Quiz 3

Problem 1 (15 points; 3 points each)**.** Decide if each of the following are true or false and provide a small proof or counterexample in each case. All vector spaces are assumed to be finite-dimensional here.

(a) ______ Given a basis $\mathcal{B} = \{\boldsymbol{v}_1, \ldots, \boldsymbol{v}_n\}$ for a vector space V and U a subspace of V, then there is $\mathcal{C} \subseteq \mathcal{B}$ that is a basis for U.

FALSE: $\mathcal{B} = \{\boldsymbol{e}_1, \boldsymbol{e}_2\}$ is a basis for $\mathbb{R}^2$ and $U = \operatorname{span}\{(1, 1)\}$ is a subspace, namely, the line with slope 1 through the origin. You cannot throw away one of $\boldsymbol{e}_1$ or $\boldsymbol{e}_2$ to get a basis for U.

(b) ______ Given a basis $\mathcal{C}$ for a subspace U of a vector space V, $\mathcal{C}$ can be extended to a basis $\mathcal{B}$ for V.

TRUE: This is one of the theorems that you have, any linearly independent set can be expanded to a basis.

(c) ______ If $\{\boldsymbol{v}_1, \ldots, \boldsymbol{v}_n\}$ is linearly independent and $\boldsymbol{v} \in \operatorname{span}(\{\boldsymbol{v}_1, \ldots, \boldsymbol{v}_n\})$, then it is possible that there are distinct $\boldsymbol{c}, \boldsymbol{b} \in \mathbb{R}^n$ such that $\boldsymbol{v} = \sum_{i=1}^n c_i \boldsymbol{v}_i = \sum_{i=1}^n b_i \boldsymbol{v}_i$.

FALSE: If such $\boldsymbol{c}$ and $\boldsymbol{b}$ exists, then $\boldsymbol{0} = \boldsymbol{v} - \boldsymbol{v} = (\sum b_i \boldsymbol{v}_i) - (\sum c_i \boldsymbol{v}_i) = \sum (b_i - c_i) \boldsymbol{v}_i$. Since $\boldsymbol{v}_i$'s are independent, $b_i - c_i = 0$ for all i, so $\boldsymbol{c} = \boldsymbol{b}$.

(d) ______ If $\{\boldsymbol{v}_1, \ldots, \boldsymbol{v}_n\}$ is linearly independent and $V = \operatorname{span}(\{\boldsymbol{v}_1, \ldots, \boldsymbol{v}_n\}) = \operatorname{span}(\{\boldsymbol{u}_1, \ldots, \boldsymbol{u}_n\})$, then $\{\boldsymbol{u}_1, \ldots, \boldsymbol{u}_n\}$ is linearly independent.

TRUE: This too is a theorem. Since $V = \operatorname{span}\{\boldsymbol{v}_1, \ldots, \boldsymbol{v}_n\}$ and $\boldsymbol{v}_i$ are independent, you know $\{\boldsymbol{v}_1, \ldots, \boldsymbol{v}_n\}$ is a basis for V and so $\dim(V) = n$. since $\operatorname{span}\{\boldsymbol{u}_1, \ldots, \boldsymbol{u}_n\}$ span V you know this set can be reduced to a basis, but any basis must have n elements, so $\{\boldsymbol{u}_1, \ldots, \boldsymbol{u}_n\}$ must already be a basis, and hence is linearly independent.

(e) ______ Apparently I only gave four of these. Free 3 points!

Problem 2 (10 pts)**.** Find a basis for span$\{\boldsymbol{u}_1, \ldots, \boldsymbol{u}_5\}$ from among the vectors $\boldsymbol{u}_1, \ldots, \boldsymbol{u}_5$, where

$$\boldsymbol{u}_1 = \begin{bmatrix}1\\2\\2\end{bmatrix} \qquad \boldsymbol{u}_2 = \begin{bmatrix}2\\5\\4\end{bmatrix} \qquad \boldsymbol{u}_3 = \begin{bmatrix}1\\3\\2\end{bmatrix} \qquad \boldsymbol{u}_4 = \begin{bmatrix}2\\7\\4\end{bmatrix} \qquad \boldsymbol{u}_5 = \begin{bmatrix}1\\1\\0\end{bmatrix}$$

Do this by building the matrix consisting of the $\boldsymbol{u}_i$'s as rows or columns (you must choose correctly) and use Gaussian elimination. This is described carefully in the notes.

Using the notation $A \sim B$ to mean A and B are related by performing elementary row operations, or $B = EA$ where E is invertible. Let $A = \begin{bmatrix}\boldsymbol{u}_1 & \cdots & \boldsymbol{u}_5\end{bmatrix}$, then

$$A = \begin{bmatrix}1 & 2 & 1 & 2 & 1\\ 2 & 5 & 3 & 7 & 1\\ 2 & 4 & 2 & 4 & 0\end{bmatrix} \sim \begin{bmatrix}1 & 2 & 1 & 2 & 1\\ 0 & 1 & 1 & 3 & -1\\ 0 & 0 & 0 & 0 & -2\end{bmatrix}$$

B is an echelon form of A with pivots in columns 1, 2, and 5. Thus $\{\boldsymbol{u}_1, \boldsymbol{u}_2, \boldsymbol{u}_5\}$ is a basis for span$\{\boldsymbol{u}_1, \ldots, \boldsymbol{u}_5\} = \mathrm{CS}(A)$.

This was all that was asked here. The rest is just something like you have on the exam.

We also know that $\boldsymbol{v}_1 = (1, 2, 1, 2, 1)$, $\boldsymbol{v}_2 = (0, 1, 1, 3, -1)$, and $\boldsymbol{v}_3 = (0, 0, 0, 0, -1)$ is a basis for $\mathrm{RS}(A)$.

By back substitution, letting $x_3 = s$ and $x_4 = t$ we have $x_5 = 0$, $x_2 = -s - 3t$, and $x_1 = -2x_2 - s - 2t = 2(s + 3t) - s - 2t = s + 4t$, so $\mathrm{NS}(A)$ consists of vectors of the form $(s + 4t, -s - 3t, s, t, 0) = s(1, -1, 1, 0, 0) + t(4, -3, 0, 1, 0)$, so $\mathrm{NS}(A) = \mathrm{span}\{\boldsymbol{v} + 4, \boldsymbol{v}_5\}$, where $\boldsymbol{v}_4 = (1, -1, 1, 0, 0)$ and $\boldsymbol{v}_5 = (4, -3, 0, 1, 0)$.

Quick check: Check that $\boldsymbol{v}_4$ and $\boldsymbol{v}_5$ are orthogonal to all rows of A.

Problem 3 (10 pts)**.** Let $c_1, c_2, \ldots, c_n$ be n distinct real numbers. Let $p_i = \prod_{\substack{j=1\\ j\neq i}}^{n} (x - c_j)/(c_i - c_j)$. Show that $\mathcal{B} = \{p_1, p_2, \ldots, p_n\}$ is a basis for P_{n-1}.

Hint: Compute $p_i(c_j)$ and look at what happens when $i = j$ and when $i \neq j$. Use this to argue the independence of $\mathcal{B}$.

I'll do the n-dimensional case. Suppose $c_1, \cdots, c_n$ are distinct reals then define $p_i \in P_{n-1}$

$$p_i = \prod_{\substack{j=1\\ j\neq i}}^{n} (x - c_j)/(c_i - c_j)$$

It is trivial to see that

$$p_i(c_j) = \begin{cases}1 & \text{if } i = j\\ 0 & \text{if } i \neq j\end{cases}$$

This shows independence since if $p = \sum_{i=1}^{n} \alpha_i p_i$, then $p_i(c_j) = \alpha_j$ so if $p = 0$, then $\alpha_j = 0$ for all j.

This all you were asked to do.

Just for fun, for the basis, $\mathcal{B} = \{p_1, p_2, \ldots, p_n\}$, $p = \sum_{i=1}^{n} p(c_i) p_i$, that is,

$$[p]_{\mathcal{B}} = \begin{bmatrix}p(c_1)\\ p(c_2)\\ \vdots\\ p(c_n)\end{bmatrix}$$

So given n points (x_i, y_i) with the x_i's distinct. The unique degree $(n-1)$-polynomial through these points is $p(x) = \sum_{i=1}^{n} y_i \cdot p_i(x)$.