Homework 6 Partial Solutions

Section 6.1

1. Find eigenvalues and basis for the corresponding eigenspaces for A:

(f)

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

$\det(A - \lambda I) = \lambda^3$ so 0 is the only eigenvalue with algebraic multiplicity 3. $\text{NS}(A - 0I) = \text{NS}(A) = \text{span}\{(1,0,0)\}$, so the geometric multiplicity is 1 and thus A is deficient.

(g)

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 2 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$

so

$$A - \lambda I = \begin{bmatrix} 1-\lambda & 1 & 1 \\ 0 & 2-\lambda & 1 \\ 0 & 0 & 1-\lambda \end{bmatrix}$$

and hence $\det(A - \lambda I) = (1-\lambda)^2(2-\lambda)$

To find the eigenspaces we just find the nullspaces for $A - \lambda I$

$\lambda = 1$:

$$A - I = \begin{bmatrix} 0 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix} \quad \text{so} \quad \text{rref}(A) = \begin{bmatrix} 0 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

and $\text{NS}(A - I) = \text{span}\{(1,0,0), (0,1,-1)\}$.

$\lambda = 2$:

$$A - 2I = \begin{bmatrix} -1 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & -1 \end{bmatrix} \quad \text{so} \quad \text{rref}(A) = \begin{bmatrix} 1 & -1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

and $\text{NS}(A - 2I) = \text{span}\{(1,1,0)\}$.

(i)

$$A = \begin{bmatrix} 4 & -5 & 1 \\ 1 & 0 & -1 \\ 0 & 1 & -1 \end{bmatrix}$$

so

$$A - \lambda A = \begin{bmatrix} 4-\lambda & -5 & 1 \\ 1 & -\lambda & -1 \\ 0 & 1 & -1-\lambda \end{bmatrix}$$

Expanding along the bottom row

$$\begin{aligned} \det(A - \lambda I) &= -\det \begin{bmatrix} 4-\lambda & 1 \\ 1 & -1 \end{bmatrix} + (-1-\lambda)\det \begin{bmatrix} 4-\lambda & -5 \\ 1 & -\lambda \end{bmatrix} \\ &= ((4-\lambda)+1) - (\lambda+1)(-\lambda(4-\lambda)+5) \\ &= 5 - \lambda - (\lambda+1)(\lambda^2 - 4\lambda + 5) \\ &= 5 - \lambda - \lambda^3 + 4\lambda^2 - 5\lambda - \lambda^2 + 4\lambda - 5 \\ &= -\lambda^3 + 3\lambda^2 - 2\lambda \\ &= -\lambda(\lambda^2 - 3\lambda + 2) = -\lambda(\lambda-2)(\lambda-1) \end{aligned}$$

You can have MATLAB help here with taking the determinant and factoring it. This only works in very special situations.

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A=[4 -5 1;1 0 -1;0 1 -1];
syms lambda;
factor(det(A - lambda*eye(3)));
```

So the eigenvalues are $2 > 1 > 0$. since there are three eigenvalues in $\mathbb{R}^3$ we know each eigenspace has dimension 1.

$\lambda = 2$:

$$A - 2I = \begin{bmatrix} 2 & -5 & 1 \\ 1 & -2 & -1 \\ 0 & 1 & -3 \end{bmatrix} \quad \text{and} \quad \text{rref}(A - 2I) = \begin{bmatrix} 1 & 0 & -7 \\ 0 & 1 & -3 \\ 0 & 0 & 0 \end{bmatrix}$$

so $E_2 = \text{NS}(A - 2I) = \text{span}\{(7, 3, 1)\}$.

$\lambda = 1$:

$$A - 2I = \begin{bmatrix} 3 & -5 & 1 \\ 1 & -1 & -1 \\ 0 & 1 & -2 \end{bmatrix} \quad \text{and} \quad \text{rref}(A - I) = \begin{bmatrix} 1 & 0 & -3 \\ 0 & 1 & -2 \\ 0 & 0 & 0 \end{bmatrix}$$

so $E_1 = \text{NS}(A - I) = \text{span}\{(3, 2, 1)\}$.

$\lambda = 0$:

$$A - 0I = \begin{bmatrix} 4 & -5 & 1 \\ 1 & 0 & -1 \\ 0 & 1 & -1 \end{bmatrix} \quad \text{and} \quad \text{rref}(A) = \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}$$

so $E_0 = \text{NS}(A - 0I) = \text{NS}(A) = \text{span}\{(1, 1, 1)\}$.

7. $\boldsymbol{x}$ is an eigenvector of $n \times n$ matrix A. $B = I - 2A + A^2 = (I - A)(I - A) = (I - A)^2$. So $B\boldsymbol{x} = (I - A)^2\boldsymbol{x} = (I - A)(I - A)\boldsymbol{x} = (I - A)(\boldsymbol{x} - A\boldsymbol{x}) = (I - A)(1 - \lambda)\boldsymbol{x} = (1 - \lambda)(I - A)\boldsymbol{x} = (1 - \lambda)^2\boldsymbol{x}$. So $(1 - \lambda)^2$ is an eigenvalue for B with same eigenvector.

9. If $A^k = 0$ and $\lambda, \boldsymbol{x}$ is an eigenvalue/eigenvector pair, then $A^k\boldsymbol{x} = \lambda^k\boldsymbol{x} = 0\boldsymbol{x} = \boldsymbol{0}$, but this means $\lambda = 0$.

10. $B = A - \alpha I$, so $\det(B - tI) = \det(A - (\alpha + t)I)$, so λ is an eigenvalue of B iff $\alpha + \lambda$ an eigenvalue of A. Given λ an eigenvalue of B, $E^B_\lambda = \mathrm{NS}(B - \lambda I) = \mathrm{NS}(A - (\alpha + \lambda)I) = E^A_{\alpha+\lambda}$.

33. Let $A, B \in \mathbb{R}^{n\times n}$, show that AB and BA have the same eigenvalues.

$$\begin{aligned}
\lambda \in \mathrm{Eig}(AB) &\implies AB\boldsymbol{x} = \lambda\boldsymbol{x} \implies BA(B\boldsymbol{x}) = \lambda(B\boldsymbol{x}) \implies \lambda \in \mathrm{Eig}(BA) \\
\lambda \in \mathrm{Eig}(BA) &\implies BA\boldsymbol{x} = \lambda\boldsymbol{x} \implies AB(A\boldsymbol{x}) = \lambda(A\boldsymbol{x}) \implies \lambda \in \mathrm{Eig}(AB)
\end{aligned}$$

34. Argument 1: Suppose $AB - BA = I$, then $AB - I = BA$, but then by (33) and (10)

$$\lambda \in \mathrm{Eig}(AB) \iff 1 + \lambda \in \mathrm{Eig}(BA) \iff \lambda \in \mathrm{Eig}(BA)$$

But then we get $\lambda \in \mathrm{Eig}(BA) \implies 1 + \lambda \in \mathrm{Eig}(BA) \implies 2 + \lambda \in \mathrm{Eig}(BA) \implies \cdots$.

Notice that this actually proves $AB - BA \neq \lambda I$, for any $\lambda \neq 0$ and even $AB - BA$ is not even similar to λI for any $\lambda > 0$.

Argument 2: If $AB - BA = I$, then

$$\mathrm{tr}(AB - BA) = \mathrm{tr}(AB) - \mathrm{tr}(BA) = \mathrm{tr}(I) = n$$

But we know $\mathrm{tr}(AB) = \mathrm{tr}(BA)$, so we get $0 = n$.

Section 6.3

1.

(e) $\det(A - \lambda I) = -(\lambda - 2)(\lambda - 1)(\lambda + 2)$ so the eigenvalues are $2 > 1 > -2$ we find the eigenspaces

$$E_2 = \mathrm{span}\{(0, 3, 1)\} \quad E_1 = \mathrm{span}\{(3, 1, 2)\} \quad E_{-2} = \mathrm{span}\{(0, -1, 1)\}$$

For example, $E_2 = \mathrm{NS}(A - 2I) = \mathrm{NS}(\mathrm{rref}(A - 2I))$.

$$\mathrm{rref}(A - 2I) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -3 \\ 0 & 0 & 0 \end{bmatrix}$$

so

$$E_2 = \mathrm{span}\{(0, 3, 1)\}$$

So we have:

$$A = S\Lambda S^{-1} = \begin{bmatrix} 0 & 3 & 0 \\ 3 & 1 & -1 \\ 1 & 2 & 1 \end{bmatrix} \begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{bmatrix} \begin{bmatrix} 0 & 3 & 0 \\ 3 & 1 & -1 \\ 1 & 2 & 1 \end{bmatrix}^{-1}$$

where S is the transition matrix from the basis of eigenvectors to the standard basis, so the columns of S are just the eigenvectors.

4. Find B so that $B^2 = A$.

(a)

$$A = \begin{bmatrix} 2 & 1 \\ -2 & -1 \end{bmatrix}$$

$$\det(A - tI) = \det \begin{bmatrix} 2-t & 1 \\ -2 & -1-t \end{bmatrix} = (2-t)(-1-t) + 2 = t^2 - t - 2 + 2 = t(t-1)$$

So the eigenvalues are 0 and 1, inspection gives

$$E_0 = \operatorname{span}\{(1,-2)\} \qquad E_1 = \operatorname{span}\{(1,-1)\}$$

So

$$A = \begin{bmatrix} 1 & 1 \\ -2 & -1 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} -1 & -1 \\ 2 & 1 \end{bmatrix}$$

This has the form SDS^{-1} so let $B = SD^{1/2}S^{-1}$, then $B^2 = SD^{1/2}S^{-1}SD^{1/2}S^{-1} = SDS^{-1} = A$. Here $D^{1/2} = D$, so $B = A$ and it is easy to check that $A^2 = A$.

Notice: For diagonal $D = \operatorname{diag}(d_1, \ldots, d_n)$, $D^r = \operatorname{diag}(d_1^r, \ldots, d_n^r)$.

(b)

$$A = \begin{bmatrix} 9 & -5 & 3 \\ 0 & 4 & 3 \\ 0 & 0 & 1 \end{bmatrix}$$

The eigenvalues are $\lambda_1 = 9 > \lambda_2 = 4 > \lambda_3 = 1$.

$$E_9 = \operatorname{NS}(A - 9I) = \operatorname{NS} \begin{bmatrix} 0 & -5 & 3 \\ 0 & -5 & 3 \\ 0 & 0 & -8 \end{bmatrix} = \operatorname{span}\{(1,0,0)\}$$

$$E_4 = \operatorname{NS}(A - 4I) = \operatorname{NS} \begin{bmatrix} 5 & -5 & 3 \\ 0 & 0 & 3 \\ 0 & 0 & -3 \end{bmatrix} = \operatorname{span}\{(1,1,0)\}$$

$$\begin{aligned} E_1 = \operatorname{NS}(A - I) &= \operatorname{NS} \begin{bmatrix} 8 & -5 & 3 \\ 0 & 3 & 3 \\ 0 & 0 & 0 \end{bmatrix} \\ &= \operatorname{NS} \begin{bmatrix} 8 & -8 & 0 \\ 0 & 3 & 3 \\ 0 & 0 & 0 \end{bmatrix} = \operatorname{span}\{(1,1,-1)\} \end{aligned}$$

Let

$$S = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & -1 \end{bmatrix} \qquad D = \begin{bmatrix} 9 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

then $A = SDS^{-1}$. Let $A^{1/n} = SD^{1/n}S^{-1}$, then

$$\begin{aligned} A^n &= (SD^{1/n}S^{-1})(SD^{1/n}S^{-1}) \cdots (SD^{1/n}S^{-1}) \\ &= SD^{1/n}(S^{-1}S)D^{1/n}(S^{-1}S) \cdots (S^{-1}S)D^{1/n}S^{-1} \\ &= SD^{1/n}ID^{1/n}I \cdots ID^{1/n}S^{-1} = S(D^{1/n})^n S^{-1} = SDS^{-1} = A \end{aligned}$$

So $B = A^{1/2} = SD^{1/2}S^{-1}$ is the matrix:

$$B = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} 3 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & -1 \end{bmatrix} = \begin{bmatrix} 3 & -1 & 1 \\ 0 & 2 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$

18. If B is diagonalizable, then $B = SDS^{-1}$, if B is similar to A, then $A = TBT^{-1}$, so $A = TSDS^{-1}T^{-1}$. Letting $U = TS$, we have $A = UDU^{-1}$.

This is sort of trivial, the whole point is that similarity is an equivalence relation. So $A \sim B \sim D \implies A \sim D$.

19. If $A = SD_AS^{-1}$ and $B = SD_BS^{-1}$, then

$$AB = SD_AS^{-1}SD_BS^{-1} = SD_AD_BS^{-1} = SD_BD_AS{-1} = SD_BS^{-1}SD_AS^{-1} = BA$$

The key point here is

$$\operatorname{diag}(d_1, \ldots, d_n) \operatorname{diag}(e_1, \ldots, e_n) = \operatorname{diag}(d_1e_1, \ldots, d_ne_n),$$

so clearly diagonal matrices commute.

31. Compute e^A

If $A = SDS^{-1}$, where $D = \operatorname{diag}(d_1, \ldots, d_n)$, then

$$e^A = \sum_{i=0}^{\infty} \frac{1}{i!} A^i = \sum_{i=0}^{\infty} \frac{1}{i!} (SDS^{-1})^i = S \left(\sum_{i=0}^{\infty} \frac{1}{i!} D^i \right) S^{-1}$$

But $\sum_{i=0}^{\infty} \frac{1}{i!} \operatorname{diag}(d_1, d_2, \ldots d_n)^i = \operatorname{diag}(\sum_{i=0}^{\infty} \frac{1}{i!} d_1^i, \ldots, \sum_{i=0}^{\infty} \frac{1}{i!} d_n^i) = \operatorname{diag}\big(e^{d_1}, \ldots, e^{d_n}\big)$

(b) $A = \left[\begin{smallmatrix} 3 & 4 \\ -2 & -3 \end{smallmatrix}\right]$ so $p_A(t) = (3-t)(-3-t) + 8 = t^2 - 1 = (t-1)(t+1)$.

$\lambda = 1$: $\operatorname{NS}(A - I) = \operatorname{NS}\left[\begin{smallmatrix} 2 & 4 \\ -2 & -4 \end{smallmatrix}\right] = \operatorname{span}\{(2, -1)\}$

$\lambda = -1$: $\operatorname{NS}(A - I) = \operatorname{NS}\left[\begin{smallmatrix} 4 & 4 \\ -2 & -2 \end{smallmatrix}\right] = \operatorname{span}\{(1, -1)\}$

Let $S = \left[\begin{smallmatrix} 2 & 1 \\ -1 & -1 \end{smallmatrix}\right]$, $S^{-1} = \frac{1}{-1}\left[\begin{smallmatrix} -1 & -1 \\ 1 & 2 \end{smallmatrix}\right] = \left[\begin{smallmatrix} 1 & 1 \\ -1 & -2 \end{smallmatrix}\right]$, and $D = \left[\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right]$, then $A = SDS^{-1}$ and

$$e^A = S \begin{bmatrix} e^1 & 0 \\ 0 & e^{-1} \end{bmatrix} S^{-1} = \begin{bmatrix} 2e^1 - e^{-1} & 2e^1 - 2e^{-1} \\ -e^1 + e^{-1} & -e^1 + 2e^{-1} \end{bmatrix}$$

(c) $A\left[\begin{smallmatrix} 1 & 1 & 1 \\ -1 & -1 & -1 \\ 1 & 1 & 1 \end{smallmatrix}\right]$ so $p_A(t) = t^2(-t+1)$ and the eigen values are 0 and 1.

$\lambda = 0$: $\operatorname{NS}(A - 0 \cdot I) = \operatorname{NS}\left[\begin{smallmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{smallmatrix}\right] = \operatorname{span}\{(-1, 1, 0), (-1, 0, 1)\}$

$\operatorname{NS}(A - 1 \cdot I) = \operatorname{NS}\left[\begin{smallmatrix} 0 & 1 & 1 \\ -1 & -2 & -1 \\ 1 & 1 & 0 \end{smallmatrix}\right] = \operatorname{NS}\left[\begin{smallmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{smallmatrix}\right] = \operatorname{span}\{(1, -1, 1)\}$

Let $S = \left[\begin{smallmatrix} -1 & -1 & 1 \\ 1 & 0 & -1 \\ 0 & 1 & 1 \end{smallmatrix}\right]$, $S^{-1} = \left[\begin{smallmatrix} 1 & 2 & 1 \\ -1 & -1 & 0 \\ 1 & 1 & 1 \end{smallmatrix}\right]$, and $D = \left[\begin{smallmatrix} 0 & & \\ & 0 & \\ & & 1 \end{smallmatrix}\right]$

$$e^A = S \begin{bmatrix} e^0 & & \\ & e^0 & \\ & & e^1 \end{bmatrix} S^{-1} = \begin{bmatrix} -1 & -1 & 1 \\ 1 & 0 & -1 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & & \\ & 1 & \\ & & e \end{bmatrix} \begin{bmatrix} 1 & 2 & 1 \\ -1 & -1 & 0 \\ 1 & 1 & 1 \end{bmatrix} = \begin{bmatrix} e & e-1 & e-1 \\ 1-e & 2-e & 1-e \\ e-1 & e-1 & e \end{bmatrix}$$

32. Solve $\boldsymbol{x}' = A\boldsymbol{x}$ and $\boldsymbol{x}(0) = \boldsymbol{x}_0$. Here we know that for $\boldsymbol{x} = e^{At}\boldsymbol{x}_0$ we have

$$\boldsymbol{x} = \left(\sum_{i=0}^{\infty} \frac{1}{i!} A^i t^i\right) \boldsymbol{x}_0$$

and

$$\begin{aligned}
\frac{d}{dt}\boldsymbol{x} &= \left(\sum_{i=0}^{\infty} \frac{1}{i!} i A^i t^{i-1}\right) \boldsymbol{x}_0 \\
&= A\left(\sum_{i=1}^{\infty} \frac{1}{(i-1)!} A^{i-1} i t^{i-1}\right) \boldsymbol{x}_0 \\
&= A\left(\sum_{i=0}^{\infty} \frac{1}{i!} A^i i t^i\right) \boldsymbol{x}_0 \\
&= A\boldsymbol{x}
\end{aligned}$$

So the solution we seek is $\boldsymbol{x} = e^{At}\boldsymbol{x}_0$. If $A = SDS^{-1}$, then $\boldsymbol{x} = Se^{Dt}S^{-1}\boldsymbol{x}_0$ and if $D = \operatorname{diag}(d_1, \ldots, d_n)$, then

$$\boldsymbol{x} = S \operatorname{diag}\left(e^{d_1 t}, \ldots, e^{d_n t}\right) S^{-1} \boldsymbol{x}_0.$$

(b) $A = \left[\begin{smallmatrix} 2 & 3 \\ -1 & -2 \end{smallmatrix}\right]$ and $\boldsymbol{x}_0 = (-4, 2)$. So the solution is

$$\boldsymbol{x} = e^{At}\boldsymbol{x}_0$$

We can diagonalize A. $\det(A - tI) = \det\left[\begin{smallmatrix} 2-t & 3 \\ -1 & -2-t \end{smallmatrix}\right] = (t^2 - 4) + 3 = t^2 - 1 = (t-1)(t+1)$

$\mathrm{NS}(A - I) = \mathrm{NS}\left[\begin{smallmatrix} 1 & 3 \\ -1 & -3 \end{smallmatrix}\right] = \operatorname{span}\{(3, -1)\}$.

$\mathrm{NS}(A + I) = \mathrm{NS}\left[\begin{smallmatrix} 3 & 3 \\ -1 & -1 \end{smallmatrix}\right] = \operatorname{span}\{(1, -1)\}$.

Let $S = \left[\begin{smallmatrix} 3 & 1 \\ -1 & -1 \end{smallmatrix}\right]$ so $A = S\left[\begin{smallmatrix} 1 & \\ & -1 \end{smallmatrix}\right]S^{-1}$ and

$$\begin{aligned}
\boldsymbol{x} &= e^{At}\boldsymbol{x}_0 \\
&= S \exp\left(\begin{bmatrix} 1 & \\ & -1 \end{bmatrix} t\right) S^{-1}\boldsymbol{x}_0 \\
&= S \begin{bmatrix} e^t & \\ & e^{-t} \end{bmatrix} S^{-1}\boldsymbol{x}_0 \\
&= \begin{bmatrix} 3 & 1 \\ -1 & -1 \end{bmatrix} \begin{bmatrix} e^t & \\ & e^{-t} \end{bmatrix} \left(\tfrac{1}{-2}\right) \begin{bmatrix} -1 & -1 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} -4 \\ 2 \end{bmatrix} \\
&= \begin{bmatrix} -3e^t - e^{-t} \\ e^t + e^{-t} \end{bmatrix}
\end{aligned}$$

So

$$\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \begin{bmatrix} -3e^t - e^{-t} \\ e^t + e^{-t} \end{bmatrix}$$

(c) This is done the same way as (b).

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & -1 \end{bmatrix}$$

So the eigenvalues are $1, 0, -1$ and

$\mathrm{NS}(A - 1 \cdot I) = \mathrm{NS}\left[\begin{smallmatrix} 0 & 1 & 1 \\ 0 & -1 & 1 \\ 0 & 0 & -2 \end{smallmatrix}\right] = \mathrm{span}\{(1,0,0)\}$.

$\mathrm{NS}(A - 0 \cdot I) = \mathrm{NS}\left[\begin{smallmatrix} 1 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & -1 \end{smallmatrix}\right] = \mathrm{span}\{(1,-1,0)\}$.

$\mathrm{NS}(A - (-1) \cdot I) = \mathrm{NS}\left[\begin{smallmatrix} 2 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{smallmatrix}\right] = \mathrm{span}\{(0,1,-1))\}$.

So

$$S = \begin{bmatrix} 1 & 1 & 0 \\ 0 & -1 & 1 \\ 0 & 0 & -1 \end{bmatrix}$$

and

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 0 \\ 0 & -1 & 1 \\ 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} e^t & & \\ & e^0 & \\ & & e^{-t} \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 \\ 0 & -1 & -1 \\ 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 3e^t - 2 \\ 2 - e^{-t} \\ e^{-t} \end{bmatrix}$$

35. Let $p(t) = \sum_{i=1}^n a_i t^i$ be the characteristic polynomial for A.

(a,b) If $A = SDS^{-1}$, then

$$p(A) = \sum_{i=1}^n a_i (SDS^{-1})^i = \sum_{i=1}^n a_i SD^i S^{-1} = S\left(\sum_{i=1}^n a_i D^i\right) S^{-1}.$$

This is because

$$A^i = (SDS^{-1})(SDS^{-1}) \cdots (SDS^{-1}) = SDIDI \cdots DS^{-1} = SD^i S^{-1}.$$

Now if $D = \mathrm{diag}(d_1, \ldots, d_n)$, then $D^i = \mathrm{diag}(d_1^i, \ldots, d_n^i)$ and

$$p(D) = \mathrm{diag}\left(\sum_{i=1}^n a_i d_1^i, \ldots, \sum_{i=1}^n a_i d_1^i\right) = \mathrm{diag}(0, \ldots, 0) = \mathbf{0}$$

since d_i is an eigenvalue of A and hence a root of $p(t)$.

(c) Assume $a_0 \neq 0$, then $A\left(-\frac{1}{a_0} \sum_{i=1}^n a_i A^{i-1}\right) = I$, So letting $q(t) = -\frac{1}{a_0} \sum_{i=1}^n a_i t^{i-1}$, we have $A^{-1} = q(A)$.