

1 True/False (100 points; 10 points each)

Problem 1.1. In class, you need only provide a T/F (make it clear!) As usual, you may earn back up to 50% of the lost points by supplying justifications afterward.

- i) True Let $S \in M_3$ be invertible, then $SM_{3 \times 3}S^{-1} = \{SAS^{-1} \mid A \in M_{3 \times 3}\}$ is a subspace of $M_{3 \times 3}$.

We must show that this set is closed under linear combinations:

$$S(\alpha_1 A_1 + \alpha_2 A_2)S^{-1} = \alpha_1 S A_1 S^{-1} + \alpha_2 S A_2 S^{-1}$$

- ii) True The map $\text{mean} : \mathbb{R}^n \rightarrow \mathbb{R}$ is linear where $\text{mean}(\mathbf{x}) = \frac{1}{n} \sum_{i=1}^n x_i$.

This is clear:

$$\text{mean}(\alpha \mathbf{x} + \beta \mathbf{y}) = \frac{1}{n} \sum_{i=1}^n (\alpha x_i + \beta y_i) = \alpha \frac{1}{n} \sum_{i=1}^n x_i + \beta \frac{1}{n} \sum_{i=1}^n y_i = \alpha \text{mean}(\mathbf{x}) + \beta \text{mean}(\mathbf{y})$$

- iii) False It is clear that for any $\mathbf{u} \in \mathbb{R}^n$, the map $l_{\mathbf{u}} : \mathbb{R}^n \rightarrow \mathbb{R}$ given by $l_{\mathbf{u}}(\mathbf{v}) = \langle \mathbf{v}, \mathbf{u} \rangle = \mathbf{u}^T \mathbf{v}$ is linear. There are however some linear $l : \mathbb{R}^n \rightarrow \mathbb{R}$ such that $l \neq l_{\mathbf{u}}$ for any $\mathbf{u} \in \mathbb{R}^n$.

Let $\mathbf{u}^T = [l]_{\mathcal{E}}$, then $l(\mathbf{v}) = [l(\mathbf{v})]_{\mathcal{E}} = [l]_{\mathcal{E}}[\mathbf{v}]_{\mathcal{E}} = \mathbf{u}^T \mathbf{v}$.

- iv) False For all linear $L : V \rightarrow W$, if $\{\mathbf{v}_1, \dots, \mathbf{v}_k\}$ is independent, then $\{L(\mathbf{v}_1), \dots, L(\mathbf{v}_k)\}$ is independent.

This is trivially false. L could just be the $\mathbf{0}$ map, that is, $L(\mathbf{v}) = \mathbf{0}$ for all $\mathbf{v} \in V$. This would be true if $\ker(L) = \{\mathbf{0}\}$.

- v) True For all linear $L : V \rightarrow W$, if $\{L(\mathbf{v}_1), \dots, L(\mathbf{v}_k)\}$ is independent, then $\{\mathbf{v}_1, \dots, \mathbf{v}_k\}$ is independent.

Suppose $\sum \alpha_i \mathbf{v}_i = \mathbf{0}$, then $L(\sum \alpha_i \mathbf{v}_i) = \sum \alpha_i L(\mathbf{v}_i) = L(\mathbf{0}) = \mathbf{0}$. Since $\{L(\mathbf{v}_1), \dots, L(\mathbf{v}_k)\}$ is independent it follows that $\alpha_i = 0$ for all i and hence that $\{\mathbf{v}_1, \dots, \mathbf{v}_k\}$ is independent.

- vi) True Let c_1, c_2, c_3, c_4 be distinct real numbers, then the polynomials $p_1(x) = x - c_1$, $p_2(x) = (x - c_1)(x - c_2)$, $p_3(x) = (x - c_1)(x - c_2)(x - c_3)$, $p_4(x) = (x - c_1)(x - c_2)(x - c_3)(x - c_4)$ is a basis for $S = \{p(x) \in \mathbb{P}_4(x) \mid p(x) = \alpha_1 x + \alpha_2 x^2 + \alpha_3 x^3 + \alpha_4 x^4\}$.

Clearly $\dim(S) = 4$ so we just need to see that $\{p_1, p_2, p_3, p_4\}$ is linearly independent. Suppose $p(x) = \alpha_1 p_1(x) + \alpha_2 p_2(x) + \alpha_3 p_3(x) + \alpha_4 p_4(x) = \mathbf{0}$. then $p(c_2) = \alpha_1(c_2 - c_1) = 0$ so $\alpha_1 = 0$. So $p(x) = \alpha_2 p_2(x) + \alpha_3 p_3(x) + \alpha_4 p_4(x) = \mathbf{0}$. Now $p(c_3) = \alpha_2(c_3 - c_1)(c_3 - c_2) = 0$ so $\alpha_2 = 0$. Thus $p(x) = \alpha_3 p_3(x) + \alpha_4 p_4(x) = \mathbf{0}$.

Now we have $p(c_4) = \alpha_3(c_4 - c_1)(c_4 - c_2)(c_4 - c_3) = 0$, so $\alpha_3 = 0$. So we have $p(x) = \alpha_4 p_4(x) = 0$. Take $d \notin \{c_1, c_2, c_3, c_4\}$, then $p(d) = \alpha_4(d - c_1)(d - c_2)(d - c_3)(d - c_4) = 0$ and so $\alpha_4 = 0$. So all $\alpha_i = 0$ and hence the set is independent and thus must span.

- vii) **True** Given any basis $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4\}$, from \mathbb{R}^4 and any four matrices $M_1, M_2, M_3, M_4 \in \mathbb{R}^{2 \times 3}$ there is a unique linear transformation $L : \mathbb{R}^4 \rightarrow \mathbb{R}^{2 \times 3}$ where $L(\mathbf{v}_i) = M_i$.

Existence: Define $L(\sum_{i=1}^4 \alpha_i \mathbf{v}_i) = \sum_{i=1}^4 \alpha_i M_i$. This is a well-defined function $L : \mathbb{R}^4 \rightarrow \mathbb{R}^{2 \times 3}$.

Showing that this is linear is just a computation: Let $\mathbf{v} = \sum_{i=1}^4 \alpha_i \mathbf{v}_i$ and $\mathbf{u} = \sum_{i=1}^4 \beta_i \mathbf{v}_i$, then

$$\begin{aligned} L(\gamma \mathbf{v} + \mathbf{u}) &= L\left(\gamma \sum_{i=1}^4 \alpha_i \mathbf{v}_i + \sum_{i=1}^4 \beta_i \mathbf{v}_i\right) = L\left(\sum_{i=1}^4 (\gamma \alpha_i + \beta_i) \mathbf{v}_i\right) \\ &= \sum_{i=1}^4 (\gamma \alpha_i + \beta_i) M_i = \gamma \sum_{i=1}^4 \alpha_i M_i + \sum_{i=1}^4 \beta_i M_i = \gamma \cdot L(\mathbf{v}) + L(\mathbf{u}) \end{aligned}$$

Uniqueness: Suppose $L' : V \rightarrow W$ is linear and sends \mathbf{v}_i to M_i , then for $\mathbf{v} \in V$, $\mathbf{v} = \sum_{i=1}^4 \alpha_i \mathbf{v}_i$ and $L'(\mathbf{v}) = L'(\sum_{i=1}^4 \alpha_i \mathbf{v}_i) = \sum_{i=1}^4 \alpha_i M_i = L(\mathbf{v})$ and thus $L = L'$.

- viii) **False** There is an invertible (one-to-one) linear map $L : \mathbb{R}^2 \rightarrow \mathbb{R}$.

$\dim(\ker(L)) + \dim(\text{Im}(L)) = 2$, but $\dim(\ker(L)) + \dim(\text{Im}(L)) \leq \dim(\ker(L)) +$

- ix) **True** Let

$$\mathcal{B} = \left\{ \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ 2 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} \right\}$$

be a basis for \mathbb{R}^3 . If $[\mathbf{v}]_{\mathcal{B}} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ then $\mathbf{v} = \begin{bmatrix} 3 \\ 3 \\ 4 \end{bmatrix}$

$$\begin{bmatrix} 3 \\ 3 \\ 4 \end{bmatrix} = (1) \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} + (1) \begin{bmatrix} 2 \\ 0 \\ 2 \end{bmatrix} + (1) \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}$$

- x) **False** Given a linear transformation $L : V \rightarrow V$ where $\dim(V) = n$ and \mathcal{B} is a basis for V , the value of $\det([L]_{\mathcal{B}})$ depends on the choice of basis \mathcal{B} .

This we discussed in class, given any other basis \mathcal{B}' we have

$$[L]_{\mathcal{B}'} = [\text{id}]_{\text{cal } \mathcal{B}, \mathcal{B}'} [L]_{\mathcal{B}, \mathcal{C}} [\text{id}]_{\mathcal{B}', \mathcal{B}}$$

Letting $S = [\text{id}]_{\text{cal } \mathcal{B}, \mathcal{B}'}$ and so $S^{-1} = [\text{id}]_{\text{cal } \mathcal{B}', \mathcal{B}}$. (A common *change of basis* scenario.) So $[L]_{\mathcal{B}'}$ and $[L]_{\mathcal{B}}$ are similar and thus have the same determinant. This way we can define $\det(L)$ for L a linear transformation, not just a matrix.

2 Long Answer (90 points)

Show all computations so that you make clear what your thought processes are.

Problem 2.1 (30 pts). Consider A given by

$$A = \begin{bmatrix} 2 & -2 & 2 & 2 & 1 \\ -4 & 2 & -5 & -3 & -4 \\ -2 & 4 & -1 & -5 & 1 \\ -4 & 6 & -3 & -7 & 0 \end{bmatrix}$$

Find a basis for each of $\text{NS}(A)$, $\text{CS}(A)$, and $\text{RS}(A)$. (10 points each)

$$\text{rref}(A) = \begin{bmatrix} 1 & 0 & \frac{3}{2} & 0 & \frac{3}{2} \\ 0 & 1 & \frac{1}{2} & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

From this we know:

$$\text{CS}(A) = \text{span}\{(-2, -4, -2, -4), (-2, 2, 4, 6), (2, -3, -5, -7)\}$$

$$\text{RS}(A) = \text{span}\{(1, 0, \frac{3}{2}, 0, \frac{3}{2}), (0, 1, \frac{1}{2}, 0, 1), (0, 0, 0, 1, 0)\}$$

Note: $\text{RS}(A)$ is not the span of the first three rows of A .

To find a basis for $\text{NS}(A)$ we are looking for solutions to $Ax = 0$. First, we have back-substitution: x_3 and x_5 are free, let $x_3 = s$ and $x_5 = t$, then

$$x_5 = t$$

$$x_4 = 0$$

$$x_3 = s$$

$$x_2 = -\frac{1}{2}s - t$$

$$x_1 = -\frac{3}{2}s - \frac{3}{2}t$$

Any vector x satisfying, $Ax = 0$ can be written as:

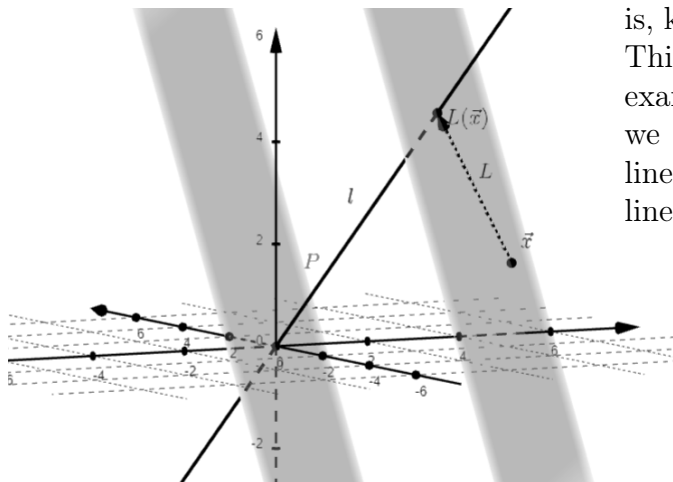
$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} -\frac{3}{2}s - \frac{3}{2}t \\ -\frac{1}{2}s - t \\ s \\ 0 \\ t \end{bmatrix} = s \begin{bmatrix} -\frac{3}{2} \\ -\frac{1}{2} \\ 1 \\ 0 \\ 0 \end{bmatrix} + t \begin{bmatrix} -\frac{3}{2} \\ -1 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

So $\{(-\frac{3}{2}, -\frac{1}{2}, 1, 0, 0), (-\frac{3}{2}, -1, 0, 0, 1)\}$ is a basis for $\text{NS}(A)$, that is,

$$\text{NS}(A) = \text{span}\{(-\frac{3}{2}, -\frac{1}{2}, 1, 0, 0), (-\frac{3}{2}, -1, 0, 0, 1)\}$$

Problem 2.2 (20 pts). Consider the map $L : \mathbb{R}^3 \rightarrow \mathbb{R}^3$ that projects a point in \mathbb{R}^3 onto the line $l : \left\{ t \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix} \mid t \in \mathbb{R} \right\}$ along the plane $P : 3x - 2y + z = 0$.

- (8 points) Find a basis \mathcal{B} for \mathbb{R}^3 so that $[L]_{\mathcal{B}}$ is simple. Give both \mathcal{B} and $[L]_{\mathcal{B}}$.
- (8 points) Next, find $[L]$ using some change of basis and the $[L]_{\mathcal{B}}$ that you found.
- (4 points) Finally, find $L((4, -4, 0))$.



Geogebra

Note: Points on P are mapped to $\mathbf{0}$, that is, $\ker(L) = P$, while points in l are fixed. This is similar to, but different from, the examples done in class. In that example, we were projecting onto a plane along a line, while here, we are projecting onto a line along a plane.

There are many choices for \mathcal{B} , I will use the two vectors $\mathbf{v}_1 = (1, 1, -1)$ and $\mathbf{v}_2 = (0, 1, 2)$ in P and $\mathbf{v}_3 = (1, -1, 2)$ in l . So

$$\mathcal{B} = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\} = \left\{ \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix} \right\}$$

and

$$[L]_{\mathcal{B}} = [[L(\mathbf{v}_1)]_{\mathcal{B}} \mid [L(\mathbf{v}_2)]_{\mathcal{B}} \mid [L(\mathbf{v}_3)]_{\mathcal{B}}] = [[\mathbf{0}]_{\mathcal{B}} \mid [\mathbf{0}]_{\mathcal{B}} \mid [\mathbf{v}_3]_{\mathcal{B}}] = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

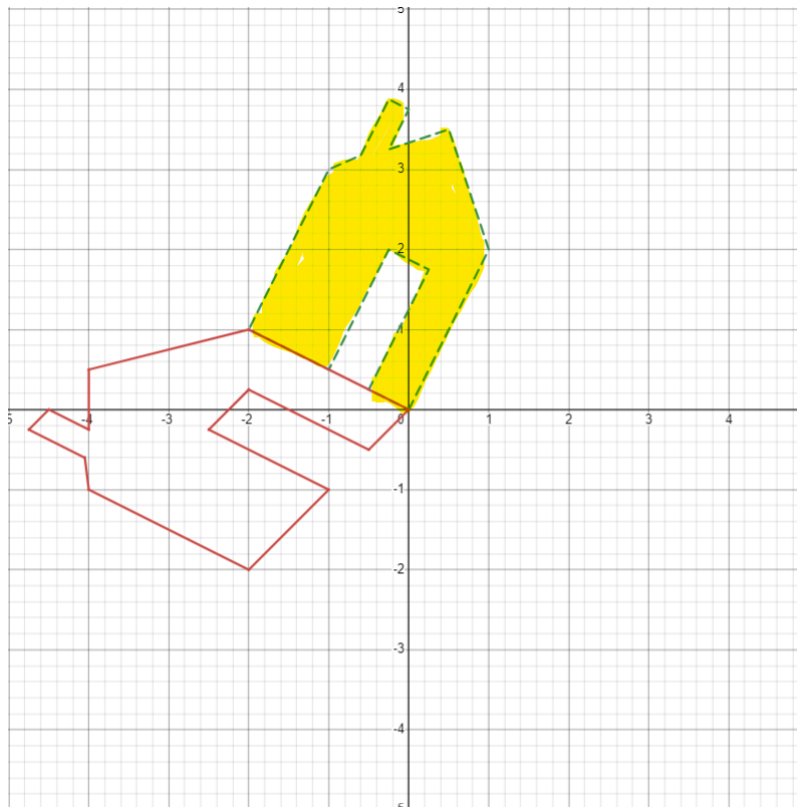
Finding $[L]$ is now trivial.

$$[L] = B[L]_{\mathcal{B}}B^{-1} = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & -1 \\ -1 & 2 & 2 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & -1 \\ -1 & 2 & 2 \end{bmatrix}^{-1} = \frac{1}{7} \begin{bmatrix} 3 & -2 & 1 \\ -3 & 2 & -1 \\ 6 & -4 & 2 \end{bmatrix}$$

and

$$L\left(\begin{bmatrix} 4 \\ -4 \\ 0 \end{bmatrix}\right) = \frac{20}{7} \begin{bmatrix} -1 \\ -1 \\ 2 \end{bmatrix}$$

Problem 2.3 (20 pts). The green (dashed/filled) house has been transformed to the red (solid) house by a linear transformation $L : \mathbb{R}^2 \rightarrow \mathbb{R}^2$.



Desmos

- i) (10 points) Find $[L]$ by first choosing basis \mathcal{G} (for the green house) and basis \mathcal{R} (for the red house) and find $[L]_{\mathcal{G}, \mathcal{R}}$.
- ii) (10 points) Find $[L]$ by using appropriate change of basis matrices together with $[L]_{\mathcal{G}, \mathcal{R}}$

There are many options here. In all cases, you might have chosen a different basis than I did, but the final matrix is the same.

(Exactly as done in class!) Take

$$\mathcal{G} = \{\mathbf{v}_1, \mathbf{v}_2\} = \left\{ \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \begin{bmatrix} -2 \\ 1 \end{bmatrix} \right\}$$

and

$$\mathcal{R} = \{\mathbf{u}_1, \mathbf{u}_2\} = \left\{ \begin{bmatrix} -2 \\ 1 \end{bmatrix}, \begin{bmatrix} -2 \\ -2 \end{bmatrix} \right\}$$

Then

$$[L]_{\mathcal{G}, \mathcal{R}} = [[L(\mathbf{v}_1)]_{\mathcal{R}} [L(\mathbf{v}_2)]_{\mathcal{R}}] = [[\mathbf{u}_1]_{\mathcal{R}} [\mathbf{u}_2]_{\mathcal{R}}] = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

and

$$\begin{aligned}[L] &= R[L]_{\mathcal{G}, \mathcal{R}} G^{-1} = R G^{-1} = \begin{bmatrix} -2 & -2 \\ 1 & -2 \end{bmatrix} \begin{bmatrix} 1 & -2 \\ 2 & 1 \end{bmatrix}^{-1} \\ &= \begin{bmatrix} -2 & -2 \\ 1 & -2 \end{bmatrix} \frac{1}{5} \begin{bmatrix} 1 & 2 \\ -2 & 1 \end{bmatrix} = \frac{1}{5} \begin{bmatrix} 2 & -6 \\ 5 & 0 \end{bmatrix}\end{aligned}$$

Problem 2.4 (20 points). Show that if $L : V \rightarrow W$ is linear and $\ker(L) = \{\mathbf{0}\}$, then for any linearly independent set $\{\mathbf{v}_1, \dots, \mathbf{v}_k\}$ from V , $\{L(\mathbf{v}_1), \dots, L(\mathbf{v}_k)\}$ is independent.

Suppose $\sum \alpha_i L(\mathbf{v}_i) = \mathbf{0}$, then we can use linearity to get

$$L\left(\sum \alpha_i \mathbf{v}_i\right) = \mathbf{0}$$

But since $\ker(L) = \{\mathbf{0}\}$ we have $\sum \alpha_i \mathbf{v}_i = \mathbf{0}$ and as the \mathbf{v}_i 's are independent we know $\alpha_i = 0$ for all i and so the $L(\mathbf{v}_i)$'s are shown independent.