Math 571 - Exam 2 (Due 11/30)

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There are 80 points here, so basically, 10 extra points. I'll take the score and use the minimum of 70 and your score as the final grade. Make your answers self-contained. If something here comes straight out of the homework, then do not "quote" the homework result as a reason. I am looking for the argument.

Question 1 (20 pts)**.** Give a reason for the non-existence of each of the following, or else provide an example. You may use theorems, but you must state the theorem, not just a reference to some theorem number.

- A continuous function $f : \mathbb{R} \to \mathbb{Q}$ that is not a constant function.

- A continuous function $f : \mathbb{Q} \to \mathbb{Z}$ that is not a constant function.

- A function $f : \mathbb{Z} \to \mathbb{R}$ that fails to be continuous.

- A continuous **onto** function $f : S_1 \to \mathbb{R}$, where $S_1 = \{(x, y) \mid x^2 + y^2 = 1\}$ is the unit circle.

Question 2 (15 pts)**.** Let $E \subset \mathbb{R}$ be bounded and $f : E \to \mathbb{R}$ be uniformly continuous. Show that $f(E)$ is also bounded. Give an example to show that continuous is not enough.

You may use, the obvious, but perhaps painful to prove fact, that if $f(E)$ is **covered** by a finite collection of bounded sets, then $f(E)$ is itself bounded.

Question 3 (15 pts)**.** (1) Let $f : \mathbb{R} \to \mathbb{R}$ be differentiable, suppose f' is bounded. Show that f is uniformly continuous.

(2) Find a uniformly continuous and differentiable function $f : \mathbb{R} \to \mathbb{R}$ whose derivative is not bounded.

Hint: The function should "wiggle" faster and faster as $|x| \to \infty$ and $f(x) \to 0$ as $|x| \to \infty$. You may use the fact that if f is continuous and both $\lim_{x\to\infty} f(x)$ and $\lim_{x\to-\infty} f(x)$ exist, then f is uniformly continuous. For some "extra bonus" you may prove this fact.

Question 4 (15 pts)**.** Let X and Y be metric spaces with X **compact** and let $f : X \to Y$ and $g : X \to Z$ be two functions, with no additional assumptions on these functions.

Suppose that for every $x \in X$, at least one of f or g is continuous at x. Show that for every $\epsilon > 0$, there is a $\delta > 0$ such that for all $x, x' \in X$:

$$d_X(x, x') < \delta \implies d_Y(f(x), f(x')) < \epsilon \text{ or } d_Z(g(x), g(x')) < \epsilon$$

This is sort of an "either/or" version of uniform continuity.

Question 5 (15 pts)**.** Let f and α be monotonically increasing bounded functions on $[a, b]$. Suppose that α is continuous at every point where f is discontinuous. Show that $f \in \mathscr{R}(\alpha)$.

You may use Theorem 4.30 in your text, and you will need to use the result of Question 4. The proof of Theorems 6.8 - 6.10 in the text should give a clue on how to proceed, but of course, the argument here is not exactly the same as any one of these alone.