Exam 2

To avoid any confusion, unless specified otherwise, vector spaces are complex vector spaces, inner-products are complex inner-products, and matrices are complex matrices. The standard inner product is $\langle \boldsymbol{u}, \boldsymbol{v} \rangle = \boldsymbol{v}^H \boldsymbol{u} = \sum_{i=1}^n \bar{\boldsymbol{v}}_i \boldsymbol{u}_i$.

Part I: True/False

Each problem is points for a total of 50 points. (5 points each.)

You do not need to justify the answers here, this is unlike the quizzes.

1. ______ If U is unitary, then U is itself unitarily diagonalizable. This means there is a unitary V so that $U = VDV^H$ where D is diagonal.

2. ______ For any diagonalizable matrix A, one can use Gram-Schmidt to find an orthogonal basis consisting of eigenvectors.

3. ______ The collection of rank k $n \times n$ matrices is a subspace of $\mathbb{R}^{n \times n}$, for $k < n$.

4. ______ If A is unitary, then $|\lambda| = 1$ for all eigenvalues λ of A.

5. ______ If $p(t)$ is a polynomial and $\boldsymbol{v}$ is an eigenvector of A with associated eigenvalue λ, then $p(A)\boldsymbol{v} = p(\lambda)\boldsymbol{v}$.

6. ______ If A and B are both $n \times n$ and $\mathcal{B}$ is a basis for $\mathbb{C}^n$ consisting of eigenvectors for both A and B, then A and B commute.

7. ______ Any matrix A can be written as a weighted sum of rank 1 matrices..

8. ______ For all Hermitian matrices A, there is a matrix B so that $B^H B = A$.

9. ______ There are linear maps $L : \mathbb{R}^5 \to \mathbb{R}^4$ such that $\dim(\ker(L)) = 2 = \dim(\operatorname{rng}(L))$.

10. ______ If A is invertible, then $ABA^{-1} = B$.

Part II: Computational (60 points)

Problem 1. (15 points) Find B so that $B^2 = A$ where

$$A = \begin{bmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{bmatrix}$$

Problem 2. (15 points) Find B so that $B^H B = A$ where A is from (1).

Problem 3. (15 points) Find the best rank 2 approximation to A from (1) with respect to $\|\cdot\|_F$.

Problem 4. (15 points) Let

$$A = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix},$$

find the characteristic polynomial and all eigenvalues, both real and complex. Explain why A is diagonalizable and compute A^{2020}. Note, I do not ask you to diagonalize A.

Part III: Theory and Proofs (45 points; 15 points each)

Pick three of the following four options. If you try all four, I will grade the first three, so if this is not what you intend, then just do three, or at least make it clear which I should grade.

Problem 1. Let S be a fixed invertible $n \times n$ matrix. Let U be the set of $n \times n$ matrices that are diagonalized by S, that is $A = SD_A S^{-1}$ for some diagonal matrix A. Either prove that that U is a subspace of $\mathbb{C}^{n \times n}$ or show that U is not a subspace of $\mathbb{C}^{n \times n}$.

Problem 2. Let A be a real $m \times n$ matrix and let $A^\dagger = V^T \Sigma^\dagger U$, where $A = U \Sigma V^T$ where U is $m \times m$, V is $n \times n$, both unitary, Σ is $m \times n$ and $\Sigma^\dagger$ is $n \times m$ have the form

$$\Sigma = \begin{bmatrix} \sigma_1 & & & & \\ & \ddots & & & \\ & & \sigma_r & & \\ & & & 0 & \\ & & & & \ddots \\ & & & & & 0 \end{bmatrix} \quad \text{and} \quad \Sigma^\dagger = \begin{bmatrix} \sigma_1^{-1} & & & & \\ & \ddots & & & \\ & & \sigma_r^{-1} & & \\ & & & 0 & \\ & & & & \ddots \\ & & & & & 0 \end{bmatrix}$$

with $\sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_r > 0$.

Show: $\hat{\boldsymbol{x}} = A^\dagger \boldsymbol{b}$ is a least-squares solution to $A\boldsymbol{x} = \boldsymbol{b}$.

Previously we used $\hat{\boldsymbol{x}} = (A^T A)^{-1} A^T \boldsymbol{b}$ for our least-squares solution, but we had the restriction that the columns of the "data" matrix A were independent, this guarantees that $\text{NS}(A) = \text{NS}(A^T A) = \{\mathbf{0}\}$. It is not hard to see that $A^\dagger = (A^T A)^{-1} A^T$ if A has linear independent columns.

Review the comments about Topic 5 DQ 2 in the Class Notes. Particularly point (2.) concerning what it means to be a least-squares solution to $A\boldsymbol{x} = \boldsymbol{b}$.

Problem 3. Prove that any complex inner-product $\langle \cdot, \cdot \rangle_V$ on a complex vector space V, there is a basis $\mathcal{U} = \{\boldsymbol{u}_1, \ldots, \boldsymbol{u}_n\}$ so that

$$\langle \boldsymbol{x}, \boldsymbol{y} \rangle_V = [\boldsymbol{y}]_{\mathcal{U}}^H [\boldsymbol{x}]_{\mathcal{U}}$$

In other words for any finite dimensional inner-product space, there is a choice of basis, so that with respect to that basis, the inner-product is represented by the standard inner-product.

Here, in case you need it, is the definition of an inner-product. All the notation here is as I always use it in my notes.

Problem 4. Use the SVD to show that any square matrix A can be written as $A = UP$ where U is unitary and P is Hermitian.