Math 571 - Exam 1

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Notation/Definition: Let (X, d) be a metric space for $A, B \subset X$ define $d(A, B) = \inf\{d(a, b) \mid a \in A \text{ and } b \in B\}$. Set $d(a, B) = d(\{a\}, B)$.

Question 1 (12 points)**.** Let (X, d) be a metric space, prove that

a) For any closed set F and $x \notin F$, $d(x, F) > 0$.

b) For any compact K and closed F with $K \cap F = \emptyset$, $d(K, F) > 0$.

c) Can the assumption that K is compact be replaced by K closed in (b)? That is, is there a metric space (X, d) and closed sets A, B so that $A \cap B = \emptyset$ and yet $d(A, B) = 0$?

Recall: In a metric space (X, d), $\operatorname{diam}(A) = \sup\{d(a, b) \mid a, b \in A\}$.

Question 2 (12 pts)**.** Let (X, d) be a metric space prove or disprove each of the following:

a) $\operatorname{diam}(A) = \operatorname{diam}(\operatorname{Cl}(A))$.

b) $\operatorname{diam}(A) = \operatorname{diam}(\operatorname{Int}(A))$.

Question 3 (12 pts)**.** Let (X, d) be a metric space and $(x_i)_{i\in\mathbb{N}}$ and $(y_i)_{i\in\mathbb{N}}$ be two Cauchy sequences. Show that $(d(x_i, y_i))_{i\in\mathbb{N}}$ converges.

For the next problem, $(x_{i_k})_{k=0}^{\infty}$ is a **subsequence** of $(x_i)_{i=0}^{\infty}$ means $i_0 < i_1 < \cdots$. A sequence $(x_i)_{i=0}^{\infty}$ is **monotone increasing** iff $x_0 \leq x_1 \leq x_2 \cdots$. Similarly define **monotone decreasing**. A sequence is **monotone** iff it is either monotone increasing or monotone decreasing.

Question 4 (12 pts)**.** Show that every infinite sequence of real numbers has a monotone subsequence that converges to $\limsup_i x_i$.

Note: The same is true for $\liminf_i x_i$.

Question 5 (Is supremum "linear"; 12 pts)**.** For $A, B \subseteq \mathbb{R}$, is it true that

i) $\sup(\alpha A) = \alpha \sup(A)$ for $\alpha \geq 0$, and

ii) $\sup(A + B) = \sup(A) + \sup(B)$.

Question 6 (Compact sets get crowded; 15 pts)**.** Show that if X is compact, then for any $\epsilon > 0$, there is $N > 0$ so that for all $S \subset X$ with $|S| \geq N$, there are two points in S whose distance is $< \epsilon$.