III Theory and Proofs (40 points; 20 points each)

Choose two of the four options. If you try more than two, I will grade only the first two, not the best two. You must decide what should be graded. These will be due 2/7 in class. Make sure your work is complete and clear. Explain your work, a proof is not just a bunch of math symbols, it is an explanation of why something is true.

Problem III.1 (20 pts)**.** If A and B are invertible $n \times n$ matrices, show that

$$(AB)^2 = A^2B^2 \iff AB = BA$$

Problem III.2 (20 pts)**.** Show that for any $m \times n$ matrix A,

$$\sum_{i=1}^{m}\sum_{j=1}^{n}\big((\boldsymbol{e}_i^m)^T A \boldsymbol{e}_j^n\big)\big(\boldsymbol{e}_i^m(\boldsymbol{e}_j^n)^T\big) = A.$$

Problem III.3 (20 pts)**.** Let A be an $n \times n$ matrix such that $AB = BA$ for all $n \times n$ matrices B. show that $A = \alpha I$ for some scalar α.

Problem III.4 (20 pts)**.** Consider the operation $\operatorname{rot}(A)$ that rotates a matrix clockwise by 90°, for example,

$$\operatorname{rot}\left(\begin{bmatrix}1 & 2\\ 3 & 4\end{bmatrix}\right) = \begin{bmatrix}3 & 1\\ 4 & 2\end{bmatrix} \text{ while text}\left(\begin{bmatrix}1 & 2 & 3\\ 4 & 5 & 6\end{bmatrix}\right) = \begin{bmatrix}4 & 1\\ 5 & 2\\ 6 & 3\end{bmatrix}$$

For $n \times n$ matrices A come up with and prove a simple formula for $\det(\operatorname{rot}(A))$ in terms of $\det(A)$.