Exam 1

- This exam covers Topics 1 - 3, Topic 4 will not be covered here.

- I will write $(a_1, a_2, \ldots, a_n)$ in place of $\begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix}$ to save space on occasion. The book writes $\left[\begin{smallmatrix} a_1 & a_2 & \cdots & a_n \end{smallmatrix}\right]^T$ for the same purpose.

- I use $\mathrm{NS}(A)$ for the null space of A, $\mathrm{RS}(A)$ for the row space of A, $\mathrm{CS}(A)$ for the column space of A. Note that $\mathrm{CS}(A) = \mathrm{rng}(A)$ is the range of A.

- When I say you can use some fact from another part of the exam, this means that you can use the fact whether or not you have completed that part of the exam correctly.

- If you have questions ask via Remind or email.

- Use only arguments that you fully understand. I am aware that you can find solutions to some of these online. I am also aware that some of these solutions use concepts and theorems far past what we have covered in Ch 1 - 3. If you use such ideas, then I will ask you to verbally explain your solution so that I can verify that you understand what you have submitted as your own work. In short, **this is an exam** and the usual expectations of academic honesty apply.

Part I: True/False (5 points each; 25 points)

For each of the following mark as (T) true or (F) false.

For the exam you do not need to provide justification.

a) _____ $A\boldsymbol{x} = \mathbf{0} \iff \text{rref}(A)\boldsymbol{x} = \mathbf{0}$

TRUE: This follows since rref(A) can be obtained from A by a sequence of elementary row operations, and elementary row operations do not change the null space of a matrices.

b) _____ $\text{tr}(AB) = \text{tr}(BA)$ for an $n \times n$ matrices A and B, where

$$\text{tr}(C) = \sum_{i=1}^{n} C_{ii} = \text{the sum of the diagonal elements of } C.$$

TRUE. This is just a computation. $(AB)_{ii} = \sum_{k=1}^{n} A_{ik}B_{ki}$, so

$$\text{tr}(AB) = \sum_{i=1}^{n}\sum_{k=1}^{n} A_{ik}B_{ki}$$

and

$$\text{tr}(BA) = \sum_{i=1}^{n}\sum_{k=1}^{n} B_{ik}A_{ki} = \sum_{k=1}^{n} A_{ki}B_{ik} = \text{tr}(AB).$$

c) _____ $\text{tr}(AB) = \text{tr}(A)\,\text{tr}(B)$ for an $n \times n$ matrices A, B, and C.

FALSE: Two random 2×2 matrices with entries from $\{-1, 0, 1\}$ are likely to work. Try this using MATLAB: round(rand(2)). The first two matrices I got this way were:

$$A = \begin{bmatrix} 0 & 1 \\ 1 & -1 \end{bmatrix} \qquad B = \begin{bmatrix} 0 & -1 \\ 0 & 0 \end{bmatrix} \quad AB = \begin{bmatrix} 0 & 0 \\ 0 & -1 \end{bmatrix}$$

So, $\text{tr}(AB) = -1$ while $\text{tr}(A)\,\text{tr}(B) = (-1)(0) = 0$.

d) _____ If W is a subspace of a vector space V, then there is a subspace U so that $V = W \oplus U$.

This notation is a little hard to find in your text: $V = U + W$ means that for all $\boldsymbol{v} \in V$, there is $\boldsymbol{u} \in U$ and $\boldsymbol{w} \in W$ so that $\boldsymbol{v} = \boldsymbol{u} + \boldsymbol{w}$. $V = U \oplus V$ means $V = U + V$ **and** $U \cap W = \{\mathbf{0}\}$, equivalently, for every $\boldsymbol{v} \in V$, there is a **unique** $\boldsymbol{u} \in U$ and $\boldsymbol{w} \in W$ so that $\boldsymbol{v} = \boldsymbol{u} + \boldsymbol{w}$.

TRUE. Let $\mathcal{B}_W$ be a basis for W and extend $\mathcal{B}_W$ to $\mathcal{B}_V$ a basis for V. Then let $U = \text{span}(\mathcal{B}_V - \cap B_W)$. It is clear that $V = W \oplus U$.

e) _____ For any $m \times n$ matrices A and B,

$$B = EA \text{ for some invertible } E \iff \text{NS}(A) = \text{NS}(B).$$

TRUE. ($\Rightarrow$) is trivial, since if $B = EA$, then $B\boldsymbol{x} = \mathbf{0} \iff EA\boldsymbol{x} = \mathbf{0} \iff A\boldsymbol{x} = E^{-1}\mathbf{0} = 0$. ($\Leftarrow$) is discussed below in the "Proofs" section.

Part II: Definitions and short answer (5 points each; 25 points)

a) Define what it means for a set of vectors $\mathcal{B} = \{\boldsymbol{v}_1, \ldots, \boldsymbol{v}_n\}$ from a real vector space V to span V.

$\{\boldsymbol{v}_1, \ldots, \boldsymbol{v}_n\}$ **spans** V iff for all $\boldsymbol{v} \in V$, $\boldsymbol{v}$ is a linear combination of the vectors in $\mathcal{B}$, that is $\boldsymbol{v} = \sum_{i=1}^{n} \alpha_i \boldsymbol{v}_i$ for some coefficients $\alpha_i \in \mathbb{R}$.

b) Define what it means for a set of vectors $\{\boldsymbol{v}_1, \ldots, \boldsymbol{v}_n\}$ from a real vector space V to be linearly independent.

A set of vectors $\mathcal{B}$ is **linearly independent** iff $\sum_{i=1}^{n} \alpha_i \boldsymbol{v}_i = \mathbf{0}$, then $\alpha_i = 0$ for all i. Equivalently, any linear combination of the vectors that gives $\mathbf{0}$ must be trivial.

There are equivalent definitions, but essentially, this is what you need. You can say something like, for all i, $\boldsymbol{v}_i \notin \operatorname{span}\{\boldsymbol{v}_1, \ldots, \boldsymbol{v}_{i-1}, \boldsymbol{v}_{i+1}, \ldots, \boldsymbol{v}_n\}$

c) What does it mean to say $(c_1, \ldots, c_n) \in \mathbb{R}^n$ represents a vector $\boldsymbol{v} \in V$ with respect to the basis $\mathcal{B} = \{\boldsymbol{v}_1, \ldots, \boldsymbol{v}_n\}$?

This means that $\boldsymbol{v} = \sum_{i=1}^{n} c_i \boldsymbol{v}_i$.

d) In general, how do you show that $\dim(V) = n$.

If A is an $m \times n$ matrix, then $n = \dim(\mathrm{RS}(A)) + \dim(\mathrm{NS}(A)) = \mathrm{rank}(A) + \mathrm{nullity}(A)$.

e) What conditions must be checked to verify that $W \subseteq V$ is a subspace of a vector space. V

Closure under addition and scalar multiplication must be checked.

Part III: Computational (15 points each; 45 point)

a) Given that A is a 3×4 matrix and

$$\mathrm{NS}(A) = \operatorname{span}\left(\left\{\left[\begin{smallmatrix}1\\-2\\1\\0\end{smallmatrix}\right], \left[\begin{smallmatrix}2\\3\\0\\1\end{smallmatrix}\right]\right\}\right)$$

compute $\mathrm{rref}(A)$. You may use the fact that

$$\mathrm{rref}(A) = \mathrm{rref}(B) \iff \mathrm{NS}(A) = \mathrm{NS}(B).$$

We know $A\boldsymbol{x} = \mathbf{0} \iff \mathrm{rref}(A)\boldsymbol{x} = \mathbf{0} \iff \boldsymbol{x} \in \mathrm{NS}(A)$. From what we are given we see $\boldsymbol{x} \in \mathrm{NS}(A)$ iff

$$\begin{bmatrix}x_1\\x_2\\x_3\\x_4\end{bmatrix} = r\begin{bmatrix}1\\-2\\1\\0\end{bmatrix} + s\begin{bmatrix}2\\3\\0\\1\end{bmatrix} = \begin{bmatrix}r+2s\\-2r+3s\\r\\s\end{bmatrix}$$

Working backwards from what we usually do we see $x_4 = s, x_3 = r$, and so $x_2 = -2x_3 + 3x_4$ and $x_1 = x_3 + 2x_4$. This gives the system

$$\begin{aligned} x_1 - x_3 - 2x_4 &= 0 \\ x_2 + 2x_3 - 3x_4 &= 0 \end{aligned}$$

This corresponds to $B\boldsymbol{x} = \mathbf{0}$ for

$$B = \begin{bmatrix}1 & 0 & -1 & -2\\0 & 1 & 2 & -3\\0 & 0 & 0 & 0\end{bmatrix}$$

B is rref and $\mathrm{NS}(B) = \mathrm{NS}(A)$, so by the assumption $B = \mathrm{rref}(A)$.

b) For the same (unknown) A used in (a) for each of $\mathrm{RS}(A)$ and $\mathrm{CS}(A)$ find a basis if possible and explain how you know that you have found a basis; if it is not possible to find a basis, then explain why it is not.

$\mathrm{RS}(A)$: Here we know $\mathrm{RS}(A) = \mathrm{RS}(\mathrm{rref}(A))$ so the non-zero rows of $\mathrm{rref}(A)$ form a basis for $\mathrm{RS}(A)$.

$\mathrm{CS}(A)$: You know that the first two columns of $\mathrm{rref}(A)$ are where the pivots are and so the first two columns of A would be a basis for $\mathrm{CS}(A)$, but you have no way of finding these and you know nothing about $\mathrm{CS}(A)$ other than $\dim(\mathrm{CS}(A)) = 2$. For example, $\mathrm{rref}(A_1) = \mathrm{rref}(A_2) = B$ for

$$A_1 = \begin{bmatrix}1 & 1 & 1 & -5\\1 & 0 & -1 & -2\\0 & 1 & 2 & -3\end{bmatrix} \text{ and } A_2 = \begin{bmatrix}0 & 1 & -1 & -2\\1 & 0 & -1 & -2\\1 & 2 & 3 & -8\end{bmatrix},$$

but

$$\mathrm{CS}(A_1) = \mathrm{span}\left(\left\{\begin{bmatrix}1\\1\\0\end{bmatrix}, \begin{bmatrix}1\\0\\1\end{bmatrix}\right\}\right) \text{ and } \mathrm{CS}(A_2) = \mathrm{span}\left(\left\{\begin{bmatrix}0\\1\\1\end{bmatrix}, \begin{bmatrix}1\\0\\2\end{bmatrix}\right\}\right),$$

and $\begin{bmatrix}1\\0\\2\end{bmatrix} \notin \mathrm{CS}(A_1)$ so $\mathrm{CS}(A_1) \neq \mathrm{CS}(A_2)$.

c) Consider the transformation $D : P_4 \to P_3$ defined by $D(p) = \frac{d}{dx}p$, so specifically, $D(c_4x^4 + c_3x^3 + c_2x^2 + c_1x^1 + c_0) = 4c_4x^3 + 3c_3x^2 + 2c_2x + c_1$. Find the matrix $[L] = [L]_{\mathcal{B},\mathcal{C}}$ for D with respect to the standard basis $\mathcal{B} = \{x^4, x^3, x^2, x, 1\}$ for P_4 and the standard basis $\mathcal{C} = \{x^3, x^2, x, 1\}$ for P_3. Represent $p = 2x^4 - x^2 + 5x - 5$ with respect to $\mathcal{B}$ and then use $[L]$ to get the representation of $D(p)$ with respect to $\mathcal{C}$.

The matrix is given by

$$[L] = \big[[D(x^4)]_{\mathcal{C}} \quad [D(x^3)]_{\mathcal{C}} \quad [D(x^2)]_{\mathcal{C}} \quad [D(x)]_{\mathcal{C}} \quad [D(1)]_{\mathcal{C}}\big]$$

Since $D(x^4) = 4x^3$, $D(x^3) = 3x^2$, $D(x^2) = 2x$, $D(x) = 1$, and $D(1) = 0$ we have

$$[D(x^4)]_{\mathcal{C}} = \begin{bmatrix}4\\0\\0\\0\end{bmatrix}, [D(x^3)]_{\mathcal{C}} = \begin{bmatrix}0\\3\\0\\0\end{bmatrix}, [D(x^2)]_{\mathcal{C}} = \begin{bmatrix}0\\0\\2\\0\end{bmatrix}, [D(x)]_{\mathcal{C}} = \begin{bmatrix}0\\0\\0\\1\end{bmatrix}, [D(1)]_{\mathcal{C}} = \begin{bmatrix}0\\0\\0\\0\end{bmatrix}$$

So

$$[L] = \begin{bmatrix}4&0&0&0&0\\0&3&0&0&0\\0&0&2&0&0\\0&0&0&1&0\end{bmatrix}$$

Now

$$[2x^4 - x^2 + 5x - 5]_{\mathcal{B}} = \begin{bmatrix}2\\0\\-2\\5\\-5\end{bmatrix}$$

and

$$[D(2x^4 - x^2 + 5x - 5)]_{\mathcal{C}} = \begin{bmatrix}4&0&0&0&0\\0&3&0&0&0\\0&0&2&0&0\\0&0&0&1&0\end{bmatrix}\begin{bmatrix}2\\0\\-1\\5\\-5\end{bmatrix} = \begin{bmatrix}(4)(2)\\(3)(0)\\(2)(-1)\\(1)(5)\end{bmatrix} = \begin{bmatrix}8\\0\\-2\\5\end{bmatrix}$$

Of course, $D(2x^4 - x^2 + 5x - 5) = 8x^3 - 2x + 5$.

Part IV: Proofs (20 points each; 60 points)

Provide complete arguments/proofs for the following.

a) Prove that $\det(A) = \det(B)$ if A and B are similar.

A and B are similar iff there is an invertible S so that $B = S^{-1}AS$. So $\det(B) = \det(S^{-1})\det(A)\det(S) = \det(S)^{-1}\det(S)\det(A) = \det(A)$.

b) Let $\boldsymbol{V} = (0, \infty) \subset \mathbb{R}$ and define vector addition by $\boldsymbol{v} \oplus \boldsymbol{u} = \boldsymbol{u}\boldsymbol{v}$ and for $\alpha \in \mathbb{R}$ define scalar multiplication by $\alpha \odot \boldsymbol{v} = \boldsymbol{v}^{\alpha}$. Show that $(V, \oplus, \odot)$ is a vector space over $\mathbb{R}$. Make sure to clearly verify all the axioms for being a vector space.

Just to be clear, if $\boldsymbol{u} = 3$ and $\boldsymbol{v} = 2$, then $\boldsymbol{u} \oplus \boldsymbol{v} = (3)(2) = 6$ and $\sqrt{2} \odot \boldsymbol{u} = 3^{\sqrt{2}}$.

Additive axioms:

- Associativity: $\boldsymbol{u} \oplus (\boldsymbol{v} \oplus \boldsymbol{w}) = \boldsymbol{u}(\boldsymbol{v}\boldsymbol{w}) = (\boldsymbol{u}\boldsymbol{v})\boldsymbol{w} = (\boldsymbol{u} \oplus \boldsymbol{v}) \oplus \boldsymbol{w}$.
- Identity: $\boldsymbol{u} \oplus 1 = (\boldsymbol{u})(1) = \boldsymbol{u} = (1)(\boldsymbol{u}) = 1 \oplus \boldsymbol{u}$. So 1 is the $\oplus$-identity.
- $\boldsymbol{u} \oplus \frac{1}{\boldsymbol{u}} = \boldsymbol{u} \cdot \frac{1}{\boldsymbol{u}} = 1 = \frac{1}{\boldsymbol{u}} \cdot \boldsymbol{u}$. So $\frac{1}{\boldsymbol{u}}$ is the $\oplus$-inverse of $\boldsymbol{u}$.
- Commutativity: $\boldsymbol{u} \oplus \boldsymbol{v} = (\boldsymbol{u})(\boldsymbol{v}) = (\boldsymbol{v})(\boldsymbol{u}) = \boldsymbol{v} \oplus \boldsymbol{u}$.

Multiplicative axioms:

- $\alpha \odot (\beta \odot \boldsymbol{u}) = (\boldsymbol{u}^{\beta})^{\alpha} = \boldsymbol{u}^{\alpha \cdot \beta} = (\alpha \cdot \beta) \odot \boldsymbol{u}$.
- $1 \odot \boldsymbol{u} = \boldsymbol{u}^1 = \boldsymbol{u}$.

Distributive axioms:

- $\alpha \odot (\boldsymbol{u} \oplus \boldsymbol{v}) = ((\boldsymbol{u})(\boldsymbol{v}))^{\alpha} = (\boldsymbol{u}^{\alpha})(\boldsymbol{v}^{\alpha}) = (\alpha \odot \boldsymbol{u}) \oplus (\alpha \odot \boldsymbol{v})$
- $(\alpha + \beta) \odot \boldsymbol{v} = \boldsymbol{v}^{\alpha+\beta} = (\boldsymbol{v}\alpha)(\boldsymbol{v}^{\beta}) = (\alpha \odot \boldsymbol{v}) \oplus (\beta \odot \boldsymbol{v})$

c) Let A be a fixed $n \times n$ matrix. Show that

$$AB = BA \text{ for all } n \times n \text{ matrices } B \iff A = \alpha I \text{ for some } \alpha$$

($\Leftarrow$): This is trivial. If $A = \alpha I$, then clearly $AB = \alpha B = B\alpha = BA$ for all B.

($\Rightarrow$): Here notice that for $k \neq l$ if $B = [b_{i,j}]$ is a matrix with $b_{l,l} \neq b_{k,k}$ and $b_{i,j} = 0$ if $(i, j) \neq (k, k)$ and $(i, j) \neq (l, l)$, then $(AB)_{k,l} = a_{k,l}b_{l,l} \neq (BA)_{k,l} = b_{k,k}a_{k,l}$ unless $a_{k,l} = 0$. Thus $a_{k,l} = 0$ for all $k \neq l$. Thus A is diagonal.

Now let B be the matrix $b_{k,l} = 1$ and $b_{i,j} = 0$ for $(i, j) \neq (k, l)$. Then $(AB)_{k,l} = a_{k,k}b_{k,l} = a_{k,k} = (BA)_{k,l} = b_{k,l}a_{l,l} = a_{l.l}$. So $a_{k,k} = a_{l,l}$ for all k, l and thus clearly, $A = \alpha I$.