

# 1 True/False (50 points; 5 points each)

**Recall:**  $A$  and  $B$  are equivalent if there is a sequence of elementary row operations leading from  $A$  to  $B$ , or equivalently,  $B = MA$  for some invertible matrix  $M$ . This is different from  $A \sim B$  ( $A$  and  $B$  are *similar* which means  $B = S^{-1}AS$  for some invertible  $S$ ).

**Problem 1.1.** In class, you need only provide a T/F (make it clear!) As usual, you may earn back up to 50% of the lost points by supplying justifications afterward.

False The collection of  $3 \times 4$  echelon matrices is a subspace of  $\mathbb{R}^{3 \times 4}$ .

Closure under scalar multiplication is ok, but it is easy to see that closure under addition fails.

True The set of  $n \times n$  matrices with all diagonal elements being 0 is a subspace of  $\mathbb{R}^{n \times n}$ .

It is clear that if  $A$  and  $B$  both have 0 diagonals, then so does  $\alpha A$  and  $A + B$ .

True Consider the map  $L : \mathbb{R}^{m \times n} \rightarrow \mathbb{R}^m$  defined by  $L(A)_i = \text{ave}(A_{i,*})$ , that is, the  $i^{\text{th}}$  entry of  $L(A)$  is the average of the  $i^{\text{th}}$  row of  $A$ .  $L$  is a linear map.

This is easy to check directly, but here is a cute argument. Let  $\mathbf{1} \in \mathbb{R}^n$  be the vector of  $n$  1's. Then

$$L(A) = \frac{1}{n}A\mathbf{1},$$

and this, being simple matrix multiplication, is clearly linear:

$$L(\alpha A + \beta B) = \frac{1}{n}(\alpha A + \beta B)\mathbf{1} = \alpha \frac{1}{n}A\mathbf{1} + \beta \frac{1}{n}B\mathbf{1} = \alpha \cdot L(A) + \beta \cdot L(B)$$

False For all linear  $L : V \rightarrow W$ , if  $\{\mathbf{v}_1, \dots, \mathbf{v}_k\}$  is independent, then  $\{L(\mathbf{v}_1), \dots, L(\mathbf{v}_k)\}$  is independent.

This is trivially false.  $L$  could just be the  $\mathbf{0}$  map, that is,  $L(\mathbf{v}) = \mathbf{0}$  for all  $\mathbf{v} \in V$ .

True For all linear  $L : V \rightarrow W$ , if  $\{L(\mathbf{v}_1), \dots, L(\mathbf{v}_k)\}$  is independent, then  $\{\mathbf{v}_1, \dots, \mathbf{v}_k\}$  is independent.

Suppose  $\sum \alpha_i \mathbf{v}_i = \mathbf{0}$ , then  $L(\sum \alpha_i \mathbf{v}_i) = \sum \alpha_i L(\mathbf{v}_i) = L(\mathbf{0}) = \mathbf{0}$ . Since  $\{L(\mathbf{v}_1), \dots, L(\mathbf{v}_k)\}$  is independent it follows that  $\alpha_i = 0$  for all  $i$  and hence that  $\{\mathbf{v}_1, \dots, \mathbf{v}_k\}$  is independent.

False There are subspaces  $V_0 = P_3 \supsetneq V_1 \supsetneq V_2 \supsetneq V_3 \supsetneq V_4 \supsetneq V_5 = \{\mathbf{0}\}$  where each  $V_i$  is a proper subspace of  $V_{i-1}$ .

Since we know  $\dim(V_0) = 4 > \dim(V_1) > \dim(V_2) > \dim(V_3) > \dim(V_4) > \dim(V_5) = 0$ , which is impossible.

True Given any basis  $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ , from  $\mathbb{R}^4$  and any four matrices  $M_1, M_2, M_3, M_4 \in \mathbb{R}^{2 \times 3}$  there is a unique linear transformation  $L : \mathbb{R}^4 \rightarrow \mathbb{R}^{2 \times 3}$  where  $L(\mathbf{v}_i) = M_i$ .

**Existence:** Define  $L(\sum_{i=1}^4 \alpha_i \mathbf{v}_i) = \sum_{i=1}^4 \alpha_i M_i$ . This is a well-defined function  $L : \mathbb{R}^4 \rightarrow \mathbb{R}^{2 \times 3}$ .

Showing that this is linear is just a computation: Let  $\mathbf{v} = \sum_{i=1}^4 \alpha_i \mathbf{v}_i$  and  $\mathbf{u} = \sum_{i=1}^4 \beta_i \mathbf{v}_i$ , then

$$\begin{aligned} L(\gamma \mathbf{v} + \mathbf{u}) &= L\left(\gamma \sum_{i=1}^4 \alpha_i \mathbf{v}_i + \sum_{i=1}^4 \beta_i \mathbf{v}_i\right) = L\left(\sum_{i=1}^4 (\gamma \alpha_i + \beta_i) \mathbf{v}_i\right) \\ &= \sum_{i=1}^4 (\gamma \alpha_i + \beta_i) W_i = \gamma \sum_{i=1}^4 \alpha_i W_i + \sum_{i=1}^4 \beta_i W_i = \gamma \cdot L(\mathbf{v}) + L(\mathbf{u}) \end{aligned}$$

**Uniqueness:** Suppose  $L' : V \rightarrow W$  is linear and sends  $\mathbf{v}_i$  to  $\mathbf{W}_i$ , then for  $\mathbf{v} \in V$ ,  $\mathbf{v} = \sum_{i=1}^4 \alpha_i \mathbf{v}_i$  and  $L'(\mathbf{v}) = L'(\sum_{i=1}^4 \alpha_i \mathbf{v}_i) = \sum_{i=1}^4 \alpha_i M_i = L(\mathbf{v})$  and thus  $L = L'$ .

True Suppose  $L : P_5 \rightarrow \mathbb{R}^4$  is linear and onto, that is,  $\text{Img}(L) = \mathbb{R}^4$ . Then  $\dim(\ker(L)) = 2$ .

Recall  $P_5$  is the space of polynomials of degree  $\leq 5$ .

$\dim(P_5) = 6$  and so  $\dim(\ker(L)) + \dim(\text{Img}(L)) = \dim(\ker(L)) + 4 = 6$ .

True Let

$$\mathcal{B} = \left\{ \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ 2 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} \right\}$$

be a basis for  $\mathbb{R}^3$ . Then for  $\mathbf{v} = \begin{bmatrix} 3 \\ 3 \\ 4 \end{bmatrix}$

$$[\mathbf{v}]_{\mathcal{B}} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} 3 \\ 3 \\ 4 \end{bmatrix} = (1) \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} + (1) \begin{bmatrix} 2 \\ 0 \\ 2 \end{bmatrix} + (1) \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}$$

False  $L : \mathbb{R}^{3 \times 3} \rightarrow \mathbb{R}^3$  is given by  $L(A) = A\mathbf{b}$  for a  $\mathbf{b} \in \mathbb{R}^3$ . If  $\mathcal{B}$  is a basis for  $\mathbb{R}^{3 \times 3}$ , then  $[L]_{\mathcal{B}} = \mathbf{b}$ .

$[L]_{\mathcal{B}}$  acts on representations of matrices wrt  $\mathcal{B}$ , it is a  $9 \times 9$  matrix, not a  $3 \times 3$  matrix.

## 2 Multiple Choice (30 points; 10 points each)

Each correct box counts for two points.

**Problem 2.1** (10 points). Which of the following are equivalent to “ $A$  is *equivalent* to  $B$ ”? Mark ‘Y’ if equivalent and ‘N’ if not.

☐ Y  $B$  results from a series of row operations from  $A$ .

☐ N  $B = AM$  for some invertible matrix  $M$ .

☐ Y  $B = MA$  for some invertible matrix  $M$ .

☐ N  $\text{CS}(A) = \text{CS}(B)$ .

☐ Y  $\text{RS}(A) = \text{RS}(B)$ .

**Problem 2.2** (10 points). Which of the following are equivalent to  $A$  is invertible for an  $n \times n$  matrix  $A$ . Mark ‘Y’ if equivalent and ‘N’ if not.

☐ Y  $A$  is equivalent to  $I$ .

☐ N  $\dim(\text{RS}(A)) = \dim(\text{CS}(A))$ .

☐ Y  $\text{NS}(A) = \{\mathbf{0}\}$ .

☐ Y  $A\mathbf{x} = \mathbf{b}$  has at least one solution for all  $\mathbf{b} \in \mathbb{R}^n$ .

☐ Y  $A\mathbf{x} = \mathbf{b}$  has a unique solution for some  $\mathbf{b}$ .

**Problem 2.3.** Which of the following implies that  $\mathcal{B} = \{\mathbf{v}_1, \dots, \mathbf{v}_n\}$  is linearly independent. Mark ‘Y’ if the property implies  $\mathcal{B}$  is independent, ‘N’ otherwise.

☐ N For every  $\mathbf{v}$  in  $V$ ,  $\mathbf{v}$  can be written as a linear combination of vectors in  $\mathcal{B}$ , i.e., there is  $\alpha_i \in \mathbb{R}$  so that  $\mathbf{v} = \sum_{i=1}^n \alpha_i \mathbf{b}_i$ .

☐ Y If  $\sum_{i=1}^n \alpha_i \mathbf{b}_i = \sum_{i=1}^n \beta_i \mathbf{b}_i$ , then  $\alpha_i = \beta_i$  for all  $i$ .

☐ Y  $\mathbf{b}_i \notin \text{span}(\mathcal{B} - \{\mathbf{b}_i\})$ , that is,  $\mathbf{b}_i$  is not a linear combination of the other vectors in  $\mathcal{B}$ .

☐ Y There is a linearly independent set  $\mathcal{C} = \{\mathbf{c}_1, \dots, \mathbf{c}_n\}$  so that  $\mathcal{C} \subset \text{span}(\mathcal{B})$ .

☐ N There is a linearly independent set  $\mathcal{C} = \{\mathbf{c}_1, \dots, \mathbf{c}_n\}$  so that  $\mathcal{B} \subset \text{span}(\mathcal{C})$ .

### 3 Computational (80 points; 20 points each)

Show all computations so that you make clear what your thought processes are.

**Problem 3.1** (20 pts). Consider  $A$  given by

$$A = \begin{bmatrix} -2 & 4 & -4 & -4 & 4 \\ -8 & 16 & -15 & -18 & 18 \\ -8 & 16 & -11 & -26 & 27 \\ -4 & 8 & -8 & -8 & 4 \end{bmatrix}$$

Find a basis for each of  $\text{NS}(A)$ ,  $\text{CS}(A)$ , and  $\text{RS}(A)$ .

$$\text{rref}(A) = \begin{bmatrix} 1 & -2 & 0 & 6 & 0 \\ 0 & 0 & 1 & -2 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

**From this we know:**

$$\text{CS}(A) = \text{span}\{(-2, -8, -8, -4), (-4, -15, -11, -8), (4, 18, 27, 4)\}$$

$$\text{RS}(A) = \text{span}\{(1, -2, 0, 6, 0), (0, 0, 1, -2, 0), (0, 0, 0, 0, 1)\}$$

Note:  $\text{RS}(A)$  is not the span of the first three rows of  $A$ .

**To find a basis for  $\text{NS}(A)$  we are looking for solutions to  $Ax = 0$ . First, we have back-substitution:**  $x_2$  and  $x_4$  are free, let  $x_2 = s$  and  $x_4 = t$ , then

$$\boxed{x_5 = 0}$$

$$\boxed{x_4 = t}$$

$$x_3 - 2t = 0 \rightarrow \boxed{x_3 = 2t}$$

$$\boxed{x_2 = s}$$

$$x_1 - 2s + 6t = 0 \rightarrow \boxed{x_1 = 2s - 6t}$$

**Any vector  $x$  satisfying,  $Ax = 0$  can be written as:**

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} 2s - 6t \\ s \\ 2t \\ t \\ 0 \end{bmatrix} = s \begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + t \begin{bmatrix} -6 \\ 0 \\ 2 \\ 1 \\ 0 \end{bmatrix}$$

So  $\{(2, 1, 0, 0, 0), (-6, 0, 2, 1, 0)\}$  is a basis for  $\text{NS}(A)$ , that is,

$$\text{NS}(A) = \text{span}\{(2, 1, 0, 0, 0), (-6, 0, 2, 1, 0)\}$$

**Problem 3.2** (20 pts). Consider  $L : P_3 \rightarrow P_6$  given by  $L(p(x)) = q(x)p(x)$  where  $q(x) = a_0 + a_1x + a_2x^2 + a_3x^3$ .

- a) (8 points) Show that  $L$  is a linear map.
- b) (8 points) Give the matrix  $[L]$  where the standard basis is used for both  $P_3$  and  $P_6$ . Just to be definite, the standard basis for  $P_k$  is  $\mathcal{E} = \{1, x, x^2, \dots, x^k\}$ .
- c) (4 points) With  $p(x) = b_0 + b_1x + b_2x^2 + b_3x^3$  compute  $[q(x)p(x)]$  using  $[L]$  and  $[p]$ .

To see that  $L$  is linear we just note that for  $p(x), h(x) \in P_3$  and  $c, d \in \mathbb{R}$  we have

$$\begin{aligned} L(c \cdot p(x) + d \cdot h(x)) &= q(x)(c \cdot p(x) + d \cdot h(x)) \\ &= c \cdot (q(x)p(x)) + d \cdot (q(x)h(x)) \\ &= c \cdot L(p(x)) + d \cdot L(h(x)) \end{aligned}$$

so linearity of  $L$  is shown.

To compute  $[L]$ , first notice that  $\dim(P_3) = 4$  and  $\dim(P_6) = 7$  so  $[L]$  is  $7 \times 4$  (a good sanity check on our solution).

$$[L] = \begin{bmatrix} [q(x)] & [q(x) \cdot x] & [q(x) \cdot x^2] & [q(x) \cdot x^3] \end{bmatrix} = \begin{bmatrix} a_0 & 0 & 0 & 0 \\ a_1 & a_0 & 0 & 0 \\ a_2 & a_1 & a_0 & 0 \\ a_3 & a_2 & a_1 & a_0 \\ 0 & a_3 & a_2 & a_1 \\ 0 & 0 & a_3 & a_2 \\ 0 & 0 & 0 & a_3 \end{bmatrix}$$

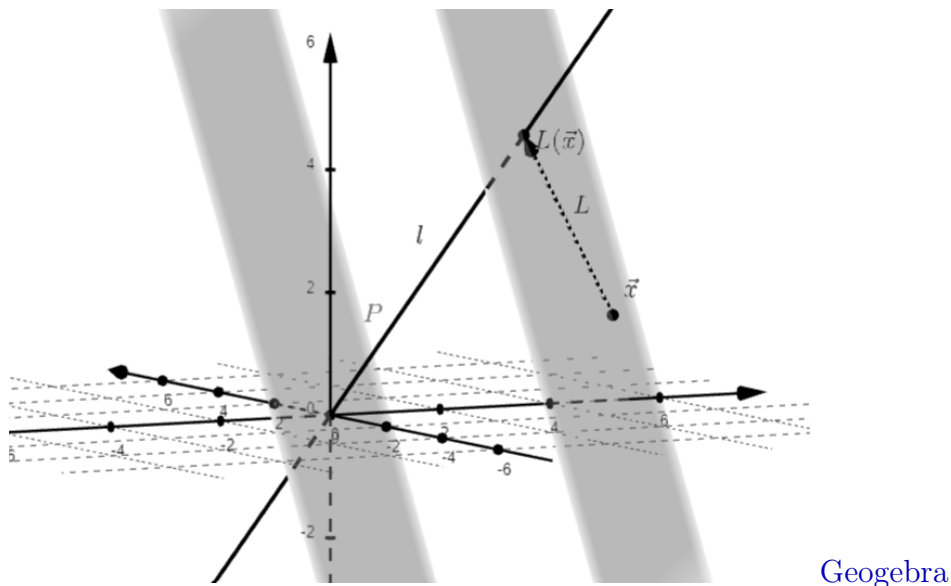
Finally,

$$[q(x)p(x)] = [L(p(x))] = [L][p(x)] = \begin{bmatrix} a_0 & 0 & 0 & 0 \\ a_1 & a_0 & 0 & 0 \\ a_2 & a_1 & a_0 & 0 \\ a_3 & a_2 & a_1 & a_0 \\ 0 & a_3 & a_2 & a_1 \\ 0 & 0 & a_3 & a_2 \\ 0 & 0 & 0 & a_3 \end{bmatrix} \begin{bmatrix} b_0 \\ b_1 \\ b_2 \\ b_3 \end{bmatrix} = \begin{bmatrix} a_0b_0 \\ a_1b_0 + a_0b_1 \\ a_2b_0 + a_1b_1 + a_0b_2 \\ a_3b_0 + a_2b_1 + a_1b_2 + a_0b_3 \\ a_3b_1 + a_2b_2 + a_1b_3 \\ a_3b_2 + a_2b_3 \\ a_3b_3 \end{bmatrix}$$

Pretty, no?

Note  $[q(x)p(x)]_i = \sum_{l=0}^i a_l b_{i-l} = \sum_{l+k=i} a_l b_k$ , which you might know from studying polynomials in an algebra class.

**Problem 3.3** (20 pts). Consider the map  $L : \mathbb{R}^3 \rightarrow \mathbb{R}^3$  that projects a point in  $\mathbb{R}^3$  onto the line  $l : \left\{ t \begin{bmatrix} -1 \\ -1 \\ 2 \end{bmatrix} \mid t \in \mathbb{R} \right\}$  along the plane  $P : 3x - 2y + z = 0$ .



Find a basis  $\mathcal{B}$  for  $\mathbb{R}^3$  so that  $[L]_{\mathcal{B}}$  is simple. Give both  $\mathcal{B}$  and  $[L]_{\mathcal{B}}$ . **(9 points for this.)** Next, find  $[L]$  using some change of basis and the  $[L]_{\mathcal{B}}$  that you found. **(9 points for this part.)** Finally, find  $L((4, -4, 0))$ . **(2 points)**

**Note:** Points on  $P$  are mapped to  $\mathbf{0}$ , that is,  $\ker(L) = P$ , while points in  $l$  are fixed.

There are many choices for  $\mathcal{B}$ , I will use the two vectors  $\mathbf{v}_1 = (1, 1, -1)$  and  $\mathbf{v}_2 = (0, 1, 2)$  in  $P$  and  $\mathbf{v}_3 = (1, -1, 2)$  in  $L$ . So

$$\mathcal{B} = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\} = \left\{ \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix} \right\}$$

and

$$[L]_{\mathcal{B}} = [[L(\mathbf{v}_1)]_{\mathcal{B}} [L(\mathbf{v}_2)]_{\mathcal{B}} [L(\mathbf{v}_3)]_{\mathcal{B}}] = [\mathbf{0}]_{\mathcal{B}} [\mathbf{0}]_{\mathcal{B}} [\mathbf{v}_3]_{\mathcal{B}}] = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

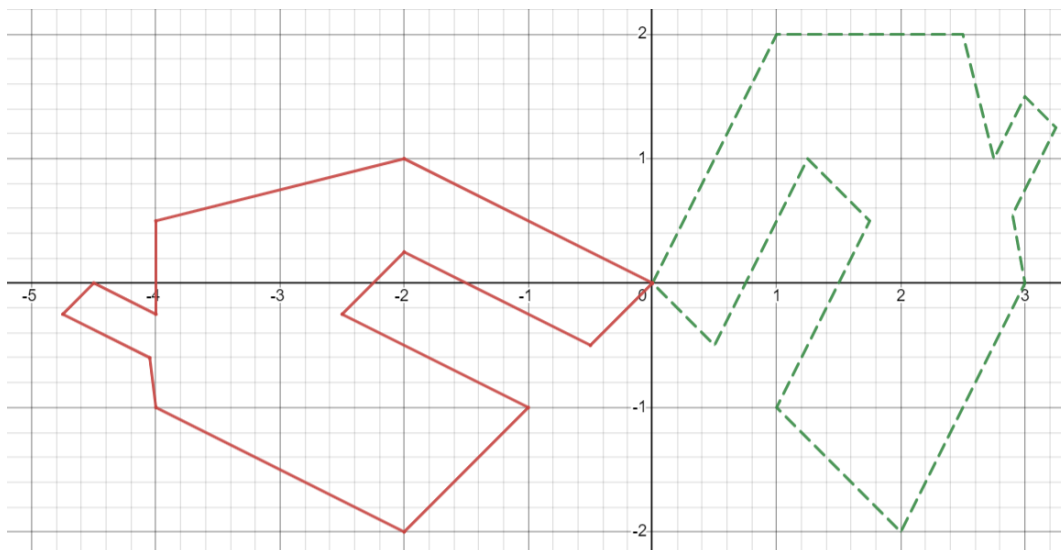
Finding  $[L]$  is now trivial.

$$[L] = B[L]_{\mathcal{B}}B^{-1} = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & -1 \\ -1 & 2 & 2 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & -1 \\ -1 & 2 & 2 \end{bmatrix}^{-1} = \frac{1}{7} \begin{bmatrix} 3 & -2 & 1 \\ -3 & 2 & -1 \\ 6 & -4 & 2 \end{bmatrix}$$

and

$$L\left(\begin{bmatrix} 4 \\ -4 \\ 0 \end{bmatrix}\right) = \frac{20}{7} \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}$$

**Problem 3.4** (20 pts). The green (dashed) house has been transformed to the red (solid) house by a linear transformation  $L : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ .



Desmos

Find  $[L]$  by first choosing basis  $\mathcal{G}$  (for the green house) and basis  $\mathcal{R}$  (for the red house) and find  $[L]_{\mathcal{G}, \mathcal{R}}$ , then use change of basis matrices to find  $[L]$ .

There are many options here; I will take

$$\mathcal{G} = \{\mathbf{v}_1, \mathbf{v}_2\} = \left\{ \begin{bmatrix} 2 \\ -2 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \end{bmatrix} \right\}$$

and

$$\mathcal{R} = \{\mathbf{u}_1, \mathbf{u}_2\} = \left\{ \begin{bmatrix} -2 \\ -2 \end{bmatrix}, \begin{bmatrix} -2 \\ 1 \end{bmatrix} \right\}$$

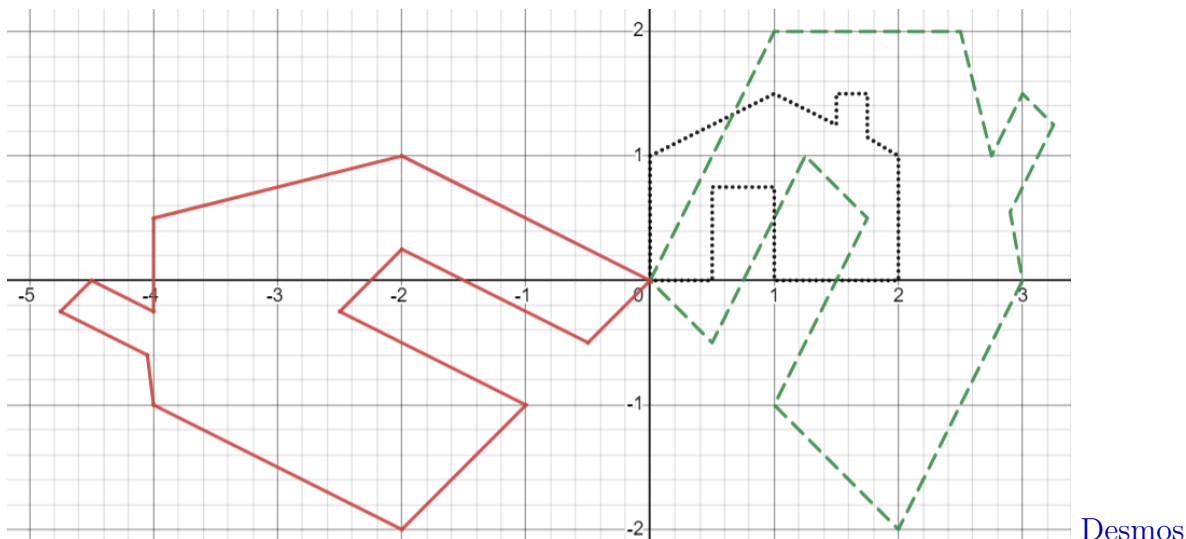
Then

$$[L]_{\mathcal{G}, \mathcal{R}} = [[L(\mathbf{v}_1)]_{\mathcal{R}} \ [L(\mathbf{v}_2)]_{\mathcal{R}}] = [[\mathbf{u}_1]_{\mathcal{R}} \ [\mathbf{u}_2]_{\mathcal{R}}] = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

and

$$[L] = R[L]_{\mathcal{G}, \mathcal{R}}G^{-1} = RG^{-1} = \begin{bmatrix} -2 & -2 \\ -2 & 1 \end{bmatrix} \begin{bmatrix} 2 & 1 \\ -2 & 2 \end{bmatrix}^{-1} = \frac{1}{3} \begin{bmatrix} 4 & -1 \\ -1 & 2 \end{bmatrix}$$

**Option 2:** There is another option here, although you need to explain what you are doing for full credit. You could view this as showing two transformations here:



You might consider the transformation from the "black" (dotted) house to the "green" (dashed) house as  $L_G$  and then the transformation from the "black" house to the "red" (solid) house  $L_R$ . Then with respect to just the standard basis. We have

$$[L_G] = \begin{bmatrix} 1 & 1 \\ -1 & 2 \end{bmatrix} \quad \text{and} \quad [L_R] = \begin{bmatrix} -1 & -2 \\ -1 & 1 \end{bmatrix}$$

Then the map from the green house to the red house with respect to the standard basis would be

$$[L] = [L_R \circ L_G^{-1}] = [L_R][L_G]^{-1} = \frac{1}{3} \begin{bmatrix} 4 & -1 \\ -1 & 2 \end{bmatrix}$$



## 4 Theory and Proofs (30 points; 10 points each)

Choose three of the four options. If you try more than three, I will grade only the first three, not the best three. You must decide what should be graded. These will be due on 3/9 in class. Make sure your work is complete and clear. Explain your work; a proof is not just a collection of math symbols, it is an explanation of why something is true.

This part is take-home. You should complete this work on your own without consulting websites, friends, the Math Center, etc.

**Problem 4.1** (10 points). Let  $V$  be a vector space with  $\dim(V) = n$  and  $U \subseteq V$  a subspace with  $\dim(U) = k$ , show that there is a subspace  $W \subseteq V$  with  $\dim(W) = l$  so that  $l + k = n$  and  $V = U \oplus W$ .

Let  $\mathcal{B} = \{\mathbf{u}_1, \dots, \mathbf{u}_k\}$  be a basis for  $U$ . Extend  $\mathcal{B}$  to a basis  $\mathcal{C}$  for  $V$ , namely,  $\mathcal{C} = \{\mathbf{u}_1, \dots, \mathbf{u}_k, \mathbf{w}_1, \dots, \mathbf{w}_l\}$  where  $n = l + k$ . Let  $W = \text{span}\{\mathbf{w}_1, \dots, \mathbf{w}_l\}$ .

Clearly,  $U + W = V$  since any  $\mathbf{v} \in V$  can be written as  $\sum_{i=1}^k \alpha_i \mathbf{u}_i + \sum_{j=1}^l \beta_j \mathbf{w}_j$ .

Suppose  $\mathbf{v} \in U \cap W$ , then  $\mathbf{v} = \sum_{i=1}^k \alpha_i \mathbf{u}_i = \sum_{j=1}^l \beta_j \mathbf{w}_j$ . But then

$$\sum_{i=1}^k \alpha_i \mathbf{u}_i - \sum_{j=1}^l \beta_j \mathbf{w}_j = \mathbf{v} - \mathbf{v} = \mathbf{0}$$

Since  $\mathcal{C}$  is independent we have that  $\alpha_i = \beta_j = 0$  for all  $i$  and  $j$ . So  $\mathbf{v} = \mathbf{0}$ . Thus  $U \cap W = \{\mathbf{0}\}$ .

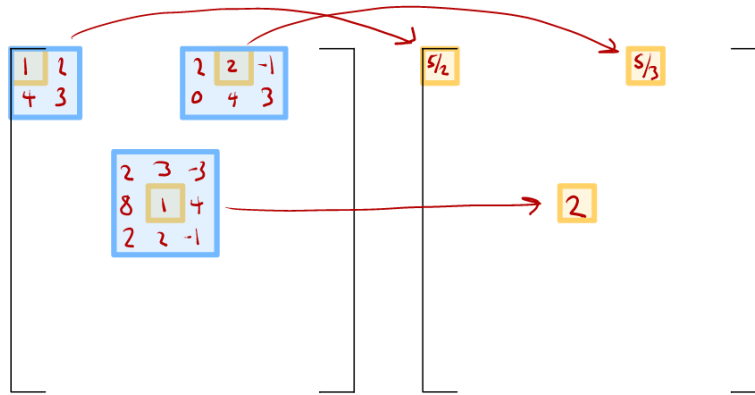
**Problem 4.2** (10 points). Show that if  $L : V \rightarrow W$  is linear and  $\ker(L) = \{\mathbf{0}\}$ , then for any linearly independent set  $\{\mathbf{v}_1, \dots, \mathbf{v}_k\}$  from  $V$ ,  $\{L(\mathbf{v}_1), \dots, L(\mathbf{v}_k)\}$  is independent.

Suppose  $\sum \alpha_i L(\mathbf{v}_i) = \mathbf{0}$ , then we can use linearity to get

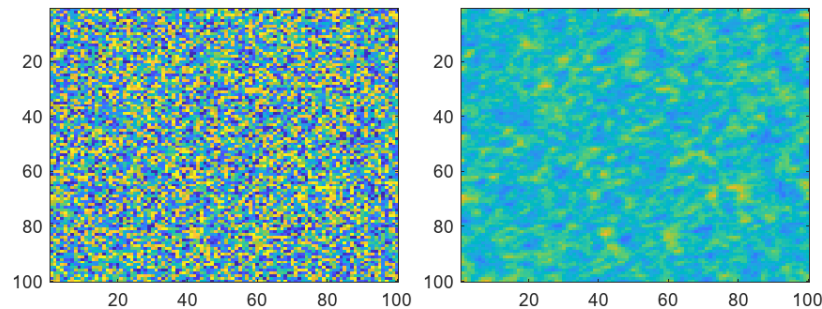
$$L\left(\sum \alpha_i \mathbf{v}_i\right) = \mathbf{0}$$

But since  $\ker(L) = \{\mathbf{0}\}$  we have  $\sum \alpha_i \mathbf{v}_i = \mathbf{0}$  and as the  $\mathbf{v}_i$ 's are independent we know  $\alpha_i = 0$  for all  $i$  and so the  $L(\mathbf{v}_i)$ 's are shown independent.

**Problem 4.3** (10 points). Consider the following operation. Given an  $m \times n$  matrix  $A$ ,  $S(A)$  will be the  $m \times n$  matrix where each entry has been replaced by the average of the entry with its neighbors.  $S$  for “smear” (often called “blur”).



Example applied to random noise (numeric value represented by color):



[MATLAB code](#)

**Main Question (7 points):** Show that  $S : \mathbb{R}^{m \times n} \rightarrow \mathbb{R}^{m \times n}$  is linear.

**Thought Question (2 points):** What do you think would happen if you repeatedly applied smearing? That is, consider  $A_1 = A$ ,  $A_2 = S(A)$ ,  $A_3 = S(S(A)) = S^2(A)$ , etc. What do you think  $S^k(A)$  would look like for large  $k$  (as an image)? What would the limiting value be?

**How might you verify your conjecture? (1 points):** Since  $S$  is linear

$$A_k = S^k(A) = \sum_{i,j} A_{i,j} S^k(E_{i,j})$$

How might you use this to verify your conjecture?

**Linearity:** To prove linearity, we need to see that  $S(\alpha A + \beta B) = \alpha \cdot S(A) + \beta \cdot S(B)$ . For this, it suffices to show that

$$S(\alpha A + \beta B)_{i,j} = \alpha \cdot S(A)_{i,j} + \beta \cdot S(B)_{i,j}$$

There are 3 cases (or four depending on how you count). Either  $(i, j)$  is at a corner and has four things to average over,  $(i, j)$  is on an edge, but not a corner, so there are six things to average over, or  $(i, j)$  is in neither of those cases and there are nine entries

to average over. (See the picture.) All three cases are essentially the same, so take the third case

$$\begin{aligned}
S(\alpha A + \beta B)_{i,j} &= \frac{1}{9} \left( \sum_{\substack{i-1 \leq \hat{i} \leq i+1 \\ j-1 \leq \hat{j} \leq j+1}} \right) (\alpha A_{\hat{i},\hat{j}} + \beta B_{\hat{i},\hat{j}}) \\
&= \alpha \frac{1}{9} \left( \sum_{\substack{i-1 \leq \hat{i} \leq i+1 \\ j-1 \leq \hat{j} \leq j+1}} A_{\hat{i},\hat{j}} \right) + \beta \left( \frac{1}{9} \sum_{\substack{i-1 \leq \hat{i} \leq i+1 \\ j-1 \leq \hat{j} \leq j+1}} B_{\hat{i},\hat{j}} \right) \\
&= \alpha \cdot S(A)_{i,j} + \beta \cdot S(B)_{i,j}
\end{aligned}$$

**Conjecture:** There might be a variety of answers here. The thing which looks reasonable to me is that as you repeatedly smear, the image just becomes a uniform color, that is all the entries in the matrix just approach a fixed value. Since you are repeatedly taking averages a good guess is that the whole thing just approached the  $m \times n$  matrix where all entries are  $\text{mean}(A)$  (the average over all entries in  $A$ .) We might write this as  $A_n \rightarrow \text{mean}(A) \mathbf{1}_{m \times n}$  where  $\mathbf{1}_{m \times n}$  is an  $m \times n$  matrix of all 1's.

If you play with the MATLAB, this seems a reasonable guess.

**Method to verify conjecture:** Since  $S^n(A) = \sum A_{i,j} S^n(E_{i,j})$ . So it would suffice to see that  $S^n(E_{i,j}) = \frac{1}{nm} \mathbf{1}_{m \times n}$ .

**Problem 4.4** (10 points). Suppose  $A$  is a  $n \times n$  matrix and  $A^{m+1} = A^m$  for some  $m$ , then already  $A^{n+1} = A^n$ .

**Hint:** This is similar, but different, to one from the exam you had for practice. You can use the same ideas. Note that  $A^{m+1} = A^m$  can be written as  $A^m(A - I) = O$ , do remember that  $AB = O$  does not mean that  $A = O$  or  $B = O$ .

There are several ways to proceed. Here is one taken from the old exam. First, notice that  $A^m(A - I) = O$  is equivalent to  $\text{NS}(A^{m+1}(A - I)) = \mathbb{R}^n$ .

Note that  $\text{NS}(A^{m+1}(A - I)) \supseteq \text{NS}(A^m(A - I))$  for all  $m$  since  $A^m(A - I)\mathbf{x} = \mathbf{0} \implies A^{m+1}(A - I)\mathbf{x} = A(A^m(A - I))\mathbf{x} = A\mathbf{0} = \mathbf{0}$ .

If  $m$  is least so that  $A^m(A - I) = O$ , then

$$\text{NS}(A - I) \subsetneq \text{NS}(A(A - I)) \subsetneq \cdots \subsetneq \text{NS}(A^m(A - I)) = \mathbb{R}^n$$

This is a strictly increasing sequence of subspaces of  $\mathbb{R}^n$ . Setting  $n_i = \dim(\text{NS}(A^i(A - I)))$  we have

$$0 \leq n_0 < n_1 < \cdots < n_m = n$$

It is clear from this that  $m \leq n$  and so  $A^n(A - I) = O$ , that is,  $A^{n+1} = A^n$ , is the fixed power.