Math 571 - Homework 4 (05.22)

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Problem 1 (R:3:8)**.** Suppose $\sum_n a_n$ converges and (b_n) is monotonic and bounded, show that $\sum_n a_n b_n$ converges.

Problem 2 (R:3:9)**.** Find the radius of convergence of the following power series.

a) $\sum_n n^3 z^n$

b) $\sum_n \frac{2^n}{n!} z^n$

c) $\sum_n \frac{2^n}{n^2} z^n$

d) $\sum_n \frac{n^3}{3^n} z^n$

Problem 3 (R:3:11)**.** Suppose $a_n > 0$, $s_n = \sum_{i=1}^n a_i$, and $\sum_i a_i = \lim_i s_i$ diverges.

a) Show that $\sum_i \frac{a_i}{a_i+1}$ diverges.

b) Show that $\frac{a_N}{s_N} + \frac{a_{N+1}}{s_{N+1}} + \cdots \frac{a_{N+k}}{s_{N+k}} \geq 1 - \frac{s_N}{s_{N+k}}$ and deduce that $\sum_i \frac{a_i}{s_i}$ diverges.

c) Show that $\frac{a_N}{s_N^2} \leq \frac{1}{s_{N-1}} - \frac{1}{s_N}$ and deduce that $\sum_i \frac{a_i}{s_i^2}$ converges.

Problem 4 (R:3:16(18)*)**.** Fix $\alpha > 1$ and $x_1 > \sqrt{\alpha}$, define

$$x_{n+1} = \frac{1}{2}\left(x_n + \frac{\alpha}{x_n}\right)$$

a) Prove that (x_n) decreases monotonically and $\lim_{n\to\infty} x_n = \sqrt{\alpha}$.

b) Let $\varepsilon_n = x_n - \sqrt{\alpha}$ be the error at the n^{th} term, show that

$$\varepsilon_{n+1} = \frac{\varepsilon_n^2}{2x_n} < \frac{\varepsilon_n^2}{2\sqrt{\alpha}}$$

Setting $\beta = 2\sqrt{\alpha}$, gives

$$\varepsilon_{n+1} < \beta\left(\frac{\varepsilon_1}{\beta}\right)^{2^n}$$

c) Choose a number $\alpha > 3$ find a bound for how many terms are need to compute $\sqrt{\alpha}$ correct to 20 decimal places where x_1 is chosen minimally so that $x_1^2 > \alpha$. Prove your answer and do the computation. You might use Python or MATLAB.

d) Replace the recursion above by

$$x_{n+1} = \frac{p-1}{p}\, x_n + \frac{\alpha}{p}\, x_n^{-p+1}$$

Discuss the behavior of (x_n) under suitable conditions. Don't bother trying to compute a recursive expression for ε_n in this case, but do prove your claims.

Problem 5. Recall the bijection, ϕ, of $\mathbb{R}^2 \cup \{\infty\}$ onto the unit sphere, S^2, in $\mathbb{R}^3$ given by $(x, y) \mapsto \left(\frac{2x}{1+x^2+y^2}, \frac{2y}{1+x^2+y^2}, \frac{1-(x^2+y^2)}{1+x^2+y^2}\right)$. This is called the stereographic projection. Let d be the induced metric on $\mathbb{R}^2$ given by $d((a, b), (c, d)) = \|\phi((a, b)) - \phi((c, d))\|_2$ in $\mathbb{R}^3$. (The (standard) Euclidean distance between points on the sphere in $\mathbb{R}^3$.)

Consider a sequence $((x_i, y_i))_{i\in\mathbb{N}}$ from $\mathbb{R}^2$ so that $\lim_i \|(x_i, y_i)\|_2^2 = \infty$.

Explain why this is a Cauchy sequence in $(\mathbb{R}^2, d)$. Does this sequence have a limit in $(\mathbb{R}^2, d)$?

Problem 6 (R:3:21*)**.** Let E_n be a descending sequence of closed subsets in a complete metric space, i.e., $E_{n+1} \subseteq E_n$. Notice that $\operatorname{diam}(E_{n+1}) \leq \operatorname{diam}(E_n)$ so $\lim_n \operatorname{diam}(E_n) = \delta$ exists in $[0, \infty]$. In each of the following cases is it true that $\bigcap_n E_n \neq \emptyset$?

a) $\delta = \infty$.

b) $0 < \delta < \infty$.

c) $\delta = 0$.

Of course, provide a proof or counter example in each case. In the case that $\bigcap_n E_n \neq \infty$, what can you say about $|\bigcap_n E_n|$.

Problem 7 (R:3:23)**.** Let (X, d) be a metric space and $(x_i)_{i\in\mathbb{N}}$ and $(y_i)_{i\in\mathbb{N}}$ be two Cauchy sequences. Show that $(d(x_i, y_i))_{i\in\mathbb{N}}$ converges.