Math 571 - Homework 6

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Problem 6.1 (R:5:8)**.** Suppose f' is continuous on $[a,b]$ and $\varepsilon > 0$. Show that there is $\delta > 0$ so that for all t such that $0 < |t-x| < \delta$ and all $a \le x \le b$

$$\left|\frac{f(t)-f(x)}{t-x} - f'(x)\right| < \varepsilon$$

This could be stated as f' is *uniform continuity* on $[a,b]$ provided f' is continuous on $[a,b]$. Does this hold for vector-valued functions?

f' is continuous on $[a,b]$ and hence uniformly continuous there since $[a,b]$ is compact. Fix $\varepsilon > 0$ and $\delta > 0$ so that $|f'(x) - f'(x')| < \varepsilon$ whenever $|x - x'| < \delta$. Let $t \in N_\delta(x)$, then MVT gives $c \in N_\delta(x)$ so that

$$\left|\frac{f(t)-f(x)}{t-x} - f'(x)\right| = \left|f'(c) - f'(x)\right| < \varepsilon$$

It $f : [a,b] \to \mathbb{R}^n$ (or $\mathbb{C}^n$) then this is still true as all of the component functions satisfy the conclusion. That is $f(x) = (f_1(x), \dots, f_n(x))$ in the real case and $f_i(x) = u_i(x) + iv_i(x)$ in the complex case.

Problem 6.2 (R:5:9)**.** Suppose f is continuous on $\mathbb{R}$, and it is known that $f'(x)$ exists for all $x \neq 0$ and $f'(x) \to 3$ as $x \to 0$. Must $f'(0)$ exist?

By MVT $\frac{f(0+h)-f(0)}{h} = f'(c)$ for c between 0 and h and so $\lim_{h\to 0} \frac{f(0+h)-f(0)}{h} = \lim_{c\to 0} f'(c) = 3$.

Problem 6.3 (R:5:11)**.** Suppose f is defined in a nbhd of x and $f''(x)$ exists. Show that

$$\lim_{h\to 0} \frac{f(x+h)+f(x-h)-2f(x)}{h^2} = f''(x)$$

Show, by example, that the above limit can exist even if $f''(x)$ does not.

Let $F(h) = f(x+h) + f(x-h)$, then $F'(0) = f'(x) - f'(x) = 0$ and $F(h) - F(0) =$

$f(x+h)+f(x-h)-2f(x)$. Let $G(h)=h^2$, then by MVT

$$\begin{aligned}\frac{f(x+h)+f(x-h)-2f(x)}{h^2} &= \frac{F(h)-F(0)}{G(h)-G(0)}\\ &= \frac{F'(c)}{G'(c)} \text{ for some } c\in N_h(0)\\ &= \frac{f'(x+c)-f'(x-c)-2f'(x)}{2c}\\ &= \frac{1}{2}\frac{f'(x+c)-f'(x)}{c}+\frac{1}{2}\frac{f'(x-c)-f'(x)}{-c}\end{aligned}$$

So

$$\begin{aligned}\lim_{h\to 0}\frac{f(x+h)+f(x-h)-2f(x)}{h^2} &= \frac{1}{2}\lim_{c\to 0}\frac{f'(x+c)-f'(x)}{c}+\frac{1}{2}\lim_{d\to 0}\frac{f'(x+d)-f'(x)}{d}\\ &= \frac{1}{2}f''(x)+\frac{1}{2}f''(x)=f''(x)\end{aligned}$$

The "symmetry" in the initial formulation gives a hint at how to find the desired counterexample. Consider $f(x)=x^2\sin\left(\frac{1}{x}\right)$ for $x\neq 0$ and $f(0)=0$. This function is odd so

$$\frac{f(0+h)+f(0-h)-2f(0)}{h^2}=0$$

$f'(x)=2x\sin\left(\frac{1}{x}\right)-\cos\left(\frac{1}{x}\right)$ for $x\neq 0$. At $x=0$ we have

$$f'(0)=\lim_{h\to 0}\frac{f(h)-f(0)}{h}=\lim_{h\to 0}h\sin\left(\frac{1}{h}\right)=0$$

Clearly, $f'(x)$ is not even continuous at $x=0$ so $f''(0)$ DNE.

Problem 6.4 (R:5:16)**.** Suppose f is twice differentiable on $(0,\infty)$ and f'' is bounded on $(0,\infty)$, and $f(x)\to 0$ as $x\to\infty$. Show that $f'(x)\to 0$ as $x\to\infty$.

We have $f(x)=f(a)+f'(a)(x-a)+\frac{f''(c)}{2}(x-a)^2$ for some c between x and a. So $f'(a)=\frac{f(x)-f(a)}{x-a}-\frac{f''(c)}{2}(x-a)$. Let $x=a+h$ so we get

$$|f'(a)|\leq\left|\frac{f(a+h)-f(a)}{h}\right|+M|h|$$

Pick $\varepsilon>0$. Fixing h we can make $Mh<\varepsilon/2$ and letting $a\to\infty$ we can make $|f(a+h)|,|f(a)|<h\varepsilon/4$ and thus

$$|f'(a)|\leq\frac{|f(a+h)|}{h}+\frac{|f(a)|}{h}+Mh<\varepsilon/4+\varepsilon/4+\varepsilon/2=\varepsilon$$

Problem 6.5 (R:5:22)**.** Let $f:\mathbb{R}\to\mathbb{R}$ be differentiable on (a,b) and continuous on $[a,b]$. A point x is a **fixed point** of f iff $f(x)=x$.

(a) Show that if $f'(t)\neq 1$ for all $t\in(a,b)$, then f can have at most one fixed point.

If there were $x,y\in\mathbb{R}$ such that $x\neq y$, $f(x)=x$, and $f(y)=y$, then from MVT, there is c between x and y so that $f(x)-f(y)=x-y=f'(c)(x-y)$, but then $f'(c)=1$.

(b) Show that $f(t) = t + (1 + e^t)^{-1}$ satisfies $|f'(t)| < 1$ and f has no fixed points.

$f'(t) = 1 - \frac{e^t}{(1+e^t)^2}$, but $0 < \frac{e^t}{(1+e^t)^2} < 1$ so $0 < f'(t) < 1$.

It can't be the case that $f(t) = t$, since $t = t + (1 + e^t)^{-1}$ would imply $(1 + e^t)^{-1} = 0$ which is false.

(c) Show that if there is $A < 1$ so that $|f'(t)| \leq A$ for all $t \in \mathbb{R}$, then f has a fixed point and moreover given any $x_0 \in \mathbb{R}$ and taking $x_{n+1} = f(x_n)$ it turns out that $x_n \to x$ and $f(x) = x$ is the unique fixed point of f.

$$|x_n - x_{n-1}| = |f(x_{n-1}) - f(x_{n-2})| = |f'(c)||x_{n-1} - x_{n-2}| \leq A|x_{n-1} - x_{n-2}|$$

Continuing this gives

$$|x_n - x_{n-1}| \leq A^{n-1}|x_1 - x_0|$$

and thus for $n > m$

$$|x_n - x_m| \leq |x_n - x_{n-1}| + \cdots + |x_{m+1} - x_m| \leq (A^{n-2} + \cdots + A^m)|x_1 - x_0|$$

Now $A^{n-1} + \cdots + A^m = A^m(A^{n-m-1} + \cdots + 1) = A^m\left(\frac{1-A^{n-m}}{1-A}\right) < A^m/(1-A)$ So for $\varepsilon > 0$, if N is chosen so that $A^N/(1-A) < \varepsilon$ and $m, n \geq N$, then

$$|x_n - x_m| < A^N/(1-A) < \varepsilon$$

Thus (x_n) is a C-seq and so $\lim_{n\to\infty} x_n = x$ exists and by continuity of f, $\lim_{n\to\infty} f(x_n) = f(x)$, but by definition $\lim_{n\to\infty} f(x_n) = \lim_{n\to\infty} x_{n+1} = x$ and thus $f(x) = x$. Uniqueness follows from (a).

Problem 6.6. Show that $f(x, y) = \sqrt{|xy|}$ is not differentiable at $(0, 0)$, but both partials $f_x(0, 0)$ and $f_y(0, 0)$ exist.

Compute

$$f_x(0,0) = \lim_{h\to 0} \frac{\sqrt{|h \cdot 0|} - \sqrt{|0 \cdot 0|}}{h} = 0$$
$$f_y(0,0) = \lim_{h\to 0} \frac{\sqrt{|0 \cdot h|} - \sqrt{|0 \cdot 0|}}{h} = 0$$

If f is differentiable at $(0, 0)$, then $D_f(0, 0)(h, k) = ah + bk$ for some a and b.

$$o_f(0,0)(h,k) = (f(0+h, 0+k) - (f(0,0)) - D_f(0,0)(h,k) = \sqrt{|hk|} - (ah + bk)$$

for some fixed a and b. This must satisfy

$$0 = \lim_{(h,k)\to 0} \frac{|(f(0+h, 0+k) - f(0,0)) - D_f(0,0)(h,k)|}{\|(h,k)\|} = \lim_{(h,k)\to 0} \frac{\sqrt{|hk|} - (ah+bk)}{\sqrt{h^2+k^2}}$$

Notice: We know that $a = \frac{\partial f}{\partial x}(0, 0) = 0$ and $b = \frac{\partial f}{\partial y}(0, 0) = 0$, but we won't use this here and derive this directly from the definition.

If you approach along $t(1,0)$ (the x-axis), then you have

$$\lim_{(h,k)\to 0} \frac{\sqrt{|hk|} - (ah+bk)}{\sqrt{h^2+k^2}} = \lim_{t\to 0} \frac{-at}{\sqrt{t^2}} = -a\,\mathrm{sgn}(t) = 0$$

so $a = 0$. Similarly along $t(0,1)$ gives $b = 0$.

So we must have $(f(0+h, 0+k) - f(0,0)) - D_f(0,0)(h,k) = \sqrt{|hk|} - (ah+bk) = \sqrt{|hk|}$.

Finally, if you let (h,k) approach $(0,0)$ along $t(1,1)$, then

$$\lim_{(h,k)\to 0} \frac{\sqrt{|hk|} - (ah+bk)}{\sqrt{h^2+k^2}} = \lim_{t\to 0} \frac{\sqrt{t^2}}{\sqrt{2t^2}} = \frac{1}{\sqrt{2}} \neq 0$$

This is a contradiction.