Name: __________________________ Exam 1 - MAT513

Warning! I can (eh ... do) make mistakes, if you think I have something wrong here, please ask.

Part I: True/False

Each problem is points for a total of 40 points. (10 problems 4 points each; 2 points for correct T/F; 2 points for correct, but brief, explanation.)

Problem 1. Decide if each of the following is true or false. For each, provide an example, counter-example, or argument as required. You may refer to a theorem if one applies.

a) False If $H < G$, the set of cosets $G/H = \{gH \mid g \in G\}$ form a group under the operation $aH + bH = abH$.

This is a basic fact that we have learned. For G/H to be a group as described, H must be normal. As an example, consider $H = \{e, f\}$ where f is a reflection in D_n and r a non-trivial rotation, then $rHrH = \{r, rf\}\{r, rf\} = \{r^2, r^2f, rfr, rfrf\} = \{e, f, r^2, r^2f\} \neq r^2H = \{r^2, r^2f\}$.

b) True $Z(G) = \bigcap_{g\in G} C(g)$.

This is just unpacking the definitions

$$\begin{aligned} x \in Z(G) &\iff \text{for all } g \in G, xg = gx \\ &\iff \text{for all } g \in G, x \in G(g) \\ &\iff x \in \bigcap_{g\in G} C(g) \end{aligned}$$

c) True For $c \in \mathbb{R}$ and $c \neq 0$, $\phi : \mathbb{R} \to \mathbb{R}$ given by $\phi(x) = cx$ is an automorphism of $(\mathbb{R}, +)$.

This is homomorphism since $\phi(a+b) = c(a+b) = ca+cb = \phi(a)+\phi(b)$ and $\phi(0) = 0$. $\ker(\phi) = \{0\}$, that is, $ca = 0 \iff a = 0$. So ϕ is 1-1. Since $\phi(a/c) = a$ we see ϕ is onto.

d) True Let $\phi : G \to H$ be a homomorphism,

$$\ker(\phi) = \{e_G\} \iff \phi \text{ is 1-1}$$

If $\phi(x) = \phi(y)$ and $\ker(\phi) = \{e_G\}$, then $e_H = \phi(x)\phi(y)^{-1} = \phi(x)\phi(y^{-1}) = \phi(xy^{-1})$ so $xy^{-1} = e_G$ and thus $x = y$, so ϕ is 1-1.

Conversely, suppose ϕ is 1-1 and $\phi(x) = e_H$, then since $\phi(e_G) = e_H$ we have $x = e_G$, so $\ker(\phi) = \{e\}$.

e) <u>False</u> S_9 has an element of order 11.

$|S_9| = 9!$ and $11 \nmid 9!$ so no element of order 11.

f) <u>False</u> There are finite groups that are not isomorphic to a subgroup of S_n for some n.

Cayley's theorem.

g) <u>False</u> There is a finite group of order n and a prime p such that $p \mid n$, but no element of G has order p.

Cauchy's Theorem.

h) <u>False</u> There is a group G and non-normal subgroup $H < G$ so that $|H| \nmid |G|$.

Langrange's Theorem.

i) <u>True</u> In S_8, $(135)(456)(567)$ is even.

$(135)(456)(567)$=$(1354)(67)$

j) <u>True</u> $[S_4 : D_4] = 3$.

$|S_4|/|D_4| = 4!/8 = 3$.

Part II: Short Answer

Each problem is 8 points for a total of 40 points. (5 problems, 8 points each)

Problem 2 (8 points)**.** Let ϕ be a homomorphism from G to H. What is the relationship between G, $\ker(\phi)$, $\mathrm{Img}(\phi) = \phi(G)$, and H. If G and H are finite, what is the relationship between $|G|$, $|\ker(\phi)|$, $\mathrm{Img}(\phi)$, and $|H|$?

$$G/\ker(\phi) \simeq \mathrm{Img}(\phi) < H$$

$$|G| = |\ker(\phi)||\mathrm{Img}(\phi)| \text{ and } |\mathrm{Img}(\phi)| \mid |H|$$

Problem 3 (8 points)**.** List the abelian groups of order 12 up to isomorphism.

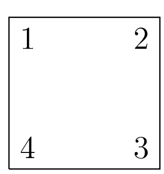
$$\mathbb{Z}_4 \times \mathbb{Z}_3 \simeq \mathbb{Z}_{12}, \text{ and } \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_3 \simeq \mathbb{Z}_2 \times \mathbb{Z}_6$$

Problem 4 (8 points)**.** What is $\mathrm{Aut}(\mathbb{Z}_{45})$ up to isomorphism, in terms of products of $\mathbb{Z}_n$'s. (Explain or show "computation.")

$$\mathrm{Aut}(\mathbb{Z}_{45}) = U(45) = U(3^3 \cdot 5) = U(3^3) \times U(5) = \mathbb{Z}_{3^2(2)} \times \mathbb{Z}_4$$

Problem 5 (8 points)**.** Show that D_4 is not a normal subgroup of S_4.

Use

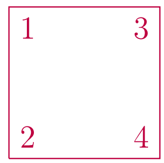


So that D_4 is generated by the rotation $R = (1234)$ and the horizontal reflection $H = (12)(34)$.

We just need to see that $\sigma D_8 \sigma^{-1} \neq D_8$ for some σ. Take $\sigma = (12)$ (or (123) or basically any $\sigma \in S_4 - D_8$, then

$$(12)(1234)(12) = (1342)$$

this would correspond to labeling like



and this cannot be achieved in D_8 since adjacent labels must stay adjacent.

Note This is a good example to keep in mind. We know that if $[G : H] = 2$, then H is normal in G. We might guess that if $[G : H]$ is prime, the same holds. This example shows that this is false. It is true that if p is the smallest prime such that $p \mid G$ and $[G : H] = p$, then H is normal.

Problem 6 (8 points)**.** What is the largest cyclic subgroup of $G = \mathbb{Z}_6 \times \mathbb{Z}_{20} \times \mathbb{Z}_{24} \times \mathbb{Z}_{45}$?

$6 = 2 \cdot 3$, $20 = 2^2 \cdot 5$, $24 = 2^3 \cdot 3$, and $45 = 3^3 \cdot 5$. So $\mathrm{lcm}(6, 20, 24, 45) = 2^3 3^3 5 = 360$. In fact $(1, 1, 1, 1)$ has this order and for any $g \in G$, $|g| = \mathrm{lcm}(|g_1|, |g_2|, |g_3|, |g_4|) \mid \mathrm{lcm}(6, 20, 24, 45)$ since $|g_1| \mid 6$, $|g_2| \mid 20$, $|g_3| \mid 24$, and $|g_4| \mid 45$.