Name: ________________________ Exam 2 - MAT345

1 True/False (50 points; 5 points each)

Recall: A and B are equivalent if there is a sequence of elementary row operations leading from A to B, or equivalently, $B = MA$ for some invertible matrix M. This is different from $A \sim B$ (A and B are *similar* which means $B = S^{-1}AS$ for some invertible S.

Problem 1.1. In class, you need only provide a T/F (make it clear!) As usual, you may earn back up to 50% of the lost points by supplying justifications afterward.

False The collection of of 3×4 echelon matrices is a subspace of $\mathbb{R}^{3\times 4}$.

Closure under scalar multiplication is ok, but it is easy to see that closure under addition fails.

True The set of $n \times n$ matrices with all diagonal elements being 0 is a subspace of $\mathbb{R}^{n\times n}$.

It is clear that if A and B both have 0 diagonals, then so does αA and $A + B$.

True Consider the map $L : \mathbb{R}^{m\times n} \to \mathbb{R}^m$ defined by $L(A)_i = \mathrm{ave}(A_{i,*})$, that is, the i^{th} entry of $L(A)$ is the average of the i^{th} row of A. L is a linear map.

This is easy to check directly, but here is a cute argument. Let $\mathbf{1} \in \mathbb{R}^n$ be the vector of n 1's. Then

$$L(A) = \tfrac{1}{n}A\mathbf{1},$$

and this, being simple matrix multiplication, is clearly linear:

$$L(\alpha A + \beta B) = \tfrac{1}{n}(\alpha A + \beta B)\mathbf{1} = \alpha\tfrac{1}{n}A\mathbf{1} + \beta\tfrac{1}{n}B\mathbf{1} = \alpha \cdot L(A) + \beta \cdot L(B)$$

False For all linear $L : V \to W$, if $\{\boldsymbol{v}_1, \ldots, \boldsymbol{v}_k\}$ is independent, then $\{L(\boldsymbol{v}_1), \ldots, L(\boldsymbol{v}_k)\}$ is independent.

This is trivially false. L could just be the $\mathbf{0}$ map, that is, $L(\boldsymbol{v}) = \mathbf{0}$ for all $\boldsymbol{v} \in V$.

True For all linear $L : V \to W$, if $\{L(\boldsymbol{v}_1), \ldots, L(\boldsymbol{v}_k)\}$ is independent, then $\{\boldsymbol{v}_1, \ldots, \boldsymbol{v}_k\}$ is independent.

Suppose $\sum \alpha_i \boldsymbol{v}_i = \mathbf{0}$, then $L(\sum \alpha_i \boldsymbol{v}_i) = \sum \alpha_i L(\boldsymbol{v}_i) = L(\mathbf{0}) = \mathbf{0}$. Since $\{L(\boldsymbol{v}_1), \ldots, L(\boldsymbol{v}_k)\}$ is independent it follows that $\alpha_i = 0$ for all i and hence that $\{\boldsymbol{v}_1, \ldots, \boldsymbol{v}_k\}$ is independent.

False There are subspaces $V_0 = P_3 \supsetneq V_1 \supsetneq V_2 \supsetneq V_3 \supsetneq V_4 \supsetneq V_5 = \{\mathbf{0}\}$ where each V_i is a proper subspace of V_{i-1}.

Since we know $\dim(V_0) = 4 > \dim(V_1) > \dim(V_2) > \dim(V_3) > \dim(V_4) > \dim(V_5) = 0$, which is impossible.

True Given any basis $\{\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3\}$, from $\mathbb{R}^4$ and any four matrices $M_1, M_2, M_3, M_4 \in \mathbb{R}^{2\times 3}$ there is a unique linear transformation $L : \mathbb{R}^4 \to \mathbb{R}^{2\times 3}$ where $L(\boldsymbol{v}_i) = M_i$.

Existence: Define $L\left(\sum_{i=1}^4 \alpha_i \boldsymbol{v}_i\right) = \sum_{i=1}^4 \alpha_i M_i$. This is a well-defined function $L : \mathbb{R}^4 \to \mathbb{R}^{2\times 3}$.

Showing that this is linear is just a computation: Let $\boldsymbol{v} = \sum_{i=1}^4 \alpha_i \boldsymbol{v}_i$ and $\boldsymbol{u} = \sum_{i=1}^4 \beta + i\boldsymbol{v}_i$, then

$$L(\gamma\boldsymbol{v} + \boldsymbol{u}) = L\left(\gamma\sum_{i=1}^4 \alpha_i\boldsymbol{v}_i + \sum_{i=1}^4 \beta_i\boldsymbol{v}_i\right) = L\left(\sum_{i=1}^4 (\gamma\alpha_i + \beta_i)\boldsymbol{v}_i\right)$$
$$= \sum_{i=1}^4 (\gamma\alpha_i + \beta_i)W_i = \gamma\sum_{i=1}^4 \alpha_i\boldsymbol{W}_i + \sum_{i=1}^4 \beta_i W_i = \gamma \cdot L(\boldsymbol{v}) + L(\boldsymbol{u})$$

Uniqueness: Suppose $L' : V \to W$ is linear and sends $\boldsymbol{v}_i$ to $\boldsymbol{W}_i$, then for $\boldsymbol{v} \in V$, $\boldsymbol{v} = \sum_{i=1}^4 \alpha_i\boldsymbol{v}_i$ and $L'(\boldsymbol{v}) = L'\left(\sum_{i=1}^4 \alpha_i\boldsymbol{v}_i\right) = \sum_{i=1}^4 \alpha_i M_i = L(\boldsymbol{v})$ and thus $L = L'$.

True Suppose $L : P_5 \to \mathbb{R}^4$ is linear and onto, that is, $\mathrm{Img}(L) = \mathbb{R}^4$. Then $\dim(\ker(L)) = 2$.

Recall P_5 is the space of polynomials of degree ≤ 5.

$\dim(P_5) = 6$ and so $\dim(\ker(L)) + \dim(\mathrm{Img}(L)) = \dim(\ker(L)) + 4 = 6$.

True Let

$$\mathcal{B} = \left\{ \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ 2 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} \right\}$$

be a basis for $\mathbb{R}^3$. Then for $\boldsymbol{v} = \begin{bmatrix} 3 \\ 3 \\ 4 \end{bmatrix}$

$$[\boldsymbol{v}]_{\mathcal{B}} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} 3 \\ 3 \\ 4 \end{bmatrix} = (1)\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} + (1)\begin{bmatrix} 2 \\ 0 \\ 2 \end{bmatrix} + (1)\begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}$$

False $L : \mathbb{R}^{3\times 3} \to \mathbb{R}^3$ is given by $L(A) = A\boldsymbol{b}$ for a $\boldsymbol{b} \in \mathbb{R}^3$. If $\mathcal{B}$ is a basis for $\mathbb{R}^{3\times 3}$, then $[L]_{\mathcal{B}} = \boldsymbol{b}$.

$[L]_{\mathcal{B}}$ acts on representations of matrices wrt $\mathcal{B}$, it is a 9×9 matrix, not a 3×3 matrix.

2 Multiple Choice (30 points; 10 points each)

Each correct box counts for two points.

Problem 2.1 (10 points)**.** Which of the following are equivalent to "A is ***equivalent*** to B"? Mark 'Y' if equivalent and 'N' if not.

$\boxed{\text{Y}}$ B results from a series of row operations from A.

$\boxed{\text{N}}$ $B = AM$ for some invertible matrix M.

$\boxed{\text{Y}}$ $B = MA$ for some invertible matrix M.

$\boxed{\text{N}}$ $\text{CS}(A) = \text{CS}(B)$.

$\boxed{\text{Y}}$ $\text{RS}(A) = \text{RS}(B)$.

Problem 2.2 (10 points)**.** Which of the following are equivalent to A is invertible for an $n \times n$ matrix A. Mark 'Y' if equivalent and 'N' if not.

$\boxed{\text{Y}}$ A is equivalent to I.

$\boxed{\text{N}}$ $\dim(\text{RS}(A)) = \dim(\text{CS}(A))$.

$\boxed{\text{Y}}$ $\text{NS}(A) = \{\mathbf{0}\}$.

$\boxed{\text{Y}}$ $A\boldsymbol{x} = \boldsymbol{b}$ has at least one solution for all $\boldsymbol{b} \in \mathbb{R}^n$.

$\boxed{\text{Y}}$ $A\boldsymbol{x} = \boldsymbol{b}$ has a unique solution for some $\boldsymbol{b}$.

Problem 2.3. Which of the following implies that $\mathcal{B} = \{\boldsymbol{v_1}, \ldots, \boldsymbol{v_n}\}$ is linearly independent. Mark 'Y' if the property implies $\mathcal{B}$ is independent, 'N' otherwise.

$\boxed{\text{N}}$ For every $\boldsymbol{v}$ in V, $\boldsymbol{v}$ can be written as a linear combination of vectors in $\mathcal{B}$, i.e., there is $\alpha_i \in \mathbb{R}$ so that $\boldsymbol{v} = \sum_{i=1}^n \alpha_i \boldsymbol{b}_i$.

$\boxed{\text{Y}}$ If $\sum_{i=1}^n \alpha_i \boldsymbol{b}_i = \sum_{i=1}^n \beta_i \boldsymbol{b}_i$, then $\alpha_i = \beta_i$ for all i.

$\boxed{\text{Y}}$ $\boldsymbol{b}_i \notin \text{span}(\mathcal{B} - \{\boldsymbol{b}_i\})$, that is, $\boldsymbol{b}_i$ is not a linear combination of the other vectors in $\mathcal{B}$.

$\boxed{\text{Y}}$ There is a linearly independent set $\mathcal{C} = \{\boldsymbol{c_1}, \ldots, \boldsymbol{c_n}\}$ so that $\mathcal{C} \subset \text{span}(\mathcal{B})$.

$\boxed{\text{N}}$ There is a linearly independent set $\mathcal{C} = \{\boldsymbol{c_1}, \ldots, \boldsymbol{c_n}\}$ so that $\mathcal{B} \subset \text{span}(\mathcal{C})$.

3 Computational (80 points; 20 points each)

Show all computations so that you make clear what your thought processes are.

Problem 3.1 (20 pts). Consider A given by

$$A = \begin{bmatrix} -2 & 4 & -4 & -4 & 4 \\ -8 & 16 & -15 & -18 & 18 \\ -8 & 16 & -11 & -26 & 27 \\ -4 & 8 & -8 & -8 & 4 \end{bmatrix}$$

Find a basis for each of NS(A), CS(A), and RS(A).

$$\mathrm{rref}(A) = \begin{bmatrix} 1 & -2 & 0 & 6 & 0 \\ 0 & 0 & 1 & -2 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

From this we know:

$$\begin{aligned} \mathrm{CS}(A) &= \mathrm{span}\{(-2,-8,-8,-4), (-4,-15,-11,-8), (4,18,27,4)\} \\ \mathrm{RS}(A) &= \mathrm{span}\{(1,-2,0,6,0), (0,0,1,-2,0), (0,0,0,0,1)\} \end{aligned}$$

Note: RS(A) is not the span of the first three rows of A.

To find a basis for NS(A) we are looking for solutions to $A\boldsymbol{x} = \mathbf{0}$. First, we have back-substitution: x_2 and x_4 are free, let $x_2 = s$ and $x_4 = t$, then

$$\boxed{x_5 = 0}$$
$$\boxed{x_4 = t}$$
$$x_3 - 2t = 0 \to \boxed{x_3 = 2t}$$
$$\boxed{x_2 = s}$$
$$x_1 - 2s + 6t = 0 \to \boxed{x_1 = 2s - 6t}$$

Any vector $\boldsymbol{x}$ satisfying, $A\boldsymbol{x} = \mathbf{0}$ can be written as:

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} 2s - 6t \\ s \\ 2t \\ t \\ 0 \end{bmatrix} = s \begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + t \begin{bmatrix} -6 \\ 0 \\ 2 \\ 1 \\ 0 \end{bmatrix}$$

So $\{(2,1,0,0,0), (-6,0,2,1,0)\}$ is a basis for NS(A), that is,

$$\mathrm{NS}(A) = \mathrm{span}\{(2,1,0,0,0), (-6,0,2,1,0)\}$$

Problem 3.2 (20 pts)**.** Consider $L : P_3 \to P_6$ given by $L(p(x)) = q(x)p(x)$ where $q(x) = a_0 + a_1x + a_2x^2 + a_3x^3$.

a) (8 points) Show that L is a linear map.

b) (8 points) Give the matrix $[L]$ where the standard basis is used for both P_3 and P_6. Just to be definite, the standard basis for P_k is $\mathcal{E} = \{1, x, x^2, \dots, x^k\}$.

c) (4 points) With $p(x) = b_0 + b_1x + b_2x^2 + b_3x^3$ compute $[q(x)p(x)]$ using $[L]$ and $[p]$.

To see that L is linear we just note that for $p(x), h(x) \in P_3$ and $c, d \in \mathbb{R}$ we have

$$\begin{aligned} L(c \cdot p(x) + d \cdot h(x)) &= q(x)(c \cdot p(x) + d \cdot h(x)) \\ &= c \cdot (q(x)p(x)) + d \cdot (q(x)(h(x)) \\ &= c \cdot L(p(x)) + d \cdot L(h(x)) \end{aligned}$$

so linearity of L is shown.

To compute $[L]$, first notice that $\dim(P_3) = 4$ and $\dim(P_6) = 7$ so $[L]$ is 7×4 (a good sanity check on our solution).

$$[L] = \big[[q(x)]\,[q(x) \cdot x]\,[q(x) \cdot x^2]\,[q(x) \cdot x^3]\big] = \begin{bmatrix} a_0 & 0 & 0 & 0 \\ a_1 & a_0 & 0 & 0 \\ a_2 & a_1 & a_0 & 0 \\ a_3 & a_2 & a_1 & a_0 \\ 0 & a_3 & a_2 & a_1 \\ 0 & 0 & a_3 & a_2 \\ 0 & 0 & 0 & a_3 \end{bmatrix}$$

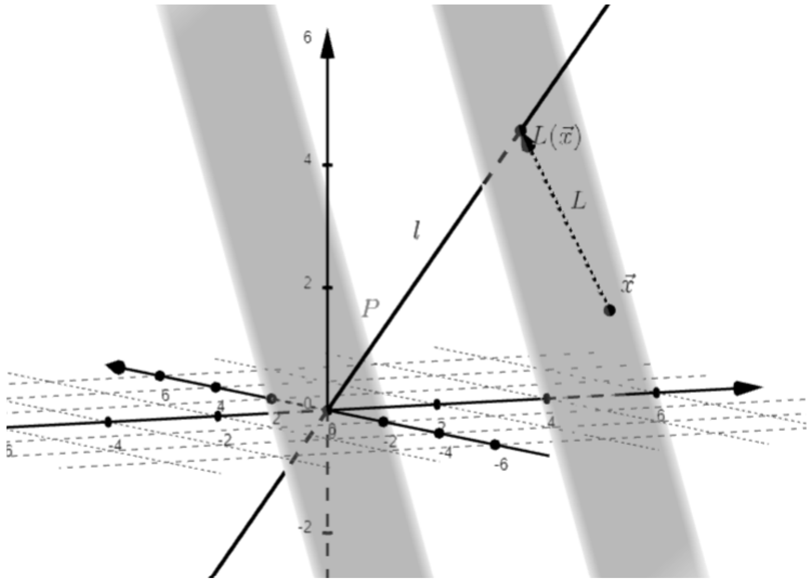
Finally,

$$[q(x)p(x)] = [L(p(x))] = [L][p(x)] = \begin{bmatrix} a_0 & 0 & 0 & 0 \\ a_1 & a_0 & 0 & 0 \\ a_2 & a_1 & a_0 & 0 \\ a_3 & a_2 & a_1 & a_0 \\ 0 & a_3 & a_2 & a_1 \\ 0 & 0 & a_3 & a_2 \\ 0 & 0 & 0 & a_3 \end{bmatrix} \begin{bmatrix} b_0 \\ b_1 \\ b_2 \\ b_3 \end{bmatrix} = \begin{bmatrix} a_0b_0 \\ a_1b_0 + a_0b_1 \\ a_2b_0 + a_1b_1 + a_0b_2 \\ a_3b_0 + a_2b_1 + a_1b_2 + a_0b_3 \\ a_3b_1 + a_2b_2 + a_1b_3 \\ a_3b_2 + a_2b_3 \\ a_3b_3 \end{bmatrix}$$

Pretty, no?

Note $[q(x)p(x)]_i = \sum_{l=0}^{i} a_l b_{i-l} = \sum_{l+k=i} a_l b_k$, which you might know from studying polynomials in an algebra class.

Problem 3.3 (20 pts)**.** Consider the map $L : \mathbb{R}^3 \to \mathbb{R}^3$ that projects a point in $\mathbb{R}^3$ onto the line $l : \left\{ t \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix} \;\middle|\; t \in \mathbb{R} \right\}$ along the plane $P : 3x - 2y + z = 0$.



Geogebra

Find a basis $\mathcal{B}$ for $\mathbb{R}^3$ so that $[L]_{\mathcal{B}}$ is simple. Give both $\mathcal{B}$ and $[L]_{\mathcal{B}}$. (**9 points for this.**) Next, find $[L]$ using some change of basis and the $[L]_{\mathcal{B}}$ that you found. (**9 points for this part.**) Finally, find $L((4,-4,0))$. (**2 points**)

Note: Points on P are mapped to $\mathbf{0}$, that is, $\ker(L) = P$, while points in l are fixed.

There are many choices for $\mathcal{B}$, I will use the two vectors $\boldsymbol{v}_1 = (1,1,-1)$ and $\boldsymbol{v}_2 = (0,1,2)$ in P and $\boldsymbol{v}_3 = (1,-1,2)$ in L. So

$$\mathcal{B} = \{\boldsymbol{v_1}, \boldsymbol{v_2}, \boldsymbol{v_3}\} = \left\{ \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix} \right\}$$

and

$$[L]_{\mathcal{B}} = \big[[L(\boldsymbol{v}_1)]_{\mathcal{B}} \, [L(\boldsymbol{v}_2)]_{\mathcal{B}} \, [L(\boldsymbol{v}_3)]_{\mathcal{B}} \big] = \big[[\mathbf{0}]_{\mathcal{B}} \, [\mathbf{0}]_{\mathcal{B}} \, [\boldsymbol{v}_3]_{\mathcal{B}} \big] = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

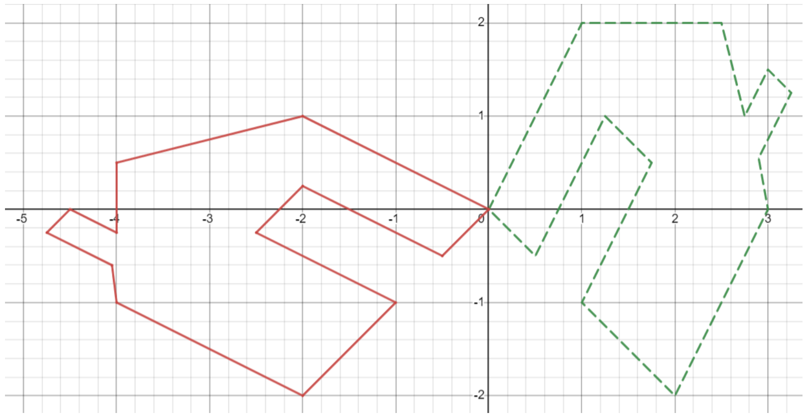
Finding $[L]$ is now trivial.

$$[L] = B[L]_{\mathcal{B}}B^{-1} = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & -1 \\ -1 & 2 & 2 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & -1 \\ -1 & 2 & 2 \end{bmatrix}^{-1} = \frac{1}{7} \begin{bmatrix} 3 & -2 & 1 \\ -3 & 2 & -1 \\ 6 & -4 & 2 \end{bmatrix}$$

and

$$L\left(\begin{bmatrix} 4 \\ -4 \\ 0 \end{bmatrix} \right) = \frac{20}{7} \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}$$

Problem 3.4 (20 pts)**.** The green (dashed) house has been transformed to the red (solid) house by a linear transformation $L : \mathbb{R}^2 \to \mathbb{R}^2$.



Desmos

Find $[L]$ by first choosing basis $\mathcal{G}$ (for the green house) and basis $\mathcal{R}$ (for the red house) and find $[L]_{\mathcal{G},\mathcal{R}}$, then use a change of basis matrices to find $[L]$.

There are many options here. In all cases, you might have chosen a different basis than I did, but the final matrix is the same.

Option 1: (Exactly as done in class!) Take

$$\mathcal{G} = \{\boldsymbol{v}_1, \boldsymbol{v}_2\} = \left\{ \begin{bmatrix} 2 \\ -2 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \end{bmatrix} \right\}$$

and

$$\mathcal{R} = \{\boldsymbol{u}_1, \boldsymbol{u}_2\} = \left\{ \begin{bmatrix} -2 \\ -2 \end{bmatrix}, \begin{bmatrix} -2 \\ 1 \end{bmatrix} \right\}$$

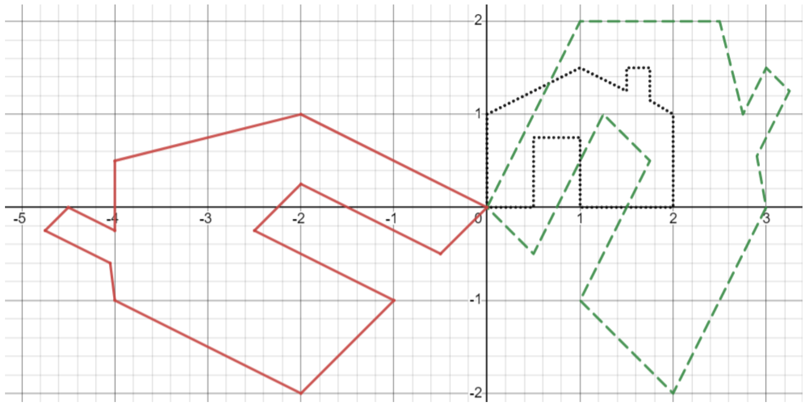
Then

$$[L]_{\mathcal{G},\mathcal{R}} = \big[[L(\boldsymbol{v}_1)]_{\mathcal{R}} \, [L(\boldsymbol{v}_2)]_{\mathcal{R}} \big] = \big[[\boldsymbol{u}_1]_{\mathcal{R}} \, [\boldsymbol{u}_2]_{\mathcal{R}} \big] = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

and

$$[L] = R[L]_{\mathcal{G},\mathcal{R}} G^{-1} = RG^{-1} = \begin{bmatrix} -2 & -2 \\ -2 & 1 \end{bmatrix} \begin{bmatrix} 2 & 1 \\ -2 & 2 \end{bmatrix}^{-1} = \tfrac{1}{3} \begin{bmatrix} 4 & -1 \\ -1 & 2 \end{bmatrix}$$

Option 2: There is another option here, although you need to explain what you are doing for full credit. You could view this as showing two transformations here:



Desmos

You might consider the transformation from the "black" (dotted) house to the "green" (dashed) house as L_G and then the transformation from the "black" house to the "red" (solid) house L_G. Then with respect to just the standard basis. We have

$$[L_G] = \begin{bmatrix} 1 & 1 \\ -1 & 2 \end{bmatrix} \quad \text{and} \quad [L_R] = \begin{bmatrix} -1 & -2 \\ -1 & 1 \end{bmatrix}$$

Then the map from the green house to the red house with respect to the standard basis would be

$$[L] = [L_R \circ L_G^{-1}] = [L_R][L_G]^{-1} = \tfrac{1}{3} \begin{bmatrix} 4 & -1 \\ -1 & 2 \end{bmatrix}$$

Option 3: Something in-between. You might have taken

$$\mathcal{G} = \left\{ \begin{bmatrix} 2 \\ -2 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \end{bmatrix} \right\}$$

and just computed

$$[L]_{\mathcal{B},E} = \big[[L(\boldsymbol{v}_1)]_{\mathcal{E}} \, [L(\boldsymbol{v}_2)]_{\mathcal{E}} \big] = \big[L(\boldsymbol{v}_1) \, L(\boldsymbol{v}_2) \big] = \begin{bmatrix} -2 & -2 \\ -2 & 1 \end{bmatrix}$$

Now then

$$[L] = [L]_{\mathcal{B},E} B^{-1} = \begin{bmatrix} -2 & -2 \\ -2 & 1 \end{bmatrix} \begin{bmatrix} 2 & 1 \\ -2 & 2 \end{bmatrix} = \tfrac{1}{3} \begin{bmatrix} 4 & -1 \\ -1 & 2 \end{bmatrix}$$

4 Theory and Proofs (30 points; 10 points each)

Choose three of the four options. If you try more than three, I will grade only the first three, not the best three. You must decide what should be graded. These will be due on 3/9 in class. Make sure your work is complete and clear. Explain your work; a proof is not just a collection of math symbols, it is an explanation of why something is true.

This part is take-home. You should complete this work on your own without consulting websites, friends, the Math Center, etc.

Problem 4.1 (10 points)**.** Let V be a vector space with $\dim(V) = n$ and $U \subseteq V$ a subspace with $\dim(U) = k$, show that there is a subspace $W \subseteq V$ with $\dim(W) = l$ so that $l + k = n$ and $V = U \oplus W$.

Let $\mathcal{B} = \{\boldsymbol{u_1, \dots, u_k}\}$ be a basis for U. Extend $\mathcal{B}$ to a basis $\mathcal{C}$ for V, namely, $\mathcal{C} = \{\boldsymbol{u_1, \dots, u_k, w_1, \dots, w_l}\}$ where $n = l + k$. Let $W = \operatorname{span}\{\boldsymbol{w_1, \dots, w_l}\}$.

Clearly, $U + W = V$ since any $\boldsymbol{v} \in V$ can be written as $\sum_{i=1}^{k} \alpha_i \boldsymbol{u}_i + \sum_{j=1}^{l} \beta_j \boldsymbol{w}_j$.

Suppose $\boldsymbol{v} \in U \cap W$, then $\boldsymbol{v} = \sum_{i=1}^{k} \alpha_i \boldsymbol{u}_i = \sum_{j=1}^{l} \beta_j \boldsymbol{w}_j$. But then

$$\sum_{i=1}^{k} \alpha_i \boldsymbol{u}_i - \sum_{j=1}^{l} \beta_j \boldsymbol{w}_j = \boldsymbol{v} - \boldsymbol{v} = \boldsymbol{0}$$

Since $\mathcal{C}$ is independent we have that $\alpha_i = \beta_j = 0$ for all i and j. So $\boldsymbol{v} = \boldsymbol{0}$. Thus $U \cap W = \{\boldsymbol{0}\}$.

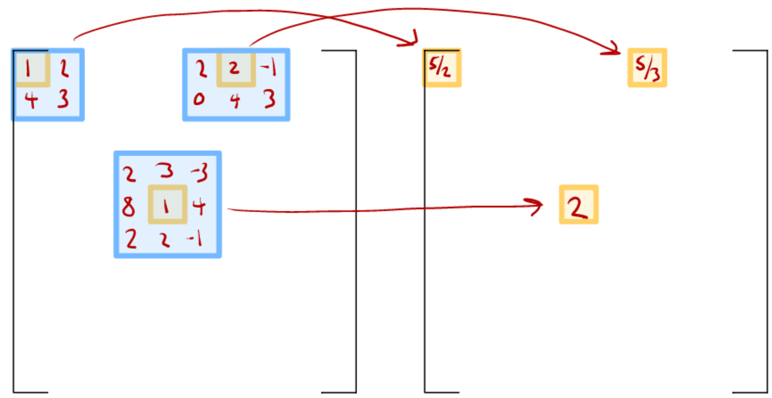
Problem 4.2 (10 points)**.** Show that if $L : V \to W$ is linear and $\ker(L) = \{\boldsymbol{0}\}$, then for any linearly independent set $\{\boldsymbol{v_1, \dots, v_k}\}$ from V, $\{L(\boldsymbol{v}_1), \dots, L(\boldsymbol{v}_k)\}$ is independent.

Suppose $\sum \alpha_i L(\boldsymbol{v}_i) = \boldsymbol{0}$, then we can use linearity to get

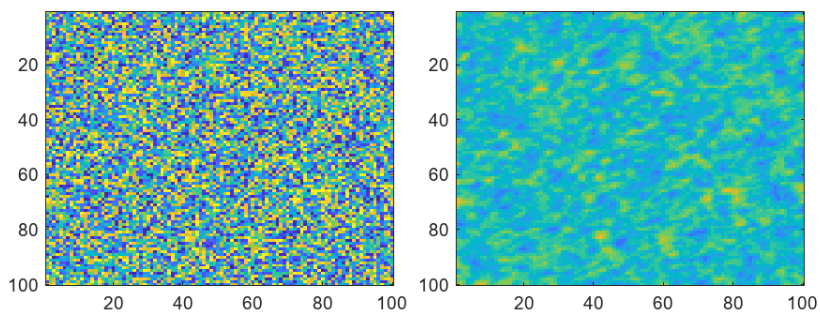
$$L\left(\sum \alpha_i \boldsymbol{v}_i\right) = \boldsymbol{0}$$

But since $\ker(L) = \{\boldsymbol{0}\}$ we have $\sum \alpha_i \boldsymbol{v}_i = \boldsymbol{0}$ and as the $\boldsymbol{v}_i$'s are independent we know $\alpha_i = 0$ for all i and so the $L(\boldsymbol{v}_i)$'s are shown independent.

Problem 4.3 (10 points)**.** Consider the following operation. Given an $m \times n$ matrix A, $S(A)$ will be the $m \times n$ matrix where each entry has been replaced by the average of the entry with its neighbors. S for "smear" (often called "blur").



Example applied to random noise (numeric value represented by color):



MATLAB code

Main Question (7 points): Show that $S : \mathbb{R}^{m\times n} \to \mathbb{R}^{m\times n}$ is linear.

Thought Question (2 points): What do you think would happen if you repeatedly applied smearing? That is, consider $A_1 = A$, $A_2 = S(A)$, $A_3 = S(S(A)) = S^2(A)$, etc. What do you think $S^k(A)$ would look like for large k (as an image)? What would the limiting value be?

How might you verify your conjecture? (1 points): Since S is linear

$$A_k = S^k(A) = \sum_{i,j} A_{i,j} S^k(E_{i,j})$$

How might you use this to verify your conjecture?

Linearity: To prove linearity, we need to see that $S(\alpha A + \beta B) = \alpha \cdot S(A) + \beta \cdot S(B)$. For this, it suffices to show that

$$S(\alpha A + \beta B)_{i,j} = \alpha \cdot S(A)_{i,j} + \beta \cdot S(B)_{i,j}$$

There are 3 cases (or four depending on how you count). Either (i, j) is at a corner and has four things to average over, (i, j) is on an edge, but not a corner, so there are six things to average over, or (i, j) is in neither of those cases and there are nine entries to average over. (See the picture.) All three cases are essentially the same, so take the third case

$$\begin{aligned} S(\alpha A+\beta B)_{i,j} &= \frac{1}{9}\left(\sum\nolimits_{\substack{i-1\le\hat{i}\le i+1\\ j-1\le\hat{j}\le j+1}}\right)(\alpha A_{\hat{i},\hat{j}}+\beta B_{\hat{i},\hat{j}}) \\ &= \alpha\frac{1}{9}\left(\sum\nolimits_{\substack{i-1\le\hat{i}\le i+1\\ j-1\le\hat{j}\le j+1}} A_{\hat{i},\hat{j}}\right)+\beta\left(\frac{1}{9}\sum\nolimits_{\substack{i-1\le\hat{i}\le i+1\\ j-1\le\hat{j}\le j+1}} B_{\hat{i},\hat{j}}\right) \\ &= \alpha\cdot S(A)_{i,j}+\beta\cdot S(B)_{i,j} \end{aligned}$$

Conjcture: There might be a variety of answers here. The thing which looks reasonable to me is that as you repeatedly smear, the image just becomes a uniform color, that is all the entries in the matrix just approach a fixed value. Since you are repeatedly taking averages a good guess is that the whole thing just approached the $m \times n$ matrix where all entries are mean(A) (the average over all entries in A.) We might write this as $A_n \to \text{mean}(A)\mathbf{1}_{m\times n}$ where $\mathbf{1}_{m\times n}$ is an $m \times n$ matrix of all 1's.

If you play with the MATLAB, this seems a reasonable guess.

Method to verify conjecture: Since $S^n(A) = \sum A_{i,j} S^n(E_{i,j})$. So it would suffice to see that $S^n(E_{i,j}) = \frac{1}{nm}\mathbf{1}_{m\times n}$.

Problem 4.4 (10 points)**.** Suppose A is a $n \times n$ matrix and $A^{m+1} = A^m$ for some m, then already $A^{n+1} = A^n$.

Hint: This is similar, but different, to one from the exam you had for practice. You can use the same ideas. Note that $A^{m+1} = A^m$ can be written as $A^m(A - I) = O$, do remember that $AB = O$ does not mean that $A = O$ or $B = O$.

There are several ways to proceed. Here is one taken from the old exam. First, notice that $A^m(A - I) = O$ is equivalent to $\text{NS}(A^{m+1}(A - I)) = \mathbb{R}^n$.

Note that $\text{NS}(A^{m+1}(A-I)) \supseteq \text{NS}(A^m(A-I))$ for all m since $A^m(A-I)\boldsymbol{x} = \mathbf{0} \implies A^{m+1}(A-I)\boldsymbol{x} = A(A^m(A-I))\boldsymbol{x} = A\mathbf{0} = \mathbf{0}$.

If m is least so that $A^m(A - I) = O$, then

$$\text{NS}(A-I) \subsetneq NS(A(A-I)) \subsetneq \cdots \subsetneq \text{NS}(A^m(A-I)) = \mathbb{R}^n$$

This is a strictly increasing sequence of subspaces of $\mathbb{R}^n$. Setting $n_i = \deg(\text{NS}(A^i(A-I))$ we have

$$0 \le n_0 < n_1 < \cdots < n_m = n$$

It is clear from this that $m \le n$ and so $A^n(A - I) = O$, that is, $A^{n+1} = A^n$, is the fixed power.