Math 571 - Exam 1 (05.22)

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NOTATION/DEFINITION: Let (X, d) be a metric space for $A, B \subset X$ define $d(A, B) = \sup\{d(a, b) \mid a \in A \text{ and } b \in B\}$. Set $d(a, B) = d(\{a\}, B)$.

Problem 1. Let (X, d) be a metric space, prove that

a) For any closed set F and $x \notin F$, $d(x, F) > 0$.

b) For any compact K and closed F with $K \cap F = \emptyset$, $d(K, F) > 0$.

Can the assumption that K is compact be dropped n (b)? That is, is there a metric space (X, d) and closed sets A, B so that $A \cap B = \emptyset$ and yet $d(A, B) = 0$?

RECALL: In a metric space (X, d), $\operatorname{diam}(A) = \sup\{d(a, b) \mid a, b \in A\}$.

Problem 2. Let (X, d) be a metric space prove or disprove each of the following:

a) $\operatorname{diam}(A) = \operatorname{diam}(\mathrm{Cl}(A))$.

b) $\operatorname{diam}(A) = \operatorname{diam}(\mathrm{Int}(A))$.

Problem 3. Let (X, d) be a metric space and $(x_i)_{i\in\mathbb{N}}$ and $(y_i)_{i\in\mathbb{N}}$ be two Cauchy sequences. Show that $(d(x_i, y_i))_{i\in\mathbb{N}}$ converges.

For the next problem, $(x_{i_k})_{k=0}^{\infty}$ is a **subsequence** of $(x_i)_{i=0}^{\infty}$ means $i_0 < i_1 < \cdots$. A sequence $(x_i)_{i=0}^{\infty}$ is **monotone increasing** iff $x_0 \leq x_1 \leq x_2 \cdots$. Similarly define **monotone decreasing**. A sequence is **monotone** iff it is either monotone increasing or monotone decreasing.

Problem 4. Show that every infinite sequence of real numbers has a monotone subsequence that converges to $\limsup_i x_i$.

NOTE: The same is true for $\liminf_i x_i$.