Math 571 - Homework 3

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Problem 3.1. Define a metric on $\mathbb{Z}$ for each integer $n > 1$ as follows. Let $s \in \mathbb{Z}$ and define $e_n(s) = \max\{a \in \mathbb{N} \mid n^a | s\}$. Set $d_n(s,t) = n^{-e_n(s-t)}$ if $s \neq t$ and $d_n(s,s) = 0$.

a) Show that $d_n : \mathbb{Z} \times \mathbb{Z} \to [0,1)$ is a metric. (Look at (c) before proving the triangle inequality.)

b) Interpret the metric, for example what does it mean to say $d_n(s,t) < \delta$ (s and t are within δ of each other.)

c) $d_n(s,t) \leq \max\{d_n(s,r), d_n(r,t)\}$. ($d_n$ is an **ultrametric**.)

Problem 3.2 (R:2:17)**.** Consider all reals in $[0,1]$ whose decimal expansion requires only the digits 3 and 5, no 0's, so there are infinitely many 3's and 5's. Call this set Y. Prove or disprove each of the following:

a) Y is dense in $[0,1]$?

b) Y nowhere dense in $[0,1]$?

c) Y is countable?

d) Y is closed?

e) Y is compact?

f) Y is perfect?

Problem 3.3 (R:2:20*)**.** If E is connected is $\mathrm{Cl}(E)$ and/or $\mathrm{Int}(E)$ necessarily connected?

Of course, give a proof or a counterexample.

Problem 3.4. Show that E is connected iff for all $p, q \in E$ there is a connected open relative to E set $A \subseteq E$ with $p, q \in A$.

Problem 3.5 (R:2:21*)**.** Prove that every convex subset of $\mathbb{R}^k$ is connected.

The original problem in Rudin is a four part problem with this being the last part. You might use the original problem as a hint/guide here.

Problem 3.6 (R:2:26)**.** Let X be a metric space in which every infinite set has a limit. Show that X is compact.

I prove this in the notes. It is an important and very useful characterization of compactness in a metric space, namely, **sequential compactness**. I do not want you to reproduce the proof I give. Use the hint from Rudin and try it the way he suggests. This builds on some problems you did last week.

Problem 3.7 (R:2:28)**.** Show that every closed set, F, in a separable metric space can be written as $P = P \cup C$ where P is perfect (perhaps empty) and C is countable.

A different hint from Rudin's: I gave you a sort of hint in class, define $F' = F - \mathrm{Iso}(F)$, recall $\mathrm{Iso}(F)$ is the set of isolated points of F. F' is called the derivative of the set F. Argue that $\mathrm{Iso}(F)$ is countable, in some natural sense F' is *closer to perfection*, since we have removed some isolated points. Notice that F' is closed. If you haven't reached perfection repeat the process. In this way you build a sequence of closed sets $F \supset F_1 \supset F_2 \cdots$ and countable sets C_i so that $F = \bigcap F_i \cup \bigcup C_i$. If $\bigcap F_i = F_\omega$ still has isolated points, continue! A transfinite recursion!