Math 571 - Homework 2 (05.22)

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Problem 1 (R:2:2*)**.** A complex number γ is **algebraic** iff γ is a root to a polynomial with integer coefficients. Prove that there are complex numbers that are not algebraic.

The following are two fun facts:

Let $\mathbb{A} \subset \mathbb{C}$ be the set of algebraic numbers.

1. $\mathbb{A}$ is a field.
2. $\mathbb{A}$ is algebraically closed, that is, if α is a root of a polynomial in $\mathbb{A}[x]$, then $\alpha \in \mathbb{A}$. So in the definition of algebraic numbers you can use any ring of coefficients R, $\mathbb{Z} \subseteq R \subseteq \mathbb{A}$.

Here is a write-up of the proofs, you should have the background to read this, but it is not an easy read.

The intent here was for you to do a counting argument. There are only countably many polynomials with integer coefficients and each has only finitely many roots, hence there are only countably many algebraic numbers.

Definition 1. A set $S \subseteq X$ is **discrete** iff every point in S is isolated.

Problem 2 (R:2:5*)**.** Prove the following for discrete $S \subset \mathbb{R}$:

a) S is countable.

For each $x \in S$ we find integer n_x so that $N_{\frac{1}{n_x}}(x) \cap (S - \{x\}) = \emptyset$. We can find $q_x \in \mathbb{Q}$ and integer $m_x > n_x$ so that $x \in N_{1m_x}(q_x)$ and $N_{1m_x}(q_x) = \{x\}$. So $x \mapsto (m_x, q_x)$ is injective, hence S is countable.

b) There is set $A \subset \mathbb{R}$ so that $\mathrm{Lim}(A) = \mathrm{Cl}(S)$.

Use the open cover from part (a) inside each $N_{m_x}(q_x)$ pick a sequence (y^x_j) so that $\lim_{i\to\infty} y^x_j = x$ and $\{y^x_i\}$ has no limit points other than x. Let $A = \bigcup_{x\in S}\{y^x_i\}$.

c) Give an example to show that we can't find a set A such that $\mathrm{Lim}(A) = S$.

Clearly, $\mathrm{Lim}(S) \subseteq \mathrm{Lim}(A)$, since $\mathrm{Lim}(\mathrm{Lim}(A)) \subseteq \mathrm{Lim}(A)$. So just take S with $\mathrm{Lim}(S) - S \neq \emptyset$.

For the following use the definition that I provided for $\mathrm{Cl}(E)$, namely, $\mathrm{Cl}(E) = \bigcap\{F \mid F \text{ is closed and } E \subseteq F\}$.

Problem 3 (R:2:6)**.** For X a metric space and $E \subseteq X$, show that

a) $\mathrm{Cl}(E) = E \cup \mathrm{Lim}(E)$.

First show $\mathrm{Cl}(E) \subseteq E \cup \mathrm{Lim}(E)$. Let $x \in \mathrm{Cl}(E)$. If $x \in E$ we are done. So assume $x \notin E$. Let O be open with $x \in O$. Towards a contradiction, suppose $o \cap E = \emptyset$, then $O^c \supset E$ and is closed and so $\mathrm{Cl}(E) \subseteq O^c$. But then $x \notin \mathrm{Cl}(E)$. A contradiction. So $O \cap E \neq \emptyset$ for all open nbhds of x, thus $x \in \mathrm{Lim}(E)$.

Next we see $E \cup \mathrm{Lim}(E) \subseteq \mathrm{Cl}(E)$. Let $F \supset E$ be closed, it suffices to see that $\mathrm{Lim}(E) \subseteq F$. Let $x \in \mathrm{Lim}(E)$. If $x \notin F$, then F^c is an open nbhd of x with $F^c \cap E = \emptyset$, so $x \notin \mathrm{Lim}(E)$, a contradiction.

b) $\mathrm{Lim}(E)$ is closed.

From (a), $\mathrm{Cl}(\mathrm{Lim}(E)) = \mathrm{Lim}(E) \cup \mathrm{Lim}(\mathrm{Lim}(E))$. So if we show $\mathrm{Lim}(\lim(E)) \subseteq \mathrm{Lim}(E)$ we are done. Let $x \in \mathrm{Lim}(\mathrm{Lim}(E))$. Let O be an open nbhd of x, then $O \cap \mathrm{Lim}(E) \neq \emptyset$. Let U be an open nbhd of $y \in O \cap \mathrm{Lim}(E)$ so that $U \subset O$. Then $U \cap E \neq \emptyset$, so $O \cap E \neq \emptyset$ and this is what was required to see that $x \in \mathrm{Lim}(E)$.

Either show or give a counterexample to $\mathrm{Lim}(E) = \mathrm{Lim}(\mathrm{Cl}(E))$.

$\mathrm{Lim}(E) \subseteq \mathrm{Lim}(\mathrm{Cl}(E))$ simply because $A \subseteq B \implies \mathrm{Lim}(A) \subseteq \mathrm{Lim}(B)$. Let $x \in \mathrm{Lim}(\mathrm{Cl}(E))$ and let O be an open nbhd of x, then $O \cap (E \cup \mathrm{Lim}(E)) \neq \emptyset$. If $O \cap E \neq \emptyset$ then we are done. Else $O \cap \mathrm{Lim}(E) \neq \emptyset$. In this case we argue as we did above. Let $y \in \mathrm{Lim}(E) \cap O$. Let $U \subset O$ be nbhd of y, then $U \cap E \neq \emptyset$, so $O \cap E \neq \emptyset$.

Problem 4 (R:2:9*)**.** Let X be a metric space, or just any topological space. Are the following true for all $E \subseteq X$?

a) $\mathrm{Int}(E)^c = \mathrm{Cl}(E^c)$.

Let's try to prove this. there are, as usual, two things to prove here.

$\mathbf{Int}(\boldsymbol{E})^{\boldsymbol{c}} \subseteq \mathbf{Cl}(\boldsymbol{E}^{\boldsymbol{c}})$: Let $x \in \mathrm{Int}(E)^c$, so $x \notin \mathrm{Int}(E)$. This means every neighborhood of x contains points in E^c. This means $x \in \mathrm{Cl}(E^c)$.

$\mathbf{Cl}(\boldsymbol{E}^{\boldsymbol{c}}) \subseteq \mathbf{Int}(\boldsymbol{E})^{\boldsymbol{c}}$: Let $x \in \mathrm{Cl}(E^c)$ so every nbhd of x meets E^c, so $x \notin \mathrm{Int}(E)$, thus $x \in \mathrm{Int}(E)^c$.

b) $\mathrm{Cl}(E) = \mathrm{Int}(E^c)^c$?

This is true and we can just apply (a) here. $\mathrm{Cl}(E) = \mathrm{Cl}((E^c)^c) = \mathrm{Int}(E^c)^c$. This clearly also gives $\mathrm{Cl}(E)^c = \mathrm{Int}(E^c)$.

c) $\mathrm{Cl}(E) = \mathrm{Cl}(\mathrm{Int}(E))$?

This is false. Just take $X = \mathbb{R}$ and $E = \mathbb{Q}$, then $\mathrm{Cl}(\mathbb{Q}) = \mathbb{R}$ but $\mathrm{Cl}(\mathrm{Int}(E)) = \mathrm{Cl}(\emptyset) = \emptyset$.

d) $\mathrm{Int}(E) = \mathrm{Int}(\mathrm{Cl}(E))$

This is just as the previous, same counterexample shows this to be false. $\mathrm{Int}(\mathbb{Q}) = \emptyset \neq \mathrm{Int}(\mathrm{Cl}(\mathbb{Q})) = \mathrm{Int}(\mathbb{R}) = \mathbb{R}$.

For each either prove the statement true or give a counterexample. For a counterexample you must provide both X and E.

An open set, E, is called a **regular open set** iff $E = \mathrm{Int}(\mathrm{Cl}(E))$. Similarly, a closed set, E, is **regular closed set** if $E = \mathrm{Cl}(\mathrm{Int}(E))$.

Let O be any open set, then ∂O is nowhere dense, that is, for all open U, there is $U' \subseteq U$ so that $\emptyset \neq U'$ and $U' \cap \partial O = \emptyset$. Let U be open and suppose $U \cap \partial O \neq \emptyset$. Let $U' = O \cap U$. Clearly, $\emptyset \neq U'$ and $U' \cap \partial O = \emptyset$, since $O \cap \partial O = \emptyset$.

Any non-empty closed nowhere-dense set, N, fails to be regular closed, and so N^c fails to be regular open. For example, the circle $S^1 \subset \mathbb{R}^2$ is the boundary of the open unit disk and thus is closed nowhere-dense, hence not regular-closed. Correspondingly, $G = \mathbb{R}^2 - S^1$ is open, but not regular open.

Definition 2. A metric space X is **separable** iff there is a countable $E \subseteq X$ with E dense in X.

Problem 5 (R:2:22)**.** Show the $\mathbb{R}^k$ is separable.

It is easy to see that $\mathbb{Q}^k$ is dense in $\mathbb{R}^k$. One way is the following, use basic open "boxes" of the form $\prod_{i=1}^k (a_i, b_i)$ for the basic open sets, instead of open balls. The fact that $\mathbb{Q} \cap (a_i, b_i) \neq \emptyset$ immediately yields that $\mathbb{Q}^k \cap \prod_{i=1}^k (a_i, b_i) \neq \emptyset$.

Definition 3. A set $\mathcal{B}$ of open sets is called a **base** for X iff for all $x \in X$ and open set U with $x \in U$, there is $V \in \mathcal{B}$ so that $x \in V \subset U$.

Problem 6 (R:2:23*)**.** Prove that a metric space is separable iff it has a countable base.

If X is separable, let S be a countable dense set. Consider $N_{\frac{1}{i}}(s)$ for $s \in S$. Let $x \in X$ and O be an open nbhd of x. Take $N_\delta(x) \subseteq O$ and $s \in S$ with $d(s, x) < \delta/4$. Then $x \in N_{\frac{1}{m}}(s) \subseteq O$ with $\frac{1}{m} < \frac{\delta}{4}$. So the sets $N_{\frac{1}{i}}(s)$ do form a countable base.

If $\{O_i \mid i \in \mathbb{N}\}$ is a countable base, then just take $s_i \in O_i$ for all i, then $S = \{s_i \mid i \in \mathbb{N}\}$ is dense.

Problem 7 (R:2:24)**.** Prove that if X is a metric space and every infinite sequence has a limit point, then X is separable. (See the hint in the text.)

For each integer $i > 0$ construct a sequence $\{x_j\}_{j=0}^{k_i}$ so that $d(x_l^i, x_k^i) \geq \frac{1}{i}$. $k_i < \infty$ for all i since otherwise there would be an infinite sequence with no limit.

Let $S = \{x_j^i \mid j \leq k_i\}$. The claim is that this set is dense in X. Let $x \in X$ and $i > 0$ an integer. $S \cap N_{\frac{1}{i}}(x) \neq \emptyset$ by construction. This is all that is required.