Linear Algebra Problems
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Vector Fields

Practice Problem 1 from "Subspaces in $\mathbb{R}^n$" section of [DZ]

Let Y^+ be the set of all vectors in $\mathbb{R}^2$ whose y components are non-negative. Is Y^+ closed under vector addition?

Solution

Yes it is. Let us choose $x_1, x_2, y_1, y_2 \in \mathbb{R}$ with $y_1, y_2 \geq 0$. Then $\vec{v}_1 = (x_1, y_1)$ and $\vec{v}_2 = (x_2, y_2)$ are both in Y^+. Let us check their sum:

$$\vec{v}_1 + \vec{v}_2 = (x_1, y_1) + (x_2, y_2) = (x_1 + x_2, y_1 + y_2) \in Y^+$$

since the sum of two non-negative numbers is a non negative numbers:

$$y_1 \geq 0 \text{ and } y_2 \geq 0 \Rightarrow y_1 + y_2 \geq 0$$

Practice Problem 2 from "Subspaces in $\mathbb{R}^n$" section of [DZ]

Let Y^+ be the set of all vectors in $\mathcal{R}^2$ whose y components are non-negative. Is Y^+ closed under scalar multiplication?

Solution

No it is not. For example, let us pick a negative scalar $\alpha = -2$ and choose an element of Y^+. Say $\vec{v} = (-3, 7) \in Y^+$. Then:

$$\alpha \cdot \vec{v} = -2\,(-3, 7) = (6, -14) \notin Y^+$$

Exercise 9.1.2 on page 469 of [KT]

Suppose you have $\mathcal{R}^2$ and the addition $(+)$ operation is defined as follows:

$$(a, b) + (c, d) = (0, b + d)$$

Scalar multiplication is defined in the usual way.Is this a vector space? Explain why or why not.

Solution

This is not a vector space since there is not a unique additive inverse. For example, the inverse of (a, b) should be $(-a, -b)$, but any ordered pair of the form $(k, -b)$ where k is an arbitrary real number would also add up to zero.

Eigenvalues & Eigenvectors

Exercise 7.1.8 on page 360 of [KT]

Suppose the A is a 3×3matrix and the following information is available:

$$A \times \begin{bmatrix} 0 \\ -1 \\ -1 \end{bmatrix} = 2 \begin{bmatrix} 0 \\ -1 \\ -1 \end{bmatrix} ; A \times \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = 1 \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} ; A \times \begin{bmatrix} -3 \\ -5 \\ -4 \end{bmatrix} = -3 \begin{bmatrix} -3 \\ -5 \\ -4 \end{bmatrix}$$

Find

$$A \times \begin{bmatrix} 2 \\ -3 \\ 3 \end{bmatrix}.$$

Solution

The problem gives us the eigenvectors and eigenvalues of A. If we can express the last vector as a linear combination of the eigenvectors of A, then we can use such linear combination to obtain the required result.
A quick way to find the linear combination is to set up an augmented matrix and rref it:

$$\begin{bmatrix} 0 & 1 & -3 & 2 \\ -1 & 1 & -5 & -3 \\ -1 & 1 & -4 & 3 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 0 & 0 & -7 \\ 0 & 1 & 0 & 20 \\ 0 & 0 & 1 & 6 \end{bmatrix}$$

Therefore

$$A \begin{bmatrix} 2 \\ -3 \\ 3 \end{bmatrix} = A \left(-7 \begin{bmatrix} 0 \\ -1 \\ -1 \end{bmatrix} + 20 \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + 6 \begin{bmatrix} -3 \\ -5 \\ -4 \end{bmatrix} \right) = -14 \begin{bmatrix} 0 \\ -1 \\ -1 \end{bmatrix} + 20 \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} - 18 \begin{bmatrix} -3 \\ -5 \\ -4 \end{bmatrix}$$

which results in:

$$A \begin{bmatrix} 2 \\ -3 \\ 3 \end{bmatrix} = \begin{bmatrix} 74 \\ 124 \\ 106 \end{bmatrix}$$

Practice Problem 9 from "Describing eigenvalue and Eigenvectors" section of [DZ]

Can an eigenvalue have multiple eigenvectors associated with it? Can an eigenvector have multiple eigenvalues associated with it? Give an Example.

Solution

Yes it is possible if it is a repeated eigenvalue. An eigenvector can only have one eigenvalue associated with it.

NOTE: add to write a matrix example

Exercise 7.1.9 on page 360 of [KT] but add to state the characteristic polynomial

State the characteristic polynomial. Find the eigenvalues and eigenvectors of the following matrix:

$$\begin{bmatrix} -6 & -92 & 12 \\ 0 & 0 & 0 \\ -2 & -31 & 4 \end{bmatrix}$$

Solution

The characteristic polynomial is $\lambda^2(\lambda + 2) = 0$.
The matrix has the following eigenvalues $\lambda_1 = -2$ and $\lambda_{2,3} = 0$. The eigenvector corresponding to $\lambda_1 = -2$ is $v_1 = (3, 0, 1)$. There is only one eigenvector for the zero eigenvalue which is $(2, 0, 1)$.

Linear Transformations

Practice Problem 6 from Describing Eigenvalues and Eigenvectors section of [DZ]

Arguing geometrically, identify the linear transformation whose standard matrix has eigenvalues $\lambda_1 = 1$ and $\lambda_2 = 1$.

Solution

These are the options

1. Vertical Shear
2. Horizontal Shear
3. counterclockwise Rotation through a 90 degrees angle
4. Reflection about a line y=mx
5. Horizontal stretch
6. Vertical stretch
7. None of the above

Correct answer is “Reflection about a line y=mx”

Exercise 5.2.1 on page 277 of [KT]

Consider the following functions which map $\mathbb{R}^n$ to $\mathbb{R}^m$.

(a) *T multiplies the j-th component of $\vec{x}$ by a nonzero number b.*

(b) *T replaces the i-th component of $\vec{x}$ with b times the j-th component added to the i-th component.*

(c) *T switches the i-th and j-th component.*

Show that these functions are linear transformations and describe their matrices A such that $T(\vec{x}) = A\vec{x}$.

Solution

(a) The matrix of T is the elementary matrix which multiplies the $j - th$ diagonal entry of the identity matrix b

(b) The matrix of T is the elementary matrix which takes b times the $j-th$ row and adds to the $i-th$ row.

(c) The matrix of T is the elementary matrix which switches the $i-th$ and the $j-th$ rows where the two components are in the $i-th$ and $j-th$ positions.

Exercise 5.2.3 on page 278 of [KT]

Suppose that T is a linear transformation such that:

$$T\begin{bmatrix}1\\2\\-6\end{bmatrix}=\begin{bmatrix}5\\1\\3\end{bmatrix};\ T\begin{bmatrix}-1\\-1\\5\end{bmatrix}=\begin{bmatrix}1\\1\\5\end{bmatrix};\ T\begin{bmatrix}0\\-1\\2\end{bmatrix}=\begin{bmatrix}5\\3\\-2\end{bmatrix}.$$

Find the matrix of T. That is find A such that $T(\vec{x})=A\vec{x}$.

Solution

$$\begin{bmatrix}5&1&5\\1&1&3\\3&5&-2\end{bmatrix}\times\begin{bmatrix}3&2&1\\2&2&1\\4&1&1\end{bmatrix}=\begin{bmatrix}37&17&11\\17&7&5\\11&14&6\end{bmatrix}$$

Linear Combinations & Basis

Problem 3 from "Additional Exercises for Chapter 3: Big Ideas About Vectors" section of [DZ]

Express the vector $\vec{v}$ as a linear combination of $\vec{u}_1$ and $\vec{u}_2$, where:

$$\vec{v} = \begin{bmatrix} 4 \\ 4 \\ -3 \end{bmatrix}; \quad \vec{u}_1 = \begin{bmatrix} 3 \\ 1 \\ -1 \end{bmatrix}; \quad \vec{u}_2 = \begin{bmatrix} 2 \\ -2 \\ 1 \end{bmatrix};$$

Solution

Set up the appropriated augmented matrix from which you can find that 2 and -1 are the coefficients of the vectors $\vec{u}_1$ and $\vec{u}_2$, respectively:

$$\vec{v} = 2\vec{u}_1 - \vec{u}_2 = 2\begin{bmatrix} 3 \\ 1 \\ -1 \end{bmatrix} - 1\begin{bmatrix} 2 \\ -2 \\ 1 \end{bmatrix} = \begin{bmatrix} 4 \\ 4 \\ -3 \end{bmatrix}.$$

Problem 5 from "Additional Exercises for Chapter 3: Big Ideas About Vectors" section of [DZ]

Here are some vectors:

$$\begin{bmatrix} 1 \\ 1 \\ -2 \end{bmatrix}; \quad \begin{bmatrix} 1 \\ 2 \\ -2 \end{bmatrix}; \quad \begin{bmatrix} 2 \\ 7 \\ -4 \end{bmatrix}; \quad \begin{bmatrix} 5 \\ 7 \\ -10 \end{bmatrix}; \quad \begin{bmatrix} 12 \\ 17 \\ -24 \end{bmatrix}$$

Describe the span of these vectors as the span of as few vectors as possible.

Solution

Problem M10 from "Linear Combination" section of [BZ], page 81 in the print version of the book.

Can $\vec{w}$ be written as a linear combination of the other 4 vectors given below?

$$\vec{w} = \begin{bmatrix} 13 \\ 15 \\ 5 \\ -17 \\ 2 \\ 25 \end{bmatrix}; \quad \vec{u}_1 = \begin{bmatrix} 2 \\ 4 \\ -3 \\ 1 \\ 2 \\ 9 \end{bmatrix}; \quad \vec{u}_2 = \begin{bmatrix} 6 \\ 3 \\ 0 \\ -2 \\ 1 \\ 4 \end{bmatrix}; \quad \vec{u}_3 = \begin{bmatrix} -5 \\ 2 \\ 1 \\ 1 \\ -3 \\ 0 \end{bmatrix}; \quad \vec{u}_4 = \begin{bmatrix} 3 \\ 2 \\ -5 \\ 7 \\ 1 \\ 3 \end{bmatrix};$$

Can any vector in $\mathbb{R}^6$ be written as a linear combination of these 4 vectors?

Solution

No, it is not possible to find such linear combination. This also proves that it is not possible to write any element of $\mathbb{R}^6$ as a linear combination of the 4 given vectors. We should have expected that result since each element of $\mathbb{R}^6$ requires 6 pieces of "information" and therefore a minimum of 6 linearly independent vectors.

Applications

REFERENCES

BZ - Beezer, R. A. (2006). A first course in linear algebra. Department of Mathematics and Computer Science. University of Puget Sound.

DZ - Davis, A. and Zachlin, P. (2022). Linear Algebra: An Interactive Introduction. Ohio Department of Higher Education.

KT - Kuttler, K. (2012). Linear algebra: theory and applications. The Saylor Foundation.