

**Problem 1** (15 points; 3 points each). Decide if each of the following is true or false.

- (a) \_\_\_\_\_ If  $A$  and  $B$  are  $n \times n$  symmetric matrices, then  $AB$  is symmetric.

This is false.  $(AB)^T = B^T A^T = BA$ , but for symmetry, we would need  $(AB)^T = AB$ , and there is no reason that  $AB = BA$ .

In fact, generating two random  $2 \times 2$  symmetric matrices will most likely give an example. Here is the first pair I generated:

$$A = \begin{bmatrix} 6 & -2 \\ -2 & -4 \end{bmatrix} \quad B = \begin{bmatrix} -4 & 4 \\ 4 & 2 \end{bmatrix} \quad AB = \begin{bmatrix} -32 & 20 \\ -8 & -16 \end{bmatrix}$$

Clearly,  $A$  and  $B$  are symmetric, while  $AB$  is not.

- (b) \_\_\_\_\_ An echelon form of a matrix is unique.

This is false. There is a unique reduced echelon form. Both

$$E_1 = \begin{bmatrix} 2 & 3 & 1 \\ 0 & 3 & 2 \end{bmatrix}$$

and

$$E_2 = \begin{bmatrix} 2 & 0 & -1 \\ 0 & 3 & 2 \end{bmatrix}$$

are both echelon and each can be got from the other by an elementary row operation.

- (c) \_\_\_\_\_ If  $A$  is a  $4 \times 2$  matrix, then  $A\mathbf{x} = \mathbf{0}$  has at least two free variables.

This is false. A  $4 \times 2$  matrix corresponds to an *overdetermined* system, usually with no solutions. Had this been a  $2 \times 4$  matrix, then there must be at least two free variables. This would be an *underdetermined* system.

- (d) \_\_\_\_\_  $A^2 - B^2 = (A - B)(A + B)$  for all square  $n \times n$  matrices  $A$  and  $B$ .

This is false.  $(A - B)(A + B) = (A - B)A + (A - B)B = A^2 - BA + AB - B^2$ , but as  $AB \neq BA$  can occur, there is no reason that  $AB - BA = 0$ .

- (e) \_\_\_\_\_ For  $A$  a  $4 \times 6$  matrix, let  $E$  be the matrix so that  $EA$  is the result of the row op  $R_1 - 3R_2 \rightarrow R_1$  applied to  $A$ . Then the first row of  $E$  is  $[1 \quad -3 \quad 0]$ .

This is false. Although I had intended it to be true, somehow, I switched 3 and 4.

I did not intend this to be a trick problem, so I will accept either answer

here.

$$\begin{bmatrix} 1 & -3 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots \end{bmatrix} \begin{bmatrix} \mathbf{a}_1 \\ \mathbf{a}_2 \\ \mathbf{a}_3 \\ \mathbf{a}_4 \end{bmatrix} = \begin{bmatrix} (1) \cdot \mathbf{a}_1 + (-3) \cdot \mathbf{a}_2 + (0) \cdot \mathbf{a}_3 + (0) \cdot \mathbf{a}_4 \\ \vdots \end{bmatrix}$$

The correct answer is  $\begin{bmatrix} 1 & -3 & 0 & 0 \end{bmatrix}$

**Problem 2** (25 points). Solve  $A\mathbf{x} = \mathbf{0}$  for

$$A = \begin{bmatrix} 4 & 8 & 5 & -3 \\ 5 & 10 & 2 & -8 \\ -3 & -6 & -4 & 2 \end{bmatrix}$$

Follow the procedure discussed in class

- (10 points) Use elementary row ops to reduce to an echelon matrix.
- (10 points) Write down the resulting triangular system and use back-substitution to solve.
- (5 points) Write out your solution as a linear combination of vectors.

Reduction to echelon form requires 3 row manipulations:

This part is 10/25 points.

$$\begin{aligned} \begin{bmatrix} 4 & 8 & 5 & -3 \\ -3 & -6 & -4 & 2 \\ 5 & 10 & 2 & -8 \end{bmatrix} &\xrightarrow{\substack{R_2 \leftarrow R_2 - (-3/4)R_1 \\ R_3 \leftarrow R_3 - (5/4)R_1}} \begin{bmatrix} 4 & 8 & 5 & -3 \\ 0 & 0 & -1/4 & -1/4 \\ 0 & 0 & -17/4 & -17/4 \end{bmatrix} \\ &\xrightarrow{R_3 \leftarrow R_3 - (17)R_2} \begin{bmatrix} 4 & 8 & 5 & -3 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \end{aligned}$$

Here  $x_2$  and  $x_4$  are free variables, so let  $x_2 = s$  and  $x_4 = t$ , then we get (by back-substitution)

$$\begin{aligned} x_3 &= -t \\ 4x_1 &= -8s - 5(-t) + 3t = -8s + 8t \rightarrow x_1 = -2s + 2t \end{aligned}$$

This is 10/25 points.

So we have

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} -2s + 2t \\ s \\ -t \\ t \end{bmatrix} = s \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + t \begin{bmatrix} 2 \\ 0 \\ -1 \\ 1 \end{bmatrix}$$

This final answer is the remaining 5/25 points.