

# Math 571 - Homework 2

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**Problem 2.1** (R:2:2\*). A complex number  $\gamma$  is **algebraic** iff  $\gamma$  is a root to a polynomial with integer coefficients. Prove that there are complex numbers that are not algebraic.

The following are two fun facts (not related to the question, but just for intellectual curiosity):

Let  $\mathbb{A} \subset \mathbb{C}$  be the set of algebraic numbers.

1.  $\mathbb{A}$  is a field.
2.  $\mathbb{A}$  is algebraically closed, that is, if  $\alpha$  is a root of a polynomial in  $\mathbb{A}[x]$ , then  $\alpha \in \mathbb{A}$ . So in the definition of algebraic numbers you can use any ring of coefficients  $R$ ,  $\mathbb{Z} \subseteq R \subseteq \mathbb{A}$ .

[Here](#) is a write-up of the proofs, you should have the background to read this, but it is not an easy read.

**Definition 1.** A set  $S \subseteq X$  is **discrete** iff every point in  $S$  is isolated.

**Problem 2.2** (R:2:5\*). Prove the following for discrete  $S \subset \mathbb{R}$ :

- a)  $\text{Lim}(S) \cap S = \emptyset$  and  $S$  is countable.
- b) There is discrete set  $A \subset \mathbb{R}$  so that  $\text{Lim}(A) = \text{Cl}(S)$ .
- c) Give an example of a discrete set  $S$  where there is no set  $A$  such that  $\text{Lim}(A) = S$ .

For the following use the definition that I provided for  $\text{Cl}(E)$ , namely,  $\text{Cl}(E) = \bigcap \{F \mid F \text{ is closed and } E \subseteq F\}$ .

**Problem 2.3** (R:2:6). For  $X$  a metric space and  $E \subseteq X$ , show that

- a)  $\text{Lim}(\text{Lim}(E)) \subseteq \text{Lim}(E)$  and equality need not obtain.
- b)  $\text{Lim}(A \cup B) = \text{Lim}(A) \cup \text{Lim}(B)$ .
- c)  $E \cup \text{Lim}(E)$  is closed and  $E \cup \text{Lim}(E) = \text{Cl}(E)$ .
- d)  $\text{Lim}(E)$  is closed and  $\text{Lim}(E) = \text{Lim}(\text{Cl}(E))$ .

**Problem 2.4** (R:2:9\*). Let  $X$  be a metric space, or just any topological space. Are the following true for all  $E \subseteq X$ ? For each either prove the statement true or give a counterexample. For a counterexample you must provide both  $X$  and  $E$ .

- a)  $\text{Int}(E)^c = \text{Cl}(E^c)$ .

b)  $\text{Cl}(E) = \text{Int}(E^c)^c$ ?

c)  $\text{Cl}(E) = \text{Cl}(\text{Int}(E))$ ?

d)  $\text{Int}(E) = \text{Int}(\text{Cl}(E))$

**Definition 2.** A metric space  $X$  is **separable** iff there is a countable  $E \subseteq X$  with  $E$  dense in  $X$ .

**Problem 2.5** (R:2:22). Show the  $\mathbb{R}^k$  is separable.

**Definition 3.** A set  $\mathcal{B}$  of open sets is called a **base** for  $X$  iff for all  $x \in X$  and open set  $U$  with  $x \in U$ , there is  $V \in \mathcal{B}$  so that  $x \in V \subset U$ .

**Problem 2.6** (R:2:23\*). Prove that a metric space is separable iff it has a countable base.

**Problem 2.7** (R:2:24). Prove that if  $X$  is a metric space and every infinite sequence has a limit point, then  $X$  is separable. (See the hint in the text.)