Math 571 - Homework 2

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Problem 2.1 (R:2:2*). A complex number γ is **algebraic** iff γ is a root to a polynomial with integer coefficients. Prove that there are complex numbers that are not algebraic.

The following are two fun facts (not related to the question, but just for intellectual curiosity):

Let $\mathbb{A} \subset \mathbb{C}$ be the set of algebraic numbers.

1. $\mathbb{A}$ is a field.
2. $\mathbb{A}$ is algebraically closed, that is, if α is a root of a polynomial in $\mathbb{A}[x]$, then $\alpha \in \mathbb{A}$. So in the definition of algebraic numbers you can use any ring of coefficients R, $\mathbb{Z} \subseteq R \subseteq \mathbb{A}$.

Here is a write-up of the proofs, you should have the background to read this, but it is not an easy read.

The intent here was for you to do a counting argument. There are only countably many polynomials with integer coefficients and each has only finitely many roots, hence there are only countably many algebraic numbers.

Definition 1. A set $S \subseteq X$ is **discrete** iff every point in S is isolated.

Problem 2.2 (R:2:5*). Prove the following for discrete $S \subset \mathbb{R}$:

a) $\mathrm{Lim}(S) \cap S = \emptyset$ and S is countable.

For each $x \in S$ we find integer n_x so that $N_{\frac{1}{n_x}}(x) \cap (S - \{x\}) = \emptyset$. We can find $q_x \in \mathbb{Q} \cap N_{\frac{1}{2n_x}}(x)$. The map $x \mapsto (n_x, q_x)$ is injective since if $x, x' \in S$ and $(n_x, q_x) = (n'_x, q_{x'})$, then $|x - x'| \leq |x - q_x| + |q_x - x'| < \frac{1}{2n_x} + \frac{1}{2n_x} = \frac{1}{n_x}$ and hence $x' \in N_{\frac{1}{n_x}}(x)$. So S is countable.

Suppose $x \in \mathrm{Lim}(S)$, then $S \cup \{x\}$ has a non-isolated point, namely, x, so $\mathrm{Lim}(S) \cap S = \emptyset$.

b) There is discrete set $A \subset \mathbb{R}$ so that $\mathrm{Lim}(A) = \mathrm{Cl}(S)$.

Proof 1: From the preceding we can write $S = \{s_i \mid i \in \mathbb{N}\}$ and get $n_0 < n_1 < \cdots$ so the $N_{\frac{1}{n_i}}(x) \cap S = \{x\}$.

Let A_i be a countable discrete set in $N_{\frac{1}{n_i}}$, with $\{s_i\} = \mathrm{Lim}(A_i)$. Namely, take a sequence of distinct points converging to s_i.

Let $A = \bigcup_{i \in \mathbb{N}} A_i$. Clearly, $S \subseteq \mathrm{Lim}(A)$ and so $\mathrm{Cl}(S) \subseteq \mathrm{Lim}(A)$.

Now we want to see that $\mathrm{Lim}(A) \subseteq \mathrm{Cl}(S)$. Let $z \in \mathrm{Lim}(A)$ and $z = \lim_i z_i$ for $z_i \in A$. If $z \in S$, there is nothing to do, so assume $z \notin S$. Each $z_i \in A_j$ for some i, if for any j, $\{z_i \mid z_i \in A_j\}$ is infinite, then clearly $z = x_j = \lim\{z_i\} z_i \in A_j$. So we see that for each j, $\{z_i \mid z_i \in A_j\}$ is finite and, in fact, the same argument shows this to be true for any infinite subset (subsequence) $\{z_{i_j} \mid j \in \mathbb{N}\}$. For any $\delta > 0$ since $\{i \mid z_i \in N_\delta(z)\}$ is infinite, we know that $\{j \mid A_j \cap N_{\delta/2}(z) \neq \emptyset\}$ is infinite. Choosing such a j so that $\frac{1}{n_j} < \delta/2$ will result in $z_k \in N_{\delta/2}(s_j)$ and $z_k \in N_{\delta/2}(z)$ so $s_j \in N_\delta(z)$. Thus $z \in \mathrm{Lim}(S)$. So $z \in S \cup \mathrm{Lim}(S) = \mathrm{Cl}(S)$.

Proof 2: This is similar but mirrors something I did in class. Let N_x be nbhds for $x \in S$ so that $N_x \cap N_y = \emptyset$ for $x, y \in S$. Let $a_i^x \to x$ for $x \in S$ where $A^x = \{a_i^x \mid i \in \mathbb{N}\} \subset N_x$ is discrete. So we "replace" each x in S with a discrete sequence in N_x with x as the limit. $A = \bigcap_{x \in S} A^x$.

Now let $z \in \mathrm{Lim}(A)$, say $z = \lim_i z_i$. If there is a fixed $x \in S$ so that infinitely many $z_i \in A^x$, then $z = x$. So the other option is $z_i = a_{j_i}^{x_i}$ and by going to a subsequence we may assume $x_i \neq x_{i'}$ for $i \neq i'$.

We want to see that $x_i \to z$. We have $d(x_i, z) \leq d(x_i, z_i) + d(z_i, z)$. For i large, we can make sure that $d(z_i, z)$ is small, but we need to make sure that $d(z_i, x_i)$ is also small. Suppose $N_x = N_{\epsilon_x}(x)$, then we know $d(x_i, z) \leq d(x_i, z_i) + d(z_i, z) < \epsilon_{x_i} + d(z_i, z)$. So we need that ϵ_{x_i} is small for large i. (This is what was built into the construction in Proof 1.)

c) Give an example of a discrete set S where there is no set A such that $\mathrm{Lim}(A) = S$.

Clearly, $\mathrm{Lim}(S) \subseteq \mathrm{Lim}(A)$, since $\mathrm{Lim}(\mathrm{Lim}(A)) \subseteq \mathrm{Lim}(A)$. So just take S with $\mathrm{Lim}(S) - S \neq \emptyset$.

For the following use the definition that I provided for $\mathrm{Cl}(E)$, namely, $\mathrm{Cl}(E) = \bigcap\{F \mid F \text{ is closed and } E \subseteq F\}$.

Problem 2.3 (R:2:6)**.** For X a metric space and $E \subseteq X$, show that

a) $\mathrm{Lim}(\mathrm{Lim}(E)) \subseteq \mathrm{Lim}(E)$ and equality need not obtain.

Let $x \, \mathrm{Lim}(\mathrm{Lim}(E))$, then for all open nbhd N of x, we have $N \cap (\mathrm{Lim}(E) - \{x\}) \neq \emptyset$. Let $y \in N \cap (\mathrm{Lim}(E) - \{x\})$. If $y \in E$, then $N \cap (E - \{x\}) \neq \emptyset$. Else $y \notin E$ and $y \in N \cap (\mathrm{Lim}(E) - \{x\})$, then as $y \in \mathrm{Lim}(E)$ we can take open nbhd N' of y so that $N' \subseteq N$, $x \notin N'$ and $N' \cap (E - \{y\}) \neq \emptyset$. But then clearly $N \cap (E - \{x\}) \neq \emptyset$. So $x \in \mathrm{Lim}(E)$.

Consider $E = \{\frac{1}{n} \mid n = 1, 2, \ldots\}$, then $\mathrm{Lim}(E) = \{0\}$ and $\mathrm{Lim}(\mathrm{Lim}(E)) = \emptyset$.

b) $\mathrm{Lim}(A \cup B) = \mathrm{Lim}(A) \cup \mathrm{Lim}(B)$.

If $x \in \mathrm{Lim}(A \cup B)$ and $x \notin \mathrm{Lim}(A)$, then there is open N so that $x \in N$ and $x \cap A - \{x\} = \emptyset$. For all open nbhd M of x with $M \subseteq N$, $M \cap (A \cup B - \{x\}) \neq \emptyset$, so $M \cap (B - \{x\}) \neq \emptyset$. Thus $x \in \mathrm{Lim}(B)$. So $x \in \mathrm{Lim}(A \cup B) \implies x \in \mathrm{Lim}(A) \wedge x \in \mathrm{Lim}(B)$ and hence $\mathrm{Lim}(A \cup B) \subseteq \mathrm{Lim}(A) \cup \mathrm{Lim}(B)$.

Clearly, $\mathrm{Lim}(A \cup B) \supseteq \mathrm{Lim}(A) \cup \mathrm{Lim}(B)$ so we have equality.

c) $E \cup \text{Lim}(E)$ is closed and $E \cup \text{Lim}(E) = \text{Cl}(E)$.

Suppose $x \notin E \cup \text{Lim}(E)$, then let N be a nbhd of x so that $N \cap (E - \{x\}) = N \cap E = \emptyset$. Lets see that $N \cap \text{Lim}(E) = \emptyset$. Suppose $y \in \mathbb{N} \cap \text{Lim}(E)$. Then N is a nbhd of y and so $N \cap (E - \{y\}) \neq \emptyset$. This contradicts $N \cap E = \emptyset$. So we see that $x \notin E \cup \text{Lim}(E)$ implies that there is a nbhd of x lying entirely outside of $E \cup \text{Lim}(E)$ and thus $(E \cup \text{Lim}(E))^c$ is open, or $E \cup \text{Lim}(E)$ is closed.

Now $E \subseteq E \cup \text{Lim}(E)$ so $\text{Cl}(E) \subseteq \text{Cl}(E \cup \text{Lim}(E)) = E \cup \text{Lim}(E)$. Conversely, $x \in \text{Lim}(E) \implies x \in \text{Cl}(E)$, since for any nbhd N of x, $N \cap (E - \{x\}) \neq \emptyset$, which is stronger than what is needed for $x \in \text{Cl}(E)$.

d) $\text{Lim}(E)$ is closed and $\text{Lim}(E) = \text{Lim}(\text{Cl}(E))$.

$$\begin{aligned}
\text{Lim}(E) &= \text{Lim}(E) \cup \text{Lim}(\text{Lim}(E)) && \text{by (a); closed by (c)} \\
&= \text{Lim}(E \cup \text{Lim}(E)) && \text{by (b)} \\
&= \text{Lim}(\text{Cl}(E)) && \text{by (c)}
\end{aligned}$$

Problem 2.4 (R:2:9*)**.** Let X be a metric space, or just any topological space. Are the following true for all $E \subseteq X$? For each either prove the statement true or give a counterexample. For a counterexample you must provide both X and E.

a) $\text{Int}(E)^c = \text{Cl}(E^c)$.

Let's try to prove this. there are, as usual, two things to prove here.

$\text{Int}(\boldsymbol{E})^{\boldsymbol{c}} \subseteq \text{Cl}(\boldsymbol{E}^{\boldsymbol{c}})$: Let $x \in \text{Int}(E)^c$, so $x \notin \text{Int}(E)$. This means every neighborhood of x contains points in E^c. This means $x \in \text{Cl}(E^c)$.

$\text{Cl}(\boldsymbol{E}^{\boldsymbol{c}}) \subseteq \text{Int}(\boldsymbol{E})^{\boldsymbol{c}}$: Let $x \in \text{Cl}(E^c)$ so every nbhd of x meets E^c, so $x \notin \text{Int}(E)$, thus $x \in \text{Int}(E)^c$.

The following two arguments came up in class:

Alternate 1:

$$\begin{aligned}
x \in \text{Cl}(E^c) &\iff \forall N\, (N \text{ a nbhd of } x \implies N \cap E^c \neq \emptyset) \\
&\iff \forall N\, (N \text{ a nbhd of } x \implies N \not\subseteq E) \\
&\iff \neg \exists N\, (N \text{ a nbhd of } x \text{ and } N \subseteq E) \\
&\iff x \notin \text{Int}(E) \\
&\iff x \in (\text{Int}(E))^c
\end{aligned}$$

Alternate 2:

$$\begin{aligned}
\mathrm{Cl}(E^c) &= \bigcap\{F \mid F \text{ is closed and } F \supseteq E^c\} \\
&= \left(\bigcup\{F^c \mid F \text{ is closed and } F \supseteq E^c\}\right)^c \\
&= \left(\bigcup\{F^c \mid F^c \text{ is open and } F^c \subseteq E\}\right)^c \\
&= \left(\bigcup\{O \mid O \text{ is open and } O \subseteq E\}\right)^c \\
&= (\mathrm{Int}(E))^c
\end{aligned}$$

b) $\mathrm{Cl}(E) = \mathrm{Int}(E^c)^c$?

This is true and we can just apply (a) here. $\mathrm{Cl}(E) = \mathrm{Cl}((E^c)^c) = \mathrm{Int}(E^c)^c$. This clearly also gives $\mathrm{Cl}(E)^c = \mathrm{Int}(E^c)$.

c) $\mathrm{Cl}(E) = \mathrm{Cl}(\mathrm{Int}(E))$?

This is false. Just take $X = \mathbb{R}$ and $E = \mathbb{Q}$, then $\mathrm{Cl}(\mathbb{Q}) = \mathbb{R}$ but $\mathrm{Cl}(\mathrm{Int}(E)) = \mathrm{Cl}(\emptyset) = \emptyset$.

d) $\mathrm{Int}(E) = \mathrm{Int}(\mathrm{Cl}(E))$

This is just as the previous, same counterexample shows this to be false. $\mathrm{Int}(\mathbb{Q}) = \emptyset \neq \mathrm{Int}(\mathrm{Cl}(\mathbb{Q})) = \mathrm{Int}(\mathbb{R}) = \mathbb{R}$.

An open set, E, is called a **regular open set** iff $E = \mathrm{Int}(\mathrm{Cl}(E))$. Similarly, a closed set, E, is **regular closed set** if $E = \mathrm{Cl}(\mathrm{Int}(E))$.

Let O be any open set, then ∂O is nowhere dense, that is, for all open U, there is $U' \subseteq U$ so that $\emptyset \neq U'$ and $U' \cap \partial O = \emptyset$. Let U be open and suppose $U \cap \partial O \neq \emptyset$. Let $U' = O \cap U$. Clearly, $\emptyset \neq U'$ and $U' \cap \partial O = \emptyset$, since $O \cap \partial O = \emptyset$.

Any non-empty closed nowhere-dense set, N, fails to be regular closed, and so N^c fails to be regular open. For example, the circle $S^1 \subset \mathbb{R}^2$ is the boundary of the open unit disk and thus is closed nowhere-dense, hence not regular-closed. Correspondingly, $G = \mathbb{R}^2 - S^1$ is open, but not regular open.

Definition 2. A metric space X is **separable** iff there is a countable $E \subseteq X$ with E dense in X.

Problem 2.5 (R:2:22)**.** Show the $\mathbb{R}^k$ is separable.

It is easy to see that $\mathbb{Q}^k$ is dense in $\mathbb{R}^k$. One way is the following, use basic open "boxes" of the form $\prod_{i=1}^k (a_i, b_i)$ for the basic open sets, instead of open balls. The fact that $\mathbb{Q} \cap (a_i, b_i) \neq \emptyset$ immediately yields that $\mathbb{Q}^k \cap \prod_{i=1}^k (a_i, b_i) \neq \emptyset$.

Definition 3. A set $\mathcal{B}$ of open sets is called a **base** for X iff for all $x \in X$ and open set U with $x \in U$, there is $V \in \mathcal{B}$ so that $x \in V \subset U$.

Problem 2.6 (R:2:23*)**.** Prove that a metric space is separable iff it has a countable base.

If X is separable, let S be a countable dense set. Consider $N_{\frac{1}{i}}(s)$ for $s \in S$. Let $x \in X$ and O be an open nbhd of x. Take $N_\delta(x) \subseteq O$ and $s \in S$ with $d(s,x) < \delta/4$. Then $x \in N_{\frac{1}{m}}(s) \subseteq O$ with $\frac{1}{m} < \frac{\delta}{4}$. So the sets $N_{\frac{1}{i}}(s)$ do form a countable base.

If $\{O_i \mid i \in \mathbb{N}\}$ is a countable base, then just take $s_i \in O_i$ for all i, then $S = \{s_i \mid i \in \mathbb{N}\}$ is dense.

Problem 2.7 (R:2:24)**.** Prove that if X is a metric space and every infinite sequence has a limit point, then X is separable. (See the hint in the text.)

For each integer $i > 0$ construct a maximal set $S_i = \{x_j^i\}_{j=0}^{k_i}$ so that $d(x_l^i, x_k^i) \geq \frac{1}{i}$. $k_i < \infty$ for all i since otherwise there would be an infinite sequence with no limit. By maximal here we mean that for any $x \in X$m $N_{\frac{1}{i}}(x) \cap S_i \neq \emptyset$, since otherwise we could add x to S_i and maintain the desired separation of elements.

Let $S = \{x_j^i \;\big|\; j \leq k_i \text{ and } i \in \mathbb{N}\}$. S is dense in X since for any $\delta > 0$ and any x, let $i \in \mathbb{N}$ so that $\frac{1}{i} < \delta$, then $S \cap N_\delta(x) \supseteq S \cap N_{\frac{1}{i}}(x) \neq \emptyset$ by construction.