Quiz 1

Problem 1 (15 points)**.** Solve the following systems simultaneously:

$$\begin{aligned} x_1 + 2x_2 &= 1 \\ 3x_1 + 4x_2 &= 0 \end{aligned} \qquad\qquad \begin{aligned} x_1 + 2x_2 &= 0 \\ 3x_1 + 4x_2 &= 1 \end{aligned}$$

Do this by forming

$$\left[\begin{array}{cc|cc} 1 & 2 & 1 & 0 \\ 3 & 4 & 0 & 1 \end{array}\right] = [\,A \mid I\,]$$

where A is the coefficient matrix. Use row operations to reduce A to reduced row echelon form Show all the steps. You end up with

$$\left[\begin{array}{cc|cc} 1 & 0 & b_{11} & b_{12} \\ 0 & 1 & b_{21} & b_{22} \end{array}\right] = [\,I \mid B\,]$$

The vectors $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} b_{11} \\ b_{21} \end{bmatrix}$ and $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} b_{12} \\ b_{22} \end{bmatrix}$ are solutions to the initial systems. Verify this by showing $AB = I$. Also show that $BA = I$.

$$\begin{aligned} \left[\begin{array}{cc|cc} 1 & 2 & 1 & 0 \\ 3 & 4 & 0 & 1 \end{array}\right] &\underset{r_2 - 3r_1 \to r_2}{\longrightarrow} \left[\begin{array}{cc|cc} 1 & 2 & 1 & 0 \\ 0 & -2 & -3 & 1 \end{array}\right] \\ &\underset{r_1 + r_2 \to r_1}{\longrightarrow} \left[\begin{array}{cc|cc} 1 & 0 & -2 & 1 \\ 0 & -2 & -3 & 1 \end{array}\right] \\ &\underset{(-1/2) r_2 \to r_2}{\longrightarrow} \left[\begin{array}{cc|cc} 1 & 0 & -2 & 1 \\ 0 & 1 & 3/2 & -1/2 \end{array}\right] \end{aligned}$$

So $B = \begin{bmatrix} -2 & 1 \\ 3/2 & -1/2 \end{bmatrix}$. The columns of B are solutions to the initial equations so

$$A \begin{bmatrix} -2 \\ 3/2 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \quad \text{and} \quad A \begin{bmatrix} 1 \\ -1/2 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

So clearly B was constructed so that $AB = I$. It is easy to verify, by multiplying the two matrices, that $BA = I$.

Problem 2 (20 points; 5 points each)**.** Consider a new operation on matrices

$$A \otimes B \overset{\text{df}}{=} AB - BA.$$

Decide if each of the following are true or false and provide a small proof or counterexample in each case.

(a) ______ $\otimes$ is commutative. For all $A, B \in \mathbb{R}^{n\times n}$, $A \otimes B = B \otimes A$.

This is false, in fact $\oplus$ is *anti-commutative*

$$A \oplus B = AB - BA = -(BA - AB) = -(B \oplus A)$$

(b) ______ $\otimes$ is associative. For all $A, B, C \in R^{n\times n}$, $(A \otimes B) \otimes C = A \otimes (B \otimes C)$.

This is false

$$\begin{aligned}(A \oplus B) \oplus C &= (A \oplus B)C - C(A \oplus B)\\ &= (AB - BA)C - C(AB - BA)\\ &= ABC - BAC - CAB + CBA\end{aligned}$$

while

$$\begin{aligned}(A \oplus B) \oplus C &= A(B \oplus C) - (B \oplus C)A\\ &= A(BC - CB) - (BC - CB)A\\ &= ABC - ACB - BCA + CBA\end{aligned}$$

This shows non-associativity already, let's go a bit further.

$$\begin{aligned}(A \oplus B) \oplus C - A \oplus (B \oplus C) &= -BAC + ACB - CAB + BCA\\ &= (ACB - CAB) - (BAC - BCA)\\ &= (AC - CA)B - B(AC - CA)\\ &= (A \oplus C)B - B(A \oplus C)\\ &= (A \oplus C) \oplus B\end{aligned}$$

So we have

$$\begin{aligned}0 &= (A \oplus B) \oplus C - A \oplus (B \oplus C) - (A \oplus C) \oplus B\\ &= (A \oplus B) \oplus C + (B \oplus C) \oplus A + (C \oplus A) \oplus B\end{aligned}$$

This is the Jacobi identity for the commutator.

(c) ______ For all $A \in \mathbb{R}^{n\times n}$, there is an $\otimes$-identity, E so that $E \otimes A = A \otimes E = A$.

This is false and can easily be seen from the anti-commutativity. Suppose E had the property that $E \oplus A - A \oplus E = A$ for all A, then

$$A = E \oplus A = -(A \oplus E) = -A$$

So $A = -A$ for all A, which is clearly false.

There are some important points to note:

- You can not prove this by example, for most choices of E and A the equality won't hold, but it does sometimes hold, for example,

$$E = \begin{bmatrix} 0 & 1/2 \\ 0 & 1 \end{bmatrix} \qquad A = \begin{bmatrix} -2 & 1 \\ -4 & 2 \end{bmatrix}$$

  You can easily check that $E \oplus A = EA - AE = A$, of course, $A \oplus E = -A$, not A.

- I is the identity for matrix multiplication, that is $AI = IA = A$ and O (zero matrix) is the identity for matrix addition, that is, $A + O = O + A = A$. Neither of these has anything to do with the question asked. Here you are looking for an identity for $\oplus$, not $+$ or $\cdot$.

(d) ______ For all $A, B \in \mathbb{R}^{n\times n}$, $(A \otimes B)^T = B^T \otimes A^T$.

This is true.

$$(A \oplus B)^T = (AB - BA)^T = (AB)^T - (BA)^T = B^T A^T - A^T B^T = B^T \oplus A^T$$

Notation: "$\stackrel{\text{df}}{=}$" means "*is defined to be*" and $\mathbb{R}^{n\times n}$ is the set/space of all $n \times n$ matrices with real entries.