Math 571 - Homework 7

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Problem 7.1 (R:5:26)**.** Suppose $f(x)$ is differentiable on $[a,b]$, $f(a)=0$, and there is a fixed A such that $|f'(x)| \leq A|f(x)|$ for all x in $[a,b]$. Show that $f(x)=0$ on $[a,b]$.

Let $d = \sup\{x \in [a,b] \mid f|_{[a,x]} = 0\}$, by continuity it is clear that $f|_{[a,d]} = 0$. If $d = b$, we are done. If $d < b$ take $0 < \delta$ so that $\delta A < 1$ and $d + \delta \leq b$. Take $e \in (d, d+\delta]$. By MVT we have $\frac{f(e)-f(d)}{e-d} = f'(\hat{d})$ so that $|f(e)| = |f'(\hat{d})||e-d| \leq |f'(\hat{d})|\delta$.

Let $M = \sup(f[d, d+\delta])$ and $M' = \sup(f'[d, d+\delta])$ we know $M' \leq AM$ by assumption. On the other hand, we have just shown that $M \leq M'\delta$, so that $M \leq M'\delta < A\delta M < M$. A contradiction!

Problem 7.2 (R:5:27)**.** Let $\phi : [a,b] \times [\alpha, \beta] \to \mathbb{R}$. A *solution to the initial–value problem* (IVP)

$$y' = \phi(x,y), \quad y(a) = c \text{ for } a \leq c \leq b$$

is a function $f : [a,b] \to [\alpha, \beta]$ satisfying

$$f(a) = c, \quad f'(x) = \phi(x, f(x)) \text{ for all } a \leq x \leq b$$

Show that if there is a constant $A \geq 0$ so that

$$|\phi(x, y_1) - \phi(x, y_2)| \leq A|y_1 - y_2| \text{ for all } x \in [a,b] \text{ and } y_1, y_2 \in [\alpha, \beta],$$

then there is at most one solution to any such IVP.

Suppose f_1 and f_2 are two such solutions, then note that by assumption

$$|f_1'(x) - f_2'(x)| = |\phi(x, f_1(x)) - \phi(x, f_2(x))| \leq A|f_1(x) - f_2(x)| \text{ for } x \in [a,b].$$

Letting $h(x) = f_1(x) - f_2(x)$ we have $h(a) = 0$, h is differentiable on $[a,b]$, and $|h'(x)| \leq A|h(x)|$ for $x \in [a,b]$. Thus by Problem 1, $h = 0$ on $[a,b]$ and thus $f_1 = f_2$.

The book points out an example $y(0) = 0$ and $y' = y^{1/2}$ on $[0,1]$. Note that this fails the hypotheses since there is no $A \geq 0$ with $|\sqrt{y}| < A|y|$ on $[0,1]$, in particular, $\lim_{y \to 0^+} \frac{\sqrt{y}}{y} = \infty$.

The book gives two solutions $y = 0$ and $y = x^2/4$. To find all solutions note

$$\begin{aligned}
\frac{y'}{y^{1/2}} &= 1 \\
y^{-1/2}\frac{dy}{dx} &= 1 \\
y^{-1/2}dy &= dx \\
\int y^{-1/2}\,dy &= \int dx \\
\frac{y^{1/2}}{1/2} + d &= x + c && (d \text{ and } c \text{ arbitrary constants}) \\
y^{1/2} &= \frac{x}{2} + C && (C \text{ an arbitrary constant}) \\
y &= \frac{x^2}{4} + 2Cx + C^2
\end{aligned}$$

If $y(0) = 0$, then $C^2 = 0$, so $C = 0$, and thus the two solutions are all.

Problem 7.3. Show that the following are equivalent for a bounded function f on $[a, b]$:

i) $f \in \mathcal{R}$, i.e., f is Riemann integrable,

ii) For all $\epsilon > 0$, there is a $\delta > 0$ such that

$$\|P\| < \delta \implies U(f, P) - L(f, P) < \epsilon$$

First, show (i) implies (ii). Let $f \in \mathcal{R}$ and $\epsilon > 0$. We have a partition P so that $U(f, P) - L(f, P) < \epsilon/2$. Take $\delta > 0$ so that $\Delta x_i > 2\delta$ for all i and so that $\delta < \frac{\epsilon}{12MN}$ where $M = \sup\{|f(x)| \mid x \in [a, b]\}$ and $N = |P|$.

Let P' be a partition with $\|P'\| < \delta$ and let $P'' = P \cup P'$, then $L(P') \le L(P'') \le U(P'') \le U(P')$ and $L(P) \le L(P'') \le U(P'') \le U(P)$. So $U(P'') - L(P'') \le U(P) - L(P) < \epsilon/2$. We want to show that $U(P') - U(P'') < \epsilon/4$ and $L(P'') - L(P') < \epsilon/4$, then

$$\begin{aligned}
U(P') - L(P') &= \big(U(P'') + (U(P') - U(P''))\big) - \big(L(P'') - (L(P'') - L(P'))\big) \\
&< (U(P'') + \epsilon/4) - (L(P'') - \epsilon/4) = (U(P'') - L(P'')) + \epsilon/2 \\
&< \epsilon/2 + \epsilon/2 = \epsilon
\end{aligned}$$

All that needs to be proved here is

$$U(P') - U(P'') < \epsilon/4, \qquad L(P'') - L(P') < \epsilon/4$$

Let $P' = a = y_0 < y_1 < \cdots < y_m = b$ and $P = a = x_0 < x_1 < \cdots < x_N = b$. For each $i = 1, 2, \dots, N-1$ there is y_{k_i} so that $x_i \in [y_{k_i-1}, y_{k_i}]$. If $x_i \in \{y_{k_i-1}, y_{k_i}\}$, then adding x_i to P' adds nothing new, so in the worst case $x_i \in (y_{k_i-1}, y_{k_i})$. Let us assume this always occurs (since this is the worst case). In this case, we have

$$\begin{aligned}
U(P') - U(P'') &= \sum_{i=1}^{N-1} \sup(f([y_{k_i-1}, y_{k_i}])(y_{k_i} - y_{k_i-1}) \\
&\qquad - \big(\sup(f([y_{k_i-1}, x_i])(x_i - y_{k_i-1}) + \sup(f([x_i, y_{k_i}])(y_{k_i} - x_i)\big) \\
&\le \sum_{i=1}^{N-1} 3M\|P'\| = 3(N-1)M\|P\| < \epsilon/4
\end{aligned}$$

The other direction (ii) implies (i) is trivial since all that is required for $f \in \mathcal{R}$ is that for all $\epsilon > 0$, there is P so that $U(f,P) - L(f,P) < \epsilon$.

Problem 7.4 (R:6:1)**.** Suppose $\alpha : [a,b] \to \mathbb{R}$ is monotonic increasing and continuous at $x_0 \in [a,b]$. consider $f : [a,b] \to \{0,1\}$ given by $f(x_0) = 1$ and $f(x) = 0$ for $x \neq x_0$. Show that $f \in \mathcal{R}(\alpha)$ and $\int_a^b f\, d\alpha = 0$.

Pick $\epsilon > 0$. Since α is continuous at x_0 take δ so that $\alpha(N_\delta(x_0)) \subseteq N_{\epsilon/2}(\alpha(x_0))$. Let $P = y_0 = a < y_1 < y_2 < y_3 = b$ where $[y_1, y_2] \subset (x_0 - \delta, x_0 + \delta)$, so that $\Delta\alpha_2 = \alpha(y_2) - \alpha(y_1) < \epsilon$. Then $M_i^{f,P} = m_i^{f,P}$ for $i \neq 2$ and $M_2^{f,P} = \sup(f([y_1, y_2])) = 1$ while $m_i^{f,P} = \inf(f([y_1, y_2])) = 0$ so that

$$U(f,P) - L(F,P) = (1-0)\Delta\alpha_2 < \epsilon$$

Problem 7.5 (R:6:2)**.** Suppose $f : [a,b] \to \mathbb{R}$ is continuous, $f \geq 0$, and $\int_a^b f\, dx = 0$, then $f = 0$.

Note that where Rudin asks you to compare with (R:6:1). You might think that these are not comparable since (R:6:1) is about $\mathcal{R}(\alpha)$ while (R:6:2) is about $\mathcal{R}$, but taking $\alpha = \mathrm{id}$ in (R:6:1) allows you to make the comparison.

This is really almost trivial. If $f \neq 0$, then $f(x) > 0$ for some $x \in [a,b]$, but then $f(x) > \delta > 0$ and so there is an open nbhd of x, $N_\delta(x) = (x-\delta, x+\delta)$ so that $f((x-\delta, x+\delta) \cap [a,b]) \subset (\delta, \infty)$. Say $(c,d) \subseteq (x-\delta, x+\delta) \cap [a,b]$, then clearly $\int_a^b f\, dx \geq \delta(d-c) > 0$.

The difference between the examples from (R:6:1) and (R:6:2) is that in the former, the function is not continuous. In fact $\int_a^b f\, dx = 0$ whenever $\{x \in [a,b] \mid f(x) \neq 0\}$ has ***measure 0***. A set Z has measure 0 whenever

$$0 = \inf\left\{ \sum_{i=1}^{\infty} (b_i - a_i) \;\middle|\; Z \subseteq \bigcup_{i=1}^{\infty} (a_i, b_i) \right\}$$

Problem 7.6 (R:6:3)**.** Define $\beta_i : [-1,1] \to [0,1]$ by $\beta_i = 0$ for $x < 0$ and $\beta_i = 1$ for $x > 0$, then $\beta_1(0) = 0$, $\beta_2(0) = 1$, and $\beta_3(0) = 1/2$. In particular β_i has a simple discontinuity at 0 with $\beta_1(0-) = \beta_1(0) = 0$ (continuous from the left), $\beta_2(0+) = \beta_2(0) = 1$ (continuous from the right), while β_3 is neither continuous from the left or right. Let $f : [-1,1] \to \mathbb{R}$ be bounded. show that

i) $f \in \mathcal{R}(\beta_1)$ iff $f(0+) = f(0)$, that is, f is continuous from the right at 0.

ii) $f \in \mathcal{R}(\beta_2)$ iff $f(0-) = f(0)$, that is, f is continuous from the left at 0.

iii) $f \in \mathcal{R}(\beta_3)$ iff f is continuous at 0.

These are all very similar. It suffices to consider partitions that include 0 so that

$$P : -1 = x_0 < x_1 < \ldots < x_k = 0 < \cdots < x_n = 1$$

where $x_k = 0$. For β_i we have

$$(\Delta\beta_i)_k = \beta_i(0) - \beta_i(x_{k-1}) = \begin{cases} 0 & i = 1 \\ 1 & i = 2 \\ 1/2 & i = 3 \end{cases}$$

and

$$(\Delta\beta_i)_{k+1} = \beta_i(k+1) - \beta_i(0) = \begin{cases} 1 & i = 1 \\ 0 & i = 2 \\ 1/2 & i = 3 \end{cases}$$

All other $(\Delta\beta_i)_j = 0$ and thus we see

$$\begin{aligned} U(f,P) - L(f,P) &= (M_k - m_k)(\Delta\beta_i)_k + (M_{k+1} - m_{k+1})(\Delta\beta_i)_{k+1} \\ &= \begin{cases} M_{k+1} - m_{k+1} & i = 1 \\ M_k - m_k & i = 2 \\ \frac{1}{2}\big((M_k - m_k) + (M_{k+1} - m_{k+1})\big) & i = 3 \end{cases} \end{aligned}$$

Now $f \in \mathcal{R}(\beta_i)$ iff for all $\epsilon > 0$ there is a P so that

$$\begin{aligned} U(f,P) - L(f,P) < \epsilon \iff& \\ &\begin{cases} M_{k+1} - m_{k+1} < \epsilon & i = 1 \\ M_k - m_k < \epsilon & i = 2 \\ \frac{1}{2}\big((M_k - m_k) + (M_{k+1} - m_{k+1})\big) < \epsilon & i = 3 \end{cases} \end{aligned}$$

Take the $i = 1$ case, this says that for all $\epsilon > 0$ there is $x_{k+1} = h > 0$ so that $\sup(f([0,h]) - \inf(f([0,h])) < \epsilon$ which says exactly that $f(0+) = f(0)$. Similarly for $i = 2$ and $i = 3$.

Problem 7.7 (R:6:10)**.** See text. This is mostly done in the notes.

Homework 8

Problem 8.8 (R:6:6)**.** Let $f : [0,1] \to \mathbb{R}$ be bounded and continuous off of the Cantor set $\mathcal{C}$. Show that $f \in \mathcal{R}$.

Recall the construction of the Cantor set. $C_0 = [0,1]$ $C_1 = [0,1] - (1/3, 2/3)$ (removing middle third). $C_2 = C_1 - (1,9) - (7/9, 8/9)$, again remove middle thirds from what was left.

Notice the lengths of what is removed: $1/3$, $1/3 + 2/9$, $1/3 + 2/9 + 4/27$, etc. Consider

$$\sum_{i=0}^{\infty} \frac{2^i}{3^{i+1}} = \frac{1}{3}\sum_{i=0}^{\infty}\left(\frac{2}{3}\right)^i = \frac{1}{3}\left(\frac{1}{1-2/3}\right) = 1$$

We can cover C_i by 2^i many disjoint intervals of length $(1/3)^i + \epsilon$ for any ϵ. Since $\mathcal{C} = \bigcap C_i$ we see that $\mathcal{C}$ has measure 0 as defined above.

Suppose f is continuous off of a measure 0 set $Z \subset [a,b]$. Let $\mathcal{O}$ be an open cover of Z by intervals (a_i, b_i) so that $\sum_i (b_i - a_i) < \epsilon$. For each $x \notin Z$ take δ_x so that $f(N_{\delta_x}(x)) \subset N_{\epsilon/2}(f(x))$. As $[a,b]$ is compact we can find a finite subcover $\{(u_i, v_i) \mid i = 1, \ldots, n\}$ so that $u_1 < a < u_2 < v_1 < u_3 < v_2 < \cdots u_n < v_{n-1} < b < v_n$ where each (u_i, v_i) is from our cover of Z or else is one of the $N_{\delta_x}(x)$.

Use $x_0 = a$, $x_i = (u_{i+1} + v_i)/2$ for $i < n$, and $x_n = b$ as the partition: $P = a = x_0 < x_1 < \cdots < b = x_n$. Let M be a bound on $|f|$ on $[a,b]$. Let $T = \{i \mid (x_{i-1}, x_i) \subseteq (a_j, b_j) \text{ for some } j\}$. Then we have

$$\begin{aligned}
U(f,P) - L(f,P) &= \sum_{i\in T}(M_i - m_i)\Delta x_i + \sum_{i\notin T}(M_i - m_i)\Delta x_i \\
&< \sum_{i\in T} 2M\Delta x_i + \sum_{i\notin T}\epsilon\Delta x_i \\
&\le 2M\epsilon + \epsilon(b-a) = \epsilon(2M + (b-a))
\end{aligned}$$

In particular whenever $p_i \ge 0$ and $\sum_{i=1}^n p_1 = 1$, then $\prod_{i=1}^n a_i^{p_i} \le \sum_{i=1}^n p_i a_i$. In particular, if $\frac{1}{p} + \frac{1}{q} = 1$, then

$$uv = (u^p)^{1/p}(v^q)^{1/q} \le \frac{u^p}{p} + \frac{v^q}{q}$$

This basically completes (a). For (b) notice

$$fg \le \frac{f^p}{p} + \frac{g^q}{q}$$

so

$$\int_a^b fg\, d\alpha \le \int_a^b \frac{f^p}{p}\, d\alpha + \int_a^b \frac{g^q}{q}\, d\alpha = \frac{1}{p} + \frac{1}{q} = 1$$

For (c) the proof is exactly as for Hölder's inequality in the notes already mentioned above. Define $\|f\|_p = \left(\int_a^b |f|^p\, d\alpha\right)^{1/p}$ provided that $|f|^p \in \mathcal{R}(\alpha)$. Let $L^p(\alpha)$ be all those bounded $f[a,b] \to \mathbb{R}$ with $\|f\|_p < \infty$. The spaces of function $L^p(\alpha)$ are normed vector spaces with norm $\|\cdot\|_p$. We want to see if $f \in L^p(\alpha)$ and $g \in L^q(\alpha)$ with $\frac{1}{p}+\frac{1}{q}=1$ we have $fg \in L^1(\alpha)$ and

$$\|fg\|_1 \leq \|f\|_p\|g\|_q \tag{$\dagger$}$$

We can replace f with $\hat{f} = \frac{f}{\|f\|_p}$ and g with $\hat{g} = \frac{g}{\|g\|_q}$, then we have $\|\hat{f}\|_p = 1 = \|\hat{q}\|_q$ and from above

$$\|\hat{f}\hat{g}\|_1 \leq 1 = \frac{\|\hat{f}\|_p^p}{p} + \frac{\|\hat{g}\|_q^q}{q}$$

But from this we have

$$\left\|\frac{f}{\|f\|_p}\frac{g}{\|g\|_q}\right\| = \frac{1}{\|f\|_p}\frac{1}{\|g\|_q}\|fg\|_1 \leq 1$$

From this ($\dagger$) follows immediately.

Problem 8.9 (Functions with only countable many discontinuities are integrable.)**.** Let f be bounded on $[a,b]$ with at most countable many discontinuities on $[a,b]$. Let $\alpha : [a,b] \to \mathbb{R}$ is monotonic increasing and α is continuous at every discontinuity of f. Show that $f \in \mathcal{R}(\alpha)$.

Hint: Fix an enumeration $S = \{s_i \mid i \in \mathbb{N}\}$ of the discontinuities of f. Fix $\epsilon > 0$ and $\epsilon_i > 0$ so that $\sum_i \epsilon_i \leq \epsilon$. Since α is continuous at s_i fix δ_i so that $\alpha(N_{\delta_i}(s_i)) \subset N_{\epsilon_i}(\alpha(s_i))$. For $x \notin S$, fix δ_x so that $f(N_{\delta_x}(x)) \subset N_\epsilon(f(x))$. Now $\mathcal{O} = \{N_{\delta_i}(s_i) \mid i \in \mathbb{N}\} \cup \{N_{\delta_x}(x) \mid x \notin S\}$ is an open cover of $[a,b]$. Apply compactness to get a finite subcover and then do something ***similar*** (not the same) as in the proof of 6.10.

Proof 1: Fix $\epsilon > 0$ and $\epsilon_i > 0$ so that $\sum_i \epsilon_i < \epsilon$. Let $S = \{s_i \mid i \in \mathbb{N}\}$ be the discontinuities of f. Since α is continuous at s_i let δ_i be so that $\alpha(N_{\delta_i}(s_i)) \subset N_{\epsilon_i}(\alpha(s_i))$. For $x \notin S$ let δ_x be chosen so that $f(N_{\delta_x})(x) \subseteq N_\epsilon(f(x))$. Let $\mathcal{O} = \{N_{\delta_i}(s_i) \mid i \in \mathbb{N}\} \cup \{N_{\delta_x}(x) \mid x \notin S\}$ be the associated open cover of $[a,b]$. Let $\mathcal{O}' \subseteq$ be a finite subcover of $[a,b]$. Novice that $\mathcal{O}'$ consists of intervals (u_i, v_i) and we may assume that $a = u_0 < u_1 < v_0 < u_2 < v_1 < u_3 < v_2 \cdots$ (a "chain"). Thus we define $x_0 = a < x_1 = (u_1+v_0)/2 < x_2 = (u_2+v_1)/2 < x_{n-1} = (u_{n-1}+v_{n-2})/2 < x_n = v_n = b$. Thus $[x_{i-1}, x_i] \subset N_{\delta_j}(s_j)$ for some j or $[x_{i-1}, x_i] \subset N_{\delta_x}(x)$ for some $x \notin S$.

Let $T = \{i \mid [x_{i-1}, x_i] \subset N_{\delta_i}(s_i) \text{ for some } s_i \in S\}$. Then letting $f|(x)| \leq M$ and $\alpha(b) - \alpha(a) = N$:

$$\begin{aligned}\sum_{i=1}^n |M_i - m_i|(\alpha(x_i) - \alpha(x_{i-1}) &= \sum_{i\in T} |M_i - m_i|(\alpha(x_i) - \alpha(x_{i-1}) + \sum_{i \notin T} |M_i - m_i|(\alpha(x_i) - \alpha(x_{i-1}) \\ &< \sum_{i \in T} 2M\epsilon_i + \sum_{i \notin T} \epsilon\alpha(x_i) - \alpha(x_{i-1}) \\ &\leq 2M\epsilon + N\epsilon = \epsilon(2M+N)\end{aligned}$$

Proof 2: (The following seems to be an option that I see commonly, but not carried out correctly. I thought I would write it out correctly here.)

Start like the above. Since α is continuous at s_i pick (a_i, b_i) satisfying:

- $(a_i, b_i) \cap (a_j, b_j) = \emptyset$. (mutually disjoint)
- $s_i \in (a_i, b_i)$.

- $\alpha((a_i,b_i)) \subseteq S_{\epsilon/2}(s_i)$ so that if $t, t' \in (a_i, b_i)$, then $|\alpha(t') - \alpha(t)| < \epsilon_i$. Where $\sum_i \epsilon_i = \epsilon$ and ϵ_i will be chosen at the end.

Let $K = [a,b] - \bigcup_i (b_i, a_i)$. K is closed and bounded, hence compact. Since f is continuous on K it is uniformly continuous and thus we can pick $\delta > 0$ so that $|x-y| < \delta \implies |f(x) - f(y)| < \epsilon$ for all $x, y \in K$.

$\mathcal{O} = \{(x - \delta/2, x + \delta/2) \mid x \in K\} \cup \{(a_i, b_i) \mid i \in \mathbb{N}\}$ is an open cover of $[a,b]$ and hence has a finite subcover $\mathcal{O}'$. Let $\mathcal{O}' = \{(u_i, v_i) \mid i < m\}$ we may assume that for no $i \neq j$ do we have $(u_i, v_i) \subset (u_j, v_j)$, as we could just toss out (u_i, v_i) in this case. So $u_0 < a < u_1 < v_0 < u_2 < v_1 < \cdots < u_{m-1} < v_{m-2} < b < v_{m-1}$. For $i = 1, \dots, m-2$ let $y_i = (u_i + v_{i-1})/2$ and set $y_0 = a$ and $y_{m-1} = b$ and let $P = \{y_i \mid i = 0, \dots, m-1\}$. Then we know for each $i = 1, \dots, m-1$ that either $[u_{i-1}, u_i] \subset (a_j, b_j)$ for some j or else $[u_{i-1}, u_i] \subset (x - \delta/2, x + \delta/2)$ for some $x \in K$.

Let $A = \{i \mid [u_{i-1}, u_i] \subset (a_{j_i}, b_{j_i})$ for some $j_i\}$, then for $i \in A$ we have $\Delta\alpha_i = \alpha(u_i) - \alpha(u_{i-1}) < \epsilon_{j_i}$ and for $i \notin A$, $|M_i^{f,P} - m_i^{f,P}| < \epsilon$. Thus

$$\begin{aligned} U(f,P,\alpha) - L(f,P,\alpha) &= \sum_{i=1}^{m-1} |M_i^{f,P} - m_i^{f,P}| \Delta\alpha_i \\ &\leq \sum_{i \in A} |M_i^{f,P} - m_i^{f,P}| \epsilon_{j_i} + \sum_{i \notin A} \epsilon \Delta\alpha_i \leq 2M\epsilon + \epsilon(\alpha(b) - \alpha(a)) \end{aligned}$$

where $M = \sup |f(x)| x \in [a,b]$.

Since M and $\alpha(b) - \alpha(a)$ are fixed constants we can make the $\epsilon(2M + \alpha(b) - \alpha(a))$ arbitrarily small. Thus $f \in \mathcal{R}(\alpha)$.

Problem 8.10 (An integrable function with uncountable many discontinuities.)**.** Let $\mathcal{C}$ be the Cantor set and f be defined by

$$f(x) = \begin{cases} 1 & x \in \mathcal{C} \\ 0 & x \notin \mathcal{C} \end{cases}$$

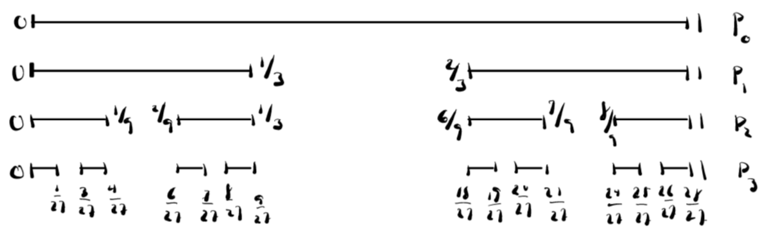
Show that $f \in \mathcal{R}$, namely, $\int_0^1 f\,dx = 0$. That f has uncountably many points of discontinuity is clear since each point of $\mathcal{C}$ is a discontinuity of f and $\mathcal{C}$ is perfect, hence uncountable.

Proof 1: The argument from Problem 7 works here. Basically, that argument showed that if $g = f$ off of a measure zero set, then $f \in \mathcal{R} \iff g \in \mathcal{R}$ and $\int_a^b f\,dx = \int_a^b g\,dx$. So here take $g = 0$ on $[0,1]$.

Proof 2: (From a student.) Notice that $L(f,P) = 0$ for any partition P of $[0,1]$. So we just need to show that $\inf_P U(f,P) = 0$.

Let P_i be the partition consisting of endpoints of the closed intervals that generate the Cantor set. So $P_0 = \{0, 1\}$, $P_1 = \{0, 1/3, 2/3, 3/3\}$, $P_2 = \{0, 1/9, 2/9, 3/9, 6/9, 7/9, 8/9, 9/9\}$, so that $P_0 \subset P_1 \subset P_2 \subset \cdots$. In P_n we have 2^{n+1} points $0 = x_0 < x_1 < \cdots < x_{2^{n+1}-1}$ and

$$\mathcal{C} \subset \bigcup_{i < 2^{n+1} \text{ even}} [x_i, x_{i+1}]$$



On $[x_i, x_{i+1}]$ for i even we have $M_i = 1$, but on $[x_i, x_{i+1}]$ for i odd we have $M_i = 0$ so

$$\begin{aligned} U(f, P_n) &= \sum_{i<2^{n+1}} (M_i - m_i) 3^{-n} \\ &= \sum_{i<2^{n+1} \text{ even}} 3^{-n} = \left(\frac{2}{3}\right)^n \end{aligned}$$

Thus we can choose n large enough to guarantee that $U(f, P_n) < \epsilon$ for any ϵ. Thus $U(f) = \inf_P U(f, P) = 0 = L(f)$ and so $f \in \mathcal{R}$ and $\int_0^1 f\, dx = 0$.