Exam 2

To avoid any confusion, unless specified otherwise, vector spaces are complex vector spaces, inner-products are complex inner-products, and matrices are complex matrices.

Part I: True/False

Each problem is points for a total of 50 points. (5 points each.)

1. ______ A is unitary iff $A^H = A^{-1}$.

2. ______ A is unitary iff A preserves inner-products, that is, $\langle \boldsymbol{x}, \boldsymbol{y} \rangle = \langle A\boldsymbol{x}, A\boldsymbol{y} \rangle$.

3. ______ If A preserves the L^2-norm, that is, $\|\boldsymbol{x}\|_2 = \|A\boldsymbol{x}\|_2$, then A preserves the inner-product.

4. ______ If A is diagonalizable and for all eigenvalues, λ of A, $|\lambda| = 1$, then A is unitary.

5. ______ If λ is an eigenvalue of A, then $\bar{\lambda}$ is an eigenvalue of A^H.

6. ______ If $\boldsymbol{v}$ is an eigenvector of A, then $\bar{\boldsymbol{v}}$ is an eigenvector of A^H.

7. ______ $\langle A, B \rangle = \mathrm{tr}(B^H A)$ is an inner product on $\mathbb{C}^{n\times n}$.

8. ______ For all Hermitian matrices A, there is a matrix B so that $B^H B = A$.

9. ______ There are linear maps $L : \mathbb{R}^5 \to \mathbb{R}^4$ such that $\dim(\ker(L)) = 2 = \dim(\mathrm{rng}(L))$.

10. ______ For $k \leq \min\{m, n\}$, the space of matrices of rank k is a subspace of $\mathbb{C}^{m\times n}$.

Part II: Computational (45 points)

Problem 1. (30 points) Find (by hand) he singular value decomposition of

$$A = \begin{bmatrix} \sqrt{2} & 1 & -\sqrt{2}/2 \\ -\sqrt{2} & 1 & \sqrt{2}/2 \\ -\sqrt{2}/2 & -1 & \sqrt{2} \\ \sqrt{2}/2 & -1 & -\sqrt{2} \end{bmatrix}$$

You should be able to complete each step by hand.

(a) Find the eigenvalues of $A^T A$, $\lambda_1 \geq \lambda_2 \geq \lambda_3 \geq 0$.

(b) Find a complete orthonormal set of eigenvectors $\{\boldsymbol{v_1}, \boldsymbol{v_2}, \boldsymbol{v_3}\}$, where $\boldsymbol{v}_i$ is an eigenvector for λ_i.

(c) Set up the 4×3 matrix Σ with $\Sigma_{ii} = \sigma_i = \sqrt{\lambda_i}$ (the i^{th} singular value) and all other $\Sigma_{ij} = 0$.

(d) Find $\boldsymbol{u}_i$ the left singular vectors. Recall $\boldsymbol{u}_i = \frac{1}{\sigma_i} A\boldsymbol{v}_i$ for $i = 1, 2, 3$ and $\boldsymbol{u}_4$ is a basis for $\mathrm{NS}(A^T)$.

(e) Let $U = \begin{bmatrix} \boldsymbol{u}_1 & \boldsymbol{u}_2 & \boldsymbol{u}_3 & \boldsymbol{u}_4 \end{bmatrix}$ and $V = \begin{bmatrix} \boldsymbol{v}_1 & \boldsymbol{v}_2 & \boldsymbol{v}_3 \end{bmatrix}$.

(f) Verify that $A = U\Sigma V^T$.

This all works out very nicely for this carefully chosen matrix A.

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Problem 2. (15 points) Find the best rank 2 approximation to A from (1) with respect to $\|\cdot\|_F$.

Part III: Theory and Proofs (45 points; 15 points each)

Choose 3: If you try all 4, I will only grade the first three.

For this section the following two definitions will be relevant for an $n \times n$ matrix A.

- A is **positive** iff for all $\boldsymbol{x}$, $\boldsymbol{x}^H A \boldsymbol{x}$ is real and non-negative.
- A is **positive-definite** iff A is positive and $\boldsymbol{x}^H A \boldsymbol{h} = \mathbf{0}$ iff $\boldsymbol{x} = \mathbf{0}$.

Problem 1. Use the Spectral Theorem to show that

- A is positive and Hermitian iff $A = B^H B$ for some matrix B.
- A is positive definite and Hermitian iff $A = B^H B$ for some B with $\mathrm{NS}(B) = \{0\}$.

In some sense B is the correct notion of the *square-root* of A.

Problem 2. Recall that an *inner-product* on V is a map $\langle\cdot,\cdot\rangle : V \times V \to \mathbb{C}$ satisfying:

- (sesqui-linearity)
  - $B(\alpha_1 \boldsymbol{x}_1 + \alpha_2 \boldsymbol{x}_2, \boldsymbol{y}) = \alpha_1 B(\boldsymbol{x}_1, \boldsymbol{y}) + \alpha_2 B(\boldsymbol{x}_2, \boldsymbol{y})$ (linear in the first position)
  - $B(\boldsymbol{x}, \beta_1 \boldsymbol{y}_1 + \beta_2 \boldsymbol{y}_2) = \bar{\beta}_1 B(\boldsymbol{x}, \boldsymbol{y}_1) + \bar{\beta}_2 B(\boldsymbol{x}, \boldsymbol{y}_2)$ (conjugate linear in the second position)
- (conjugate symmetry) $\langle \boldsymbol{x}, \boldsymbol{y}\rangle = \overline{\langle \boldsymbol{y}, \boldsymbol{x}\rangle}$.
- (positive-definite) If $\boldsymbol{x} \neq \boldsymbol{0}$, then $\langle \boldsymbol{x}, \boldsymbol{x}\rangle \in (0, \infty)$. In particular, $\langle \boldsymbol{x}, \boldsymbol{x}\rangle$ is a non-negative real number and $\langle \boldsymbol{x}, \boldsymbol{x}\rangle = 0 \iff \boldsymbol{x} = \boldsymbol{0}$.

Show that for any basis $\mathcal{C} = \{\boldsymbol{c}_1, \ldots, \boldsymbol{c}_n\}$ of V and inner-product $\langle\cdot,\cdot\rangle : V \times V \to \mathbb{C}$, there is a matrix representation of $\langle\cdot,\cdot\rangle$ as

$$\langle \boldsymbol{x}, \boldsymbol{y}\rangle = [y]_{\mathcal{C}}^{H} A \boldsymbol{x}_{\mathcal{C}}$$

where A is a positive-definite Hermitian matrix.

Problem 3. Without using anything from the above two results show that for any complex inner-product $\langle \cdot, \cdot \rangle_V$ on a complex vector space V, there is a basis $\mathcal{U} = \{\boldsymbol{u}_1, \ldots, \boldsymbol{u}_n\}$ so that

$$\langle \boldsymbol{x}, \boldsymbol{y} \rangle_V = [\boldsymbol{y}]_{\mathcal{U}}^H [\boldsymbol{x}]_{\mathcal{U}}$$

In other words for any finite dimensional inner-product space, there is a choice of basis, so that with respect to that basis, the inner-product is represented by the standard inner-product.

Problem 4. Use the SVD to show that any square matrix A can be written as $A = UP$ where U is unitary and P is Hermitian.