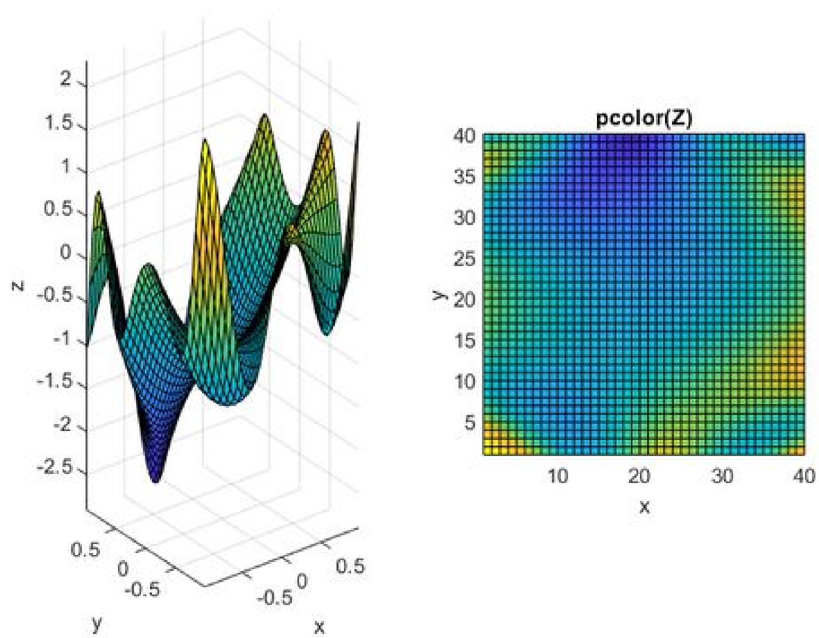
SVD Decomposition of a Surface and Dimension Reduction

Low Rank Approximation of a Surface

The code for this example is here.

Consider a surface lying over the square patch in the x, y-plane given by $[-1, 1] \times [-1, 1]$. The z-coordinates are produced by a function $z = f(x, y)$. We can almost view this as an infinite matrix, for example, at row $y = .25$, and column $x = -.15$, there is the entry $z = f(-.15, .25)$. Discretize $[-1, 1]$ by splitting it into 40 sub-intervals all of width $2/40 = 0.05$ and get a 40×40 matrix of $z_{i,j}$ values corresponding to $f(x_i, y_j)$ where (x_i, y_j) is the center of the $(i, j)^{\text{th}}$ square area, $A_{i,j}$ with area $dA_{i,j} = (0.05)^2$. Here $x_i = (-1 + 0.025) + i \cdot 0.05 = -0.975 + i \cdot 0.05$. Similarly $y_j = -0.975 + j \cdot 0.05$. The function taken for this example is $f(x, y) = \sin(2\pi xy^2 - y) - \cos(4\pi x^2 y + x) + x^2 - y^3$. The 40×40 matrix of values will be called Z, and MATLAB can provide a surface plot together with a coloring of the matrix that represents the z values.

This is all very easy to achieve and plot in MATLAB. Here is code taken from the provided MATLAB live script SVDProject.mlx:



Here is what Z looks like as a matrix:

$$Z = \begin{bmatrix} 2.3070 & 2.1675 & 1.9471 & 1.6540 & \cdots \\ 2.1036 & 1.8324 & 1.5112 & 1.1539 & \cdots \\ 1.6833 & 1.3435 & 0.9860 & 0.6241 & \cdots \\ 1.1526 & 0.8045 & 0.4634 & 0.1390 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}$$

It is important to understand how the indices in the Z-matrix correspond to the x, y-plane so here is X and Y are

$$X = \begin{bmatrix} -0.9750 & -0.9250 & -0.8750 & -0.8250 & \cdots \\ -0.9750 & -0.9250 & -0.8750 & -0.8250 & \cdots \\ -0.9750 & -0.9250 & -0.8750 & -0.8250 & \cdots \\ -0.9750 & -0.9250 & -0.8750 & -0.8250 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}$$

and

$$Y = \begin{bmatrix} -0.9750 - 0.9750 - 0.9750 - 0.9750 & \cdots & & & & \\ -0.9250 - 0.9250 - 0.9250 - 0.9250 & \cdots & & & & \\ -0.8750 - 0.8750 - 0.8750 - 0.8750 & \cdots & & & & \\ -0.8250 - 0.8250 - 0.8250 - 0.8250 & \cdots & & & & \\ \vdots & \vdots & \vdots & \vdots & \ddots & \end{bmatrix}$$

So $X(i,j) = x_j$ and $Y(i,j) = y_i$ so $z(x_j, y_i) = z(X(i,j), Y(i,j)) = Z(i,j)$.

Let $Z = U\Sigma V^T$ be the SVD decomposition of Z so that $Z = \sum_{i=1}^{k} \sigma_i \boldsymbol{u}_i \boldsymbol{v}_i^T$, where $k = \text{rank}(Z)$.

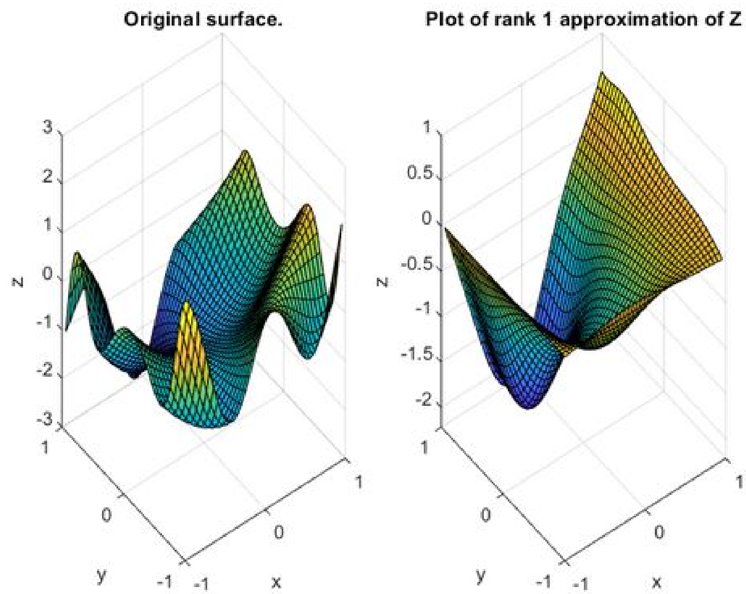
Let $Z(r)$ be the "best" rank r approximation of Z, this is given by

$$Z(r) = \sum_{i=1}^{r} \sigma_i \boldsymbol{u}_i \boldsymbol{v}_i^T$$

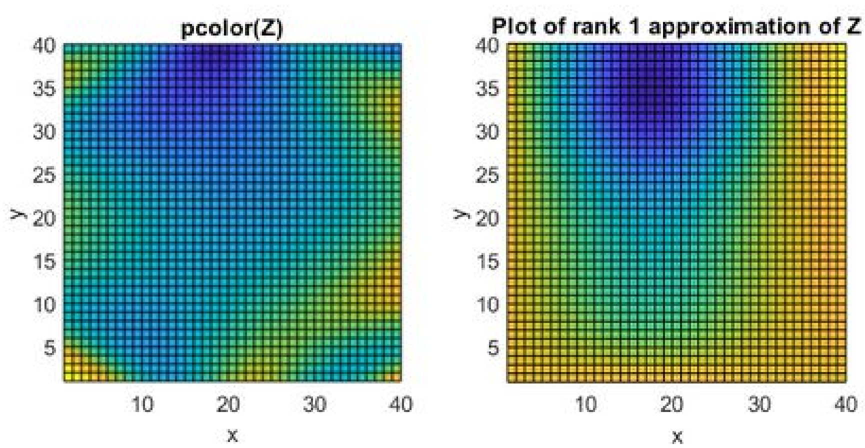
Think about the interpretation of $\boldsymbol{u}_i \boldsymbol{v}_i^T$ here. These are rank-one matrices, each of which can be viewed as a surface/image, and the original image surface/image Z is the weighted sum of these rank-one surfaces/images. The rank-one surface $\boldsymbol{u}_i \boldsymbol{v}_i^T$ is very simple to interpret. All horizontal slices look like v_i, itself viewed as a function of "x" and u_i viewed as a function of "y." So it is as if $f_i(x,y) = u_i(y)v_i(x)$ is a separable function of two variables, and $f(x,y) = \sum_i \sigma_i f_i(x,y) = \sum_i \sigma_i u_i(y) v_i(x)$. Then the *rank-r* approximation to f is $f_r(x,y) = \sum_{i=1}^{r} \sigma_i u_i(y) v_i(x)$ for $r \leq k = \text{rank}(Z)$.

The rank 1 approximation

Here are the images of the rank one approximations to Z:

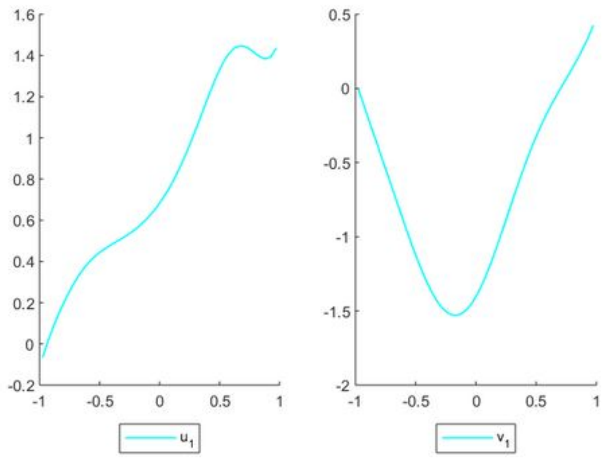


Original surface compared to the rank 1 approximation $\sigma_1 \boldsymbol{u}_1 \boldsymbol{v}_1^T$

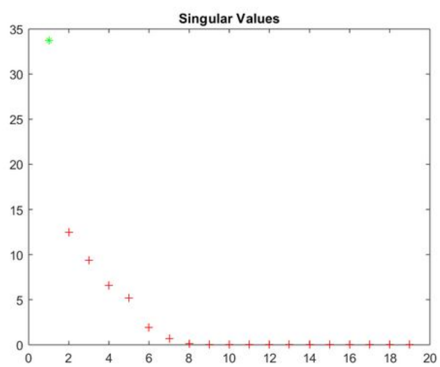


Coloring of the two matrices Z and $\sigma_1 \boldsymbol{u}_1 \boldsymbol{v}_1^T$ (the "best" rank 1 approximation to Z).

Here are the images of the two functions generating these, so think of this as $f_1(x, y) = \sigma_1 u_1(x) v_1(y)$.



The scree plot gives some idea about how well the rank-one surface approximates the original.



We can measure the percentage of the *information* or *variance* in Z captured by our rank r approximation $Z(r)$ as

$$\frac{\sum_{i=1}^{r} \sigma_i^2}{\sum_{i=1}^{k} \sigma_i^2}$$

For this rank 1 approximation we have

$$\frac{\sigma_1^2}{\sum_{i=1}^{k} \sigma_i^2} = 78.07\%$$

so about 78.07% of the "information" in the original surface is captured by this rank 1 approximation.

As another familiar measure for how well the rank-one approximation $Z(1) = \sigma_1 u_1 v_1^T$ would be to take the L^2 distance between the surfaces. This would be

$$\|f(x, y) - f_1(x_y)\|^2 = \iint_A (f(x, y) - f_1(x, y))^2 \, dx \, dy$$

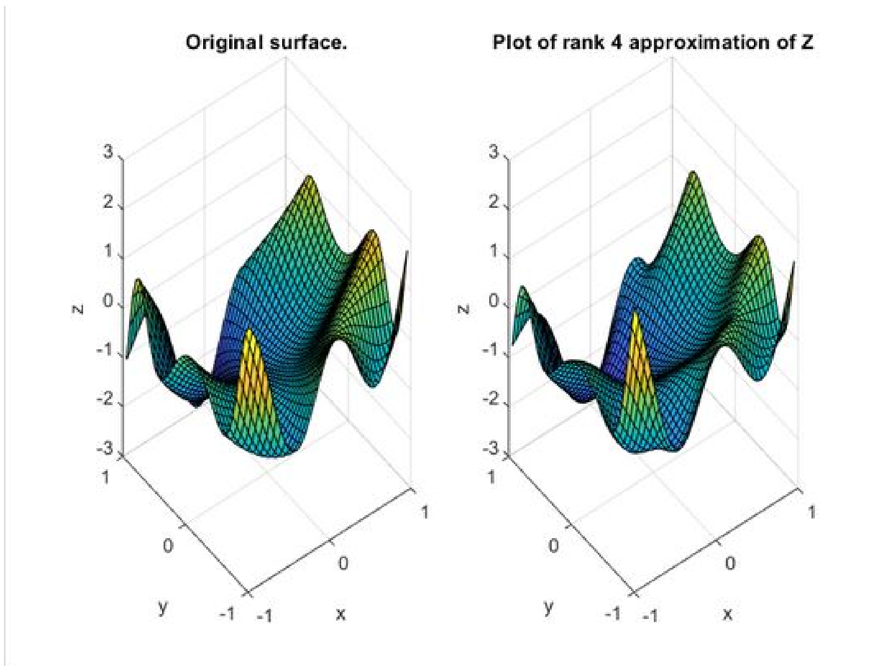
which is approximated here by:

$$L^2\text{-error: } \sum_i \sum_j (Z_{i,j} - Z(1)_{i,j})\, dA_{i,j} = \sum_i \sum_j (Z_{i,j} - \sigma_i u_1(i) v_1^T(j))^2\, dA_{i,j}$$

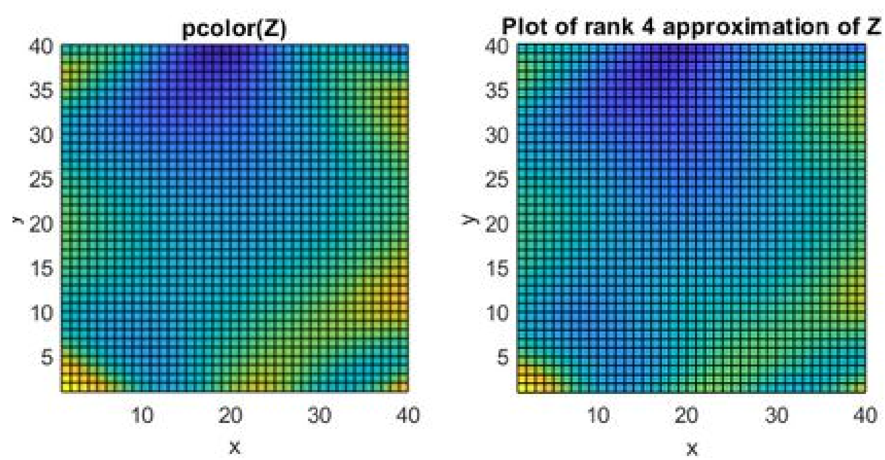
For the rank-one approximation, we get the L^2-error to be 0.7999.

Rank 4 Approximation.

Above, the rank-one approximation was investigated. Here, we look at the ran-4 approximation. The rank 4 approximation $Z(4) = \sigma_1 \boldsymbol{u}_1 \boldsymbol{v}_1^T + \sigma_2 \boldsymbol{u}_2 \boldsymbol{v}_2^T + \sigma_3 \boldsymbol{u}_3 \boldsymbol{v}_3^T + \sigma_4 \boldsymbol{u}_4 \boldsymbol{v}_4^T$ captures about 97.736% of the information! Here are plots of the rank-4 approximation of Z vs Z itself.

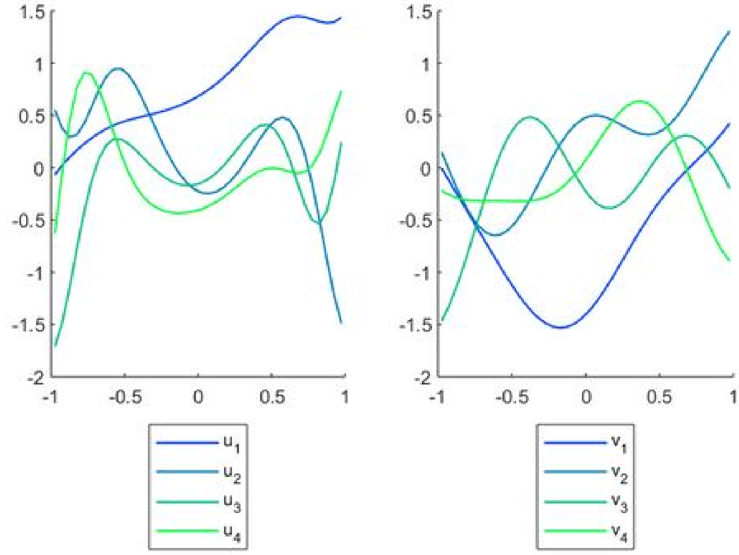


Original surface compared to $Z(4)$ (the rank 4 approximation)



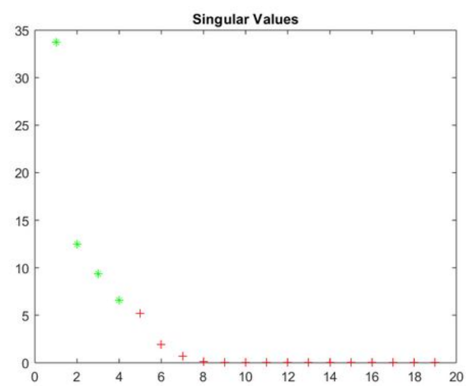
Here are the coloring of the two matrices Z and $Z(4)$ (the "best" rank 4 approximation to Z).

It is clear that more details of the original surface are captured here.



Plot of $\boldsymbol{u}_1, \boldsymbol{u}_2, \boldsymbol{u}_3, \boldsymbol{u}_4$ (the horizontal slices) and $\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3, \boldsymbol{v}_4$ (the vertical slices).

The following is again the scree plot of the singular values showing the relative weights of the 4 singular values used.



$$\frac{\sigma_1^2 + \sigma_2^2 + \sigma_3^2 + \sigma_4^2}{\sum_{i=1}^{k} \sigma_i^2} = 0.9785 = 97.85\%$$

So about 97.85% of the original information from the surface is contained in the rank 4 approximation.

The L^2-error is

$$\sum_{\substack{1 \le i \le 41 \\ 1 \le j \le 41}} (Z(i,j) - Z(4)(i,j))^2 \, dA = 0.07828$$

Which is significantly smaller than the rank-one L^2-error.

Image Reduction Project

A surface can simply be interpreted as a matrix of numbers; similarly, a gray image can be interpreted as a matrix of "gray values," which, if one was so inclined, could itself be interpreted as a surface. Now that we have investigated low-rank approximations of a surface, let's consider low-rank approximations of an image. This immediately leads to the idea of dimension reduction and one method of *compressing* images.

The file "SVDImRed.mlx" has code that reads in an image from the web or your computer, converts the image to a gray image which is just a matrix like Z above, in this case the matrix is called "GrayImg." An SVD decomposition is created GrayImg $= USV^T$. You can view the image reconstructed exactly as described above for the reconstruction of the surface by using the command "show(M,N)" which will reconstruct the image using the M^{th} through N^{th} singular values/singular vectors, i.e.,

$$\text{GrayImg}(M, N) = \sum_{i=M}^{N} \sigma_i \boldsymbol{u}_i \boldsymbol{v}_i^T$$

Here, you should complete the following tasks and prepare a report on what you find.

- Look at the first ten rank 1 reconstructions, that is, run show(i,i) for each i for values $i = 1$ through $i = 10$. Comment on what you think the i^{th} rank 1 image (singular value) is capturing from the original image. Next, look at the rank 10 reconstruction with show(1,10). Comment on what you notice. You might also try looking at show(5,20) and other combinations.
- How much data is in the original image? Each gray value is a 1-byte = 8-bit positive integer value between 0 and 255. So, each pixel requires 1 byte of information.
- How much data must be stored for the rank 10 reconstruction of the image? What is the percent reduction in data storage?
- What percentage of the variance is accounted for in the rank 10 reconstruction?
- What rank reconstruction is required to capture 99% of the initial variance (information)? What is the percent reduction in storage for this rank? Discuss how this image looks compared to the original.

Recall that the variance explained is

$$\text{variance explained} = \frac{\sum_{i=M}^{N} \sigma_i^2}{\sum_{i=1}^{k} \sigma_i^2},$$

where k is the rank of the matrix corresponding to the original image.

Here is a sample project. Your work does not need to be exactly like this. This is simply one example of what you might do and notice.