Homework 3 Partial Solutions

Section 3.1

8. This questions is about arbitrary vectors, these could be vectors in $\mathbb{R}^n$ but it could also be the space of matrices $\mathbb{R}^{n\times m}$, could be the space of continuous functions on the unit interval into $\mathbb{R}$, $C([0,1],\mathbb{R})$, etc. So you must argue generally using axioms of vector spaces.

$$\begin{aligned} x + y &= x + z \\ (-x) + (x + y) &= (-x) + (x + z) && \text{(A4)} \\ (-x + x) + y &= (-x + x) + z && \text{(A2)} \\ 0 + y &= 0 + z && \text{(A4)} \\ y &= z && \text{(A3)} \end{aligned}$$

13. There are various ways to see that this is not a vector space. One way is to notice that there is no 0 element!

What element a of $\mathbb{R}$ would satisfy $\max(a, r) = r$ for all $r \in \mathbb{R}$? For $r \geq 0$, $a = 0$ would suffice, but what would work for $r < 0$? If $a \oplus r = r$ for $r < 0$, then $a < r$. But then $a < r$ for all $r \in \mathbb{R}$!

14. Let $V = \mathbb{Z}$ and define scalar multiplication by

$$\alpha \cdot_V n = \lfloor \alpha \rfloor \cdot n \tag{1}$$

$$n +_V m = n + m \tag{2}$$

Is this a vector space?

All the additive axioms clearly hold since these are true of integer arithmetic.

The problem here is $\alpha \cdot_V (\beta \cdot_V n) = (\alpha \cdot \beta) \cdot_V n$. For example:

$$.5 \cdot_V (2 \cdot_V n) = 0 \cdot (2 \cdot n) = 0$$

while

$$(.5 \cdot 2) \cdot_V n = 1 \cdot_V n = 1 \cdot n = n$$

Section 3.2

2.

(a) This is not a subspace because $(0,0)^T \notin S$.

(b) This is a subspace.

- If $(a,b,c) \in S$, then $\alpha(a,b,c)^T \in S$, since, $a = b = c$ implies $\alpha a = \alpha b = \alpha c$.
- If $(a,b,c)^T, (A,B,C)^T \in S$, then $a + A = b + B = c + C$, so $(a,b,c)^T + (A,B,C)^T \in S$.

Thus S is closed under scalar multiplication and addition and is a subspace.

(c) This is a subspace. Do just like (b), but use the property $x_1 = x_2 + x_3$. Another way is to notice that $S = NS(A)$ where $A = \begin{bmatrix} 1 & -1 & -1 \end{bmatrix}$. (We could have done this with (b) as well.)

(d) This is not a subspace $(1,2,1)^T$ and $(4,1,1)^T$ are in S, but the sum $(5,3,2)^T \notin S$

4.

(a) $\text{rref}(A) = I_2$ so $\text{NS}(A) = \text{span}\{\mathbf{0}\}$.

(b) $\text{rref}(A) = \begin{bmatrix} 1 & 2 & -3 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$ so $A\boldsymbol{x} = \mathbf{0}$ is equivalent to

$$\begin{aligned} x_1 + 2x_2 - 3x_3 &= 0 \\ x_4 &= 0 \end{aligned}$$

Let $x_2 = s$ and $x_3 = t$, then we have:

$$\begin{aligned} x_1 &= -2s + 3t \\ x_2 &= s \\ x_3 &= t \\ x_4 &= 0 \end{aligned}$$

which is the same as

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = s \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + t \begin{bmatrix} 3 \\ 0 \\ 1 \\ 0 \end{bmatrix}$$

So $\text{NS}(A) = \text{span}\{(-1,1,0,0)^T, (3,0,1,0)^T\}$.

(c) $\text{rref}(A) = \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}$ so this has x_3 as a free variable. Let $x_3 = t$, then

$$\begin{aligned} x_1 &= t \\ x_2 &= t \end{aligned}$$

is the resulting system so an element of $\mathrm{NS}(A)$ is of the form

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} t \\ t \\ t \end{bmatrix} = t \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

so $\mathrm{NS}(A) = \mathrm{span}\{(1,1,1)^T\}$.

(d) Just as an example of using MATLAB

```
A=[1 1 -1 2; 2 2 -3 1; -1 -1 0 -5]
rref(A)
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$\mathrm{rref}(A) = \begin{bmatrix} 1 & 1 & 0 & 5 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ so x_2 and x_4 are the non-pivot, hence free variables. Let $x_2 = s$ and $x_4 = t$, then the system becomes

$$\begin{aligned} x_1 &= -s - 5t \\ x_3 &= -3t \end{aligned}$$

So we have $\boldsymbol{x} \in \mathrm{NS}(A)$ iff

$$\boldsymbol{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} -s - 5t \\ s \\ -3t \\ t \end{bmatrix} = s \begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \end{bmatrix} + t \begin{bmatrix} -5 \\ 0 \\ -3 \\ 1 \end{bmatrix}$$

and thus

$$\mathrm{NS}(A) = \mathrm{span}\left\{ \begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -5 \\ 0 \\ -3 \\ 1 \end{bmatrix} \right\}$$

8. A is fixed.

- $\mathbf{0}A = A\mathbf{0}$ so $\mathbf{0} \in S$
- Let $B, C \in S$, then $BA = AB$ and $CA = AC$ so $(B+C)A = BA + CA = AB + AC = A(B+C)$ and hence $B + C \in S$.
- Let $B \in S$, then $(\alpha B)A = \alpha(BA) = \alpha(AB) = A(\alpha B)$, so $\alpha B \in S$.

11. Just put the vectors in as columns, or rows, of a matrix A. Find $\mathrm{rref}(A)$. If there are two non-zero rows, that is $\mathrm{rank}(A) = 2$, then the set is a basis. for example, given $B = \{(2,1)^T, (3,2)^T\}$ for $A = \begin{bmatrix} 2 & 3 \\ 1 & 2 \end{bmatrix}$ (I put the vectors in as columns). $\mathrm{rref}(A) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, so B spans $\mathbb{R}^2$. (You could just compute $\mathrm{rank}(A)$ in MATLAB.

13. If $A = \begin{bmatrix} \boldsymbol{x}_1 & \boldsymbol{x}_2 \end{bmatrix}$, then $x \in \mathrm{span}\{\boldsymbol{x}_1, \boldsymbol{x}_2\}$ iff $A\boldsymbol{z} = \boldsymbol{x}$ has a solution, similar for $\boldsymbol{y}$. So for $\boldsymbol{x}$ just try to solve

$$\begin{bmatrix} -1 & 3 \\ 2 & 4 \\ 3 & 2 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} = \begin{bmatrix} 2 \\ 6 \\ 6 \end{bmatrix}$$

Since

$$\operatorname{rref}\left(\begin{bmatrix} -1 & 3 & 2 \\ 2 & 4 & 6 \\ 3 & 2 & 6 \end{bmatrix}\right) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

this has no solution. Recall this was an augmented matrix and the last row means $0z_1 + 0z_2 = 1$ which is nonsense.

17.

(a) Adding a vector to a spanning set leaves it a spanning set. This is clear since if $S \subset S' \subset V$ are sets of vectors in a vector space V, then clearly $\operatorname{span}(S) \subset \operatorname{span}(S')$. But if $\operatorname{span}(S) = V$, i.e., S is a spanning set, then $V \subset \operatorname{span}(S) \subset \operatorname{span}(S') \subset V$ so these must all be the same.

(b) Removing a vector from a spanning set may, or may not, leave it as a spanning set. If it is a minimal spanning set (a basis), then removing a vector will mean that what is left is no longer spanning.

Section 3.3

2. Again just write these vectors down as the rows of a matrix A. If $\operatorname{rref}(A)$ has any 0 rows, then the vectors are not independent, otherwise they are. For example:

$$\operatorname{rref}\left(\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 2 & 3 \end{bmatrix}\right) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

So these vectors are not independent.

5. (This is sort of the opposite of the spanning case.)

(a) Adding vectors to a linearly independent set can obviously mess up independence. (Just add a linear combination of the original vectors.) For example, if $S \subset \mathbb{R}^n$ is linearly independent, then $S \cup \{\mathbf{0}\}$ is not.

(b) Clearly removing a vector from a linearly independent set cannot mess up linear independence.

Specifically if $S = \{\boldsymbol{v_1}, \ldots, \boldsymbol{v_n}\}$ and $S' \subset S$, say $S' = \{\boldsymbol{v_{i_1}}, \ldots, \boldsymbol{v_{i_k}}\}$ and $c_{i_1}\boldsymbol{v_{i_1}} + \cdots + c_{i_k}\boldsymbol{v_{i_k}} = \mathbf{0}$ is a linear combination of elements of S', then this is trivially also a linear combination of elements of S and hence by the independence of S we have $c_{i_1} = \cdots = c_{i_k} = 0$. So S' is linearly independent.

8. Determine whether the following are independent in P_3.

(a) $\{1, x^2, x^2 - 2\}$ is not independent as $x^2 - 2 = -2 \cdot 1 + 1 \cdot x^2$, so $x^2 - 2$ is a linear combination of 1 and x^2.

(c) $\{x + 2, x + 1, x^2 - 1\}$ relative to the standard (ordered) basis for P_3, $\{1, x, x^2\}$, this is equivalent to asking if $\{(2, 1, 0), (1, 1, 0), (-1, 0, 1)\}$ is linearly independent. Clearly,

$$\begin{bmatrix} 2 & 1 & -1 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 0 \\ 0 & -1 & -1 \\ 0 & 0 & 1 \end{bmatrix}$$

so $\{x+2, x+1, x^2-1\}$ is linearly independent.

(d) $\{x+2, x^2-1\}$ is independent since $\{x+2, x+1, x^2-1\}$ is linearly independent, by (c).

9. Show the following sets are linearly independent in $C([0,1])$

(a) $\sin(\pi x)$ and $\cos(\pi x)$

One interesting way here is to note that $\langle f, g\rangle = \int_0^1 f \cdot g\, dx$ is an inner-product on $C([0,1])$ and $\langle \sin(\pi x), \cos(\pi x)\rangle = 0$, so actually, these two functions are orthogonal!

A less interesting way is to note that if $a\sin(\pi x) + b\cos(\pi x) = 0$ (the 0 function), then letting $x = 0$ gives $a\sin(0) + b\cos(0) = b = 0$ and letting $x = 1/2$ gives $a\sin(\pi/2) + b\cos(\pi/2) = a = 0$ so $a = b = 0$ and hence the two functions are independent.

(b) $x^{3/2}$ and $x^{5/2}$

Suppose $ax^{3/2} + bx^{5/2} = 0$ for all $x \in [0,1]$, then for $x = 1$ we have $a + b = 0$ and for $x = 1/4$ we have $a(1/2)^3 + b(1/2)^5 = 0$ so $a + b(1/2)^2 = 0$ hence $a + b/4 = 0$ or equivalently $4a + b = 0$. Solving

$$\begin{aligned} 4a + b &= 0 \\ a + b &= 0 \end{aligned}$$

gives $a = b = 0$. So These are independent.

(c) 1, $x^x - e^{-x}$ and $e^x + e^{-x}$

Again suppose $h(x) = a + b(e^x - e^{-x}) + c(e^x + e^{-x}) = 0$. It is easy to see $h(0) = a + 2c = 0$, $h'(0) = 2b = 0$ and $h''(0) = 2c = 0$. So clearly, $a = b = c = 0$ as desired.

(d) e^x, e^{-x} and e^{2x}

This is like (c), Assume $h(x) = ae^x + be^{-x} + ce^{2x}$, then $h'(x) = ae^x - be^{-x} + 2ce^{2x}$ and $h''(x) = ae^e + be^{-x} + 4e^{2x}$ and so

$$\begin{aligned} h(0) &= a + b + c \ \ = 0 \\ h'(0) &= a - b + 2c = 0 \\ h''(0) &= a + b + 4c = 0 \end{aligned}$$

It is easy to check that this has the unique solution $a = b = c = 0$.

10. It turns out here that 1, $\cos(x)$, and $\sin^2(x/2)$ are linearly dependent and this is from one of the half-angle formulas,

$$\cos(x) = \cos^2(x/2) - \sin^2(x/2) = 1 - 2\sin^2(x/2)$$

.

16. Show that the columns of A are linearly independent iff $\mathrm{NS}(A) = \{\mathbf{0}\}$.

Suppose A is $m \times n$ so $A = \begin{bmatrix} \boldsymbol{a_1} & \cdots & \boldsymbol{a_n} \end{bmatrix}$ with $\boldsymbol{a_i} \in \mathbb{R}^m$ the $\mathrm{i^{th}}$ column of A. Then

$$A\boldsymbol{x} = x_1\boldsymbol{a_1} + \cdots + x_n\boldsymbol{a_n}$$

is an arbitrary linear combination of the columns of A and so.

(if) Assume $\mathrm{NS}(A) = \{\mathbf{0}\}$, then $x_1\boldsymbol{a_1} + \cdots + x_n\boldsymbol{a_n} = 0$ iff $A\boldsymbol{x} = \mathbf{0}$ iff $\boldsymbol{x} = \mathbf{0}$, that is $x_1 = x_2 = \cdots x_n = 0$. So the columns of A are linearly independent since the only linear combination giving $\mathbf{0}$ is the trivial combination.

(only-if) Assume the columns of A are linearly independent, then $A\boldsymbol{x} = \mathbf{0}$ would mean the $x_1\boldsymbol{a_1} + \cdots + x_n\boldsymbol{a_n} = 0$ so by linear independence, $x_1 = x_2 = \cdots = 0$ and hence $A\boldsymbol{x} = \mathbf{0}$ implies $\boldsymbol{x} = \mathbf{0}$ so $\mathrm{NS}(A) = \{\mathbf{0}\}$.

17. Suppose $\mathrm{NS}(A) = \{\mathbf{0}\}$ and $\boldsymbol{x_1}, \boldsymbol{x_2}, \ldots, \boldsymbol{x_k}$ are linearly independent. Suppose also

$$\alpha_1 A\boldsymbol{x_1} + \alpha_2 A\boldsymbol{x_2} + \cdots + \alpha_k A\boldsymbol{x_k} = 0,$$

then

$$\mathbf{0} = \alpha_1 A\boldsymbol{x}_1 + \cdots + \alpha_k A\boldsymbol{x}_k = A(\alpha_1\boldsymbol{x}_1 + \cdots + \alpha_k\boldsymbol{x}_k)$$

so $\alpha_1\boldsymbol{x}_1 + \cdots + \alpha_k\boldsymbol{x}_k \in \mathrm{NS}(A) = \{\mathbf{0}\}$ and thus

$$\alpha_1\boldsymbol{x}_1 + \cdots + \alpha_k\boldsymbol{x}_k = \mathbf{0}$$

But the x_i's are linearly independent so $a_1 = a_2 = \cdots = a_k = 0$. but this is what we needed to see that $A\boldsymbol{x_1}, A\boldsymbol{x_2}, \ldots, A\boldsymbol{x_k}$ is linearly independent.

Section 3.4

5.

(a) Let A be the matrix whose columns are the three vectors given

$$A = \begin{bmatrix} 2 & 3 & 2 \\ 1 & -1 & 6 \\ 3 & 4 & 4 \end{bmatrix}$$

The given vectors are linearly independent iff $\mathrm{NS}(A) = \{\mathbf{0}\}$, since

$$\mathrm{NS}(A) = \{\mathbf{0}\} \text{ iff } A\boldsymbol{x} = \mathbf{0} \text{ implies } \boldsymbol{x} = \mathbf{0},$$

but the right hand side here says precisely that the only linear combination of the columns that yields $\mathbf{0}$ is the trivial combination, that is all coefficients are 0.

$$\operatorname{rref} A = \begin{bmatrix} 1 & 0 & 4 \\ 0 & 1 & -2 \\ 0 & 0 & 0 \end{bmatrix}$$

Clearly, this has a nontrivial null space, in fact,

$$\mathrm{NS}(A) = \mathrm{span}\{(-4, 2, 1))\}$$

So $-4\boldsymbol{x_1} + 2\boldsymbol{x_2} + \boldsymbol{x_3} = \mathbf{0}$, where these were the given vectors. (Easy for the reader to check. Do it!)

(b) Clearly $\boldsymbol{x_1}$ and $\boldsymbol{x_2}$ are linearly independent, since there is no $r \in \mathbb{R}$ such that $r\boldsymbol{x_1} = \boldsymbol{x_2}$.

(c) Let $S = \text{span}\{\boldsymbol{x_1}, \boldsymbol{x_2}, \boldsymbol{x_3}\}$, then (a) and (b) together show $2 \leq \dim(S) < 3$ so $\dim(S) = 2$.

(d) A 2-dimensional subspace of $\mathbb{R}^3$ is a plane.

alternate solution

$$\begin{bmatrix} 3 & -3 & -6 \\ -2 & 2 & 4 \\ 3 & 4 & 4 \end{bmatrix} \sim \begin{bmatrix} 1 & -1 & -2 \\ 0 & 0 & 0 \\ 0 & 7 & 10 \end{bmatrix} \sim \begin{bmatrix} 1 & -1 & -2 \\ 0 & 7 & 10 \\ 0 & 0 & 0 \end{bmatrix}$$

So a basis for $V = \text{span}\{\boldsymbol{x}_1, \boldsymbol{x}_2, \boldsymbol{x}_3\}$ is given by $\{\boldsymbol{x}_2, \boldsymbol{x}_2\}$. So $\dim(V) = 2$ and V is a plane in $\mathbb{R}^3$.

7. $(a + b, a - b + 2c, b, c) = a(1, 1, 0, 0) + b(1, -1, 1, 0) + c(0, 2, 0, 1)$

It is easy to see that $\{(1, 1, 0, 0), (1, -1, 1, 0), (0, 2, 0, 1)\}$ is independent so $\dim(S) = 3$.

8.

(a) No, two non co-linear vectors span a plane not all of $\mathbb{R}^3$

(b) X must be linearly independent. We can be more specific here. If A has columns $\boldsymbol{x_1} = (1, 1, 1)$, $\boldsymbol{x_2} = (3, -1, 4)$, and $\boldsymbol{x}_3 = (a_1, a_2, a_3)$, then X is linearly independent iff any of the following hold

- $\text{NS}(A) = \{\mathbf{0}\}$
- $\det(A) = 0$
- $\text{rref}(A) = I_3$

Any one of these can be used to characterize the x_3 that are allowed, but geometrically we know that the set of these vectors is ALL vectors not in the plane spanned by $\boldsymbol{x}_1$ and $\boldsymbol{x}_2$.

(c) Any vector not in the plane spanned by $\boldsymbol{x}, \boldsymbol{x}_2$ will work, say $\boldsymbol{x}_3 = (1, 0, 0)^T$

13. $\cos(2x) = 2\cos^2(x) - 1$, so $\dim(\text{span}\{\cos(2x), \cos^2(x), 1\}) = 2$.

Section 3.5

1. Find the transition matrix from the basis $\mathcal{U} = \{\boldsymbol{u}_1, \boldsymbol{u}_2\}$ to the standard basis. This I would also denote $[\text{id}]_{\mathcal{U},\mathcal{E}}$, where $\text{id} : \mathbb{R}^2 \to \mathbb{R}^2$ is just the identity transformation.

(a) $U = \begin{bmatrix} \boldsymbol{u}_1 & \boldsymbol{u}_2 \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$

(b) $U = \begin{bmatrix} \boldsymbol{u}_1 & \boldsymbol{u}_2 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 2 & 5 \end{bmatrix}$

(c) $U = \begin{bmatrix} \boldsymbol{u}_1 & \boldsymbol{u}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$

2. This is just the opposite of (1), find the transition matrix from the standard basis to the basis $\mathcal{U} = \{\boldsymbol{u}_1, \boldsymbol{u}_2\}$, that is find $[\mathrm{id}]_{\mathcal{E},\mathcal{B}}$.

Letting U be the matrix from (1), here the matrix we desire is U^{-1}, so

(a) $U^{-1} = \begin{bmatrix} \boldsymbol{u}_1 & \boldsymbol{u}_2 \end{bmatrix}^{-1} = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}^{-1} = \frac{1}{2}\begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}$

(b) $U^{-1} = \begin{bmatrix} \boldsymbol{u}_1 & \boldsymbol{u}_2 \end{bmatrix}^{-1} = \begin{bmatrix} 1 & 2 \\ 2 & 5 \end{bmatrix}^{-1} = \begin{bmatrix} 5 & -2 \\ -2 & 1 \end{bmatrix}$

(c) $U^{-1} = \begin{bmatrix} \boldsymbol{u}_1 & \boldsymbol{u}_2 \end{bmatrix}^{-1} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}^{-1} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$

3.

(a) The transition matrix for $\mathcal{V} = \{\boldsymbol{v}_1, \boldsymbol{v}_2\} \to \{\boldsymbol{e}_1, \boldsymbol{e}_2\}$ is $V = \begin{bmatrix} 3 & 4 \\ 2 & 3 \end{bmatrix}$. So the transformation matrix from $\mathcal{V} = \{\boldsymbol{v}_1, \boldsymbol{v}_2\} \to \mathcal{U} = \{\boldsymbol{u}_1, \boldsymbol{u}_2\}$ is $U^{-1}V$, where U is as in 1.

(a) $U^{-1}V = \begin{bmatrix} 2.5 & 3.5 \\ -0.5 & -0.5 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 5 & 7 \\ -1 & -1 \end{bmatrix}$

(b) $U^{-1}V = \begin{bmatrix} 11 & 14 \\ -4 & -5 \end{bmatrix}$

(c) $U^{-1}V = \begin{bmatrix} 2 & 3 \\ 3 & 4 \end{bmatrix}$

6. Let $\mathcal{U} = \{\boldsymbol{u}_1, \boldsymbol{u}_2, \boldsymbol{u}_3\} = \{(1,1,1), (1,2,2), (1,3,4)\}$ and $\mathcal{V} = \{\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3\} = \{(4,6,7), (0,1,1), (0,1,2)\}$.

(a) Find transition matrix from $\mathcal{V}$ to $\mathcal{U}$.

This is

$$[\mathrm{id}]_{\mathcal{V},\mathcal{U}} = [\mathrm{id} \circ \mathrm{id}]_{\mathcal{V},\mathcal{U}} = [\mathrm{id}]_{\mathcal{E},\mathcal{U}}[\mathrm{id}]_{\mathcal{V},\mathcal{E}} = U^{-1}V = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 2 \\ 1 & 3 & 4 \end{bmatrix}^{-1} \begin{bmatrix} 4 & 0 & 0 \\ 6 & 1 & 1 \\ 7 & 1 & 2 \end{bmatrix} = \begin{bmatrix} 1 & -1 & -2 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$$

(b) Find the $\mathcal{U}$ representation of $\boldsymbol{v} = 2\boldsymbol{v}_1 + 3\boldsymbol{v}_2 - 4\boldsymbol{v}_3$.

We see $[\boldsymbol{v}]_{\mathcal{V}} = \begin{bmatrix} 2 \\ 3 \\ -4 \end{bmatrix}$ and

$$[\boldsymbol{v}]_{\mathcal{U}} = [\mathrm{id}]_{\mathcal{V},\mathcal{U}}[\boldsymbol{v}]_{\mathcal{V}} = \begin{bmatrix} 1 & -1 & -2 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 2 \\ 3 \\ -4 \end{bmatrix} = \begin{bmatrix} 7 \\ 5 \\ -2 \end{bmatrix}$$

So $\boldsymbol{v} = 7\boldsymbol{u}_2 + 5\boldsymbol{u}_2 - 2\boldsymbol{u}_3$.

You should check this:

$$2\boldsymbol{v}_1 + 3\boldsymbol{v}_2 - 4\boldsymbol{v}_3 = \begin{bmatrix} 4 & 0 & 0 \\ 6 & 1 & 1 \\ 7 & 1 & 2 \end{bmatrix} \begin{bmatrix} 2 \\ 3 \\ -4 \end{bmatrix} = \begin{bmatrix} 8 \\ 11 \\ 9 \end{bmatrix}$$

$$7\boldsymbol{u}_2 + 5\boldsymbol{u}_2 - 2\boldsymbol{u}_3 = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 2 \\ 1 & 3 & 4 \end{bmatrix} \begin{bmatrix} 7 \\ 5 \\ -2 \end{bmatrix} = \begin{bmatrix} 8 \\ 11 \\ 9 \end{bmatrix}$$

10. Find transition matrix from the basis $\mathcal{B} = \{1, x, x^2\}$ for $\mathbb{P}_3$ to $\mathcal{C} = \{1, 1+x, 1+x+x^2\}$. The transformation matrix from from $\mathcal{C}$ to $\mathcal{B}$ is easy:

$$[1]_{\lfloor} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \quad [1+x]_{\mathcal{B}} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \quad [1+x+x^2]_{\mathcal{B}} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

So we have

$$[\mathrm{id}]_{\mathcal{C},\mathcal{B}} = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$

The matrix in the other direction, from $\mathcal{B}$ to $\mathcal{C}$ is just the inverse

$$[\mathrm{id}]_{\mathcal{B},\mathcal{C}} = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix}$$

As an application, write $p = 3 - 2x + 4x^2$ in the $\mathcal{C}$ basis. $[p]_{\mathcal{B}} = (3, -2, 4)$ so

$$[p]_{\mathcal{C}} = \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 3 \\ -2 \\ 4 \end{bmatrix} = \begin{bmatrix} 5 \\ -6 \\ 4 \end{bmatrix}$$

hence

$$3 - 2x + 4x^2 = 5 - 6(1+x) + 4(1+x+x^2)$$

Section 3.6

1. Let A denote the matrix given

(a) $\mathrm{rref}(A) = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ so

So letting $x_3 = t$ we get $x_2 = 0$ and $x_1 = -2t$ and the solutions $A\boldsymbol{x} = \boldsymbol{0}$ are all those $\boldsymbol{x} = t \begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix}$

$$\begin{aligned}
\mathrm{RS}(A) &= \mathrm{span}\{(1,0,2)^T,(0,1,0)^T\}\\
\mathrm{CS}(A) &= \mathrm{span}\{(1,2,4)^T,(3,1,7)^T\}\\
\mathrm{NS}(A) &= \mathrm{span}\{(-2,0,1)^T\}
\end{aligned}$$

Remarks: The non zero rows of rref(A) are a basis for RS(A). The columns of A that are pivot columns of rref(A) are a basis for CS(A).

(b) $\mathrm{rref}(A) = \begin{bmatrix} 1 & 0 & 0 & 0\\ 0 & 1 & 0 & -2/7\\ 0 & 0 & 1 & 10/7 \end{bmatrix}$

So setting $x_4 = t$ we get $A\boldsymbol{x} = \mathbf{0}$ at $\boldsymbol{x} = t\begin{bmatrix} 0\\ 2/7\\ -10/7\\ 1\end{bmatrix}$ and so

$$\begin{aligned}
\mathrm{RS}(A) &= \mathrm{span}\{(1,0,0,0)^T,(0,1,0,-2/7)^T,(0,0,0,1,10/7)^T\}\\
\mathrm{CS}(A) &= \mathrm{span}\{(-3,1,3)^T,(1,2,4)^T,(3,-1,5)^T\}\\
\mathrm{NS}(A) &= \mathrm{span}\{(0,2/7,-10/7,1)^T\}
\end{aligned}$$

6. If $\boldsymbol{b}$ is in CS(A) and the columns of A are independent, then $A\boldsymbol{x} = \boldsymbol{b}$ has a solution, since this is what $\boldsymbol{b} \in \mathrm{CS}(A)$ means and the solution is unique. For is $A\boldsymbol{x}' = A\boldsymbol{x} = \boldsymbol{b}$, then $A\boldsymbol{x}' - A\boldsymbol{x} = A(\boldsymbol{X}' - \boldsymbol{x}) = \mathbf{0}$, but this means $\boldsymbol{x}' - \boldsymbol{x} = \mathbf{0}$, since the columns of A are independent and hence $\boldsymbol{x} = \boldsymbol{x}'$.

9.

(a) If A is 6×5 and $\dim(\mathrm{NS}(A)) = 2$, then since $\mathbb{R}^5 = \mathrm{RS}(A) \oplus \mathrm{NS}(A)$ we have $5 = \dim \mathrm{RS}(A) + 2$ so $\dim \mathrm{RS}(A) = 3$.

(b) If B is 6×5, then as above $5 = \dim \mathrm{NS}(A) + \dim \mathrm{RS}(A) = \dim \mathrm{NS}(A) + \mathrm{rank}(A) = \dim \mathrm{NS}(A) + 4$, so $\dim \mathrm{NS}(A) = 1$.

14. From U read off the solutions to $A\boldsymbol{x} = 0$, i.e. $\mathrm{NS}(A) = \mathrm{NS}(U)$ as

$$\begin{bmatrix} x_1\\ x_2\\ x_3\\ x_4\end{bmatrix} = \begin{bmatrix} -2s-t\\ -s-4t\\ s\\ t\end{bmatrix} = s\begin{bmatrix} -2\\ -1\\ 1\\ 0\end{bmatrix} + t\begin{bmatrix} -1\\ -4\\ 0\\ 1\end{bmatrix} = s\cdot\boldsymbol{u}_1 + t\cdot\boldsymbol{u}_2$$

Now we know $A(s\boldsymbol{u}_1 + t\boldsymbol{u}_2) = \mathbf{0}$ so in particular, $A\boldsymbol{u}_1 = A\boldsymbol{u}_2 = 0$ and if $A = \begin{bmatrix} a_1 & \boldsymbol{a}_2 & \boldsymbol{a}_3 & \boldsymbol{a}_4 \end{bmatrix}$, then

$$\begin{aligned}
A\boldsymbol{u}_1 &= -2\boldsymbol{a}_1 - \boldsymbol{a}_2 + \boldsymbol{a}_3 = \mathbf{0}\\
A\boldsymbol{u}_2 &= -\boldsymbol{a}_1 - 4\boldsymbol{a}_2 + \boldsymbol{a}_4 = \mathbf{0}
\end{aligned}$$

so

$$
\begin{aligned}
\boldsymbol{a}_3 &= 2\boldsymbol{a}_1 + \boldsymbol{a}_2 \\
\boldsymbol{a}_4 &= \boldsymbol{a}_1 + 4\boldsymbol{a}_2
\end{aligned}
$$