

Matrix Mult Example

$$\begin{bmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 4 \\ 3 & 2 \\ 2 & -1 \end{bmatrix}$$

(A)

$$\left[\begin{bmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 3 \\ 2 \end{bmatrix} \right] \left[\begin{bmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{bmatrix} \begin{bmatrix} 4 \\ 2 \\ -1 \end{bmatrix} \right]$$

$$= \left[\begin{bmatrix} 1 \\ 3 \\ 2 \end{bmatrix}(1) + \begin{bmatrix} 2 \\ 2 \\ 1 \end{bmatrix}(3) + \begin{bmatrix} 3 \\ 1 \\ 1 \end{bmatrix}(2) \quad \begin{bmatrix} 1 \\ 3 \\ 2 \end{bmatrix}(4) + \begin{bmatrix} 2 \\ 2 \\ 1 \end{bmatrix}(2) + \begin{bmatrix} 3 \\ 1 \\ 1 \end{bmatrix}(-1) \right]$$

$$= \left[\begin{bmatrix} 1 \\ 3 \\ 2 \end{bmatrix} + \begin{bmatrix} 6 \\ 6 \\ 2 \end{bmatrix} + \begin{bmatrix} 6 \\ 2 \\ 1 \end{bmatrix} \quad \begin{bmatrix} 4 \\ 12 \\ 2 \end{bmatrix} + \begin{bmatrix} 4 \\ 4 \\ 2 \end{bmatrix} + \begin{bmatrix} -3 \\ -1 \\ 1 \end{bmatrix} \right]$$

$$= \boxed{\begin{bmatrix} 13 & 5 \\ 11 & 15 \end{bmatrix}}$$

(B)

$$\begin{bmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 4 \\ 3 & 2 \\ 2 & -1 \end{bmatrix} = \begin{bmatrix} (1)(1) + 2(3) + 3(2) \\ (3)(1) + 2(3) + 1(2) \end{bmatrix} = \begin{bmatrix} 13 \\ 11 \end{bmatrix} = \boxed{\begin{bmatrix} 13 & 5 \\ 11 & 15 \end{bmatrix}}$$

(C)

$$\begin{bmatrix} (1 \cdot 3) \begin{bmatrix} 1 \\ 3 \end{bmatrix} & (1 \cdot 2 \cdot 3) \begin{bmatrix} 4 \\ 2 \\ -1 \end{bmatrix} \\ (3 \cdot 2 \cdot 1) \begin{bmatrix} 1 \\ 3 \end{bmatrix} & (3 \cdot 2 \cdot 1) \begin{bmatrix} 4 \\ 2 \\ -1 \end{bmatrix} \end{bmatrix} = \boxed{\begin{bmatrix} 13 & 5 \\ 11 & 15 \end{bmatrix}}$$

Matrix Algebra

Transpose: $(\mathbf{A}^T)_{ij} = A_{ji}$

$$\begin{bmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{bmatrix}^T = \begin{bmatrix} 1 & 3 \\ 2 & 2 \\ 3 & 1 \end{bmatrix}$$

Common uses (Inner prod)

$$\mathbf{v} \cdot \mathbf{u} = \mathbf{u}^T \mathbf{v}$$

$$(1 \ 2 \ 3) \cdot (2 \ -1 \ 1) =$$

$$\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \cdot \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix} = [2 \ -1 \ 1] \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

$$(7 \times 3) \times (3 \times 1) = 1 \times 1$$

Rank 1 (outer product)

$$u \otimes v = u v^T$$

$$\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \otimes \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} [2 -1 1]$$

$$= \begin{bmatrix} 2 & -1 & 1 \\ 4 & -2 & 2 \\ 6 & -3 & 3 \end{bmatrix}$$

$A \rightarrow$ symmetric iff $A^T = A$

Algebraic Rules

We will see these again. These say
 $\mathbb{R}^{m \times n}$ is a vector-space

- ① $A + B = B + A$ (commutativity of addition)
- ② $(A + B) + C = A + (B + C)$ (associativity)
- ③ $A + O = O + A = A$ (additive identity)
- ④ $A \cdot A = O$ (additive inverse)

- ⑤ $\alpha(\beta A) = (\alpha\beta)A$
- ⑥ $\alpha(A + B) = \alpha A + \alpha B$
- ⑦ $(\alpha + \beta)A = \alpha A + \beta A$
- ⑧ $1 \cdot A = A; 0 \cdot A = O$

Note. In general $AB \neq BA$

This is clear for non-squares
but even for squares

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} -1 & 1 \\ -1 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} = \begin{bmatrix} -2 & -2 \\ 2 & 2 \end{bmatrix}$$

- ⑨ $(AB)C = A(BC)$ (associativity)
- ⑩ $A(B+C) = AB + AC$
- $(A+B)C = AC + BC$

- ⑪ $(A+B)^T = A^T + B^T$
- ⑫ $(AB)^T = B^T A^T$

Other things to watch for:

$$AB = O \rightarrow A = O \text{ or } B = O$$

$$\text{Ex. } \begin{bmatrix} 1 & 1 \\ 2 & 2 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$$AC = AD \rightarrow C = D \quad (\text{cancellation fails})$$

Let's prove ⑨ and ⑫

Proof of ⑨: $((AB)C)_{ij} = \sum_{p=1}^k (AB)_{ir} C_{pj}$

$A \text{ } m \times n$

$B \text{ } n \times k$

$C \text{ } k \times l$

\therefore $\sum_{p=1}^k \left[\sum_{q=1}^n (A_{iq} B_{qp}) C_{pj} \right] = \sum_{p=1}^k \left[\sum_{q=1}^n ((A_{iq} B_{qp}) C_{pj}) \right]$

$= \sum_{p=1}^k \left[\sum_{q=1}^n (A_{iq} (B_{qp} C_{pj})) \right] = \sum_{q=1}^n \left[A_{iq} \sum_{p=1}^k (B_{qp} C_{pj}) \right]$

We will have a better way to see this later.

$$= \sum_{q=1}^n A_{iq} (BC)_{qj} = (A(BC))_{ij}$$

(12) $(AB)_{ij} = (AB)_{ji} = \sum_{p=1}^n A_{ip} B_{pj} = \sum_{p=1}^n B_{ip}^T A_{pj}^T = (B^T A^T)_{ij}$

A $m \times n$
 B $n \times l$

Powers, identity, polynomials

Square Matrices

For square matrix A , $A^k = \underbrace{A \cdot A \cdots A}_k$

$$A^k A^s = A^{k+s}$$

$$(A^k)^s = A^{(ks)}$$

$$I_n = \begin{bmatrix} 1 & & & \\ & \ddots & & 0 \\ & & \ddots & \\ 0 & & & 1 \end{bmatrix}$$

$$I A = A I = A \quad (I \text{ is multiplicative identity})$$

Define $A^0 = I$

Inverses Let A be a square $n \times n$ matrix. $B = A^{-1}$

iff $BA = AB = I$. (*)

Note. If B, B' both satisfy (*), then

$B = B(AD^{-1}) = (BA)D^{-1} = B' \Rightarrow B = B'$, i.e. A^{-1} is unique
matrix satisfying (*).

A is called non-singular if A^{-1} exists else A is called

singular

$$(AB)^{-1} = B^{-1}A^{-1} : (B^{-1}A^{-1})(AB) = I = (AB)(B^{-1}A^{-1})$$

Special Case: Diagonal Matrices Forgot this!

$$\begin{bmatrix} d_{11} & & & \\ & d_{22} & & \\ & & \ddots & \\ & & & d_{nn} \end{bmatrix} \begin{bmatrix} c_{11} & & & \\ & \ddots & & 0 \\ & & \ddots & \\ 0 & & & c_{nn} \end{bmatrix} = \begin{bmatrix} d_{11}c_{11} & & & \\ & d_{22}c_{22} & & \\ & & \ddots & \\ & & & d_{nn}c_{nn} \end{bmatrix}$$

$$\text{diag}(d_{11}, \dots, d_{nn}) \text{ diag}(c_{11}, \dots, c_{nn}) = \text{diag}(d_{11}c_{11}, \dots, d_{nn}c_{nn})$$

• If $A = \begin{bmatrix} a_{11} & 0 & \dots \\ 0 & \ddots & \dots \\ 0 & \dots & a_{nn} \end{bmatrix} = \text{diag}(a_{11}, \dots, a_{nn})$, then $A^k = \text{diag}(a_{11}^k, \dots, a_{nn}^k) = \begin{bmatrix} a_{11}^k & & & \\ & \ddots & & \\ & & \ddots & \\ & & & a_{nn}^k \end{bmatrix}$

• If $a_{ii} \neq 0$, then $\text{diag}(a_{ii}, \dots, a_{ii})^{-1} = \text{diag}(a_{ii}^{-1}, \dots, a_{ii}^{-1})$

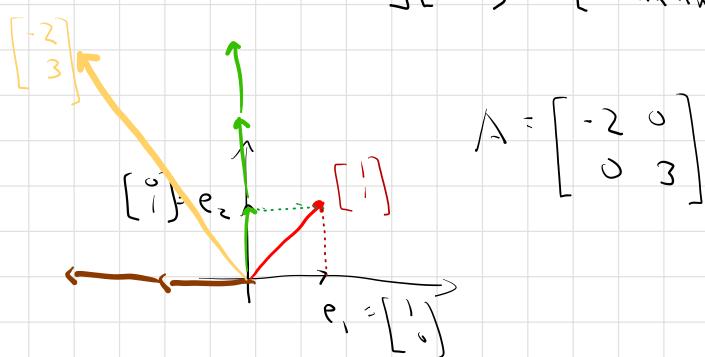
$$\begin{bmatrix} a_{ii} & 0 & \dots \\ 0 & \ddots & \dots \\ 0 & \dots & a_{ii} \end{bmatrix} \cdot \begin{bmatrix} 1/a_{ii} & 0 & \dots \\ 0 & \ddots & \dots \\ 0 & \dots & 1/a_{ii} \end{bmatrix} = \begin{bmatrix} 1 & 0 & \dots \\ 0 & \ddots & \dots \\ 0 & \dots & 1 \end{bmatrix}$$

For A diagonal A is singular iff one of the diagonal elements of A is 0.

$$Ax = \begin{bmatrix} a_{11} & 0 & \dots \\ 0 & \ddots & \dots \\ 0 & \dots & a_{nn} \end{bmatrix} \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} a_{11}x_1 \\ \vdots \\ a_{nn}x_n \end{bmatrix}$$

so A stretches

each $e_i = \begin{bmatrix} 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \end{bmatrix} e_i$ by a_{ii}



Elementary Matrices

We will code G-J elimination by a sequence of matrix ops.

$$R_i \leftrightarrow R_j \quad \text{do this to } I \quad E_{ij}^I \quad (E_{ij}^I)^{-1} = E_{ji}^I$$

$$cR_i \rightarrow R_i \quad \text{do this to } I \quad E_{ci}^I \quad (E_{ci}^I)^{-1} = E_{ci}^I$$

$$R_j + cR_i \rightarrow R_j \quad \text{do this to } I \quad E_{c,i,j}^I \quad (E_{c,i,j}^I)^{-1} = E_{-c,i,j}^I$$

Suppose we start with A and use G-J to solve $(m \times n)$

$Ax = b$ so we do a sequence of row ops on $(A|b)$ (reduced row echelon form)

\triangleright op₁, op₂, ... \rightarrow B

$$\underbrace{E_1 \ E_2 \ \dots}_{m \times m}$$

$$E_1 \triangleright E_2 \triangleright \dots$$

$$(E_1 E_2 \dots E_n) A (\underbrace{E_1 E_2 \dots E_n}_{m \times m})$$

$$\Delta x = b \rightarrow \underbrace{M \Delta x}_{U} = Mb$$

Instead of solving $Ax = b$ solve $Ux = Mb$ (back subs)

$$\left[\begin{array}{cccc|cccc} 4 & 8 & 2 & 5 & 1 & 0 & 0 & 0 \\ 5 & 10 & -4 & 5 & 0 & 1 & 0 & 0 \\ -3 & -6 & -2 & -3 & 0 & 0 & 1 & 0 \\ 5 & 10 & 1 & 5 & 0 & 0 & 0 & 1 \end{array} \right] \xrightarrow{\frac{(-1)}{5}/\frac{(1)}{13}} \left[\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 \\ \frac{5}{4} & 1 & 0 & 0 \\ -\frac{3}{4} & \frac{1}{13} & 1 & 0 \\ \frac{5}{4} & \frac{3}{13} & 0 & 1 \end{array} \right] = L$$

$$E_1 R_2 - \frac{5}{4} R_1 \rightarrow R_2$$

$$\left[\begin{array}{cccc|cccc} 4 & 8 & 2 & -6 & 1 & 0 & 0 & 0 \\ 0 & 0 & -\frac{13}{4} & -\frac{3}{2} & -\frac{5}{4} & 1 & 0 & 0 \\ 0 & 0 & -\frac{1}{4} & \frac{3}{4} & \frac{3}{4} & 0 & 1 & 0 \\ 0 & 0 & -\frac{3}{4} & -\frac{5}{4} & 0 & 0 & 0 & 1 \end{array} \right] \xrightarrow{\frac{(2)}{2}/\frac{(1)}{13}} \left[\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 \\ -\frac{5}{4} & 1 & 0 & 0 \\ \frac{3}{4} & 0 & 1 & 0 \\ -\frac{3}{4} & 0 & 0 & 1 \end{array} \right]$$

$$E_2 R_3 + \frac{3}{4} R_1 \rightarrow R_3$$

$$E_3 R_4 - \frac{1}{4} R_1 \rightarrow R_4$$

$$E_4 R_3 - \frac{1}{13} R_1 \rightarrow R_3$$

$$E_5 R_4 - \frac{1}{12} R_1 \rightarrow R_4$$

$$\left[\begin{array}{cccc|cccc} 4 & 8 & 2 & -6 & 1 & 0 & 0 & 0 \\ 0 & 0 & -\frac{13}{4} & -\frac{3}{2} & -\frac{5}{4} & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{11}{13} & -\frac{1}{13} & 1 & 0 \\ 0 & 0 & 0 & 0 & -\frac{25}{12} & -\frac{3}{13} & 0 & 1 \end{array} \right]$$

$$Y_{\text{new}} \leftarrow \text{combine } \underbrace{E_1 E_2 E_3}_{G_1} \left[\begin{array}{cccc} 1 & 0 & 0 & 0 \\ -\frac{5}{4} & 1 & 0 & 0 \\ \frac{3}{4} & 0 & 1 & 0 \\ -\frac{3}{4} & 0 & 0 & 1 \end{array} \right]$$

$$E_6 R_4 + \frac{25}{12} R_3 \rightarrow R_4$$

$$\left[\begin{array}{cccc|cccc} 4 & 8 & 2 & -6 & 1 & 0 & 0 & 0 \\ 0 & 0 & -\frac{13}{4} & -\frac{3}{2} & -\frac{5}{4} & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{11}{13} & -\frac{1}{13} & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -\frac{7}{22} & \frac{25}{12} \end{array} \right]$$

$$M = E_6 E_5 E_4 E_3 E_2 E_1 \quad L = M^{-1} = E_1^{-1} E_2^{-1} E_3^{-1} E_4^{-1} E_5^{-1} E_6^{-1}$$

$$MA = U \rightarrow M^{-1} MA = M^{-1} U = LU$$

$$M = G_3 G_2 G_1$$

Solve $Ax = b \leftrightarrow$ Solve $\underbrace{Ux = Mb}_{\text{fwd back sub}} = L^T b$

$$M^{-1} = G_1^{-1} G_2^{-1} G_3^{-1}$$

$$\left[\begin{array}{ccc} 1 & 0 & 0 \\ \frac{5}{4} & 1 & 0 \\ -\frac{3}{4} & 0 & 1 \end{array} \right] \left[\begin{array}{ccc} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & \frac{1}{13} & 0 \end{array} \right] \left[\begin{array}{ccc} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{array} \right] = \left[\begin{array}{ccc} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{array} \right]$$

$$G_1^{-1} \quad (G_2^{-1} G_3^{-1}) \quad L$$

$$\left[\begin{array}{ccc} 1 & 0 & 0 \\ \frac{5}{4} & 1 & 0 \\ -\frac{3}{4} & 0 & 1 \end{array} \right] \left[\begin{array}{ccc} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & \frac{1}{13} & 0 \end{array} \right] \left[\begin{array}{ccc} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{array} \right] = \left[\begin{array}{ccc} 1 & 0 & 0 \\ \frac{5}{4} & 1 & 0 \\ -\frac{3}{4} & 0 & 1 \end{array} \right] \left[\begin{array}{ccc} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{array} \right] = \left[\begin{array}{ccc} 1 & 0 & 0 \\ \frac{5}{4} & 1 & 0 \\ -\frac{3}{4} & 0 & 1 \end{array} \right]$$

$A \rightarrow$ invertible $\Leftrightarrow A^{-1}X = I$ for some X

$$[A | I] \xrightarrow{\text{elem. ops}} [I | X] \quad X = A^{-1}$$

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \\ 2 & 1 & 3 \end{bmatrix}$$

$$\left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 2 & 1 & 3 & 0 & 1 & 0 \\ 2 & 1 & 3 & 0 & 0 & 1 \end{array} \right] \xrightarrow{R_1 - 3R_2 \rightarrow R_1} \left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & -4 & -8 & -3 & 1 & 0 \\ 0 & -3 & -3 & -2 & 0 & 1 \end{array} \right]$$

$$\begin{array}{l} -\frac{1}{4}R_2 \rightarrow R_2 \\ \xrightarrow{-\frac{1}{3}R_3 \rightarrow R_3} \left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & 1 & 2 & \frac{3}{4} & -\frac{1}{4} & 0 \\ 0 & 1 & 1 & \frac{2}{3} & 0 & -\frac{1}{3} \end{array} \right] \xrightarrow{R_3 - R_2 \rightarrow R_3} \left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & 1 & 2 & \frac{3}{4} & -\frac{1}{4} & 0 \\ 0 & 0 & -1 & -\frac{1}{12} & \frac{1}{4} & -\frac{1}{3} \end{array} \right] \end{array}$$

$$\begin{array}{l} R_2 + 2R_3 \rightarrow R_2 \\ \xrightarrow{R_1 + 2R_3 \rightarrow R_1} \left[\begin{array}{ccc|ccc} 1 & 2 & 0 & \frac{3}{4} & \frac{3}{4} & -1 \\ 0 & 1 & 0 & \frac{7}{12} & \frac{1}{4} & -\frac{2}{3} \\ 0 & 0 & 1 & -\frac{1}{12} & \frac{1}{4} & -\frac{1}{3} \end{array} \right] \xrightarrow{R_1 - 2R_2 \rightarrow R_1} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -\frac{5}{12} & \frac{1}{4} & \frac{1}{3} \\ 0 & 1 & 0 & \frac{7}{12} & \frac{1}{4} & -\frac{2}{3} \\ 0 & 0 & 1 & -\frac{1}{12} & \frac{1}{4} & -\frac{1}{3} \end{array} \right] \end{array}$$

$$A^{-1} = \begin{bmatrix} -\frac{5}{12} & \frac{1}{4} & \frac{1}{3} \\ \frac{7}{12} & \frac{1}{4} & -\frac{2}{3} \\ \frac{1}{12} & -\frac{1}{4} & \frac{1}{3} \end{bmatrix} = \frac{1}{12} \begin{bmatrix} -5 & 3 & 4 \\ 7 & 3 & -8 \\ 1 & -3 & 4 \end{bmatrix}$$

$$\text{Check: } \frac{1}{12} \begin{bmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \\ 2 & 1 & 3 \end{bmatrix} \begin{bmatrix} -5 & 3 & 4 \\ 7 & 3 & -8 \\ 1 & -3 & 4 \end{bmatrix} = \frac{1}{12} \begin{bmatrix} 12 & 0 & 0 \\ 0 & 12 & 0 \\ 0 & 0 & 12 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Equivalences of invertibility

- ① A is invertible (there is B so that $AB = BA = I$)
- ② There is a B s.t. $BA = I$ or $AB = I$
- ③ A is of full rank
- ④ $\text{rref}(A) = I$
- ⑤ $Ax = 0$ has a unique solution
- ⑥ A is non-singular