

Homework 3 Partial Solutions

Section 3.1

8. This questions is about arbitrary vectors, these could be vectors in \mathbb{R}^n but it could also be the space of matrices $\mathbb{R}^{n \times m}$, could be the space of continuous functions on the unit interval into \mathbb{R} , $C([0, 1], \mathbb{R})$, etc. So you must argue generally using axioms of vector spaces.

$$x + y = x + z$$

$$(-x) + (x + y) = (-x) + (x + z) \quad (\text{A4})$$

$$(-x + x) + y = (-x + x) + z \quad (\text{A2})$$

$$0 + y = 0 + z \quad (\text{A4})$$

$$y = z \quad (\text{A3})$$

13. There are various ways to see that this is not a vector space. One way is to notice that there is no 0 element!

What element a of \mathbb{R} would satisfy $\max(a, r) = r$ for all $r \in \mathbb{R}$? For $r \geq 0$, $a = 0$ would suffice, but what would work for $r < 0$? If $a \oplus r = r$ for $r < 0$, then $a < r$. But then $a < r$ for all $r \in \mathbb{R}$!

14. Let $V = \mathbb{Z}$ and define scalar multiplication by

$$\alpha \cdot_V n = \lfloor \alpha \rfloor \cdot n \quad (1)$$

$$n +_V m = n + m \quad (2)$$

Is this a vector space?

All the additive axioms clearly hold since these are true of integer arithmetic.

The problem here is $\alpha \cdot_V (\beta \cdot_V n) = (\alpha \cdot \beta) \cdot_V n$. For example:

$$.5 \cdot_V (2 \cdot_V n) = 0 \cdot (2 \cdot n) = 0$$

while

$$(.5 \cdot 2) \cdot_V n = 1 \cdot_V n = 1 \cdot n = n$$

Section 3.2

2.

(a) This is not a subspace because $(0, 0)^T \notin S$.

(b) This is a subspace.

- If $(a, b, c) \in S$, then $\alpha(a, b, c)^T \in S$, since, $a = b = c$ implies $\alpha a = \alpha b = \alpha c$.
- If $(a, b, c)^T, (A, B, C)^T \in S$, then $a + A = b + B = c + C$, so $(a, b, c)^T + (A, B, C)^T \in S$.

Thus S is closed under scalar multiplication and addition and is a subspace.

(c) This is a subspace. Do just like (b), but use the property $x_1 = x_2 + x_3$. Another way is to notice that $S = NS(A)$ where $A = \begin{bmatrix} 1 & -1 & -1 \end{bmatrix}$. (We could have done this with (b) as well.)

(d) This is not a subspace $(1, 2, 1)^T$ and $(4, 1, 1)^T$ are in S , but the sum $(5, 3, 2)^T \notin S$

4.

(a) $\text{rref}(A) = I_2$ so $NS(A) = \text{span}\{\mathbf{0}\}$.

(b) $\text{rref}(A) = \begin{bmatrix} 1 & 2 & -3 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$ so $A\mathbf{x} = \mathbf{0}$ is equivalent to

$$\begin{aligned} x_1 + 2x_2 - 3x_3 &= 0 \\ x_4 &= 0 \end{aligned}$$

Let $x_2 = s$ and $x_3 = t$, then we have:

$$\begin{aligned} x_1 &= -2s + 3t \\ x_2 &= s \\ x_3 &= t \\ x_4 &= 0 \end{aligned}$$

which is the same as

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = s \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + t \begin{bmatrix} 3 \\ 0 \\ 1 \\ 0 \end{bmatrix}$$

So $NS(A) = \text{span}\{(-1, 1, 0, 0)^T, (3, 0, 1, 0)^T\}$.

(c) $\text{rref}(A) = \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}$ so this has x_3 as a free variable. Let $x_3 = t$, then

$$\begin{aligned} x_1 &= t \\ x_2 &= t \end{aligned}$$

is the resulting system so an element of $\text{NS}(A)$ is of the form

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} t \\ t \\ t \end{bmatrix} = t \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

so $\text{NS}(A) = \text{span}\{(1, 1, 1)^T\}$.

(d) Just as an example of using MATLAB

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1 A=[1 1 -1 2; 2 2 -3 1; -1 -1 0 -5]
2 rref(A)
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$\text{rref}(A) = \begin{bmatrix} 1 & 1 & 0 & 5 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ so x_2 and x_4 are the non-pivot, hence free variables. Let $x_2 = s$ and $x_4 = t$, then the system becomes

$$\begin{aligned} x_1 &= -s - 5t \\ x_3 &= -3t \end{aligned}$$

So we have $\mathbf{x} \in \text{NS}(A)$ iff

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} -s - 5t \\ s \\ -3t \\ t \end{bmatrix} = s \begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \end{bmatrix} + t \begin{bmatrix} -5 \\ 0 \\ -3 \\ 1 \end{bmatrix}$$

and thus

$$\text{NS}(A) = \text{span} \left\{ \begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -5 \\ 0 \\ -3 \\ 1 \end{bmatrix} \right\}$$

8. A is fixed.

- $\mathbf{0}A = A\mathbf{0}$ so $\mathbf{0} \in S$
- Let $B, C \in S$, then $BA = AB$ and $CA = AC$ so $(B + C)A = BA + CA = AB + AC = A(B + C)$ and hence $B + C \in S$.
- Let $B \in S$, then $(\alpha B)A = \alpha(BA) = \alpha(AB) = A(\alpha B)$, so $\alpha B \in S$.

11. Just put the vectors in as columns, or rows, of a matrix A . Find $\text{rref}(A)$. If there are two non-zero rows, that is $\text{rank}(A) = 2$, then the set is a basis. for example, given $B = \{(2, 1)^T, (3, 2)^T\}$ for $A = \begin{bmatrix} 2 & 3 \\ 1 & 2 \end{bmatrix}$ (I put the vectors in as columns). $\text{rref}(A) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, so B spans \mathbb{R}^2 . (You could just compute $\text{rank}(A)$ in MATLAB.

13. If $A = [\mathbf{x}_1 \quad \mathbf{x}_2]$, then $\mathbf{x} \in \text{span}\{\mathbf{x}_1, \mathbf{x}_2\}$ iff $A\mathbf{z} = \mathbf{x}$ has a solution, similar for \mathbf{y} . So for \mathbf{x} just try to solve

$$\begin{bmatrix} -1 & 3 \\ 2 & 4 \\ 3 & 2 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} = \begin{bmatrix} 2 \\ 6 \\ 6 \end{bmatrix}$$

Since

$$\text{rref} \left(\begin{bmatrix} -1 & 3 & 2 \\ 2 & 4 & 6 \\ 3 & 2 & 6 \end{bmatrix} \right) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

This has no solution. Recall this was an augmented matrix, and the last row means $0z_1 + 0z_2 = 1$, which is nonsense.

17. Here is what this question is getting at. Suppose you take $\mathbf{b} \in \text{CS}(A) = \text{Img}(A)$ so $\mathbf{b} = A\mathbf{x}$ for some \mathbf{x} . Then if $A\mathbf{x}' = \mathbf{b}$ also, we see that $A\mathbf{x} - A\mathbf{x}' = A(\mathbf{x} - \mathbf{x}') = \mathbf{0}$ so $\mathbf{x} - \mathbf{x}' \in \text{NS}(A)$.

It is also clear that if $\mathbf{z} \in \text{NS}(A)$ and $A\mathbf{x} = \mathbf{b}$, then $A(\mathbf{x} + \mathbf{z}) = A\mathbf{x} + A\mathbf{z} = \mathbf{b} + \mathbf{0} = \mathbf{b}$.

From these two facts we see that if \mathbf{x} is **any** solution to the system $A\mathbf{x} = \mathbf{b}$, then the set of **all** solutions is

$$\mathbf{x} + \text{NS}(A) = \{\mathbf{x} + \mathbf{z} \mid A\mathbf{z} = \mathbf{0}\}$$

18.

(a) Adding a vector to a spanning set leaves it a spanning set. This is clear since if $S \subset S' \subset V$ are sets of vectors in a vector space V , then clearly $\text{span}(S) \subset \text{span}(S')$. But if $\text{span}(S) = V$, i.e., S is a spanning set, then $V \subset \text{span}(S) \subset \text{span}(S') \subset V$ so these must all be the same.

(b) Removing a vector from a spanning set may, or may not, leave it as a spanning set. If it is a minimal spanning set (a basis), then removing a vector will mean that what is left is no longer spanning.

Section 3.3

2. Again just write these vectors down as the rows of a matrix A . If $\text{rref}(A)$ has any 0 rows, then the vectors are not independent, otherwise they are. For example:

$$\text{rref} \left(\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 2 & 3 \end{bmatrix} \right) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

So these vectors are not independent.

5. (This is sort of the opposite of the spanning case.)

(a) Adding vectors to a linearly independent set can obviously mess up independence. (Just add a linear combination of the original vectors.) For example, if $S \subset \mathbb{R}^n$ is linearly independent, then $S \cup \{\mathbf{0}\}$ is not.

(b) Clearly removing a vector from a linearly independent set cannot mess up linear independence.

Specifically if $S = \{\mathbf{v}_1, \dots, \mathbf{v}_n\}$ and $S' \subset S$, say $S' = \{\mathbf{v}_{i_1}, \dots, \mathbf{v}_{i_k}\}$ and $c_{i_1}\mathbf{v}_{i_1} + \dots + c_{i_k}\mathbf{v}_{i_k} = \mathbf{0}$ is a linear combination of elements of S' , then this is trivially also a linear combination of elements of S and hence by the independence of S we have $c_{i_1} = \dots = c_{i_k} = 0$. So S' is linearly independent.

8. Determine whether the following are independent in P_3 .

(a) $\{1, x^2, x^2 - 2\}$ is not independent as $x^2 - 2 = -2 \cdot 1 + 1 \cdot x^2$, so $x^2 - 2$ is a linear combination of 1 and x^2 .

(c) $\{x + 2, x + 1, x^2 - 1\}$ relative to the standard (ordered) basis for P_3 , $\{1, x, x^2\}$, this is equivalent to asking if $\{(2, 1, 0), (1, 1, 0), (-1, 0, 1)\}$ is linearly independent. Clearly,

$$\begin{bmatrix} 2 & 1 & -1 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 0 \\ 0 & -1 & -1 \\ 0 & 0 & 1 \end{bmatrix}$$

so $\{x + 2, x + 1, x^2 - 1\}$ is linearly independent.

(d) $\{x + 2, x^2 - 1\}$ is independent since $\{x + 2, x + 1, x^2 - 1\}$ is linearly independent, by (c).

9. Show the following sets are linearly independent in $C([0, 1])$

(a) $\sin(\pi x)$ and $\cos(\pi x)$

One interesting way here is to note that $\langle f, g \rangle = \int_0^1 f \cdot g \, dx$ is an inner-product on $C([0, 1])$ and $\langle \sin(\pi x), \cos(\pi x) \rangle = 0$, so actually, these two functions are orthogonal!

A less interesting way is to note that if $a \sin(\pi x) + b \cos(\pi x) = 0$ (the 0 function), then letting $x = 0$ gives $a \sin(0) + b \cos(0) = b = 0$ and letting $x = 1/2$ gives $a \sin(\pi/2) + b \cos(\pi/2) = a = 0$ so $a = b = 0$ and hence the two functions are independent.

(b) $x^{3/2}$ and $x^{5/2}$

Suppose $ax^{3/2} + bx^{5/2} = 0$ for all $x \in [0, 1]$, then for $x = 1$ we have $a + b = 0$ and for $x = 1/4$ we have $a(1/2)^3 + b(1/2)^5 = 0$ so $a + b(1/2)^2 = 0$ hence $a + b/4 = 0$ or equivalently $4a + b = 0$. Solving

$$\begin{aligned} 4a + b &= 0 \\ a + b &= 0 \end{aligned}$$

gives $a = b = 0$. So These are independent.

(c) $1, x^x - e^{-x}$ and $e^x + e^{-x}$

Again suppose $h(x) = a + b(e^x - e^{-x}) + c(e^x + e^{-x}) = 0$. It is easy to see $h(0) = a + 2c = 0$, $h'(0) = 2b = 0$ and $h''(0) = 2c = 0$. So clearly, $a = b = c = 0$ as desired.

(d) e^x, e^{-x} and e^{2x}

This is like (c), Assume $h(x) = ae^x + be^{-x} + ce^{2x}$, then $h'(x) = ae^x - be^{-x} + 2ce^{2x}$ and $h''(x) = ae^x + be^{-x} + 4e^{2x}$ and so

$$\begin{aligned} h(0) &= a + b + c = 0 \\ h'(0) &= a - b + 2c = 0 \\ h''(0) &= a + b + 4c = 0 \end{aligned}$$

It is easy to check that this has the unique solution $a = b = c = 0$.

10. It turns out here that 1 , $\cos(x)$, and $\sin^2(x/2)$ are linearly dependent and this is from one of the half-angle formulas,

$$\cos(x) = \cos^2(x/2) - \sin^2(x/2) = 1 - 2\sin^2(x/2)$$

.

16. Show that the columns of A are linearly independent iff $\text{NS}(A) = \{\mathbf{0}\}$.

Suppose A is $m \times n$ so $A = [\mathbf{a}_1 \ \cdots \ \mathbf{a}_n]$ with $\mathbf{a}_i \in \mathbb{R}^m$ the i^{th} column of A . Then

$$A\mathbf{x} = x_1\mathbf{a}_1 + \cdots + x_n\mathbf{a}_n$$

is an arbitrary linear combination of the columns of A and so.

(if) Assume $\text{NS}(A) = \{\mathbf{0}\}$, then $x_1\mathbf{a}_1 + \cdots + x_n\mathbf{a}_n = \mathbf{0}$ iff $A\mathbf{x} = \mathbf{0}$ iff $\mathbf{x} = \mathbf{0}$, that is $x_1 = x_2 = \cdots = x_n = 0$. So the columns of A are linearly independent since the only linear combination giving $\mathbf{0}$ is the trivial combination.

(only-if) Assume the columns of A are linearly independent, then $A\mathbf{x} = \mathbf{0}$ would mean the $x_1\mathbf{a}_1 + \cdots + x_n\mathbf{a}_n = \mathbf{0}$ so by linear independence, $x_1 = x_2 = \cdots = 0$ and hence $A\mathbf{x} = \mathbf{0}$ implies $\mathbf{x} = \mathbf{0}$ so $\text{NS}(A) = \{\mathbf{0}\}$.

17. Suppose $\text{NS}(A) = \{\mathbf{0}\}$ and $\mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_k$ are linearly independent. Suppose also

$$\alpha_1 A\mathbf{x}_1 + \alpha_2 A\mathbf{x}_2 + \cdots + \alpha_k A\mathbf{x}_k = \mathbf{0},$$

then

$$\mathbf{0} = \alpha_1 A\mathbf{x}_1 + \cdots + \alpha_k A\mathbf{x}_k = A(\alpha_1 \mathbf{x}_1 + \cdots + \alpha_k \mathbf{x}_k)$$

so $\alpha_1 \mathbf{x}_1 + \cdots + \alpha_k \mathbf{x}_k \in \text{NS}(A) = \{\mathbf{0}\}$ and thus

$$\alpha_1 \mathbf{x}_1 + \cdots + \alpha_k \mathbf{x}_k = \mathbf{0}$$

But the \mathbf{x}_i 's are linearly independent so $\alpha_1 = \alpha_2 = \cdots = \alpha_k = 0$. but this is what we needed to see that $A\mathbf{x}_1, A\mathbf{x}_2, \dots, A\mathbf{x}_k$ is linearly independent.

Section 3.4

5.

(a) Let A be the matrix whose columns are the three vectors given

$$A = \begin{bmatrix} 2 & 3 & 2 \\ 1 & -1 & 6 \\ 3 & 4 & 4 \end{bmatrix}$$

The given vectors are linearly independent iff $\text{NS}(A) = \{\mathbf{0}\}$, since

$$\text{NS}(A) = \{\mathbf{0}\} \text{ iff } A\mathbf{x} = \mathbf{0} \text{ implies } \mathbf{x} = \mathbf{0},$$

but the right hand side here says precisely that the only linear combination of the columns that yields $\mathbf{0}$ is the trivial combination, that is all coefficients are 0.

$$\text{rref } A = \begin{bmatrix} 1 & 0 & 4 \\ 0 & 1 & -2 \\ 0 & 0 & 0 \end{bmatrix}$$

Clearly, this has a nontrivial null space, in fact,

$$\text{NS}(A) = \text{span}\{(-4, 2, 1)\}$$

So $-4\mathbf{x}_1 + 2\mathbf{x}_2 + \mathbf{x}_3 = \mathbf{0}$, where these were the given vectors. (Easy for the reader to check. Do it!)

(b) Clearly \mathbf{x}_1 and \mathbf{x}_2 are linearly independent, since there is no $r \in \mathbb{R}$ such that $r\mathbf{x}_1 = \mathbf{x}_2$.

(c) Let $S = \text{span}\{\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3\}$, then (a) and (b) together show $2 \leq \dim(S) < 3$ so $\dim(S) = 2$.

(d) A 2-dimensional subspace of \mathbb{R}^3 is a plane.

alternate solution

$$\begin{bmatrix} 3 & -3 & -6 \\ -2 & 2 & 4 \\ 3 & 4 & 4 \end{bmatrix} \sim \begin{bmatrix} 1 & -1 & -2 \\ 0 & 0 & 0 \\ 0 & 7 & 10 \end{bmatrix} \sim \begin{bmatrix} 1 & -1 & -2 \\ 0 & 7 & 10 \\ 0 & 0 & 0 \end{bmatrix}$$

So a basis for $V = \text{span}\{\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3\}$ is given by $\{\mathbf{x}_2, \mathbf{x}_3\}$. So $\dim(V) = 2$ and V is a plane in \mathbb{R}^3 .

$$7. (a+b, a-b+2c, b, c) = a(1, 1, 0, 0) + b(1, -1, 1, 0) + c(0, 2, 0, 1)$$

It is easy to see that $\{(1, 1, 0, 0), (1, -1, 1, 0), (0, 2, 0, 1)\}$ is independent so $\dim(S) = 3$.

8.

(a) No, two non co-linear vectors span a plane not all of \mathbb{R}^3

(b) X must be linearly independent. We can be more specific here. If A has columns $\mathbf{x}_1 = (1, 1, 1)$, $\mathbf{x}_2 = (3, -1, 4)$, and $\mathbf{x}_3 = (a_1, a_2, a_3)$, then X is linearly independent iff any of the following hold

- $\text{NS}(A) = \{\mathbf{0}\}$
- $\det(A) = 0$
- $\text{rref}(A) = I_3$

Any one of these can be used to characterize the x_3 that are allowed, but geometrically we know that the set of these vectors is ALL vectors not in the plane spanned by \mathbf{x}_1 and \mathbf{x}_2 .

(c) Any vector not in the plane spanned by \mathbf{x}, \mathbf{x}_2 will work, say $\mathbf{x}_3 = (1, 0, 0)^T$

$$13. \cos(2x) = 2\cos^2(x) - 1, \text{ so } \dim(\text{span}\{\cos(2x), \cos^2(x), 1\}) = 2.$$

Section 3.5

1. Find the transition matrix from the basis $\mathcal{U} = \{\mathbf{u}_1, \mathbf{u}_2\}$ to the standard basis. This I would also denote $[\text{id}]_{\mathcal{U}, \mathcal{E}}$, where $\text{id} : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ is just the identity transformation.

(a) $U = [\mathbf{u}_1 \quad \mathbf{u}_2] = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$

(b) $U = [\mathbf{u}_1 \quad \mathbf{u}_2] = \begin{bmatrix} 1 & 2 \\ 2 & 5 \end{bmatrix}$

(c) $U = [\mathbf{u}_1 \quad \mathbf{u}_2] = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$

2. This is just the opposite of (1), find the transition matrix from the standard basis to the basis $\mathcal{U} = \{\mathbf{u}_1, \mathbf{u}_2\}$, that is find $[\text{id}]_{\mathcal{E}, \mathcal{B}}$.

Letting U be the matrix from (1), here the matrix we desire is U^{-1} , so

(a) $U^{-1} = [\mathbf{u}_1 \quad \mathbf{u}_2]^{-1} = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}^{-1} = \frac{1}{2} \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}$

(b) $U^{-1} = [\mathbf{u}_1 \quad \mathbf{u}_2]^{-1} = \begin{bmatrix} 1 & 2 \\ 2 & 5 \end{bmatrix}^{-1} = \begin{bmatrix} 5 & -2 \\ -2 & 1 \end{bmatrix}$

(c) $U^{-1} = [\mathbf{u}_1 \quad \mathbf{u}_2]^{-1} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}^{-1} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$

3.

(a) The transition matrix for $\mathcal{V} = \{\mathbf{v}_1, \mathbf{v}_2\} \rightarrow \{\mathbf{e}_1, \mathbf{e}_2\}$ is $V = \begin{bmatrix} 3 & 4 \\ 2 & 3 \end{bmatrix}$. So the transformation matrix from $\mathcal{V} = \{\mathbf{v}_1, \mathbf{v}_2\} \rightarrow \mathcal{U} = \{\mathbf{u}_1, \mathbf{u}_2\}$ is $U^{-1}V$, where U is as in 1.

(a) $U^{-1}V = \begin{bmatrix} 2.5 & 3.5 \\ -0.5 & -0.5 \end{bmatrix} = \frac{1}{2} \begin{bmatrix} 5 & 7 \\ -1 & -1 \end{bmatrix}$

(b) $U^{-1}V = \begin{bmatrix} 11 & 14 \\ -4 & -5 \end{bmatrix}$

(c) $U^{-1}V = \begin{bmatrix} 2 & 3 \\ 3 & 4 \end{bmatrix}$

6. Let $\mathcal{U} = \{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\} = \{(1, 1, 1), (1, 2, 2), (1, 3, 4)\}$ and $\mathcal{V} = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\} = \{(4, 6, 7), (0, 1, 1), (0, 1, 2)\}$.

(a) Find transition matrix from \mathcal{V} to \mathcal{U} .

This is

$$[\text{id}]_{\mathcal{V}, \mathcal{U}} = [\text{id} \circ \text{id}]_{\mathcal{V}, \mathcal{U}} = [\text{id}]_{\mathcal{E}, \mathcal{U}} [\text{id}]_{\mathcal{V}, \mathcal{E}} = U^{-1}V = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 2 \\ 1 & 3 & 4 \end{bmatrix}^{-1} \begin{bmatrix} 4 & 0 & 0 \\ 6 & 1 & 1 \\ 7 & 1 & 2 \end{bmatrix} = \begin{bmatrix} 1 & -1 & -2 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$$

(b) Find the \mathcal{U} representation of $\mathbf{v} = 2\mathbf{v}_1 + 3\mathbf{v}_2 - 4\mathbf{v}_3$.

We see $[\mathbf{v}]_{\mathcal{V}} = \begin{bmatrix} 2 \\ 3 \\ -4 \end{bmatrix}$ and

$$[\mathbf{v}]_{\mathcal{U}} = [\text{id}]_{\mathcal{V}, \mathcal{U}} [\mathbf{v}]_{\mathcal{V}} = \begin{bmatrix} 1 & -1 & -2 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 2 \\ 3 \\ -4 \end{bmatrix} = \begin{bmatrix} 7 \\ 5 \\ -2 \end{bmatrix}$$

So $\mathbf{v} = 7\mathbf{u}_2 + 5\mathbf{u}_2 - 2\mathbf{u}_3$.

You should check this:

$$\begin{aligned} 2\mathbf{v}_1 + 3\mathbf{v}_2 - 4\mathbf{v}_3 &= \begin{bmatrix} 4 & 0 & 0 \\ 6 & 1 & 1 \\ 7 & 1 & 2 \end{bmatrix} \begin{bmatrix} 2 \\ 3 \\ -4 \end{bmatrix} = \begin{bmatrix} 8 \\ 11 \\ 9 \end{bmatrix} \\ 7\mathbf{u}_2 + 5\mathbf{u}_2 - 2\mathbf{u}_3 &= \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 2 \\ 1 & 3 & 4 \end{bmatrix} \begin{bmatrix} 7 \\ 5 \\ -2 \end{bmatrix} = \begin{bmatrix} 8 \\ 11 \\ 9 \end{bmatrix} \end{aligned}$$

10. Find transition matrix from the basis $\mathcal{B} = \{1, x, x^2\}$ for \mathbb{P}_3 to $\mathcal{C} = \{1, 1+x, 1+x+x^2\}$. The transformation matrix from \mathcal{C} to \mathcal{B} is easy:

$$[1]_{\mathcal{C}} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \quad [1+x]_{\mathcal{C}} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \quad [1+x+x^2]_{\mathcal{C}} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

So we have

$$[\text{id}]_{\mathcal{C}, \mathcal{B}} = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$

The matrix in the other direction, from \mathcal{B} to \mathcal{C} is just the inverse

$$[\text{id}]_{\mathcal{B}, \mathcal{C}} = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix}$$

As an application, write $p = 3 - 2x + 4x^2$ in the \mathcal{C} basis. $[p]_{\mathcal{B}} = (3, -2, 4)$ so

$$[p]_{\mathcal{C}} = \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 3 \\ -2 \\ 4 \end{bmatrix} = \begin{bmatrix} 5 \\ -6 \\ 4 \end{bmatrix}$$

hence

$$3 - 2x + 4x^2 = 5 - 6(1+x) + 4(1+x+x^2)$$

Section 3.6

1. Let A denote the matrix given

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 1 & 4 \\ 4 & 7 & 8 \end{bmatrix}$$

(a) $\text{rref}(A) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ so $\text{rank}(A) = 3$ and thus $\text{RS}(A) = \mathbb{R}^3$, $\text{CS}(A) = \mathbb{R}^3$, and $\text{NS}(A) = \{\mathbf{0}\}$. So we can take the standard basis for \mathbb{R}^3 as a basis for $\text{CS}(A)$ and $\text{NS}(A)$.

(b) $\text{rref}(A) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & -2/7 \\ 0 & 0 & 1 & 10/7 \end{bmatrix}$

So setting $x_4 = t$ we get $A\mathbf{x} = \mathbf{0}$ at $\mathbf{x} = t \begin{bmatrix} 0 \\ 2/7 \\ -10/7 \\ 1 \end{bmatrix}$ and so

$$\text{RS}(A) = \text{span}\{(1, 0, 0, 0)^T, (0, 1, 0, -2/7)^T, (0, 0, 1, 10/7)^T\}$$

$$\text{CS}(A) = \text{span}\{(-3, 1, 3)^T, (1, 2, 4)^T, (3, -1, 5)^T\}$$

$$\text{NS}(A) = \text{span}\{(0, 2/7, -10/7, 1)^T\}$$

6. How many solutions to $A\mathbf{x} = \mathbf{b}$ will there be if $\mathbf{b} \in \text{CS}(A)$ and the columns of A are dependent?

Since $\mathbf{b} \in \text{CS}(A)$ we know that there is an \mathbf{x}_0 so that $A\mathbf{x}_0 = \mathbf{b}$, so there is at least one solution. If $\mathbf{z} \in \text{NS}(A)$, then $A(\mathbf{x}_0 + \mathbf{z}) = A\mathbf{x}_0 + A\mathbf{z} = \mathbf{b} + \mathbf{0} = \mathbf{b}$. Conversely, if $A\mathbf{x}' = \mathbf{b}$, then $A(\mathbf{x}_0 - \mathbf{x}') = \mathbf{0}$, so $\mathbf{x}_0 - \mathbf{x}' \in \text{NS}(A)$ and clearly, $\mathbf{x}' = \mathbf{x}_0 - (\mathbf{x}_0 - \mathbf{x}')$, so $\mathbf{x}' = \mathbf{x}_0 + \mathbf{z}$ for some $\mathbf{z} \in \text{NS}(A)$. Thus the set of all solutions to $A\mathbf{x} = \mathbf{b}$ is given by $\mathbf{x}_0 + \text{NS}(A)$.

Since the rows of A are independent, $\text{NS}(A) \neq \{\mathbf{0}\}$ and hence there are infinitely many solutions to $A\mathbf{x} = \mathbf{b}$.

9.

(a) If A is 6×5 and $\dim(\text{NS}(A)) = 2$, then since $\mathbb{R}^5 = \text{RS}(A) \oplus \text{NS}(A)$ we have $5 = \dim \text{RS}(A) + 2$ so $\dim \text{RS}(A) = 3$.

(b) If B is 6×5 , then as above $5 = \dim \text{NS}(A) + \dim \text{RS}(A) = \dim \text{NS}(A) + \text{rank}(A) = \dim \text{NS}(A) + 4$, so $\dim \text{NS}(A) = 1$.

14. From U read off the solutions to $A\mathbf{x} = \mathbf{0}$, i.e. $\text{NS}(A) = \text{NS}(U)$ as

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} -2s - t \\ -s - 4t \\ s \\ t \end{bmatrix} = s \begin{bmatrix} -2 \\ -1 \\ 1 \\ 0 \end{bmatrix} + t \begin{bmatrix} -1 \\ -4 \\ 0 \\ 1 \end{bmatrix} = s \cdot \mathbf{u}_1 + t \cdot \mathbf{u}_2$$

Now we know $A(s\mathbf{u}_1+t\mathbf{u}_2) = \mathbf{0}$ so in particular, $A\mathbf{u}_1 = A\mathbf{u}_2 = \mathbf{0}$ and if $A = \begin{bmatrix} a_1 & \mathbf{a}_2 & \mathbf{a}_3 & \mathbf{a}_4 \end{bmatrix}$, then

$$A\mathbf{u}_1 = -2\mathbf{a}_1 - \mathbf{a}_2 + \mathbf{a}_3 = \mathbf{0}$$

$$A\mathbf{u}_2 = -\mathbf{a}_1 - 4\mathbf{a}_2 + \mathbf{a}_4 = \mathbf{0}$$

so

$$\mathbf{a}_3 = 2\mathbf{a}_1 + \mathbf{a}_2$$

$$\mathbf{a}_4 = \mathbf{a}_1 + 4\mathbf{a}_2$$