Quiz 6

Question 1 (15 points; 3 points each)**.** Decide if each of the following are true or false and provide a justification or counterexample in each case. A justification could consist of a theorem from the text. All vector spaces are assumed to be finite-dimensional here. All vector spaces are now over $\mathbb{C}$ unless otherwise stated.

(a) ______ If all eigenvalues of A are 0, then $A = 0$.

This is false. For example

$$\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$$

has only 0 as an eigenvalue.

(b) ______ All $n \times n$ matrices are diagonalizable.

This is false. A shear matrix is a typical example

$$\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$$

You must have a complete basis of eigenvectors.

(c) ______ If $p(x)$ is a polynomial and A is an $n \times n$ matrix, then

$$p(A) = \begin{bmatrix} p(a_{1,1}) & \cdots & p(a_{1,n}) \\ \vdots & \ddots & \vdots \\ p(a_{n,1}) & \cdots & p(a_{n,n}) \end{bmatrix}$$

This is false. Even for $p(X) = x^2$. In general, it is not true that $(A^n)_{i,j} = A(i,j)^2$.

This is true for diagonal matrices and even for diagonalizable matrices. This is part of the reason diagonalizable matrices are so important. See the Class Notes for a set of slides on this.

(d) ______ If $p(x)$ is a polynomial and $A = S^{-1}DS$ where $D = \operatorname{diag}(d_1, \ldots, d_n)$, then

$$p(A) = S^{-1} \operatorname{diag}(p(d_1), \ldots, p(d_n))S$$

Here

$$\operatorname{diag}(d_1, \ldots, d_n) = \begin{bmatrix} d_1 & 0 & \cdots & 0 \\ 0 & d_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & d_n \end{bmatrix}$$

This is true and as mentioned above is one of the key reasons that diagonalizable matrices are so important.

(e) ______ If A is upper-triangular, then the eigenvalues of A are exactly the diagonal elements of A.

This is true and is a simple calculation.

Question 2 (5 points)**.** Let $A = \begin{bmatrix} 5/4 & -3/4 \\ -3/4 & 5/4 \end{bmatrix}$, write $A = U\Lambda U^{-1}$ where U is unitary, columns are orthonormal basis for $\mathbb{R}^2$ and $\Lambda = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}$ with $\lambda_1 > \lambda_2$.

Recall: $U^{-1} = U^T$ for unitary U.

Find the eigenvalues: $\det(A - tI) = (5/4 - t)^2 - (3/4)^2 = (5/4 - t - 3/4)(5/4 - t + 3/4) = (1/2 - t)(2 - t)$ so $\lambda_1 = 2 > \lambda_2 = 1/2$ and

$$\Lambda = \begin{bmatrix} 2 & 0 \\ 0 & 1/2 \end{bmatrix}$$

Find the eigenspace for λ_1:

$$\mathrm{NS}\left(\begin{bmatrix} -3/4 & -3/4 \\ -3/4 & -3/4 \end{bmatrix}\right) = \mathrm{NS}\left(\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}\right)$$

So a basis is given by $\boldsymbol{v_1} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$.

Similarly a basis for the eigenspace of λ_2 is $\boldsymbol{v_2} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$.

Now $\boldsymbol{v_1}$ and $\boldsymbol{v_2}$ are already orthogonal so we need only normalize these to get

$$\boldsymbol{u_1} = \begin{bmatrix} 1/\sqrt{2} \\ -1/\sqrt{2} \end{bmatrix} \qquad \boldsymbol{u_2} = \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix}$$

So

$$U = \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ -1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}$$

and

$$\begin{bmatrix} 5/4 & -3/4 \\ -3/4 & 5/4 \end{bmatrix} = \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ -1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix} \begin{bmatrix} 2 & 0 \\ 0 & 1/2 \end{bmatrix} \begin{bmatrix} 1/\sqrt{2} & -1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}$$

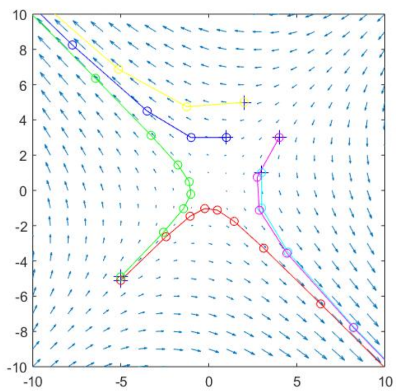
Question 3 (5 points)**.** Suppose the matrix A from Question 2 is used to transform points in the plane iteratively. That is, given a point $\boldsymbol{v}$, consider the sequence $\boldsymbol{v_n} = A^n\boldsymbol{v}$. Letting $U = \begin{bmatrix} \boldsymbol{u_1} & \boldsymbol{u_2} \end{bmatrix}$ so that $\boldsymbol{u_i}$ is an eigenvector associated to λ_i and letting $\boldsymbol{v} = c_1\boldsymbol{u_1} + c_2\boldsymbol{u_2}$ what is a simple expressions for a_n and b_n so that $\boldsymbol{v_n} = A^n\boldsymbol{v} = a_n\boldsymbol{u_1} + b_n\boldsymbol{u_2}$.

We have

$$
\begin{aligned}
A(c_1\boldsymbol{u_1} + c_2\boldsymbol{u_2}) &= c_1A\boldsymbol{u_1} + c_2A\boldsymbol{u_2} = \lambda_1c_1\boldsymbol{u_1} + \lambda_2c_2\boldsymbol{u_2} \\
A^2(c_1\boldsymbol{u_1} + c_2\boldsymbol{u_2}) &= A(\lambda_1c_1\boldsymbol{u_1} + \lambda_2c_2\boldsymbol{u_2}) = \lambda_1c_1A(\boldsymbol{u_1}) + \lambda_2c_2A(\boldsymbol{u_2}) = \lambda_1^2c_1\boldsymbol{u}_1 + \lambda_2^2c_2\boldsymbol{u}_2 \\
&\vdots \\
A^n(c_1\boldsymbol{u_1} + c_2\boldsymbol{u_2}) &= \lambda_1^nc_1\boldsymbol{u_1} + \lambda_2^nc_2\boldsymbol{u_2} = 2^nc_1\boldsymbol{u_1} + (1/2)^nc_2\boldsymbol{u_2}
\end{aligned}
$$

So $a_n = 2^nc_1$ and $b_n = (1/2)^nc_2$.

This is all I expected. For a visual, here is an image of a few points traced out by this action and the code that generated the image.



Here is the code that generated the image, just for fun. (download code)

```
clear;
close all;
clc;

A = 1/4*[5 -3; -3 5];
x = -10:1:10;
y = x;
[X,Y] = meshgrid(x,y);
Z = A*[X(:)'; Y(:)'];
u = Z(1,:);
v = Z(2,:);
fig = figure();
quiver(X(:)',Y(:)',u-X(:)',v-Y(:)')
axis([-10 10 -10 10]);
daspect([1 1 1]);
hold on;


x = [  -5    -5    1  3  2  4];
y = [-5.1  -4.9    3  1  5  3];

plot(x,y,"b+",'MarkerSize',10);
```

```

[M N] = size(x);

for i = 1:N
    p(:,:,i) = [x(i);y(i)];
end

colors =["red","green","blue","cyan","yellow","magenta"]

for i=2:10
    for j=1:N
        R(:,:,j) = A*p(:,i-1,j);
    end
    p = cat(2,p,R);
    for k=1:N
        plot(p(1,i-1:i,k),p(2,i-1:i,k),'Color',colors(k),...
            'Marker','o');
    end
    drawnow;
    pause(.5);
end
```