Exam 1 – Math 215

Problem 1 (30 points; 3 points each)**.** Decide if each of the following are true or false. You do not need to justify your choice here.

(a) _____ $(p \to q) \leftrightarrow (\neg p \wedge q)$ is a tautology.

FALSE: What is true is $(p \to q) \leftrightarrow (\neg p \vee q)$.

(b) _____ $(p \to q) \leftrightarrow (q \to p)$ is a tautology.

FALSE: $q \to p$ is the converse and an implication is not equivalent to its converse.

(c) _____ $\exists x P(x) \wedge \exists x Q(x) \equiv \exists x\, (P(x) \wedge Q(x))$.

FALSE: Let $P(x)$ be "$x = 1$" and $Q(x)$ be "$x = 0$."

(d) _____ $\exists x P(x) \wedge \exists x Q(x) \equiv \exists x \exists y\, (P(x) \wedge Q(y))$.

TRUE: The following are equivalent

- $\exists x \mathbb{P}(x) \wedge \exists x Q(x)$.
- There is some a and some b so that $P(a) \wedge Q(b)$.
- $\exists x \exists y\, (P(x) \wedge Q(y))$

(e) _____ $\neg(\forall x P(x) \wedge \exists x Q(x)) \equiv \exists x \neg P(x) \vee \forall x \neg Q(x)$

TRUE: The following are equivalent

- $\neg(\forall x P(x) \wedge \exists x Q(x))$
- $\neg \forall x P(x) \vee \neg \exists x Q(x)$
- $\exists x \neg P(x) \vee \forall x \neg Q(x)$

(f) _____ $A \subseteq B \implies C - A \subseteq C - B$

FALSE: For example, $C \neq \emptyset$, $A = \emptyset$, and $B = C$, then $A \subseteq B$ yet $C - A = C \not\subseteq C - B = \emptyset$.

(g) _____ $P(\{\emptyset, \{\emptyset\}\}) = \{\emptyset, \{\emptyset\}, \{\emptyset, \{\emptyset\}\}\}$

FALSE: The set $\{\emptyset, \{\emptyset\}\}$ has two elements, so $P(\{\emptyset, \{\emptyset\}\})$ must have $2^2 = 4$.

(h) _____ $f : \mathbb{Z} \times \mathbb{Z}^+ \to \mathbb{Q}$ given by $f(n, m) = n/m$ is onto or surjective.

TRUE

(i) _____ $g : \mathbb{Q} \to \mathbb{Z} \times \mathbb{Z}^+$ given by $g(q) = (n, m)$ iff $q = n/m$ where n and m have no common factors is 1-1 or injective.

TRUE

(j) _____ With f and g as in (h) and (i), $f \circ g : \mathbb{Q} \to \mathbb{Q}$ is the identity on $\mathbb{Q}$, so $f = g^{-1}$.

FALSE: It is true that for $q \in \mathbb{Q}$, if $q = n/m$ where n and m have no common factors and $m > 0$, then $(f \circ g)(q) = f(g(n/m)) = f(n/m) = n/m = q$ so $f \circ g = \mathrm{id}_{\mathbb{Q}}$, but for $f = g^{-1}$ to be true, it must also be the case that $g \circ f = \mathrm{id}_{\mathbb{Z}\times\mathbb{Z}^+}$ and this is not true, for example, $(g \circ f)(2, 4) = g(f(2, 4)) = g(2/4) = (1, 2)$.

Problem 2 (Multiple Choice; 40 points; 4 points each)**.** You may select any number of choices, 0 – 4. You get one point per each correct item, meaning if the item should be selected you get a point, if it should not be selected you get a point.

(a) Which of the following are propositions?

- ⊗ The rat was a spy.
- ○ Wait, wait, don't tell me.
- ○ This sentence is false.
- ○ $x + 2 = 4$.

(b) Which of the conditionals are true?

- ⊗ $5 > 3 \to 5$ is prime.
- ○ $5 > 3 \to 5$ is not prime
- ⊗ $5 < 3 \to 5$ is prime
- ⊗ $5 < 3 \to 5$ is not prime.

(c) Which biconditionals are true for arbitrary integer n?

- ○ $5 > n \leftrightarrow 25 > n^2$.
- ○ $5 > n \leftrightarrow 5n > n^2$.
- ⊗ $5 > n \leftrightarrow 25 > 5n$.
- ○ $5 > n \leftrightarrow 1/5 < 1/n$.

(d) Which are equivalent to $p \to q$?

- ⊗ $\neg q \to \neg p$.
- ⊗ $\neg p \vee q$.
- ⊗ $(p \wedge \neg q) \to F$.
- ⊗ $T \to (p \to q)$.

(e) Which of the following are contingencies?

- ○ $(p \wedge q) \vee (\neg p \wedge q) \vee (p \wedge \neg q) \vee (\neg p \wedge \neg q)$
- ⊗ $(p \to q) \wedge (q \to r) \wedge (r \to \neg p)$
- ⊗ $(p \wedge q \wedge \neg r) \vee (\neg p \wedge q \wedge r) \vee (p \wedge \neg q \wedge r)$
- ⊗ $(p \vee q \vee \neg r) \wedge (\neg p \vee \neg q \vee r) \wedge (p \vee q \vee r)$.

(f) Which of the following say that there are exactly two things satisfying P(x):

- ○ $\exists x P(x) \wedge \exists y P(y) \wedge \neg \exists z P(z)$.
- ⊗ $\exists x \exists y (P(x) \wedge P(y) \wedge x \neq y) \wedge \neg \exists x \exists y \exists z (P(x) \wedge P(y) \wedge P(z) \wedge x \neq y \wedge x \neq z \wedge y \neq z)$.
- ⊗ $\exists x \exists y \big(P(x) \wedge P(y) \wedge x \neq y \wedge \forall z \big(P(z) \to (z = x \vee z = y)\big)\big)$
- ○ $\forall x \forall y \forall z \big(P(x) \wedge P(y) \wedge P(z) \to (x = y \vee x = z \vee y = z)\big)$.

(g) f is continuous at a means $\lim_{x\to a} f(x) = f(a)$ and this is defined as

$$\forall \epsilon > 0 \exists \delta > 0 \forall x \big(0 < |x-a| < \delta \to |f(x) - f(a)| < \epsilon\big)$$

Which of the following imply that f is not continuous at a?

○ $\exists \epsilon > 0 \exists \delta > 0 \exists x \big(0 < |x-a| < \delta \wedge |f(x) - f(a)| \geq \epsilon\big)$
○ $\exists \epsilon > 0 \forall \delta > 0 \exists x \big(0 < |x-a| < \delta \to |f(x) - f(a)| \geq \epsilon\big)$
⊗ $\exists \epsilon > 0 \forall \delta > 0 \exists x \big(0 < |x-a| < \delta \wedge |f(x) - f(a)| \geq \epsilon\big)$
○ $\exists \epsilon > 0 \forall \delta > 0 \exists x \big(0 < |x-a| < \delta \to |f(x) - f(a)| < \epsilon\big)$.

(h) Which of the following imply $A = B$?

⊗ $A \subseteq B \wedge B \subseteq A$.
⊗ $A - B = \emptyset \wedge A \cup B = A$.
⊗ $A \cap B = B \wedge A \cup B = B$.
⊗ $A - B = \emptyset = B - A$.

(i) Which of the following are always true where all sets are subsets of a universal set U?

⊗ $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$.
○ $\overline{A \cup (B \cap C)} = (\overline{A} \cup \overline{B}) \cap (\overline{A} \cup \overline{C})$.
○ $\overline{A \cup (B \cap C)} = \overline{A} \cup (\overline{B} \cap \overline{C})$.
⊗ $\overline{A \cup B} = \overline{A} \cap \overline{B}$.

(j) Which of the following are countably infinite?

○ The set of points on the unit interval $(0, 1)$.
⊗ The set of all polynomials with integer coefficients.
○ The set of all grains of sand on the Earth.
⊗ The set of finite strings of letters in the standard alphabet.

Problem 3 (Short answer; 40 points; 10 points each). Choose four of the five problems, I will grade the first four chosen, so if you do all five and get 1, 2, 3, and 5 correct but 4 wrong, you will score 30/40, since I will have graded 1 - 4. It is your job to decide which four I grade.

(a) Show by any method that the following is a tautology.

$$((p \land \neg q) \to F) \leftrightarrow (p \to q)$$

You can use a truth table here, or use equivalences. I will do the latter:

$$\begin{aligned}
((p \land \neg q) \to F) &\equiv \neg(p \land \neg q) \lor F && a \to b \equiv \neg a \lor b \\
&\equiv (\neg p \lor q) \lor F && \neg(a \land b) = \neg a \lor \neg b \text{ and } \neg\neg b \equiv b \\
&\equiv \neg p \lor q && a \lor F \equiv a \\
&\equiv p \to q && a \to b \equiv \neg a \lor b \text{ again}
\end{aligned}$$

(b) Write down a sentence using quantifiers and logical connectives which asserts that $P(x)$ has at most one items satisfying it.

We can say this by saying P is satisfied by nothing, or by exactly one thing. So the following works

$$\forall x \neg P(x) \lor \exists x (P(x) \land \forall y (P(y) \to x = y))$$

(c) Give a compound proposition in p, q, and r that is true when exactly two of p, q, or r are true.

This is just

$$(p \land q \land \neg r) \lor (p \land \neg q \land r) \lor (\neg p \land q \land r)$$

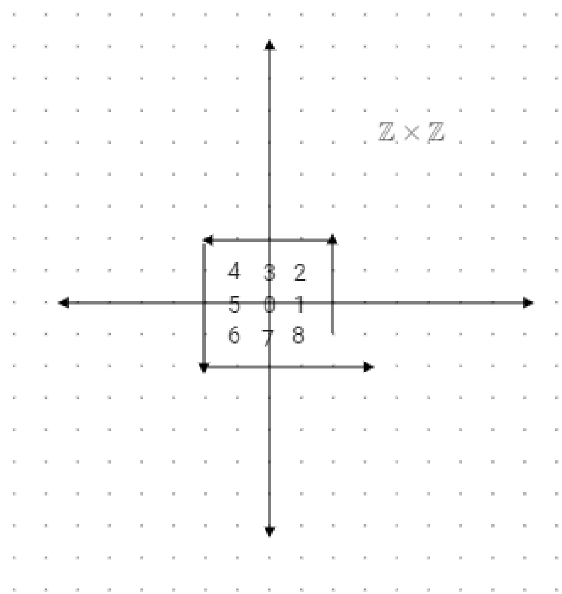
I did not ask for this, but here is more: Clearly if exactly two of p, q, r are true, then exactly one of the disjuncts is true and if there are less than two of p, q, r true, or all three are true, then all disjuncts are false. This is a simple case of the *disjunctive normal form*, DNF, that we have discussed.

(d) Explain why proof by contradiction is valid. That is, you want to prove $p \to q$ and to do this you prove $(p \land \neg q) \to F$, that is, assuming p and $\neg q$ you derive a contradiction.

If you did (a), there is essentially nothing to do here. If you assume p and $\neg q$ and derive a contradiction, then by definition of a valid argument, you have shown that $(p \land \neg q) \to F$ holds. From (a) we know this is equivalent to $p \to q$.

(e) Explain why $|\mathbb{Z} \times \mathbb{Z}| \leq |\mathbb{Z}|$ by providing an injection $f : \mathbb{Z} \times \mathbb{Z} \to \mathbb{Z}$.

There are many ways to accomplish this, here is one injection $f : \mathbb{Z} \times \mathbb{Z} \to \mathbb{N}$:



Problem 4 (Free response; 60 points; 20 points each)**.** Select three of the following four to complete. As above, you must make clear which three you choose.

(a) Either prove or disprove the following: For finite sets A and B,

$$\mathcal{P}(A \times B) \neq \{C \times D \mid C \in \mathcal{P}(A) \wedge D \in \mathcal{P}(B)\}.$$

It is clear that for $C \in \mathcal{P}(A) \wedge D \in \mathcal{P}(B)$, then $C \times D \in \mathcal{P}(A \times D)$. So the issue is in the other direction. Here we need only produce an example of $E \subseteq A \times B$ so that E is not a *rectangle*, i.e., $E \neq C \times D$. This is easy, for example take $E = \{(a, b), (a', b')\}$ where $a \neq a'$ and $b \neq b'$. If $E = C \times D$, then $C = \{a, a'\}$ and $d = \{b, b'\}$, but then $C \times D = \{(a, b), (a, b'), (a', b), a', b')\} \neq E$.

(b) Use the rules of inference to provide an argument that from the premises $\forall x(P(x) \to Q(x))$ and $\forall x(Q(x) \to R(x))$ the conclusion $\forall x(P(x) \to R(x))$ follows. Make sure to indicate the rules of inference used and to what they are applied.

$\forall x(P(x) \to Q(x))$	Given
$\forall x(Q(x) \to R(x))$	Given
$P(a) \to Q(a)$ arbitrary a	Universal instantiation
$Q(a) \to R(a)$ arbitrary a	Universal instantiation
$P(a) \to R(a)$ arbitrary a	$((p \to q) \wedge (q \to r)) \to (p \to r)$
$\underline{\forall x(P(x) \to R(x))}$	Universal Generalization
$\therefore \forall x(P(x) \to R(x))$	

(c) Prove that there are 100 consecutive integers that are not perfect squares. Is your proof direct/indirect? Is it constructive/nonconstructive?

This is easy to provide direct constructive proof of. Consider $(100+1)^2 = 100^2 + 2 \cdot 100 + 1$ so $101^2 - 100^2 = 201$. There are 200 numbers between 100^2 and 101^2 and clearly none of them can be perfect squares.

(d) Prove the triangle inequality for real numbers, $|x| + |y| \geq |x + y|$. What methods do you use? Indirect/direct proof? Proof by cases? Etc.

Here we can use proof by cases.

Case 1 $(x \geq 0 \wedge y \geq 0)$: In this case $x + y = |x| + |y| = |x + y| = x + y$.

Case 2 $(x < 0 \wedge y < 0)$: In this case $|x| + |y| = (-x) + (-y) = -(x + y) = |x + y|$.

Case 3 (otherwise). Without loss of generality assume $x < 0 \wedge y \geq 0$, then $|x|+|y| = (-x)+y > x+y$ and $|x|+|y| = (-x)+y > x+(-y) = -(x+y)$. So $|x|+|y| > |x+y|$.

Notice that in cases (1) and (2) we actually get equality.