Math 571 - Homework 5 (05.22)

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Notation: For $f : X \to Y$ and $E \subseteq X$ set $f(X) = \{f(e) \mid e \in E\}$, this is called the *image of E under f*.

Problem 1 (R:4:2*)**.** Let $f : X \to Y$ be continuous. Let $E \subseteq X$, show that $f(\mathrm{Cl}(E)) \subseteq \mathrm{Cl}(f(E))$. By example show that this containment can be proper, that is $\mathrm{Cl}(f(E)) \nsubseteq f(\mathrm{Cl}(E))$ can hold.

You may take X and Y to be metric if you want, but this is not relevant.

Definition Let $f : E \subset X \to Y$, the graph of f is the set $\mathrm{Graph}(f) = \{(x, f(x) \mid x \in E\} \subseteq X \times Y$.

Problem 2. Let $f : E \subset X \to Y$ be continuous where Y is Hausdorff, show that $\mathrm{Graph}(f)$ is closed in $E \times Y$.

Problem 3 (R:4:6)**.** Suppose $f : E \subseteq X \to Y$ and E is compact. Suppose further that X and Y are Hausdorff (or metric if you prefer). Show that f is continuous on E iff $\mathrm{Graph}(f)$ is compact.

Hint: You may use the fact that if K and H are compact, then $K \times H$ is compact and that If K is compact and $C \subseteq K$ is closed, then C is compact. (Both of these are in notes and book.)

Problem 4. Let $f : E \subset X \to Y$ where both X and Y are metric spaces with Y complete. suppose f is uniformly continuous on E, show that there is a unique continuous extension $\hat{f} : \mathrm{Cl}(E) \to Y$. Moreover, $\hat{f}$ remains uniformly continuous.

Definition: A set $E \subset X$ has the *Bolzano-Weierstrass property* iff every sequence in X has a convergent subsequence.

Problem 5. Show that if $E \subseteq X$ has the Bolzano-Weierstrass property, then

a) $\mathrm{Cl}(E)$ also has Bolzano-Weierstrass property.

b) If X is metric, then E is bounded.

c) For X metric E has the Bolzano-Weierstrass property iff $\mathrm{Cl}(E)$ is compact.

Problem 6 (R:4:8*)**.** Let $f : E \subseteq X \to Y$ be uniformly continuous on E where E has the Bolzano-Weierstrass property and Y is complete. Show that f is bounded on E, that is $f(E)$ is bounded in Y.

Argue, from this, that if X is any Euclidean space and $E \subseteq X$ bounded. If f is continuous on E, then $f(E)$ is bounded.

Problem 7 (R:4:19)**.** Show that if $f : \mathbb{R} \to \mathbb{R}$ satisfies the intermediate value theorem and $f^{-1}(r) = \{x \mid f(x) = r\}$ is closed for $r \in \mathbb{Q}$, then f is continuous. (See the text for a hint. $\mathbb{Q}$ here could be replaced by any dense set.)