Name: ______________________ Quiz 3 - MAT345

1 True/False (15 points; 3 points each)

Problem 1.1. Include your reasoning/justification for your choices here.

False Let A be an $m \times n$ matrix and $b \in \mathbb{R}^m$, there is always at most one least squares solution to $Ax = b$.

This is, in general false. For any least square solution, $\hat{x}$, any $x \in \hat{x} + \mathrm{NS}(A)$ is also a least squares solution.

True Let A be an $m \times n$ matrix and $b \in \mathbb{R}^m$, there is always at least one least squares solution to $Ax = b$.

Let $\hat{b} = \mathrm{proj}^{\perp}_{\mathrm{CS}(A)}(b)$, then $\hat{b} \in \mathrm{CS}(A)$ so $\hat{b} = A\hat{x}$ for some $\hat{x}$ and $A\hat{x} = \hat{b}$. This is what it means to be a least-squares solution.

True Let A be an $m \times n$ matrix and $b \in \mathbb{R}^m$. There is always a unique $\hat{b}$ so that if $\hat{x}$ is a least squares solution to $Ax = b$, then $A\hat{x} = \hat{b}$.

This is true exactly as the preceding.

False Let A be an $m \times n$ matrix and $b \in \mathbb{R}^m$. If $\hat{x}$ is a least squares solution to $Ax = b$, then it is always true that $A\hat{x} = b$.

By definition $A\hat{x} = \hat{b}$ where $\hat{b} = \mathrm{proj}_{\mathrm{CS}(A)}(b)$, so this is false when $\hat{b} \neq b$, i.e., any time $b \notin \mathrm{CS}(A)$. Also, if you had "T" above, it is a good indication that "F" should be here:)

2 Multiple Choice (10 points; 5 points each)

Each correct box counts for two points.

Problem 2.1 (5 points)**.** For $\langle \cdot, \cdot \rangle : V \times V \to \mathbb{C}$ a (complex) inner product, which of the following always hold? Put "Y" if the condition is necessary and "N" if it is not.

$\boxed{\text{Y}}$ $\langle c_1 v_1 + c_2 v_2, u \rangle = c_1 \langle v_1, u \rangle + c_2 \langle v_2, u \rangle$.

$\boxed{\text{Y}}$ $\langle v, v \rangle \in [0, \infty)$.

$\boxed{\text{N}}$ $\langle u, v \rangle + \langle v, u \rangle = 2 \langle v, u \rangle$.

$\boxed{\text{Y}}$ If $\langle u, v \rangle = 0$, then $\|u\|^2 + \|v\|^2 = \|u + v\|^2$, where $\|u\|^2 = \langle u, u \rangle$.

$\boxed{\text{N}}$ If $\|u\|^2 + \|v\|^2 = \|u + v\|^2$, then $\langle u, v \rangle = 0$.

Problem 2.2 (5 points)**.** Which of the following are equivalent to "$\hat{x}$ is a least-squares solution to $Ax = b$"? Put "Y" if the condition is equivalent; otherwise, put "N."

$\boxed{\text{Y}}$ $\|A\hat{x} - b\| \leq \|Ax - b\|$ for all x.

$\boxed{\text{Y}}$ $b - A\hat{x} \perp Ax$ for all x.

$\boxed{\text{Y}}$ $A\hat{x} = \hat{b}$ where $\hat{b} = \text{proj}^{\perp}_{\text{CS}(A)}(b)$.

$\boxed{\text{Y}}$ $A^T A \hat{x} = A^T b$.

$\boxed{\text{Y}}$ If A is orthogonal, i.e., the columns of A are all unit vectors and mutually orthogonal, then $\hat{x} = A^T b$

3 Take Home Computational (15 points)

Show all computations so that you make clear what your thought processes are. When you use technology, ensure you are clear about what you are doing and using.

Problem 3.1 (10 pts)**.** I used a function unknown to you, $g(x)$, to generate data and then added noise. Here is a MATLAB template Quiz3.mlx and a data file Quiz3.mat.

Your task is to find the function that best approximates (in the least-squares sense based on the data) the unknown $g(x)$ built out of the following basis functions:

$$1, \quad x, \quad x^2, \quad \sqrt{x}$$

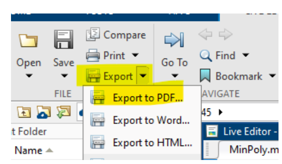
That is you are looking for the coefficients c_1, c_2, c_3, c_4 so that if

$$\hat{g}(x) = c_1 + c_2 x + c_3 x^2 + c_4 \sqrt{x},$$

then $\|\hat{g} - Y\| = \sum_{i=1}^{N} (\hat{g}(x_i) - y_i)^2$ is minimized.

Think of $g(x)$ as some rule or physical law, the data as a set of measurements at the various values x_i (which might be distance, speed, time, whatever), where the measurements have some random error. Your $\hat{g}(x)$ is then the best **model** you can find based on the basis functions $\{1, x, x^2, \sqrt{x}\}$.

In a more realistic scenario, I would provide you with several sets of data, and you would compare the error of your model to data other than the set you used to construct (train) the model.

Submit your completed Quiz3.mlx file or export it as a PDF: 

Here is a solution as MATLAB and one as PDF.