

Part I: True/False

Each problem is points for a total of 60 points. (10 problems 6 points each; 3 points for correct T/F; 3 points for correct explanation.)

Problem 1. Decide if each of the following is true or false. For each, provide an example or counter-example or an argument as required. You may refer to a theorem if that applies.

- a) False Let V be a vector space and $S = \{\mathbf{v}_1, \dots, \mathbf{v}_n\}$ such that $\text{span}(S) = V$. S can be extended to a basis for V .

Any spanning set can be restricted to a basis. Any expansion $T \supsetneq S$ would definitely not be linearly independent and hence not a basis.

- b) True Suppose \mathcal{B} is a basis for V , then for any vector $\mathbf{v} \notin \mathcal{B}$, $\mathcal{B} \cup \{\mathbf{v}\}$ is dependent.

$\mathbf{v} \in \text{span}(\mathcal{B})$ since \mathcal{B} is a basis, but then $\mathbf{v} = \alpha_1 \mathbf{v}_1 + \dots + \alpha_k \mathbf{v}_k$ for some scalars $\alpha_i \neq 0$ and $\mathbf{v}_i \in \mathcal{B}$. But then

$$\mathbf{v} - (\alpha_1 \mathbf{v}_1 + \dots + \alpha_k \mathbf{v}_k) = \mathbf{0}$$

is a non-trivial linear combination of vectors, so $\mathcal{B} \cup \{\mathbf{v}\}$ is dependent.

- c) False $U = \{(x, y) \in \mathbb{R}^2 \mid x \text{ and } y \text{ have the same sign}\}$ is a subspace of \mathbb{R}^2 .

Not closed under addition: $(2, 2) + (-3, -1) = (-1, 1)$.

- d) False The map $L : \mathbb{R}^2 \rightarrow \mathbb{R}$ given by $L(x_1, x_2) = |x_1 - x_2|$ is linear.

There are many ways to see that this is not linear. one is just that $L((-1)(x_1, x_2)) = L(-x_1, -x_2) = |(-x_1) - (-x_2)| = |x_2 - x_1| = x_1 - x_2 \neq (-1)L(x_1, x_2)$.

- e) True The evaluation map at c , $e_c : P \rightarrow \mathbb{R}$ given by $e_c(p(x)) = p(c)$ is linear where P is the vector space of all polynomials with real coefficients.

This is true. Clearly

$$e_c(\alpha_1 p_1(x) + \alpha_2 p_2(x)) = \alpha_1 p_1(c) + \alpha_2 p_2(c) = \alpha_1 e_c(p_1(x)) + \alpha_2 e_c(p_2(x)).$$

- f) False There are subspaces $V_0 = \mathbb{R}^4 \supsetneq V_1 \supsetneq V_2 \supsetneq V_3 \supsetneq V_4 \supsetneq V_5 = \{\mathbf{0}\}$ where each V_i is a proper subspace of V_{i-1} .

Since we know $\dim(V_0) = 4 > \dim(V_1) > \dim(V_2) > \dim(V_3) > \dim(V_4) > \dim(V_5) = 0$, which is impossible.

- g) True Given any three linearly independent vectors $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$, from \mathbb{R}^3 and any three vectors $\{p_1(x), p_2(x), p_3(x)\}$ from P_6 (polynomials of degree 6), there is a unique linear function $L : \mathbb{R}^3 \rightarrow P_6$ satisfying $L(\mathbf{v}_i) = p_i(x)$, for $i = 1, 2, 3$.

This is true, any linear map $L : V \rightarrow W$ is completely determined by where the basis vectors are mapped

- h) True Suppose $L : \mathbb{R}^{2 \times 3} \rightarrow \mathbb{R}^4$ is linear and onto, that is, $\text{Img}(L) = \mathbb{R}^4$. Then $\dim(\ker(L)) = 2$.

Recall $\mathbb{R}^{2 \times 3}$ is the space of 2×3 matrices.

$\dim(\mathbb{R}^{2 \times 3}) = 6$ and so $\dim(\ker(L)) + \dim(\text{Img}(L)) = \dim(\ker(L)) + 4 = 6$.

- i) True Let $\mathcal{B} = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ be a basis for V and suppose $\mathbf{v} = \alpha_1 \mathbf{v}_1 + \alpha_2 \mathbf{v}_2 + \alpha_3 \mathbf{v}_3$. Then

$$[\mathbf{v}]_{\mathcal{B}} = \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{bmatrix}$$

This is trivially true as this is the definition of $[\mathbf{v}]_{\mathcal{B}}$.

- j) False $L : \mathbb{R}^{3 \times 3} \rightarrow \mathbb{R}^{3 \times 3}$ is given by $L(A) = BA$ for a 3×3 matrix B . If \mathcal{B} is a basis for $\mathbb{R}^{3 \times 3}$, then $[L]_{\mathcal{B}} = B$.

$[L]_{\mathcal{B}}$ acts on representations of matrices wrt \mathcal{B} , it is a 9×9 matrix, not a 3×3 matrix.

Part II: Computational (80 points)

Show all computations so that you make clear what your thought processes are.

Problem 2 (20 pts). Consider A given by

$$A = \begin{bmatrix} 1 & 2 & -4 & 3 & 2 \\ -3 & -6 & 14 & -13 & -3 \\ 0 & 0 & 3 & -6 & 4 \\ 2 & 4 & -7 & 4 & 5 \end{bmatrix}$$

Find a basis for each of $\text{NS}(A)$, $\text{CS}(A)$, and $\text{RS}(A)$.

Hint: This should require exactly one (not two or three) reduction of a matrix to echelon form.

Gauss-Jordan elimination to get echelon form:

$$\begin{aligned} \begin{bmatrix} 1 & 2 & -4 & 3 & 2 \\ -3 & -6 & 14 & -13 & -3 \\ 0 & 0 & 2 & -4 & 3 \\ 2 & 4 & -7 & 4 & 5 \end{bmatrix} &\xrightarrow{\substack{R_2+3R_1 \rightarrow R_2 \\ R_4-2R_1 \rightarrow R_4}} \begin{bmatrix} 1 & 2 & -4 & 3 & 2 \\ 0 & 0 & 2 & -4 & 3 \\ 0 & 0 & 2 & -4 & 3 \\ 0 & 0 & 1 & -2 & 1 \end{bmatrix} \\ &\xrightarrow{R_2 \leftrightarrow R_4} \begin{bmatrix} 1 & 2 & -4 & 3 & 2 \\ 0 & 0 & 1 & -2 & 1 \\ 0 & 0 & 2 & -4 & 3 \\ 0 & 0 & 2 & -4 & 3 \end{bmatrix} \\ &\xrightarrow{\substack{R_3-3R_2 \rightarrow R_3 \\ R_4-2R_2 \rightarrow R_4}} \begin{bmatrix} 1 & 2 & -4 & 3 & 2 \\ 0 & 0 & 1 & -2 & 1 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \\ &\xrightarrow{R_4-R_3 \rightarrow R_4} \begin{bmatrix} 1 & 2 & -4 & 3 & 2 \\ 0 & 0 & 1 & -2 & 1 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \end{aligned}$$

From this we know:

$$\text{CS}(A) = \text{span}\{(1, -3, 0, 2), (-4, 14, 3, -7), (2, -3, 4, 5)\}$$

$$\text{RS}(A) = \text{span}\{(1, 2, -4, 3, 2), (0, 0, 1, -2, 1), (0, 0, 0, 0, 1)\}$$

Note: $\text{RS}(A)$ is not the span of the first three rows of A .

To find a basis for $\text{NS}(A)$ we are looking for solutions to $Ax = 0$. First we

have back-substitution: x_2 and x_4 are free, let $x_2 = s$ and $x_4 = t$, then

$$\begin{aligned} x_5 &= 0 \\ x_4 &= t \\ x_3 - 2t &= 0 \rightarrow x_3 = 2t \\ x_2 &= s \\ x_1 + 2s - 4(2t) + 3t &= 0 \rightarrow x_1 = -2s + 5t \end{aligned}$$

Any vector x satisfying, $Ax = 0$ can be written as:

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} -2s + 5t \\ s \\ 2t \\ t \\ 0 \end{bmatrix} = s \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + t \begin{bmatrix} 5 \\ 0 \\ 2 \\ 1 \\ 0 \end{bmatrix}$$

So $\{(-2, 1, 0, 0, 0), (5, 0, 2, 1, 0)\}$ is a basis for $\text{NS}(A)$, that is,

$$\text{NS}(A) = \text{span}\{(-2, 1, 0, 0, 0), (5, 0, 2, 1, 0)\}$$

Problem 3 (20 pts). Let $L : \mathbb{R}^{3 \times 2} \rightarrow \mathbb{R}^{2 \times 2}$ given by $L(A) = DA$ where

$$D = \begin{bmatrix} 1 & 2 & 0 \\ 2 & -1 & 1 \end{bmatrix}$$

- a) (8 points) Show that L is a linear map.
- b) (12 points) Give the matrix $[L]_{\mathcal{B}, \mathcal{C}}$ in terms of the basis \mathcal{B} for $\mathbb{R}^{3 \times 2}$ and \mathcal{C} for $\mathbb{R}^{2 \times 2}$ given by:

$$\begin{aligned} \mathcal{B} &= \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix} \right\} \\ \mathcal{C} &= \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\} \end{aligned}$$

$$\begin{aligned} [L]_{\mathcal{B}, \mathcal{C}} &= \left[[DB^{1,1}]_{\mathcal{C}} [DB^{1,2}]_{\mathcal{C}} \cdots [DB^{3,2}]_{\mathcal{C}} \right] \\ &= \left[\begin{bmatrix} 1 & 0 \\ 2 & 0 \end{bmatrix}_{\mathcal{C}} \begin{bmatrix} 0 & 1 \\ 0 & 2 \end{bmatrix}_{\mathcal{C}} \begin{bmatrix} -2 & 0 \\ -1 & 0 \end{bmatrix}_{\mathcal{C}} \begin{bmatrix} 0 & -2 \\ 0 & -1 \end{bmatrix}_{\mathcal{C}} \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}_{\mathcal{C}} \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}_{\mathcal{C}} \right] \\ &= \begin{bmatrix} 1 & 0 & 2 & 0 & 0 & 0 \\ 0 & 1 & 0 & 2 & 0 & 0 \\ 2 & 0 & -1 & 0 & 1 & 0 \\ 0 & 2 & 0 & -1 & 0 & 1 \end{bmatrix} \end{aligned}$$

Problem 4 (20 pts). Consider the map $L : \mathbb{R}^3 \rightarrow \mathbb{R}^3$ that maps any point in \mathbb{R}^3 onto the plane spanned by $(1, -2, 1)$ and $(2, 0, -2)$ in such a way that points in the plane are fixed and which maps $(1, 1, 1)$ to $(0, 0, 0)$.

- a) (7 points) Find $[L]_{\mathcal{B}}$ for $\mathcal{B} = \{(1, -2, 1), (2, 0, -2), (1, 1, 1)\}$.
- b) (5 points) Find the change of basis matrix $[\text{id}]_{\mathcal{B}, \mathcal{E}}$ (from the basis \mathcal{B} to the standard basis.)
- c) (8 points) Find the matrix for L wrt the standard basis using the first two parts. (Give me the decomposition: $[\text{id}]_{\mathcal{B}, \mathcal{E}}[L]_{\mathcal{B}}[\text{id}]_{\mathcal{E}, \mathcal{B}}$ as well as the resulting matrix.

$L((1, -2, 1)) = (1, -2, 1)$ so $[L(1, -2, 1)]_{\mathcal{B}} = (1, 0, 0)$, similarly $[L(2, 0, -2)]_{\mathcal{B}} = (0, 1, 0)$, and $[L(1, 1, 1)]_{\mathcal{B}} = (0, 0, 0)$. So

$$[L]_{\mathcal{B}} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

The change of basis matrix from \mathcal{B} to \mathcal{E} is

$$[\text{id}]_{\mathcal{B}, \mathcal{E}} = B = \begin{bmatrix} 1 & 2 & 1 \\ -2 & 0 & 1 \\ 1 & -2 & 1 \end{bmatrix}$$

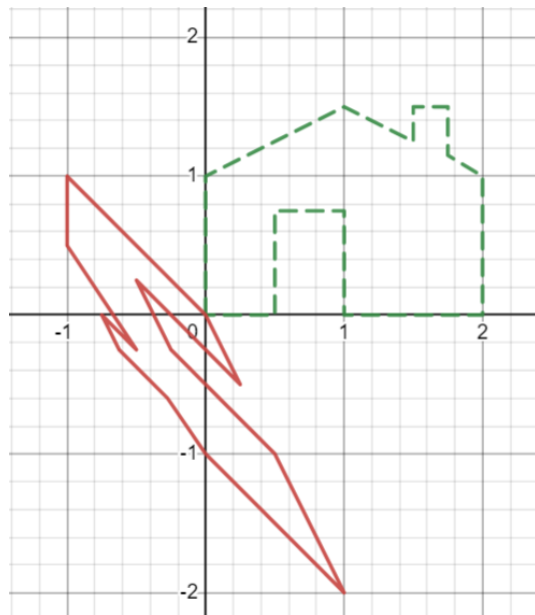
and

$$[\text{id}]_{\mathcal{E}, \mathcal{B}} = B^{-1}$$

So

$$\begin{aligned} [L] &= B[L]_{\mathcal{B}}B^{-1} = \begin{bmatrix} 1 & 2 & 1 \\ -2 & 0 & 1 \\ 1 & -2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1/6 & -1/3 & 1/6 \\ 1/4 & 0 & -1/4 \\ 1/3 & 1/3 & 1/3 \end{bmatrix} \\ &= \begin{bmatrix} 2/3 & -1/3 & -1/3 \\ -1/3 & 2/3 & -1/3 \\ -1/3 & -1/3 & 2/3 \end{bmatrix} \end{aligned}$$

Problem 5 (20 pts). The green (dashed) house has been transformed to the red (solid) house by a linear transformation $L : \mathbb{R}^2 \rightarrow \mathbb{R}^2$.



Desmos

- a) What is $L(\mathbf{e}_1)$?
- b) What is $L(\mathbf{e}_2)$?
- c) What is $[L]$?

Clearly

$$L(\mathbf{e}_1) = \begin{bmatrix} 1/2 \\ -2 \end{bmatrix} \quad L(\mathbf{e}_2) = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

So

$$[L] = [L(\mathbf{e}_1) \ L(\mathbf{e}_2)] = \begin{bmatrix} 1/2 & -1 \\ -1 & 1 \end{bmatrix}$$

Part III: Theory and Proofs (30 points; 10 points each)

Choose three of the five options. If you try all five, I will grade the first three, not the best three. You must decide what should be graded.

This part is take-home. You should complete this work on your own without consulting websites, friends, the Math Center, etc.

Problem 6 (10 points). Suppose S is an independent set of vectors from a vector space V , then

$$S \cup \{\mathbf{v}\} \text{ is dependent} \iff \mathbf{v} \in \text{span}(S).$$

(\Leftarrow) $\mathbf{v} \in \text{span}(S)$ means that $\mathbf{v} = \alpha_1 \mathbf{v}_1 + \cdots + \alpha_k \mathbf{v}_k$ for some scalars α_i and $\mathbf{v}_i \in S$. Clearly then

$$\mathbf{v} - (\alpha_1 \mathbf{v}_1 + \cdots + \alpha_k \mathbf{v}_k) = \mathbf{0}$$

so $S \cup \{\mathbf{v}\}$ is dependent since we have written $\mathbf{0}$ as a non-trivial linear combination of vectors from $S \cup \{\mathbf{v}\}$.

(\Rightarrow) $S \cup \{\mathbf{v}\}$ is dependent so $\mathbf{v} = \alpha_1 \mathbf{v}_1 + \cdots + \alpha_k \mathbf{v}_k = \mathbf{0}$ for some scalars $\alpha_i \neq 0$ and $\mathbf{v}_i \in S \cup \{\mathbf{v}\}$. Since S is independent, it must be that \mathbf{v} is one of the \mathbf{v}_i 's. WLOG suppose $\mathbf{v} = \mathbf{v}_1$, then

$$\mathbf{v} = -\frac{1}{\alpha_1}(\alpha_2 \mathbf{v}_2 + \cdots + \alpha_k \mathbf{v}_k)$$

and so $\mathbf{v} \in \text{span}(S)$.

Problem 7 (10 points). Show that if $L : V \rightarrow W$ is linear and $\{L(\mathbf{v}_1), L(\mathbf{v}_2), L(\mathbf{v}_3)\}$ is linearly independent, then $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is linearly independent.

More generally, if $L : V \rightarrow W$ is linear, then the pre-image of S , $L^{-1}(S) = \{\mathbf{v} \mid L(\mathbf{v}) \in S\}$ is linearly independent for any linearly independent set S .

Let

$$\alpha_1 \mathbf{v}_1 + \alpha_2 \mathbf{v}_2 + \alpha_3 \mathbf{v}_3 = \mathbf{0},$$

then

$$L(\alpha_1 \mathbf{v}_1 + \alpha_2 \mathbf{v}_2 + \alpha_3 \mathbf{v}_3) = \alpha_1 L(\mathbf{v}_1) + \alpha_2 L(\mathbf{v}_2) + \alpha_3 L(\mathbf{v}_3) = L(\mathbf{0}) = \mathbf{0}$$

so by the independence of $\{L(\mathbf{v}_1), L(\mathbf{v}_2), L(\mathbf{v}_3)\}$ we have $\alpha_1 = \alpha_2 = \alpha_3 = 0$ and thus $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is linearly independent.

Problem 8 (10 points). Suppose $A = [\mathbf{a}_1 \ \mathbf{a}_2 \ \mathbf{a}_3 \ \mathbf{a}_4 \ \mathbf{a}_5]$ is a 4×5 matrix and

$$\text{NS}(A) = \text{span}\{(-2, 1, 0, 0, 0), (5, 0, 2, 1, 0)\}$$

Find $\text{rref}(A)$ and explain how you know that what you have found is $\text{rref}(A)$.

We know a typical element of $\text{NS}(A)$ is of the form $(x_1, x_2, x_3, x_4, x_5) = (-2s + 5t, s, 2t, t, 0)$ and since $A\mathbf{x} = \mathbf{0}$ can be written as a linear combination of columns of A we know

$$(-2s + 5t)\mathbf{a}_1 + s\mathbf{a}_2 + 2t\mathbf{a}_3 + t\mathbf{a}_4 + 0\mathbf{a}_5 = \mathbf{0}$$

Letting $s = 1$ and $t = 0$ we get $-2\mathbf{a}_1 + \mathbf{a}_2 = \mathbf{0}$ and letting $s = 0$ and $t = 1$ we get $5\mathbf{a}_1 + 2\mathbf{a}_3 + \mathbf{a}_4 = \mathbf{0}$. Thus we have

$$\mathbf{a}_2 = 2\mathbf{a}_1 \quad \text{and} \quad \mathbf{a}_4 = -5\mathbf{a}_1 - 2\mathbf{a}_3$$

Thus we get

$$[\mathbf{a}_1 \ \mathbf{a}_3 \ \mathbf{a}_5] \begin{bmatrix} 1 & 2 & 0 & -5 & 0 \\ 0 & 0 & 1 & -2 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} = [\mathbf{a}_1 \ \mathbf{a}_2 \ \mathbf{a}_3 \ \mathbf{a}_4 \ \mathbf{a}_5] = A$$

We know $\text{rank}(A) = 3 = 5 - \dim(\text{NS}(A))$ so $\{\mathbf{a}_1, \mathbf{a}_3, \mathbf{a}_5\}$ are linearly independent vectors in \mathbb{R}^4 . Let $\mathbf{b} \in \mathbb{R}^4$ be so that $\{\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3, \mathbf{b}\}$ is a basis and let $M = [\mathbf{a}_1 \ \mathbf{a}_3 \ \mathbf{a}_5 \ \mathbf{b}]$, then M is invertible and

$$M \begin{bmatrix} 1 & 2 & 0 & -5 & 0 \\ 0 & 0 & 1 & -2 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} = MR = A$$

So A is equivalent to R . But R is in RREF form so $R = \text{rref}(A)$, since there is only one RREF matrix equivalent to A .

Note: Recall A and B are equivalent if B can be formed from a sequence of elementary row operations applied to A ; equivalently, A and B are equivalent iff $B = MA$ for some invertible M . We know

$$A \text{ and } B \text{ are equivalent} \implies \text{NS}(A) = \text{NS}(B).$$

It turns out that for matrices of the same size

$$A \text{ is equivalent to } B \iff \text{NS}(A) = \text{NS}(B)$$

To see this it suffices to show that

$$\text{NS}(A) = \text{NS}(B) \implies \text{rref}(A) = \text{rref}(B).$$

The above basically does this argument by showing that $\text{rref}(A)$ can be computed from a basis for $\text{NS}(A)$.

Problem 9 (10 points). Suppose A is a 5×5 matrix and $A^n = O$ for some n , then $A^5 = O$.

There are several ways to proceed. Here is one. Note that $\text{NS}(A^{m+1}) \supseteq \text{NS}(A^m)$ for all m since $A^m \mathbf{x} = \mathbf{0} \implies A^{m+1} \mathbf{x} = A(A^m \mathbf{x}) = \mathbf{0}$.

If $\text{NS}(A^{m+1}) = \text{NS}(A^m)$, then $\text{NS}(A^{m+k}) = \text{NS}(A^m)$ for all k . To see this, suppose $\text{NS}(A^{m+k}) = \text{NS}(A^m)$, then

$$A^{m+k+1} \mathbf{x} = \mathbf{0} \iff A^{m+k}(A\mathbf{x}) = \mathbf{0} \quad (\text{by assumption})$$

$$\iff A^m(A\mathbf{x}) = \mathbf{0} \quad (1)$$

$$\iff A^{m+1} \mathbf{x} = \mathbf{0} \quad (2)$$

$$\iff A^m \mathbf{x} = \mathbf{0} \quad (3)$$

This means that we have the following situation

$$\text{NS}(A) \subsetneq \text{NS}(A^2) \subsetneq \dots \text{NS}(A^{m-1}) \subsetneq \text{NS}(A^m) = \text{NS}(A^n) \text{ for all } n \geq m$$

Since $0 < \dim(\text{NS}(A)) < \dim(\text{NS}(A^2)) < \dots < \dim(\text{NS}(A^m)) \leq 5$ we know $m \leq 5$.

If $A^n = O$ for any n , then $\text{NS}(A^n) = \mathbb{R}^5$. But the first place where $\text{NS}(A^n) = \mathbb{R}^5$ will be for $n \leq 5$ and so $A^5 = O$.

Problem 10 (10 points). For A and B are $n \times n$ matrices. Show that

$$AB \text{ is invertible} \iff \text{both } A \text{ and } B \text{ are invertible}$$

(\Leftarrow) **case:** If A and B are invertible, then AB is invertible, since $(AB)^{-1} = B^{-1}A^{-1}$.

(\Rightarrow) **case (Proof 1 using NS)** If B is not invertible, then $\text{NS}(B) \neq \{\mathbf{0}\}$, but $B\mathbf{x} = \mathbf{0} \implies A(B\mathbf{x}) = (AB)\mathbf{x} = \mathbf{0}$, so $\text{NS}(AB) \neq \{\mathbf{0}\}$ and hence AB is not invertible.

If B is invertible, but A is not, then again let $\mathbf{x} \in \text{NS}(A)$, since B is invertible, $\mathbf{x} = B\mathbf{y}$ for some \mathbf{y} , in fact, $\mathbf{y} = B^{-1}\mathbf{x}$. But then, $A(B\mathbf{y}) = (AB)\mathbf{y} = \mathbf{0}$ and so $\text{NS}(AB) \neq \{\mathbf{0}\}$, so again AB is not invertible.

So if either A or B is not invertible, then neither is AB , and hence if AB is invertible, then both A and B must be invertible.

(\Rightarrow) **case (Proof 2 using det)** Suppose AB is invertible, then $0 \neq \det(AB) = \det(A)\det(B)$ so $\det(A) \neq 0 \neq \det(B)$ and so A and B are invertible.

(\Rightarrow) **case (Proof 3 using algebra.)** Suppose AB is invertible, then $A(B(AB)^{-1}) = (AB)(AB)^{-1} = I$ so $A^{-1} = B(AB)^{-1}$ and $B^{-1} = (AB)^{-1}A$ for a similar reason.

Note: This actually uses that $E = F^{-1}$ iff $EF = I$ **or** $FE = I$, whereas the actual definition has "**and**" not "**or**." To prove this, one usually uses one of the above arguments.