Name: ____________________ Exam 1 - MAT345

I True/False (60 points; 6 points each)

Each problem is points for a total of 50 points. (5 points each and one free point.) In class, you only provide the T/F.

Corrections: If you choose to make corrections for 50% back on this section, then you must provide reasons for ALL of these, not just the ones that you miss. A reason might be as simple as, "by Theorem ...," or it might require an example or counterexample. In any case, some correct reason or counterexample must be provided.

Problem I.1 (50 points; 5 points each)**.** Decide if each of the following is true or false.

1. True If $A\boldsymbol{x} = \boldsymbol{b}$ has a unique solution for some $\boldsymbol{b}$, then $A\boldsymbol{x} = \boldsymbol{c}$ has at most one solution for any $\boldsymbol{c}$.

   There are several ways to verify this. One way is to appeal to facts about Gaussian elimination. If there is a unique solution to $A\boldsymbol{x} = \boldsymbol{b}$, then there are no free variables. This is also the case when trying to solve $A\boldsymbol{x} = \boldsymbol{c}$ and hence there would be at most one solution.

   An argument that is essentially the same is as follows. The solution set to $A\boldsymbol{x} = \boldsymbol{b}$ is $\boldsymbol{x_b} + \mathrm{NS}(A)$ where $\boldsymbol{x_b}$ is any specific solution to $A\boldsymbol{x} = \boldsymbol{b}$. So if there is a unique solution to $A\boldsymbol{x} = \boldsymbol{b}$, then $\mathrm{NS}(A) = \{\boldsymbol{0}\}$. So if there is a solution $\boldsymbol{x_c}$ to $A\boldsymbol{x} = \boldsymbol{c}$, then the set of all solutions is $\boldsymbol{x_c} + \mathrm{NS}(A) = \boldsymbol{x_c} + \{\boldsymbol{0}\} = \boldsymbol{x_c}$, so the solution is unique if one exists at all.

2. False Diagonal $n \times n$ matrices commute with arbitrary $n \times n$ matrices, that is, for any $n \times n$ diagonal D, $DA = AD$ for all $n \times n$ matrices A.

   To see this is false, you may just produce an example, like

   $$\begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 2 \\ 3 & 3 \end{bmatrix}$$

   while

   $$\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix} = \begin{bmatrix} 2 & 3 \\ 2 & 3 \end{bmatrix}$$

   Generally, if $DA = AD$ for all A, then $D = dI$ for some d. (Fun exercise.)

3. True For A an $m \times n$ matrix $(\boldsymbol{e}_i^m)^T A \boldsymbol{e}_j^n = A_{i,j}$.

   $A\boldsymbol{e}_j^n$ is the j^{th} column of A and $(\boldsymbol{e}_i^m)^T(A\boldsymbol{e}_j^n)$ is thus the i^{th} entry in the j^{th} column of A and hence is $A_{i,j}$.

   Alternatively, $(\boldsymbol{e}_i^m)^T A$ is the i^{th} row of A and so $((\boldsymbol{e}_i^m)^T A)e_j^n$ is the j^{th} entry in the i^{th} column of A, which again is the $(i,j)^{\text{th}}$ entry of A.

4. False Let A and B be $n \times n$ matrices, if $(A-B)(A+B) = O$, then either $A = B$ or $A = -B$.

   One simple example would be given by:

$$A = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \text{ and } B = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

5. True If $A^2 - I$ is invertible, then $A - I$ and $A + I$ must also both be invertible.

$$A^2 - I = A^2 - I^2 = (A-I)(A+I)$$

   and AB is invertible iff A and B are invertible.

6. False If A is equivalent to B, then $\det(A) = \det(B)$.

   Type I and Type II operations generally alter the determinant.

7. True If A and B are equivalent matrices, then $\mathrm{NS}(A) = \mathrm{NS}(B)$.

   This is basically the whole point of equivalence. If B is obtained by a sequence of elementary row operations from A, then $A\boldsymbol{x} = \mathbf{0} \iff B\boldsymbol{x} = \mathbf{0}$. Elementary row operations were defined just to ensure this property.

8. False Consider the operation flip(A) that "flips" a matrix horizontally, so for example

$$\mathrm{flip}\left(\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}\right) = \begin{bmatrix} 2 & 1 \\ 4 & 3 \end{bmatrix} \text{ while } \mathrm{flip}\left(\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}\right) = \begin{bmatrix} 3 & 2 & 1 \\ 6 & 5 & 4 \end{bmatrix}$$

   For any $n \times n$ matrix A, $\det(\mathrm{flip}(A)) = -\det(A)$.

   The point is to consider how many "swaps" of columns are required. If $n = 2k$ or $n = 2k+1$, then k swaps are required, so for k even $\det(\mathrm{flip}(A)) = \det(A)$, e.g., $n = 1, 4, 5, 8, 9, \ldots$, however, if k is odd, then $\det(\mathrm{flip}(A)) = -\det(A)$, e.g., $2, 3, 6, 7, \ldots$.

9. True We have used in class that AB is invertible iff both A and B are invertible, but never proved this. The following is a valid proof of this fact.

$$\begin{aligned} AB \text{ is invertible} &\iff \det(AB) \neq 0 \\ &\iff \det(A)\det(B) \neq 0 \\ &\iff \det(A) \neq 0 \text{ and } \det(B) \neq 0 \\ &\iff A \text{ is invertible and } B \text{ is invertible} \end{aligned}$$

   Yes, this is clearly a simple and valid proof and we have proved the relevant facts in class, namely, $\det(AB) = \det(A)\det(B)$ and $\det(A) \neq 0 \iff A$ is invertible.

10. False Cramer's rule is the most efficient way to solve a system of n equations and n unknowns and it works even when Gaussian elimination fails.

    Everything said is just the opposite of the truth. For an $n \times n$ system, Gaussian elimination takes about n^3 operations whereas Cramer's rule takes something like $n!$. So for a 10×10 system on the order of 1,000 operations are required for elimination, ok, that is a lot you say, but Cramer's rule requires on the order of 1,307,674,368,000 many operations!

    Moreover, Cramer's rule only works when there is a unique solution whereas Gaussian elimination provides the entire parameterized family of solutions.

II Computational (90 points)

Show all computations so that you make clear what your thought processes are.

Problem II.1 (20 pts)**.** Let

$$A = \begin{bmatrix} 2 & 0 & 0 \\ 3 & 2 & 5 \\ 3 & -3 & -1 \\ -2 & 0 & 1 \end{bmatrix}; \qquad B = \begin{bmatrix} 4 & 5 & -1 & -3 \\ 2 & -4 & 3 & 0 \\ -1 & 0 & 3 & 0 \end{bmatrix}$$

1. Express the fourth row of AB as a linear combination of rows of B.

$$(-2)\begin{bmatrix} 4 & 5 & -1 & 3 \end{bmatrix} + (0)\begin{bmatrix} 2 & -4 & 3 & 0 \end{bmatrix} + (1)\begin{bmatrix} -1 & 0 & 3 & 0 \end{bmatrix} = \begin{bmatrix} -9 & -10 & 5 & -6 \end{bmatrix}$$

2. Express the second column of AB as a linear combination of the columns of A.

$$(5)\begin{bmatrix} 2 \\ 3 \\ 3 \\ -2 \end{bmatrix} + (-4)\begin{bmatrix} 0 \\ 2 \\ -3 \\ 0 \end{bmatrix} + (0)\begin{bmatrix} 0 \\ 5 \\ -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 10 \\ 7 \\ 27 \\ -10 \end{bmatrix}$$

3. Express $(AB)_{1,2}$ as a product of a row of A and a column of B.

$$(AB)_{1,2} = \begin{bmatrix} 2 & 0 & 0 \end{bmatrix} \begin{bmatrix} 5 \\ -4 \\ 0 \end{bmatrix} = 10$$

Problem II.2 (30 pts)**.** Solve $A\boldsymbol{x} = \boldsymbol{b}$ where

$$A = \begin{bmatrix} -1 & -1 & 1 & 0 & 1 \\ -5 & -7 & 1 & -4 & 9 \\ -4 & -10 & -8 & -12 & 17 \\ 2 & -8 & -22 & -20 & 15 \end{bmatrix} \qquad \boldsymbol{b} = \begin{bmatrix} 2 \\ 24 \\ 48 \\ 72 \end{bmatrix}$$

1. (15 points) Use row operations (show all work and indicate operations) to reduce A to an echelon form. (This should work out very nicely - no fractions required..)
2. (10 points) Use back-substitution to solve the resulting system. Make sure to indicate which variables are free.
3. (5 points) Write your solution as a linear combination of vectors.

Gauss-Jordan elimination to get echelon form:

$$\left[\begin{array}{ccccc|c} -1 & -1 & 1 & 0 & 1 & 2 \\ -5 & -7 & 1 & -4 & 9 & 24 \\ -4 & -10 & -8 & -12 & 17 & 48 \\ 2 & -8 & -22 & -20 & 15 & 72 \end{array}\right] \underset{\substack{R_2-5R_1\to R_2 \\ R_3-4R_1\to R_3 \\ R_4+2R_1\to R_4}}{\longrightarrow} \left[\begin{array}{ccccc|c} -1 & -1 & 1 & 0 & 1 & 2 \\ 0 & -2 & -4 & -4 & 4 & 14 \\ 0 & -6 & -12 & -12 & 13 & 40 \\ 0 & -10 & -20 & -20 & 17 & 76 \end{array}\right]$$

$$\underset{\substack{R_3-2R_2\to R_3 \\ R_4-5R_2\to R_4}}{\longrightarrow} \left[\begin{array}{ccccc|c} -1 & -1 & 1 & 0 & 1 & 2 \\ 0 & -2 & -4 & -4 & 4 & 14 \\ 0 & 0 & 0 & 0 & 1 & -2 \\ 0 & 0 & 0 & 0 & -3 & 6 \end{array}\right]$$

$$\underset{R_4+3R_3\to R_4}{\longrightarrow} \left[\begin{array}{ccccc|c} -1 & -1 & 1 & 0 & 1 & 2 \\ 0 & -2 & -4 & -4 & 4 & 14 \\ 0 & 0 & 0 & 0 & 1 & -2 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

Back-substitution: x_3 and x_4 are free.

$$\boxed{x_5 = -2}$$

$$\boxed{x_4 = \alpha}$$

$$\boxed{x_3 = \beta}$$

$$-2x_2 - 4\beta - 4\alpha + 4(-2) = 14 \to -2x_2 = 22 + 4\beta + 4\alpha \to \boxed{x_2 = -2\alpha - 2\beta - 11}$$

$$-x_1 - (-2\alpha - 2\beta - 11) + \beta - 2 = 2 \to \boxed{x_1 = 2\alpha + 3\beta + 7}$$

Solution as a linear combination of vectors:

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} 2\alpha + 3\beta + 7 \\ -2\alpha - 2\beta - 11 \\ \beta \\ \alpha \\ -2 \end{bmatrix} = \begin{bmatrix} 7 \\ -11 \\ 0 \\ 0 \\ -2 \end{bmatrix} + \alpha \begin{bmatrix} 2 \\ -2 \\ 0 \\ 1 \\ 0 \end{bmatrix} + \beta \begin{bmatrix} 3 \\ -2 \\ 1 \\ 0 \\ 0 \end{bmatrix}$$

Problem II.3 (20 pts)**.** Use Cramer's rule to find x_4, where

$$\begin{bmatrix} 3 & -2 & 0 & 3 \\ -1 & 3 & 0 & 3 \\ 0 & 2 & 0 & 2 \\ 2 & 1 & 3 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 9 \\ 5 \\ 4 \\ 2 \end{bmatrix}$$

Note: These determinants should work out very nicely if you chose how you expand carefully.

Let

$$A = \begin{bmatrix} 3 & -2 & 0 & 3 \\ -1 & 3 & 0 & 3 \\ 0 & 2 & 0 & 2 \\ 2 & 1 & 3 & 0 \end{bmatrix} \text{ and } B = \begin{bmatrix} 3 & -2 & 0 & 9 \\ -1 & 3 & 0 & 5 \\ 0 & 2 & 0 & 4 \\ 2 & 1 & 3 & 2 \end{bmatrix}$$

so that B is obtained by replacing the 4th column of A by $\left[\begin{smallmatrix} 9 \\ 5 \\ 4 \\ 2 \end{smallmatrix}\right]$. Then

$$x_4 = \frac{\det(B)}{\det(A)}$$

where, by expanding along the 3rd column of A we have

$$\begin{aligned} \det(A) &= (-3)\det \begin{bmatrix} 3 & -2 & 3 \\ -1 & 3 & 3 \\ 0 & 2 & 2 \end{bmatrix} \\ &= (-3)\left((-2)\det \begin{bmatrix} 3 & 3 \\ -1 & 3 \end{bmatrix} + (2)\det \begin{bmatrix} 3 & -2 \\ -1 & 3 \end{bmatrix}\right) \\ &= (-3)\big((-2)(9+3) + (2)(9-2)\big) \\ &= (-3)(-2)(12-7) = 30 \end{aligned}$$

and by expanding again along the 3rd column of B

$$\begin{aligned} \det(B) &= (-3)\det \begin{bmatrix} 3 & -2 & 9 \\ -1 & 3 & 5 \\ 0 & 2 & 4 \end{bmatrix} \\ &= (-3)\left((-2)\det \begin{bmatrix} 3 & 9 \\ -1 & 5 \end{bmatrix} + (4)\det \begin{bmatrix} 3 & -2 \\ -1 & 3 \end{bmatrix}\right) \\ &= (-3)\big((-2)(15+9) + 4(9-2)\big) = (-3)(-2)(24-2(7)) = 60 \end{aligned}$$

So

$$x_4 = \frac{60}{30} = 2$$

Problem II.4 (20 pts)**.** Consider

$$\begin{aligned} A = \begin{bmatrix} 1 & 2 & -4 & 3 & 2 \\ 2 & 4 & -7 & 4 & 5 \\ -3 & -6 & 14 & -13 & -3 \end{bmatrix} &\underset{\substack{R_2-2R_1\to R_2 \\ R_3+3R_1\to R_3}}{\longrightarrow} \begin{bmatrix} 1 & 2 & -4 & 3 & 2 \\ 0 & 0 & 1 & -2 & 1 \\ 0 & 0 & 2 & -4 & 3 \end{bmatrix} \\ &\underset{R_3-2R_2\to R_3}{\longrightarrow} \begin{bmatrix} 1 & 2 & -4 & 3 & 2 \\ 0 & 0 & 1 & -2 & 1 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} = U \end{aligned}$$

Write A in the form LU where L is lower-triangular with 1's on the diagonal, and U is the Echelon matrix given.

$$L = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ -3 & 2 & 1 \end{bmatrix}$$

and

$$A = \begin{bmatrix} 1 & 2 & -4 & 3 & 2 \\ 2 & 4 & -7 & 4 & 5 \\ -3 & -6 & 14 & -13 & -3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ -3 & 2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & -4 & 3 & 2 \\ 0 & 0 & 1 & -2 & 1 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} = LU$$

III Theory and Proofs (40 points; 20 points each)

Choose two of the four options. If you try more than two, I will grade only the first two, not the best two. You must decide what should be graded. These will be due 2/7 in class. Make sure your work is complete and clear. Explain your work, a proof is not just a bunch of math symbols, it is an explanation of why something is true.

Problem III.1 (20 pts)**.** If A and B are invertible $n \times n$ matrices, show that

$$(AB)^2 = A^2B^2 \iff AB = BA$$

($\Leftarrow$) Assume $AB = BA$, then

$$(AB)^2 = (AB)(AB) = A(BA)B = A(AB)B = (AA)(BB) = A^2B^2$$

So $AB = BA \implies (AB)^2 = A^2B^2$ as desired.

($\Rightarrow$) Suppose $(AB)^2 = A^2B^2$, then

$$\begin{gathered}(AB)(AB) = A^2B^2 = AABB \\ \Downarrow \\ (A^{-1}A)BA(BB^{-1}) = (A^{-1}A)AB(BB^{-1}) \\ \Downarrow \\ IBAI = IABI \\ \Downarrow \\ BA = AB\end{gathered}$$

So $(AB)^2 = A^2B^2 \implies AB = BA$.

Problem III.2 (20 pts)**.** Show that for any $m \times n$ matrix A,

$$\sum_{i=1}^{m}\sum_{j=1}^{n}\big((\boldsymbol{e}_i^m)^T A\boldsymbol{e}_j^n\big)\big(\boldsymbol{e}_i^m(\boldsymbol{e}_j^n)^T\big) = A.$$

The point here is that $(\boldsymbol{e}_i^m)^T A\boldsymbol{e}_j^n = A_{i,j}$ (see discussion of I.3) and $e_i^m(\boldsymbol{e}_j^n)^T$ is the $m \times n$ matrix with a 1 in the $(i,j)^{\text{th}}$ position and 0's everywhere else, call this matrix $\boldsymbol{E}_{i,j}^{m\times n}$, these make up the *standard basis* of $\mathbb{R}^{m\times n}$.

Problem III.3 (20 pts)**.** Let A be an $n \times n$ matrix such that $AB = BA$ for all $n \times n$ matrices B. show that $A = \alpha I$ for some scalar α.

Consider the matrix $E_{i,j}$ which has a 1 in the $(i,j)^{\text{th}}$ position and 0's everywhere else.

$$E_{i,j} = \boldsymbol{e}_i \boldsymbol{e}_j^T = \begin{array}{c} \begin{array}{ccccccc} & & & j & & & \end{array} \\ \begin{bmatrix} \mathbf{0} & \cdots & \mathbf{0} & \boldsymbol{e}_i & 0 & \cdots & \mathbf{0} \end{bmatrix} \end{array} = \begin{array}{cc} \begin{bmatrix} \mathbf{0} \\ \vdots \\ \mathbf{0} \\ \boldsymbol{e}_j \\ \mathbf{0} \\ \vdots \\ \mathbf{0} \end{bmatrix} & \begin{array}{c} \\ \\ \\ i \\ \\ \\ \\ \end{array} \end{array}$$

So

$$E_{i,j}A = \boldsymbol{e}_i(\boldsymbol{e}_j^T A) = \boldsymbol{e}_i A_{j,*} = \begin{array}{cc} \begin{bmatrix} \mathbf{0} \\ \vdots \\ \mathbf{0} \\ A_{j,*} \\ \mathbf{0} \\ \vdots \\ \mathbf{0} \end{bmatrix} & \begin{array}{c} \\ \\ \\ i \\ \\ \\ \\ \end{array} \end{array}$$

and

$$AE_{i,j} = (A\boldsymbol{e}_i)\boldsymbol{e}_j^T = A_{*,i}\boldsymbol{e}_j^T = \begin{array}{c} \begin{array}{ccccccc} & & & j & & & \end{array} \\ \begin{bmatrix} \mathbf{0} & \cdots & \mathbf{0} & A_{*,i} & \mathbf{0} & \cdots & \mathbf{0} \end{bmatrix} \end{array}$$

To be clear, $AE_{i,j}$ is the matrix with 0's everywhere except the j^{th} column where we find $A_{*,i}$, the i^{th} column of A. Similarly, $E_{i,j}A$ is the matrix with all rows 0's except for the i^{th} row, where we find $A_{j,*}$.

Since $E_{i,j}A = AE_{j,i}$ we have that

$$(E_{i,j}A)_{i,m} = A_{j,m} = (AE_{i,j})_{i,m} = \begin{cases} 0 & m \neq j \\ A_{i,i} & m = j \end{cases}$$

But this shows that $A_{j,m} = 0$ when $j \neq m$ and that $A_{j,j} = A_{i,i}$ for all i, but this means that $A = \alpha I$ for $\alpha = A_{1,1}$.

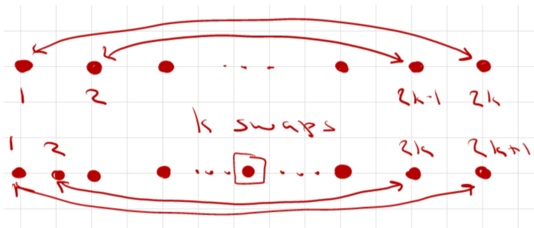
Problem III.4 (20 pts)**.** Consider the operation $\operatorname{rot}(A)$ that rotates a matrix clockwise by 90°, for example,

$$\operatorname{rot}\left(\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}\right) = \begin{bmatrix} 3 & 1 \\ 4 & 2 \end{bmatrix} \text{ while text}\left(\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}\right) = \begin{bmatrix} 4 & 1 \\ 5 & 2 \\ 6 & 3 \end{bmatrix}$$

For $n \times n$ matrices A come up with and prove a simple formula for $\det(\operatorname{rot}(A))$ in terms of $\det(A)$.

If you think about it $\mathrm{rot}(A)$ is like A^T in that it turns the rows of A into the columns of $\mathrm{rot}(A)$, just in the opposite order. (Note that the rows of $\mathrm{rot}(A)$ are the columns of A, but all are reversed.)

Clearly, the $\det(\mathrm{rot}(A)) = \pm\det(A)$ since the columns of $\mathrm{rot}(A)$ are just the rows of A. To determine the sign we just need to consider how many column exchanges are required on $\mathrm{rot}(A)$ to transform it to A^T. If $n = 2k$ or $n = 2k+1$ exactly k column exchanges are required.



So

$$\det(\mathrm{rot}(A)) = (-1)^k \det(A^T) = (-1)^k \det(A)$$

You can define $\mathrm{flip}(A)$ as we did in the T/F section and then see that $\mathrm{rot}(A) = \mathrm{flip}(A^T)$ and so as $\det(\mathrm{flip}(A)) = (-1)^k \det(A)$ we have $\det(\mathrm{rot}(A)) = \det(\mathrm{flip}(A^T)) = (-1)^k \det(A^T) = (-1)^k \det(A)$.