Matrix Mult Example

$$\boxed{\begin{bmatrix}1&2&3\\3&2&1\end{bmatrix}\begin{bmatrix}1&4\\3&2\\2&-1\end{bmatrix}}$$

A

$$\left[\begin{bmatrix}1&2&3\\3&2&1\end{bmatrix}\begin{bmatrix}1\\3\\2\end{bmatrix} \quad \begin{bmatrix}1&2&3\\3&2&1\end{bmatrix}\begin{bmatrix}4\\2\\-1\end{bmatrix}\right]$$

$$= \left[\begin{bmatrix}1\\3\end{bmatrix}(1)+\begin{bmatrix}2\\2\end{bmatrix}(3)+\begin{bmatrix}3\\1\end{bmatrix}(2) \quad \begin{bmatrix}1\\3\end{bmatrix}(4)+\begin{bmatrix}2\\2\end{bmatrix}(2)+\begin{bmatrix}3\\1\end{bmatrix}(-1)\right]$$

$$= \left[\begin{bmatrix}1\\3\end{bmatrix}+\begin{bmatrix}6\\6\end{bmatrix}+\begin{bmatrix}6\\2\end{bmatrix} \quad \begin{bmatrix}4\\12\end{bmatrix}+\begin{bmatrix}4\\4\end{bmatrix}+\begin{bmatrix}-3\\-1\end{bmatrix}\right]$$

$$= \boxed{\begin{bmatrix}13&5\\11&15\end{bmatrix}}$$

B

$$\begin{bmatrix}[1\ 2\ 3]\begin{bmatrix}1&4\\3&2\\2&-1\end{bmatrix}\\ [3\ 2\ 1]\begin{bmatrix}1&4\\3&2\\2&-1\end{bmatrix}\end{bmatrix} = \begin{bmatrix}(1)[1\ 4]+2[3\ 2]+3[2\ -1]\\ 3[1\ 4]+2[3\ 2]+(1)[2\ -1]\end{bmatrix}$$

$$= \begin{bmatrix}[1\ 4]+[6\ 4]+[6\ -3]\\ [3\ 12]+[6\ 4]+[2\ -1]\end{bmatrix} = \boxed{\begin{bmatrix}13&5\\11&15\end{bmatrix}}$$

C

$$\begin{bmatrix}[1\ 2\ 3]\begin{bmatrix}1\\3\\2\end{bmatrix} & [1\ 2\ 3]\begin{bmatrix}4\\2\\-1\end{bmatrix}\\ [3\ 2\ 1]\begin{bmatrix}1\\3\\2\end{bmatrix} & [3\ 2\ 1]\begin{bmatrix}4\\2\\-1\end{bmatrix}\end{bmatrix} = \boxed{\begin{bmatrix}13&5\\11&15\end{bmatrix}}$$

Matrix Algebra

Transpose: $(A^T)_{ij} = A_{ji}$

$$\begin{bmatrix}1&2&3\\3&2&1\end{bmatrix}^T = \begin{bmatrix}1&3\\2&2\\3&1\end{bmatrix}$$

Common uses (Inner prod)

$v \cdot u = u^T v$

$(1\ 2\ 3)\cdot(2\ {-1}\ 1) =$

$$\begin{bmatrix}1\\2\\3\end{bmatrix}\cdot\begin{bmatrix}2\\-1\\1\end{bmatrix} = [2\ {-1}\ 1]\begin{bmatrix}1\\2\\3\end{bmatrix}$$

$(1\times 3)\times(3\times 1) = 1\times 1$

Rank 1 (outer product)

$u \otimes v = u v^T$

$$\begin{bmatrix}1\\2\\3\end{bmatrix} \otimes \begin{bmatrix}2\\-1\\1\end{bmatrix} = \begin{bmatrix}1\\2\\3\end{bmatrix}\begin{bmatrix}2 & -1 & 1\end{bmatrix}$$

$$= \begin{bmatrix}2 & -1 & 1\\4 & -2 & 2\\6 & -3 & 3\end{bmatrix}$$

A is symmetric iff $A^T = A$

Algebraic Rules

① $A + B = B + A$ (commutativity of addition)
② $(A+B)+C = A+(B+C)$ (associativity)
③ $A + 0 = 0 + A = A$ (additive identity)
④ $A - A = 0$ (additive inverse)

We will see these again. These say $\mathbb{R}^{m\times n}$ is a vector-space

⑤ $\alpha(\beta A) = (\alpha\beta)A$
⑥ $\alpha(A+B) = \alpha A + \alpha B$ } Distributive
⑦ $(\alpha+\beta)A = \alpha A + \beta A$
⑧ $1 \cdot A = A$; $0 \cdot A = 0$

Note. In general $AB \neq BA$

This is clear for non-square but even for square

$$\begin{bmatrix}1 & 2\\3 & 4\end{bmatrix}\begin{bmatrix}1 & -1\\-1 & 1\end{bmatrix} = \begin{bmatrix}-1 & 1\\-1 & 1\end{bmatrix}$$

$$\begin{bmatrix}1 & -1\\-1 & 1\end{bmatrix}\begin{bmatrix}1 & 2\\3 & 4\end{bmatrix} = \begin{bmatrix}-2 & -2\\2 & 2\end{bmatrix}$$

⑨ $(AB)C = A(BC)$ (associativity)
⑩ $A(B+C) = AB + AC$ } (distributive)
$(A+B)C = AC + BC$

⑪ $(A+B)^T = A^T + B^T$
⑫ $(AB)^T = B^T A^T$

Other things to watch for:

$\boxed{AB = 0 \not\Rightarrow A = 0 \text{ or } B = 0}$ Eg. $\begin{bmatrix}1 & 1\\2 & 2\end{bmatrix}\begin{bmatrix}1 & -1\\-1 & 1\end{bmatrix} = \begin{bmatrix}0 & 0\\0 & 0\end{bmatrix}$

$\boxed{AC = AD \not\Rightarrow C = D}$ (cancellation fails)

Let's prove ⑨ and ⑫

Proof of ⑨: $((AB)C)_{ij} = \sum_{p=1}^{k} (AB)_{ip} C_{pj}$

A $m\times n$
B $n\times k$
C $k\times l$

$$= \sum_{p=1}^{k}\left[\left(\sum_{q=1}^{n} A_{iq}B_{qp}\right)C_{pj}\right] \overset{\text{dist}}{=} \sum_{p=1}^{k}\left[\sum_{q=1}^{n}\left((A_{iq}B_{qp})C_{pj}\right)\right]$$

$$\overset{\text{assoc.}}{=} \sum_{p=1}^{k}\left[\sum_{q=1}^{n}\left(A_{iq}(B_{qp}C_{pj})\right)\right] \overset{\text{dist}}{=} \sum_{q=1}^{n}\left[A_{iq}\sum_{p=1}^{k}(B_{qp}C_{pj})\right]$$

We will have a better way to see this later.

$$= \sum_{q=1}^{n} A_{iq} (BC)_{qj} = (A(BC))_{ij}$$

(12) $$(AB)^T_{ij} = (AB)_{ji} = \sum_{p=1}^{n} A_{jp} B_{pi} = \sum_{p=1}^{n} B^T_{ip} A^T_{pj} = (B^T A^T)_{ij}$$

A $m \times n$
B $n \times k$

Powers, identity, polynomials — Square Matrices

For square matrix A, $A^k = \underbrace{A \cdot A \cdots A}_{k}$

$$\boxed{\begin{aligned} A^k A^s &= A^{k+s} \\ (A^k)^s &= A^{(ks)} \end{aligned}}$$

$$I_{n \times n} = \begin{bmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{bmatrix}$$

$IA = AI = A$ (I is multiplicative identity)

Define $A^0 = I$

Inverses Let A be a square $n \times n$ matrix. $B = A^{-1}$ iff $BA = AB = I$. (*)

<u>Note</u>. If B, B' both satisfy (*), then
$B = B(AB') = (BA)B' = B'$ so $B = B'$, i.e. A^{-1} is unique matrix satisfying (*).

A is called **non-singular** if A^{-1} exists else A is called **singular**

$(AB)^{-1} = B^{-1}A^{-1}$: $(B^{-1}A^{-1})(AB) = I = (AB)(B^{-1}A^{-1})$

I forgot this!

Special Case: Diagonal Matrices

$$\begin{bmatrix} d_{11} & & & 0 \\ & d_{22} & & \\ & & \ddots & \\ 0 & & & d_{nn} \end{bmatrix} \begin{bmatrix} c_{11} & & 0 \\ & \ddots & \\ 0 & & c_{nn} \end{bmatrix} = \begin{bmatrix} d_{11}c_{11} & & & 0 \\ & d_{22}c_{22} & & \\ & & \ddots & \\ 0 & & & d_{nn}c_{nn} \end{bmatrix}$$

$\mathrm{diag}(d_{11}, \ldots, d_{nn})\, \mathrm{diag}(c_{11}, \ldots, c_{nn}) =$
$\mathrm{diag}(d_{11}c_{11}, \ldots, d_{nn}c_{nn})$

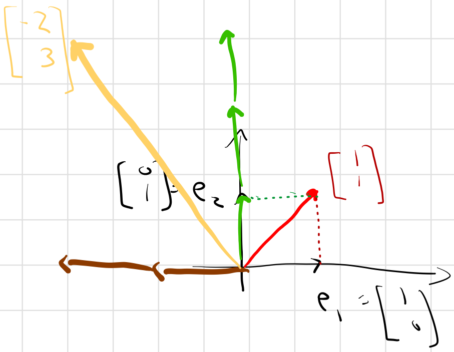
- If $A = \begin{bmatrix} a_{11} & & 0 \\ & \ddots & \\ 0 & & a_{nn} \end{bmatrix} = \mathrm{diag}(a_{11}, \ldots, a_{nn})$, then $A^k = \mathrm{diag}(a_{11}^k, \ldots, a_{nn}^k)$
$$= \begin{bmatrix} a_{11}^k & & \\ & \ddots & \\ & & a_{nn}^k \end{bmatrix}$$

- If $a_{ii} \neq 0$, then $\mathrm{diag}(a_{11} \cdots a_{nn})^{-1} = \mathrm{diag}(a_{11}^{-1} \cdots a_{nn}^{-1})$
$$\begin{bmatrix} a_{11} & & 0 \\ & \ddots & \\ 0 & & a_{nn} \end{bmatrix}^{-1} = \begin{bmatrix} 1/a_{11} & & 0 \\ & \ddots & \\ 0 & & 1/a_{nn} \end{bmatrix} = \begin{bmatrix} a_{11}^{-1} & & \\ & \ddots & \\ & & a_{nn}^{-1} \end{bmatrix}$$

For A diagonal A is singular iff one of the diagonal elements of A is 0.

$$Ax = \begin{bmatrix} a_{11} & & 0 \\ & \ddots & \\ 0 & & a_{nn} \end{bmatrix} \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} a_{11}x_1 \\ \vdots \\ a_{nn}x_n \end{bmatrix}$$

so A stretches each $e_i = \begin{bmatrix} 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \end{bmatrix} \leftarrow i$ by a_{ii}



$$A = \begin{bmatrix} -2 & 0 \\ 0 & 3 \end{bmatrix}$$

Elementary Matrices

We will code G-J elemination by a sequence of matrix ops.

$R_i \leftrightarrow R_j$	do this to I	E^{I}_{ij}	$(E^{I}_{ij})^{-1} = E^{I}_{ij}$
$cR_i \to R_i$	do this to I	$E^{II}_{c,i}$	$(E^{II}_{ci})^{-1} = E_{c^{-1},i}$
$R_j + cR_i \to R_j$	do this to I	$E^{III}_{c,i,j}$	$(E^{III}_{c,i,j})^{-1} = E^{III}_{-c,i,j}$

Suppose we start with A and use G-J to solve
$\underset{(m \times n)}{A}x = b$ so we do a sequence of row ops on $(A|b)$

$A \xrightarrow{op_1, op_2, \ldots} B$ (echelon or RREF)
$\underbrace{E_1 \quad E_2 \cdots}_{m \times m}$

$$A \quad b$$
$$E_1 A \quad E_1 b$$
$$\vdots$$
$$(E_n E_{n-1} \cdots E_1) A \quad (\overbrace{E_n E_{n-1} \cdots E_1}^{M} b)$$

$$Ax = b \longrightarrow \underbrace{MA}_{U}x = Mb$$

Instead of solving $Ax = b$ solve $Ux = Mb$ (back subs)

$$\left[\begin{array}{cccc|cccc} \boxed{4} & 8 & 2 & 5 & 1 & 0 & 0 & 0 \\ \boxed{5} & 10 & -4 & 5 & 0 & 1 & 0 & 0 \\ -3 & -6 & -2 & -3 & 0 & 0 & 1 & 0 \\ 5 & 10 & 1 & 5 & 0 & 0 & 0 & 1 \end{array}\right] \qquad \begin{bmatrix} 1 & 0 & 0 & 0 \\ 5/4 & 1 & 0 & 0 \\ -3/4 & 1/13 & 1 & 0 \\ 5/4 & 3/13 & -23/22 & 1 \end{bmatrix} = L$$

$(-1/2)/(-13/2)$

$$\left(-23/26\right)/\left(11/13\right) = \frac{(-23)(13)}{(11)(26)} = -23/22$$

$E_1\ R_2 - \frac{5}{4}R_1 \to R_2$

$E_2\ R_3 + \frac{3}{4}R_1 \to R_3$

$E_3\ R_4 - \frac{5}{4}R_1 \to R_4$

$$\left[\begin{array}{ccccc|cccc} 4 & 8 & 2 & -6 & 5 & 1 & 0 & 0 & 0 \\ 0 & 0 & \boxed{-13/2} & -3/2 & -5/4 & -5/4 & 1 & 0 & 0 \\ 0 & 0 & \boxed{-1/2} & -1/2 & 3/4 & 3/4 & 0 & 1 & 0 \\ 0 & 0 & -3/2 & -3/2 & -5/4 & -5/4 & 0 & 0 & 1 \end{array}\right]$$

$E_4\ R_3 - 1/13\, R_2 \to R_3$

$E_5\ R_4 - 3/13\, R_2 \to R_4$

$$\left[\begin{array}{ccccc|cccc} 4 & 8 & 2 & -6 & 5 & 1 & 0 & 0 & 0 \\ 0 & 0 & -13/2 & -3/2 & -5/4 & -5/4 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & \boxed{11/13} & 11/13 & -1/13 & 1 & 0 \\ 0 & 0 & 0 & 0 & \boxed{-23/26} & -23/26 & -3/13 & 0 & 1 \end{array}\right]$$

$E_6\ R_4 + \frac{23}{22}R_3 \to R_4$

$$\left[\begin{array}{ccccc|cccc} \multicolumn{5}{c|}{U} & \multicolumn{4}{c}{M} \\ 4 & 8 & 2 & -6 & 5 & 1 & 0 & 0 & 0 \\ 0 & 0 & -13/2 & -3/2 & -5/4 & -5/4 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 11/13 & 11/13 & -1/13 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -7/22 & 23/22 & 1 \end{array}\right]$$

$M = E_6E_5E_4E_3E_2E_1 \qquad L = M^{-1} = E_1^{-1}E_2^{-1}E_3^{-1}E_4^{-1}E_5^{-1}E_6^{-1}$

$$MA = U \to M^{-1}MA = M^{-1}U = LU$$

$$\text{Solve } Ax = b \longleftrightarrow \text{Solve } \underbrace{Ux = Mb = L^{-1}b}_{\text{fast back subs}}$$

A is invertible $\longleftrightarrow AX = I$ for some X

$$[A \mid I] \xrightarrow[\text{elem ops}]{} [I \mid X] \qquad X = A^{-1}$$

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \\ 2 & 1 & 3 \end{bmatrix}$$

$$\left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 3 & 2 & 1 & 0 & 1 & 0 \\ 2 & 1 & 3 & 0 & 0 & 1 \end{array}\right] \xrightarrow[R_3 - 2R_1 \to R_3]{R_2 - 3R_1 \to R_2} \left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & -4 & -8 & -3 & 1 & 0 \\ 0 & -3 & -3 & -2 & 0 & 1 \end{array}\right]$$

$$\xrightarrow[-\frac{1}{3}R_3 \to R_3]{-\frac{1}{4}R_2 \to R_2} \left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & 1 & 2 & 3/4 & -1/4 & 0 \\ 0 & 1 & 1 & 2/3 & 0 & -1/3 \end{array}\right] \xrightarrow{R_3 - R_2 \to R_3} \left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & 1 & 2 & 3/4 & -1/4 & 0 \\ 0 & 0 & -1 & -1/12 & 1/4 & -1/3 \end{array}\right]$$

$$\xrightarrow[R_1 + 3R_3 \to R_1]{R_2 + 2R_3 \to R_2} \left[\begin{array}{ccc|ccc} 1 & 2 & 0 & 3/4 & 3/4 & -1 \\ 0 & 1 & 0 & 7/12 & 1/4 & -2/3 \\ 0 & 0 & 1 & -1/12 & 1/4 & -1/3 \end{array}\right] \xrightarrow{R_1 - 2R_2 \to R_1} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -5/12 & 1/4 & 1/3 \\ 0 & 1 & 0 & 7/12 & 1/4 & -2/3 \\ 0 & 0 & -1 & -1/12 & 1/4 & -1/3 \end{array}\right]$$

$$A^{-1} = \begin{bmatrix} -5/12 & 1/4 & 1/3 \\ 7/12 & 1/4 & -2/3 \\ 1/12 & -1/4 & 1/3 \end{bmatrix} = \frac{1}{12}\begin{bmatrix} -5 & 3 & 4 \\ 7 & 3 & -8 \\ 1 & -3 & 4 \end{bmatrix}$$

Check. $$\frac{1}{12}\begin{bmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \\ 2 & 1 & 3 \end{bmatrix}\begin{bmatrix} -5 & 3 & 4 \\ 7 & 3 & -8 \\ 1 & -3 & 4 \end{bmatrix} = \frac{1}{12}\begin{bmatrix} 12 & 0 & 0 \\ 0 & 12 & 0 \\ 0 & 0 & 12 \end{bmatrix} = \begin{bmatrix} 1 & 0 & \\ 0 & 1 & \\ & & 1 \end{bmatrix}$$

The following are equivalent for square A

1. A is invertible
2. A is product of elem matrices
3. $\operatorname{rank}(A) = n$ (full rank)
4. $\operatorname{rref}(A) = I$