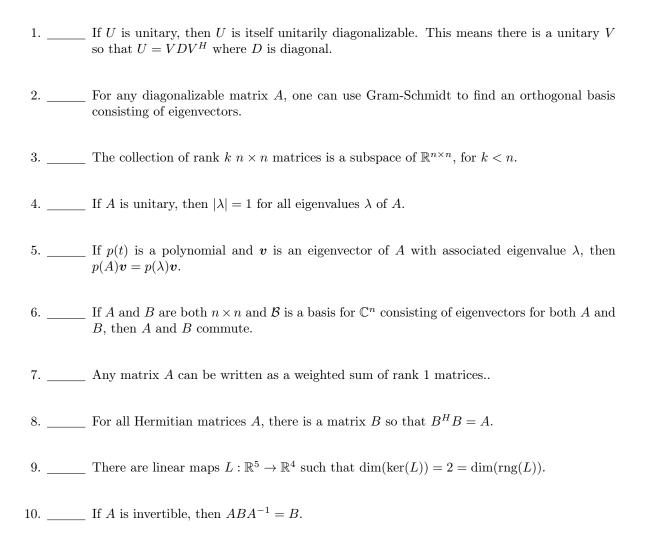
## Exam 2

To avoid any confusion, unless specified otherwise, vector spaces are complex vector spaces, inner-products are complex inner-products, and matrices are complex matrices. The standard inner product is  $\langle \boldsymbol{u}, \boldsymbol{v} \rangle = \boldsymbol{v}^H \boldsymbol{u} = \sum_{i=1}^n \bar{v}_i \boldsymbol{u}_i$ .

## Part I: True/False

Each problem is points for a total of 50 points. (5 points each.)

You do not need to justify the answers here, this is unlike the quizzes.



## Part II: Computational (60 points)

**Problem** 1. (15 points) Find B so that  $B^2 = A$  where

$$A = \begin{bmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{bmatrix}$$

**Problem** 2. (15 points) Find B so that  $B^HB = A$  where A is from (1).

**Problem** 3. (15 points) Find the best rank 2 approximation to A from (1) with respect to  $\|\cdot\|_F$ .

**Problem** 4. (15 points) Let

$$A = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix},$$

find the characteristic polynomial and all eigenvalues, both real and complex. Explain why A is diagonalizable and compute  $A^{2020}$ . Note, I do not ask you to diagonalize A.

## Part III: Theory and Proofs (45 points; 15 points each)

Pick three of the following four options. If you try all four, I will grade the first three, so if this is not what you intend, then just do three, or at least make it clear which I should grade.

**Problem** 1. Let S be a fixed invertible  $n \times n$  matrix. Let U be the set of  $n \times n$  matrices that are diagonalized by S, that is  $A = SD_AS^{-1}$  for some diagonal matrix A. Either prove that that U is a subspace of  $\mathbb{C}^{n \times n}$  or show that U is not a subspace of  $\mathbb{C}^{n \times n}$ .

**Problem** 2. Let A be a real  $m \times n$  matrix and let  $A^{\dagger} = V^T \Sigma^{\dagger} U$ , where  $A = U \Sigma V^T$  where U is  $m \times m$ , V is  $n \times n$ , both unitary,  $\Sigma$  is  $m \times n$  and  $\Sigma^{\dagger}$  is  $n \times m$  have the form

with  $\sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_r > 0$ .

Show that  $\hat{x} = A^{\dagger} b$  is a least-squares solution to Ax = b. Further show that  $A^{\dagger} = (A^T A)^{-1} A^T$  if A has linear independent columns, i.e.,  $\operatorname{rank}(A) = n$ .

Review the comments about Topic 5 DQ 2 in the Class Notes. Particularly point (2.) concerning what it means to be a least-squares solution to Ax = b.)

**Problem** 3. Prove that any complex inner-product  $\langle \cdot, \cdot \rangle_V$  on a complex vector space V, there is a basis  $\mathcal{U} = \{u_1, \dots, u_n\}$  so that

$$\langle oldsymbol{x}, oldsymbol{y} 
angle_V = [oldsymbol{y}]_\mathcal{U}^H [oldsymbol{x}]_\mathcal{U}$$

In other words for any finite dimensional inner-product space, there is a choice of basis, so that with respect to that basis, the inner-product is represented by the standard inner-product.

Here, in case you need it, is the definition of an inner-product. All the notation here is as I always use it in my notes.

