Quiz 6

Question 1 (15 points; 3 points each)**.** Decide if each of the following are true or false and provide a justification or counterexample in each case. A justification could consist of a theorem from the text. All vector spaces are assumed to be finite-dimensional here. All vector spaces are now over $\mathbb{C}$ unless otherwise stated.

(a) ______ If the characteristic polynomial of a 4×4 matrix is $p(t) = (t-1)^2 t^2$, then there must be an invertible matrix S so that $A = SDS^{-1}$ where

$$D = \operatorname{diag}(1,1,0,0) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

This is false. For example,

$$A = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

(b) ______ If A and B are $n \times n$ matrices and λ_A and λ_B are eigenvalues for A and B respectively with respect to the same eigenvector $\boldsymbol{v}$, then $AB\boldsymbol{v} = BA\boldsymbol{v}$.

This is true. This is a trivial computation

$$AB\boldsymbol{v} = A\lambda_B\boldsymbol{v} = \lambda_B A\boldsymbol{v} = \lambda_B\lambda_A\boldsymbol{v}.$$

Similarly, $BA\boldsymbol{v} = \lambda_A\lambda_B\boldsymbol{v}$ and since $\lambda_A\lambda_B = \lambda_B\lambda_A$ it is clear that the assertion is true.

(c) ______ If A and B are $n \times n$ matrices and λ_A is an eigenvalue of A and λ_B is an eigenvalue of B, then $\lambda_A\lambda_B$ is an eigenvalue of AB.

This is false. This would be true if there is a $\boldsymbol{v}$ that is simultaneously an eigenvector for A and λ_A and B and λ_B. A counterexample can be found even for the simplest of matrices:

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}$$

Then 2 is an eigenvalue for bot A and B, but

$$AB = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$$

so clearly $4 = 2 \cdot 2$ is not an eigenvalue for AB.

(d) _____ If A and B are diagonalizable $n \times n$ matrices, then AB is diagonalizable.

This is false, a counterexample suffices

$$\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$$

The two on the LHS are diagonalizable since they each have two distinct eigenvalues, the RGS is the typical example of a non-diagonalizable matrix. The only eigenvalue is 1, it has algebraic degree 2, but $E_1 = \text{span}\{(0,1)\}$ so the geometric degree is 1.

(e) _____ Suppose A is diagonalizable, then e^A is diagonalizable and e^λ is an eigenvalue of e^A iff λ is and eigenvalue of A.

One direction is trivial, if λ is an eigenvalue of A, then this is a simple calculation. If $A = SDS^{-1}$, then $e^A = Se^DS^{-1}$ and $e^{\text{diag}(d_1,\ldots,d_n)} = \text{diag}(e^{d_1}, \ldots, e^{d_n})$. So e^λ is an eigenvalue of e^A.

For the converse we use the fact that we have a full basis of eigenvectors for e^A. That is if $\{\lambda_1, \ldots, \lambda_m\}$ are eigenvalues of A and E_{λ_i} is the eigenspace associated to λ_i, then we know that $E_{\lambda_i} \subseteq E_{e^{\lambda_i}}$ (the eigenspace for e^A w.r.t. e^{λ_i}. But $\mathbb{R}^n = E_{\lambda_1} \oplus \cdots \oplus E_{\lambda_m}$ and thus $\mathbb{R}^n = E_{e^{\lambda_1}} \oplus \cdots \oplus E_{e^{\lambda_m}}$, thus the e^{λ_i} are ALL of the eigenvalues for e^A. Thus if μ is an eigenvalue for e^A, then $\mu = e^{\lambda_i}$ for some i.

Question 2 (10 points)**.** Let $A = \begin{bmatrix} 0 & 0 & 2 \\ 0 & 3 & 0 \\ 2 & 0 & 0 \end{bmatrix}$, write $A = U\Lambda U^{-1}$ where U is unitary, columns are orthonormal basis for $\mathbb{R}^3$ and $\Lambda = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix}$ with $\lambda_1 > \lambda_2 > \lambda_3$.

Recall: $U^{-1} = U^T$ for unitary U.

Find the eigenvalues: $\det(A - tI) = -(t-3)(t-2)(t+2)$ so the eigenvalues are $\lambda_1 = 3 > \lambda_2 = 2 > \lambda_3 = -2$.

Find the eigenspace for $\lambda_1 = 3$:

$$\text{NS}\left(\begin{bmatrix} -3 & 0 & 2 \\ 0 & 0 & 0 \\ 2 & 0 & -3 \end{bmatrix}\right) = \text{span}\left(\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}\right).$$

Find the eigenspace for $\lambda_=2$:

$$\text{NS}\left(\begin{bmatrix} -2 & 0 & 2 \\ 0 & 1 & 0 \\ 2 & 0 & -2 \end{bmatrix}\right) = \text{span}\left(\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}\right).$$

Find the eigenspace for $\lambda_= -2$:

$$\text{NS}\left(\begin{bmatrix} 2 & 0 & 2 \\ 0 & 1 & 0 \\ 2 & 0 & 2 \end{bmatrix}\right) = \text{span}\left(\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}\right).$$

$$U = \begin{bmatrix} 0 & 1/\sqrt{2} & 1/\sqrt{2} \\ 1 & 0 & 0 \\ 0 & 1/\sqrt{2} & -1/\sqrt{2} \end{bmatrix}$$

and

$$\begin{bmatrix} 0 & 0 & 2 \\ 0 & 3 & 0 \\ 2 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1/\sqrt{2} & 1/\sqrt{2} \\ 1 & 0 & 0 \\ 0 & 1/\sqrt{2} & -1/\sqrt{2} \end{bmatrix} \begin{bmatrix} 3 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & -2 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \\ 1/\sqrt{2} & 0 & 1/\sqrt{2} \\ 1/\sqrt{2} & 0 & -1/\sqrt{2} \end{bmatrix}$$

Question 3 (10 points)**.** Suppose the matrix $A = \frac{1}{12}\begin{bmatrix} 5 & -1 \\ -1 & 5 \end{bmatrix}$ is used to transform points in the plane iteratively. That is, given a point $\boldsymbol{v}$, consider the sequence $\boldsymbol{v_n} = A^n\boldsymbol{v}$. Letting $U = \begin{bmatrix} \boldsymbol{u_1} & \boldsymbol{u_2} \end{bmatrix}$ so that $\boldsymbol{u_i}$ is an eigenvector associated to λ_i and letting $\boldsymbol{v} = c_1\boldsymbol{u_1} + c_2\boldsymbol{u_2}$ what is a simple expressions for a_n and b_n so that $\boldsymbol{v_n} = A^n\boldsymbol{v} = a_n\boldsymbol{u_1} + b_n\boldsymbol{u_2}$.

After a little work, you have $A = UDU^{-1}$ where

$$U = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} \quad \text{and} \quad D = \begin{bmatrix} 1/3 & 0 \\ 0 & 1/2 \end{bmatrix}$$

$$\begin{aligned} A(c_1\boldsymbol{u_1} + c_2\boldsymbol{u_2}) &= c_1A\boldsymbol{u_1} + c_2A\boldsymbol{u_2} = \lambda_1c_1\boldsymbol{u_1} + \lambda_2c_2\boldsymbol{u_2} \\ A^2(c_1\boldsymbol{u_1} + c_2\boldsymbol{u_2}) &= A(\lambda_1c_1\boldsymbol{u_1} + \lambda_2c_2\boldsymbol{u_2}) = \lambda_1c_1A(\boldsymbol{u_1}) + \lambda_2c_2A(\boldsymbol{u_2}) = \lambda_1^2c_1\boldsymbol{u}_1 + \lambda_2^2c_2\boldsymbol{u}_2 \\ &\vdots \\ A^n(c_1\boldsymbol{u_1} + c_2\boldsymbol{u_2}) &= \lambda_1^nc_1\boldsymbol{u_1} + \lambda_2^nc_2\boldsymbol{u_2} = (1/3)^nc_1\boldsymbol{u_1} + (1/2)^nc_2\boldsymbol{u_2} \end{aligned}$$

So $a_n = (1/3)^nc_1$ and $b_n = (1/2)^nc_2$.