Name: ____ Quiz 2 - MAT345

Problem 1 (20 points; 4 points each)**.** Decide if each of the following are true or false and provide a small proof or counterexample in each case. All vector spaces are assumed to be finite-dimensional here.

(a) False Given a basis $\mathcal{B} = \{\boldsymbol{v}_1, \ldots, \boldsymbol{v}_n\}$ for a vector space V and U a subspace of V, then there is $\mathcal{C} \subseteq \mathcal{B}$ that is a basis for U.

(b) True Given a basis $\mathcal{C}$ for a subspace U of a vector space V, $\mathcal{C}$ can be extended to a basis $\mathcal{B}$ for V.

(c) False If $\{\boldsymbol{v}_1, \ldots, \boldsymbol{v}_n\}$ is linearly independent and $\boldsymbol{v} \in \operatorname{span}(\{\boldsymbol{v}_1, \ldots, \boldsymbol{v}_n\})$, then it is possible that there are distinct $\boldsymbol{c}, \boldsymbol{b} \in \mathbb{R}^n$ such that $\boldsymbol{v} = \sum_{i=1}^n c_i \boldsymbol{v}_i = \sum_{i=1}^n b_i \boldsymbol{v}_i$.

(d) True If $\{\boldsymbol{v}_1, \ldots, \boldsymbol{v}_n\}$ is linearly independent and $V = \operatorname{span}(\{\boldsymbol{v}_1, \ldots, \boldsymbol{v}_n\}) = \operatorname{span}(\{\boldsymbol{u}_1, \ldots, \boldsymbol{u}_n\})$, then $\{\boldsymbol{u}_1, \ldots, \boldsymbol{u}_n\}$ is linearly independent.

(e) False Suppose V is a vector space and $U \subseteq V$ is a subspace. For any $\boldsymbol{v} \in V$, there is a **unique** $\boldsymbol{u} \in U$ so that $\boldsymbol{v} = \boldsymbol{u} + (\boldsymbol{v} - \boldsymbol{u})$, that is, there is a unique "projection" of V into U.

Problem 2 (10 pts)**.** Show that the collection, U, of upper triangular 3×3 matrices is a subspace of $\mathbb{R}^{3\times 3}$ (the space of all 3×3 matrices). Give a basis $\mathcal{B}$ for U and for $\boldsymbol{v} = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 4 & 5 \\ 0 & 0 & 6 \end{bmatrix}$, give $[\boldsymbol{v}]_{\mathcal{B}}$.

To show that U is a subspace we need only show that $\alpha\boldsymbol{v}+\boldsymbol{u} \in U$ for $\boldsymbol{v}, \boldsymbol{u} \in U$. So let $\boldsymbol{u} = \begin{bmatrix} u_{11} & u_{12} & u_{13} \\ 0 & u_{22} & u_{23} \\ 0 & 0 & u_{33} \end{bmatrix}$ and let $\boldsymbol{v} = \begin{bmatrix} v_{11} & v_{12} & v_{13} \\ 0 & v_{22} & v_{23} \\ 0 & 0 & v_{33} \end{bmatrix}$, then $\alpha\boldsymbol{u}+\boldsymbol{v} = \begin{bmatrix} \alpha u_{11}+v_{11} & \alpha u_{12}+v_{12} & \alpha u_{13}+v_{13} \\ 0 & \alpha u_{22}+v_{22} & \alpha u_{23}+v_{23} \\ 0 & 0 & \alpha u_{33}+v_{33} \end{bmatrix} \in U$.

A basis is clearly given by $E^{ij}_{lk} = 1$ if $i=j$ and $l=k$ and $j \le i$ and 0 otherwise. So $E^{11} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$, $E^{12} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$, etc. This basis has six elements, so $\dim(U) = 6$.

With this basis, clearly $\boldsymbol{v} = E^{11} + 2E^{12} + 3E^{13} + 4E^{22} + 5E^{23} + 6E^{33}$.

Problem 3. Find a basis for $\operatorname{span}\{\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3, \boldsymbol{v}_4, \boldsymbol{v}_5\}$ where

$$\boldsymbol{v}_1 = \begin{bmatrix} 1 \\ 3 \\ -3 \\ 2 \end{bmatrix}, \boldsymbol{v}_2 = \begin{bmatrix} 1 \\ 4 \\ 0 \\ 4 \end{bmatrix}, \boldsymbol{v}_3 = \begin{bmatrix} 2 \\ 4 \\ -12 \\ 0 \end{bmatrix}, \boldsymbol{v}_4 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 2 \end{bmatrix}, \boldsymbol{v}_5 = \begin{bmatrix} -1 \\ -2 \\ 8 \\ 4 \end{bmatrix}$$

Method 1: Let

$$A = \begin{bmatrix} 1 & 1 & 2 & 0 & -1 \\ 3 & 4 & 4 & 0 & -2 \\ -3 & 0 & -12 & 1 & 8 \\ 2 & 4 & 0 & 2 & 4 \end{bmatrix}$$

$$A \underset{\substack{R_2-3R_1\to R_2 \\ R_3+3R_1\to R_3 \\ R_4-2R_1\to R_4}}{\Longrightarrow} \begin{bmatrix} 1 & 1 & 2 & 0 & -1 \\ 0 & 1 & -2 & 0 & 1 \\ 0 & 3 & -6 & 1 & 5 \\ 0 & 2 & -4 & 2 & 6 \end{bmatrix} \underset{\substack{R_3-3R_2\to R_3 \\ R_4-2R_2\to R_4}}{\Longrightarrow} \begin{bmatrix} 1 & 1 & 2 & 0 & -1 \\ 0 & 1 & -2 & 0 & 1 \\ 0 & 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 2 & 4 \end{bmatrix} \underset{R_4-2R_3\to R_4}{\Longrightarrow} \begin{bmatrix} \boxed{1} & 1 & 2 & 0 & -1 \\ 0 & \boxed{1} & -2 & 0 & 1 \\ 0 & 0 & 0 & \boxed{1} & 2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

So $\mathcal{B} = \{\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_4\}$ is a basis. (This is all you need.)

In fact, from our CR decomposition, we know

$$A = \begin{bmatrix} 1 & 1 & 0 \\ 3 & 4 & 0 \\ -3 & 0 & 1 \\ 2 & 2 & 2 \end{bmatrix} \begin{bmatrix} \boxed{1} & 1 & 2 & 0 & -1 \\ 0 & \boxed{1} & -2 & 0 & 1 \\ 0 & 0 & 0 & \boxed{1} & 2 \end{bmatrix}$$

So we know $\boldsymbol{v}_3 = 2\boldsymbol{v}_2 - 2\boldsymbol{v}_2$ and $\boldsymbol{v}_5 = -\boldsymbol{v}_1 + \boldsymbol{v}_2 + 2\boldsymbol{v}_3$.

Method 2: Let

$$B = \begin{bmatrix} 1 & 3 & -3 & 2 \\ -1 & -2 & 8 & 4 \\ 0 & 0 & 1 & 2 \\ 1 & 4 & 0 & 4 \\ 2 & 4 & -12 & 0 \end{bmatrix}$$

Then eliminate:

$$B \underset{\substack{R_2+R_1\to R_2 \\ R_4-R_1\to R_4 \\ R_5-2R_1\to R_5}}{\Longrightarrow} \begin{bmatrix} 1 & 3 & -3 & 2 \\ 0 & 1 & 5 & 6 \\ 0 & 0 & 1 & 2 \\ 0 & 1 & 3 & 2 \\ 0 & -2 & -6 & -4 \end{bmatrix} \underset{\substack{R_4-R_2\to R_4 \\ R_5+R_2\to R_5}}{\Longrightarrow} \begin{bmatrix} 1 & 3 & -3 & 2 \\ 0 & 1 & 5 & 6 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & -2 & -4 \\ 0 & 0 & 4 & 8 \end{bmatrix} \underset{\substack{R_4+2R_3\to R_4 \\ R_5-4R_3\to R_5}}{\Longrightarrow} \begin{bmatrix} 1 & 3 & -3 & 2 \\ 0 & 1 & 5 & 6 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} = B'$$

So $\text{span}\{\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3, \boldsymbol{v}_4, \boldsymbol{v}_5\} = \text{RS}(B) = \text{RS}(B') = \text{span}\{(1,3,-3,2),(0,1,5,6),(0,0,1,2)\}$. So a basis for $\text{span}\{\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3, \boldsymbol{v}_4, \boldsymbol{v}_5\}$ is $\mathcal{B}' = \{\boldsymbol{u}_1, \boldsymbol{u}_2, \boldsymbol{u}_3\}$ where

$$\boldsymbol{u}_1 = \begin{bmatrix} 1 \\ 3 \\ -3 \\ 2 \end{bmatrix}, \boldsymbol{u}_2 = \begin{bmatrix} 0 \\ 1 \\ 5 \\ 6 \end{bmatrix}, \boldsymbol{u}_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 2 \end{bmatrix}$$