Quiz 6

Question 1 (15 points; 3 points each)**.** Decide if each of the following are true or false and provide a justification or counterexample in each case. A justification could consist of a theorem from the text. All vector spaces are assumed to be finite-dimensional here. All vector spaces are now over $\mathbb{C}$ unless otherwise stated.

(a) ______ If all eigenvalues of A are 0, then $A = 0$.

(b) ______ All $n \times n$ matrices are diagonalizable.

(c) ______ If $p(x)$ is a polynomial and A is an $n \times n$ matrix, then

$$p(A) = \begin{bmatrix} p(a_{1,1}) & \cdots & p(a_{1,n}) \\ \vdots & \ddots & \vdots \\ p(a_{n,1}) & \cdots & p(a_{n,n}) \end{bmatrix}$$

(d) ______ If $p(x)$ is a polynomial and $A = S^{-1}DS$ where $D = \operatorname{diag}(d_1, \ldots, d_n)$, then

$$p(A) = S^{-1} \operatorname{diag}(p(d_1), \ldots, p(d_n))S$$

Here

$$\operatorname{diag}(d_1, \ldots, d_n) = \begin{bmatrix} d_1 & 0 & \cdots & 0 \\ 0 & d_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & d_n \end{bmatrix}$$

(e) ______ If A is upper-triangular, then the eigenvalues of A are exactly the diagonal elements of A.

Question 2 (5 points)**.** Let $A = \begin{bmatrix} 5/4 & -3/4 \\ -3/4 & 5/4 \end{bmatrix}$, write $A = U\Lambda U^{-1}$ where U is unitary, columns are orthonormal basis for $\mathbb{R}^2$ and $\Lambda = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}$ with $\lambda_1 > \lambda_2$.

Recall: $U^{-1} = U^T$ for unitary U.

Question 3 (5 points)**.** Suppose the matrix A from Question 2 is used to transform points in the plane iteratively. That is, given a point $\boldsymbol{v}$, consider the sequence $\boldsymbol{v_n} = A^n\boldsymbol{v}$. Letting $U = \begin{bmatrix}\boldsymbol{u_1} & \boldsymbol{u_2}\end{bmatrix}$ so that $\boldsymbol{u_i}$ is an eigenvector associated to λ_i and letting $\boldsymbol{v} = c_1\boldsymbol{u_1} + c_2\boldsymbol{u_2}$ what is a simple expressions for a_n and b_n so that $\boldsymbol{v_n} = A^n\boldsymbol{v} = a_n\boldsymbol{u_1} + b_n\boldsymbol{u_2}$.