Exam 1

This exam covers Topics 1 - 3, Topic 4 will not be covered here.

Part I: True/False (5 points each; 25 points)

For each of the following mark as true or false.

a) _____ If A and B are $n \times n$ lower triangular matrices, then AB is also lower triangular.

b) _____ If W is a subspace of a vector space V and $\mathcal{B}$ is a basis for W, then B can be extended to a basis for V.

c) _____ If W is a subspace of a vector space V and $\mathcal{B}$ is a basis for V, then B can be restricted to a basis for W.

d) _____ Let A be an $n \times n$ matrix, then $\det(A^2 + I) \geq 0$.

e) _____ For $n \times n$ matrices A and B, define $A \otimes B = AB - BA$. The operator $\otimes$ is associative.

Part II: Definitions and Theorems (5 points each; 25 points)

a) Define what it means for a set of vectors to be linearly independent.

b) State the Rank-Nullity Theorem.

c) If B arises from a matrix A by elementary row operations, what is the relationship between $\mathrm{NS}(A)$ and $\mathrm{NS}(B)$?

d) If V is a vector space with subspaces U and W, define $U + W$. Is $U + W$ also a subspace of V?

e) Give the definition of "A is an invertible matrix."

Part III: Computational (15 points each; 45 point)

a) Use row ops to find the echelon form of

$$A = \begin{bmatrix} 1 & 2 & 2 & -2 & 2 \\ -2 & 0 & -4 & 1 & -10 \\ 1 & 2 & -1 & 0 & 3 \end{bmatrix}$$

Make sure to write out your steps and indicate the row ops at each step.

b) Use the echelon matrix found above to find a basis for RS(A), NS(A), and CS(A). For RS(A) and CS(A) choose the basis from among the rows and columns of A.

c) Show that skew-symmetric 3×3 matrices form as subspace of all 3×3 matrices and find a basis for this subspace.

Part IV: Proofs (20 points each; 60 points) - Choose three!

Provide complete arguments/proofs for three of the following. If you try more than three, I will just grade the first three, so pick three your best three! If you want to ask me about these, please do.

a) It is easy to see that if A is invertible, then $AB = I$ or $BA = I \implies B = A^{-1}$. This shows the inverse is unique if it exists. Now use this fact to prove:

   Let A and B be square matrices with $AB = I$. Then A is invertible and $B = A^{-1}$.

b) It is easy to see that if $\boldsymbol{x} \in \mathbb{R}^n$ is viewed as a column vector, then $\boldsymbol{x}^T\boldsymbol{x} \geq 0$ and $\boldsymbol{x}^T\boldsymbol{x} = 0 \iff \boldsymbol{x} = 0$. This is because $\boldsymbol{x}^T\boldsymbol{x} = \sum_{i=1}^n x_i^2$. Use this fact to show

$\mathrm{NS}(A) = \mathrm{NS}(A^T A)$ for any matrix A.

c) If A is an $m \times n$ matrix and $A\boldsymbol{x} = \mathbf{0}$ for all $\boldsymbol{x} \in \mathbb{R}^n$, then $A = \mathbf{0}$, the all 0 matrix.

d) This one has three steps to help you out.

i) Show $\mathrm{NS}(A^{m+1}) \supseteq \mathrm{NS}(A^m)$ for all m.

ii) Show that if $\mathrm{NS}(A^{m+1}) = \mathrm{NS}(A^m)$, then $\mathrm{NS}(A^n) = \mathrm{NS}(A^m)$ for all $n \geq m$.

iii) If A is an $n \times n$ matrix and $A^{n+1} = \mathbf{0}$, then $A^n = \mathbf{0}$.