Exam 2

To avoid any confusion, unless specified otherwise, vector spaces are complex vector spaces, inner-products are complex inner-products, and matrices are complex matrices. The standard inner product is $\langle \boldsymbol{u}, \boldsymbol{v} \rangle = \boldsymbol{v}^H \boldsymbol{u} = \sum_{i=1}^{n} \bar{\boldsymbol{v}}_i \boldsymbol{u}_i$. Keep in mind that $A^H = A^T$ for real matrices and symmetric = Hermitian for real matrices.

Part I: True/False

Each problem is points for a total of 50 points. (5 points each.)

You do not need to justify the answers here, this is unlike the quizzes.

1. ______ If U is unitary, then U is itself unitarily diagonalizable. This means there is a unitary V so that $U = VDV^H$ where D is diagonal.

2. ______ For any diagonalizable matrix A, one can use Gram-Schmidt to find an orthogonal basis consisting of eigenvectors.

3. ______ The collection of rank k $n \times n$ matrices is a subspace of $\mathbb{R}^{n \times n}$, for $k < n$.

4. ______ If A is unitary, then $|\lambda| = 1$ for all eigenvalues λ of A.

5. ______ If $p(t)$ is a polynomial and $\boldsymbol{v}$ is an eigenvector of A with associated eigenvalue λ, then $p(A)\boldsymbol{v} = p(\lambda)\boldsymbol{v}$.

6. ______ If A and B are both $n \times n$ and $\mathcal{B}$ is a basis for $\mathbb{C}^n$ consisting of eigenvectors for both A and B, then A and B commute.

7. ______ Any matrix A can be written as a weighted sum of rank 1 matrices..

8. ______ For all Hermitian matrices A, there is a matrix B so that $B^H B = A$.

9. ______ If A is an $m \times n$ matrix, then $\text{rng}(A) \oplus \text{NS}(A^T) = \mathbb{R}^m$

10. ______ If A is an invertible $n \times n$ matrix, then $ABA^{-1} = B$ for all $n \times n$ matrices B.

Part II: Computational (60 points)

P1. (15 points) Find B so that $B^2 = A$ where

$$A = \begin{bmatrix} 13 & -5 & 5 \\ -8 & 10 & -8 \\ -3 & -3 & 5 \end{bmatrix}$$

P2. (15 points) Find B so that $B^H B = A$ where

$$A = \begin{bmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{bmatrix}$$

P3. (15 points) Find the best rank 2 approximation to A from (2) with respect to $\|\cdot\|_F$.

P4. (15 points) Let

$$A = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix},$$

find the characteristic polynomial and all eigenvalues, both real and complex. Explain why A is diagonalizable and compute A^{2020}. Note, I do not ask you to diagonalize A.

Part III: Theory and Proofs (45 points; 15 points each)

Pick three of the following four options. If you try all four, I will grade the first three, so if this is not what you intend, then just do three, or at least make it clear which I should grade.

P1. Let $L : V \to V$ be a linear transformation and let $\mathcal{B} = \{\boldsymbol{v}_1, \ldots, \boldsymbol{v}_n\}$ be a basis. Show that $[L]_{\mathcal{B}}$ is upper triangular iff $L(\boldsymbol{v}_i) \in \operatorname{span}\{\boldsymbol{v}_1, \ldots, \boldsymbol{v}_i\}$ for all i.

P2. Let $L : \mathbb{R}^n \to \mathbb{R}^n$ be a linear transformation with $L^2 = L$ and for all $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{R}^n$, $\langle \boldsymbol{x}, L(\boldsymbol{y}) \rangle = \langle L(\boldsymbol{x}), \boldsymbol{y} \rangle$. Let $U = \operatorname{rng}(L)$. (See this for information on inner products.)

(a) Show that $L(\boldsymbol{x})$ is the orthogonal projection of $\boldsymbol{x}$ onto U, that is, show that $\boldsymbol{x} - L(\boldsymbol{x}) \perp U$ for all $\boldsymbol{x} \in \mathbb{R}^n$.

(b) Use (a) to show that $\|\boldsymbol{x} - L(\boldsymbol{x})\|_2^2 = \min\{\|\boldsymbol{x} - L(\boldsymbol{y})\|_2^2 \mid \boldsymbol{y} \in \mathbb{R}^n\}$.

P3. Any quadratic $q(x_1,\ldots,x_n)$ in n variables $\boldsymbol{x} = x_1,\ldots,x_n$ can be written as

$$q(\boldsymbol{x}) = \boldsymbol{x}^T Q\boldsymbol{x} + P\boldsymbol{x} + c$$

where Q is $n \times n$ and symmetric, P is $1 \times n$, and $c \in \mathbb{R}$. This is trivial $Q_{ii} =$ the coefficient on x_i^2, $Q_{ij} = Q_{ji} = \frac{1}{2}($ the coefficient on $x_i x_j)$, while $P_{1i} =$ the coefficient on x_i, and c is the constant term.

Example: Consider $q(x_1,x_2,x_3) = 7x_1^2+10x_2^2+19x_3^2+28x_1x_2+8x_1x_3-20x_2x_3+2x_2-3x_2+x_3+5$. Then

$$q(\boldsymbol{x}) = \begin{bmatrix} x_1 & x_2 & x_3 \end{bmatrix} \begin{bmatrix} 7 & 14 & 4 \\ 14 & 10 & -10 \\ 4 & -10 & 19 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 2 & -3 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + 5 = \boldsymbol{x}^T Q\boldsymbol{x} + P\boldsymbol{x} + c$$

Explain how the Spectral Theorem can be used to show that there is an orthonormal basis $\mathcal{C} = \{\boldsymbol{u}_1,\ldots,\boldsymbol{u}_n\}$ so that if standard coordinates are replaced with coordinates relative to $\mathcal{C}$, i.e., $\boldsymbol{y} = [\boldsymbol{x}]_{\mathcal{C}}$, then $q(\boldsymbol{x}) = q'(\boldsymbol{y}) = \boldsymbol{y}^T D\boldsymbol{y} + P'\boldsymbol{y} + c$ where D is diagonal. Thus all *cross-terms*, terms of the form $y_i y_j$ for $i \neq j$, have been eliminated.

Use this to find $q'(\boldsymbol{y})$ for the example $q(\boldsymbol{x})$ above.

To save you some work: $Q = UDU^T$ where

$$U = \begin{bmatrix} -2/3 & -1/3 & 2/3 \\ 2/3 & -2/3 & 1/3 \\ 1/3 & 2/3 & 2/3 \end{bmatrix} \text{ and } D = \begin{bmatrix} -9 & & \\ & 27 & \\ & & 18 \end{bmatrix}$$

P4. Use the SVD to show that any square matrix A can be written as $A = UP$ where U is unitary and P is Hermitian.