Exam 1

This exam covers Topics 1 - 3, Topic 4 will not be covered here.

Part I: True/False (5 points each; 25 points)

For each of the following mark as true or false.

a) _____ If A and B are $n \times n$ lower triangular matrices, then AB is also lower triangular.

This is true and in fact the diagonal is the product of the diagonals.

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ 0 & a_{22} & a_{23} & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & a_{nn} \end{bmatrix} \begin{bmatrix} b_{11} & b_{12} & b_{13} & \cdots & b_{1n} \\ 0 & b_{22} & b_{23} & \cdots & b_{2n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & b_{nn} \end{bmatrix} = \begin{bmatrix} a_{11}b_{11} & d_{12} & d_{13} & \cdots & d_{1n} \\ 0 & a_{22}b_{22} & d_{23} & \cdots & d_{2n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & a_{nn}b_{nn} \end{bmatrix}$$

b) _____ If W is a subspace of a vector space V and $\mathcal{B}_W$ is a basis for W, then there is a unique subspace U so that $V = W \oplus U$ and a basis $\mathcal{B}_U$ for U so that $\mathcal{B}_V = \mathcal{B}_W + \mathcal{B}_U$ is a basis for V.

This is false, there may be many, infinitely many such U. Just take $\mathbb{R}^2$ and $W = \text{span}\{(0,1)\}$, the y-axis, then $U = \text{span}\{(a,b)\}$, the line containing (a,b). The only condition is that W and U are not the same line, that is $a \neq 0$.

c) _____ If W is a subspace of a vector space V and $\mathcal{B}$ is a basis for V, then B can be restricted to a basis for W.

This is not true. Let $W = \text{span}\{(1,1)\} \subseteq \mathbb{R}^2 = V$. The standard basis for $\mathbb{R}^2$ can not be restricted to a basis for W.

d) _____ Let A be an $n \times n$ matrix over $\mathbb{C}$, then $\det(\bar{A}) = \det(A)$, where $\bar{A}_{i,j} = \overline{A_{i,j}}$. Here, $\bar{z} = a - ib$ when $z - a + ib$, the *complex conjugate of* z.

This is true and can be seen in several ways. Induction using expansion along a row or column works. For $n = 1$, this is trivial, $\det[\,\alpha\,] = \det[\,\bar{\alpha}\,]$. If A is $(n+1) \times (n+1)$

and we know the proposition holds for $n \times n$ matrices, then

$$\begin{aligned}
\det(\bar{A}) &= \sum_{i=1}^{n} \bar{a}_{i,j}(-1)^{i+j} \det(\bar{M}_{i,j}) \\
&= \sum_{i=1}^{n} \bar{a}_{i,j}(-1)^{i+j} \overline{\det(M_{i,j})} && \text{(induction hypothesis)} \\
&= \overline{\sum_{i=1}^{n} a_{i,j}(-1)^{i+j} \det(M_{i,j})} \\
&= \overline{\det(A)}
\end{aligned}$$

e) _____ For $n \times n$ matrices A and B, define $A \otimes B = AB - BA$. The operator $\otimes$ is not associative or commutative.

This is true. Failure of commutativity is trivial since $A \otimes B = -(B \otimes A)$, this is *anti-commutative.*

$$\begin{aligned}
A \otimes (B \otimes C) &= A(B \otimes C) - (B \otimes C)A \\
&= A(BC - CB) - (BC - CB)A \\
&= ABC - ACB - BCA + CBA
\end{aligned}$$

while

$$\begin{aligned}
(A \otimes B) \otimes C &= (A \otimes B)C - C(A \otimes B) \\
&= (AB - BA)C - C(AB - BA) \\
&= ABC - BAC - CAB + CBA
\end{aligned}$$

So

$$\begin{aligned}
A \otimes (B \otimes C) - (A \otimes B) \otimes C &= -ACB - BCA - (-BAC - CAB) \\
&= (CA - AC)B - B(CA - AC) \\
&= (C \otimes A) \otimes B
\end{aligned}$$

which in general is not $\mathbf{0}$.

Notice $A \otimes (B \otimes C) = -(B \otimes C) \otimes A$ so we have

$$-(B \otimes C) \otimes A - (A \otimes B) \otimes C = (C \otimes A) \otimes B$$

or equivalently

$$(A \otimes B) \otimes C + (B \otimes C) \otimes A + (C \otimes A) \otimes B = 0$$

This is an important identity in quantum mechanics called the Jacobi identity.

Part II: Definitions and Theorems (5 points each; 25 points)

a) Define what it means for a set of vectors $\mathcal{B} = \{\boldsymbol{v}_1, \ldots, \boldsymbol{v}_n\}$ from a real vector space V to span V.

$\{\boldsymbol{v}_1, \ldots, \boldsymbol{v}_n\}$ **spans** V iff for all $\boldsymbol{v} \in V$, $\boldsymbol{v}$ is a linear combination of the vectors in $\mathcal{B}$, that is $\boldsymbol{v} = \sum_{i=1}^{n} \alpha_i \boldsymbol{v}_i$ for some coefficients $\alpha_i \in \mathbb{R}$.

b) Define what it means for a set of vectors $\{\boldsymbol{v}_1, \ldots, \boldsymbol{v}_n\}$ from a real vector space V to be linearly independent.

A set of vectors $\mathcal{B}$ is **linearly independent** iff $\sum_{i=1}^{n} \alpha_i \boldsymbol{v}_i = \mathbf{0}$, then $\alpha_i = 0$ for all i. Equivalently, any linear combination of the vectors that gives $\mathbf{0}$ must be trivial.

There are equivalent definitions, but essentially, this is what you need. You can say something like, for all i, $\boldsymbol{v}_i \notin \operatorname{span}\{\boldsymbol{v}_1, \ldots, \boldsymbol{v}_{i-1}, \boldsymbol{v}_{i+1}, \ldots, \boldsymbol{v}_n\}$

c) Define what it means for a set of vectors $\mathcal{B} = \{\boldsymbol{v}_1, \ldots, \boldsymbol{v}_n\}$ to be a basis for a vector space V.

$\mathcal{B}$ has must be a linearly independent and span V.

d) State the Rank-Nullity Theorem.

If A is an $m \times n$ matrix, then $n = \dim(\mathrm{RS}(A)) + \dim(\mathrm{NS}(A)) = \operatorname{rank}(A) + \operatorname{nullity}(A)$.

e) If B arises from a matrix A by elementary row operations, what is the relationship between $\mathrm{NS}(A)$ and $\mathrm{NS}(B)$?

$\mathrm{NS}(A) = \mathrm{NS}(B)$

This is because $A\boldsymbol{x} = \boldsymbol{b} \iff B\boldsymbol{x} = \boldsymbol{b}$, i.e., the systems of equations are equivalent. So $A\boldsymbol{x} = \mathbf{0} \iff B\boldsymbol{x} = \mathbf{0}$ and thus $\boldsymbol{x} \in \mathrm{NS}(A) \iff \boldsymbol{x} \in \mathrm{NS}(B)$.

Part III: Computational (15 points each; 45 point)

a) Use row ops to find an echelon form of

$$A = \begin{bmatrix} 1 & 2 & 2 & -2 & 2 \\ 2 & 4 & 1 & -2 & 5 \\ 1 & 2 & -1 & 0 & 3 \end{bmatrix}$$

Make sure to write out your steps and indicate the row ops at each step.

$$A \xrightarrow[R_3 - R_1 \to R_3]{R_2 - 2R_1 \to R_2} \begin{bmatrix} 1 & 2 & 2 & -2 & 2 \\ 0 & 0 & -3 & 2 & 1 \\ 0 & 0 & -3 & 2 & 1 \end{bmatrix} \xrightarrow{R_3 - R_2 \to R_3} \begin{bmatrix} 1 & 2 & 2 & -2 & 2 \\ 0 & 0 & -3 & 2 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

b) Use the echelon matrix found above to find a basis for RS(A), NS(A), and CS(A). Give a brief reason for your choice.

Without a justification, you might just have a lucky guess and I will not accept this. Your justification can be short and use facts from the text or from the notes that I have provided.

A basis for RS(A) is given by $\{(1, 2, 2, -2, 2), (0, 0, -3, 2, 1)\}$.

Justification: Take the non-zero rows of the echelon form.

A basis for CS(A) is given by columns 1 and 3 of A, that is, $\{(1, 2, 1), (2, 1, -1)\}$

Justification: These correspond to the pivot columns and we know this is a basis.

For NS(A) we perform back substitution, letting $x_2 = r$, $x_4 = s$, and $x_5 = t$, so

$$-3x_3 = -2s - t$$

so

$$\begin{aligned} x_3 &= (2/3)s + (1/3)t \\ x_1 &= -2r - 2x_3 + 2s - 2t \\ &= -2r - 2((2/3)s + (1/3)t) + 2s - 2t \\ &= -2r + 2/3s - 8/3t \end{aligned}$$

So a typical element of NS(A) is

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} -2r + (2/3)s - (8/3)t \\ r \\ (2/3)s + (1/3)t \\ s \\ t \end{bmatrix} = r \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + s \begin{bmatrix} 2/3 \\ 0 \\ 2/3 \\ 1 \\ 0 \end{bmatrix} + t \begin{bmatrix} -8/3 \\ 0 \\ 1/3 \\ 0 \\ 1 \end{bmatrix}$$

A basis is

$$\{(-2, 1, 0, 0, 0), (2/3, 0, 2/3, 1, 0), (-8/3, 0, 1/3, 0, 1)\}$$

c) Show that skew-symmetric 3×3 matrices form as subspace of all 3×3 matrices and find a basis for this subspace.

These matrices look like
$$\begin{bmatrix} 0 & a & b \\ -a & 0 & c \\ -b & -c & 0 \end{bmatrix}$$
So a basis is
$$\left\{ \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix} \right\}$$
If you think about it the symmetric matrices have a similar basis of size 9, so $M^{3\times 3} = \text{Sym}^{3\times 3} \oplus \text{skewSym}^{3\times 3}$ and the basis for both are mutually orthogonal.

Part IV: Proofs (20 points each; 60 points) - Choose three!

Provide complete arguments/proofs for three of the following. If you try more than three, I will just grade the first three, so pick three your best three! If you want to ask me about these, please do.

a) A is invertible iff there exists a matrix B so that $AB = BA = I$. It is simple to show that:

   (i) If A is invertible and $AB = I$, then $BA = I$ as well and B is the unique such matrix.

   (ii) If A is invertible and $BA = I$, then $AB = I$ as well and B is the unique such matrix.

   This shows that if A is invertible, then there is a unique matrix B such that $AB = I$ or $BA = I$. Call this unique matrix A^{-1}.

   The goal here is to show that the assumption *"A is invertible"* is not needed in (i) or (ii).

   Prove: Let A and B be square matrices with $AB = I$. Show that A is invertible and hence $B = A^{-1}$.

   You may refer to Theorem 1.5.2 or Theorem 2.2.2, but be clear and complete in your argument.

   Proof 1: Show that $\text{NS}(B) = \{\mathbf{0}\}$ and hence B is invertible and from above $A = B^{-1}$, but then clearly A is invertible too.

   Clearly, $\boldsymbol{x} \in \text{NS}(B) \implies \boldsymbol{x} \in \text{NS}(AB) = \text{NS}(I) = \{\mathbf{0}\}$, so
$$\{\mathbf{0}\} \subseteq \text{NS}(B) \subseteq \text{NS}(AB) = \{\mathbf{0}\}$$

so $\mathrm{NS}(B) = \{\mathbf{0}\}$. So B is invertible and $AB = I$, so $A = B^{-1}$.

Proof 2: $\det(AB) = \det(A)\det(B) = 1$, so $\det(A) \neq 0$, hence A is invertible.

b) **Prove:** $\mathrm{NS}(A) = \mathrm{NS}(A^T A)$ for any matrix A.

You have actually done this already in the homework, but you may also use the easy fact that $\boldsymbol{x}^T\boldsymbol{x} > 0$ for $\boldsymbol{x} \neq \mathbf{0}$.

As in (a), clearly, $\mathrm{NS}(A) \subseteq \mathrm{NS}(A^T A)$, we just need the other direction.

Suppose $\boldsymbol{x} \in \mathrm{NS}(A^T A)$, then $A^T A\boldsymbol{x} = \mathbf{0}$, so $\boldsymbol{x}^T(A^T A\boldsymbol{x}) = 0$. But $\boldsymbol{x}^T(A^T A\boldsymbol{x}) = (\boldsymbol{x}^T A^T)(A\boldsymbol{x}) = (A\boldsymbol{x})^T(A\boldsymbol{x})$. As mentioned, is $(A\boldsymbol{x})^T(A\boldsymbol{x}) = 0$ iff $A\boldsymbol{x} = \mathbf{0}$. So $\boldsymbol{x} \in \mathrm{NS}(A)$ as needed.

c) **Prove:** If A and B are $m \times n$ matrices such that $A\boldsymbol{x} = B\boldsymbol{x}$ for all $\boldsymbol{x} \in \mathbb{R}^n$, then $A = B$.

The hypothesis is equivalent to $(A - B)\boldsymbol{x} = \mathbf{0}$ for all $\boldsymbol{x}$ and the conclusion is equivalent to $A - B = \mathbf{0}$.

It suffices to prove:

$$\text{If } A\boldsymbol{x} = \mathbf{0} \text{ for all } \boldsymbol{x}\text{, then } A = \mathbf{0}.$$

This is simple, say $A = \begin{bmatrix} \boldsymbol{a}_1 & \boldsymbol{a}_2 & \cdots & \boldsymbol{a}_n \end{bmatrix}$ (the columns of A). Then $A\boldsymbol{e}_j = \boldsymbol{a}_j = \mathbf{0}$. But then $\boldsymbol{a}_j(i) = A_{ij} = 0$ for all $1 \leq i, j \leq n$. So $A = \mathbf{0}$ (the all 0 matrix).

d) **Prove:** If A is an $n \times n$ matrix and $A^k = \mathbf{0}$ for any k, then $A^n = \mathbf{0}$.

Proof 1: To do this show

i) Show $\mathrm{NS}(A^{m+1}) \supseteq \mathrm{NS}(A^m)$ for all m.

ii) Show that if $\mathrm{NS}(A^{m+1}) = \mathrm{NS}(A^m)$, then $\mathrm{NS}(A^n) = \mathrm{NS}(A^m)$ for all $n \geq m$.

It is clear that $\mathrm{NS}(A^{m+1}) \supseteq \mathrm{NS}(A^m)$, since $A^m\boldsymbol{x} = \mathbf{0} \implies A(A^m\boldsymbol{x}) = \mathbf{0} \implies A^{m+1}\boldsymbol{x} = \mathbf{0}$. So (i) is shown,

For (ii) suppose $\mathrm{NS}(A^m) = \mathrm{NS}(A^{m+1})$, then $A^{m+2}\boldsymbol{x} = \mathbf{0} \implies A^{m+1}(A\boldsymbol{x}) = \mathbf{0} \implies A^m(A\boldsymbol{x}) = \mathbf{0} \implies A^{m+1}\boldsymbol{x} = \mathbf{0}$. So $\mathrm{NS}(A^{m+2}) \subseteq \mathrm{NS}(A^{m+1})$, but then $\mathrm{NS}(A^{m+2}) = \mathrm{NS}(A^{m+1}) = \mathrm{NS}(A^m)$. Now just keep going to get $\mathrm{NS}(A^k) = \mathrm{NS}(A^m)$ for all $k \geq m$.

This means we have

$$\mathrm{NS}(A^0) \subsetneq \mathrm{NS}(A^1) \subsetneq \mathrm{NS}(A^2) \subsetneq \cdots \subsetneq \mathrm{NS}(A^{m-1}) \subsetneq \mathrm{NS}(A^m) = \mathrm{NS}(A^{m+1}) = \cdots$$

The m at which $\mathrm{NS}(A^k) = \mathrm{NS}(A^m)$ for all $m \geq k$ must itself be $\leq n$.

If $A^k = \mathbf{0}$ for any k, then $\mathrm{NS}(A^k) = \mathbb{R}^n$ is maximal and thus $m \leq k$ and $\mathrm{NS}(A^m) = \mathbb{R}^n$. Since $m \leq n$, $\mathrm{NS}(A^n) = \mathbb{R}^n$ and so $A^n = \mathbf{0}$.

Proof 2: You can use induction. To do this we need to prove something that sounds slightly stronger:

P_n : For any $n \times n$ matrix A, if $A^m = \mathbf{0}$ for any $m > n$, then $A^n = 0$.

base case: (n = 1) If $A^m = [a]^m = [a^m] = [0]$, for m>1, then $a = 0$, so $A^1 = [a] = [0]$ as needed.

inductive step: Suppose P_{n-1}: For any $m > n-1$, $A^m = \mathbf{0} \implies A^{n-1} = \mathbf{0}$ for all $(n-1) \times (n-1)$ matrices. We want to prove P_n.

Assume A is an $n \times n$ matrix and $A^m = 0$ for some $m > n$. Notice that $\ker(A) \neq \{\mathbf{0}\}$, since if $\ker(A) = \{0\}$, then $A : \mathbb{R}^n \to \mathbb{R}^n$ is injective and thus A^m is also injective, so $\ker(A^m) = \{\mathbf{0}\}$. This obviously contradicts $A^m = \mathbf{0}$.

Let $\boldsymbol{v}_1 \in \ker(A)$ and let $\mathcal{B} = \{\boldsymbol{v}_1, \boldsymbol{v}_2, \dots, \boldsymbol{v}_n\}$ be a basis for $\mathbb{R}^n$. So letting $B = \begin{bmatrix} \boldsymbol{v}_1 & \boldsymbol{v}_2 & \cdots & \boldsymbol{v}_n \end{bmatrix}$ we have

$$A = BA'B^{-1}$$

where

$$A' = \begin{bmatrix} 0 & a'_{12} & a'_{13} & \cdots & a'_{1n} \\ 0 & & & & \\ \vdots & & & \hat{A} & \\ 0 & & & & \end{bmatrix}$$

where $\hat{A}$ is the indicated $(n-1) \times (n-1)$ submatrix of A'.

A' is the matrix of $L(\boldsymbol{x}) = A\boldsymbol{x}$ with respect to the basis $\mathcal{B}$. Notice that $A^m = \mathbf{0}$ means $L^m(\boldsymbol{x}) = \mathbf{0}$ for all $\boldsymbol{x}$ and hence $A'\boldsymbol{x} = \mathbf{0}$ for all $\boldsymbol{x}$, a finally this means $A'^m = \mathbf{0}$.

Notice that A' has the block form

$$\begin{bmatrix} 0 & \boldsymbol{b}^T \\ \mathbf{0} & \hat{A} \end{bmatrix}$$

and

$$(A')^2 = \begin{bmatrix} 0 & \boldsymbol{b}^T \\ \mathbf{0} & \hat{A} \end{bmatrix} \begin{bmatrix} 0 & \boldsymbol{b}^T \\ \mathbf{0} & \hat{A} \end{bmatrix} = \begin{bmatrix} 0 & \boldsymbol{b}^T \hat{A} \\ \mathbf{0} & \hat{A}^2 \end{bmatrix}$$

By a second, obvious induction,

$$(A')^k = \begin{bmatrix} 0 & \boldsymbol{b}^T \hat{A}^{k-1} \\ \mathbf{0} & \hat{A}^k \end{bmatrix}$$

We assume $\hat{A'}^m = \mathbf{0}$ so $\hat{A}^m = \mathbf{0}$ and by induction $\hat{A}^{n-1} = \mathbf{0}$ and thus

$$(A')^n = \begin{bmatrix} 0 & \boldsymbol{b}^T \hat{A}^{n-1} \\ \mathbf{0} & \hat{A}^n \end{bmatrix} = \mathbf{0}$$