Quiz 1

Problem 1 (15 points; 3 points each)**.** Decide if each of the following are true or false and provide a justification or counterexample in each case. A justification could consist of a theorem from the text. All vector spaces are assumed to be finite-dimensional here.

(a) _____ Given a matrix with integer entries, it is possible to use elementary operations I – III with only integer constants in II and III. So that the final echelon matrix and all matrices along the way have only integer entries.

This is true. Using the definition of echelon form that does not require the pivots to have the value 1, this is true. Clearly if a type II, $R_i \leftarrow cR_i$, or type III operation, $R_i \leftarrow R_i + cR_j$, and the original R_i and R_j consists of integers, then so does the new R_i.

Note: An example does not suffice for an argument here.

(b) _____ There is only one echelon form of a matrix.

This is false. There is a unique reduced echelon form. Both

$$E_1 = \begin{bmatrix} 2 & 3 & 1 \\ 0 & 3 & 2 \end{bmatrix}$$

and

$$E_2 = \begin{bmatrix} 2 & 0 & -1 \\ 0 & 3 & 2 \end{bmatrix}$$

are both echelon and each can be got from the other by an elementary row operation.

(c) _____ The inverse of an elementary row operation is another elementary row operation of exactly the same type.

This is true. Just go through the cases.

Type I The inverse of "swap row i and row j" is just "swap row i and row j." In this case the operation is its own inverse.

Type II The inverse of "multiply row i by c" is "multiply row i by c^{-1}."

Type III The inverse of add "$c \cdot \text{row}_i$ to row_j and replace row_j" is "add $-c \cdot \text{row}_i$ to row_j and replace row_j"

This is a theorem in the text, so you can refer to that Theorem 1.5.1.

(d) _____ If $AC = BC$, then $A = B$.

This is false. We have

$$AC = BC \iff AC - BC = 0 \iff (A - B)C = 0$$

So basically the claim is equivalent to $AB = 0 \iff A = 0$ or $B = 0$ and this is false. Easy example:

$$\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ 1 & -1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

(e) _____ If A and B are invertible $n \times n$ matrices, then

$$((AB)^T)^{-1} = ((AB)^{-1})^T = (A^T)^{-1}(B^T)^{-1} = (A^{-1})^T(B^{-1})^T$$

This is true. First off

$$((AB)^T)^{-1} = (B^T A^T)^{-1} = (A^T)^{-1}(B^T)^{-1}$$

Both $(\cdot)^{-1}$ and $(\cdot)^T$ swap the order of multiplication.

For the next bit, we need $(A^T)^{-1} = (A^{-1})^T$, this we argue by

$$I = (A^{-1}A)^T = A^T(A^{-1})^T$$

but we know that $AB = I \implies B = A^{-1}$ for A and B $n \times n$ matrices. This we will discuss more later. In particular

$$A^T(A^{-1})^T = I \implies (A^{-1})^T = (A^T)^{-1}$$

Problem 2 (10 points)**.** Write down the augmented matrix associated to this system and then use the elementary row operations I and III to find an equivalent echelon, upper-triangular, form of this matrix. (See the note below.) Show every step of the reduction and indicate what operation you use. Use only integer constants in II – III.

Note: The book requires the leading entries in non-zero rows be 1's. Many, including myself, drop this and just describe the pivot as the leading non-zero entry. If you like, call this more general form, "*upper-triangular*."

Eliminate all the non-pivot leading coefficients in column 1 first, then work on column 2, etc. You can combine all operations for one column in a single step, so this should require 2 or 3 steps depending on how you write things.

$$\begin{aligned} 4x_1 + x_2 - x_3 + 3x_4 &= 5 \\ -4x_1 - 4x_2 + x_3 &= 4 \\ -2x_1 + 4x_2 + 4x_3 - 4x_4 &= -12 \end{aligned}$$

The augmented matrix is

$$\left[\begin{array}{cccc|c} 4 & 1 & -1 & 3 & 5 \\ -4 & -4 & 1 & 0 & 4 \\ -2 & 4 & 4 & -4 & -12 \end{array}\right]$$

Reduction to echelon form requires 3 row manipulations:

$$\begin{aligned}
\left[\begin{array}{cccc|c} 4 & 1 & -1 & 3 & 5 \\ -4 & -4 & 1 & 0 & 4 \\ -2 & 4 & 4 & -4 & -12 \end{array}\right] &\xrightarrow[R_3 \leftarrow 2R_3]{} \left[\begin{array}{cccc|c} 4 & 1 & -1 & 3 & 5 \\ -4 & -4 & 1 & 0 & 4 \\ -4 & 8 & 8 & -8 & -24 \end{array}\right] \\
&\xrightarrow[\substack{R_2 \leftarrow R_1 + R_2 \\ R_3 \leftarrow R_1 + R_3}]{} \left[\begin{array}{cccc|c} 4 & 1 & -1 & 3 & 5 \\ 0 & -3 & 0 & 3 & 9 \\ 0 & 9 & 7 & -5 & -19 \end{array}\right] \\
&\xrightarrow[R_3 \leftarrow R_3 + 3R_2]{} \left[\begin{array}{cccc|c} 4 & 1 & -1 & 3 & 5 \\ 0 & -3 & 0 & 3 & 9 \\ 0 & 0 & 7 & 4 & 8 \end{array}\right]
\end{aligned}$$

Problem 3 (10 points)**.** Using your echelon form above re-write the initial system as an equivalent triangular system. Let the variable that correspond to non-pivot element be the independent variable and solve for the remaining three variables in terms of this one. This is the "**back substitution**" step. Finally, write the solution set as $\{t\boldsymbol{v} + \boldsymbol{u} \,|\, t \in \mathbb{R}\}$ for some $\boldsymbol{v}, \boldsymbol{u} \in \mathbb{R}^4$. This way it is clear that the solution set is a line in $\mathbb{R}^4$.

The pivots are 4, -3, and 7 with corresponding pivot (dependent) variables x_1, x_2, and x_3, the only independent variable being x_4.

So let $t \in \mathbb{R}$ be arbitrary and set $x_4 = t$ to give the diagonal system

$$\begin{aligned} 4x_1 + x_2 - x_3 &= 5 - 3t \\ -3x_2 &= 9 - 3t \\ 7x_3 &= 8 - 4t \end{aligned}$$

Back substitution gives

$$\begin{aligned} x_2 &= t - 3 \\ x_3 &= \frac{1}{7}(8 - 4t) \end{aligned}$$

and

$$\begin{aligned} 4x_1 + (t-3) - \frac{1}{7}(8 - 4t) &= 5 - 3t \\ 28x_1 = 35 - 21t - 7t + 21 - 4t + 8 &= 64 - 32t \\ x_1 &= \frac{16}{7} - \frac{8}{7}t \end{aligned}$$

This gives

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 16/7 \\ -3 \\ 8/7 \\ 0 \end{bmatrix} + t \begin{bmatrix} -8/7 \\ 1 \\ -4/7 \\ 1 \end{bmatrix}$$

This shows that the solution set is a straight line in $\mathbb{R}^4$.