

Exam 2

To avoid any confusion, unless specified otherwise, vector spaces are complex vector spaces, inner-products are complex inner-products, and matrices are complex matrices. The standard inner product is $\langle \mathbf{u}, \mathbf{v} \rangle = \mathbf{v}^H \mathbf{u} = \sum_{i=1}^n \bar{v}_i u_i$. Keep in mind that $A^H = A^T$ for real matrices and symmetric = Hermitian for real matrices.

Part I: True/False

Each problem is points for a total of 50 points. (5 points each.)

You do not need to justify the answers here, this is unlike the quizzes.

1	2	3	4	5	6	7	8	9	10
T	F	F	T	T	T	T	F	F	F

1. _____ If U is unitary, then U is itself unitarily diagonalizable. This means there is a unitary V so that $U = VDV^H$ where D is diagonal.

This is true. $U^H U = U U^H = I$, so U is normal, hence unitarily diagonalizable.

2. _____ For any diagonalizable matrix A , one can use Gram-Schmidt to find an orthogonal basis consisting of eigenvectors.

This is false. You must first have that the eigenspaces for different eigenvalues are orthogonal.

3. _____ The collection of rank k $n \times n$ matrices is a subspace of $\mathbb{R}^{n \times n}$, for $k < n$.

This is false, in fact SVD shows how to write any matrix as a sum of rank 1 matrices.

4. _____ If A is unitary, then $|\lambda| = 1$ for all eigenvalues λ of A .

This is true. Let λ be an eigenvalue, with unit eigenvector \mathbf{v} . then $\langle A\mathbf{v}, A\mathbf{v} \rangle = \langle \lambda\mathbf{v}, \lambda\mathbf{v} \rangle = \bar{\lambda}\lambda\|\mathbf{v}\|_2^2 = |\lambda|^2 = (A\mathbf{v})^H(A\mathbf{v}) = \mathbf{v}^H(A^H A)\mathbf{v} = \mathbf{v}^H I \mathbf{v} = \|\mathbf{v}\|_2^2 = 1$. So $|\lambda|^2 = 1$.

5. _____ If $p(t)$ is a polynomial and \mathbf{v} is an eigenvector of A with associated eigenvalue λ , then $p(A)\mathbf{v} = p(\lambda)\mathbf{v}$.

This is true and trivial. $p(x) = \sum_{i=1}^k a_i x^i$, so $p(A)\mathbf{v} = \sum_{i=1}^k a_i A^i \mathbf{v} = \sum_{i=1}^k a_i \lambda^i \mathbf{v} = p(\lambda)\mathbf{v}$

6. _____ If A and B are both $n \times n$ and \mathcal{B} is a basis for \mathbb{C}^n consisting of eigenvectors for both A and B , then A and B commute.

This is true. $AB = (SD_A S^{-1})(SB_B S^{-1}) = AD_A D_B S^{-1} = SD_B D_A S^{-1} = (SD_B S^{-1})(SD_A S^{-1}) = BA$.

7. _____ Any matrix A can be written as a weighted sum of rank 1 matrices..

This is true and is essentially one of the statements of the SVD. $U\Sigma V^T = \sum_{i=1}^r \sigma_i u_i v_i^T$ where $r = \text{rank}(A)$. Each $u_i v_i^T$ is an $m \times n$ rank-1 matrix.

8. _____ For all Hermitian matrices A , there is a matrix B so that $B^H B = A$.

This is false. A variant that is true is given in the first problem in part III. The point is that $B^H B$ is not only Hermitian, but also positive.

9. _____ If A is an $m \times n$ matrix, then $\text{rng}(A) \oplus \text{NS}(A^T) = \mathbb{R}^m$

This is true. You have previously proved that $\text{RS}(A) \oplus \text{NS}(A) = \mathbb{R}^n$. You just apply this result to A^T noting that $\text{RS}(A^T) = \text{CS}(A) = \text{rng}(A)$.

10. _____ If A is an invertible $n \times n$ matrix, then $ABA^{-1} = B$ for all $n \times n$ matrices B .

This is false. You have shown that the only $n \times n$ matrices that commute with all other $n \times n$ matrices are the diagonal matrices.

Part II: Computational (60 points)

P1. (15 points) Find B so that $B^2 = A$ where

$$A = \begin{bmatrix} 13 & -5 & 5 \\ -8 & 10 & -8 \\ -3 & -3 & 5 \end{bmatrix}$$

This is like 6.3 #4.

First diagonalize A .

Find the eigenvalues:

$$\begin{aligned} \det \left(\begin{bmatrix} 13-\lambda & -5 & 5 \\ -8 & 10-\lambda & -8 \\ -3 & -3 & 5-\lambda \end{bmatrix} \right) &= (13-\lambda)((10-\lambda)(5-\lambda)-24) - (-5)((-8)(5-\lambda)-24) + (5)((24+(3)(10-\lambda)) \\ &= (13-\lambda)(26-15\lambda+\lambda^2) + (5)(-64+8\lambda) + (5)(54-3\lambda) \\ &= (13-\lambda)(\lambda-13)(\lambda-2) + 5(-10+5\lambda) \\ &= (13-\lambda)(\lambda-13)(\lambda-2) + 25(-2+\lambda) \\ &= (\lambda-2)[(13-\lambda)(\lambda-13)+25] \\ &= (\lambda-2)(5-(13-\lambda))(5+(13-\lambda)) \\ &= -(\lambda-2)(\lambda-8)(\lambda-18) \end{aligned}$$

So the eigenvalues are $\lambda_1 = 18 > \lambda_2 = 8 > \lambda_3 = 2$.

This means $A = S \begin{bmatrix} 18 & & \\ & 8 & \\ & & 2 \end{bmatrix} S^{-1}$ and so $B = S \begin{bmatrix} \sqrt{18} & & \\ & \sqrt{8} & \\ & & \sqrt{2} \end{bmatrix} S^{-1}$ will be our matrix, where $S = [\mathbf{v}_1 \ \mathbf{v}_2 \ \mathbf{v}_3]$ where \mathbf{v}_i is an eigenvector for λ_i .

Find eigenspaces:

$$E_{18} = \text{NS} \left(\begin{bmatrix} -5 & -5 & 5 \\ -8 & -8 & -8 \\ -3 & -3 & -13 \end{bmatrix} \right) = \text{NS} \left(\begin{bmatrix} -1 & -1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \right) = \text{span} \left(\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} \right)$$

$$E_8 = \text{NS} \left(\begin{bmatrix} 5 & -5 & 5 \\ -8 & 2 & -8 \\ -3 & -3 & -3 \end{bmatrix} \right) = \text{NS} \left(\begin{bmatrix} 1 & -1 & 1 \\ -4 & 1 & -4 \\ -1 & -1 & -1 \end{bmatrix} \right) = \text{NS} \left(\begin{bmatrix} 1 & -1 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \right) = \text{span} \left(\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} \right)$$

$$E_2 = \text{NS} \left(\begin{bmatrix} 11 & -5 & 5 \\ -8 & 8 & -8 \\ -3 & -3 & 3 \end{bmatrix} \right) = \text{NS} \left(\begin{bmatrix} 11 & -5 & 5 \\ -1 & 1 & -1 \\ -1 & -1 & 1 \end{bmatrix} \right) = \text{NS} \left(\begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix} \right) = \text{span} \left(\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \right)$$

So here we could use $S = \begin{bmatrix} 1 & 1 & 0 \\ -1 & 0 & 1 \\ 0 & -1 & 1 \end{bmatrix}$

$$B = SDS^{-1} = \begin{bmatrix} 1 & 1 & 0 \\ -1 & 0 & 1 \\ 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} 3\sqrt{2} & & \\ & 2\sqrt{2} & \\ & & \sqrt{2} \end{bmatrix} \begin{bmatrix} 1/2 & -1/2 & 1/2 \\ 1/2 & 1/2 & -1/2 \\ 1/2 & 1/2 & 1/2 \end{bmatrix} = \frac{\sqrt{2}}{2} \begin{bmatrix} 5 & -1 & 1 \\ -2 & 4 & -2 \\ -1 & 1 & 3 \end{bmatrix}$$

P2. (15 points) Find B so that $B^H B = A$ where

$$A = \begin{bmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{bmatrix}$$

This is like 6.4 #14.

First diagonalize A .

Find the eigenvalues:

$\det\left(\begin{bmatrix} 1-\lambda & -1 & 0 \\ -1 & 2-\lambda & -1 \\ 0 & -1 & 1-\lambda \end{bmatrix}\right) = (1-\lambda)((2-\lambda)(1-\lambda)-1) - (-1)((-1)(1-\lambda)-0) = (1-\lambda)(1-3\lambda+\lambda^2) - (1-\lambda) = (1-\lambda)((1-3\lambda+\lambda^2)-1) = (1-\lambda)(-3\lambda+\lambda^2) = (1-\lambda)(\lambda)(-3+\lambda)$. So the eigenvalues are $\lambda_1 = 3 > \lambda_2 = 1 > \lambda_3 = 0$.

This means $A = S \begin{bmatrix} 3 & & \\ & 1 & \\ & & 0 \end{bmatrix} S^{-1}$ and so $B = S \begin{bmatrix} \sqrt{3} & & \\ & 1 & \\ & & 0 \end{bmatrix} S^{-1}$ will be our matrix, where $S = [\mathbf{v}_1 \ \mathbf{v}_2 \ \mathbf{v}_3]$ where \mathbf{v}_i is an eigenvector for λ_i .

Find eigenspaces:

$$E_3 = \text{NS}\left(\begin{bmatrix} -2 & -1 & 0 \\ -1 & -1 & -1 \\ 0 & -1 & -2 \end{bmatrix}\right) = \text{NS}\left(\begin{bmatrix} -1 & -1 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}\right) = \text{span}\left(\begin{bmatrix} -1 \\ -2 \\ 1 \end{bmatrix}\right)$$

$$E_1 = \text{NS}\left(\begin{bmatrix} 0 & -1 & 0 \\ -1 & -1 & -1 \\ 0 & -1 & 0 \end{bmatrix}\right) = \text{NS}\left(\begin{bmatrix} -1 & 2 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}\right) = \text{span}\left(\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}\right)$$

$$E_0 = \text{NS}(A) = \text{NS}\left(\begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}\right) = \text{span}\left(\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}\right)$$

$$S = \begin{bmatrix} 1/\sqrt{6} & 1/\sqrt{2} & 1/\sqrt{3} \\ -2/\sqrt{6} & 0 & 1/\sqrt{3} \\ 1/\sqrt{6} & -1/\sqrt{2} & 1/\sqrt{3} \end{bmatrix}$$

this is unitary so $S^{-1} = S^T$ and finally

Let $B = D^{1/2} S^H$ where $A = B^H B = S D S^H$ just as above. So

$$B = \begin{bmatrix} \sqrt{3} & & \\ & 1 & \\ & & 0 \end{bmatrix} \begin{bmatrix} 1/\sqrt{6} & -2/\sqrt{6} & 1/\sqrt{6} \\ 1/\sqrt{2} & 0 & 1/\sqrt{2} \\ 1/\sqrt{3} & 1/\sqrt{3} & 1/\sqrt{3} \end{bmatrix} = \frac{\sqrt{2}}{2} \begin{bmatrix} 1 & -2 & 1 \\ 1 & 0 & -1 \\ 0 & 0 & 0 \end{bmatrix}$$

P3. (15 points) Find the best rank 2 approximation to A from (2) with respect to $\|\cdot\|_F$.

This is like 6.5 #4.

You know $\text{rank}(A) = 2$ so the best rank 2 approximation of A is A , but if you just plug into the computation, you get the following:

You already have the SVD of $A = U \Sigma V^T = S D S^T$, so $U = V$ in this case and $D = \Sigma$. Now the best rank-2 approximation of A is thus (using MATLAB type notation)

$$\begin{aligned} C &= S(:, 1:2) D(1:2, 1:2) S^T(1:2, :) \\ &= \begin{bmatrix} 1/\sqrt{6} & 1/\sqrt{2} \\ -2/\sqrt{6} & 0 \\ 1/\sqrt{6} & -1/\sqrt{2} \end{bmatrix} \begin{bmatrix} 3 & \\ & 1 \end{bmatrix} \begin{bmatrix} 1/\sqrt{6} & -2/\sqrt{6} & 1/\sqrt{6} \\ 1/\sqrt{2} & 0 & 1/\sqrt{2} \end{bmatrix} \\ &= \begin{bmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{bmatrix} = A \end{aligned}$$

P4. (15 points) Let

$$A = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix},$$

find the characteristic polynomial and all eigenvalues, both real and complex. Explain why A is diagonalizable and compute A^{2020} . Note, I do not ask you to diagonalize A .

Find eigenvalues:

$\det\left(\begin{bmatrix} -\lambda & 0 & 1 \\ 1 & -\lambda & 0 \\ 0 & 1 & -\lambda \end{bmatrix}\right) = -\lambda(-\lambda)^2 - (-1) = -\lambda^3 + 1$, so the roots are 1, $e^{i\frac{2\pi}{3}} = -\frac{1}{2} + i\frac{\sqrt{3}}{2}$, and $e^{i\frac{4\pi}{3}} = -\frac{1}{2} - i\frac{\sqrt{3}}{2}$.

Compute A^{2020} :

We see $2020 = 673 \cdot 3 + 1$, so $\lambda_i^{2020} = (\lambda_i^3)^{673} \cdot \lambda_i = \lambda_i$. So $S^{2020} = SD^{2020}S^{-1} = S \begin{bmatrix} \lambda_1^{2020} & & \\ & \lambda_2^{2020} & \\ & & \lambda_3^{2020} \end{bmatrix} S^{-1} = S \begin{bmatrix} \lambda_1 & & \\ & \lambda_2 & \\ & & \lambda_3 \end{bmatrix} S^{-1} = A$.

Note we actually don't need to know the eigenvalues, just that $\lambda^3 = 1$.

Alternatively, you might just compute that $A^3 = I$, so $A^{2020} = I^{637}A = A$.

Part III: Theory and Proofs (45 points; 15 points each)

Pick three of the following four options. If you try all four, I will grade the first three, so if this is not what you intend, then just do three, or at least make it clear which I should grade.

- P1. Let $L : V \rightarrow V$ be a linear transformation and let $\mathcal{B} = \{\mathbf{v}_1, \dots, \mathbf{v}_n\}$ be a basis. Show that $[L]_{\mathcal{B}}$ is upper triangular iff $L(\mathbf{v}_i) \in \text{span}\{\mathbf{v}_1, \dots, \mathbf{v}_i\}$ for all i .

This is an “if and only if” so there are two things to do.

(\implies) Assume $[L]_{\mathcal{B}}$ is upper-triangular. To make notation simpler suppose $[L]_{\mathcal{B}} = A$ and a_{ij} is the ij^{th} entry in A . Then $[L]_{\mathcal{B}} = [[L(\mathbf{v}_1)]_{\mathcal{B}} \cdots [L(\mathbf{v}_n)]_{\mathcal{B}}]$ and since A is upper-triangular

$$[L(\mathbf{v}_i)]_{\mathcal{B}} = \begin{bmatrix} a_{1i} \\ \vdots \\ a_{ii} \\ 0 \\ \vdots \\ 0 \end{bmatrix} \quad (\dagger)$$

and so $L(\mathbf{v}_i) = \sum_{j=1}^i a_{ji} \mathbf{v}_j \in \text{span}\{\mathbf{v}_1, \dots, \mathbf{v}_i\}$.

(\impliedby) Suppose $L(\mathbf{v}_i) \in \text{span}\{\mathbf{v}_1, \dots, \mathbf{v}_i\}$ for all i , then $L(\mathbf{v}_i) = \sum_{j=1}^i a_{ji} \mathbf{v}_j$ and thus (\dagger) holds here too, so $[L]_{\mathcal{B}}$ is upper-triangular.

- P2. Let $L : \mathbb{R}^n \rightarrow \mathbb{R}^n$ be a linear transformation with $L^2 = L$ and for all $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$, $\langle \mathbf{x}, L(\mathbf{y}) \rangle = \langle L(\mathbf{x}), \mathbf{y} \rangle$. Let $U = \text{rng}(L)$. (See [this](#) for information on inner products.)

- (a) Show that $L(\mathbf{x})$ is the orthogonal projection of \mathbf{x} onto U , that is, show that $\mathbf{x} - L(\mathbf{x}) \perp U$ for all $\mathbf{x} \in \mathbb{R}^n$.
- (b) Use (a) to show that $\|\mathbf{x} - L(\mathbf{x})\|_2^2 = \min\{\|\mathbf{x} - L(\mathbf{y})\|_2^2 \mid \mathbf{y} \in \mathbb{R}^n\}$.

Proof of (a): We must show $\langle \mathbf{x} - L(\mathbf{x}), L(\mathbf{y}) \rangle = 0$, we have

$$\begin{aligned} \langle \mathbf{x} - L(\mathbf{x}), L(\mathbf{y}) \rangle &= \langle \mathbf{x}, L(\mathbf{y}) \rangle - \langle L(\mathbf{x}), L(\mathbf{y}) \rangle \\ &= \langle \mathbf{x}, L(\mathbf{y}) \rangle - \langle \mathbf{x}, L(L(\mathbf{y})) \rangle && (\text{Since } \langle L(\mathbf{x}), \mathbf{z} \rangle = \langle \mathbf{x}, L(\mathbf{z}) \rangle.) \\ &= \langle \mathbf{x}, L(\mathbf{y}) \rangle - \langle \mathbf{x}, L(\mathbf{y}) \rangle && (\text{Since } L(L(\mathbf{y})) = L(\mathbf{y}).) \\ &= 0 \end{aligned}$$

This shows that $\mathbf{x} - L(\mathbf{x}) \perp U$.

Proof of (b): Let $\mathbf{y} \in \mathbb{R}^n$ be arbitrary, then

$$\begin{aligned} \|\mathbf{x} - L(\mathbf{y})\|_2^2 &= \|\mathbf{x} - L(\mathbf{x}) + L(\mathbf{x}) - L(\mathbf{y})\|_2^2 \\ &= \|\mathbf{x} - L(\mathbf{x})\|_2^2 + \|L(\mathbf{x} - \mathbf{y})\|_2^2 \\ (\text{By Pythagorean Theorem: } \mathbf{x} - L(\mathbf{x}) &\perp L(\mathbf{x} - \mathbf{y}) \text{ by (a) since } L(\mathbf{x} - \mathbf{y}) \in U) \\ &\leq \|\mathbf{x} - L(\mathbf{x})\|_2^2 && (\text{Since } \|L(\mathbf{x} - \mathbf{y})\|_2^2 \geq 0) \end{aligned}$$

Recall: (Pythagorean Theorem) If V is an inner-product space and $\mathbf{u} \perp \mathbf{v}$, then $\|\mathbf{u} + \mathbf{v}\|_2^2 = \|\mathbf{u}\|_2^2 + \|\mathbf{v}\|_2^2$.

Proof:

$$\begin{aligned} \|\mathbf{u} + \mathbf{v}\|_2^2 &= \langle \mathbf{u} + \mathbf{v}, \mathbf{u} + \mathbf{v} \rangle \\ &= \langle \mathbf{u} + \mathbf{v}, \mathbf{u} \rangle + \langle \mathbf{u} + \mathbf{v}, \mathbf{v} \rangle \\ &= \langle \mathbf{u}, \mathbf{u} \rangle + \langle \mathbf{v}, \mathbf{u} \rangle + \langle \mathbf{u}, \mathbf{v} \rangle + \langle \mathbf{v}, \mathbf{v} \rangle \\ &= \|\mathbf{u}\|_2^2 + 0 + 0 + \|\mathbf{v}\|_2^2 \end{aligned}$$

Note that given $U \subseteq \mathbb{R}^n$ there is a unique orthogonal projection onto U . Suppose L and L' are as above. We know that for all $\mathbf{x} \in \mathbb{R}^n$,

$$\|\mathbf{x} - L(\mathbf{x})\|_2^2 = \min\{\|\mathbf{x} - \mathbf{u}\|_2^2 \mid \mathbf{u} \in U\} = \|\mathbf{x} - L'(\mathbf{x})\|_2^2$$

But

$$\|\mathbf{x} - L(\mathbf{x})\|_2^2 = \|\mathbf{x} - L'(\mathbf{x}) + L'(\mathbf{x}) - L(\mathbf{x})\|_2^2 = \|\mathbf{x} - L'(\mathbf{x})\|_2^2 + \|L'(\mathbf{x}) - L(\mathbf{x})\|_2^2 = \|\mathbf{x} - L'(\mathbf{x})\|_2^2$$

and this means that $\|L'(\mathbf{x}) - L(\mathbf{x})\|_2^2 = 0$ so $L(\mathbf{x}) = L'(\mathbf{x})$ for all $\mathbf{x} \in \mathbb{R}^n$.

P3. Any quadratic $q(x_1, \dots, x_n)$ in n variables $\mathbf{x} = x_1, \dots, x_n$ can be written as

$$q(\mathbf{x}) = \mathbf{x}^T Q \mathbf{x} + P \mathbf{x} + c$$

where Q is $n \times n$ and symmetric, P is $1 \times n$, and $c \in \mathbb{R}$. This is trivial $Q_{ii} =$ the coefficient on x_i^2 , $Q_{ij} = Q_{ji} = \frac{1}{2}$ (the coefficient on $x_i x_j$), while $P_{1i} =$ the coefficient on x_i , and c is the constant term.

Example: Consider $q(x_1, x_2, x_3) = 7x_1^2 + 10x_2^2 + 19x_3^2 + 28x_1x_2 + 8x_1x_3 - 20x_2x_3 + 2x_2 - 3x_3 + 5$. Then

$$q(\mathbf{x}) = \begin{bmatrix} x_1 & x_2 & x_3 \end{bmatrix} \begin{bmatrix} 7 & 14 & 4 \\ 14 & 10 & -10 \\ 4 & -10 & 19 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 2 & -3 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + 5 = \mathbf{x}^T Q \mathbf{x} + P \mathbf{x} + c$$

Explain how the Spectral Theorem can be used to show that there is an orthonormal basis $\mathcal{C} = \{\mathbf{u}_1, \dots, \mathbf{u}_n\}$ so that if standard coordinates are replaced with coordinates relative to \mathcal{C} , i.e., $\mathbf{y} = [\mathbf{x}]_{\mathcal{C}}$, then $q(\mathbf{x}) = q'(\mathbf{y}) = \mathbf{y}^T D \mathbf{y} + P' \mathbf{y} + c$ where D is diagonal. Thus all *cross-terms*, terms of the form $y_i y_j$ for $i \neq j$, have been eliminated.

Use this to find $q'(\mathbf{y})$ for the example $q(\mathbf{x})$ above.

To save you some work: $Q = U D U^T$ where

$$U = \begin{bmatrix} -2/3 & -1/3 & 2/3 \\ 2/3 & -2/3 & 1/3 \\ 1/3 & 2/3 & 2/3 \end{bmatrix} \text{ and } D = \begin{bmatrix} -9 & & \\ & 27 & \\ & & 18 \end{bmatrix}$$

By the Spectral Theorem or SVD there is an orthonormal U so that $Q = U D U^T$ for D diagonal. Let $\mathcal{C} = \{\mathbf{u}_1, \dots, \mathbf{u}_n\}$ where \mathbf{u}_i is the i^{th} column of U , then $U^T = U^{-1}$ is the change of basis matrix from \mathcal{E} to \mathcal{C} and so $\mathbf{y} = U^T \mathbf{x} = U^{-1} \mathbf{x} = [\mathbf{x}]_{\mathcal{C}}$. Then $q(\mathbf{x}) = \mathbf{x}^T (U D U^T) \mathbf{x} + P \mathbf{x} + c = (\mathbf{x}^T U) D (U^T \mathbf{x}) + P U (U^T \mathbf{x}) + c = \mathbf{y}^T D \mathbf{y} + P' \mathbf{y} + c = q'(\mathbf{y})$. There are no terms in $q'(\mathbf{y})$ of the form $y_i y_j$ for $i \neq j$, i.e., no cross terms.

P4. Use the SVD to show that any square matrix A can be written as $A = U P$ where U is unitary and P is Hermitian.

Let $A = V \Sigma W^H$ as in SVD and let $U = V W^H$, this is unitary since both V and W are unitary. So

$$A = (V W^H (W \Sigma W^H)) = U P$$

where $P = W \Sigma W^H$. This P is clearly Hermitian.

Note: If we think of matrices like generalizations of complex numbers, then Hermitian matrices are like the reals, $P^H = P$ is like $\bar{z} = z$ for $z \in \mathbb{C}$. A unitary is "like" a rotation, so here we represent A as a rotation followed by a "real." this is like writing $z = e^{i\theta} r$, the polar form of a complex number.