Name: ______________________________ Quiz 2 (Make Up)- MAT345

Problem 1 (20 points; 4 points each)**.** Decide if each of the following are true or false and provide a small proof or counterexample in each case. All vector spaces are assumed to be finite-dimensional here.

(a) True Given a basis $\mathcal{B} = \{\boldsymbol{b}_1, \ldots, \boldsymbol{b}_n\}$ for a vector space V of dimension n. There are only finitely many subspaces U of V so that U has a basis which is a subset of $\mathcal{B}$.

There are exactly 2^n subsets of $\mathcal{B}$ and these correspond to all of the acceptable subspaces.

(b) False Given a spanning set $\mathcal{S} = \{\boldsymbol{v}_1, \ldots, \boldsymbol{v}_k\}$ for a vector space V and a subspace U of V. $\mathcal{S}$ can be reduced (by throwing out some vectors) to a basis for U.

It is easy to produce an example where this is false. For example, $\mathcal{S} = \{\boldsymbol{e}_1, \boldsymbol{e}_2\}$ spans $\mathbb{R}^2$, but no subset of $\mathcal{S}$ spans the subspace $U = \operatorname{span}\{(1, 1)\}$, i.e., the line $x = y$.

We can use the previous. There are infinitely many subspaces of V, only finitely many of which have a basis from $\mathcal{S}$, so there are infinitely many subspaces that are not spanned by vectors from $\mathcal{S}$.

(c) True If $\boldsymbol{v} \in \operatorname{span}\{\boldsymbol{v}_1, \ldots, \boldsymbol{v}_n\}$, it is possible that there are distinct $\boldsymbol{c}, \boldsymbol{b} \in \mathbb{R}^n$ such that $\boldsymbol{v} = \sum_{i=1}^n c_i \boldsymbol{v}_i = \sum_{i=1}^n b_i \boldsymbol{v}_i$.

There would be many (infinitely) ways to represent a single vector $\boldsymbol{v}$ unless the given set of vectors were independent.

(d) True If $\{\boldsymbol{v}_1, \ldots, \boldsymbol{v}_n\}$ spans a vector space V and $\{\boldsymbol{u}_1, \ldots, \boldsymbol{u}_n\} \subseteq V$ is independent. Then $\{\boldsymbol{u}_1, \ldots, \boldsymbol{u}_n\}$ spans V.

That the $\boldsymbol{v}_i$'s span V tells us that $\dim(V) \leq n$. That the $\boldsymbol{u}_i$'s are independent tells us that $\dim(V) \geq n$. Together we know that $\dim(V) = n$, and hence both the given sets of vectors must be a basis.

(e) False Given U and W subspaces of a vector space V so that $U + W = V$ ($U + W = \{\boldsymbol{u} + \boldsymbol{w} \mid \boldsymbol{u} \in U \text{ and } \boldsymbol{w} \in W\}$), then for every $\boldsymbol{v} \in V$, there is a unique pair $\boldsymbol{u} \in U, \boldsymbol{w} \in W$ so that $u + w = v$.

U and W could be two planes in $V = \mathbb{R}^3$ that intersect in a line L, so $U \cap V = L$. Any $\boldsymbol{u} \in U$ can be written as $(\boldsymbol{u} - \boldsymbol{l}) + \boldsymbol{l} = \boldsymbol{u} + \boldsymbol{0}$ for any $\boldsymbol{l} \in L$. So there are infinitely many ways to write $\boldsymbol{u}$ as an element of $U + W$.

Note: If $U \cap W = \{\boldsymbol{0}\}$, then the decomposition becomes unique.

Problem 2 (10 pts)**.** A square matrix A is called **anti-symmetric** if $A^T = -A$.

a) Show that the anti-symmetric 3×3 matrices form a subspace of all 3×3 matrices.

b) Give a basis, $\mathcal{B}$, for the 3×3 anti-symmetric matrices.

c) Give representation $[\boldsymbol{v}]_{\mathcal{B}}$ for $\boldsymbol{v} = \begin{bmatrix} 0 & 1 & 2 \\ -1 & 0 & 3 \\ -2 & -3 & 0 \end{bmatrix}$ with respect to the basis that you gave.

Denote by U the set of 3×3 anti-symmetric matrices. Clearly, $O = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \in U$. Next, we need to show that U is closed under scalar multiplication and addition. This is done by just taking arbitrary elements of U and computing:

$$\begin{bmatrix} 0 & a & b \\ -a & 0 & c \\ -b & -c & 0 \end{bmatrix} + \begin{bmatrix} 0 & A & B \\ -A & 0 & C \\ -B & -C & 0 \end{bmatrix} = \begin{bmatrix} 0 & a+A & b+B \\ -(a+A) & 0 & c+C \\ -(b+B) & -(c+C) & 0 \end{bmatrix}$$

and

$$\alpha \begin{bmatrix} 0 & a & b \\ -a & 0 & c \\ -b & -c & 0 \end{bmatrix} = \begin{bmatrix} 0 & \alpha a & \alpha b \\ -\alpha a & 0 & \alpha c \\ -\alpha b & -\alpha c & 0 \end{bmatrix}$$

A basis is clearly given by

$$\mathcal{B} = \left\{ \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix} \right\}$$

With this basis, clearly

$$\boldsymbol{v} = \begin{bmatrix} 0 & 1 & 2 \\ -1 & 0 & 3 \\ -2 & -3 & 0 \end{bmatrix} = (1)\begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} + (2)\begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{bmatrix} + (3)\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix}$$

So $[\boldsymbol{v}]_{\mathcal{B}} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$

Problem 3. Find a basis for $\operatorname{span}\{\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3, \boldsymbol{v}_4, \boldsymbol{v}_5\}$ from the given vectors

$$\boldsymbol{v}_1 = \begin{bmatrix} 1 \\ 2 \\ -2 \\ -3 \end{bmatrix}, \boldsymbol{v}_2 = \begin{bmatrix} 2 \\ 4 \\ -4 \\ -6 \end{bmatrix}, \boldsymbol{v}_3 = \begin{bmatrix} 0 \\ 1 \\ -2 \\ 4 \end{bmatrix}, \boldsymbol{v}_4 = \begin{bmatrix} -3 \\ -4 \\ 2 \\ 17 \end{bmatrix}, \boldsymbol{v}_5 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ -3 \end{bmatrix}$$

Let

$$A = \begin{bmatrix} 1 & 2 & 0 & -3 & 0 \\ 2 & 4 & 1 & -4 & 0 \\ -2 & -4 & -2 & 2 & 1 \\ -3 & -6 & 4 & 17 & -3 \end{bmatrix}$$

$$A \underset{\substack{R_2-2R_1\to R_2 \\ R_3+2R_1\to R_3 \\ R_4+3R_1\to R_4}}{\Longrightarrow} \begin{bmatrix} 1 & 2 & 0 & -3 & 0 \\ 0 & 0 & 1 & 2 & 0 \\ 0 & 0 & -2 & -4 & 1 \\ 0 & 0 & 4 & 8 & -3 \end{bmatrix} \underset{\substack{R_3+2R_2\to R_3 \\ R_4-4R_2\to R_4}}{\Longrightarrow} \begin{bmatrix} 1 & 2 & 0 & -3 & 0 \\ 0 & 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & -3 \end{bmatrix} \underset{R_4+3R_3\to R_4}{\Longrightarrow} \begin{bmatrix} \boxed{1} & 2 & 0 & -3 & 0 \\ 0 & 0 & \boxed{1} & 2 & 0 \\ 0 & 0 & 0 & 0 & \boxed{1} \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

So $\mathcal{B} = \{\boldsymbol{v}_1, \boldsymbol{v}_3, \boldsymbol{v}_5\}$ is a basis. (This is all you need.)

In fact, from our CR decomposition, we know

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ -2 & -2 & 1 \\ -3 & 4 & -3 \end{bmatrix} \begin{bmatrix} \boxed{1} & 2 & 0 & -3 & 0 \\ 0 & 0 & \boxed{1} & 2 & 0 \\ 0 & 0 & 0 & 0 & \boxed{1} \end{bmatrix}$$

So we know $\boldsymbol{v}_2 = 2\boldsymbol{v}_1$ and $\boldsymbol{v}_4 = -3\boldsymbol{v}_1 + 2\boldsymbol{v}_3$.