Exam 1

This exam covers Topics 1 - 3, Topic 4 will not be covered here.

Part I: True/False (5 points each; 25 points)

For each of the following mark as true or false.

a) _____ If A and B are $n \times n$ lower triangular matrices, then AB is also lower triangular.

b) _____ If W is a subspace of a vector space V and $\mathcal{B}_W$ is a basis for W, then there is a unique subspace U so that $V = W \oplus U$ and a basis $\mathcal{B}_U$ for U so that $\mathcal{B}_V = \mathcal{B}_W + \mathcal{B}_U$ is a basis for V.

c) _____ If W is a subspace of a vector space V and $\mathcal{B}$ is a basis for V, then B can be restricted to a basis for W.

d) _____ Let A be an $n \times n$ matrix over $\mathbb{C}$, then $\det(\bar{A}) = \det(A)$, where $\bar{A}_{i,j} = \overline{A_{i,j}}$. Here, $\bar{z} = a - ib$ when $z - a + ib$, the *complex conjugate of* z.

e) _____ For $n \times n$ matrices A and B, define $A \otimes B = AB - BA$. The operator $\otimes$ is not associative or commutative.

Part II: Definitions and Theorems (5 points each; 25 points)

a) Define what it means for a set of vectors $\mathcal{B} = \{\boldsymbol{v}_1, \ldots, \boldsymbol{v}_n\}$ from a real vector space V to span V.

b) Define what it means for a set of vectors $\{\boldsymbol{v}_1, \ldots, \boldsymbol{v}_n\}$ from a real vector space V to be linearly independent.

c) Define what it means for a set of vectors $\mathcal{B} = \{\boldsymbol{v}_1, \ldots, \boldsymbol{v}_n\}$ to be a basis for a vector space V.

d) State the Rank-Nullity Theorem.

e) If B arises from a matrix A by elementary row operations, what is the relationship between $\mathrm{NS}(A)$ and $\mathrm{NS}(B)$?

Part III: Computational (15 points each; 45 point)

a) Use row ops to find an echelon form of

$$A = \begin{bmatrix} 1 & 2 & 2 & -2 & 2 \\ 2 & 4 & 1 & -2 & 5 \\ 1 & 2 & -1 & 0 & 3 \end{bmatrix}$$

Make sure to write out your steps and indicate the row ops at each step.

b) Use the echelon matrix found above to find a basis for $\mathrm{RS}(A)$, $\mathrm{NS}(A)$, and $\mathrm{CS}(A)$. Give a brief reason for your choice.

Without a justification, you might just have a lucky guess and I will not accept this. Your justification can be short and use facts from the text or from the notes that I have provided.

c) Show that skew-symmetric 3×3 matrices form as subspace of all 3×3 matrices and find a basis for this subspace.

Part IV: Proofs (20 points each; 60 points) - Choose three!

Provide complete arguments/proofs for three of the following. If you try more than three, I will just grade the first three, so pick three your best three! If you want to ask me about these, please do.

a) A is invertible iff there exists a matrix B so that $AB = BA = I$. It is simple to show that:

   (i) If A is invertible and $AB = I$, then $BA = I$ as well and B is the unique such matrix.

   (ii) If A is invertible and $BA = I$, then $AB = I$ as well and B is the unique such matrix.

   This shows that if A is invertible, then there is a unique matrix B such that $AB = I$ or $BA = I$. Call this unique matrix A^{-1}.

   The goal here is to show that the assumption *"A is invertible"* is not needed in (i) or (ii).

   Prove: Let A and B be square matrices with $AB = I$. Show that A is invertible and hence $B = A^{-1}$.

   You may refer to Theorem 1.5.2 or Theorem 2.2.2, but be clear and complete in your argument.

b) **Prove:** $\mathrm{NS}(A) = \mathrm{NS}(A^T A)$ for any matrix A.

You have actually done this already in the homework, but you may also use the easy fact that $\boldsymbol{x}^T\boldsymbol{x} > 0$ for $\boldsymbol{x} \neq \mathbf{0}$.

c) **Prove:** If A and B are $m \times n$ matrices such that $A\boldsymbol{x} = B\boldsymbol{x}$ for all $\boldsymbol{x} \in \mathbb{R}^n$, then $A = B$.

d) **Prove:** If A is an $n \times n$ matrix and $A^k = \mathbf{0}$ for any k, then $A^n = \mathbf{0}$.