Math 571 - Exam 2 (Due 11/30)

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There are 80 points here, so basically, 10 extra points. I'll take the score and use the minimum of 70 and your score as the final grade. Make your answers self-contained. If something here comes straight out of the homework, then do not "quote" the homework result as a reason. I am looking for the argument.

Question 1 (20 pts)**.** Give a reason for the non-existence of each of the following, or else provide an example. You may use theorems, but you must state the theorem, not just a reference to some theorem number.

- A continuous function $f : \mathbb{R} \to \mathbb{Q}$ that is not a constant function.

  If $f(\mathbb{R}) = E \subset \mathbb{Q}$ is not a singleton, then it is not connected and this is a contradiction.

- A continuous function $f : \mathbb{Q} \to \mathbb{Z}$ that is not a constant function.

  It is simple enough to choose such a function whose range is $\{0, 1\}$, for example, $f(r) = 0$ if $r < \sqrt{2}$ and $f(r) = 1$ if $r > \sqrt{2}$.

  For fun, let's find an onto example. Take $(s_i)_{i\in\mathbb{Z}}$ to be irrational numbers such that $i < j \implies s_i < s_j$. Then simply map (s_i, s_{i+1}) to $i \in \mathbb{Z}$. For each i, $f^{-1}(i) = (s_i, s_{i+1}) \cap \mathbb{Q}$ is *clopen* (closed and open) and $f^{-1}(i) \cap f^{-1}(j) = \emptyset$. Let $A \subset Z$ (every subset of $\mathbb{Z}$ is open), then $f^{-1}(A) = \bigcup_{i\in A} f^{-1}(i)$. This is a union of open sets hence open.

- A function $f : \mathbb{Z} \to \mathbb{R}$ that fails to be continuous.

  Every function from $\mathbb{Z}$ to $\mathbb{R}$ is continuous, since every subset of $\mathbb{Z}$ is open and hence $f^{-1}(O)$ is open for every open $O \subset \mathbb{R}$.

- A continuous **onto** function $f : S_1 \to \mathbb{R}$, where $S_1 = \{(x, y) \mid x^2 + y^2 = 1\}$ is the unit circle.

  S_1 is compact and $\mathbb{R}$ is not so $f(S_1) = \mathbb{R}$ is not possible. In fact, S_1 is compact and connected, hence $f(S_1)$ is compact and connected and thus must be a finite closed interval $[a, b]$ for some real numbers a, b with $a < b$.

Question 2 (15 pts)**.** Let $E \subset \mathbb{R}$ be bounded and $f : E \to \mathbb{R}$ be uniformly continuous. Show that $f(E)$ is also bounded. Give an example to show that continuous is not enough.

You may use the obvious, but perhaps painful to prove fact, that if $f(E)$ is **covered** by a finite collection of bounded sets, then $f(E)$ is itself bounded.

Proof 1: For $\epsilon > 0$ there is $\delta > 0$ such that for all $x, x' \in E$, $|x-x'| < \delta \implies |f(x)-f(x')| < \epsilon$. Since E is bounded, $\mathrm{Cl}(E)$ is also bounded, for example, $E \subseteq N_M(0) = (-M, M)$ for some

$M > 0$ and so $\mathrm{Cl}(E) \subseteq [-M, M]$. Since $\mathrm{Cl}(E)$ is closed and bounded, and as we are working in $\mathbb{R}$, we know $\mathrm{Cl}(E)$ is compact. Let $\mathcal{O} = \{N_\delta(x) \mid x \in E\}$. Since for any $x \in \mathrm{Cl}(E)$, there is $y \in E$ so that $x \in N_\delta(y)$ we have that $\mathcal{O}$ is a cover of $\mathrm{Cl}(E)$. By the compactness of $\mathrm{Cl}(E)$ we can find a finite subset of $\mathcal{O}$, say $\mathcal{O}' = \{N_\delta(x_1), \ldots, N_\delta(x_n)\}$, that covers $\mathrm{Cl}(E)$. Letting $B_i = N_\delta(x_i) \cap E$ the B_i's form a finite cover of E. Note that if $x, y \in B_i$, then $|x - x_i|, |y - x_i| < \delta$ and so $|f(x) - f(x_i)|, |f(y) - f(x_i)| < \epsilon$. By the triangle inequality, $|x - y| < 2\epsilon$ and thus the sets $f(B_i) \subseteq N_{2\epsilon}(x_i)$. So $\{f(B_1), \ldots, f(B_n)\}$ is a finite cover of $f(E)$ by bounded sets. Hence, $f(E)$ is bounded as it is covered by finitely many bounded sets.

Proof 2: Suppose $f(E)$ is unbounded and choose $x_i \in E$ so that $|f(x_i)| \to \infty$, say $|f(x_i)| > i$ for all i. The sequence (x_i) is bounded, being a subset of the bounded set E, and thus by the Bolzano-Weierstrass Theorem there is a subsequence that converges to a limit, this limit might be in $\mathrm{Cl}(E) - E$, but that would not affect the argument. We can just replace x_i by this subsequence so we may assume that $x_i \to x$. This means that x_i is a Cauchy sequence. But we know that the ***uniformly continuous*** image of a Cauchy-sequence is itself Cauchy. Thus $(f(x_i))$ is Cauchy, but all Cauchy sequences are bounded, so we have a contradiction. Thus the assumption that $f(E)$ is unbounded is false.

An example: To see that continuity is not enough, let $E = (0, 1)$ and $f(x) = 1/x$. Clearly, $f(E) = (1, \infty)$ and f is continuous on E.

Question 3 (15 pts)**.** (1) Let $f : \mathbb{R} \to \mathbb{R}$ be differentiable, suppose f' is bounded. Show that f is uniformly continuous.

(2) Find a uniformly continuous and differentiable function $f : \mathbb{R} \to \mathbb{R}$ whose derivative is not bounded.

Hint: The function should "wiggle" faster and faster as $|x| \to \infty$ and $f(x) \to 0$ as $|x| \to \infty$. You may use the fact that if f is continuous and both $\lim_{x\to\infty} f(x)$ and $\lim_{x\to-\infty} f(x)$ exist, then f is uniformly continuous. For some "extra bonus" you may prove this fact.

Let $M > 0$ be a bound so $|f'(x)| < M$ for all $x \in \mathbb{R}$. By MVT for any $a < b$ we have $f(b) - f(a) = f'(t)(b - a)$ for some $t \in (a, b)$. So $|f(b) - f(a)| < M|b - a|$ and thus for $\epsilon > 0$, let $\delta = \frac{\epsilon}{M}$, then for x, x' with $|x - x'| < \delta$, we have $|f(x) - f(x')| < M|x - x'| < M\delta = \epsilon$.

Example 1: A simple example is:

$$f(x) = \begin{cases} \frac{\sin(x^3)}{x} & x \neq 0 \\ 0 & x = 0 \end{cases}$$

Clearly, this function is continuous given that $\lim_{h\to 0} \frac{\sin(h^3)}{h} = 0$. (Use L'Hospital). Since $\lim_{x\to\infty} f(x)$ and $\lim_{x\to-\infty} f(x)$ both exist, f is is uniformly continuous.

It is also clear that $f(x)$ is differentiable for $x \neq 0$. For $x = 0$ we need to compute:

$$\lim_{h\to 0} \frac{f(h) - f(0)}{h} = \lim_{h\to 0} \frac{\sin(h^2)}{h} \overset{\text{l'H}}{=} \lim_{h\to 0} \frac{3h^2 \cos(h^3)}{1} = 0$$

For $x \neq 0$ we have $f'(x) = \frac{\cos(x^3)(3x^2)}{x} - \frac{\sin(x^3)}{x^2} = 3x\cos(x^3) - \frac{\sin(x^3)}{x^2}$. As $x \to \infty$. it is clear that $f'(x)$ is unbounded. Here is a plot: LINK.

Example 2: Another, related, example is

$$g(x) = \begin{cases} x\sin(1/x^3) & x \neq 0 \\ 0 & x = 0 \end{cases}$$

This function is clearly continuous has limits at $\pm\infty$, hence uniformly continuous. The derivative is clearly not bounded. Here is a plot: LINK

Note the two examples are related by $g(x) = f(1/x)$

Note: You must do something to show that your example is uniformly continuous. For example $\sin(x^2)$ is not uniformly continuous. Here is a "proof by picture". Note how the intervals in the shaded area grow smaller. The following is useful for showing a function is uniformly continuous.

Fact: Suppose $f : \mathbb{R} \to \mathbb{R}$ is continuous and $\lim_{x\to\infty} f(x) = L$ and $\lim_{x\to-\infty} f(x) = M$, then f is uniformly continuous. We want to see that f is uniformly continuous. Fix $\epsilon > 0$. There is N so that for all $x > N$, $|f(x) - L| < \epsilon/2$ and $|f(-x) - M| < \epsilon/2$. Notice from this it is clear that for $x > x' > N$ or $x < x' < -N$ we have $|f(x) - f(x')| < \epsilon$. Now f is uniformly continuous on $[-N-1, N+1]$ and so there is $1/2 > \delta > 0$ so that $|x - x'| < \delta \implies |f(x) - f(x')| < \epsilon$. It follows that $|x - x'| < \delta$ either puts x and x' in $(-\infty, N)$, $(-N-1/2, N+1/2)$, or (N, ∞) and thus $|x - x'| < \delta \implies |f(x) - f(x')| < \epsilon$ for any x and x'.

Question 4 (15 pts)**.** Let X and Y be metric spaces with X **compact** and let $f : X \to Y$ and $g : X \to Z$ be two functions, with no additional assumptions on these functions.

Suppose that for every $x \in X$, at least one of f or g is continuous at x. Show that for every $\epsilon > 0$, there is a $\delta > 0$ such that for all $x, x' \in X$:

$$d_X(x, x') < \delta \implies d_Y(f(x), f(x')) < \epsilon \text{ or } d_Z(g(x), g(x')) < \epsilon$$

This is sort of an "either/or" version of uniform continuity.

Proof 1: Fix $\epsilon > 0$. For each $x \in X$, fix $\delta_x > 0$, so that either

$$d_X(x, x') < \delta_x \implies d_Y(f(x), f(x')) < \epsilon$$

or

$$d_X(x, x') < \delta_x \implies d_Z(g(x), g(x')) < \epsilon$$

In the first case, say x is f-good, and in the second, say x is g-good. Notice that x can be g-good and f-good simultaneously.

Let $\mathscr{O} = \{N_{\delta_x/2}(x) \mid x \in X\}$ is an open cover of X. Let $\{N_{\delta_{x_i}/2}(x_i) \mid i = 1, 2, \ldots, n\}$ be a finite subcover. Let $\delta = \min\{\delta_i \mid i = 1, \ldots, n\}/2$. Then for $x, x' \in X$, if $d_X(x, x') < \delta$, we have $x \in N_{\delta_{x_i}/2}(x_i)$ for some i and $d_X(x, x') < \delta_{x_i}/2$ so $d_X(x', x_i) < d_X(x', x) + d_X(x, x_i) < \delta_{x_i}$. If x_i is f-good, then we have $d_Y(f(x), f(x')) < \epsilon$, else $d_Z(g(x), g(x')) < \epsilon$. This is what we wanted to prove.

Proof 2: Suppose for some $\varepsilon > 0$ for all $\delta > 0$ there is x, x' with $d_X(x, x') < \delta$ and $d_Y(f(x), f(x')) \geq \varepsilon$ and $d_Z(g(x), g(x')) \geq \varepsilon$. Choose sequences x_i and x_i' so that $d_X(x_i, x_i') < \delta_i$ where $\delta_i \to 0$ and so that $d_Y(f(x_i), f(x_i')) \geq \varepsilon$ and $d_Z(g(x_i), g(x_i')) \geq \varepsilon$. As X is compact, there is a convergent subsequence x_{i_k} of x_i. Let $x_{i_k} \to x$, clearly as $d(x'_{i_k}, x_{i_k}) \to 0$ we also have $x'_{i_k} \to x$.

By assumption, either f or g is continuous at x. If f is continuous at x, then $f(x_{i_k}) \to f(x)$ and $f(x'_{i_k}) \to f(x)$. But then $d_Y(f(x_{i_k}), f(x'_{i_k})) \leq d_Y(f(x), f(x_{i_k}) + d_Y(f(x), f(x_{i_k}))$ can be made arbitrarily small contradicting our assumption.

Question 5 (15 pts)**.** Let f and α be monotonically increasing bounded functions on $[a, b]$. Suppose that α is continuous at every point where f is discontinuous. Show that $f \in \mathscr{R}(\alpha)$.

You may use Theorem 4.30 in your text, and you will need to use the result of Question 4. The proof of Theorems 6.8 - 6.10 in the text should give a clue on how to proceed, but of course, the argument here is not exactly the same as any one of these alone.

Let $\epsilon > 0$. By the Question 4, there is $\delta > 0$ so that:

$$|x - x'| < \delta \implies |f(x) - f(x')| < \epsilon \text{ or } |\alpha(x) - \alpha(x')| < \epsilon$$

Let P be a partition $a = x_0 < x_1 < \cdots < x_n = b$ with $x_i - x_{i-1} < \delta$ for $i = 1, 2, \ldots, x_n$, then

$$\begin{aligned}
U(P, \alpha, f) - L(P, \alpha, f) &= \sum_{i=1}^{n} (M_i^{P,f} - m_i^{P,f})(\alpha(x_i) - \alpha(x_{i-1})) \\
&= \sum_{i=1}^{n} (f(x_i) - f(x_{i-1}))(\alpha(x_i) - \alpha(x_{i-1})) \\
&\leq \sum_{i \in A} \epsilon(\alpha(x_i) - \alpha(x_{i-1})) + \sum_{i \in B} (f(x_i) - f(x_{i-1}))\epsilon \\
&\leq \sum_{i=1}^{n} \epsilon(\alpha(x_i) - \alpha(x_{i-1})) + \sum_{i=1}^{n} (f(x_i) - f(x_{i-1}))\epsilon \\
&= \epsilon(\alpha(b) - \alpha(a)) + (f(b) - f(a))\epsilon \qquad \text{(monotonicity used)} \\
&= \epsilon((f(b) - f(a)) + (\alpha(b) - \alpha(a)))
\end{aligned}$$

where A is the set of i so that $|f(x_i) - f(x_{i-1})| < \epsilon$ and B is the rest, so for $i \in B$ we have $|\alpha(x_i) - \alpha(x_{i-1})| < \epsilon$ By replacing the original ϵ by $\frac{\epsilon}{(f(b)-f(a))+(\alpha(b)-\alpha(a))}$, we have what we want.