Math 571 - Homework 7

Richard Ketchersid

Problem 7.1 (R:5:26)**.** Suppose $f(x)$ is differentiable on $[a, b]$, $f(a) = 0$, and there is a fixed A such that $|f'(x)| \leq A|f(x)|$ for all x in $[a, b]$. Show that $f(x) = 0$ on $[a, b]$.

Proof 1: Let $Z = \{x \in [a, b] \mid \forall y \leq x(f(y) = 0)\}$. If $Z = [a, b]$, then we are done. Else let $c \in [a, b] - Z$, then there is $y \leq c$ with $f(y) \neq 0$. Say $f(y) > 0$, then $f(y) > \delta > 0$ for some δ and we get an open nbhd of y, say, $N_r(y)$ so that $f(N_r(y)) \subseteq (\delta, \infty)$. This shows $c \in (y, b]$ and $(y, b] \cap Z = \emptyset$. So c is in an open nbhd of $[a, b]$ disjoint from Z. We have shown that for $c \notin Z$, there is an open nbhd of c disjoint from Z, thus Z is closed.

Let $z^* = \sup(Z) < b$ and choose $c > z^*$ so that $c - z^* < 1/A$. Say $|f(c)| = M$ look at $C = \{c \mid |f(c)| \geq M\}$, this set is closed and clearly $c \geq c^* = \inf(C) > z^*$ and $|f(c^*)| = M$. Assume $f(c^*) = M$, or alternatively, $f(c^*) = -M$, and the same argument works with obvious modifications.

By MVT there is $d \in (z^*, c^*)$ so that

$$f(c^*) - f(z^*) = f(c^*) - 0 = M = f'(d)(c^* - z^*) \leq A(c^* - z^*)f(d) < f(d)$$

But this means we have $d \in (z^*, c^*)$ with $f(d) > M$ contradicting the choice of c^*. What leads to this contradiction? It was the assumption that $Z \neq [a, b]$, we conclude that $Z = [a, b]$.

Proof 2: (As hint in text using a student solution.) Fix N so that $A(b-1)/N < 1$ and let $x_i = a + i(b - a)/N$. So $x_0 = a$ and $x_n = b$. Suppose f is 0 on $[x_0, x_i]$ we will see that f must then be 0 on $[x_0, x_{i+1}]$.

Let $M_0 = \sup(f([x_i, x_{i+1}])$ and $M_1 = \sup(f'([x_i, x_{i+1}])$. Notice that $M_1 \leq AM_0$ by assumption. We know for $x \in [x_i, x_{i+1}]$ that $f(x) - f(x_i) = f(x) = f'(c)(x_{i+1} - x_i)$ by MVT. So $f(x) \leq M_1(x_{i+1} - x_i) \leq AM_0(x_{i+1} - x_i)$, but this means $M_0 \leq M_0A(x_{i+1} - x_i)$ and this is non-sense as $M_0A(x_{i+1} - x_i) < M_0$ unless $M_0 = 0$. So $M_0 = 0$ and thus f is 0 on $[x_i, x_{i+1}]$.

Problem 7.2 (R:5:27)**.** Let $\phi : [a, b] \times [\alpha, \beta] \to \mathbb{R}$. A *solution to the initial–value problem* (IVP)

$$y' = \phi(x, y), \quad y(a) = c \text{ for } a \leq c \leq b$$

is a function $f : [a, b] \to [\alpha, \beta]$ satisfying

$$f(a) = c, \quad f'(x) = \phi(x, f(x)) \text{ for all } a \leq x \leq b$$

Show that if there is a constant $A \geq 0$ so that

$$|\phi(x, y_1) - \phi(x, y_2)| \leq A|y_1 - y_2| \text{ for all } x \in [a, b] \text{ and } y_1, y_2 \in [\alpha, \beta],$$

then there is at most one solution to any such IVP.

Suppose f_1 and f_2 are two such solutions, then note that by assumption

$$|f_1'(x) - f_2'(x)| = |\phi(x, f_1(x)) - \phi(x, f_2(x))| \leq A|f_1(x) - f_2(x)| \text{ for } x \in [a, b].$$

Letting $h(x) = f_1(x) - f_2(x)$ we have $h(a) = 0$, h is differentiable on $[a, b]$, and $|h'(x)| \leq A|h(x)|$ for $x \in [a, b]$. Thus by Problem 1, $h = 0$ on $[a, b]$ and thus $f_1 = f_2$.

The book points out an example $y(0) = 0$ and $y' = y^{1/2}$ on $[0, 1]$. Note that this fails the hypotheses since there is no $A \geq 0$ with $|\sqrt{y}| < A|y|$ on $[0, 1]$, in particular, $\lim_{y \to 0^+} \frac{\sqrt{y}}{y} = \infty$. The book gives two solutions $y = 0$ and $y = x^2/4$. To find all solutions note

$$\begin{aligned}
\frac{y'}{y^{1/2}} &= 1 \\
y^{-1/2}\frac{dy}{dx} &= 1 \\
y^{-1/2}dy &= dx \\
\int y^{-1/2}\,dy &= \int dx \\
\frac{y^{1/2}}{1/2} + d &= x + c && (d \text{ and } c \text{ arbitrary constants}) \\
y^{1/2} &= \frac{x}{2} + C && (C \text{ an arbitrary constant}) \\
y &= \frac{x^2}{4} + 2Cx + C^2
\end{aligned}$$

If $y(0) = 0$, then $C^2 = 0$, so $C = 0$, and thus the two solutions are all.

Problem 7.3. Show that the following are equivalent for a bounded function f on $[a, b]$:

i) $f \in \mathcal{R}$, i.e., f is Riemann integrable,

ii) For all $\epsilon > 0$, there is a $\delta > 0$ such that

$$\|P\| < \delta \implies U(f, P) - L(f, P) < \epsilon$$

First, show (i) implies (ii). Let $f \in \mathcal{R}$ and $\epsilon > 0$. We have a partition P so that $U(f, P) - L(f, P) < \epsilon/2$. Take $\delta > 0$ so that $\Delta x_i > 2\delta$ for all i and so that $\delta < \frac{\epsilon}{12MN}$ where $M = \sup\{|f(x)| \mid x \in [a, b]\}$ and $N = |P|$.

Let P' be a partition with $\|P'\| < \delta$ and let $P'' = P \cup P'$, then $L(P') \leq L(P'') \leq U(P'') \leq U(P')$ and $L(P) \leq L(P'') \leq U(P'') \leq U(P)$. So $U(P'') - L(P'') \leq U(P) - L(P) < \epsilon/2$. We want to show that $U(P') - U(P'') < \epsilon/4$ and $L(P'') - L(P') < \epsilon/4$, then

$$\begin{aligned}
U(P') - L(P') &= \big(U(P'') + (U(P') - U(P''))\big) - \big(L(P'') - (L(P'') - L(P'))\big) \\
&< (U(P'') + \epsilon/4) - (L(P'') - \epsilon/4) = (U(P'') - L(P'')) + \epsilon/2 \\
&< \epsilon/2 + \epsilon/2 = \epsilon
\end{aligned}$$

All that needs to be proved here is

$$U(P') - U(P'') < \epsilon/4, \qquad L(P'') - L(P') < \epsilon/4$$

Let $P' = a = y_0 < y_1 < \cdots < y_m = b$ and $P = a = x_0 < x_1 < \cdots < x_N = b$. For each $i = 1, 2, \ldots, N-1$ there is y_{k_i} so that $x_i \in [y_{k_i-1}, y_{k_i}]$. If $x_i \in \{y_{k_i-1}, y_{k_i}\}$, then adding x_i to P' adds nothing new, so in the worst case $x_i \in (y_{k_i-1}, y_{k_i})$. Let us assume this always occurs (since this is the worst case). In this case, we have

$$\begin{aligned} U(P') - U(P'') = \sum_{i=1}^{N-1} &\sup(f([y_{k_i-1}, y_{k_i}])(y_{k_i} - y_{k_i-1}) \\ &- \big(\sup(f([y_{k_i-1}, x_i])(x_i - y_{k_i-1}) + \sup(f([x_i, y_{k_i}])(y_{k_i} - x_i)\big) \\ \le \sum_{i=1}^{N-1} &3M\|P'\| = 3(N-1)M\|P\| < \epsilon/4 \end{aligned}$$

The other direction (ii) implies (i) is trivial since all that is required for $f \in \mathcal{R}$ is that for all $\epsilon > 0$, there is P so that $U(f,P) - L(f,P) < \epsilon$.

Problem 7.4 (R:6:1)**.** Suppose $\alpha : [a,b] \to \mathbb{R}$ is monotonic increasing and continuous at $x_0 \in [a,b]$. consider $f : [a,b] \to \{0,1\}$ given by $f(x_0) = 1$ and $f(x) = 0$ for $x \neq x_0$. Show that $f \in \mathcal{R}(\alpha)$ and $\int_a^b f\,d\alpha = 0$.

Pick $\epsilon > 0$. Since α is continuous at x_0 take δ so that $\alpha(N_\delta(x_0)) \subseteq N_{\epsilon/2}(\alpha(x_0))$. Let $P = y_0 = a < y_1 < y_2 < y_3 = b$ where $[y_1, y_2] \subset (x_0 - \delta, x_0 + \delta)$, so that $\Delta\alpha_2 = \alpha(y_2) - \alpha(y_1) < \epsilon$. Then $M_i^{f,P} = m_i^{f,P}$ for $i \neq 2$ and $M_2^{f,P} = \sup(f([y_1, y_2])) = 1$ while $m_i^{f,P} = \inf(f([y_1, y_2])) = 0$ so that

$$U(f,P) - L(F,P) = (1-0)\Delta\alpha_2 < \epsilon$$

Problem 7.5 (R:6:2)**.** Suppose $f : [a,b] \to \mathbb{R}$ is continuous, $f \geq 0$, and $\int_a^b f\,dx = 0$, then $f = 0$.

Note that where Rudin asks you to compare with (1), there might be the thought that these do not compare since (1) is about $\mathcal{R}(\alpha)$ while (2) is about $\mathcal{R}$, but taking $\alpha = \mathrm{id}$ in (1) allows you to make the comparison.

This is really almost trivial. If $f \neq 0$, then $f(x) > 0$ for some $x \in [a,b]$, but then $f(x) > \delta > 0$ and so there is an open nbhd of x, $N_\delta(x) = (x-\delta, x+\delta)$ so that $f((x-\delta, x+\delta) \cap [a,b]) \subset (\delta, \infty)$. Say $(c,d) \subseteq (x-\delta, x+\delta) \cap [a,b]$, then clearly $\int_a^b f\,dx \geq \delta(d-c) > 0$.

The difference in the example from R:6:1 and R:6:2 is clearly that in R:6:1, the function is not continuous. In fact $\int_a^b f\,dx = 0$ whenever $\{x \in [a,b] \mid f(x) \neq 0\}$ has ***measure 0***. A set Z has measure 0 whenever

$$0 = \inf\left\{ \sum_{i=1}^{\infty} (b_i - a_i) \;\middle|\; Z \subseteq \bigcup_{i=1}^{\infty} (a_i, b_i) \right\}$$

Problem 7.6 (R:6:3)**.** Define $\beta_i : [-1,1] \to [0,1]$ by $\beta_i = 0$ for $x < 0$ and $\beta_i = 1$ for $x > 0$, then $\beta_1(0) = 0$, $\beta_2(0) = 1$, and $\beta_3(0) = 1/2$. In particular β_i has a simple discontinuity at 0 with $\beta_1(0-) = \beta_1(0) = 0$ (continuous from the left), $\beta_2(0+) = \beta_2(0) = 1$ (continuous from the right), while β_3 is neither continuous from the left or right. Let $f : [-1,1] \to \mathbb{R}$ be bounded. show that

i) $f \in \mathcal{R}(\beta_1)$ iff $f(0+) = f(0)$, that is, f is continuous from the right at 0.

ii) $f \in \mathcal{R}(\beta_2)$ iff $f(0-) = f(0)$, that is, f is continuous from the left at 0.

iii) $f \in \mathcal{R}(\beta_3)$ iff f is continuous at 0.

These are all very similar. It suffices to consider partitions that include 0 so that $P = -1 = x_0 < x_1 < \dots x_n = 1$ an where $x_k = 0$. For β_i we have

$$(\Delta\beta_i)_k = \beta_i(0) - \beta_i(x_{k-1}) = \begin{cases} 0 & i = 1 \\ 1 & i = 2 \\ 1/2 & i = 3 \end{cases}$$

and

$$(\Delta\beta_i)_{k+1} = \beta_i(k+1) - \beta_i(0) = \begin{cases} 1 & i = 1 \\ 0 & i = 2 \\ 1/2 & i = 3 \end{cases}$$

All other $(\Delta\beta_i)_j = 0$ and thus we see

$$\begin{aligned} U(f,P) - L(f,P) &= (M_k - m_k)(\Delta\beta_i)_k + (M_{k+1} - m_{k+1})(\Delta\beta_i)_{k+1} \\ &= \begin{cases} M_{k+1} - m_{k+1} & i = 1 \\ M_k - m_k & i = 2 \\ 1/2\big((M_k - m_k) + (M_{k+1} - m_{k+1}\big) & i = 3 \end{cases} \end{aligned}$$

Now $f \in \mathcal{R}(\beta_i)$ iff for all $\epsilon > 0$ there is a P so that

$$\begin{aligned} &U(f,P) - L(f,P) < \epsilon \iff \\ &\qquad \begin{cases} M_{k+1} - m_{k+1} < \epsilon & i = 1 \\ M_k - m_k < \epsilon & i = 2 \\ 1/2\big((M_k - m_k) + (M_{k+1} - m_{k+1}\big) < \epsilon & i = 3 \end{cases} \end{aligned}$$

Take the $i = 1$ case, this says that for all $\epsilon > 0$ there is $x_{k+1} = h > 0$ so that $\sup(f([0,h]) - \inf(f([0,h])) < \epsilon$ which says exactly that $f(0+) = f(0)$. Similarly for $i = 2$ and $i = 3$.

Problem 7.7 (R:6:6)**.** Let $f : [0,1] \to \mathbb{R}$ be bounded and continuous off of the Cantor set $\mathcal{C}$. Show that $f \in \mathcal{R}$.

Recall the construction of the Cantor set. $C_0 = [0,1]$ $C_1 = [0,1] - (1/3, 2/3)$ (removing middle third). $C_2 = C_1 - (1,9) - (7/9, 8/9)$, again remove middle thirds from what was left.

Notice the lengths of what is removed: $1/3$, $1/3 + 2/9$, $1/3 + 2/9 + 4/27$, etc. Consider

$$\sum_{i=0}^{\infty} \frac{2^i}{3^{i+1}} = \frac{1}{3}\sum_{i=0}^{\infty}\left(\frac{2}{3}\right)^i = \frac{1}{3}\left(\frac{1}{1 - 2/3}\right) = 1$$

We can cover C_i by 2^i many disjoint intervals of length $(1/3)^i + \epsilon$ for any ϵ. Since $\mathcal{C} = \bigcap C_i$ we see that $\mathcal{C}$ has measure 0 as defined above.

Suppose f is continuous off of a measure 0 set $Z \subset [a,b]$. Let $\mathcal{O}$ be an open cover of Z by intervals (a_i, b_i) so that $\sum_i (b_i - a_i) < \epsilon$. For each $x \notin Z$ take δ_x so that $f(N_{\delta_x}(x)) \subset$

$N_{\epsilon/2}(f(x))$. As $[a,b]$ is compact we can find a finite subcover $\{(u_i,v_i) \mid i=1,\ldots,n\}$ so that $u_1 < a < u_2 < v_1 < u_3 < v_2 < \cdots u_n < v_{n-1} < b < v_n$ where each (u_i,v_i) is from our cover of Z or else is one of the $N_{\delta_x}(x)$.

Use $x_0 = a$, $x_i = (u_{i+1}+v_i)/2$ for $i < n$, and $x_n = b$ as the partition: $P = a = x_0 < x_1 < \cdots < b = x_n$. Let M be a bound on $|f|$ on $[a,b]$. Let $T = \{i \mid (x_{i-1},x_i) \subseteq (a_j,b_j) \text{ for some } j\}$. Then we have

$$\begin{aligned}
U(f,P) - L(f,P) &= \sum_{i\in T}(M_i - m_i)\Delta x_i + \sum_{i\notin T}(M_i - m_i)\Delta x_i \\
&< \sum_{i\in T} 2M\Delta x_i + \sum_{i\notin T}\epsilon\Delta x_i \\
&\le 2M\epsilon + \epsilon(b-a) = \epsilon(2M + (b-a))
\end{aligned}$$

Problem 7.8 (R:6:10)**.** See text. This is mostly done in the notes.

In particular whenever $p_i \ge 0$ and $\sum_{i=1}^n p_1 = 1$, then $\prod_{i=1}^n a_i^{p_i} \le \sum_{i=1}^n p_i a_i$. In particular, if $\frac{1}{p} + \frac{1}{q} = 1$, then

$$uv = (u^p)^{1/p}(v^q)^{1/q} \le \frac{u^p}{p} + \frac{v^q}{q}$$

This basically completes (a). For (b) notice

$$fg \le \frac{f^p}{p} + \frac{g^q}{q}$$

so

$$\int_a^b fg\,d\alpha \le \int_a^b \frac{f^p}{p}\,d\alpha + \int_a^b \frac{g^q}{q}\,d\alpha = \frac{1}{p} + \frac{1}{q} = 1$$

For (c) the proof is exactly as for Hölder's inequality in the notes already mentioned above. Define $\|f\|_p = \left(\int_a^b |f|^p\,d\alpha\right)^{1/p}$ provided that $|f|^p \in \mathcal{R}(\alpha)$. Let $L^p(\alpha)$ be all those bounded $f[a,b] \to \mathbb{R}$ with $\|f\|_p < \infty$. The spaces of function $L^p(\alpha)$ are normed vector spaces with norm $\|\cdot\|_p$. We want to see if $f \in L^p(\alpha)$ and $g \in L^q(\alpha)$ with $\frac{1}{p} + \frac{1}{q} = 1$ we have $fg \in L^1(\alpha)$ and

$$\|fg\|_1 \le \|f\|_p\|g\|_q \tag{†}$$

We can replace f with $\hat{f} = \frac{f}{\|f\|_p}$ and g with $\hat{g} = \frac{g}{\|g\|_q}$, then we have $\|\hat{f}\|_p = 1 = \|\hat{g}\|_q$ and from above

$$\|\hat{f}\hat{g}\|_1 \le 1 = \frac{\|\hat{f}\|_p^p}{p} + \frac{\|\hat{g}\|_q^q}{q}$$

But from this we have

$$\left\|\frac{f}{\|f\|_p}\frac{g}{\|g\|_q}\right\| = \frac{1}{\|f\|_p}\frac{1}{\|g\|_q}\|fg\|_1 \le 1$$

From this (†) follows immediately.

Problem 7.9 (Functions with only countable many discontinuities are integrable.)**.** Let f be bounded on $[a,b]$ with at most countable many discontinuities on $[a,b]$. Let $\alpha : [a,b] \to \mathbb{R}$ is monotonic increasing and α is continuous at every discontinuity of f. Show that $f \in \mathcal{R}(\alpha)$.

Hint: Fix an enumeration $S = \{s_i \mid i \in \mathbb{N}\}$ of the discontinuities of f. Fix $\epsilon > 0$ and $\epsilon_i > 0$ so that $\sum_i \epsilon_i \le \epsilon$. Since α is continuous at s_i fix δ_i so that $\alpha(N_{\delta_i}(s_i)) \subset N_{\epsilon_i}(\alpha(s_i))$. For $x \notin S$, fix δ_x so that $f(N_{\delta_x}(x)) \subset N_\epsilon(f(x))$. Now $\mathcal{O} = \{N_{\delta_i}(s_i) \mid i \in \mathbb{N}\} \cup \{N_{\delta_x}(x) \mid x \notin S\}$ is an open cover of $[a, b]$. Apply compactness to get a finite subcover and then do something ***similar*** (not the same) as in the proof of 6.10.

Proof 1: Fix $\epsilon > 0$ and $\epsilon_i > 0$ so that $\sum_i \epsilon_i < \epsilon$. Let $S = \{s_i \mid i \in \mathbb{N}\}$ be the discontinuities of f. Since α is continuous at s_i let δ_i be so that $\alpha(N_{\delta_i}(s_i)) \subset N_{\epsilon_i}(\alpha(s_i))$. For $x \notin S$ let δ_x be chosen so that $f(N_{\delta_x})(x) \subseteq N_\epsilon(f(x))$. Let $\mathcal{O} = \{N_{\delta_i}(s_i) \mid i \in \mathbb{N}\} \cup \{N_{\delta_x}(x) \mid x \notin S\}$ be the associated open cover of $[a, b]$. Let $\mathcal{O}' \subseteq$ be a finite subcover of $[a, b]$. Novice that $\mathcal{O}'$ consists of intervals (u_i, v_i) and we may assume that $a = u_0 < u_1 < v_0 < u_2 < v_1 < u_3 < v_2 \cdots$ (a "chain"). Thus we define $x_0 = a < x_1 = (u_1 + v_0)/2 < x_2 = (u_2 + v_1)/2 < x_{n-1} = (u_{n-1} + v_{n-2})/2 < x_n = v_n = b$. Thus $[x_{i-1}, x_i] \subset N_{\delta_j}(s_j)$ for some j or $[x_{i-1}, x_i] \subset N_{\delta_x}(x)$ for some $x \notin S$.

Let $T = \{i \mid [x_{i-1}, x_i] \subset N_{\delta_i}(s_i) \text{ for some } s_i \in S\}$. Then letting $f|(x)| \le M$ and $\alpha(b) - \alpha(a) = N$:

$$\begin{aligned}
\sum_{i=1}^{n} |M_i - m_i|(\alpha(x_i) - \alpha(x_{i-1}) &= \sum_{i \in T} |M_i - m_i|(\alpha(x_i) - \alpha(x_{i-1}) + \sum_{i \notin T} |M_i - m_i|(\alpha(x_i) - \alpha(x_{i-1}) \\
&< \sum_{i \in T} 2M\epsilon_i + \sum_{i \notin T} \epsilon\alpha(x_i) - \alpha(x_{i-1}) \\
&\le 2M\epsilon + N\epsilon = \epsilon(2M + N)
\end{aligned}$$

Proof 2: (The following seems to be an option that I see commonly, but not carried out correctly. I thought I would write it out correctly here.)

Start like the above. Since α is continuous at s_i pick (a_i, b_i) satisfying:

- $(a_i, b_i) \cap (a_j, b_j) = \emptyset$. (mutually disjoint)
- $s_i \in (a_i, b_i)$.
- $\alpha((a_i, b_i)) \subseteq S_{\epsilon/2}(s_i)$ so that if $t, t' \in (a_i, b_i)$, then $|\alpha(t') - \alpha(t)| < \epsilon_i$. Where $\sum_i \epsilon_i = \epsilon$ and ϵ_i will be chosen at the end.

Let $K = [a, b] - \bigcup_i (b_i, a_i)$. K is closed and bounded, hence compact. Since f is continuous on K it is uniformly continuous and thus we can pick $\delta > 0$ so that $|x - y| < \delta \implies |f(x) - f(y)| < \epsilon$ for all $x, y \in K$.

$\mathcal{O} = \{(x - \delta/2, x + \delta/2) \mid x \in K\} \cup \{(a_i, b_i) \mid i \in \mathbb{N}\}$ is an open cover of $[a, b]$ and hence has a finite subcover $\mathcal{O}'$. Let $\mathcal{O}' = \{(u_i, v_i) \mid i < m\}$ we may assume that for no $i \ne j$ do we have $(u_i, v_i) \subset (u_j, v_j)$, as we could just toss out (u_i, v_i) in this case. So $u_0 < a < u_1 < v_0 < u_2 < v_1 < \cdots < u_{m-1} < v_{m-2} < b < v_{m-1}$. For $i = 1, \ldots, m-2$ let $y_i = (u_i + v_{i-1})/2$ and set $y_0 = a$ and $y_{m-1} = b$ and let $P = \{y_i \mid i = 0, \ldots, m-1\}$. Then we know for each $i = 1, \ldots, m-1$ that either $[u_{i-1}, u_i] \subset (a_j, b_j)$ for some j or else $[u_{i-1}, u_i] \subset (x - \delta/2, x + \delta/2)$ for some $x \in K$.

Let $A = \{i \mid [u_{i-1}, u_i] \subset (a_{j_i}, b_{j_i}) \text{ for some } j_i\}$, then for $i \in A$ we have $\Delta\alpha_i = \alpha(u_i) - \alpha(u_{i-1}) <$

ϵ_{j_i} and for $i \notin A$, $|M_i^{f,P} - m_i^{f,P}| < \epsilon$. Thus

$$\begin{aligned} U(f,P,\alpha) - L(f,P,\alpha) &= \sum_{i=1}^{m-1} |M_i^{f,P} - m_i^{f,P}|\Delta\alpha_i \\ &\leq \sum_{i\in A} |M_i^{f,P} - m_i^{f,P}|\epsilon_{j_i} + \sum_{i\notin A} \epsilon\Delta\alpha_i \leq 2M\epsilon + \epsilon(\alpha(b) - \alpha(a)) \end{aligned}$$

where $M = \sup |f(x)| x \in [a,b]$.

Since M and $\alpha(b) - \alpha(a)$ are fixed constants we can make the $\epsilon(2M + \alpha(b) - \alpha(a))$ arbitrarily small. Thus $f \in \mathcal{R}(\alpha)$.

Problem 7.10 (An integrable function with uncountable many discontinuities.)**.** Let $\mathcal{C}$ be the Cantor set and f be defined by

$$f(x) = \begin{cases} 1 & x \in \mathcal{C} \\ 0 & x \notin \mathcal{C} \end{cases}$$

Show that $f \in \mathcal{R}$, namely, $\int_0^1 f\,dx = 0$. That f has uncountably many points of discontinuity is clear since each point of $\mathcal{C}$ is a discontinuity of f and $\mathcal{C}$ is perfect, hence uncountable.

Proof 1: The argument from Problem 7 works here. Basically, that argument showed that if $g = f$ off of a measure zero set, then $f \in \mathcal{R} \iff g \in \mathcal{R}$ and $\int_a^b f\,dx = \int_a^b g\,dx$. So here take $g = 0$ on $[0,1]$.

Proof 2: (From a student.) Let P_i be the partition that breaks $[0,1]$ into 3^i many pieces. Let C_i be the i^{th} approximation of the Cantor set, so $C_0 = [0,1]$, $C_1 = [0,1/3] \cup [2/3,1]$, etc. In the picture, the black part represents the closed sets C_i.

Notice that on each of these sections $M_j - m_j = 1$ since there are always points outside $\mathcal{C}$ in any interval, yet the endpoints are always in $\mathcal{C}$. The same argument holds for the orange segments. In P_i, 2^i pieces of the partition are black. The orange segments satisfy a recurrence, namely, if there are a_i in P_i, then there are $2 \cdot a_i + 2^i$ in P_{i+1} since each black gives one orange and each orange splits into two orange. So $a_{i+1} = 2a_i + 2^i$. You can check that $a_i = i2^{i-1}$. In particular, $(0)(2^{-1}) = 0$ and $(i+1)2^i = i2^i + 2^i = 2(i2^{i-1}) + 2^i = 2a_i + 2^i$ as required. So there are $i2^{i-1} + 2^i = (i+2)2^{i-1}$ many elements of the partition where $M_i - m_i = 1$ so that

$$U(f,P) - L(f,P) = \frac{(i+2)}{3}\left(\frac{2}{3}\right)^{i-1} \to 0 \text{ as } i \to \infty$$

The following is for a future class, but it came up here so I wanted to record it. Let $\alpha : [a,b] \to \mathbb{R}$ and say $Z \subseteq [a,b]$ has α-measure 0 iff for all $\epsilon > 0$ there is (a_i, b_i) so that $Z \subseteq \bigcup_{i=0}^{\infty}(a_i, b_i)$

and $\sum_{i=0}^{\infty} \alpha(b_i) - \alpha(a_i) < \epsilon$. The argument above works for $\mathcal{R}(\alpha)$ with α-measure zero replacing measure 0.

Notice that if Z has α-measure zero and $z \in Z$, then α is continuous at z. To see this let $\epsilon > 0$ and take $\{(a_i, b_i) \mid i \in \mathbb{N}\}$ covering Z with $\sum_i \alpha(b_i) - \alpha(a_i) < \epsilon$. Then $z \in (a_i, b_i)$ and clearly $\alpha((a_i, b_i)) \subset N_\epsilon(\alpha(z))$, since $\alpha(b_i) - \alpha(a_i) < \epsilon$. So if Z is the set of discontinuities of f, then α must be continuous at each $z \in Z$.

Problem 7.11. Show that if $f[a, b] \to \mathbb{R}$ is bounded and $Z = \{x \mid f \text{ is discontinuous at } x\}$ is countable α is continuous at each in Z, then Z has α-measure zero.

Problem 7.12 (Generalization of Problem 7)**.** Show that if $f : [a, b] \to \mathbb{R}$ is bounded and $\alpha : [a, b] \to \mathbb{R}$ is monotonic increasing with f discontinuous on a set Z of α-measure zero with α continuous at each point in Z, then $f \in \mathcal{R}(\alpha)$.

Problem 7.13. Let $f, g : [a, b] \to \mathbb{R}$ and let $Z = \{z \mid g(z) \neq f(z)\}$. If Z has α-measure zero show that

i) $f \in \mathcal{R}(\alpha) \iff g \in \mathcal{R}(\alpha)$

ii) If $f \in \mathcal{R}(\alpha)$, then $\int_a^b f \, d\alpha = \int_a^b g \, d\alpha$