Name: ____________________________ Exam 2 - MAT345

Part I: True/False

Each problem is points for a total of 60 points. (10 problems 6 points each; 3 points for correct T/F; 3 points for correct explanation.)

Problem 1. Decide if each of the following is true or false. For each, provide an example or counter-example or an argument as required. You may refer to a theorem if that applies.

a) ______ Let V be a vector space and $S = \{\boldsymbol{v}_1, \ldots, \boldsymbol{v}_n\}$ such that $\operatorname{span}(S) = V$. S can be extended to a basis for V.

b) ______ Suppose $\mathcal{B}$ is a basis for V, then for any vector $\boldsymbol{v} \notin \mathcal{B}$, $\mathcal{B} \cup \{\boldsymbol{v}\}$ is dependent.

c) ______ $U = \{(x, y) \in \mathbb{R}^2 \mid x \text{ and } y \text{ have the same sign}\}$ is a subspace of $\mathbb{R}^2$.

d) ______ The map $L : \mathbb{R}^2 \to \mathbb{R}$ given by $L(x_1, x_2) = |x_1 - x_2|$ is linear.

e) ______ The evaluation map at c, $e_c : P \to \mathbb{R}$ given by $e_c(p(x)) = p(c)$ is linear where P is the vector space of all polynomials with real coefficients.

f) ______ There are subspaces $V_0 = \mathbb{R}^4 \supsetneqq V_1 \supsetneqq V_2 \supsetneqq V_3 \supsetneqq V_4 \supsetneqq V_5 = \{\mathbf{0}\}$ where each V_i is a proper subspace of V_{i-1}.

g) ______ Given any three linearly independent vectors $\{\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3\}$,from $\mathbb{R}^3$ and any three vectors $\{p_1(x), p_2(x), p_3(x)\}$ from P_6 (polynomials of degree 6), there is a unique linear function $L : \mathbb{R}^3 \to P_6$ satisfying $L(\boldsymbol{v}_i) = p_i(x)$, for $i = 1, 2, 3$.

h) ______ Suppose $L : \mathbb{R}^{2\times 3} \to \mathbb{R}^4$ is linear and onto, that is, $\operatorname{Img}(L) = \mathbb{R}^4$. Then $\dim(\ker(L)) = 2$.

Recall $\mathbb{R}^{2\times 3}$ is the space of 2×3 matrices.

i) ______ Let $\mathcal{B} = \{\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3\}$ be a basis for V and suppose $\boldsymbol{v} = \alpha_1\boldsymbol{v}_1 + \alpha_2\boldsymbol{v}_2 + \alpha_3\boldsymbol{v}_3$. Then

$$[\boldsymbol{v}]_{\mathcal{B}} = \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{bmatrix}$$

j) ______ $L : \mathbb{R}^{3\times 3} \to \mathbb{R}^{3\times 3}$ is given by $L(A) = BA$ for a 3×3 matrix B. If $\mathcal{B}$ is a basis for $\mathbb{R}^{3\times 3}$, then $[L]_{\mathcal{B}} = B$.

Part II: Computational (80 points)

Show all computations so that you make clear what your thought processes are.

Problem 2 (20 pts)**.** Consider A given by

$$A = \begin{bmatrix} 1 & 2 & -4 & 3 & 2 \\ -3 & -6 & 14 & -13 & -3 \\ 0 & 0 & 3 & -6 & 4 \\ 2 & 4 & -7 & 4 & 5 \end{bmatrix}$$

Find a basis for each of $\mathrm{NS}(A)$, $\mathrm{CS}(A)$, and $\mathrm{RS}(A)$.

Hint: This should require exactly one (not two or three) reduction of a matrix to echelon form.

Workspace

Problem 3 (20 pts)**.** Let $L : \mathbb{R}^{3\times 2} \to \mathbb{R}^{2\times 2}$ given by $L(A) = DA$ where

$$D = \begin{bmatrix} 1 & 2 & 0 \\ 2 & -1 & 1 \end{bmatrix}$$

a) (8 points) Show that L is a linear map.

b) (12 points) Give the matrix $[L]_{\mathcal{B},\mathcal{C}}$ in terms of the basis $\mathcal{B}$ for $\mathbb{R}^{3\times 2}$ and $\mathcal{C}$ for $\mathbb{R}^{2\times 2}$ given by:

$$\mathcal{B} = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}$$
$$\mathcal{C} = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}$$

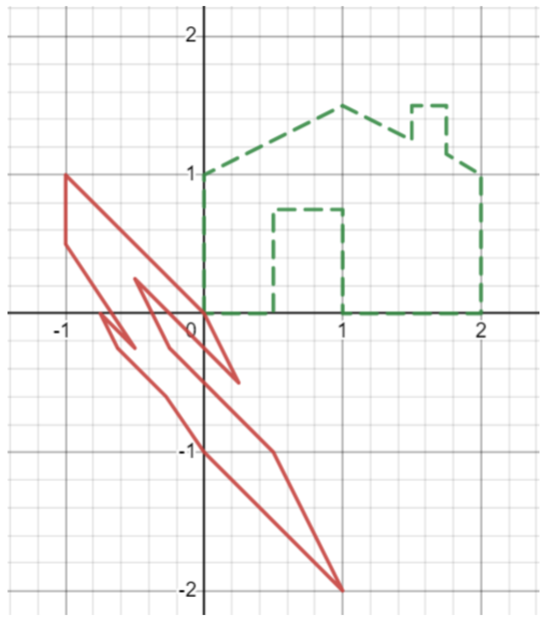
Problem 4 (20 pts)**.** Consider the map $L : \mathbb{R}^3 \to \mathbb{R}^3$ that maps any point in $\mathbb{R}^3$ onto the plane spanned by $(1, -2, 1)$ and $(2, 0, -2)$ in such a way that points in the plane are fixed and which maps $(1, 1, 1)$ to $(0, 0, 0)$.

a) (7 points) Find $[L]_{\mathcal{B}}$ for $\mathcal{B} = \{(1, -2, 1), (2, 0, -2), (1, 1, 1)\}$.

b) (5 points) Find the change of basis matrix $[\mathrm{id}]_{\mathcal{B},E}$ (from the basis $\mathcal{B}$ to the standard basis.)

c) (8 points) Find the matrix for L wrt the standard basis using the first two parts. (Give me the decomposition: $[\mathrm{id}]_{\mathcal{B},\mathcal{E}}[L]_{\mathcal{B}}[\mathrm{id}]_{\mathcal{E},\mathcal{B}}$ as well as the resulting matrix.

Problem 5 (20 pts)**.** The green (dashed) house has been transformed to the red (solid) house by a linear transformation $L : \mathbb{R}^2 \to \mathbb{R}^2$.



Desmos

a) What is $L(\boldsymbol{e}_1)$?

b) What is $L(\boldsymbol{e}_2)$?

c) What is $[L]$?

Part III: Theory and Proofs (30 points; 10 points each)

Choose three of the five options. If you try all five, I will grade the first three, not the best three. You must decide what should be graded.

This part is take-home. You should complete this work on your own without consulting websites, friends, the Math Center, etc.

Problem 6 (10 points)**.** Suppose S is an independent set of vectors from a vector space V, then

$$S \cup \{\boldsymbol{v}\} \text{ is dependent} \iff \boldsymbol{v} \in \operatorname{span}(S).$$

Problem 7 (10 points)**.** Show that if $L : V \to W$ is linear and $\{L(\boldsymbol{v}_1), L(\boldsymbol{v}_2), L(\boldsymbol{v}_3)\}$ is linearly independent, then $\{\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3\}$ is linearly independent.

Problem 8 (10 points)**.** Suppose $A = \begin{bmatrix} \boldsymbol{a}_1 \, \boldsymbol{a}_2 \, \boldsymbol{a}_3 \, \boldsymbol{a}_4 \, \boldsymbol{a}_5 \end{bmatrix}$ is a 4×5 matrix and

$$\mathrm{NS}(A) = \mathrm{span}\{(-2, 1, 0, 0, 0), (5, 0, 2, 1, 0)\}$$

Find $\mathrm{rref}(A)$ and explain how you know that what you have found is $\mathrm{rref}(A)$.

Problem 9 (10 points)**.** Suppose A is a 5×5 matrix and $A^n = O$ for some n, then $A^5 = O$.

Problem 10 (10 points)**.** For A and B are $n \times n$ matrices. Show that

$$AB \text{ is invertible} \iff \text{both } A \text{ and } B \text{ are invertible}$$