

Math 571 - Homework 5

Richard Ketchersid

Notation: For $f : X \rightarrow Y$ and $E \subseteq X$ set $f(E) = \{f(e) \mid e \in E\}$, this is called the *image of E under f* .

Problem 5.1 (R:4:2*). Let $f : X \rightarrow Y$ be continuous. Let $E \subseteq X$, show that $f(\text{Cl}(E)) \subseteq \text{Cl}(f(E))$. By example, show that this containment can be proper, that is $f(\text{Cl}(E)) \not\subseteq \text{Cl}(f(E))$ can hold.

Definition Let $f : E \subseteq X \rightarrow Y$, the graph of f is the set $\text{Graph}(f) = \{(x, f(x)) \mid x \in E\} \subseteq X \times Y$.

Problem 5.2. Let $f : E \subseteq X \rightarrow Y$ be continuous where Y is Hausdorff, show that $\text{Graph}(f)$ is closed in $E \times Y$.

Problem 5.3 (R:4:6). Suppose $f : E \subseteq X \rightarrow Y$ and E is compact. Suppose further that X and Y are Hausdorff (or metric if you prefer). Show that f is continuous on E iff $\text{Graph}(f)$ is compact.

Hint: You may use the fact that if K and H are compact, then $K \times H$ is compact and that if K is compact and $C \subseteq K$ is closed, then C is compact. (Both of these are in notes and book.)

Problem 5.4. Let $f : E \subseteq X \rightarrow Y$ where both X and Y are metric spaces with Y complete. suppose f is uniformly continuous on E , show that there is a unique continuous extension $\hat{f} : \text{Cl}(E) \rightarrow Y$. Moreover, \hat{f} remains uniformly continuous.

Definition: A set $E \subseteq X$ has the *Bolzano-Weierstrass property* iff every sequence in E has a convergent subsequence.

Problem 5.5 (R:4:8*). Let $f : E \subseteq X \rightarrow Y$ be uniformly continuous on E where E has the Bolzano-Weierstrass property and Y is complete. Show that f is bounded on E , that is, $f(E)$ is bounded in Y .

Problem 5.6 (R:4:19). Show that if $f : \mathbb{R} \rightarrow \mathbb{R}$ satisfies the intermediate value theorem and $f^{-1}(r) = \{x \mid f(x) = r\}$ is closed for $r \in \mathbb{Q}$, then f is continuous. (See the text for a hint. \mathbb{Q} here could be replaced by any dense set.)