Homework 1 Partial Solutions

Homework 1 problems:

Section 1.1

8. Use elimination and back substitution to solve the given system:

$$\begin{aligned} x_1 + 2x_2 - 2x_3 &= 1 \\ 2x_1 + 5x_2 + x_3 &= 9 \\ x_1 + 3x_2 + 4x_3 &= 9 \end{aligned}$$

Elimination on the augmented system looks like:

$$\left[\begin{array}{ccc|c} 1 & 2 & -2 & 1 \\ 2 & 5 & 1 & 9 \\ 1 & 3 & 4 & 9 \end{array}\right] \underset{\substack{r_2 - 2r_1 \to r_2 \\ r_3 - r_1 \to r_3}}{\longrightarrow} \left[\begin{array}{ccc|c} 1 & 2 & -2 & 1 \\ 0 & 1 & 5 & 7 \\ 0 & 1 & 6 & 8 \end{array}\right] \underset{r_3 - r_2 \to r_3}{\longrightarrow} \left[\begin{array}{ccc|c} 1 & 2 & -2 & 1 \\ 0 & 1 & 5 & 7 \\ 0 & 0 & 1 & 1 \end{array}\right]$$

This gives the triangular system

$$\begin{aligned} x_1 + 2x_2 - 2x_3 &= 1 \\ x_2 + 5x_3 &= 7 \\ x_3 &= 1 \end{aligned}$$

Back substitution gives:

$$\begin{aligned} x_2 &= 7 - 5(1) = 2 \\ x_1 &= 1 + 2(1) - 2(2) = -1 \end{aligned}$$

So we have the solution $(-1, 2, 1)$. (Check this in the initial system!)

9.

(a) Suppose $m_1 \neq m_2$. Assume $m_1 \neq 0$. (If $m_1 = 0$, then $m_2 \neq 0$ by our assumption so we could just swap the rolls below.) Multiply the first equation by $-m_2/m_1$ add to the second and replace the second. This yields the new system:

$$\begin{aligned} -m_1x_1 + \qquad\qquad x_2 &= b_1 \\ (1 - m_2/m_1)x_2 &= b_2 - (m_2/m_1)b_1 \end{aligned}$$

Now you can use back substitution:

$$\begin{aligned} x_2 &= \frac{b_2 - (m_2/m_1)b_1}{1 - m_2/m_1} = \frac{m_1b_2 - m_2b_1}{m_1 - m_2} && \text{(ok since } m_1 \neq m_2) \\ x_1 &= -\frac{b_1 - x_2}{m_1} = \frac{2m_2b_1 - m_1(b_1 + b_2)}{m_1(m_1 - m_2)} \end{aligned}$$

So we have a unique solution.

(b) If $m_1 = m_2$, then the system is just

$$\begin{aligned} -m_1x_1 + x_2 &= b_1 \\ -m_1x_1 + x_2 &= b_2 \end{aligned}$$

This is equivalent to the single equation $-m_1x_1 + x_2 = b_1$ if $b_1 = b_2$, else you get $0 = b_2 - b_1 \neq 0$ which cannot happen.

(c) For (a) saying $m_1 \neq m_2$ is equivalent to saying that the two lines have different slopes and hence have a unique point of intersection. For (b), if $m_1 = m_2$, the lines are parallel, they are the same line if $b_1 = b_2$, else they are distinct parallel lines and hence never intersect.

Section 1.2

5.

(c)

$$\begin{aligned} x_1 + x_2 &= 0 \\ 2x_1 + 3x_2 &= 0 \\ 3x_1 - 2x_2 &= 0 \end{aligned}$$

Gaussian Elimination looks like

$$\left[\begin{array}{cc|c} 1 & 1 & 0 \\ 2 & 3 & 0 \\ 3 & -2 & 0 \end{array}\right] \xrightarrow[\substack{r_2 - 2r_1 \to r_2 \\ r_3 - 3r_1 \to r_3}]{} \left[\begin{array}{cc|c} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & -5 & 0 \end{array}\right] \xrightarrow[r_3 + 5r_2 \to r_3]{} \left[\begin{array}{cc|c} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

This results in the triangular system

$$\begin{aligned} x_1 + x_2 &= 0 \\ x_2 &= 0 \end{aligned}$$

Which is solved by back substitution to give: $x_2 = 0$, and $x_1 = -x_2 = 0$.

(d)

$$\begin{aligned} 3x_1 + 2x_2 - x_3 &= 4 \\ x_1 - 2x_2 + 2x_3 &= 1 \\ 11x_1 + 2x_2 + x_3 &= 14 \end{aligned}$$

Gaussian Elimination looks like

$$\left[\begin{array}{ccc|c} 3 & 2 & -1 & 4 \\ 1 & -2 & 2 & 1 \\ 11 & 2 & 1 & 14 \end{array}\right] \xrightarrow[r_1 \leftrightarrow r_2]{} \left[\begin{array}{ccc|c} 1 & -2 & 2 & 1 \\ 3 & 2 & -1 & 4 \\ 11 & 2 & 1 & 14 \end{array}\right] \xrightarrow[\substack{r_2 - 3r_1 \to r_2 \\ r_3 - 11r_1 \to r_3}]{}$$
$$\left[\begin{array}{ccc|c} 1 & -2 & 2 & 1 \\ 0 & 8 & -7 & 1 \\ 0 & 24 & -21 & 3 \end{array}\right] \xrightarrow[r_3 - 3r_2 \to r_3]{} \left[\begin{array}{ccc|c} 1 & -2 & 2 & 1 \\ 0 & 8 & -7 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

This is equivalent to the "triangular system"

$$\begin{aligned} x_1 - 2x_2 + 2x_3 &= 1 \\ 8x_2 - 7x_3 &= 1 \end{aligned}$$

So x_1 and x_2 are the two "pivot variables" an x_3 is the "free variable." Let $x_3 = t$ be any value, then we use back substitution to get:

$$\begin{aligned} x_2 &= 1/8 + 7/8t \\ x_1 - 2(1/8 + 7/8t) + 2t &= 1 \\ x_1 &= 5/4 - 1/4t \end{aligned}$$

So

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 5/4 - 1/4t \\ 1/8 + 7/8t \\ t \end{bmatrix} = \begin{bmatrix} 5/4 \\ 1/8 \\ 0 \end{bmatrix} + t \begin{bmatrix} -1/4 \\ 7/8 \\ 1 \end{bmatrix}$$

(e)

$$\begin{aligned} 2x_1 + 3x_2 + x_3 &= 1 \\ x_1 + x_2 + x_3 &= 3 \\ 3x_1 + 4x_2 + 2x_3 &= 4 \end{aligned}$$

Elimination on the augmented system looks like:

$$\left[\begin{array}{ccc|c} 2 & 3 & 1 & 1 \\ 1 & 1 & 1 & 3 \\ 3 & 4 & 2 & 4 \end{array}\right] \xrightarrow[r_1 \leftrightarrow r_2]{} \left[\begin{array}{ccc|c} 1 & 1 & 1 & 3 \\ 2 & 3 & 1 & 1 \\ 3 & 4 & 2 & 4 \end{array}\right] \xrightarrow[\substack{r_2 - 2r_1 \to r_2 \\ r_3 - 3r_1 \to r_3}]{} \left[\begin{array}{ccc|c} 1 & 1 & 1 & 3 \\ 0 & 1 & -1 & -5 \\ 0 & 1 & -1 & -5 \end{array}\right] \xrightarrow[r_3 - r_2 \to r_3]{} \left[\begin{array}{ccc|c} 1 & 1 & 1 & 3 \\ 0 & 1 & -1 & -5 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

The echelon form of the matrix coefficient matrix is now shown.

This system is consistent. if we let $x_3 = t$, then by back substitution get

$$\begin{aligned} x_3 &= t \\ x_2 &= -5 + t \\ x_1 &= 3 - t - (-5 + t) = 8 - 2t \end{aligned}$$

so the solution set is

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 8 \\ -5 \\ 0 \end{bmatrix} + t \begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix}$$

8. For what values of a does the following have a unique solution?

$$\left[\begin{array}{ccc|c} 1 & 2 & 1 & 1 \\ -1 & 4 & 3 & 2 \\ 2 & -2 & a & 3 \end{array}\right]$$

$$\left[\begin{array}{ccc|c} 1 & 2 & 1 & 1 \\ -1 & 4 & 3 & 2 \\ 2 & -2 & a & 3 \end{array}\right] \underset{\substack{r_2+r_1\to r_2 \\ r_3-2r_1\to r_3}}{\longrightarrow} \left[\begin{array}{ccc|c} 1 & 2 & 1 & 1 \\ 0 & 6 & 4 & 3 \\ 0 & -6 & a-2 & 1 \end{array}\right]$$

$$\underset{r_3+r_2\to r_3}{\longrightarrow} \left[\begin{array}{ccc|c} 1 & 2 & 1 & 3 \\ 0 & 6 & 4 & -5 \\ 0 & 0 & a+2 & 4 \end{array}\right]$$

All that is required to get a unique solution is that $a + 2 \neq 0$ or $a \neq -2$.

11. Solve

$$\begin{aligned} x_1 + 2x_2 &= 2 \\ 3x_1 + 7x_2 &= 8 \end{aligned} \qquad\qquad \begin{aligned} x_1 + 2x_2 &= 1 \\ 3x_1 + 7x_2 &= 7 \end{aligned}$$

Set this up as:

$$\left[\begin{array}{cc|cc} 1 & 2 & 2 & 1 \\ 3 & 7 & 8 & 7 \end{array}\right] \underset{r_2-3r_3\to r_3}{\longrightarrow} \left[\begin{array}{cc|cc} 1 & 2 & 2 & 1 \\ 0 & 1 & 2 & 4 \end{array}\right] \underset{r_1-2r_2\to r_1}{\longrightarrow} \left[\begin{array}{cc|cc} 1 & 0 & -2 & -7 \\ 0 & 1 & 2 & 4 \end{array}\right]$$

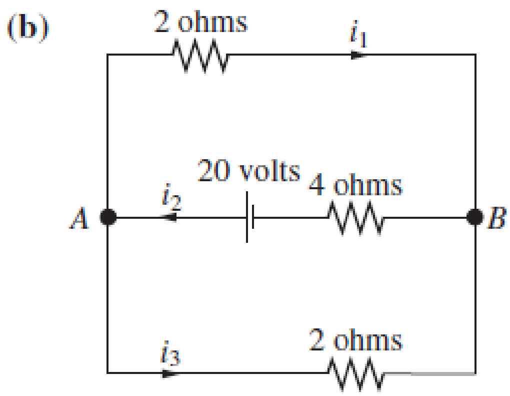
So the solutions are

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -2 \\ 2 \end{bmatrix} \qquad \text{and} \qquad \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -7 \\ 4 \end{bmatrix}$$

respectively. As usual, you should check this.

22

(b)



From this we have

$$
\begin{aligned}
i_1 - i_2 + i_3 &= 0 && \text{(node B)}\\
-i_1 + i_2 - i_3 &= 0 && \text{(node A)}\\
2i_1 + 4i_2 &= 20 && \text{(top loop)}\\
4i_2 + 2i_3 &= 20 && \text{(bottom loop)}
\end{aligned}
$$

This gives

$$
\left[\begin{array}{ccc|c} 1 & -1 & 1 & 0\\ -1 & 1 & -1 & 0\\ 2 & 4 & 0 & 20\\ 0 & 4 & 2 & 20 \end{array}\right]
\xrightarrow[\substack{r_2+r_1\to r_2\\ r_3-2r_1\to r_3}]{}
\left[\begin{array}{ccc|c} 1 & -1 & 1 & 0\\ 0 & 0 & 0 & 0\\ 0 & 6 & -2 & 20\\ 0 & 4 & 2 & 20 \end{array}\right]
$$

$$
\xrightarrow[\substack{r_2\leftrightarrow r_4\\ 1/2r_2\to r_2\\ 1/2r_3\to r_3}]{}
\left[\begin{array}{ccc|c} 1 & -1 & 1 & 0\\ 0 & 2 & 1 & 10\\ 0 & 3 & -1 & 10\\ 0 & 0 & 0 & 0 \end{array}\right]
$$

$$
\xrightarrow[r_3-3/2r_2\to r_3]{}
\left[\begin{array}{ccc|c} 1 & -1 & 1 & 0\\ 0 & 2 & 1 & 10\\ 0 & 0 & -5/2 & -5\\ 0 & 0 & 0 & 0 \end{array}\right]
$$

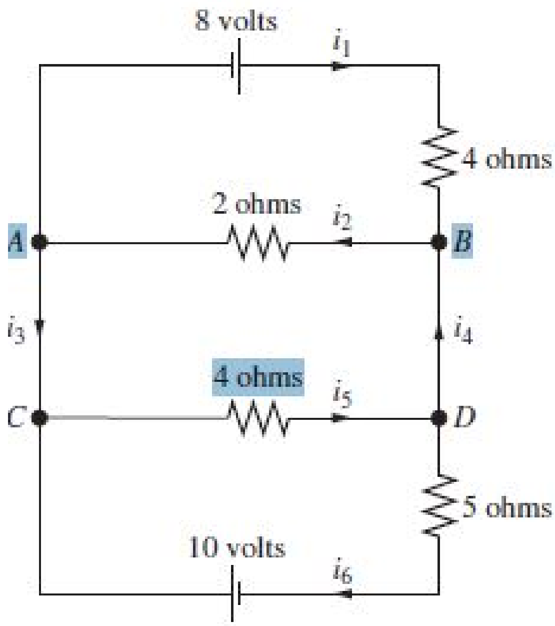
This gives the triangular system

$$
\begin{aligned}
i_1 - i_2 + i_3 &= 0\\
2i_2 + i_3 &= 10\\
-5/2 i_3 &= -5
\end{aligned}
$$

So

$$
\begin{aligned}
i_3 = 2 &\Longrightarrow i_3 = 2\\
2i_2 + 2 = 10 &\Longrightarrow i_2 = 4\\
i_1 - 4 + 2 = 0 &\Longrightarrow i_1 = 2
\end{aligned}
$$

(c)



The equations here are

$$
\begin{aligned}
-i_1 + i_2 - i_3 &= 0 && \text{(Node A)}\\
i_1 - 1_2 + i_4 &= 0 && \text{(Node B)}\\
i_3 - i_5 + i_6 &= 0 && \text{(Node C)}\\
-i_4 + i_5 - i_6 &= 0 && \text{(Node D)}\\
4i_1 + 2i_2 &= 8 && \text{(Top Loop)}\\
4i_5 + 5i_6 &= 10 && \text{(Bottom Node)}
\end{aligned}
$$

The augmented matrix is

$$
\left[\begin{array}{cccccc|c}
-1 & 1 & -1 & 0 & 0 & 0 & 0\\
1 & -1 & 0 & 1 & 0 & 0 & 0\\
0 & 0 & 1 & 0 & -1 & 1 & 0\\
0 & 0 & 0 & -1 & 1 & -1 & 0\\
4 & 2 & 0 & 0 & 0 & 0 & 8\\
0 & 0 & 0 & 0 & 4 & 5 & 10
\end{array}\right]
$$

solving gives

$$\begin{bmatrix} 1/2 \\ 3 \\ 5/2 \\ 5/2 \\ 5/2 \\ 0 \end{bmatrix}$$

Section 1.3

6. This is just computational. Do this by hand, but this is also good to verify in MATLAB.

7. This is just computational. Do this by hand, but this is also good to verify in MATLAB.

13.

(a) Say the variables are x_1, x_2, x_3, x_4, x_5 the variables x_2, x_4 and x_5 will be independent since the columns 2,4, and 5 do not contain pivot elements. The others can be solved in terms of those. Let s, t, u be arbitrary real numbers and set $x_2 = s$, $x_4 = t$, and $x_5 = u$, then

$$\begin{aligned} x_1 &= -2 - 2s - 3t - u \\ x_2 &= s \\ x_3 &= 5 - 2t - 4u \\ x_4 &= t \\ x_5 &= u \end{aligned}$$

We can write this nicely as

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} -2 \\ 0 \\ 5 \\ 0 \\ 0 \end{bmatrix} + s \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + t \begin{bmatrix} -3 \\ 0 \\ -2 \\ 1 \\ 0 \end{bmatrix} + u \begin{bmatrix} -1 \\ 0 \\ -4 \\ 0 \\ 1 \end{bmatrix} \tag{1}$$

(b)

We know

$$A\mathbf{x} = \mathbf{b} \iff \begin{bmatrix} 1 & 2 & 0 & 3 & 1 \\ 0 & 0 & 1 & 2 & 4 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \mathbf{x} = \begin{bmatrix} -2 \\ 5 \\ 0 \\ 0 \end{bmatrix}$$

We also know that

$$\mathbf{x} = \begin{bmatrix} -2 \\ 0 \\ 5 \\ 0 \\ 0 \end{bmatrix}$$

is a solution, namely the solution where $s = t = u = 0$

So $A\mathbf{x} = -2\mathbf{a_1} + 5\mathbf{a_3} = -2 \begin{bmatrix} 1 \\ 1 \\ 3 \\ 4 \end{bmatrix} + 5 \begin{bmatrix} 2 \\ -1 \\ 1 \\ 3 \end{bmatrix} = \begin{bmatrix} 8 \\ -7 \\ -1 \\ 7 \end{bmatrix} = \mathbf{b}$

16. $A^T = -A$ implies $a_{ii} = -a_{ii}$ so $a_{ii} = 0$.

Section 1.4

8. Check that $A^2 = I$, so $A^{2n+1} = A$ and $A^{2n} = I$.

10. Assume $A^T = A$ and $B^T = B$.

(a) Symmetric since: $(A + B)^T = A^T + B^T = A + B$

(b) Symmetric since: $(A^2)^T = A^T A^T = AA = A^2$

(c) Not necessarily transitive since: $(AB)^T = B^T A^T = BA$. Symmetry would only happen if $AB = BA$, i.e., A and B commute.

Example: Let $A = \left[\begin{smallmatrix} 1 & 1 \\ 1 & 1 \end{smallmatrix}\right]$ and $B = \left[\begin{smallmatrix} 1 & -1 \\ -1 & -1 \end{smallmatrix}\right]$, notice both A and B are symmetric. $AB = \left[\begin{smallmatrix} 0 & -2 \\ 0 & -2 \end{smallmatrix}\right]$ is not symmetric so $(AB)^T \neq AB$.

(d) Symmetric since: $(ABA)^T = A^T B^T A^T = ABA$

(e) Symmetric since: $(AB + BA)^T = (AB)^T + (BA)^T = B^T A^T + A^T B^T = BA + AB =$

$AB + BA$

(f) Not symmetric since: $(AB - BA)^T = (AB)^T - (BA)^T = B^T A^T - A^T B^T = BA - AB = -(AB - BA)$. So $AB - BA$ is symmetric iff $AB - BA = 0$, that is iff A and B commute.

The same example as in (c) works here.

22.

$$
\begin{aligned}
RR^T &= \begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix} \begin{bmatrix} \cos(\theta) & \sin(\theta) \\ -\sin(\theta) & \cos(\theta) \end{bmatrix} \\
&= \begin{bmatrix} \cos^2(\theta) + \sin^2(\theta) & \cos(\theta)\sin(\theta) - \sin(\theta)\cos(\theta) \\ \sin(\theta)\cos(\theta) - \cos(\theta)\sin(\theta) & \sin^2(\theta) + \cos^2(\theta) \end{bmatrix} \\
&= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}
\end{aligned}
$$

23. $H = I - 2\mathbf{u}\mathbf{u}^T$, so

$$
\begin{aligned}
H^2 &= (I - 2\mathbf{u}\mathbf{u}^T)(I - 2\mathbf{u}\mathbf{u}^T) \\
&= I^2 - 2\mathbf{u}\mathbf{u}^T I - 2I\mathbf{u}\mathbf{u}^T + 4\mathbf{u}\mathbf{u}^T\mathbf{u}\mathbf{u}^T \\
&= I^2 - 4\mathbf{u}\mathbf{u}^T + 4\mathbf{u}\mathbf{u}^T\mathbf{u}\mathbf{u}^T \\
&= I^2 - 4\mathbf{u}\mathbf{u}^T + 4\mathbf{u}(1)\mathbf{u}^T \\
&= I^2 - 4\mathbf{u}\mathbf{u}^T + 4\mathbf{u}\mathbf{u}^T \\
&= I^2 \\
&= I
\end{aligned}
$$

27. Assume $A^2 = I$ and let $B = \frac{1}{2}(I + A)$ and $C = \frac{1}{2}(I - A)$. Note

$$B^2 = \frac{1}{4}(I^2 + IA + AI + A^2) = \frac{1}{4}(I + 2A + I) = \frac{1}{4}(2)(I + A) = B$$

Similarly, $C^2 = C$.

30.

$$(A + A^T)^T = A^T + (A^T)^T = A^T + A = A + A^T$$

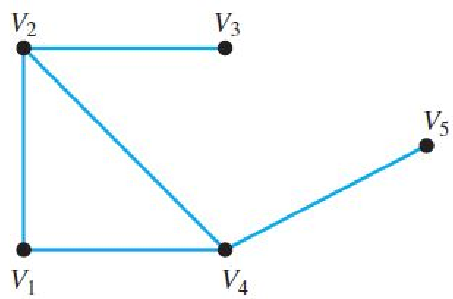
So $A + A^T$ is symmetric. Similarly,

$$(A - A^T)^T = A^T - (A^T)^T = A^T - A = -(A - A^T)$$

So $A - A^T$ is skew-symmetric.

Since $A = \frac{1}{2}(A + A^T) + \frac{1}{2}(A - A^T)$ we see A can be written as symmetric + skew-symmetric.

33.



(a) Adjacency Matrix, A:

$$A = \begin{bmatrix} 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$

(b)

$$A^2 = \begin{bmatrix} 2 & 1 & 1 & 1 & 1 \\ 1 & 3 & 0 & 1 & 1 \\ 1 & 0 & 1 & 1 & 0 \\ 1 & 1 & 1 & 3 & 0 \\ 1 & 1 & 0 & 0 & 1 \end{bmatrix}$$

There are 6 walks starting at V_1 and that is the sum of column 1 or row 1 (the matrix is symmetric). $V_1V_2V_1$, $V_1V_4V_1$, V_1, V_2V_3, $V_1V_2V_4$, $V_1V_4V_2$, and $V_1V_4V_5$.

(c)

$$A^2 = \begin{bmatrix} 2 & 4 & 1 & 4 & 1 \\ 4 & 2 & 3 & 5 & 1 \\ 1 & 3 & 0 & 1 & 1 \\ 4 & 5 & 1 & 2 & 3 \\ 1 & 1 & 1 & 3 & 0 \end{bmatrix}$$

There are 5 walks of length 3 from V_2 to V_4 and 1 of length 2 so 6 altogether.

Section 1.5

8. find the LU decomposition of the following matrices:

(b) $\begin{bmatrix} 2 & 4 \\ -2 & 1 \end{bmatrix}$

$$\begin{bmatrix} 2 & 4 \\ -2 & 1 \end{bmatrix} \underset{r_2+(1)r_1\to r_2;L_{2,1}=-1}{\longrightarrow} \begin{bmatrix} 2 & 4 \\ 0 & 5 \end{bmatrix} = U$$

The $(2,1)$ position of L is -1 so

$$L = \begin{bmatrix} 1 & 0 \\ -1 & 0 \end{bmatrix}$$

It is easy to see that

$$\begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 2 & 4 \\ 0 & 5 \end{bmatrix} = \begin{bmatrix} 2 & 4 \\ -2 & 1 \end{bmatrix}$$

(d) $\begin{bmatrix} -2 & 1 & 2 \\ 4 & 1 & -2 \\ -6 & -3 & 4 \end{bmatrix}$

$$\begin{bmatrix} -2 & 1 & 2 \\ 4 & 1 & -2 \\ -6 & -3 & 4 \end{bmatrix} \underset{\substack{r_2+(2)r_1\to r_2;L_{2,1}=-2 \\ r_3+(-3)r_1\to r_3;L_{3,1}=3}}{\longrightarrow} \begin{bmatrix} -2 & 1 & 2 \\ 0 & 3 & 2 \\ 0 & -6 & -2 \end{bmatrix}$$

$$\underset{r_3+(2)r_2\to r_3;L_{3,2}=-2}{\longrightarrow} \begin{bmatrix} -2 & 1 & 2 \\ 0 & 3 & 2 \\ 0 & 0 & 2 \end{bmatrix} = U$$

We see

$$L = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 3 & -2 & 1 \end{bmatrix}$$

Again, it is easy to check that

$$LU = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 3 & -2 & 1 \end{bmatrix} \begin{bmatrix} -2 & 1 & 2 \\ 0 & 3 & 2 \\ 0 & 0 & 2 \end{bmatrix} = \begin{bmatrix} -2 & 1 & 2 \\ 4 & 1 & -2 \\ -6 & -3 & 4 \end{bmatrix}$$

19. (a) If U is upper triangular with non-zero diagonal, then clearly U can be row reduced to I. Since U is row equivalent to I, U is invertible. (Theorem 1.5.2).

(b) **Proof 1:** For this, essentially look into the proof of 1.5.2. Let $E_n E_{n-1} \cdots E_1 U = I$ where E_i is an elementary matrix. Since U starts as upper-triangular (u.t.) it is clear that only type

II and III operations are needed and the type II here are of the form $R_i - c \cdot R_j \to R_i$ where $i < j$, so c goes in the $(i,j)^{\text{th}}$ spot and hence this matrix is u.t. So all the E_i are u.t. Thus $E_n E_{n-1} \cdots E_1 = U^{-1}$ is u.t.

Proof 2: We can show that of $AB = C$ with B and C u.t. and B invertible, then A is u.t. This will clearly imply what we want. Suppose A is not u.t. and let i be least such that there is $j < i$ with $A_{i,j} \neq 0$. Let j be the least $< i$ so that $A_{i,j} = 0$. Then $0 = C_{i,j} = \sum_{k \leq n} A_{i,k} B_{k,j} = \sum_{k \leq j} A_{i,k} B_{k,j}$, that is,

$$0 = A_{i,1} B_{1,j} + \cdots + A_{i,j} B_{j,j} = 0 + \cdots 0 + A_{i,j} B_{j,j} = A_{i,j} B_{j,j} = 0$$

This is a contradiction to the choice of $A_{i,j}$ and the fact that B is invertible, hence $B_{j,j} \neq 0$, since B is u.t.

Proof 3: This is the easiest proof (it is essentially proof 1, just with row ops instead of elementary matrices.)

Suppose consider starting with $\left[U \,\middle|\, I \right]$ and applying row ops to get to $\left[I \,\middle|\, U^{-1} \right]$. In the first step, we just divide each row by the diagonal element:

$$\left[\begin{array}{ccccc|ccccc} u_{1,1} & u_{1,2} & u_{1,3} & \cdots & & 1 & 0 & 0 & \cdots & \\ 0 & u_{2,2} & u_{2,3} & \cdots & & 0 & 1 & 0 & \cdots & \\ \vdots & & \ddots & & \vdots & \vdots & & \ddots & & \vdots \\ 0 & 0 \cdots & u_{n,n} & & & 0 & 0 & \cdots & & 1 \end{array}\right] \longrightarrow \left[\begin{array}{ccccc|ccccc} 1 & \frac{u_{1,2}}{u_{1,1}} & \frac{u_{1,3}}{u_{1,1}} & \cdots & & \frac{1}{u_{1,1}} & 0 & 0 & \cdots & \\ 0 & 1 & \frac{u_{2,3}}{u_{2,2}} & \cdots & & 0 & \frac{1}{u_{2,2}} & 0 & \cdots & \\ \vdots & & \ddots & & \vdots & \vdots & & \ddots & & \vdots \\ 0 & 0 & \cdots & & 1 & 0 & 0 & \cdots & & \frac{1}{u_{n,n}} \end{array}\right]$$

The rest of the row ops will proceed right-to-left and bottom-to-top. The 0's in the lower left of the RHS will never be touched, and neither will the diagonal elements. So we also see that $U^{-1}_{i,i} = \frac{1}{U_{i,i}}$. (Proof 1 shows this additional fact as well.)

28. (a) There is nothing to do except unpack matrix multiplication, which is done below for (b).

(b) For this

$$V\boldsymbol{c} = c_0 \begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix} + c_1 \begin{bmatrix} x_1 \\ \vdots \\ x_{n+1} \end{bmatrix} + \cdots + c_n \begin{bmatrix} x_i^n \\ \vdots \\ x_{n+1}^n \end{bmatrix} = \begin{bmatrix} p(x_1) \\ \vdots \\ p(x_{n+1}) \end{bmatrix} = \begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix}$$

But then we have $n+1$ distinct roots to a polynomial of degree n, which is a contradiction unless $p(x) \equiv 0$, i.e., p is the constant 0 polynomial.

32. This is false, for example let $A = B = I$ and $C = -I$, then clearly A is row equivalent to B and C, but $B + C = \mathbf{0}$ (the all 0 matrix). It is not true that A is row equivalent to $\mathbf{0}$, else $I = \mathbf{0}$ (a version of $1 = 0$).