Name: ______________________________ Exam 1 - MAT345

Part I: True/False

Each problem is points for a total of 50 points. (7 points each and one free point.)

Problem 1 (50 points; 5 points each)**.** Decide if each of the following is true or false.

(a) True If A and B commute, then so do A^T and B^T.

$$A^T B^T = (BA)^T = (AB)^T = B^T A^T$$

(b) True For any invertible matrix A, $(A^T)^{-1} = (A^{-1})^T$.

$$I = I^T = (AA^{-1})^T = (A^{-1})^T A^T$$

Similarly, $A^T(A^{-1})^T = I$ and so $(A^T)^{-1} = (A^{-1})^T$.

(c) False For all $n \times n$ matrices A and B, $\det(A + B) = \det(A) + \det(B)$

$$\det(I + (-I)) = \det(O) = 0 \neq \det(I) + \det(-I) = 1 + 1 = 2$$

(d) False For all $n \times n$ matrices A, $\det(cA) = c \cdot \det(A)$

$\det(cA) = c^n \det(A)$ when A is $n \times n$.

(e) True For all $n \times n$ matrices A and B, $\det(AB) = \det(BA)$.

$$\det(AB) = \det(A)\det(B) = \det(B)\det(A) = \det(BA)$$

(f) False If

$$\text{rref}(A) = \begin{bmatrix} 1 & 2 & 0 & 1 \\ 0 & 0 & 1 & 2 \end{bmatrix},$$

then the solutions of $A\boldsymbol{x} = \mathbf{0}$ are given by

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = s \begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + t \begin{bmatrix} 1 \\ 0 \\ 2 \\ 1 \end{bmatrix}$$

Apply back substitution to the system $\text{rref}(A)\boldsymbol{x} = \mathbf{0}$. We have here x_2 and x_4 are free so laet $x_2 = s$ and $x_4 = t$, then we have

$$\boxed{x_4 = t}$$
$$x_3 + 2t = 0 \rightarrow \boxed{x_3 = -2t}$$
$$\boxed{x_2 = s}$$
$$x_1 + 2s + t = 0 \rightarrow \boxed{x_1 = -2s - t}$$

so

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} -2s - t \\ s \\ -2t \\ t \end{bmatrix} = s \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + t \begin{bmatrix} -1 \\ 0 \\ -2 \\ 1 \end{bmatrix}$$

(g) False If A is an $m \times n$ matrix, then in the expression $A\boldsymbol{x} = \boldsymbol{b}$, $\boldsymbol{x}$ represents m variables, or a vector in $\mathbb{R}^m$, and $\boldsymbol{b}$ is a vector in $\mathbb{R}^n$.

$\boldsymbol{x}$ must be $n \times 1$ for $A\boldsymbol{x}$ to make sense, so $\boldsymbol{x} \in \mathbb{R}^n$, not $\mathbb{R}^m$. similarly, $\boldsymbol{b} \in \mathbb{R}^m$, since $A\boldsymbol{x}$ is $m \times 1$.

Part II: Computational (80 points)

Show all computations so that you make clear what your thought processes are.

Problem 2 (20 pts)**.** Let

$$A = \begin{bmatrix} 4 & 5 & -1 & -3 \\ 2 & -4 & 3 & 0 \\ -1 & 0 & 3 & 0 \end{bmatrix}; \qquad B = \begin{bmatrix} 2 & 0 & 0 \\ 3 & 2 & 5 \\ 3 & -3 & -1 \\ -2 & 0 & 1 \end{bmatrix}$$

1. Express the third row of AB as a linear combination of rows of B.

$$(-1)\begin{bmatrix} 2 & 0 & 0 \end{bmatrix} + (3)\begin{bmatrix} 3 & -3 & -1 \end{bmatrix} = \begin{bmatrix} 7 & -9 & -3 \end{bmatrix}$$

2. Express the second column of AB as a linear combination of the columns of A.

$$(2)\begin{bmatrix} 5 \\ -4 \\ 0 \end{bmatrix} + (-3)\begin{bmatrix} -1 \\ 3 \\ 3 \end{bmatrix} = \begin{bmatrix} 13 \\ -17 \\ -9 \end{bmatrix}$$

3. Express $(AB)_{1,2}$ as a product of a row of A and a column of B.

$$(AB)_{1,2} = \begin{bmatrix} 4 & 5 & -1 & -3 \end{bmatrix} \begin{bmatrix} 0 \\ 2 \\ -3 \\ 0 \end{bmatrix} = 13$$

Problem 3 (20 pts)**.** Solve $A\boldsymbol{x} = \boldsymbol{b}$ where

$$A = \begin{bmatrix} 1 & 2 & -4 & 3 & 2 \\ 2 & 4 & -7 & 4 & 5 \\ -3 & -6 & 14 & -13 & -3 \end{bmatrix} \qquad \boldsymbol{b} = \begin{bmatrix} 4 \\ 8 \\ -11 \end{bmatrix}$$

1. (8 points) Use row operations (show all work and indicate operations) to reduce A to an echelon form. (This should work out very nicely - no fractions required..)
2. (7 points) Use back-substitution to solve the resulting system. Make sure to indicate which variables are free.
3. (5 points) Write your solution as a linear combination of vectors.

Gauss-Jordan elimination to get echelon form:

$$\left[\begin{array}{ccccc|c} 1 & 2 & -4 & 3 & 2 & 4 \\ 2 & 4 & -7 & 4 & 5 & 8 \\ -3 & -6 & 14 & -13 & -3 & -11 \end{array}\right] \xrightarrow[R_3+3R_1\to R_3]{R_2-2R_1\to R_2} \left[\begin{array}{ccccc|c} 1 & 2 & -4 & 3 & 2 & 4 \\ 0 & 0 & 1 & -2 & 1 & 0 \\ 0 & 0 & 2 & -4 & 3 & 1 \end{array}\right]$$

$$\xrightarrow[R_2-2R_1\to R_2]{} \left[\begin{array}{ccccc|c} 1 & 2 & -4 & 3 & 2 & 4 \\ 0 & 0 & 1 & -2 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 \end{array}\right]$$

Back-substitution: x_2 and x_4 are free, let $x_2 = s$ and $x_4 = t$, then

$$\boxed{x_5 = 1}$$
$$\boxed{x_4 = t}$$
$$x_3 - 2t + 1 = 0 \to \boxed{x_3 = 2t - 1}$$
$$\boxed{x_2 = s}$$
$$x_1 + 2s - 4(2t-1) + 3t + 2 = 4 \to \boxed{x_1 = -2s + 5t - 2}$$

Solution as a linear combination of vectors:

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} -2s + 5t - 6 \\ s \\ 2t - 1 \\ t \\ 1 \end{bmatrix} = \begin{bmatrix} -2 \\ 0 \\ -1 \\ 0 \\ 1 \end{bmatrix} + s \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + t \begin{bmatrix} 5 \\ 0 \\ 2 \\ 1 \\ 0 \end{bmatrix}$$

Problem 4 (20 pts)**.** Use Cramer's rule to find x_3, where

$$\begin{bmatrix} 1 & 2 & -4 & 3 \\ 0 & -6 & -4 & 0 \\ 0 & 3 & 0 & 0 \\ 2 & 5 & -3 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \\ 1 \\ 3 \end{bmatrix}$$

Note: These determinants should work out very nicely if you chose how you expand carefully.

Let

$$A = \begin{bmatrix} 1 & 2 & -4 & 3 \\ 0 & -6 & -4 & 0 \\ 0 & 3 & 0 & 0 \\ 2 & 5 & -3 & 4 \end{bmatrix} \text{ and } B = \begin{bmatrix} 1 & 2 & 2 & 3 \\ 0 & -6 & 0 & 0 \\ 0 & 3 & 1 & 0 \\ 2 & 5 & 3 & 4 \end{bmatrix}$$

so that B is obtained by replacing the 3^{rd} column of A by $\begin{bmatrix} 2 \\ 0 \\ 1 \\ 3 \end{bmatrix}$. Then

$$x_3 = \frac{\det(B)}{\det(A)}$$

where, by expanding along the 3^{rd} row of A we have

$$\det(A) = (-3)\det\begin{bmatrix} 1 & -4 & 3 \\ 0 & -4 & 0 \\ 2 & -3 & 4 \end{bmatrix} = (-3)(-4)\det\begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix} = (-3)(-4)(4-6) = -24$$

and by expanding along 2^{nd} row of B

$$\det(B) = (-6)\det\begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 0 \\ 2 & 3 & 4 \end{bmatrix} = (-6)(1)\det\begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix} = (-6)(1)(4-6) = 12$$

So

$$x_3 = \frac{12}{-24} = -\frac{1}{2}$$

Problem 5 (20 pts)**.** Write A in the form LU where L is lower-triangular with 1's on the diagonal, and U is upper-triangular for

$$A = \begin{bmatrix} 2 & -2 & 1 \\ 4 & -1 & 1 \\ -6 & 12 & -4 \end{bmatrix}$$

$$\begin{bmatrix} 2 & -2 & 1 \\ 4 & -1 & 1 \\ -6 & 12 & -4 \end{bmatrix} \xrightarrow[\substack{R_2-(\mathbf{2})R_1\to R_2 \\ R_3-(\mathbf{-3})R_1\to R_3}]{} \begin{bmatrix} 2 & -2 & 1 \\ 0 & 3 & -1 \\ 0 & 6 & -1 \end{bmatrix} \qquad L = \begin{bmatrix} 1 & 0 & 0 \\ \mathbf{2} & 1 & 0 \\ \mathbf{-3} & 0 & 1 \end{bmatrix}$$

$$\xrightarrow[R_3-(\mathbf{2})R_2\to R_3]{} \begin{bmatrix} 2 & -2 & 1 \\ 0 & 3 & -1 \\ 0 & 0 & 1 \end{bmatrix} = U \quad L = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ -3 & \mathbf{2} & 1 \end{bmatrix}$$

So

$$\begin{bmatrix} 2 & -2 & 1 \\ 4 & -1 & 1 \\ -6 & 12 & -4 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ -3 & 2 & 1 \end{bmatrix} \begin{bmatrix} 2 & -2 & 1 \\ 0 & 3 & -1 \\ 0 & 0 & 1 \end{bmatrix}$$

Part III: Theory and Proofs (60 points; 20 points each)

Choose three of the five options. If you try all five, I will grade the first three, not the best three. You must decide what should be graded.

Problem 6. Show that for any symmetric $n \times n$ matrices A and B that $AB + BA$ is symmetric.

$$(AB + BA)^T = (AB)^T + (BA)^T = B^T A^T + A^T B^T = BA + AB = AB + BA$$

Problem 7 (20 pts)**.** For A and B invertible $n \times n$ matrices, prove

$$((AB)^T)^{-1} = ((AB)^{-1})^T.$$

You may use the fact we have already discussed that for any invertible matrix A, $(A^T)^{-1} = (A^{-1})^T$.

$$((AB)^T)^{-1} = (B^T A^T)^{-1} = (A^T)^{-1}(B^T)^{-1} = (A^{-1})^T(B^{-1})^T = (B^{-1}A^{-1})^T = ((AB)^{-1})^T$$

Problem 8 (20 pts)**.** Let A be an $n \times m$ matrix, show that

$$A = O \text{ (the zero matrix)} \iff A\boldsymbol{x} = \mathbf{0} \text{ for all } \boldsymbol{x}$$

One direction is trivial. Clearly, if $A = O$, then $A\boldsymbol{x} = \mathbf{0}$ for all $\boldsymbol{x}$.

Argument 1: For the other direction recall that $A\boldsymbol{e}_i$ is the i^{th} column of A and since $A\boldsymbol{e}_i = \mathbf{0}$, the i^{th} column of A is just the 0-vector. But if all columns of A are all 0's, then A is all 0's, i.e., $A = O$.

Argument 2: Let $R = \text{rref}(A)$, we have $A\boldsymbol{x} = \mathbf{0} \iff R\boldsymbol{x} = \mathbf{0}$, so $R\boldsymbol{x} = \mathbf{0}$ for all $\boldsymbol{x} \in \mathbb{R}^m$. If $R = O$, then since R comes from A by row-operations, and hence A can be got from R by row operations, we know $A = O$. So it suffices to show that $R = O$.

Suppose $R \neq O$, then there is a non-zero column with a pivot in R. Say the i^{th} column is such and the i^{th} column is $\boldsymbol{e}_k$. Then $R\boldsymbol{e}_i = \boldsymbol{e}_k \neq \mathbf{0}$, which is a contradiction. (**Proof by contradiction.**)

Problem 9 (20 pts)**.** Let A be an $m \times n$ matrix, show that for any $\boldsymbol{x} \in \mathbb{R}^n$,

$$A\boldsymbol{x} = \mathbf{0} \iff A^T A\boldsymbol{x} = \mathbf{0}.$$

Hint: There are several ways to do this, but you might use that if $A^T A\boldsymbol{x} = \mathbf{0}$, then $\boldsymbol{x}^T A^T A\boldsymbol{x} = (A\boldsymbol{x})^T(A\boldsymbol{x}) = \mathbf{0}$. When can $\boldsymbol{y}^T\boldsymbol{y} = \mathbf{0}$?

One direction is trivial. If $A\boldsymbol{x} = \mathbf{0}$, then clearly, $A^T A\boldsymbol{x} = \mathbf{0}$.

So assume $A^T A\boldsymbol{x} = \mathbf{0}$, then $(\boldsymbol{x}^T A^T)(A\boldsymbol{x}) = (A\boldsymbol{x})^T(A\boldsymbol{x}) = \mathbf{0}$. But since $\boldsymbol{y}^T\boldsymbol{y} = \sum_{i=1}^n y_i^2$ we see that

$$\boldsymbol{y}^T\boldsymbol{y} = \mathbf{0} \iff \boldsymbol{y} = \mathbf{0}$$

Thus $A\boldsymbol{x} = \mathbf{0}$ as desired.

Problem 10 (20 pts)**.** Let A be an 3×5 matrix given by rows as:

$$A = \begin{bmatrix} \boldsymbol{a}_1 & \boldsymbol{a}_2 \cdots & \boldsymbol{a}_5 \end{bmatrix}$$

Let

$$\text{rref}(A) = \begin{bmatrix} 1 & -2 & 0 & 3 & 0 \\ 0 & 0 & 1 & -4 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

Explain how we know that $\boldsymbol{a}_2 = -2\boldsymbol{a}_1$ and $\boldsymbol{a}_4 = 3\boldsymbol{a}_1 - 4\boldsymbol{a}_3$ and hence that

$$A = \begin{bmatrix} \boldsymbol{a}_1 & \boldsymbol{a}_3 & \boldsymbol{a}_5 \end{bmatrix} \begin{bmatrix} 1 & -2 & 0 & 3 & 0 \\ 0 & 0 & 1 & -4 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

Hint: What is the solution to $A\boldsymbol{x} = \mathbf{0}$? This can be read off of rref(A).

Recall

$$A\boldsymbol{x} = \mathbf{0} \iff \text{rref}(A)\boldsymbol{x} = \mathbf{0}$$

We have that x_2 and x_4 are free variables and from rref(A) we read off the solutions to $A\boldsymbol{x} = \mathbf{0}$ as

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} 2s - 3t \\ s \\ 4t \\ t \\ 0 \end{bmatrix}$$

Thus

$$A\boldsymbol{x} = \mathbf{0} \iff (2s - 3t)\boldsymbol{a}_1 + s\boldsymbol{a}_2 + (4t)\boldsymbol{a}_3 + t\boldsymbol{a}_4 + 0\mathbf{5} = \mathbf{0}$$

Setting $s = 1$ and $t = 0$ gives

$$2\boldsymbol{a}_1 + \boldsymbol{a}_2 = \mathbf{0} \text{ so } \boldsymbol{a}_2 = -2\boldsymbol{a}_1$$

Setting $s = 0$ and $t = 1$ gives

$$-3\boldsymbol{a}_1 + 4\boldsymbol{a}_3 + \boldsymbol{a}_4 = \mathbf{0}, \text{ so } \boldsymbol{a}_4 = 3\boldsymbol{a}_1 - 4\boldsymbol{a}_3$$

These give

$$\begin{bmatrix} \boldsymbol{a}_1 & \boldsymbol{a}_3 & \boldsymbol{a}_5 \end{bmatrix} \begin{bmatrix} 1 & -2 & 0 & 3 & 0 \\ 0 & 0 & 1 & -4 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} \boldsymbol{a}_1 & -2\boldsymbol{a}_1 & \boldsymbol{a}_3 & 3\boldsymbol{a}_1 - 4\boldsymbol{a}_3 & \boldsymbol{a}_5 \end{bmatrix} = A$$