Exam 1 – Math 215

1 True/False

Problem 1.1 (60 points; 6 points each)**.** Decide if each of the following are true or false. You do not need to justify your choice here.

(a) True The proposition $\neg(p \vee \neg(q \vee p))$ is equivalent to $\neg p \wedge q$.

(b) False Up to propositional equivalence, there are only 2^3 compound propositions using the atomic propositional variables, p, q, and r.

(c) True $\neg\exists x(P(x) \to Q(x)) \equiv \forall x P(x) \wedge \neg\exists x Q(x))$

(d) False $\exists x P(x) \wedge \exists x Q(x) \equiv \exists x(P(x) \wedge Q(x))$.

(e) True There is a finite model of $\exists x(x = x)$ (something exists), $\forall x \exists y(N(x, y) \wedge x \neq y)$ (for everything there is a different thing *next to* it), $\forall x, y, z(N(x, y) \wedge N(y, z) \to N(x, z))$ (being *next to* is transitive).

(f) True $A \subseteq B \implies C - B \subseteq C - A$

(g) False For all sets A, $A \subseteq \mathcal{P}(A)$ (the powerset of A),

(h) True The set of all polynomials with rational coefficients, denoted $\mathbb{Q}[x]$, is countable.

(i) False The set of irrationals is uncountable.

(j) False In the inductive step of a proof by induction of $\forall n \geq 0 P(n)$, one may assume that for all $n \geq 0$, $P(n) \to P(n + 1)$.

2 Free Response

100 points total, 20 points each

Problem 2.1. Prove or give a counter-example to the following statement:

The product of two irrational numbers is irrational.

Just consider $\sqrt{2}$, $\sqrt{2} \cdot \sqrt{2} = 2$. So, two irrationals can multiply to give a rational. For an example with two distinct numbers, we can use $a = \sqrt{3} + \sqrt{2}$ and $b = \sqrt{3} - \sqrt{2}$, so that $a \cdot b = (\sqrt{3})^2 - (\sqrt{2})^2 = 3 - 2 = 1$.

Problem 2.2. Prove that the sum of a rational number and an irrational number is irrational.

Suppose r is irrational, s is rational, and $r+s=t$ is rational. Then $r = t-s$ is rational, contrary to the assumption on r.

Problem 2.3. Prove:

$$p \wedge (q_1 \vee q_2 \vee \cdots \vee q_n) \equiv (p \wedge q_1) \vee (p \wedge q_2) \vee \cdots \vee (q \wedge q_n)$$

You may use induction (strong or weak), the well-ordering principle, or infinite decent. Make clear what you are doing, for example, if you use induction, make the base case(s) and inductive step clear.

The thing we are trying to prove is $P(n)$ for $n \geq 1$ where $P(n)$ is exactly as given

$$P(n) : p \wedge (q_1 \vee q_2 \vee \cdots \vee q_n) \equiv (p \wedge q_1) \vee (p \wedge q_2) \vee \cdots \vee (q \wedge q_n)$$

Base Case ($n = 1$):

$$P(1) : p \wedge q_1 = p \wedge q_1$$

is trivially true.

Inductive Step: Suppose $P(n)$. We must prove $P(n+1)$:

$$\begin{aligned} p \wedge (q_1 \vee q_2 \vee \cdots \vee q_n \vee q_{n+1}) &\equiv p \wedge ((q_1 \vee \cdots \vee q_n) \vee q_{n+1}) \\ &\equiv p \wedge (q_1 \vee \cdots \vee q_n) \vee (p \wedge q_{n+1}) \\ &\equiv ((p \wedge q_1) \vee (p \wedge q_2) \vee \cdots \vee (q \wedge q_n)) \vee (p \wedge q_{n+1}) \\ &\equiv (p \wedge q_1) \vee (p \wedge q_2) \vee \cdots \vee (q \wedge q_n) \vee (p \wedge q_{n+1}) \end{aligned}$$

Thus $P(n+1)$ holds.

By induction, $\forall n \geq 1 P(n)$.

Problem 2.4. Explain why the set of finite sequences of rational numbers, $\mathbb{Q}^{<\infty} = \{(q_1, q_2, \ldots, q_n) \mid n \in \mathbb{N}\}$, is countable.

Use facts you know about countability, including the countability of $\mathbb{Q}$ and facts about which set operations preserve countability, e.g., the countable union of countable sets is countable, etc. Make clear which facts you are using.

You might recognize that this is related to one of the T/F questions.

$\mathbb{Q}$ is countable and hence $\mathbb{Q}^n$ is countable. (You can just use this or you can argue that $\mathbb{Q} \times \mathbb{Q}$ is countable, $(\mathbb{Q} \times \mathbb{Q}) \times \mathbb{Q}$ is countable, etc. (This is really in induction.)

Now $\mathbb{Q}^{<\infty} = \bigcup_{i=1}^{\infty} \mathbb{Q}^i$ is a countable union of countable sets and hence is countable.

Finally, and not part of this problem, the map taking a tuple, $(q_0, \ldots, q_n) \mapsto q_0 + q_1 x + q_2 x^2 + \cdots + q_n x^n$ is *clearly bijection between* $Q^{<\infty}$ *and* $\mathbb{Q}[x]$.

Problem 2.5. Show that if $a_1, a_2, \ldots, a_n$ are real numbers and $a = \frac{1}{n}\sum_{i=1}^{n} a_i$ (the average), then for some i, $a_i \geq a$.

There are many ways to proceed. Induction is an option, or a direct argument. Here is one way. Suppose $a_i < a$ for all i, then $\sum_{i=1}^{n} a_i < \sum_{i=1}^{n} a = n \cdot a$, but then $\frac{1}{n}\sum_{i=1}^{n} a_i < a$ contradicting the definition of a.