues and eigenvectors are, at least geometrically.

Problem 3.3 (20 pts)**.** Suppose

$$A = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1/2 & 0 & 0 \\ 0 & -1/3 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 0 \end{bmatrix}^{-1}$$

Describe the "long-term" behavior of $A^n \boldsymbol{v}$ for an arbitrary point $\boldsymbol{v} \in \mathbb{R}^3$. More specifically, in the limit as $n \to \infty$ what happens to $A^n \boldsymbol{v}$.

Note: Depending on where $\boldsymbol{v}$ is in $\mathbb{R}^3$, there might be different long-term behavior.

For any $\boldsymbol{v}$, we may write $\boldsymbol{v} = a\boldsymbol{v}_1 + b\boldsymbol{v}_2 + c\boldsymbol{v}_3$ where

$$\boldsymbol{v}_1 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \quad \boldsymbol{v}_2 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \quad \boldsymbol{v}_3 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$$

and we see that

$$A^n \boldsymbol{v} = (1/2)^n a\boldsymbol{v}_1 + (-1/3)^n b\boldsymbol{v}_2 + (1)^n c\boldsymbol{v}_3$$

and since both $(1/2)^n$ and $(-1/3)^n$ approach 0 as n gets large, $A^n\boldsymbol{v}$ approaches $c\boldsymbol{v}$. All points are "attracted" to the line L, in particular $\lim_{n\to\infty} A^n\boldsymbol{v} = c\boldsymbol{v}_3 = \left[\begin{smallmatrix} c \\ c \\ 0 \end{smallmatrix}\right]$.

You can calculate c to get

$$\begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 0 \end{bmatrix}^{-1} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 1 & -1 & 1 \\ -1 & 1 & 1 \\ 1 & 1 & -1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

So $c = \frac{1}{2}(x+y-z)$ and so

$$A^n \begin{bmatrix} x \\ y \\ z \end{bmatrix} \to \frac{1}{2}(x+y-z)\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$$

For the next two problems, let

$$A = \begin{bmatrix} 3/2 & 3/2 \\ 3/2 & 3/2 \\ -1 & 1 \\ -1 & 1 \end{bmatrix}.$$

Problem 3.4 (20 pts)**.** Unitarily diagonalize the 2×2 matrix A^TA. That is, find V a unitary (real) matrix and diagonal Λ so that $A = V\Lambda V^T$. Make sure that $\Lambda = \left[\begin{smallmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{smallmatrix}\right]$ where $\lambda_1 \geq \lambda_2$.

Check your answer! Make sure that your V is unitary and check that $A^TA = V\Lambda V^T$. You need these in the next step so you want to double-check here to make sure that they are correct.

$$A^TA = \begin{bmatrix} 13/2 & 5/2 \\ 5/2 & 13/2 \end{bmatrix}$$

So $p_A(t) = (13/2 - t)^2 - (5/2)^2$ and

$$\begin{aligned} p_A(t) = 0 &\iff (13-2t)^2 - 5^2 = 0 \\ &\iff 4t^2 - 52t + (13^2 - 5^2) = 0 \\ &\iff 4t^2 - 52t + 12^2 = 0 \\ &\iff t^2 - 13t + 36 = (t-9)(t-4) = 0 \end{aligned}$$

So the eigenvalues are 9 and 4. So

$$\Lambda = \begin{bmatrix} 9 & 0 \\ 0 & 4 \end{bmatrix}$$

Now find the basis for the eigenspaces

$$E_9 = \text{NS}(A - 9I) = \text{NS}\begin{bmatrix} -5/2 & 5/2 \\ -5/2 & 5/2 \end{bmatrix} = \text{span}\left\{\begin{bmatrix} 1 \\ 1 \end{bmatrix}\right\}$$

and

$$E_4 = \mathrm{NS}(A - 4I) = \mathrm{NS}\begin{bmatrix} 5/2 & 5/2 \\ 5/2 & 5/2 \end{bmatrix} = \mathrm{span}\left\{ \begin{bmatrix} -1 \\ 1 \end{bmatrix} \right\}$$

So

$$V = \begin{bmatrix} 1/\sqrt{2} & -1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}$$

Problem 3.5 (20 pts)**.** Find the SVD for A (same A as in the previous problem.) Explain how you get the singular values and the left singular vectors.

Check! When done, you should have unitary 4×4 matrix U, a diagonal 4×2 matrix Σ, and the unitary 2×2 matrix V (from above) so that $A = U\Sigma V^T$.

The singular values are just $\sigma_1 = \sqrt{9}$ and $\sigma_2 = \sqrt{4} = 2$ and so we know from the previous problem that

$$\Sigma = \begin{bmatrix} 3 & 0 \\ 0 & 2 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \text{ and } V = \begin{bmatrix} 1/\sqrt{2} & -1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}$$

We have

$$\boldsymbol{u}_1 = \frac{A\boldsymbol{v}_1}{\sigma_1} = \frac{1}{3}\begin{bmatrix} 3/\sqrt{2} \\ 3/\sqrt{2} \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \\ 0 \\ 0 \end{bmatrix}$$

and

$$\boldsymbol{u}_2 = \frac{A\boldsymbol{v}_1}{\sigma_2} = \frac{1}{2}\begin{bmatrix} 0 \\ 0 \\ 2/\sqrt{2} \\ 2/\sqrt{2} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix}$$

For $\boldsymbol{u}_3$ and $\boldsymbol{u}_4$ we need an orthonormal basis for $\mathrm{NS}(A)$

$$\begin{aligned} \mathrm{NS}(A^T) &= \mathrm{NS}\begin{bmatrix} 3/2 & 3/2 & -1 & -1 \\ 3/2 & 3/2 & 1 & 1 \end{bmatrix} = \mathrm{NS}\begin{bmatrix} 3/2 & 3/2 & -1 & -1 \\ 0 & 0 & 2 & 2 \end{bmatrix} \\ &= \mathrm{NS}\begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix} = \mathrm{span}\left\{ \begin{bmatrix} 1 \\ -1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ -1 \\ 1 \end{bmatrix} \right\} \end{aligned}$$

So

$$\boldsymbol{u}_3 = \begin{bmatrix} 1/\sqrt{2} \\ -1/\sqrt{2} \\ 0 \\ 0 \end{bmatrix} \text{ and } \boldsymbol{u}_4 = \begin{bmatrix} 0 \\ 0 \\ 1/\sqrt{2} \\ -1/\sqrt{2} \end{bmatrix}$$

and so

$$U = \begin{bmatrix} 1/\sqrt{2} & 0 & 1/\sqrt{2} & 0 \\ 1/\sqrt{2} & 0 & -1/\sqrt{2} & 0 \\ 0 & 1/\sqrt{2} & 0 & 1/\sqrt{2} \\ 0 & 1/\sqrt{2} & 0 & -1/\sqrt{2} \end{bmatrix}$$

So

$$\begin{aligned} A &= \begin{bmatrix} 1/\sqrt{2} & 0 & 1/\sqrt{2} & 0 \\ 1/\sqrt{2} & 0 & -1/\sqrt{2} & 0 \\ 0 & 1/\sqrt{2} & 0 & 1/\sqrt{2} \\ 0 & 1/\sqrt{2} & 0 & -1/\sqrt{2} \end{bmatrix} \begin{bmatrix} 3 & 0 \\ 0 & 2 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ -1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix} \\ &= \frac{1}{2} \begin{bmatrix} 1 & 0 & 1 & 0 \\ 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & -1 \end{bmatrix} \begin{bmatrix} 3 & 0 \\ 0 & 2 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} \\ &= \frac{1}{2} \begin{bmatrix} 3 & 0 \\ 3 & 0 \\ 0 & 2 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} \\ &= \frac{1}{2} \begin{bmatrix} 3 & 3 \\ 3 & 3 \\ -2 & 2 \\ -2 & 2 \end{bmatrix} \text{Correct!} \end{aligned}$$