

## Homework 5 Partial Solutions

### Homework 5 Problems

#### 5.1

1.

(a)

$\cos(\theta) = \frac{\mathbf{w}^T \mathbf{v}}{\|\mathbf{w}\| \|\mathbf{v}\|} = 1$  so  $\mathbf{w}$  and  $\mathbf{v}$  are in the same direction.

This is clear since  $3(2, 1, 3) = (6, 3, 9)$ .

5.  $y = 2x$  is the same as  $U = \text{span}\{\mathbf{u}\}$ , where  $\mathbf{u} = (1, 2)$ . The point in  $U$  closest to  $\mathbf{v} = (5, 2)$  is the projection  $P_U \mathbf{v}$  where  $P_U$  is the projection map onto  $U$ , this has matrix  $P_U = A(A^T A)^{-1} A^T$ , where  $A = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$  and  $U = \text{CS}(A)$ .

Notice,  $A(A^T A)^{-1} A^T \mathbf{v} = \mathbf{u}(\mathbf{u}^T \mathbf{u})^{-1}(\mathbf{u}^T \mathbf{v}) = \frac{\langle \mathbf{v}, \mathbf{u} \rangle}{\langle \mathbf{u}, \mathbf{u} \rangle} \mathbf{u} = \text{proj}_{\mathbf{u}}(\mathbf{v})$ , which is how the text defines the projection of  $\mathbf{v}$  onto  $\mathbf{u}$ .

$$P_U = \frac{1}{5} \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}$$

so  $P_U \mathbf{v} = \frac{1}{5}(9, 18)$  or if you like  $\text{proj}_{\mathbf{u}}(\mathbf{v}) = \frac{9}{5}(1, 2)$ .

13. Let  $\mathbf{v}$  and  $\mathbf{u}$  be vectors in any inner product space. We have

$$\begin{aligned} \|\mathbf{v} + \mathbf{u}\|^2 &= \langle \mathbf{v} + \mathbf{u}, \mathbf{v} + \mathbf{u} \rangle \\ &= \langle \mathbf{v}, \mathbf{v} + \mathbf{u} \rangle + \langle \mathbf{u}, \mathbf{v} + \mathbf{u} \rangle \\ &= \langle \mathbf{v}, \mathbf{v} \rangle + \langle \mathbf{v}, \mathbf{u} \rangle + \langle \mathbf{u}, \mathbf{v} \rangle + \langle \mathbf{u}, \mathbf{u} \rangle \\ &= \|\mathbf{v}\|^2 + \|\mathbf{u}\|^2 + 2\langle \mathbf{v}, \mathbf{u} \rangle \\ &\leq \|\mathbf{v}\|^2 + \|\mathbf{u}\|^2 + 2\|\mathbf{v}\|\|\mathbf{u}\| && \text{(Cauchy's Theorem)} \\ &= (\|\mathbf{v}\| + \|\mathbf{u}\|)^2 \end{aligned}$$

So  $\|\mathbf{v} + \mathbf{u}\| \leq \|\mathbf{v}\| + \|\mathbf{u}\|$ .

Equality will hold when  $\langle \mathbf{v}, \mathbf{u} \rangle = 0$ , this would mean

$$(\langle \mathbf{u}, \mathbf{v} \rangle)^2 = \langle \mathbf{v}, \mathbf{v} \rangle \langle \mathbf{u}, \mathbf{u} \rangle$$

Assuming  $\|\mathbf{u}\| \neq 0 \neq \|\mathbf{v}\|$ , then we have

$$\begin{aligned}\langle \|\mathbf{v}\|\mathbf{u} - \|\mathbf{u}\|\mathbf{v}, \|\mathbf{v}\|\mathbf{u} - \|\mathbf{u}\|\mathbf{v} \rangle &= \|\mathbf{v}\|\langle \mathbf{u}, \|\mathbf{v}\|\mathbf{u} - \|\mathbf{u}\|\mathbf{v} \rangle - \|\mathbf{u}\|\langle \mathbf{v}, \|\mathbf{v}\|\mathbf{u} - \|\mathbf{u}\|\mathbf{v} \rangle \\ &= \|\mathbf{v}\|^2\langle \mathbf{u}, \mathbf{u} \rangle - \|\mathbf{v}\|\|\mathbf{u}\|\langle \mathbf{u}, \mathbf{v} \rangle - \|\mathbf{u}\|\|\mathbf{v}\|\langle \mathbf{v}, \mathbf{u} \rangle + \|\mathbf{u}\|^2\langle \mathbf{v}, \mathbf{v} \rangle \\ &= 2\|\mathbf{v}\|^2\|\mathbf{v}\|^2 - 2(\|\mathbf{u}\|\|\mathbf{v}\|)^2 = 0\end{aligned}$$

So  $\|\mathbf{v}\|\mathbf{u} - \|\mathbf{u}\|\mathbf{v} = \mathbf{0}$ , but this means that  $\mathbf{u}$  and  $\mathbf{v}$  differ by a scalar multiple.

**18.**

(a) Show that

$$p_{\mathbf{y}}(\mathbf{x}) = \frac{\langle \mathbf{x}, \mathbf{y} \rangle}{\langle \mathbf{y}, \mathbf{y} \rangle} \mathbf{y}$$

is the orthogonal projection of  $\mathbf{x}$  onto  $\mathbf{y}$ . That is,  $\mathbf{x} - p_{\mathbf{y}}(\mathbf{x}) \perp \mathbf{y}$ .

Notice the usual inner product is  $\langle \mathbf{x}, \mathbf{y} \rangle = \mathbf{y}^T \mathbf{x}$ , so this does solve the problem.

Clearly

$$\langle \mathbf{x} - p_{\mathbf{y}}(\mathbf{x}), \mathbf{y} \rangle = \langle \mathbf{x}, \mathbf{y} \rangle - \frac{\langle \mathbf{x}, \mathbf{y} \rangle}{\langle \mathbf{y}, \mathbf{y} \rangle} \langle \mathbf{y}, \mathbf{y} \rangle = 0$$

(b) In arguing for (13) above we see  $\|\mathbf{u} + \mathbf{v}\|^2 = \|\mathbf{u}\|^2 + \|\mathbf{v}\|^2 \iff \langle \mathbf{u}, \mathbf{v} \rangle = 0$ , so here we have

$$\|\mathbf{x}\|^2 = \|\mathbf{x} - p_{\mathbf{y}}(\mathbf{x}) + p_{\mathbf{y}}(\mathbf{x})\|^2 = \|\mathbf{x} - p_{\mathbf{y}}(\mathbf{x})\|^2 + \|p_{\mathbf{y}}(\mathbf{x})\|^2 = 8^2 + 6^2 = 10^2$$

So  $\|\mathbf{x}\| = 10$ .

## 5.2

**1.**

(d) Here

$$A = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 1 & 1 & 2 & 2 \end{bmatrix} \quad \text{and} \quad \text{rref}(A) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

So we read off a basis for the four subspaces:

RS( $A$ ) = CS( $A^T$ ): A basis is  $\{(1, 0, 0, 0), (0, 1, 0, 0), (0, 0, 1, 1)\}$  (the non-zero rows of  $\text{rref}(A)$ )

CS( $A$ ): A basis is  $\{(1, 0, 0, 1), (0, 1, 0, 1), (0, 1, 1, 2)\}$  (the pivot columns of  $A$ ).

NS( $A$ ): A basis is  $\{(0, 0, 0, 1)\}$  since we see that the solutions to  $A\mathbf{x} = \mathbf{0}$  are of the form  $\mathbf{x} = t(0, 0, -1, 1)$ .

Note: NS( $A$ )  $\perp$  CS( $A^T$ ) as we know must happen.

**5.** If  $A$  is  $3 \times 2$  and of rank 2, then RS( $A$ ) is a 2-dimensional subspace of  $\mathbb{R}^2$ , hence RS( $A$ ) =  $\mathbb{R}^2$ .  $\mathbb{R}^3 = \text{NS}(A^T) \oplus \text{RS}(A^T)$  and RS( $A^T$ ) = CS( $A$ ) is a 2-dimensional subset of  $\mathbb{R}^3$ , a plane. So RS( $A^T$ ) is a plane in  $\mathbb{R}^3$  and NS( $A^T$ ) is the line normal to that plane.

**13.**

(a) Let  $\mathbf{x} \in \text{NS}(A^T A)$ .  $A\mathbf{x} \in \text{rng}(A)$  by definition of  $\text{rng}(A)$ , no assumption necessary. Of course  $\mathbf{x} \in \text{NS}(A^T A)$  means  $(A^T A)\mathbf{x} = \mathbf{0}$ , but  $(A^T A)\mathbf{x} = A^T(A\mathbf{x}) = \mathbf{0} \implies A\mathbf{x} \in \text{NS}(A^T)$ .

(b) For any  $\mathbf{x}$ ,  $\mathbf{x} \in \text{NS}(A) \implies A\mathbf{x} = \mathbf{0} \implies A^T A\mathbf{x} = \mathbf{0}$ . So

$$\text{NS}(A) \subseteq \text{NS}(A^T A)$$

So let  $\mathbf{x} \in \text{NS}(A^T A)$ , then  $A\mathbf{x} \in \text{NS}(A^T) \cap \text{rng}(A)$  by (a). But  $\text{rng}(A) = \text{CS}(A) = \text{RS}(A^T)$  and so  $A\mathbf{x} \in \text{NS}(A^T) \cap \text{RS}(A^T)$ . But  $\text{NS}(A^T) \perp \text{RS}(A^T)$  so  $\text{NS}(A^T) \cap \text{RS}(A^T) = \{\mathbf{0}\}$ . Thus  $A\mathbf{x} = \mathbf{0}$  and  $\mathbf{x} \in \text{NS}(A)$ . So we have

$$\text{NS}(A^T A) \subseteq \text{NS}(A)$$

So we have  $\text{NS}(A^T A) = \text{NS}(A)$  as desired.

(c) If  $A$  is  $m \times n$ , then  $A^T A$  is  $n \times n$  and the rank-nullity theorem gives

$$\text{rank}(A) + \dim(\text{NS}(A)) = n = \text{rank}(A^T A) + \dim(\text{NS}(A^T A))$$

By (b),  $\dim(\text{NS}(A)) = \dim(\text{NS}(A^T A))$  so  $\text{rank}(A) = \text{rank}(A^T A)$ .

(d) If  $A$  has independent columns, then  $\text{rank}(A) = n = \text{rank}(A^T A)$  so  $A^T A$  has full rank and is invertible.

**15.** By definition  $W = U \oplus V$  iff  $W = U + V$  for any  $\mathbf{w} \in W$ , there are unique  $\mathbf{u}$  and  $\mathbf{v}$  so that  $\mathbf{w} = \mathbf{u} + \mathbf{v}$ .

If  $\mathbf{z} \in U \cap V$  and  $\mathbf{z} \neq \mathbf{0}$ , then  $\mathbf{z} = \mathbf{u} + \mathbf{v}$ , where there are two options  $\mathbf{u} = \mathbf{z}$  and  $\mathbf{v} = \mathbf{0}$  or  $\mathbf{u} = \mathbf{0}$  and  $\mathbf{v} = \mathbf{z}$ .

**16.** Let  $\text{rank}(A) = k$  and  $\{\mathbf{x}_1, \dots, \mathbf{x}_k\}$  be a basis for  $\text{RS}(A) = \text{rng}(A^T)$ . Then to show that  $\{A\mathbf{x}_1, \dots, A\mathbf{x}_k\}$  is a basis for  $\text{rng}(A) = \text{CS}(A)$  we only need to check independence, since we know  $\dim(\text{RS}(A)) = \dim(\text{CS}(A)) = \text{rank}(A)$ . Suppose  $\sum_{i=1}^k \alpha_i A\mathbf{x}_i = \mathbf{0}$ , then  $A\left(\sum_{i=1}^k \alpha_i \mathbf{x}_i\right) = \mathbf{0}$ , so  $\sum_{i=1}^k \alpha_i \mathbf{x}_i \in \text{NS}(A)$ . But  $\text{NS}(A) \cap \text{rng}(A^T) = \{\mathbf{0}\}$  and thus  $\sum_{i=1}^k \alpha_i \mathbf{x}_i = \mathbf{0}$  and so  $\alpha_i = 0$  for all  $i$ .

## 5.3

**3.**

(b) Find first the orthogonal projection of  $\mathbf{b}$  onto  $\text{CS}(A)$ . The first two columns of  $A$  are a basis for  $\text{CS}(A)$  so we can use those two to find the projection  $\hat{\mathbf{b}} = B(B^T B)^{-1} B^T \mathbf{b}$  where  $B = A(:, 1:2)$ . We find  $\hat{\mathbf{b}} = (3, 1, 4)^T$  now we can ask what  $\hat{\mathbf{x}}$  satisfy  $A\hat{\mathbf{x}} = (3, 1, 4)^T$ . Solving this you find solutions have the form

$$\hat{\mathbf{x}} = \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} + t \begin{bmatrix} -2 \\ -1 \\ 1 \end{bmatrix}$$

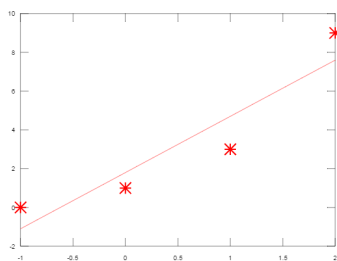
5. Trying to find  $(\alpha, \beta)^T$  so that

$$\underbrace{\begin{bmatrix} 1 & -1 \\ 1 & 0 \\ 1 & 1 \\ 1 & 2 \end{bmatrix}}_A \underbrace{\begin{bmatrix} \alpha \\ \beta \end{bmatrix}}_{\hat{x}} = \underbrace{\begin{bmatrix} 0 \\ 1 \\ 3 \\ 9 \end{bmatrix}}_b$$

For this  $\begin{bmatrix} \alpha \\ \beta \end{bmatrix} = (A^T A)^{-1} A^T \mathbf{b}$  and we get:

$$\begin{bmatrix} \alpha \\ \beta \end{bmatrix} = \begin{bmatrix} 1.8 \\ 2.9 \end{bmatrix}$$

So the equation is  $y = 2.9x + 1.8$ .



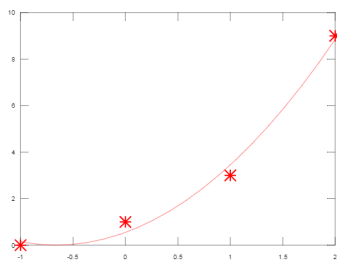
6. Trying to find  $(\alpha, \beta)^T$  so that

$$\underbrace{\begin{bmatrix} 1 & -1 & 1 \\ 1 & 0 & 0 \\ 1 & 1 & 1 \\ 1 & 2 & 4 \end{bmatrix}}_A \underbrace{\begin{bmatrix} \alpha \\ \beta \\ \gamma \end{bmatrix}}_{\hat{x}} = \underbrace{\begin{bmatrix} 0 \\ 1 \\ 3 \\ 9 \end{bmatrix}}_b$$

For this  $\begin{bmatrix} \alpha \\ \beta \end{bmatrix} = (A^T A)^{-1} A^T \mathbf{b}$  and we get:

$$\begin{bmatrix} \alpha \\ \beta \\ \gamma \end{bmatrix} = \begin{bmatrix} 0.55 \\ 1.65 \\ 1.25 \end{bmatrix}$$

So the equation is  $y = 1.25x^2 + 1.65x + 0.55$ .



**13.** We know that if  $A$  is  $m \times n$  and  $\mathbf{b} \in \mathbb{R}^m$ , then there is a unique  $\hat{\mathbf{b}} \in \text{rng}(A) = \text{CS}(A)$  so that  $\hat{\mathbf{b}} - \mathbf{b} \perp \text{rng}(A)$  and hence  $\|\hat{\mathbf{b}} - \mathbf{b}\|$  is minimal among all  $\|\mathbf{y} - \mathbf{b}\|$  for all  $\mathbf{y} \in \text{rng}(A)$ .

You should understand why  $\hat{\mathbf{b}} = A(A^T A)^{-1} A^T \mathbf{b}$ . So that  $\hat{\mathbf{x}} = (A^T A)^{-1} A^T \mathbf{b}$  is such that  $\|A\hat{\mathbf{x}} - \mathbf{b}\|$  is minimized and what you know is  $A^T A\hat{\mathbf{x}} = A^T \mathbf{b}$ .

We are assuming that  $\hat{\mathbf{x}}$  is a least-squares solution to  $A\mathbf{x} = \mathbf{b}$ , so  $A^T A\hat{\mathbf{x}} = A^T \mathbf{b}$ . There are two things to prove:

(1) If  $\mathbf{y}$  is a least-squares solution to  $A\mathbf{x} = \mathbf{b}$ , then  $\mathbf{y} = \hat{\mathbf{x}} + \mathbf{z}$  for  $\mathbf{z} \in \text{NS}(A)$ .

We know  $A^T A\mathbf{y} = A^T \mathbf{b} = A^T A\hat{\mathbf{x}}$  so  $A^T A(\mathbf{y} - \hat{\mathbf{x}}) = \mathbf{0}$ . Let  $\mathbf{z} = \mathbf{y} - \hat{\mathbf{x}}$ , then  $\mathbf{z} \in \text{NS}(A^T A) = \text{NS}(A)$  and  $\mathbf{y} = \hat{\mathbf{x}} + \mathbf{z}$  as desired.

(2) If  $\mathbf{y} = \hat{\mathbf{x}} + \mathbf{z}$  for some  $\mathbf{z} \in \text{NS}(A)$ , then  $\mathbf{y}$  is a least square-solution to  $A\mathbf{x} = \mathbf{b}$ .

$A^T A(\hat{\mathbf{x}} + \mathbf{z}) = A^T A\hat{\mathbf{x}} + A^T A\mathbf{z} = \mathbf{b} + \mathbf{0} = \mathbf{b}$ . So  $\hat{\mathbf{x}} + \mathbf{z}$  is a least-squares solution to  $A\mathbf{x} = \mathbf{b}$ .

**14**  $A$  is  $m \times n$ ,  $B$  is  $n \times r$ . and  $C = AB$ .

(a)  $\mathbf{x} \in \text{NS}(B) \implies B\mathbf{x} = \mathbf{0} \implies AB\mathbf{x} = \mathbf{0} \implies \mathbf{x} \in \text{NS}(C)$ , so  $\text{NS}(B) \subseteq \text{NS}(C)$ .

(b)  $\text{NS}(C) \oplus \text{NS}(C)^\perp = \mathbb{R}^r = \text{NS}(B) \oplus \text{NS}(B)^\perp$ , so  $\text{rng}(C^T) = \text{CS}(C^T) = \text{NS}(C)^\perp \subseteq \text{NS}(B)^\perp = \text{CS}(B^T) = \text{rng}(B^T)$ .

## 5.4

4. Given

$$A = \begin{bmatrix} 1 & 2 & 2 \\ 1 & 0 & 2 \\ 3 & 1 & 1 \end{bmatrix} \quad B = \begin{bmatrix} -4 & 1 & 1 \\ -3 & 3 & 2 \\ 1 & -2 & -2 \end{bmatrix}$$

(a)

$$\begin{aligned} \langle A, B \rangle &= (1)(-4) + (2)(1) + (2)(1) + (1)(-3) + (0)(3) \\ &\quad + (2)(2) + (3)(1) + (1)(-2) + (1)(-2) = 0 \end{aligned}$$

So  $A$  and  $B$  are orthogonal.

(b)

$$\|A\|_F^2 = (1)^2 + (2)^2 + (2)^2 + (1)^2 + (0)^2 + (2)^2 + (3)^2 + (1)^2 + (1)^2 = 25$$

(c)

$$\|A\|_F^2 = (-4)^2 + (1)^2 + (1)^2 + (-3)^2 + (3)^2 + (2)^2 + (1)^2 + (-2)^2 + (-2)^2 = 49$$

(d)  $\|A + B\|_F^2 = \|A\|_F^2 + \|B\|_F^2 = 25 + 49 = 74$ , since  $A \perp_F B$ .

8.

(a)

$$\cos(\theta) = \frac{\langle 1, x \rangle}{\|1\| \cdot \|x\|} = \frac{\int_0^1 (1 \cdot x) dx}{\left(\int_0^1 1^2 dx\right)^{1/2} \cdot \left(\int_0^1 x^2 dx\right)^{1/2}} = \frac{1/2}{(1)^{1/2} \cdot (1/3)^{1/2}} = \sqrt{3}/2$$

So  $\theta = \pi/6$ . Of course, this really doesn't mean anything, the relevant thing here is  $\frac{\langle 1, x \rangle}{\|1\| \cdot \|x\|}$ .

(b)  $p = p_x(1) = \frac{\langle 1, x \rangle}{\langle x, x \rangle} x = \frac{\int_0^1 x dx}{\int_0^1 x^2 dx} = \frac{1/2}{1/3} x = (3/2)x$ .

Check that  $1 - p \perp p$ ,  $\langle p, 1 - p \rangle = \int_0^1 (3/2)x(1 - (3/2)x) dx = \int_0^1 (3/2)x - (3/2)^2 x^2 dx = (3/2)(1/2) - (1/3)(3/2)^2 = 0$ !

(c)

$$\begin{aligned}\|1\|^2 &= \langle 1, 1 \rangle = \int_0^1 1 dx = 1 \\ \|p\|^2 &= \|(3/2)x\|^2 = (3/2)^2 \langle x, x \rangle = \int_0^1 x^2 dx = (3/2)^2 \frac{1}{3} = 3/4 \\ \|1 - p\|^2 &= \int_0^1 (1 - (3/2)x)^2 dx = (2/3) \int_1^{-1/2} u^2 du \\ &= (2/3)(1/3)((-1/2)^3 - (1^3)) = (2/3)(1/3)(-9/8) = 1/4\end{aligned}$$

Now

$$\|p\|^2 + \|1 - p\|^2 = 3/4 + 1/4 = 1 = \|1\|^2 = \|p + (1 - p)\|^2$$

Verifying Pythagoras.

**10.** Let  $x_i = (i - 3)/2$  for  $i = 1, \dots, 5$ . Define  $\langle p, q \rangle = \sum_{i=1}^5 p(x_i)q(x_i)$  for  $p, q \in \mathbb{P}_5[x]$ .

$$\langle x, x^2 \rangle = \sum_{i=1}^5 (i - 3)/2 \cdot (i - 3)/2^2 = \frac{1}{8} \sum_{i=1}^5 (i - 3)^3 = \frac{1}{8}((-2)^3 + (-1)^3 + (0)^3 + (1)^3 + (2)^3) = 0$$

So  $x \perp x^2$  as desired.

## 5.5

**29** Use  $\langle f, g \rangle = \int_{-1}^1 f \cdot g dx$

(a)  $\langle 1, x \rangle = \int_{-1}^1 1 \cdot x dx = \frac{1}{2}x^2 \Big|_{-1}^1 = 0$ . So  $1 \perp x$ .

(b)

$$\begin{aligned}\|1\|^2 &= \langle 1, 1 \rangle = \int_{-1}^1 1 \cdot 1 dx = x \Big|_{-1}^1 = 2 \\ \|x\|^2 &= \langle x, x \rangle = \int_{-1}^1 x^2 dx = \frac{1}{3}x^3 \Big|_{-1}^1 = \frac{2}{3}\end{aligned}$$

(c) To find the best approximation to  $x^{1/3}$  we project  $x^{1/3}$  onto  $\text{span}\{1, x\}$

$$p_1(x^{1/3}) = \frac{\langle x^{1/3}, 1 \rangle}{\langle 1, 1 \rangle} 1 = \frac{\int_{-1}^1 x^{1/3} dx}{2} = \frac{(3/4)x^{4/3} \Big|_{-1}^1}{2} = 0$$

$$p_x(x^{1/3}) = \frac{\langle x^{1/3}, x \rangle}{\langle x, x \rangle} x = \frac{\int_{-1}^1 x^{4/3} dx}{(2/3)} x = \frac{(3/7)x^{7/3} \Big|_{-1}^1}{(2/3)} x = 2(3/7)(3/2)x = 9/7 x$$

So the projection of  $x^{1/3}$  onto  $\text{span}\{1, x\}$  is  $0 \cdot 1 + 9/7 \cdot x = 9/7 x$

[Here is a Desmos](#) graph to illustrate this.

### 30

(a) Let  $f_1 = 1$  and  $f_2 = 2x - 1$ ,  $\langle f_1, f_2 \rangle = \int_0^1 f_1 \cdot f_2 dx = \int_0^1 (2x - 1) dx = x^2 - x \Big|_0^1 = 0$ , so  $f_1 \perp f_2$ .

(b)  $\|f_1\|_2^2 = \langle f_1, f_1 \rangle = \int_0^1 f_1^2 dx = \int_0^1 (1)^2 dx = x \Big|_0^1 = 1$  and  $\|f_2\|_2^2 = \int_0^1 (2x - 1)^2 dx = \frac{1}{2} \int_{-1}^1 u^2 du = \frac{1}{2} \frac{u^3}{3} \Big|_{-1}^1 = \frac{1}{3}$ . So  $\|f_1\| = 1$  and  $\|f_2\| = \frac{1}{\sqrt{3}}$ , so the unit vector in the direction of  $f_2$  is  $\sqrt{3}f_2$ .

(c) The projection of  $g(x) = \sqrt{x}$  onto  $\text{span}\{f_1, f_2\}$  is  $\langle g, f_1 \rangle \cdot f_1 + \langle g, \sqrt{3}f_2 \rangle \cdot \sqrt{3}f_2 = \langle g, f_1 \rangle \cdot f_1 + \langle g, f_2 \rangle \cdot 3f_2$ . We have

$$\begin{aligned} \langle g, f_1 \rangle &= \int_0^1 x^{1/2} dx = \frac{2}{3} x^{3/2} \Big|_0^1 = \frac{2}{3} \\ \langle g, f_2 \rangle &= \int_0^1 x^{1/2} (2x - 1) dx \\ &= \int_0^1 2x^{3/2} - x^{1/2} dx \\ &= \left( 2 \cdot \frac{2}{5} x^{5/2} - \frac{2}{3} x^{3/2} \right) \Big|_0^1 = \frac{2}{15} \end{aligned}$$

So the projection of  $g(x) = \sqrt{x}$  onto  $\text{span}\{f_1, f_2\}$  is

$$\hat{g} = \frac{2}{3} \cdot f_1 + \frac{2}{15} \cdot 3 \cdot f_2 = \frac{2}{3} + \frac{2}{5}(2x - 1) = \frac{4}{5}x + \frac{4}{15}$$

See demo [here](#)

## 1 5.6

4. The strategy here is simple:

- Start with  $\mathcal{V} = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\} = \{1, x, x^2\}$ .
- $\mathbf{u}_1 = \mathbf{v}_1$
- $\mathbf{q}_1 = \mathbf{u}_1 / \|\mathbf{u}_1\|$
- $\mathbf{u}_2 = \mathbf{v}_2 - \langle \mathbf{v}_2, \mathbf{q}_1 \rangle \mathbf{q}_1$

- $\mathbf{q}_2 = \mathbf{u}_2 / \|\mathbf{u}_2\|$
- $\mathbf{u}_3 = \mathbf{v}_3 - \langle \mathbf{v}_3, \mathbf{q}_1 \rangle \mathbf{q}_1 - \langle \mathbf{v}_3, \mathbf{q}_2 \rangle \mathbf{q}_2$
- $\mathbf{q}_3 = \mathbf{u}_3 / \|\mathbf{u}_3\|$

That is the idea, here are the details. This is how all simple Gram-Schmidt proceeds.

Choose  $\mathbf{u}_1 = 1$ , then  $\langle 1, 1 \rangle = \int_0^1 1 \, dx = 1$  so this is already normalized and so set  $\mathbf{q}_1 = \mathbf{u}_1$ .

Set  $\mathbf{u}_2 = x - \langle x, \mathbf{q}_1 \rangle \mathbf{q}_1 = x - \int_0^1 x \, dx = x - \frac{1}{2}$ . Now  $\|\mathbf{u}_2\|^2 = \int_0^1 \left(x - \frac{1}{2}\right)^2 \, dx = \int_0^1 x^2 - x + \frac{1}{4} \, dx = \frac{1}{3}x^3 - \frac{1}{2}x^2 + \frac{1}{4}x \Big|_{x=0}^{x=1} = \frac{1}{3} - \frac{1}{2} + \frac{1}{4} = \frac{1}{12}$ . So  $\mathbf{q}_2 = \sqrt{12} \left(x - \frac{1}{2}\right) = \sqrt{3}(2x - 1)$ .

Finally,  $\mathbf{u}_3 = x^2 - \langle x^2, \mathbf{q}_2 \rangle \mathbf{q}_2 - \langle x^2, \mathbf{q}_1 \rangle \mathbf{q}_1$ . We have  $\langle x^2, \mathbf{q}_2 \rangle = \int_0^1 \sqrt{3}(2x - 1)x^2 \, dx = \sqrt{3} \left(\frac{1}{2}x^4 - \frac{1}{3}x^3\right) \Big|_{x=0}^{x=1} = \frac{1}{2\sqrt{3}}$ . So  $\langle x^2, \mathbf{q}_2 \rangle \mathbf{q}_2 = \left(x - \frac{1}{2}\right)$ . Also,  $\langle x^2, \mathbf{q}_1 \rangle = \int_0^1 x^2 \, dx = \frac{1}{3}$ , so  $\mathbf{u}_3 = x^2 - \left(x - \frac{1}{2}\right) - \frac{1}{3} = x^2 - x + \frac{1}{6}$ .

We have  $\|\mathbf{u}_3\|^2 = \int_0^1 \left(x^2 - x + \frac{1}{6}\right)^2 \, dx = \int_0^1 x^4 - 2x^3 + \frac{4}{3}x^2 - \frac{1}{3}x + \frac{1}{36} \, dx = \frac{1}{180}$  and so  $\mathbf{q}_3 = \sqrt{5}(6x^2 - 6x + 1)$ .

[A SageCell page that does computations](#)

5. Let

(a)

$$A = \begin{bmatrix} 2 & 1 \\ 1 & 1 \\ 2 & 1 \end{bmatrix} = [\mathbf{a}_1 \quad \mathbf{a}_2]$$

Let

$$\begin{aligned} \mathbf{q}_1 &= \mathbf{a}_1 = (2, 1, 2)^T \\ \mathbf{q}_2 &= \mathbf{a}_2 - \frac{\langle \mathbf{a}_2, \mathbf{q}_1 \rangle}{\langle \mathbf{q}_1, \mathbf{q}_1 \rangle} \mathbf{q}_1 \\ &= (1, 1, 1)^T - \frac{\langle (2, 1, 2)^T, (1, 1, 1)^T \rangle}{\langle (2, 1, 2)^T, (2, 1, 2)^T \rangle} (2, 1, 2)^T \\ &= (1/9)(-1, 4, -1)^T \end{aligned}$$

Check that this is orthogonal to  $(2, 1, 2)^T$ .

Now just normalize these

$$\begin{aligned} \hat{\mathbf{q}}_1 &= (2/3, 1/3, 2/3)^T \\ \hat{\mathbf{q}}_2 &= (1/(3\sqrt{2}))(-1, 4, -1)^T \end{aligned}$$

So

$$Q = \begin{bmatrix} 2/3 & -1/(3\sqrt{2}) \\ 1/3 & 4/(3\sqrt{2}) \\ 2/3 & -1/(3\sqrt{2}) \end{bmatrix}$$

$$R = Q^T A = \begin{bmatrix} \langle \hat{\mathbf{q}}_1, \mathbf{a}_1 \rangle & \langle \hat{\mathbf{q}}_1, \mathbf{a}_2 \rangle \\ 0 & \langle \hat{\mathbf{q}}_2, \mathbf{a}_2 \rangle \end{bmatrix} = \begin{bmatrix} 3 & 5/3 \\ 0 & 2/(3\sqrt{2}) \end{bmatrix}.$$



Check:  $A = QR$ .

7. We know  $\text{CS}(A) \perp \text{NS}(A)$  where  $A = \begin{bmatrix} 1 & 1 & 1 & -1 \\ 1 & 1 & 3 & 5 \end{bmatrix}$ . To find  $\ker(A)$  start with  $\text{rref}(A) = \begin{bmatrix} 1 & 1 & 0 & -4 \\ 0 & 0 & 1 & 3 \end{bmatrix}$ . So letting  $x_2 = s$  and  $x_4 = t$  we have

$$\mathbf{x} \in \ker(A) \iff \mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = s \begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \end{bmatrix} + t \begin{bmatrix} 4 \\ 0 \\ -3 \\ 1 \end{bmatrix}$$

so a basis for  $\ker(A) = \text{span}\{(-1, 1, 0, 0)^T, (4, 0, -3, 1)^T\}$

Check that these are indeed orthogonal to the given vectors.

Now use GS

$$\mathbf{q}_3 = (-1, 1, 0, 0)^T$$

$$\mathbf{q}_4 = (4, 0, -3, 1)^T - \frac{\langle (4, 0, -3, 1)^T, (-1, 1, 0, 0)^T \rangle}{\langle (-1, 1, 0, 0)^T, (-1, 1, 0, 0)^T \rangle} (-1, 1, 0, 0)^T = (2, 2, -3, 1)^T$$

Now just normalize to make these into unit vectors.

$$\hat{\mathbf{q}}_3 = \frac{1}{\sqrt{2}} \begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \end{bmatrix} \quad \hat{\mathbf{q}}_4 = \frac{1}{3\sqrt{2}} \begin{bmatrix} 2 \\ 2 \\ -3 \\ 1 \end{bmatrix}$$

8. Use Gram-Schmidt to find orthonormal basis for  $\text{span}\{\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3\}$  where

$$\mathbf{x}_1 = \begin{bmatrix} 4 \\ 2 \\ 2 \\ 1 \end{bmatrix} \quad \mathbf{x}_2 = \begin{bmatrix} 2 \\ 0 \\ 0 \\ 2 \end{bmatrix} \quad \mathbf{x}_3 = \begin{bmatrix} 1 \\ 1 \\ -1 \\ 1 \end{bmatrix}$$

$$\mathbf{q}_1 = \mathbf{x}_1$$

$$\begin{aligned} \mathbf{q}_2 &= \mathbf{x}_2 - \frac{\langle \mathbf{q}_1, \mathbf{x}_2 \rangle}{\langle \mathbf{q}_1, \mathbf{q}_1 \rangle} \mathbf{q}_1 \\ &= (2, 0, 0, 2) - \frac{10}{25} (4, 2, 2, 1) \\ &= (2/5, -4/5, -4/5, 8/5) = (2/5)(1, -2, -2, 4) \end{aligned}$$

$$\begin{aligned} \mathbf{q}_3 &= \mathbf{x}_3 - \frac{\langle \mathbf{x}_3, \mathbf{q}_1 \rangle}{\langle \mathbf{q}_1, \mathbf{q}_1 \rangle} \mathbf{q}_1 - \frac{\langle \mathbf{x}_3, \mathbf{q}_2 \rangle}{\langle \mathbf{q}_2, \mathbf{q}_2 \rangle} \mathbf{q}_2 \\ &= (1, 1, -1, 1) - \frac{5}{25} (4, 2, 2, 1) - \frac{(2/5)(5)}{(2/5)^2(25)} (2/5)(1, -2, -2, 4) \\ &= (1, 1, -1, 1) - (4/5, 2/5, 2/5, 1/5) - (1/5, -2/5, -2/5, 4/5) \\ &= (0, 1, -1, 0) = (0, 1, -1, 0) \end{aligned}$$

So the final normalized orthonormal basis is

$$\mathbf{q}_1 = \begin{pmatrix} 1 \\ 5 \end{pmatrix} \begin{bmatrix} 4 \\ 2 \\ 2 \\ 1 \end{bmatrix} \quad \mathbf{q}_2 = \begin{pmatrix} 1 \\ 5 \end{pmatrix} \begin{bmatrix} 1 \\ -2 \\ -2 \\ 4 \end{bmatrix} \quad \mathbf{q}_3 = \begin{pmatrix} 1 \\ 2 \end{pmatrix}^{1/2} \begin{bmatrix} 0 \\ 1 \\ -1 \\ 0 \end{bmatrix}$$

9. The modified Gram-Schmidt looks like:

First pass:

$$\begin{aligned} \mathbf{q}_1 &= \mathbf{x}_1 \\ \mathbf{q}_2 &= \mathbf{x}_2 - \frac{\langle \mathbf{q}_1, \mathbf{x}_2 \rangle}{\langle \mathbf{q}_1, \mathbf{q}_1 \rangle} \mathbf{q}_1 \\ &= (2, 0, 0, 2) - \frac{10}{25} (4, 2, 2, 1) \\ &= (2/5, -4/5, -4/5, 8/5) = (2/5)(1, -2, -2, 4) \\ \mathbf{q}_3 &= \mathbf{x}_3 - \frac{\langle \mathbf{x}_3, \mathbf{q}_1 \rangle}{\langle \mathbf{q}_1, \mathbf{q}_1 \rangle} \mathbf{q}_1 \\ &= (1, 1, -1, 1) - \frac{5}{25} (4, 2, 2, 1) \\ &= (1/5, 3/5, -7/5, 4/5) = (1/5)(1, 3, -7, 4) \end{aligned}$$

Now  $\mathbf{q}_2$  and  $\mathbf{q}_3$  are orthogonal to  $\mathbf{q}_1$ .

Second pass:

$$\begin{aligned} \mathbf{q}'_1 &= \mathbf{q}_1 \\ \mathbf{q}'_2 &= \mathbf{q}_2 = (2/5)(1, -2, -2, 4) \\ \mathbf{q}'_3 &= \mathbf{q}_3 - \frac{\langle \mathbf{q}_3, \mathbf{q}_2 \rangle}{\langle \mathbf{q}_2, \mathbf{q}_2 \rangle} \mathbf{q}_2 \\ &= (1/5)(1, 3, -7, 4) - \frac{(1/5)(2/5)25}{(2/5)^2(25)} (2/5)(1, -2, -2, 4) \\ &= (1/5)(1, 3, -7, 4) - (1/5)(1, -2, -2, 4) \\ &= (1/5)(0, 5, -5, 0) = (0, 1, -1, 0) \end{aligned}$$

Now just normalize and get the same answer as in (8).

**14. and 15.** Let  $\mathcal{B}$  be a basis for  $W = U \cap V$  and let  $\mathcal{B}_U \supseteq \mathcal{B}$  be a basis for  $U$  that extends  $\mathcal{B}_W$ . Extend  $\mathcal{B}_U$  to  $\mathcal{B}_{V+U}$  a basis for  $V + U$ . For each  $\mathbf{z} \in \mathcal{B}_{V+U} - \lfloor_U$ ,  $\mathbf{z} = \mathbf{v}_z + \mathbf{u}_z$  and as  $\mathbf{z} \notin U$  we know  $\mathbf{v}_z \neq 0$  and if  $\mathcal{C} = \langle \mathbf{w}_z \mid \mathbf{z} \in \mathcal{B}_{V+U} - \lfloor_U \rangle$ , then clearly  $\mathcal{B}_U \cup \mathcal{C}$  is still independent and spans  $V + U$ . So we may assume  $\mathcal{B}_{U+V} - \mathcal{B}_U \subset V$ .

Claim:  $\mathcal{B}_V \stackrel{\text{def}}{=} \mathcal{B}_W \cup (\mathcal{B}_{U+V} - \mathcal{B}_U)$  is a basis for  $V$ .

Independence is for free since  $\mathcal{B}_{V+U}$  is independent. So we must show that  $\mathcal{B}_V$  spans  $V$ . Let  $\mathbf{v} \in V$  and  $\mathbf{v} = \mathbf{w} + \mathbf{u} + \mathbf{z}$  where  $\mathbf{w} \in \text{span}(\mathcal{B}_W)$ ,  $\mathbf{u} \in \text{span}(\mathcal{B}_U - \mathcal{B}_W)$ , and  $\mathbf{z} \in \text{span}(\mathcal{B}_{U+V} - \mathcal{B}_U) \subseteq V$ .

If  $\mathbf{u} \neq \mathbf{0}$ , then  $\mathbf{u} = \mathbf{v} - (\mathbf{w} + \mathbf{z}) \in V$ , but then  $\mathbf{u} \in U \cap V = W$  which is impossible since then a non-zero linear combination from  $\mathcal{B}_U - \mathcal{B}_W$  is also a linear combination from  $\mathcal{B}_W$ , contradicting the independence of  $\mathcal{B}_U$ .

So  $\mathbf{v} = \mathbf{w} + \mathbf{z} \in \text{span}(\mathcal{B}_V)$ . The

This shows  $|\mathcal{B}_U \cup \mathcal{B}_V| = |\mathcal{B}_U| + |\mathcal{B}_V| - |\mathcal{B}_U \cap \mathcal{B}_V|$  so  $\dim(U + V) = \dim(U) + \dim(V) - \dim(U \cap V)$ .