Part IV: Proofs (15 points each; 60 points)

a) Let A be an $n \times n$ matrix. Prove that

$$AB = BA \text{ for all } B \text{ iff } A = \alpha I_n$$

where I_n is the $n \times n$ identity matrix and $\alpha \in \mathbb{R}$.

b) **Prove:** For an invertible $n \times n$ matrix. Show that A^{-1} is symmetric if A is symmetric.

c) **Prove:** If $V = U + W$, then

These are exactly the same, not two different things.

$$
\begin{aligned}
& U \cap W = \{\mathbf{0}\} \\
\boxed{\begin{array}{c}\iff \\ \iff\end{array}}\ & \begin{array}{l}\text{every } \boldsymbol{v} \in V \text{ can be written uniquely as } \boldsymbol{v} = \boldsymbol{u} + \boldsymbol{w} \text{ for some } \boldsymbol{u} \in U \text{ and } \boldsymbol{v} \in V \\ (\forall \boldsymbol{v} \in V)(\forall \boldsymbol{u}, \boldsymbol{u}' \in U)(\forall \boldsymbol{w}, \boldsymbol{w}' \in W)(\boldsymbol{v} = \boldsymbol{u} + \boldsymbol{w} = \boldsymbol{u}' + \boldsymbol{w}' \implies \boldsymbol{u} = \boldsymbol{u}' \text{ and } \boldsymbol{w} = \boldsymbol{w}')\end{array}
\end{aligned}
$$

Remark: $V = U \oplus W$ is defined as $V = U + W$ and $U \cap W = \{\mathbf{0}\}$. This result means that $V = U \oplus W$ iff every $\boldsymbol{v} \in V$ is uniquely decomposed as $\boldsymbol{v} = \boldsymbol{u} + \boldsymbol{w}$ where $\boldsymbol{u} \in U$ and $\boldsymbol{w} \in W$. For this reason, this is often taken as the definition of $V = U \oplus W$.

d) **Prove:** Suppose $\mathbb{R}^n = U \oplus W$ and A and B are matrices so that

- $\operatorname{rng}(A) = U$, $A^2 = A$, and $\boldsymbol{x} - A\boldsymbol{x} \in W$ for all $\boldsymbol{x} \in \mathbb{R}^n$,
- $\operatorname{rng}(B) = U$, $B^2 = B$, and $\boldsymbol{x} - B\boldsymbol{x} \in W$ for all $\boldsymbol{x} \in \mathbb{R}^n$.

Show that $A = B$.