Exam 1

This exam covers Topics 1 - 3, Topic 4 will not be covered here.

Part I: True/False (5 points each; 25 points)

For each of the following mark as true or false.

a) _____ $\text{tr}(AB) = \text{tr}(BA)$ for an $n \times n$ matrices A and B, where $\text{tr}(C) \stackrel{\text{df}}{=} \sum_{i=1}^{n} C_{ii}$, the sum of the diagonal of C.

This is true. This is just a computation. $(AB)_{ii} = \sum_{k=1}^{n} A_{ik}B_{ki}$, so

$$\text{tr}(AB) = \sum_{i=1}^{n}\sum_{k=1}^{n} A_{ik}B_{ki}$$

and

$$\text{tr}(BA) = \sum_{i=1}^{n}\sum_{k=1}^{n} B_{ik}A_{ki} = \sum_{k=1}^{n} A_{ki}B_{ik} = \text{tr}(AB).$$

b) _____ $\text{tr}(ABC) = \text{tr}(BAC)$ for an $n \times n$ matrices A, B, and C.

Interestingly, this is false, as an example can show. In fact, generating any three random 2×2 matrices with entries from $\{-1, 0, 1\}$ are likely to work. Try this using MATLAB: round(2*rand(2)−1). The first three matrices I got this way were:

$$A = \begin{bmatrix} 0 & 1 \\ 1 & -1 \end{bmatrix} \qquad B = \begin{bmatrix} 0 & -1 \\ 0 & 0 \end{bmatrix} \qquad C = \begin{bmatrix} -1 & 0 \\ 1 & 0 \end{bmatrix}$$

So

$$ABC = \begin{bmatrix} 0 & 0 \\ -1 & 0 \end{bmatrix} \qquad BAC = \begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix}$$

So clearly, $\text{tr}(ABC) \neq \text{tr}(BAC)$.

What is true in general is

$$\text{tr}(A_1 A_2 \ldots A_k) = \text{tr}(A_2 A_3 \ldots A_k A_1)$$

That is, if you cycle the first factor to the end, then the trace is unaffected.

c) _____ If W is a subspace of a vector space V, then there is a subspace U so that $V = W \oplus U$.

This is true. Let $\mathcal{B}_W$ be a basis for W and extend $\mathcal{B}_W$ to $\mathcal{B}_V$ a basis for V. Then let $U = \operatorname{span}(\mathcal{B}_V - \cap B_W)$. It is clear that $V = W \oplus U$.

d) _____ If W is a subspace of a vector space V and $\mathcal{B}$ is a basis for V, then B can be restricted to a basis for W.

This is false. Let $W = \operatorname{span}\{(1,1)\} \subseteq \mathbb{R}^2 = V$. The standard basis for $\mathbb{R}^2$ can not be restricted to a basis for W.

e) _____ If $B = EA$ where E is invertible, then $\mathrm{NS}(A) = \mathrm{NS}(B)$.

This is true. Clearly, $A\boldsymbol{x} = \mathbf{0} \implies EA\boldsymbol{x} = B\boldsymbol{x} = \mathbf{0}$, so $\mathrm{NS}(A) \subseteq \mathrm{NS}(B)$. Conversely, $B\boldsymbol{x} = \mathbf{0} \implies EA\boldsymbol{x} = \mathbf{0} \implies A\boldsymbol{x} = E^{-1}\mathbf{0} = \mathbf{0}$. So $\mathrm{NS}(B) \subseteq \mathrm{NS}(A)$.

Part II: Definitions and Theorems (5 points each; 25 points)

a) Define what it means for a set of vectors $\mathcal{B} = \{\boldsymbol{v}_1, \ldots, \boldsymbol{v}_n\}$ from a real vector space V to span V.

$\{\boldsymbol{v}_1, \ldots, \boldsymbol{v}_n\}$ **spans** V iff for all $\boldsymbol{v} \in V$, $\boldsymbol{v}$ is a linear combination of the vectors in $\mathcal{B}$, that is $\boldsymbol{v} = \sum_{i=1}^{n} \alpha_i \boldsymbol{v}_i$ for some coefficients $\alpha_i \in \mathbb{R}$.

b) Define what it means for a set of vectors $\{\boldsymbol{v}_1, \ldots, \boldsymbol{v}_n\}$ from a real vector space V to be linearly independent.

A set of vectors $\mathcal{B}$ is **linearly independent** iff $\sum_{i=1}^{n} \alpha_i \boldsymbol{v}_i = \mathbf{0}$, then $\alpha_i = 0$ for all i. Equivalently, any linear combination of the vectors that gives $\mathbf{0}$ must be trivial.

There are equivalent definitions, but essentially, this is what you need. You can say something like, for all i, $\boldsymbol{v}_i \notin \operatorname{span}\{\boldsymbol{v}_1, \ldots, \boldsymbol{v}_{i-1}, \boldsymbol{v}_{i+1}, \ldots, \boldsymbol{v}_n\}$

c) Define what it means for a set of vectors $\mathcal{B} = \{\boldsymbol{v}_1, \ldots, \boldsymbol{v}_n\}$ to be a basis for a vector space V.

$\mathcal{B}$ has must be a linearly independent and span V.

d) State the Rank-Nullity Theorem.

If A is an $m \times n$ matrix, then $n = \dim(\mathrm{RS}(A)) + \dim(\mathrm{NS}(A)) = \mathrm{rank}(A) + \mathrm{nullity}(A)$.

e) What conditions must be checked to verify that $W \subseteq V$ is a subspace of a vector space. V

Closure under addition and scalar multiplication must be checked.

Part III: Computational (15 points each; 45 point)

a) Use row ops to find an echelon form of

$$A = \begin{bmatrix} 1 & 2 & 2 & -2 & 2 \\ 2 & 4 & 1 & -2 & 5 \\ 1 & 2 & -1 & 0 & 3 \end{bmatrix}$$

Make sure to write out your steps and indicate the row ops at each step.

$$A \xrightarrow[R_3 - R_1 \to R_3]{R_2 - 2R_1 \to R_2} \begin{bmatrix} 1 & 2 & 2 & -2 & 2 \\ 0 & 0 & -3 & 2 & 1 \\ 0 & 0 & -3 & 2 & 1 \end{bmatrix} \xrightarrow{R_3 - R_2 \to R_3} \begin{bmatrix} 1 & 2 & 2 & -2 & 2 \\ 0 & 0 & -3 & 2 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

b) Use the echelon matrix found above to find a basis for RS(A), NS(A), and CS(A). Give a brief reason for your choice.

Without a justification, you might just have a lucky guess and I will not accept this. Your justification can be short and use facts from the text or from the notes that I have provided.

A basis for RS(A) is given by $\{(1, 2, 2, -2, 2), (0, 0, -3, 2, 1)\}$.

Justification: Take the non-zero rows of the echelon form.

A basis for CS(A) is given by columns 1 and 3 of A, that is, $\{(1, 2, 1), (2, 1, -1)\}$

Justification: These correspond to the pivot columns and we know this is a basis.

For NS(A) we perform back substitution, letting $x_2 = r$, $x_4 = s$, and $x_5 = t$, so

$$-3x_3 = -2s - t$$

so

$$\begin{aligned} x_3 &= (2/3)s + (1/3)t \\ x_1 &= -2r - 2x_3 + 2s - 2t \\ &= -2r - 2((2/3)s + (1/3)t) + 2s - 2t \\ &= -2r + 2/3s - 8/3t \end{aligned}$$

So a typical element of NS(A) is

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} -2r + (2/3)s - (8/3)t \\ r \\ (2/3)s + (1/3)t \\ s \\ t \end{bmatrix} = r \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + s \begin{bmatrix} 2/3 \\ 0 \\ 2/3 \\ 1 \\ 0 \end{bmatrix} + t \begin{bmatrix} -8/3 \\ 0 \\ 1/3 \\ 0 \\ 1 \end{bmatrix}$$

A basis is

$$\{(-2, 1, 0, 0, 0), (2/3, 0, 2/3, 1, 0), (-8/3, 0, 1/3, 0, 1)\}$$

c) Show that the upper-triangular $n \times n$ matrices form a subspace of all $n \times n$ matrices and find a basis for this subspace.

Part IV: Proofs (15 points each; 60 points)

Provide complete arguments/proofs for the following.

a) **Prove:** Let A and B be square matrices with $AB = I$. Show that A is invertible.

You may refer to Theorem 1.5.2 or Theorem 2.2.2, but be clear and complete in your argument.

Proof of invertibility 1: Show that $\mathrm{NS}(B) = \{\mathbf{0}\}$ and hence B is invertible and from above $A = B^{-1}$, but then clearly A is invertible too.

Clearly, $\boldsymbol{x} \in \mathrm{NS}(B) \implies \boldsymbol{x} \in \mathrm{NS}(AB) = \mathrm{NS}(I) = \{\mathbf{0}\}$, so

$$\{\mathbf{0}\} \subseteq \mathrm{NS}(B) \subseteq \mathrm{NS}(AB) = \{\mathbf{0}\}$$

so $\mathrm{NS}(B) = \{\mathbf{0}\}$. So B is invertible and $AB = I$, so $A = B^{-1}$.

Proof of invertibility 2: $\det(AB) = \det(A)\det(B) = 1$, so $\det(A) \neq 0$, hence A is invertible.

b) **Prove:** Let A be an $m \times n$ matrix, $\mathbb{R}^n = \mathrm{NS}(A) \oplus \mathrm{RS}(A)$.

You can use the rank-nullity theorem and just argue that $\mathrm{NS}(A) \cap \mathrm{RS}(A) = \{\mathbf{0}\}$. You know then that $\dim(\mathrm{RS}(A)) + \dim(\mathrm{NS}(A)) = n$ so if you have $\mathcal{B}$ a basis for $\mathrm{RS}(A)$ and $\mathcal{C}$ a basis for $\mathrm{NS}(A)$, then $\mathcal{B} \cup \mathcal{C}$ has size n and is linearly independent, hence is a basis for $\mathbb{R}^n$. (There is a little bit that I am leaving to the reader here.)

c) **Prove:** If A and B are $m \times n$ matrices such that $A\boldsymbol{x} = B\boldsymbol{x}$ for all $\boldsymbol{x} \in \mathbb{R}^n$, then $A = B$.

The hypothesis is equivalent to $(A - B)\boldsymbol{x} = \mathbf{0}$ for all $\boldsymbol{x}$ and the conclusion is equivalent to $A - B = \mathbf{0}$.

It suffices to prove:

$$\text{If } A\boldsymbol{x} = \mathbf{0} \text{ for all } \boldsymbol{x}, \text{ then } A = \mathbf{0}.$$

This is simple, say $A = \begin{bmatrix} \boldsymbol{a}_1 & \boldsymbol{a}_2 & \cdots & \boldsymbol{a}_n \end{bmatrix}$ (the columns of A). Then $A\boldsymbol{e}_j = \boldsymbol{a}_j = \mathbf{0}$. But then $\boldsymbol{a}_j(i) = A_{ij} = 0$ for all $1 \leq i, j \leq n$. So $A = \mathbf{0}$ (the all 0 matrix).

d) **Prove:** If A is an $n \times n$ matrix and $A^k = \mathbf{0}$ for any k, then $A^n = \mathbf{0}$.

Proof 1: To do this show

i) Show $\mathrm{NS}(A^{m+1}) \supseteq \mathrm{NS}(A^m)$ for all m.

ii) Show that if $\mathrm{NS}(A^{m+1}) = \mathrm{NS}(A^m)$, then $\mathrm{NS}(A^n) = \mathrm{NS}(A^m)$ for all $n \geq m$.

It is clear that $\text{NS}(A^{m+1}) \supseteq \text{NS}(A^m)$, since $A^m\boldsymbol{x} = \mathbf{0} \implies A(A^m\boldsymbol{x}) = \mathbf{0} \implies A^{m+1}\boldsymbol{x} = \mathbf{0}$. So (i) is shown,

For (ii) suppose $\text{NS}(A^m) = \text{NS}(A^{m+1})$, then $A^{m+2}\boldsymbol{x} = \mathbf{0} \implies A^{m+1}(A\boldsymbol{x}) = \mathbf{0} \implies A^m(A\boldsymbol{x}) = \mathbf{0} \implies A^{m+1}\boldsymbol{x} = \mathbf{0}$. So $\text{NS}(A^{m+2}) \subseteq \text{NS}(A^{m+1})$, but then $\text{NS}(A^{m+2}) = \text{NS}(A^{m+1}) = \text{NS}(A^m)$. Now just keep going to get $\text{NS}(A^k) = \text{NS}(A^m)$ for all $k \geq m$.

This means we have

$$\text{NS}(A^0) \subsetneq \text{NS}(A^1) \subsetneq \text{NS}(A^2) \subsetneq \cdots \subsetneq \text{NS}(A^{m-1}) \subsetneq \text{NS}(A^m) = \text{NS}(A^{m+1}) = \cdots$$

The m at which $\text{NS}(A^k) = \text{NS}(A^m)$ for all $m \geq k$ must itself be $\leq n$.

If $A^k = \mathbf{0}$ for any k, then $\text{NS}(A^k) = \mathbb{R}^n$ is maximal and thus $m \leq k$ and $\text{NS}(A^m) = \mathbb{R}^n$. Since $m \leq n$, $\text{NS}(A^n) = \mathbb{R}^n$ and so $A^n = \mathbf{0}$.

Proof 2: You can use induction. To do this we need to prove something that sounds slightly stronger:

P_n : For any $n \times n$ matrix A, if $A^m = \mathbf{0}$ for any $m > n$, then $A^n = 0$.

base case: (n = 1) If $A^m = [a]^m = [a^m] = [0]$, for m>1, then $a = 0$, so $A^1 = [a] = [0]$ as needed.

inductive step: Suppose P_{n-1}: For any $m > n-1$, $A^m = \mathbf{0} \implies A^{n-1} = \mathbf{0}$ for all $(n-1) \times (n-1)$ matrices. We want to prove P_n.

Assume A is an $n \times n$ matrix and $A^m = 0$ for some $m > n$. Notice that $\ker(A) \neq \{\mathbf{0}\}$, since if $\ker(A) = \{0\}$, then $A : \mathbb{R}^n \to \mathbb{R}^n$ is injective and thus A^m is also injective, so $\ker(A^m) = \{\mathbf{0}\}$. This obviously contradicts $A^m = \mathbf{0}$.

Let $\boldsymbol{v}_1 \in \ker(A)$ and let $\mathcal{B} = \{\boldsymbol{v}_1, \boldsymbol{v}_2, \ldots, \boldsymbol{v}_n\}$ be a basis for $\mathbb{R}^n$. So letting $B = \begin{bmatrix}\boldsymbol{v}_1 & \boldsymbol{v}_2 & \cdots & \boldsymbol{v}_n\end{bmatrix}$ we have

$$A = BA'B^{-1}$$

where

$$A' = \begin{bmatrix} 0 & a'_{12} & a'_{13} & \cdots & a'_{1n} \\ 0 & & & & \\ \vdots & & \hat{A} & & \\ 0 & & & & \end{bmatrix}$$

where $\hat{A}$ is the indicated $(n-1) \times (n-1)$ submatrix of A'.

A' is the matrix of $L(\boldsymbol{x}) = A\boldsymbol{x}$ with respect to the basis $\mathcal{B}$. Notice that $A^m = \mathbf{0}$ means $L^m(\boldsymbol{x}) = \mathbf{0}$ for all $\boldsymbol{x}$ and hence $A'\boldsymbol{x} = \mathbf{0}$ for all $\boldsymbol{x}$, a finally this means $A'^m = \mathbf{0}$.

Notice that A' has the block form

$$\begin{bmatrix} 0 & \boldsymbol{b}^T \\ \mathbf{0} & \hat{A} \end{bmatrix}$$

and

$$(A')^2 = \begin{bmatrix} 0 & \boldsymbol{b}^T \\ \mathbf{0} & \hat{A} \end{bmatrix} \begin{bmatrix} 0 & \boldsymbol{b}^T \\ \mathbf{0} & \hat{A} \end{bmatrix} = \begin{bmatrix} 0 & \boldsymbol{b}^T \hat{A} \\ \mathbf{0} & \hat{A}^2 \end{bmatrix}$$

By a second, obvious induction,

$$(A')^k = \begin{bmatrix} 0 & \boldsymbol{b}^T \hat{A}^{k-1} \\ \mathbf{0} & \hat{A}^k \end{bmatrix}$$

We assume ${\hat{A}'}^m = \mathbf{0}$ so $\hat{A}^m = \mathbf{0}$ and by induction $\hat{A}^{n-1} = \mathbf{0}$ and thus

$$(A')^n = \begin{bmatrix} 0 & \boldsymbol{b}^T \hat{A}^{n-1} \\ \mathbf{0} & \hat{A}^n \end{bmatrix} = \mathbf{0}$$