# Math 571 - Homework 5 (05.22)

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**Notation:** For $f : X \to Y$ and $E \subseteq X$ set $f(X) = \{f(e) \mid e \in E\}$, this is called the *image of $E$ under $f$*.

**Problem 1** (R:4:2*)**.** Let $f : X \to Y$ be continuous. Let $E \subseteq X$, show that $f(\mathrm{Cl}(E)) \subseteq \mathrm{Cl}(f(E))$. By example show that this containment can be proper, that is $\mathrm{Cl}(f(E)) \nsubseteq f(\mathrm{Cl}(E))$ can hold.

You may take $X$ and $Y$ to be metric if you want, but this is not relevant. Let $y \in f(\mathrm{Cl}(X))$, so $y = f(x)$ for $x \in \mathrm{Cl}(E)$. Let $O$ be an open nbhd of $y$ and let $U$ be an open nbhd of $x$ so that $f(U) \subset O$. Since $x \in \mathrm{Cl}(E)$ we have $U \cap E \neq \emptyset$. Let $e \in U \cap E$, then $f(e) \in f(U) \cap f(E) \subseteq O \cap f(E)$. So we have shown that for any open nbhd $O$ of $y$, $y \cap f(E) \neq \emptyset$, thus $y \in \mathrm{Cl}(f(E))$.

Consider $f : \mathbb{R} \to \mathbb{R}$ given by $f(x) = \frac{1}{1+x^2}$. So $f(\mathbb{R}) = (0, 1] \subsetneq \mathrm{Cl}(f(\mathbb{R})) = \mathrm{Cl}((0, 1]) = [0, 1]$.

**Definition** Let $f : E \subset X \to Y$, the graph of $f$ is the set $\mathrm{Graph}(f) = \{(x, f(x) \mid x \in E\} \subseteq X \times Y$.

**Problem 2.** Let $f : E \subset X \to Y$ be continuous where $Y$ is Hausdorff, show that $\mathrm{Graph}(f)$ is closed in $E \times Y$.

**(Proof 1) Hausdorff Case:** Let $(x, y) \in E \times Y - \mathrm{Graph}(f)$. So $f(x) = y' \neq y$. Let $O$ be an open nbhds $O$ and $O'$ of $y$ and $y'$ respectively so that $O \cap O' = \emptyset$. (Here we use the Hausdorff property.) Let $U$ be an open nbhd of $x$ so that $f(U \cap E) \subseteq O'$. I claim that $(U \times O) \cap \mathrm{Graph}(f) = \emptyset$. Suppose that $(\tilde{x}, \tilde{y}) \in (U \cap O) \cap \mathrm{Graph}(f)$, then $f(\tilde{x}) = \tilde{y}$, so $f(U \cap E) \cap O \neq \emptyset$, contradicting $f(U \cap E) \subseteq O'$ and $O' \cap O = \emptyset$.

**(Proof 2) Metric Case:** Suppose $((x_i, f(x_i))$ is a convergent sequence in $E \times Y$, that is $((x_i, f(x_i)) \to (x, y)$. In particular, $x_i \to x \in E$ and as $f$ is sequentially continuous $f(x_i) \to f(x)$, thus $y = f(x)$ and we see $\mathrm{Graph}(f)$ is sequentially closed, hence closed.

**Problem 3** (R:4:6)**.** Suppose $f : E \subseteq X \to Y$ and $E$ is compact. Suppose further that $X$ and $Y$ are Hausdorff (or metric if you prefer). Show that $f$ is continuous on $E$ iff $\mathrm{Graph}(f)$ is compact.

**Hint:** You may use the fact that if $K$ and $H$ are compact, then $K \times H$ is compact and that If $K$ is compact and $C \subseteq K$ is closed, then $C$ is compact. (Both of these are in notes and book.)

**(Proof 1) Hausdorff Case:** Assume $f$ is continuous, then $f(E) \subset Y$ is compact and $\mathrm{Graph}(f) \subset E \times f(E)$ is closed, hence $\mathrm{Graph}(f)$ is a closed subset of the compact set $E \times f(E)$

and hence compact.

Consider the map $F : E \to \text{Graph}(f)$ given by $F(x) = (x, f(x))$.

**Claim**: $F$ is continuous iff $f$ is continuous.

**Proof of Claim**: This follows from showing that for $U \subset X$ open and $V \subset Y$ open

$$F^{-1}((U \times V) \cap \text{Graph}(g)) = f^{-1}(V) \cap U. \qquad (\dagger)$$

This shows that the pullback by $F$ for all basic open sets in $\text{Graph}(g)$ are open in $E$ iff the pullback by $f$ of all open subsets of $Y$ are open in $E$, which when unpacked says $F$ is continuous iff $f$ is continuous. Checking $(\dagger)$ is an easy exercise.

So we need only show now that $F$ is continuous. But $F^{-1}$ is just projection $(x, f(x)) \mapsto x$ and this is continuous. Since $\text{Graph}(f)$ is compact and $X$ is Hausdorff, $F^{-1}$ is a closed map, and hence $F$ is continuous. (See here.)

**(Proof 2) Metric Case:** Suppose $\text{Graph}(f)$ is compact, hence sequentially compact. Suppose $x_i \in E$ and $x_i \to x \in E$. Consider $((x_i, y_i))$ in $\text{Graph}(f)$ we know there is a convergent subsequence $((x_{n_i}, y_{n_i}))_i \to (x, y) \in \text{Graph}(f)$. But then $\lim_i x_{n_i} = \lim_i x_i = x$ and $y = f(x)$ and $y_{n_i} \to y$, so $f(x_{n_i}) \to y$.

Suppose $y_i \not\to y$ as $i \to \infty$, then there is $y' \neq y$ and subsequence $((x_{m_i}, y_{m_i}))_i \to (x, y') \in \text{Graph}(f)$. But then $f(x) = y = y'$ which is a contradiction. so $y_i \to y$, that is $f(x_i) \to f(x)$. Thus $f$ is sequentially continuous and hence continuous.

The other direction is easier. Suppose $f$ is continuous and $((x_i, y_i))$ is a sequence from $\text{Graph}(f)$. then $x_{n_i} \to x \in E$ for some subsequence $x_{n_i}$ since $E$ is sequentially compact. But then $f(x_{n_i}) \to f(x)$ and so $((x_{n_i}, y_{n_i}))_i \to (x, y) = (x, f(x)) \in \text{Graph}(f)$. So $\text{Graph}(f)$ is sequentially compact, hence compact.

**Problem 4.** Let $f : E \subset X \to Y$ where both $X$ and $Y$ are metric spaces with $Y$ complete. suppose $f$ is uniformly continuous on $E$, show that there is a unique continuous extension $\hat{f} : \text{Cl}(E) \to Y$. Moreover, $\hat{f}$ remains uniformly continuous.

**Existence**: Let $x \in \text{Cl}(E) - E$ so that $x$ is a limit point of $E$, then $x = \lim_i x_i$ for $(x_i)$ a sequence from $E$. Since $(x_i)$ is a Cauchy sequence and $f$ is uniformly continuous, $(f(x_i))$ is Cauchy and thus has a limit $y$. To see that $x \mapsto y$ defines an extension of $f$ we must see that $y$ is independent of the particular sequence $(x_i)$ chosen and that $y = f(x)$ for $x \in E$. The second follows from the first trivially, since letting $x_i = x$ for all $i$, $(x_i)$ is a Cauchy sequence converging to $x$. Suppose $(x_i')$ is another sequence from $E$ with $\lim_i x_i' = x$. Then the sequence $(z_i)$ where $z_{2i} = x_i$ and $z_{2i+1} = x_i'$ is a sequence from $E$ converging to $x$ and clearly $(f(x_i))$ and $(f(x_i'))$ are both Cauchy subsequences of the Cauchy sequence $(f(z_i))$, thus all of these must have the same limit $y$.

To see that $\hat{f}$ is uniform continuous, let $\epsilon > 0$ take $\delta$ that witnesses uniform continuity on $E$, so for all $x, x' \in E$, $d^X(x, x') < \delta \implies d^Y(f(x), f(x')) < \epsilon/2$. Suppose $x, x' \in \text{Cl}(E)$ and $d^X(x, x') < \delta$. Take $u, u' \in E$ with $d^Y(f(u), \hat{f}(x)) < \epsilon/4$, $d^Y(f(u'), \hat{f}(x')) < \epsilon/4$, and $d^X(u, u') < \delta$, then $d^Y(\hat{f}(x), \hat{f}(x')) \leq d^Y(\hat{f}(x), f(u)) + d^Y(f(u), f(u')) + d^Y(\hat{f}(x'), f(u')) < \epsilon$.

**Uniqueness**: Suppose $g : \text{Cl}(E) \to Y$ is continuous and $f = g|_E$, then we must show that $g = \hat{f}$. This is trivial since if $x \in E$ there is nothing to do. If $x \notin E$, then $x = \lim_i x_i$ for $x_i \in E$, so $g(x) = \lim_i g(x_i) = \lim_i f(x_i) = \hat{f}(x)$.

**Definition**: A set $E \subset X$ has the *Bolzano-Weierstrass property* iff every sequence in $X$ has a convergent subsequence.

**Problem 5.** Show that if $E \subseteq X$ has the Bolzano-Weierstrass property, then

a) $\mathrm{Cl}(E)$ also has Bolzano-Weierstrass property.

Let $x_i \in \mathrm{Cl}(E)$, then for each $i$ there is $x_i' \in E$ so that $d^X(x_i, x_i') < 1/i$. Then $x_i'$ has a convergent subsequence $(x'_{n_i})$ and it is clear that $(x_{n_i})$ also converges (to the same limit).

b) If $X$ is metric, then $E$ is bounded.

If $E$ is unbounded, then it is simple to choose a sequence $x_i \in E$ so that $d^X(x_i, x_j) > 1$ for all $i, j$. But then this sequence has no convergent subsequence.

c) For $X$ metric $E$ has the Bolzano-Weierstrass property iff $\mathrm{Cl}(E)$ is compact.

$\mathrm{Cl}(E)$ is sequentially compact, hence compact.

**Problem 6** (R:4:8*)**.** Let $f : E \subseteq X \to Y$ be uniformly continuous on $E$ where $E$ has the Bolzano-Weierstrass property and $Y$ is complete. Show that $f$ is bounded on $E$, that is $f(E)$ is bounded in $Y$.

**Proof 1**: From problem 4 we can extend $f$ to $\hat{f} : \mathrm{Cl}(E) \to Y$ and from problem 5, $\mathrm{Cl}(E)$ is compact. So $\hat{f}(\mathrm{Cl}(E))$ is compact hence bounded in $Y$ and so $f(E)$ is bounded.

**Proof 2**: (You don't actually need Problem 5 or the stuff about compactness.) Suppose $f(E)$ is unbounded. Then get $x_i \in E$ so that $d^Y(x_i, x_j) \geq 1$. By assumption there is a convergent and hence Cauchy subsequence of $(x_i)$, say $(x_{n_i})$. By uniform continuity of $f$, $(f(x_{n_i}))$ is a Cauchy sequence in $Y$. But this is a contradiction.

**Problem 7** (R:4:19)**.** Show that if $f : \mathbb{R} \to \mathbb{R}$ satisfies the intermediate value theorem and $f^{-1}(r) = \{x \mid f(x) = r\}$ is closed for $r \in \mathbb{Q}$, then $f$ is continuous. (See the text for a hint. $\mathbb{Q}$ here could be replaced by any dense set.)

Suppose $f$ fails to be continuous at $x$. Fix $\epsilon > 0$ such that for all $\delta > 0$, there is some $x' \in (x - \delta, x + \delta)$ so that $f(x') \notin (f(x) - \epsilon, f(x) + \epsilon)$. We can then choose a sequence $x_i \to x$ so the for all $i$, $f(x_i) \notin N_\epsilon(f(x))$. We may assume WLOG $f(x_i) \leq f(x) - \epsilon < f(x)$ for all $i$ since either infinitely many of the $x_i$ satisfy this or else $f(x) < f(x) + \epsilon \leq f(x_i)$ and the proof would be the same in each case. Fix $r \in (f(x) - \epsilon, f(x)) \cap \mathbb{Q}$. So we can get $t_i \in N_{|x_i - x|}(x)$ so that $f(t_i) = r$ using the IVT property. But then it is clear that $t_i \to x$ and as $t_i \in f^{-1}(r)$ for all $i$ we have $x \in f^{-1}(r)$. But then $f(x) = r$ and this is a contradiction.