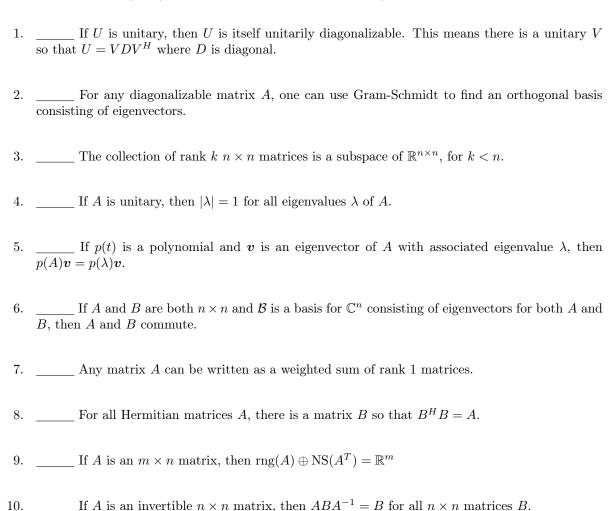
## Exam 2

To avoid any confusion, unless specified otherwise, vector spaces are complex vector spaces, inner-products are complex inner-products, and matrices are complex matrices. The standard inner product is  $\langle \boldsymbol{u}, \boldsymbol{v} \rangle = \boldsymbol{v}^H \boldsymbol{u} = \sum_{i=1}^n \bar{\boldsymbol{v}}_i \boldsymbol{u}_i$ . Keep in mind that  $A^H = A^T$  for real matrices and symmetric = Hermitian for real matrices.

## Part I: True/False

Each problem is points for a total of 50 points. (5 points each.)

You do not need to justify the answers here, this is unlike the quizzes.



## Part II: Computational (60 points)

P1. (15 points) Find B so that  $B^2 = A$  where

$$A = \begin{bmatrix} 13 & -5 & 5 \\ -8 & 10 & -8 \\ -3 & -3 & 5 \end{bmatrix}$$

P2. (15 points) Find B so that  $B^HB=A$  where

$$A = \begin{bmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{bmatrix}$$

P3. (15 points) Find the best rank 2 approximation to A from (2) with respect to  $\|\cdot\|_F$ .

P4. (15 points) Let

$$A = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix},$$

find the characteristic polynomial and all eigenvalues, both real and complex. Explain why A is diagonalizable and compute  $A^{2020}$ . Note, I do not ask you to diagonalize A.

## Part III: Theory and Proofs (45 points; 15 points each)

Pick three of the following four options. If you try all four, I will grade the first three, so if this is not what you intend, then just do three, or at least make it clear which I should grade.

P1. Let  $L: V \to V$  be a linear transformation and let  $\mathcal{B} = \{v_1, \dots, v_n\}$  be an ordered basis. Show that  $[L]_{\mathcal{B}}$  is upper triangular iff  $L(v_i) \in \operatorname{span}\{v_1, \dots, v_i\}$  for all i.

- P2. Let P be an  $n \times n$  symmetric matrix that satisfies  $P^2 = P$ .
  - (a) Let U = rng(P) = CS(P). Show that  $\boldsymbol{v} P\boldsymbol{v} \perp U$ .

(b) Use (a) to see that  $\|\boldsymbol{v} - P\boldsymbol{v}\|_2^2 = \min\{\|\boldsymbol{v} - \boldsymbol{u}\|_2^2 \mid \boldsymbol{u} \in U\}$ . That is  $P\boldsymbol{v}$  is the vector in U closest to  $\boldsymbol{v}$ .

P3. Let  $\lambda_1, \lambda_2, \dots, \lambda_n$  be the complete set of n eigenvalues of an  $n \times n$  matrix A. Show that

$$\det(A) = \prod_{i=1}^{n} \lambda_i = \lambda_1 \cdot \lambda_2 \cdot \dots \cdot \lambda_n.$$

Note: You do not know that A is diagonalizable, there might fail to be a basis of eigenvectors.

P4. Use the SVD to show that any square matrix A can be written as A = UP where U is unitary and P is Hermitian.