Name: ________________________ Exam 1 - MAT345

I True/False (60 points; 6 points each)

Each problem is points for a total of 50 points. (5 points each and one free point.) In class, you only provide the T/F.

Corrections: If you choose to make corrections for 50% back on this section, then you must provide reasons for ALL of these, not just the ones that you miss. A reason might be as simple as, "by Theorem ...," or it might require an example or counterexample. In any case, some correct reason or counterexample must be provided.

Problem I.1 (50 points; 5 points each)**.** Decide if each of the following is true or false.

1. True If $A\boldsymbol{x} = \boldsymbol{b}$ has a unique solution for some $\boldsymbol{b}$, then $A\boldsymbol{x} = \boldsymbol{c}$ has at most one solution for any $\boldsymbol{c}$.

2. False Diagonal $n \times n$ matrices commute with arbitrary $n \times n$ matrices, that is, for any $n \times n$ diagonal D, $DA = AD$ for all $n \times n$ matrices A.

3. True For A an $m \times n$ matrix $(\boldsymbol{e}_i^m)^T A \boldsymbol{e}_j^n = A_{i,j}$.

4. False Let A and B be $n \times n$ matrices, if $(A - B)(A + B) = O$, then either $A = B$ or $A = -B$.

5. True If $A^2 - I$ is invertible, then $A - I$ and $A + I$ must also both be invertible.

6. False If A is equivalent to B, then $\det(A) = \det(B)$.

7. True If A and B are equivalent matrices, then $\text{NS}(A) = \text{NS}(B)$.

8. False Consider the operation flip(A) that "flips" a matrix horizontally, so for example

$$\text{flip}\left(\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}\right) = \begin{bmatrix} 2 & 1 \\ 4 & 3 \end{bmatrix} \text{ while } \text{flip}\left(\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}\right) = \begin{bmatrix} 3 & 2 & 1 \\ 6 & 5 & 4 \end{bmatrix}$$

For any $n \times n$ matrix A, $\det(\text{flip}(A)) = -\det(A)$.

9. True We have used in class that AB is invertible iff both A and B are invertible, but never proved this. The following is a valid proof of this fact.

$$\begin{aligned} AB \text{ is invertible } &\iff \det(AB) \neq 0 \\ &\iff \det(A)\det(B) \neq 0 \\ &\iff \det(A) \neq 0 \text{ and } \det(B) \neq 0 \\ &\iff A \text{ is invertible and } B \text{ is invertible} \end{aligned}$$

10. False Cramer's rule is the most efficient way to solve a system of n equations and n unknowns and it works even when Gaussian elimination fails.

II Computational (90 points)

Show all computations so that you make clear what your thought processes are.

Problem II.1 (20 pts). Let

$$A = \begin{bmatrix} 2 & 0 & 0 \\ 3 & 2 & 5 \\ 3 & -3 & -1 \\ -2 & 0 & 1 \end{bmatrix}; \qquad B = \begin{bmatrix} 4 & 5 & -1 & -3 \\ 2 & -4 & 3 & 0 \\ -1 & 0 & 3 & 0 \end{bmatrix}$$

1. Express the fourth row of AB as a linear combination of rows of B.

$$(-2)\begin{bmatrix} 4 & 5 & -1 & 3 \end{bmatrix} + (0)\begin{bmatrix} 2 & -4 & 3 & 0 \end{bmatrix} + (1)\begin{bmatrix} -1 & 0 & 3 & 0 \end{bmatrix} = \begin{bmatrix} -9 & -10 & 5 & -6 \end{bmatrix}$$

2. Express the second column of AB as a linear combination of the columns of A.

$$(5)\begin{bmatrix} 2 \\ 3 \\ 3 \\ -2 \end{bmatrix} + (-4)\begin{bmatrix} 0 \\ 2 \\ -3 \\ 0 \end{bmatrix} + (0)\begin{bmatrix} 0 \\ 5 \\ -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 10 \\ 7 \\ 27 \\ -10 \end{bmatrix}$$

3. Express $(AB)_{1,2}$ as a product of a row of A and a column of B.

$$(AB)_{1,2} = \begin{bmatrix} 2 & 0 & 0 \end{bmatrix}\begin{bmatrix} 5 \\ -4 \\ 0 \end{bmatrix} = 10$$

Problem II.2 (30 pts). Solve $A\boldsymbol{x} = \boldsymbol{b}$ where

$$A = \begin{bmatrix} -1 & -1 & 1 & 0 & 1 \\ -5 & -7 & 1 & -4 & 9 \\ -4 & -10 & -8 & -12 & 17 \\ 2 & -8 & -22 & -20 & 15 \end{bmatrix} \qquad \boldsymbol{b} = \begin{bmatrix} 2 \\ 24 \\ 48 \\ 72 \end{bmatrix}$$

1. (15 points) Use row operations (show all work and indicate operations) to reduce A to an echelon form. (This should work out very nicely - no fractions required..)
2. (10 points) Use back-substitution to solve the resulting system. Make sure to indicate which variables are free.
3. (5 points) Write your solution as a linear combination of vectors.

Gauss-Jordan elimination to get echelon form:

$$\left[\begin{array}{ccccc|c} -1 & -1 & 1 & 0 & 1 & 2 \\ -5 & -7 & 1 & -4 & 9 & 24 \\ -4 & -10 & -8 & -12 & 17 & 48 \\ 2 & -8 & -22 & -20 & 15 & 72 \end{array}\right] \xrightarrow[\substack{R_2-5R_1\to R_2\\ R_3-4R_1\to R_3\\ R_4+2R_1\to R_4}]{} \left[\begin{array}{ccccc|c} -1 & -1 & 1 & 0 & 1 & 2 \\ 0 & -2 & -4 & -4 & 4 & 14 \\ 0 & -6 & -12 & -12 & 13 & 40 \\ 0 & -10 & -20 & -20 & 17 & 76 \end{array}\right]$$

$$\xrightarrow[\substack{R_3-2R_2\to R_3\\ R_4-5R_2\to R_4}]{} \left[\begin{array}{ccccc|c} -1 & -1 & 1 & 0 & 1 & 2 \\ 0 & -2 & -4 & -4 & 4 & 14 \\ 0 & 0 & 0 & 0 & 1 & -2 \\ 0 & 0 & 0 & 0 & -3 & 6 \end{array}\right]$$

$$\xrightarrow[R_4+3R_3\to R_4]{} \left[\begin{array}{ccccc|c} -1 & -1 & 1 & 0 & 1 & 2 \\ 0 & -2 & -4 & -4 & 4 & 14 \\ 0 & 0 & 0 & 0 & 1 & -2 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

Back-substitution: x_3 and x_4 are free.

$$\boxed{x_5 = -2}$$

$$\boxed{x_4 = \alpha}$$

$$\boxed{x_3 = \beta}$$

$$-2x_2 - 4\beta - 4\alpha + 4(-2) = 14 \to -2x_2 = 22 + 4\beta + 4\alpha \to \boxed{x_2 = -2\alpha - 2\beta - 11}$$

$$-x_1 - (-2\alpha - 2\beta - 11) + \beta - 2 = 2 \to \boxed{x_1 = 2\alpha + 3\beta + 7}$$

Solution as a linear combination of vectors:

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} 2\alpha + 3\beta + 7 \\ -2\alpha - 2\beta - 11 \\ \beta \\ \alpha \\ -2 \end{bmatrix} = \begin{bmatrix} 7 \\ -11 \\ 0 \\ 0 \\ -2 \end{bmatrix} + \alpha \begin{bmatrix} 2 \\ -2 \\ 0 \\ 1 \\ 0 \end{bmatrix} + \beta \begin{bmatrix} 3 \\ -2 \\ 1 \\ 0 \\ 0 \end{bmatrix}$$

Problem II.3 (20 pts)**.** Use Cramer's rule to find x_4, where

$$\begin{bmatrix} 3 & -2 & 0 & 3 \\ -1 & 3 & 0 & 3 \\ 0 & 2 & 0 & 2 \\ 2 & 1 & 3 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 9 \\ 5 \\ 4 \\ 2 \end{bmatrix}$$

Note: These determinants should work out very nicely if you chose how you expand carefully.

Let

$$A = \begin{bmatrix} 3 & -2 & 0 & 3 \\ -1 & 3 & 0 & 3 \\ 0 & 2 & 0 & 2 \\ 2 & 1 & 3 & 0 \end{bmatrix} \text{ and } B = \begin{bmatrix} 3 & -2 & 0 & 9 \\ -1 & 3 & 0 & 5 \\ 0 & 2 & 0 & 4 \\ 2 & 1 & 3 & 2 \end{bmatrix}$$

so that B is obtained by replacing the 4^{th} column of A by $\left[\begin{smallmatrix} 9 \\ 5 \\ 4 \\ 2 \end{smallmatrix}\right]$. Then

$$x_4 = \frac{\det(B)}{\det(A)}$$

where, by expanding along the 3^{rd} column of A we have

$$\begin{aligned}
\det(A) &= (-3)\det \begin{bmatrix} 3 & -2 & 3 \\ -1 & 3 & 3 \\ 0 & 2 & 2 \end{bmatrix} \\
&= (-3)\left((-2)\det \begin{bmatrix} 3 & 3 \\ -1 & 3 \end{bmatrix} + (2)\det \begin{bmatrix} 3 & -2 \\ -1 & 3 \end{bmatrix}\right) \\
&= (-3)\big((-2)(9+3) + (2)(9-2)\big) \\
&= (-3)(-2)(12-7) = 30
\end{aligned}$$

and by expanding again along the 3^{rd} column of B

$$\begin{aligned}
\det(B) &= (-3)\det \begin{bmatrix} 3 & -2 & 9 \\ -1 & 3 & 5 \\ 0 & 2 & 4 \end{bmatrix} \\
&= (-3)\left((-2)\det \begin{bmatrix} 3 & 9 \\ -1 & 5 \end{bmatrix} + (4)\det \begin{bmatrix} 3 & -2 \\ -1 & 3 \end{bmatrix}\right) \\
&= (-3)\big((-2)(15+9) + 4(9-2)\big) = (-3)(-2)(24-2(7)) = 60
\end{aligned}$$

So

$$x_4 = \frac{60}{30} = 2$$

Problem II.4 (20 pts). Consider

$$\begin{aligned}
A = \begin{bmatrix} 1 & 2 & -4 & 3 & 2 \\ 2 & 4 & -7 & 4 & 5 \\ -3 & -6 & 14 & -13 & -3 \end{bmatrix} &\underset{\substack{R_2-2R_1\to R_2 \\ R_3+3R_1\to R_3}}{\longrightarrow} \begin{bmatrix} 1 & 2 & -4 & 3 & 2 \\ 0 & 0 & 1 & -2 & 1 \\ 0 & 0 & 2 & -4 & 3 \end{bmatrix} \\
&\underset{R_3-2R_2\to R_3}{\longrightarrow} \begin{bmatrix} 1 & 2 & -4 & 3 & 2 \\ 0 & 0 & 1 & -2 & 1 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} = U
\end{aligned}$$

Write A in the form LU where L is lower-triangular with 1's on the diagonal, and U is the Echelon matrix given.

$$L = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ -3 & 2 & 1 \end{bmatrix}$$

and

$$A = \begin{bmatrix} 1 & 2 & -4 & 3 & 2 \\ 2 & 4 & -7 & 4 & 5 \\ -3 & -6 & 14 & -13 & -3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ -3 & 2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & -4 & 3 & 2 \\ 0 & 0 & 1 & -2 & 1 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} = LU$$

III Theory and Proofs (40 points; 20 points each)

Choose two of the four options. If you try more than two, I will grade only the first two, not the best two. You must decide what should be graded. These will be due 2/7 in class. Make sure your work is complete and clear. Explain your work, a proof is not just a bunch of math symbols, it is an explanation of why something is true.

Problem III.1 (20 pts)**.** If A and B are invertible $n \times n$ matrices, show that

$$(AB)^2 = A^2B^2 \iff AB = BA$$

Problem III.2 (20 pts)**.** Show that for any $m \times n$ matrix A,

$$\sum_{i=1}^{m}\sum_{j=1}^{n}\left((\boldsymbol{e}_i^m)^T A \boldsymbol{e}_j^n\right)\left(\boldsymbol{e}_i^m(\boldsymbol{e}_j^n)^T\right) = A.$$

Problem III.3 (20 pts)**.** Let A be an $n \times n$ matrix such that $AB = BA$ for all $n \times n$ matrices B. show that $A = \alpha I$ for some scalar α.

Problem III.4 (20 pts)**.** Consider the operation $\operatorname{rot}(A)$ that rotates a matrix clockwise by $90°$, for example,

$$\operatorname{rot}\left(\begin{bmatrix}1 & 2\\ 3 & 4\end{bmatrix}\right) = \begin{bmatrix}3 & 1\\ 4 & 2\end{bmatrix} \text{ while text}\left(\begin{bmatrix}1 & 2 & 3\\ 4 & 5 & 6\end{bmatrix}\right) = \begin{bmatrix}4 & 1\\ 5 & 2\\ 6 & 3\end{bmatrix}$$

For $n \times n$ matrices A come up with and prove a simple formula for $\det(\operatorname{rot}(A))$ in terms of $\det(A)$.