# Homework 1 Partial Solutions

**Homework 1 problems:**

## Section 1.1

**8.** Use elimination and back substitution to solve the given system:

$$\begin{aligned} x_1 + 2x_2 - 2x_3 &= 1 \\ 2x_1 + 5x_2 + x_3 &= 9 \\ x_1 + 3x_2 + 4x_3 &= 9 \end{aligned}$$

Elimination on the augmented system looks like:

$$\left[\begin{array}{ccc|c} 1 & 2 & -2 & 1 \\ 2 & 5 & 1 & 9 \\ 1 & 3 & 4 & 9 \end{array}\right] \underset{\substack{r_2-2r_1\to r_2 \\ r_3-r_1\to r_3}}{\longrightarrow} \left[\begin{array}{ccc|c} 1 & 2 & -2 & 1 \\ 0 & 1 & 5 & 7 \\ 0 & 1 & 6 & 8 \end{array}\right] \underset{r_3-r_2\to r_3}{\longrightarrow} \left[\begin{array}{ccc|c} 1 & 2 & -2 & 1 \\ 0 & 1 & 5 & 7 \\ 0 & 0 & 1 & 1 \end{array}\right]$$

This gives the triangular system

$$\begin{aligned} x_1 + 2x_2 - 2x_3 &= 1 \\ x_2 + 5x_3 &= 7 \\ x_3 &= 1 \end{aligned}$$

Back substitution gives:

$$\begin{aligned} x_2 &= 7 - 5(1) = 2 \\ x_1 &= 1 + 2(1) - 2(2) = -1 \end{aligned}$$

So we have the solution $(-1, 2, 1)$. (Check this in the initial system!)

**9.**

**(a)** Suppose $m_1 \neq m_2$. Assume $m_1 \neq 0$. (If $m_1 = 0$, then $m_2 \neq 0$ by our assumption so we could just swap the rolls below.) Multiply the first equation by $-m_2/m_1$ add to the second and replace the second. This yields the new system:

$$\begin{aligned} -m_1x_1 + \qquad\qquad\qquad x_2 &= b_1 \\ (1 - m_2/m_1)x_2 &= b_2 - (m_2/m_1)b_1 \end{aligned}$$

Now you can use back substitution:

$$\begin{aligned} x_2 &= \frac{b_2 - (m_2/m_1)b_1}{1 - m_2/m_1} = \frac{m_1b_2 - m_2b_1}{m_1 - m_2} && \text{(ok since } m_1 \neq m_2\text{)} \\ x_1 &= -\frac{b_1 - x_2}{m_1} = \frac{2m_2b_1 - m_1(b_1 + b_2)}{m_1(m_1 - m_2)} \end{aligned}$$

So we have a unique solution.

**(b)** If $m_1 = m_2$, then the system is just

$$\begin{aligned} -m_1x_1 + x_2 &= b_1 \\ -m_1x_1 + x_2 &= b_2 \end{aligned}$$

This is equivalent to the single equation $-m_1x_1+x_2 = b_1$ if $b_1 = b_2$, else you get $0 = b_2-b_1 \neq 0$ which cannot happen.

**(c)** For (a) saying $m_1 \neq m_2$ is equivalent to saying that the two lines have different slopes and hence have a unique point of intersection. For (b), if $m_1 = m_2$, the lines are parallel, they are the same line if $b_1 = b_2$, else they are distinct parallel lines and hence never intersect.

## Section 1.2

**5.**

**(c)**

$$\begin{aligned} x_1 + x_2 &= 0 \\ 2x_1 + 3x_2 &= 0 \\ 3x_1 - 2x_2 &= 0 \end{aligned}$$

Gaussian Elimination looks like

$$\left[\begin{array}{cc|c} 1 & 1 & 0 \\ 2 & 3 & 0 \\ 3 & -2 & 0 \end{array}\right] \underset{\substack{r_2-2r_1\to r_2 \\ r_3-3r_1\to r_3}}{\longrightarrow} \left[\begin{array}{cc|c} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & -5 & 0 \end{array}\right] \underset{r_3+5r_2\to r_3}{\longrightarrow} \left[\begin{array}{cc|c} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

This results in the triangular system

$$\begin{aligned} x_1 + x_2 &= 0 \\ x_2 &= 0 \end{aligned}$$

Which is solved by back substitution to give: $x_2 = 0$, and $x_1 = -x_2 = 0$.

**(d)**

$$\begin{aligned} 3x_1 + 2x_2 - x_3 &= 4 \\ x_1 - 2x_2 + 2x_3 &= 1 \\ 11x_1 + 2x_2 + x_3 &= 14 \end{aligned}$$

Gaussian Elimination looks like

$$\left[\begin{array}{ccc|c} 3 & 2 & -1 & 4 \\ 1 & -2 & 2 & 1 \\ 11 & 2 & 1 & 14 \end{array}\right] \xrightarrow[r_1 \leftrightarrow r_2]{} \left[\begin{array}{ccc|c} 1 & -2 & 2 & 1 \\ 3 & 2 & -1 & 4 \\ 11 & 2 & 1 & 14 \end{array}\right] \xrightarrow[\substack{r_2 - 3r_1 \to r_2 \\ r_3 - 11r_1 \to r_3}]{}$$
$$\left[\begin{array}{ccc|c} 1 & -2 & 2 & 1 \\ 0 & 8 & -7 & 1 \\ 0 & 24 & -21 & 3 \end{array}\right] \xrightarrow[r_3 - 3r_2 \to r_3]{} \left[\begin{array}{ccc|c} 1 & -2 & 2 & 1 \\ 0 & 8 & -7 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

This is equivalent to the "triangular system"

$$\begin{aligned} x_1 - 2x_2 + 2x_3 &= 1 \\ 8x_2 - 7x_3 &= 1 \end{aligned}$$

So $x_1$ and $x_2$ are the two "pivot variables" an $x_3$ is the "free variable." Let $x_3 = t$ be any value, then we use back substitution to get:

$$\begin{aligned} x_2 &= 1/8 + 7/8t \\ x_1 - 2(1/8 + 7/8t) + 2t &= 1 \\ x_1 &= 5/4 - 1/4t \end{aligned}$$

So

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 5/4 - 1/4t \\ 1/8 + 7/8t \\ t \end{bmatrix} = \begin{bmatrix} 5/4 \\ 1/8 \\ 0 \end{bmatrix} + t \begin{bmatrix} -1/4 \\ 7/8 \\ 1 \end{bmatrix}$$

**(e)**

$$\begin{aligned} 2x_1 + 3x_2 + x_3 &= 1 \\ x_1 + x_2 + x_3 &= 3 \\ 3x_1 + 4x_2 + 2x_3 &= 4 \end{aligned}$$

Elimination on the augmented system looks like:

$$\left[\begin{array}{ccc|c} 2 & 3 & 1 & 1 \\ 1 & 1 & 1 & 3 \\ 3 & 4 & 2 & 4 \end{array}\right] \xrightarrow[r_1 \leftrightarrow r_2]{} \left[\begin{array}{ccc|c} 1 & 1 & 1 & 3 \\ 2 & 3 & 1 & 1 \\ 3 & 4 & 2 & 4 \end{array}\right] \xrightarrow[\substack{r_2 - 2r_1 \to r_2 \\ r_3 - 3r_1 \to r_3}]{} \left[\begin{array}{ccc|c} 1 & 1 & 1 & 3 \\ 0 & 1 & -1 & -5 \\ 0 & 1 & -1 & -5 \end{array}\right] \xrightarrow[r_3 - r_2 \to r_3]{} \left[\begin{array}{ccc|c} 1 & 1 & 1 & 3 \\ 0 & 1 & -1 & -5 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

The echelon form of the matrix coefficient matrix is now shown.

This system is consistent. if we let $x_3 = t$, then by back substitution get

$$\begin{aligned} x_3 &= t \\ x_2 &= -5 + t \\ x_1 &= 3 - t - (-5 + t) = 8 - 2t \end{aligned}$$

so the solution set is

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 8 \\ -5 \\ 0 \end{bmatrix} + t \begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix}$$

**8.** For what values of $a$ does the following have a unique solution?

$$\left[\begin{array}{ccc|c} 1 & 2 & 1 & 1 \\ -1 & 4 & 3 & 2 \\ 2 & -2 & a & 3 \end{array}\right]$$

$$\left[\begin{array}{ccc|c} 1 & 2 & 1 & 1 \\ -1 & 4 & 3 & 2 \\ 2 & -2 & a & 3 \end{array}\right] \underset{\substack{r_2+r_1\to r_2 \\ r_3-2r_1\to r_3}}{\longrightarrow} \left[\begin{array}{ccc|c} 1 & 2 & 1 & 1 \\ 0 & 6 & 4 & 3 \\ 0 & -6 & a-2 & 1 \end{array}\right]$$
$$\underset{r_3+r_2\to r_3}{\longrightarrow} \left[\begin{array}{ccc|c} 1 & 2 & 1 & 3 \\ 0 & 6 & 4 & -5 \\ 0 & 0 & a+2 & 4 \end{array}\right]$$

All that is required to get a unique solution is that $a + 2 \neq 0$ or $a \neq -2$.

**11.** Solve

$$\begin{aligned} x_1 + 2x_2 &= 2 \\ 3x_1 + 7x_2 &= 8 \end{aligned} \qquad\qquad \begin{aligned} x_1 + 2x_2 &= 1 \\ 3x_1 + 7x_2 &= 7 \end{aligned}$$

Set this up as:

$$\left[\begin{array}{cc|cc} 1 & 2 & 2 & 1 \\ 3 & 7 & 8 & 7 \end{array}\right] \underset{r_2-3r_3\to r_3}{\longrightarrow} \left[\begin{array}{cc|cc} 1 & 2 & 2 & 1 \\ 0 & 1 & 2 & 4 \end{array}\right] \underset{r_1-2r_2\to r_1}{\longrightarrow} \left[\begin{array}{cc|cc} 1 & 0 & -2 & -7 \\ 0 & 1 & 2 & 4 \end{array}\right]$$

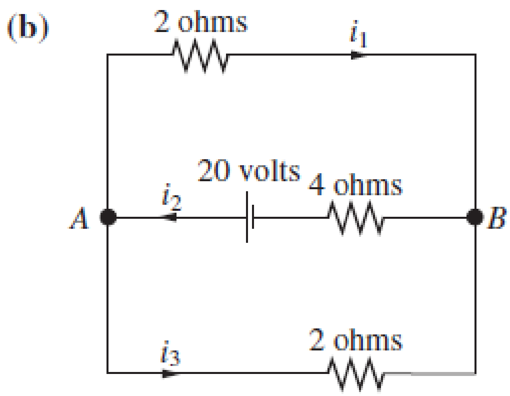
So the solutions are

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -2 \\ 2 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -7 \\ 4 \end{bmatrix}$$

respectively. As usual, you should check this.

**22**

**(b)**




From this we have

$$\begin{aligned} 1_1 - i_2 + i_3 &= 0 && \text{(node B)} \\ -i_1 + i_2 - i_3 &= 0 && \text{(node A)} \\ 2i_1 + 4i_2 &= 20 && \text{(top loop)} \\ 4i_2 + 2i_3 &= 20 && \text{(bottom loop)} \end{aligned}$$

This gives

$$\left[\begin{array}{ccc|c} 1 & -1 & 1 & 0 \\ -1 & 1 & -1 & 0 \\ 2 & 4 & 0 & 20 \\ 0 & 4 & 2 & 20 \end{array}\right] \underset{\substack{r_2+r_1\to r_2 \\ r_3-2r_1\to r_3}}{\longrightarrow} \left[\begin{array}{ccc|c} 1 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 6 & -2 & 20 \\ 0 & 4 & 2 & 20 \end{array}\right]$$

$$\underset{\substack{r_2\leftrightarrow r_4 \\ 1/2r_2\to r_2 \\ 1/2r_3\to r_3}}{\longrightarrow} \left[\begin{array}{ccc|c} 1 & -1 & 1 & 0 \\ 0 & 2 & 1 & 10 \\ 0 & 3 & -1 & 10 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$$\underset{r_3-3/2r_2\to r_3}{\longrightarrow} \left[\begin{array}{ccc|c} 1 & -1 & 1 & 0 \\ 0 & 2 & 1 & 10 \\ 0 & 0 & -5/2 & -5 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

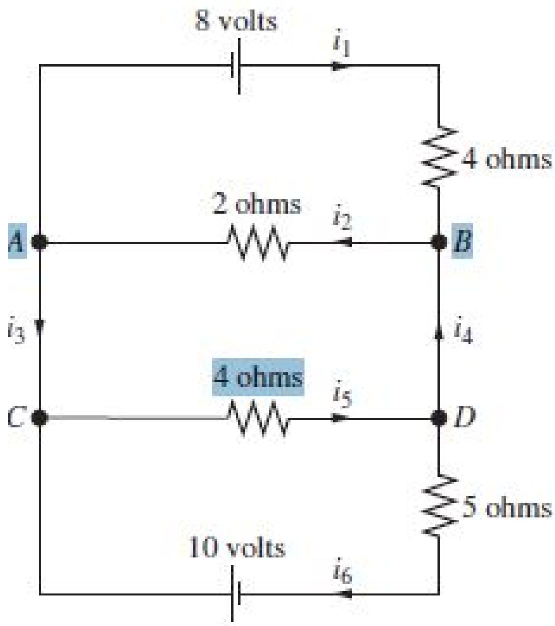
This gives the triangular system

$$\begin{aligned} i_1 - i_2 + i_3 &= 0 \\ 2i_2 + i_3 &= 10 \\ -5/2i_3 &= -5 \end{aligned}$$

So

$$\begin{aligned} i_3 = 2 &\Longrightarrow i_3 = 2 \\ 2i_2 + 2 = 10 &\Longrightarrow i_2 = 4 \\ i_1 - 4 + 2 = 0 &\Longrightarrow i_1 = 2 \end{aligned}$$

**(c)**




The equations here are

$$
\begin{aligned}
-i_1 + i_2 - i_3 &= 0 && \text{(Node A)}\\
i_1 - 1_2 + i_4 &= 0 && \text{(Node B)}\\
i_3 - i_5 + i_6 &= 0 && \text{(Node C)}\\
-i_4 + i_5 - i_6 &= 0 && \text{(Node D)}\\
4i_1 + 2i_2 &= 8 && \text{(Top Loop)}\\
4i_5 + 5i_6 &= 10 && \text{(Bottom Node)}
\end{aligned}
$$

The augmented matrix is

$$
\left[\begin{array}{cccccc|c}
-1 & 1 & -1 & 0 & 0 & 0 & 0\\
1 & -1 & 0 & 1 & 0 & 0 & 0\\
0 & 0 & 1 & 0 & -1 & 1 & 0\\
0 & 0 & 0 & -1 & 1 & -1 & 0\\
4 & 2 & 0 & 0 & 0 & 0 & 8\\
0 & 0 & 0 & 0 & 4 & 5 & 10
\end{array}\right]
$$

solving gives

$$\begin{bmatrix} 1/2 \\ 3 \\ 5/2 \\ 5/2 \\ 5/2 \\ 0 \end{bmatrix}$$

# Section 1.3

**6.** This is just computational. Do this by hand, but this is also good to verify in MATLAB.

**7.** This is just computational. Do this by hand, but this is also good to verify in MATLAB.

**13.**

**(a)** Say the variables are $x_1, x_2, x_3, x_4, x_5$ the variables $x_2$, $x_4$ and $x_5$ will be independent since the columns 2,4, and 5 do not contain pivot elements. The others can be solved in terms of those. Let $s, t, u$ be arbitrary real numbers and set $x_2 = s$, $x_4 = t$, and $x_5 = u$, then

$$\begin{aligned} x_1 &= -2 - 2s - 3t - u \\ x_2 &= s \\ x_3 &= 5 - 2t - 4u \\ x_4 &= t \\ x_5 &= u \end{aligned}$$

We can write this nicely as

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} -2 \\ 0 \\ 5 \\ 0 \\ 0 \end{bmatrix} + s \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + t \begin{bmatrix} -3 \\ 0 \\ -2 \\ 1 \\ 0 \end{bmatrix} + u \begin{bmatrix} -1 \\ 0 \\ -4 \\ 0 \\ 1 \end{bmatrix} \tag{1}$$

**(b)**

We know

$$A\mathbf{x} = \mathbf{b} \iff \begin{bmatrix} 1 & 2 & 0 & 3 & 1 \\ 0 & 0 & 1 & 2 & 4 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \mathbf{x} = \begin{bmatrix} -2 \\ 5 \\ 0 \\ 0 \end{bmatrix}$$

We also know that

$$\mathbf{x} = \begin{bmatrix} -2 \\ 0 \\ 5 \\ 0 \\ 0 \end{bmatrix}$$

is a solution, name ly the solution where $s = t = u = 0$

So $A\mathbf{x} = -2\mathbf{a_1} + 5\mathbf{a_3} = -2 \begin{bmatrix} 1 \\ 1 \\ 3 \\ 4 \end{bmatrix} + 5 \begin{bmatrix} 2 \\ -1 \\ 1 \\ 3 \end{bmatrix} = \begin{bmatrix} 8 \\ -7 \\ -1 \\ 7 \end{bmatrix} = \mathbf{b}$

**16.** $A^T = -A$ implies $a_{ii} = -a_{ii}$ so $a_{ii} = 0$.

## Section 1.4

**8.** Check that $A^2 = I$, so $A^{2n+1} = A$ and $A^{2n} = I$.

**10.** Assume $A^T = A$ and $B^T = B$.

**(a)** Symmetric since: $(A + B)^T = A^T + B^T = A + B$

**(b)** Symmetric since: $(A^2)^T = A^T A^T = AA = A^2$

**(c)** Not necessarily transitive since: $(AB)^T = B^T A^T = BA$. Symmetry would only happen if $AB = BA$, i.e., $A$ and $B$ commute.

Example: Let $A = \left[\begin{smallmatrix} 1 & 1 \\ 1 & 1 \end{smallmatrix}\right]$ and $B = \left[\begin{smallmatrix} 1 & -1 \\ -1 & -1 \end{smallmatrix}\right]$, notice both $A$ and $B$ are symmetric. $AB = \left[\begin{smallmatrix} 0 & -2 \\ 0 & -2 \end{smallmatrix}\right]$ is not symmetric so $(AB)^T \neq AB$.

**(d)** Symmetric since: $(ABA)^T = A^T B^T A^T = ABA$

**(e)** Symmetric since: $(AB + BA)^T = (AB)^T + (BA)^T = B^T A^T + A^T B^T = BA + AB =$

$AB + BA$

**(f)** Not symmetric since: $(AB - BA)^T = (AB)^T - (BA)^T = B^T A^T - A^T B^T = BA - AB = -(AB - BA)$. So $AB - BA$ is symmetric iff $AB - BA = 0$, that is iff $A$ and $B$ commute.

The same example as in (c) works here.

**22.**

$$\begin{aligned} RR^T &= \begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix} \begin{bmatrix} \cos(\theta) & \sin(\theta) \\ -\sin(\theta) & \cos(\theta) \end{bmatrix} \\ &= \begin{bmatrix} \cos^2(\theta) + \sin^2(\theta) & \cos(\theta)\sin(\theta) - \sin(\theta)\cos(\theta) \\ \sin(\theta)\cos(\theta) - \cos(\theta)\sin(\theta) & \sin^2(\theta) + \cos^2(\theta) \end{bmatrix} \\ &= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \end{aligned}$$

**23.** $H = I - 2\mathbf{u}\mathbf{u}^T$, so

$$\begin{aligned} H^2 &= (I - 2\mathbf{u}\mathbf{u}^T)(I - 2\mathbf{u}\mathbf{u}^T) \\ &= I^2 - 2\mathbf{u}\mathbf{u}^T I - 2I\mathbf{u}\mathbf{u}^T + 4\mathbf{u}\mathbf{u}^T\mathbf{u}\mathbf{u}^T \\ &= I^2 - 4\mathbf{u}\mathbf{u}^T + 4\mathbf{u}\mathbf{u}^T\mathbf{u}\mathbf{u}^T \\ &= I^2 - 4\mathbf{u}\mathbf{u}^T + 4\mathbf{u}(1)\mathbf{u}^T \\ &= I^2 - 4\mathbf{u}\mathbf{u}^T + 4\mathbf{u}\mathbf{u}^T \\ &= I^2 \\ &= I \end{aligned}$$

**27.** Assume $A^2 = I$ and let $B = \frac{1}{2}(I + A)$ and $C = \frac{1}{2}(I - A)$. Note

$$B^2 = \frac{1}{4}(I^2 + IA + AI + A^2) = \frac{1}{4}(I + 2A + I) = \frac{1}{4}(2)(I + A) = B$$

Similarly, $C^2 = C$.

**30.**

$$(A + A^T)^T = A^T + (A^T)^T = A^T + A = A + A^T$$

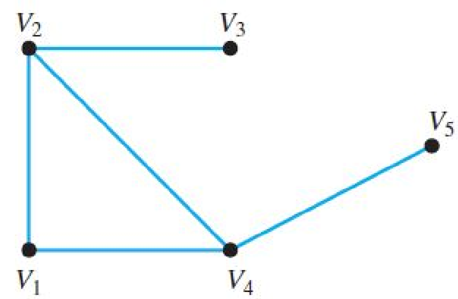
So $A + A^T$ is symmetric. Similarly,

$$(A - A^T)^T = A^T - (A^T)^T = A^T - A = -(A - A^T)$$

So $A - A^T$ is skew-symmetric.

Since $A = \frac{1}{2}(A + A^T) + \frac{1}{2}(A - A^T)$ we see $A$ can be written as symmetric + skew-symmetric.

**33.**




(a) Adjacency Matrix, $A$:

$$A = \begin{bmatrix} 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$

(b)

$$A^2 = \begin{bmatrix} 2 & 1 & 1 & 1 & 1 \\ 1 & 3 & 0 & 1 & 1 \\ 1 & 0 & 1 & 1 & 0 \\ 1 & 1 & 1 & 3 & 0 \\ 1 & 1 & 0 & 0 & 1 \end{bmatrix}$$

There are 6 walks starting at $V_1$ and that is the sum of column 1 or row 1 (the matrix is symmetric). $V_1V_2V_1$, $V_1V_4V_1$, $V_1, V_2V_3$, $V_1V_2V_4$, $V_1V_4V_2$, and $V_1V_4V_5$.

(c)

$$A^2 = \begin{bmatrix} 2 & 4 & 1 & 4 & 1 \\ 4 & 2 & 3 & 5 & 1 \\ 1 & 3 & 0 & 1 & 1 \\ 4 & 5 & 1 & 2 & 3 \\ 1 & 1 & 1 & 3 & 0 \end{bmatrix}$$

There are 5 walks of length 3 from $V_2$ to $V_4$ and 1 of length 2 so 6 altogether.

## Section 1.5

**8.** find the $LU$ decomposition of the following matrices:

**(b)** $\begin{bmatrix} 2 & 4 \\ -2 & 1 \end{bmatrix}$

$$\begin{bmatrix} 2 & 4 \\ -2 & 1 \end{bmatrix} \underset{r_2+(1)r_1\to r_2; L_{2,1}=-1}{\longrightarrow} \begin{bmatrix} 2 & 4 \\ 0 & 5 \end{bmatrix} = U$$

The $(2,1)$ position of $L$ is $-1$ so

$$L = \begin{bmatrix} 1 & 0 \\ -1 & 0 \end{bmatrix}$$

It is easy to see that

$$\begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 2 & 4 \\ 0 & 5 \end{bmatrix} = \begin{bmatrix} 2 & 4 \\ -2 & 1 \end{bmatrix}$$

**(d)** $\begin{bmatrix} -2 & 1 & 2 \\ 4 & 1 & -2 \\ -6 & -3 & 4 \end{bmatrix}$

$$\begin{bmatrix} -2 & 1 & 2 \\ 4 & 1 & -2 \\ -6 & -3 & 4 \end{bmatrix} \underset{\substack{r_2+(2)r_1\to r_2; L_{2,1}=-2 \\ r_3+(-3)r_1\to r_3; L_{3,1}=3}}{\longrightarrow} \begin{bmatrix} -2 & 1 & 2 \\ 0 & 3 & 2 \\ 0 & -6 & -2 \end{bmatrix}$$

$$\underset{r_3+(2)r_2\to r_3; L_{3,2}=-2}{\longrightarrow} \begin{bmatrix} -2 & 1 & 2 \\ 0 & 3 & 2 \\ 0 & 0 & 2 \end{bmatrix} = U$$

We see

$$L = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 3 & -2 & 1 \end{bmatrix}$$

Again, it is easy to check that

$$LU = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 3 & -2 & 1 \end{bmatrix} \begin{bmatrix} -2 & 1 & 2 \\ 0 & 3 & 2 \\ 0 & 0 & 2 \end{bmatrix} = \begin{bmatrix} -2 & 1 & 2 \\ 4 & 1 & -2 \\ -6 & -3 & 4 \end{bmatrix}$$

**19.** (a) If $U$ is upper triangular with non-zero diagonal, then clearly $U$ can be row reduced to $I$. Since $U$ is row equivalent to $I$, $U$ is invertible. (Theorem 1.5.2).

(b) **Proof 1:** For this, essentially look into the proof of 1.5.2. Let $E_n E_{n-1} \cdots E_1 U = I$ where $E_i$ is an elementary matrix. Since $U$ starts as upper-triangular (u.t.) it is clear that only type

II and III operations are needed and the type II here are of the form $R_i - c \cdot R_j \to R_i$ where $i < j$, so $c$ goes in the $(i, j)^{\text{th}}$ spot and hence this matrix is u.t. So all the $E_i$ are u.t. Thus $E_n E_{n-1} \cdots E_1 = U^{-1}$ is u.t.

**Proof 2:** We can show that of $AB = C$ with $B$ and $C$ u.t. and $B$ invertible, then $A$ is u.t. This will clearly imply what we want. Suppose $A$ is not u.t. and let $i$ be least such that there is $j < i$ with $A_{i,j} \neq 0$. Let $j$ be the least $< i$ so that $A_{i,j} = 0$. Then $0 = C_{i,j} = \sum_{k \leq n} A_{i,k} B_{k,j} = \sum_{k \leq j} A_{i,k} B_{k,j}$, that is,

$$0 = A_{i,1} B_{1,j} + \cdots + A_{i,j} B_{j,j} = 0 + \cdots 0 + A_{i,j} B_{j,j} = A_{i,j} B_{j,j} = 0$$

This is a contradiction to the choice of $A_{i,j}$ and the fact that $B$ is invertible, hence $B_{j,j} \neq 0$, since $B$ is u.t.

**28.** (a) There is nothing to do except unpack matrix multiplication, which is done below for (b).

(b) For this

$$V\boldsymbol{c} = c_0 \begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix} + c_1 \begin{bmatrix} x_1 \\ \vdots \\ x_{n+1} \end{bmatrix} + \cdots + c_n \begin{bmatrix} x_i^n \\ \vdots \\ x_{n+1}^n \end{bmatrix} = \begin{bmatrix} p(x_1) \\ \vdots \\ p(x_{n+1}) \end{bmatrix} = \begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix}$$

But then we have $n + 1$ distinct roots to a polynomial of degree n, which is a contradiction unless $p(x) \equiv 0$, i.e., $p$ is the constant 0 polynomial.

**32.** This is false, for example let $A = B = I$ and $C = -I$, then clearly $A$ is row equivalent to $B$ and $C$, but $B + C = \mathbf{0}$ (the all 0 matrix). It is not true that $A$ is row equivalent to $\mathbf{0}$, else $I = \mathbf{0}$ (a version of $1 = 0$).