Name: ____________________ Exam 2 - MAT345

1 True/False (50 points; 5 points each)

Recall: A and B are equivalent if there is a sequence of elementary row operations leading from A to B, or equivalently, $B = MA$ for some invertible matrix M. This is different from $A \sim B$ (A and B are *similar* which means $B = S^{-1}AS$ for some invertible S.

Problem 1.1. In class, you need only provide a T/F (make it clear!) As usual, you may earn back up to 50% of the lost points by supplying justifications afterward.

False The collection of of 3×4 echelon matrices is a subspace of $\mathbb{R}^{3\times 4}$.

Closure under scalar multiplication is ok, but it is easy to see that closure under addition fails.

True The set of $n \times n$ matrices with all diagonal elements being 0 is a subspace of $\mathbb{R}^{n\times n}$.

It is clear that if A and B both have 0 diagonals, then so does αA and $A + B$.

True Consider the map $L : \mathbb{R}^{m\times n} \to \mathbb{R}^m$ defined by $L(A)_i = \text{ave}(A_{i,*})$, that is, the i^{th} entry of $L(A)$ is the average of the i^{th} row of A. L is a linear map.

This is easy to check directly, but here is a cute argument. Let $\mathbf{1} \in \mathbb{R}^n$ be the vector of n 1's. Then

$$L(A) = \tfrac{1}{n}A\mathbf{1},$$

and this, being simple matrix multiplication, is clearly linear:

$$L(\alpha A + \beta B) = \tfrac{1}{n}(\alpha A + \beta B)\mathbf{1} = \alpha\tfrac{1}{n}A\mathbf{1} + \beta\tfrac{1}{n}B\mathbf{1} = \alpha \cdot L(A) + \beta \cdot L(B)$$

False For all linear $L : V \to W$, if $\{\boldsymbol{v}_1, \dots, \boldsymbol{v}_k\}$ is independent, then $\{L(\boldsymbol{v}_1), \dots, L(\boldsymbol{v}_k)\}$ is independent.

This is trivially false. L could just be the $\mathbf{0}$ map, that is, $L(\boldsymbol{v}) = \mathbf{0}$ for all $\boldsymbol{v} \in V$.

True For all linear $L : V \to W$, if $\{L(\boldsymbol{v}_1), \dots, L(\boldsymbol{v}_k)\}$ is independent, then $\{\boldsymbol{v}_1, \dots, \boldsymbol{v}_k\}$ is independent.

Suppose $\sum \alpha_i \boldsymbol{v}_i = \mathbf{0}$, then $L(\sum \alpha_i \boldsymbol{v}_i) = \sum \alpha_i L(\boldsymbol{v}_i) = L(\mathbf{0}) = \mathbf{0}$. Since $\{L(\boldsymbol{v}_1), \dots, L(\boldsymbol{v}_k)\}$ is independent it follows that $\alpha_i = 0$ for all i and hence that $\{\boldsymbol{v}_1, \dots, \boldsymbol{v}_k\}$ is independent.

False There are subspaces $V_0 = P_3 \supsetneqq V_1 \supsetneqq V_2 \supsetneqq V_3 \supsetneqq V_4 \supsetneqq V_5 = \{\mathbf{0}\}$ where each V_i is a proper subspace of V_{i-1}.

Since we know $\dim(V_0) = 4 > \dim(V_1) > \dim(V_2) > \dim(V_3) > \dim(V_4) > \dim(V_5) = 0$, which is impossible.

True Given any basis $\{\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3\}$, from $\mathbb{R}^4$ and any four matrices $M_1, M_2, M_3, M_4 \in \mathbb{R}^{2\times 3}$ there is a unique linear transformation $L : \mathbb{R}^4 \to \mathbb{R}^{2\times 3}$ where $L(\boldsymbol{v}_i) = M_i$.

Existence: Define $L\left(\sum_{i=1}^4 \alpha_i \boldsymbol{v}_i\right) = \sum_{i=1}^4 \alpha_i M_i$. This is a well-defined function $L : \mathbb{R}^4 \to \mathbb{R}^{2\times 3}$.

Showing that this is linear is just a computation: Let $\boldsymbol{v} = \sum_{i=1}^4 \alpha_i \boldsymbol{v}_i$ and $\boldsymbol{u} = \sum_{i=1}^4 \beta + i\boldsymbol{v}_i$, then

$$L(\gamma\boldsymbol{v} + \boldsymbol{u}) = L\left(\gamma \sum_{i=1}^4 \alpha_i \boldsymbol{v}_i + \sum_{i=1}^4 \beta_i \boldsymbol{v}_i\right) = L\left(\sum_{i=1}^4 (\gamma\alpha_i + \beta_i)\boldsymbol{v}_i\right)$$
$$= \sum_{i=1}^4 (\gamma\alpha_i + \beta_i) W_i = \gamma \sum_{i=1}^4 \alpha_i \boldsymbol{W}_i + \sum_{i=1}^4 \beta_i W_i = \gamma \cdot L(\boldsymbol{v}) + L(\boldsymbol{u})$$

Uniqueness: Suppose $L' : V \to W$ is linear and sends $\boldsymbol{v}_i$ to $\boldsymbol{W}_i$, then for $\boldsymbol{v} \in V$, $\boldsymbol{v} = \sum_{i=1}^4 \alpha_i \boldsymbol{v}_i$ and $L'(\boldsymbol{v}) = L'\left(\sum_{i=1}^4 \alpha_i \boldsymbol{v}_i\right) = \sum_{i=1}^4 \alpha_i M_i = L(\boldsymbol{v})$ and thus $L = L'$.

True Suppose $L : P_5 \to \mathbb{R}^4$ is linear and onto, that is, $\mathrm{Img}(L) = \mathbb{R}^4$. Then $\dim(\ker(L)) = 2$.

Recall P_5 is the space of polynomials of degree ≤ 5.

$\dim(P_5) = 6$ and so $\dim(\ker(L)) + \dim(\mathrm{Img}(L)) = \dim(\ker(L)) + 4 = 6$.

True Let

$$\mathcal{B} = \left\{ \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ 2 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} \right\}$$

be a basis for $\mathbb{R}^3$. Then for $\boldsymbol{v} = \begin{bmatrix} 3 \\ 3 \\ 4 \end{bmatrix}$

$$[\boldsymbol{v}]_{\mathcal{B}} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} 3 \\ 3 \\ 4 \end{bmatrix} = (1)\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} + (1)\begin{bmatrix} 2 \\ 0 \\ 2 \end{bmatrix} + (1)\begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}$$

False $L : \mathbb{R}^{3\times 3} \to \mathbb{R}^3$ is given by $L(A) = A\boldsymbol{b}$ for a $\boldsymbol{b} \in \mathbb{R}^3$. If $\mathcal{B}$ is a basis for $\mathbb{R}^{3\times 3}$, then $[L]_{\mathcal{B}} = \boldsymbol{b}$.

$[L]_{\mathcal{B}}$ acts on representations of matrices wrt $\mathcal{B}$, it is a 9×9 matrix, not a 3×3 matrix.

2 Multiple Choice (30 points; 10 points each)

Each correct box counts for two points.

Problem 2.1 (10 points)**.** Which of the following are equivalent to "A is ***equivalent*** to B"? Mark 'Y' if equivalent and 'N' if not.

$\boxed{\text{Y}}$ B results from a series of row operations from A.

$\boxed{\text{N}}$ $B = AM$ for some invertible matrix M.

$\boxed{\text{Y}}$ $B = MA$ for some invertible matrix M.

$\boxed{\text{N}}$ $\text{CS}(A) = \text{CS}(B)$.

$\boxed{\text{Y}}$ $\text{RS}(A) = \text{RS}(B)$.

Problem 2.2 (10 points)**.** Which of the following are equivalent to A is invertible for an $n \times n$ matrix A. Mark 'Y' if equivalent and 'N' if not.

$\boxed{\text{Y}}$ A is equivalent to I.

$\boxed{\text{N}}$ $\dim(\text{RS}(A)) = \dim(\text{CS}(A))$.

$\boxed{\text{Y}}$ $\text{NS}(A) = \{\mathbf{0}\}$.

$\boxed{\text{Y}}$ $A\boldsymbol{x} = \boldsymbol{b}$ has at least one solution for all $\boldsymbol{b} \in \mathbb{R}^n$.

$\boxed{\text{Y}}$ $A\boldsymbol{x} = \boldsymbol{b}$ has a unique solution for some $\boldsymbol{b}$.

Problem 2.3. Which of the following implies that $\mathcal{B} = \{\boldsymbol{v_1}, \ldots, \boldsymbol{v_n}\}$ is linearly independent. Mark 'Y' if the property implies $\mathcal{B}$ is independent, 'N' otherwise.

$\boxed{\text{N}}$ For every $\boldsymbol{v}$ in V, $\boldsymbol{v}$ can be written as a linear combination of vectors in $\mathcal{B}$, i.e., there is $\alpha_i \in \mathbb{R}$ so that $\boldsymbol{v} = \sum_{i=1}^n \alpha_i \boldsymbol{b}_i$.

$\boxed{\text{Y}}$ If $\sum_{i=1}^n \alpha_i \boldsymbol{b}_i = \sum_{i=1}^n \beta_i \boldsymbol{b}_i$, then $\alpha_i = \beta_i$ for all i.

$\boxed{\text{Y}}$ $\boldsymbol{b}_i \notin \text{span}(\mathcal{B} - \{\boldsymbol{b}_i\})$, that is, $\boldsymbol{b}_i$ is not a linear combination of the other vectors in $\mathcal{B}$.

$\boxed{\text{Y}}$ There is a linearly independent set $\mathcal{C} = \{\boldsymbol{c_1}, \ldots, \boldsymbol{c_n}\}$ so that $\mathcal{C} \subset \text{span}(\mathcal{B})$.

$\boxed{\text{N}}$ There is a linearly independent set $\mathcal{C} = \{\boldsymbol{c_1}, \ldots, \boldsymbol{c_n}\}$ so that $\mathcal{B} \subset \text{span}(\mathcal{C})$.

3 Computational (80 points; 20 points each)

Show all computations so that you make clear what your thought processes are.

Problem 3.1 (20 pts)**.** Consider A given by

$$A = \begin{bmatrix} -2 & 4 & -4 & -4 & 4 \\ -8 & 16 & -15 & -18 & 18 \\ -8 & 16 & -11 & -26 & 27 \\ -4 & 8 & -8 & -8 & 4 \end{bmatrix}$$

Find a basis for each of NS(A), CS(A), and RS(A).

$$\text{rref}(A) = \begin{bmatrix} 1 & -2 & 0 & 6 & 0 \\ 0 & 0 & 1 & -2 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

From this we know:

$$\begin{aligned} \text{CS}(A) &= \text{span}\{(-2,-8,-8,-4), (-4,-15,-11,-8), (4,18,27,4)\} \\ \text{RS}(A) &= \text{span}\{(1,-2,0,6,0), (0,0,1,-2,0), (0,0,0,0,1)\} \end{aligned}$$

Note: RS(A) is not the span of the first three rows of A.

To find a basis for NS(A) we are looking for solutions to $A\boldsymbol{x} = \mathbf{0}$. First, we have back-substitution: x_2 and x_4 are free, let $x_2 = s$ and $x_4 = t$, then

$$\boxed{x_5 = 0}$$
$$\boxed{x_4 = t}$$
$$x_3 - 2t = 0 \to \boxed{x_3 = 2t}$$
$$\boxed{x_2 = s}$$
$$x_1 - 2s + 6t = 0 \to \boxed{x_1 = 2s - 6t}$$

Any vector $\boldsymbol{x}$ satisfying, $A\boldsymbol{x} = \mathbf{0}$ can be written as:

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} 2s - 6t \\ s \\ 2t \\ t \\ 0 \end{bmatrix} = s\begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + t\begin{bmatrix} -6 \\ 0 \\ 2 \\ 1 \\ 0 \end{bmatrix}$$

So $\{(2,1,0,0,0), (-6,0,2,1,0)\}$ is a basis for NS(A), that is,

$$\text{NS}(A) = \text{span}\{(2,1,0,0,0), (-6,0,2,1,0)\}$$

Problem 3.2 (20 pts)**.** Consider $L : P_3 \to P_6$ given by $L(p(x)) = q(x)p(x)$ where $q(x) = a_0 + a_1x + a_2x^2 + a_3x^3$.

a) (8 points) Show that L is a linear map.

b) (8 points) Give the matrix $[L]$ where the standard basis is used for both P_3 and P_6. Just to be definite, the standard basis for P_k is $\mathcal{E} = \{1, x, x^2, \ldots, x^k\}$.

c) (4 points) With $p(x) = b_0 + b_1x + b_2x^2 + b_3x^3$ compute $[q(x)p(x)]$ using $[L]$ and $[p]$.

To see that L is linear we just note that for $p(x), h(x) \in P_3$ and $c, d \in \mathbb{R}$ we have

$$\begin{aligned} L(c \cdot p(x) + d \cdot h(x)) &= q(x)(c \cdot p(x) + d \cdot h(x)) \\ &= c \cdot (q(x)p(x)) + d \cdot (q(x)(h(x)) \\ &= c \cdot L(p(x)) + d \cdot L(h(x)) \end{aligned}$$

so linearity of L is shown.

To compute $[L]$, first notice that $\dim(P_3) = 4$ and $\dim(P_6) = 7$ so $[L]$ is 7×4 (a good sanity check on our solution).

$$[L] = \big[[q(x)]\,[q(x)\cdot x]\,[q(x)\cdot x^2]\,[q(x)\cdot x^3]\big] = \begin{bmatrix} a_0 & 0 & 0 & 0 \\ a_1 & a_0 & 0 & 0 \\ a_2 & a_1 & a_0 & 0 \\ a_3 & a_2 & a_1 & a_0 \\ 0 & a_3 & a_2 & a_1 \\ 0 & 0 & a_3 & a_2 \\ 0 & 0 & 0 & a_3 \end{bmatrix}$$

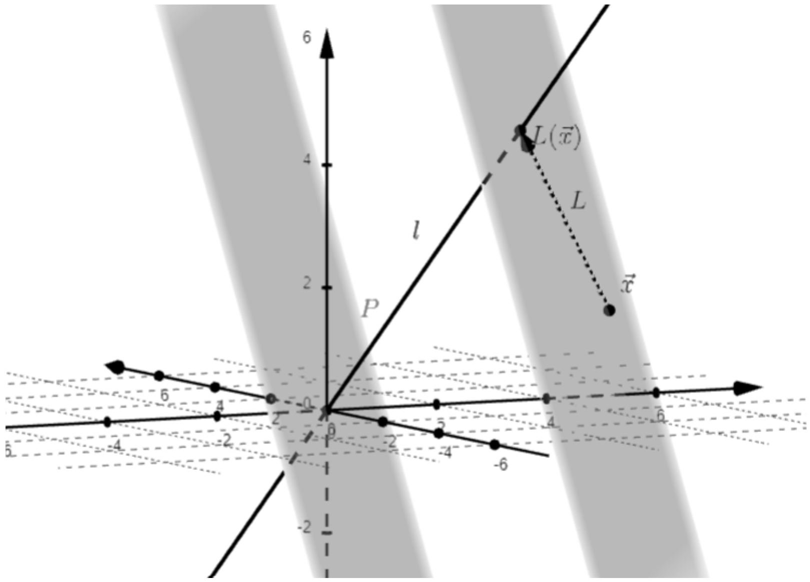
Finally,

$$[q(x)p(x)] = [L(p(x))] = [L][p(x)] = \begin{bmatrix} a_0 & 0 & 0 & 0 \\ a_1 & a_0 & 0 & 0 \\ a_2 & a_1 & a_0 & 0 \\ a_3 & a_2 & a_1 & a_0 \\ 0 & a_3 & a_2 & a_1 \\ 0 & 0 & a_3 & a_2 \\ 0 & 0 & 0 & a_3 \end{bmatrix} \begin{bmatrix} b_0 \\ b_1 \\ b_2 \\ b_3 \end{bmatrix} = \begin{bmatrix} a_0b_0 \\ a_1b_0 + a_0b_1 \\ a_2b_0 + a_1b_1 + a_0b_2 \\ a_3b_0 + a_2b_1 + a_1b_2 + a_0b_3 \\ a_3b_1 + a_2b_2 + a_1b_3 \\ a_3b_2 + a_2b_3 \\ a_3b_3 \end{bmatrix}$$

Pretty, no?

Note $[q(x)p(x)]_i = \sum_{l=0}^{i} a_l b_{i-l} = \sum_{l+k=i} a_l b_k$, which you might know from studying polynomials in an algebra class.

Problem 3.3 (20 pts)**.** Consider the map $L : \mathbb{R}^3 \to \mathbb{R}^3$ that projects a point in $\mathbb{R}^3$ onto the line $l : \left\{ t \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix} \;\middle|\; t \in \mathbb{R} \right\}$ along the plane $P : 3x - 2y + z = 0$.



Geogebra

Find a basis $\mathcal{B}$ for $\mathbb{R}^3$ so that $[L]_\mathcal{B}$ is simple. Give both $\mathcal{B}$ and $[L]_\mathcal{B}$. (**9 points for this.**) Next, find $[L]$ using some change of basis and the $[L]_\mathcal{B}$ that you found. (**9 points for this part.**) Finally, find $L((4,-4,0))$. (**2 points**)

Note: Points on P are mapped to $\mathbf{0}$, that is, $\ker(L) = P$, while points in l are fixed.

There are many choices for $\mathcal{B}$, I will use the two vectors $\boldsymbol{v}_1 = (1,1,-1)$ and $\boldsymbol{v}_2 = (0,1,2)$ in P and $\boldsymbol{v}_3 = (1,-1,2)$ in L. So

$$\mathcal{B} = \{\boldsymbol{v_1}, \boldsymbol{v_2}, \boldsymbol{v_3}\} = \left\{ \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix} \right\}$$

and

$$[L]_\mathcal{B} = \big[[L(\boldsymbol{v}_1)]_\mathcal{B} \, [L(\boldsymbol{v}_2)]_\mathcal{B} \, [L(\boldsymbol{v}_3)]_\mathcal{B} \big] = \big[[\mathbf{0}]_\mathcal{B} \, [\mathbf{0}]_\mathcal{B} \, [\boldsymbol{v}_3]_\mathcal{B} \big] = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

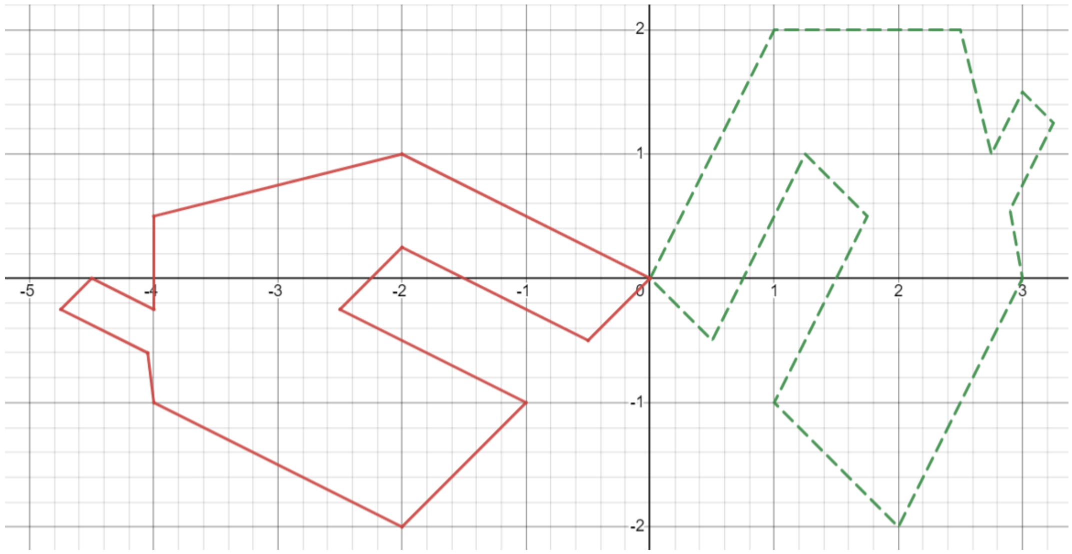
Finding $[L]$ is now trivial.

$$[L] = B[L]_\mathcal{B}B^{-1} = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & -1 \\ -1 & 2 & 2 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & -1 \\ -1 & 2 & 2 \end{bmatrix}^{-1} = \frac{1}{7} \begin{bmatrix} 3 & -2 & 1 \\ -3 & 2 & -1 \\ 6 & -4 & 2 \end{bmatrix}$$

and

$$L\left(\begin{bmatrix} 4 \\ -4 \\ 0 \end{bmatrix} \right) = \frac{20}{7} \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}$$

Problem 3.4 (20 pts)**.** The green (dashed) house has been transformed to the red (solid) house by a linear transformation $L : \mathbb{R}^2 \to \mathbb{R}^2$.



Desmos

Find $[L]$ by first choosing basis $\mathcal{G}$ (for the green house) and basis $\mathcal{R}$ (for the red house) and find $[L]_{\mathcal{G},\mathcal{R}}$, then use change of basis matrices to find $[L]$.

There are many options here; I will take

$$\mathcal{G} = \{\boldsymbol{v}_1, \boldsymbol{v}_2\} = \left\{ \begin{bmatrix} 2 \\ -2 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \end{bmatrix} \right\}$$

and

$$\mathcal{R} = \{\boldsymbol{u}_1, \boldsymbol{u}_2\} = \left\{ \begin{bmatrix} -2 \\ -2 \end{bmatrix}, \begin{bmatrix} -2 \\ 1 \end{bmatrix} \right\}$$

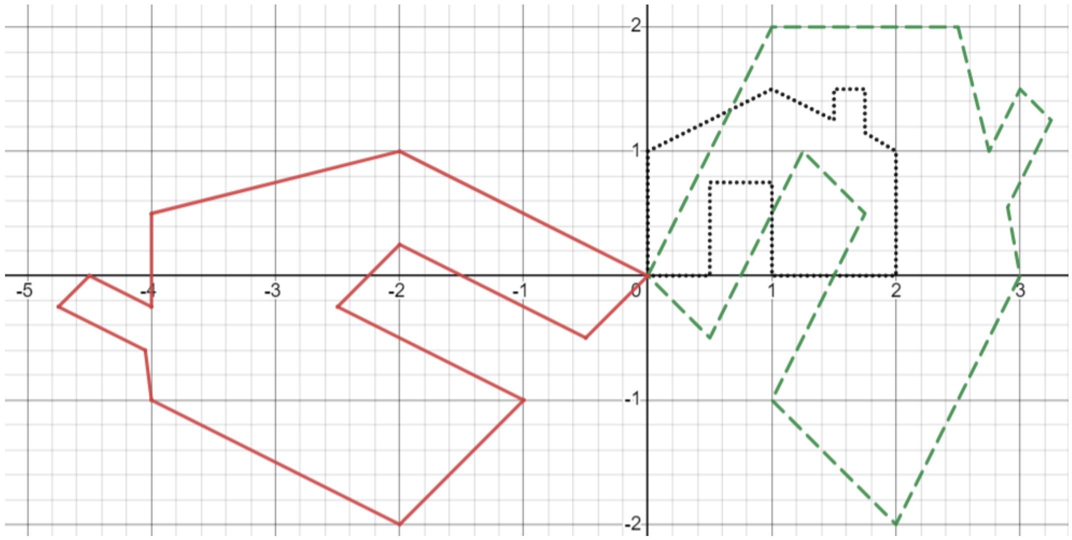
Then

$$[L]_{\mathcal{G},\mathcal{R}} = \left[[L(\boldsymbol{v}_1)]_{\mathcal{R}} \, [L(\boldsymbol{v}_2)]_{\mathcal{R}} \right] = \left[[\boldsymbol{u}_1]_{\mathcal{R}} \, [\boldsymbol{u}_2]_{\mathcal{R}} \right] = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

and

$$[L] = R[L]_{\mathcal{G},\mathcal{R}} G^{-1} = RG^{-1} = \begin{bmatrix} -2 & -2 \\ -2 & 1 \end{bmatrix} \begin{bmatrix} 2 & 1 \\ -2 & 2 \end{bmatrix}^{-1} = \frac{1}{3} \begin{bmatrix} 4 & -1 \\ -1 & 2 \end{bmatrix}$$

Option 2: There is another option here, although you need to explain what you are doing for full credit. You could view this as showing two transformations here:



Desmos

You might consider the transformation from the "black" (dotted) house to the "green" (dashed) house as L_G and then the transformation from the "black" house to the "red" (solid) house L_G. Then with respect to just the standard basis. We have

$$[L_G] = \begin{bmatrix} 1 & 1 \\ -1 & 2 \end{bmatrix} \quad \text{and} \quad [L_R] = \begin{bmatrix} -1 & -2 \\ -1 & 1 \end{bmatrix}$$

Then the map from the green house to the red house with respect to the standard basis would be

$$[L] = [L_R \circ L_G^{-1}] = [L_R][L_G]^{-1} = \tfrac{1}{3}\begin{bmatrix} 4 & -1 \\ -1 & 2 \end{bmatrix}$$

4 Theory and Proofs (30 points; 10 points each)

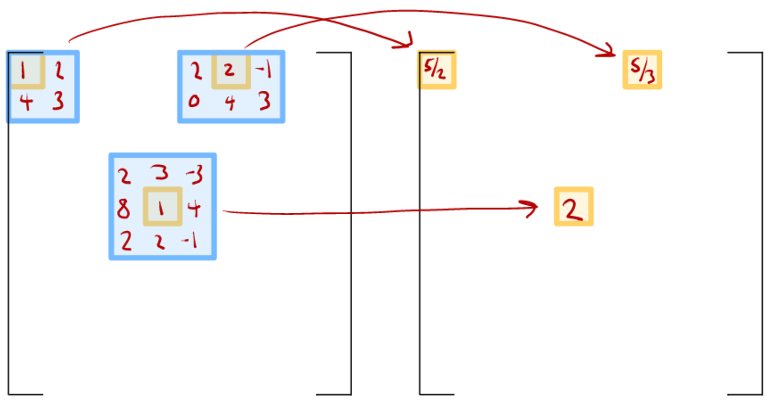
Choose three of the four options. If you try more than three, I will grade only the first three, not the best three. You must decide what should be graded. These will be due on 3/9 in class. Make sure your work is complete and clear. Explain your work; a proof is not just a collection of math symbols, it is an explanation of why something is true.

This part is take-home. You should complete this work on your own without consulting websites, friends, the Math Center, etc.

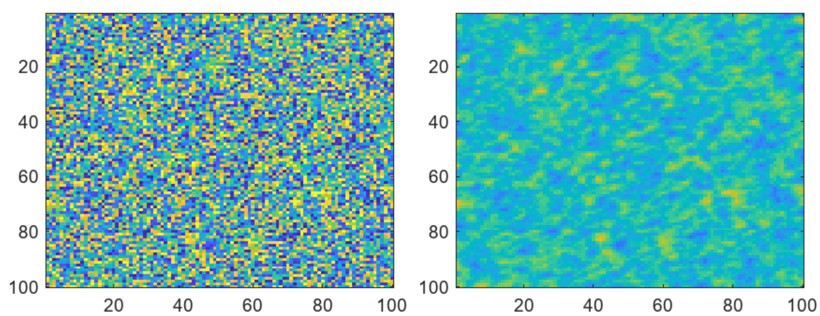
Problem 4.1 (10 points)**.** Let V be a vector space with $\dim(V) = n$ and $U \subseteq V$ a subspace with $\dim(U) = k$, show that there is a subspace $W \subseteq V$ with $\dim(W) = l$ so that $l + k = n$ and $V = U \oplus W$.

Problem 4.2 (10 points)**.** Show that if $L : V \to W$ is linear and $\ker(L) = \{\mathbf{0}\}$, then for any linearly independent set $\{\boldsymbol{v_1, \ldots, v_k}\}$ from V, $\{L(\boldsymbol{v}_1), \ldots, L(\boldsymbol{v}_k)\}$ is independent.

Problem 4.3 (10 points)**.** Consider the following operation. Given an $m \times n$ matrix A, $S(A)$ will be the $m \times n$ matrix where each entry has been replaced by the average of the entry with its neighbors. S for "smear" (often called "blur").



Example applied to random noise (numeric value represented by color):



MATLAB code

Main Question (7 points): Show that $S : \mathbb{R}^{m\times n} \to \mathbb{R}^{m\times n}$ is linear.

Thought Question (2 points): What do you think would happen if you repeatedly applied smearing? That is, consider $A_1 = A$, $A_2 = S(A)$, $A_3 = S(S(A)) = S^2(A)$, etc. What do you think $S^k(A)$ would look like for large k (as an image)? What would the limiting value be?

How might you verify your conjecture? (1 points): Since S is linear

$$A_k = S^k(A) = \sum_{i,j} A_{i,j} S^k(E_{i,j})$$

How might you use this to verify your conjecture?

Problem 4.4 (10 points)**.** Suppose A is a $n \times n$ matrix and $A^{m+1} = A^m$ for some m, then already $A^{n+1} = A^n$.

Hint: This is similar, but different, to one from the exam you had for practice. You can use the same ideas. Note that $A^{m+1} = A^m$ can be written as $A^m(A - I) = O$, do remember that $AB = O$ does not mean that $A = O$ or $B = O$.