Math 571 - Exam 1 (05.22)

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Problem 1 (R:5:26)**.** See text.

Problem 2 (R:5:27)**.** See text.

Problem 3. Show that the following are equivalent for a bounded function f on $[a, b]$:

i) $f \in \mathcal{R}$, i.e., f is Riemann integrable,

ii) For all $\epsilon > 0$, there is a $\delta > 0$ such that

$$\|P\| < \delta \implies U(f, P) - L(f, P) < \epsilon$$

Problem 4 (R:6:1)**.** See text.

Problem 5 (R:6:2)**.** See text. Note that where Rudin asks you to compare with (1) there might be the thought that these do not compare since (1) is about $\mathcal{R}(\alpha)$ while (2) is about $\mathcal{R}$, but taking $\alpha = \mathrm{id}$ in (1) allows you to make the comparison.

Problem 6 (R:6:3)**.** See text.

Problem 7 (R:6:6)**.** See text.

Problem 8 (Functions with only countable many discontinuities are integrable.)**.** Let f be bounded on $[a, b]$ with at most countable many discontinuities on $[a, b]$. Let $\alpha : [a, b] \to \mathbb{R}$ is monotonic increasing and α is continuous at every discontinuity of f. Show that $f \in \mathcal{R}(\alpha)$.

Hint: Fix an enumeration $S = \{s_i \mid i \in \mathbb{N}\}$ of the discontinuities of f. Fix $\epsilon > 0$ and $\epsilon_i > 0$ so that $\sum_i \epsilon_i \leq \epsilon$. Since α is continuous at s_i fix δ_i so that $\alpha(N_{\delta_i}(s_i)) \subset N_{\epsilon_i}(\alpha(s_i))$. For $x \notin S$, fix δ_x so that $f(N_{\delta_x}(x)) \subset N_\epsilon(f(x))$. Now $\mathcal{O} = \{N_{\delta_i}(s_i) \mid i \in \mathbb{N}\} \cup \{N_{\delta_x}(x) \mid x \notin S\}$ is an open cover of $[a, b]$. Apply compactness to get a finite subcover and then do something ***similar*** (not the same) as in the proof of 6.10.

Problem 9 (An integrable function with uncountable many discontinuities.)**.** Let $\mathcal{C}$ be the Cantor set and f be defined by

$$f(x) = \begin{cases} 1 & x \in \mathcal{C} \\ 0 & x \notin \mathcal{C} \end{cases}$$

Show that $f \in \mathcal{R}$, namely, $\int_0^1 f \, dx = 0$. That f has uncountably many points of discontinuity is clear since each point of $\mathcal{C}$ is a discontinuity of f and $\mathcal{C}$ is perfect, hence uncountable.