Homework 6 Solutions

Ch 16: 25, 27, 35, 37, 57, 58, 63, 64 - 66 (these are all related), 67, 68

25. If $x-2$ is a factor of $p(x) = x^4 - 2x - 2$, then $p(2) = 0$, $p(2) = 10 \bmod p = 0$ so $p = 2$ and $p = 5$.

27. (Used hint from the book here.) $U(p)$ is abelian of order $p-1$, if $U(p)$ were not cyclic, then by the fundamental theorem of abelian groups, for some q prime, $q \mid p-1$, there is $H \simeq \mathbb{Z}_q \times \mathbb{Z}_q < (U(p), \cdot, 1)$ (the multiplicative group). Let $\phi : \mathbb{Z}_q \times \mathbb{Z}_q \simeq H$ and let $x_{a,b} = \phi(a,b) \in U(p)$, then $x_{a,b}^q = 1$ and so $p(x) = x^q - 1$ has q^2 many solutions, which we know is impossible.

35. Show that $p(x) = x^3 - 2x^2 - 9$ has a root in every field. $p(3) = 3^3 - 2(3^2) - 9 = 3(3^2) - 2(3^2) - 3^2 = (3-2-1)(3^2) = 0$. So 3 is a root in any field. In $\mathbb{Z}_2$, $3 = 1$ and in $\mathbb{Z}^3$, $3 = 0$, but the argument still holds.

37. Let F be a field and $I = \{f(x) \in F[x] \mid f(1) = 0 \text{ and } f(2) = 0\}$. Find $g(x) \in F[x]$ so that $I = (g(x))$.

Let $g(x) = (x-1)(x-2) = x^2 - 3x + 2$, then $(g(x)) = \{f(x)(x-1)(x-2) \mid f(x) \in F[x]\}$. Clearly, $(g) \subseteq I$, conversely, the division algorithm shows that if $f(x) \in I$, then $f(x) = f'(x)(x-1)(x-2)$ for some $f'(x)$.

57. Show that in $\mathbb{Z}_p[x]$, $x^{p-1} - 1 = \prod_{a=1}^{p-1}(x-a)$.

This is because $a^{p-1} = 1$ in $\mathbb{Z}_p$ for all $a \in U(p) = \{1, \cdots, p-1\}$. Thus each element is a root of $x^{p-1} - 1$, and so the factorization follows.

58. (Wilson's Theorem) For every integer $n > 1$, $(n-1)! \bmod n = n-1$ iff n is prime.

If n is prime, then

$$x^{n-1} - 1 = (x-1)(x^{n-2} + x^{n-3} + \cdots + 1) = (x-1)(x-2)\cdots(x-(n-1))$$

So

$$x^{n-2} + x^{n-3} + \cdots + 1 = (x-2)(x-3)\cdots(x-(n-1)) \bmod n$$

Evaluating both sides at $x = 1$ gives

$$n-1 = (-1)(-2)\cdots(-(n-1)) = (n-1)(n-2)\cdots(1) = (n-1)! \bmod n$$

Conversely, if $n = s \cdot t$ is not prime, then $n \mid (n-1)!$ so $(n-1)! = 0 \bmod n$.

63. For a field that properly contains the field of complex numbers, the first thing that comes to mind is the quotient field of $\mathbb{C}[x]$. That is the field of rational functions over $\mathbb{C}$.

64. If I is an ideal of R show that $I[x]$ is an ideal of $R[x]$. It is clear that $I[x]$ is closed under addition. For the multiplicative closure a little effort is required, consider $p(x) \in I[x]$ with coefficients $a_i \in I$ and $q(x) \in R[x]$ with coefficients $b_i \in R$, then the coefficient of x^i in $p(x)q(x)$ is

$$c_i = \sum_{j=0}^{i} a_j b_{i-j} \in I$$

So $p(x)q(x) \in I[x]$.

65. $2\mathbb{Z}$ is a maximal ideal in $\mathbb{Z}$, since $\mathbb{Z}/2\mathbb{Z} \simeq \mathbb{Z}_2$ is a field. But, $\mathbb{Z}[x]/2\mathbb{Z}[x] \simeq \mathbb{Z}_2[x]$ is an integral domain, but not a field.

66. Show that if I is a prime ideal of R (commutative and unitary), then $I[x]$ is a prime ideal of $R[x]$.

If I is prime, then R/I is an integral domain. Now $R[x]/I[x] \simeq (R/I)[x]$ and since R/I is an integral domain, so is $R/I[x]$.

Note To prove $R[x]/I[x] \simeq (R/I)[x]$ define the map $\phi : R[x] \to (R/I)[x]$ by $\sum_{i=1}^{n} r_i x^i \mapsto \sum_{i=1}^{n} (r_i/I)x^i$. It is easy to see that this is a homomorphism and is surjective. Now show that $\ker(\phi) = I[x]$.

67. Show that $x = 1$ is the only solution to $x^{25} - 1$ in $\mathbb{Z}_{37}$.

For $x^{25} = 1$ in $U(37)$ we know that $|x| \mid 25 = 5^2$, on the other hand, $|x| \mid |U(37)| = 36 = 6^2$. Only $\gcd(36, 25) = 1$ so $|x| = 1$ and hence $x = 1$.

68. Show that $\mathbb{Q}[x]/)(x^2 - 2) \simeq \mathbb{Q}[\sqrt{2}]$.

There are several ways to do this. Here is one. Define $\phi : \mathbb{Q}[x] \to \mathbb{Q}[\sqrt{2}]$ by $x \mapsto \sqrt{2}$ and everything else maps as must be. A little effort verifies this to be a homomorphism and onto. So suppose $\phi(p(x)) = 0$, then $\sqrt{2}$ is a root of $p(x)$. We know $\overline{p(\sqrt{2})} = \bar{p}(\overline{\sqrt{2}}) = p(-\sqrt{2}) = 0$ as well, so $x^2 - 2 \mid p(x)$ and thus $\ker(\phi) = (x^2 - 2)$.

Note Here as usual $\overline{a + b\sqrt{2}} = a - b\sqrt{2}$.

Ch 17: 7, 12, 14, 15, 19, 28, 38, 39, 40

7. Suppose $r + 1/r$ is an odd integer, show that r is irrational.

Let n be an integer and consider $2n + 1 = x + 1/x$ or $x^2 - (2n+1)x + 1 = 0$. If r is rational, then this must factor over $\mathbb{Q}$. But if this factors over $\mathbb{Q}$, then it factors over $\mathbb{Z}$ as $(x-p)(x-q)$ with $p, q \in \mathbb{Z}$ so that either $p = q = 1$ or $p = q = -1$ and $2n + 1 = p + q = \pm 2$.

12. Construct a field of order 27.

Consider $x^3 + 2x + 1$. This has no root in $\mathbb{Z}_3$, so it is irreducible in $\mathbb{Z}_3[x]$ and hence $\mathbb{Z}_3[x]/(x^3 + x + 1)$ is a field, since $\mathbb{Z}_3[x]$ is a PID. The classes of $\mathbb{Z}_3[x]$ are given by $ax^2 + bx + c$ with $a, b, c \in \mathbb{Z}[3]$ so there are $3^3 = 27$ elements.

14. Which of the following are irreducible over $\mathbb{Q}$?

a. $x^5 + 9x^4 + 12x^2 + 6$: This is irreducible over $\mathbb{Q}$ since since $3 \nmid 1$, $3 \mid 9, 12, 6$, and $3^2 \nmid 6$.

b. x^4+x+1: x^4+x+1 has no linear factors since the only possible roots are ± 1. If it factors into quadratics, then we must have $x^4+x+1 = (x^2+ax+1)(x^2+bx+1) = x^4+(a+b)x^3+(ab+2)x^2+(a+b)x+1$. But then $a+b=1$ and $a+b=0$, so this can't happen either.

c. x^4+3x^2+3: This is like (a.). $3 \nmid 1$, $3 \mid 0, 3, 3$, and $3^2 \nmid 3$.

d. x^5+5x+1: Let's see if this is irreducible in $\mathbb{Z}_2[x]$. There are no linear factors, no roots in $\mathbb{Z}_2$. If there is a quadratic factor it must be one of $x^2, x^2+1, x^2+x, x^2+x+1$. Each of x^2, x^2+1, x^2+x have roots in $\mathbb{Z}_2$ so these can't be a factor. So x^2+x+1 is the only option. Actually, this does not work as $(x^2+x+1)(x^3+x^2-1) = x^5+5x+1$ in $\mathbb{Z}_2[x]$.

We can try $\mathbb{Z}_3[x]$. The quadratic factor would have to be x^2+ax+b and the only of those that do not have a root in $\mathbb{Z}_3$ are x^2+1, x^2+x+2, and x^2+2x+2 (see Example 8 in text). We can try long division with these and see that none divide evenly, so x^5+5x+1 is irreducible in $\mathbb{Z}_3[x]$.

e. $(5/2)x^5+(9/2)x^4+15x^2+6x+3/14$: $(1/14)(35x^5+63x^4+210x^2+84x+3)$. Again, as above $3 \nmid 35$, $3 \mid 63, 210, 84, 3$, and $3^2 \nmid 3$. So the polynomial is irreducible.

15. Consider $\mathbb{Z}_2[x]/(x^3+x+1)$.

$$(x^2+x)^2 = x^2(x+1)^2 = x^2(x^2+2x+1) = x^2(x^2+1) = x(x^3+x) = x(-1) = -x \bmod (x^3+x+1)$$

and noting that $1 = -1 = x^3+x$ we can divide x^3+x by x^2+x and get $x+1$.

$$(x^2+x)(x+1) = x^3+2x^2+x = x^3+x = -1 = 1$$

So $(x^2+x)^{-1} = x+1$.

19. Consider $F = \mathbb{Z}_7[x]/(x^2+2)$. x^2+2 has no roots in $\mathbb{Z}_7$ so x^2+2 is irreducible and $\mathbb{Z}_7[x]/(x^2+2)$ is a field.

$$\begin{aligned}
x^1 &= x, & x^2 &= -2 = 5, \\
x^3 &= 5x, & x^4 &= 5^2 = 25 = 4, \\
x^5 &= 4x, & x^6 &= 4x^2 = 20 = 6, \\
x^7 &= 6x, & x^8 &= 6x^2 = 30 = 2, \\
x^9 &= 2x, & x^{10} &= 2x^2 = 10 = 3, \\
x^{11} &= 3x, & x^{12} &= 15 = 1
\end{aligned}$$

So $|x| = 12$

$$\begin{aligned}
(x+1) &= x+1, & (x+1)^2 &= 2x+6, \\
(x+1)^3 &= x+2, & (x+1)^4 &= 3x, \\
(x+1)^5 &= 3x+1, & (x+1)^6 &= 4x+2, \\
(x+1)^7 &= 6x+1, & (x+1)^8 &= 3 \\
(x+1)^9 &= 3x+3, & (x+1)^{10} &= 6x+4 \\
(x+1)^{11} &= 3x+6, & (x+1)^{12} &= 2x \\
&\vdots & &\vdots
\end{aligned}$$

I got tired of this one so I made python do it for me. We see here that $U(F)$ is cyclic and $(x+1)$ is a primitive 48^{th} root of unity.

28. (a) and (b) seem to be asking the same thing as the quadratic monic polynomials are just those polynomials of the form x^2+ax+b. These are irreducible so long as they have no root in $\mathbb{Z}_p$. That is $x^2+ax+b \neq (x-m)(x-n) = x^2-(m+n)x+mn$ for any $m,n \in \mathbb{Z}_p$. There are $p(p-1)/2$ of the form $(x-m)(x-n)$ where $m \neq n$ and p where $m=n$ for a total of $p(p-1)/2+p$ many reducible monomial quadratics and thus $p^2-(p(p-1)/2+p) = p^2-(p^2-p)/2+p = p^2/2+p/2 = (p)(p+1)/2$ irreducible.

38. If $x^{p-1}-x^{p-2}+\cdots-x+1 = p(-x) = (-x)^{p-1}+(-x)^{p-2}+\cdots+(-x)^1+1 = f(x)g(x)$ with $\deg(g), deg(f) > 0$. Then $p(x) = p(--x) = x^{p-1}+x^{p-2}+\cdots+x+1 = f(-x)g(-x) = f_1(x)g_2(x)$. But this contradicts the irreducibility of the cyclotomic polynomial.

39. The evaluation map is obviously a homomorphism. Let $f(x) \in \ker(\phi)$. If $p(x) \nmid f(x)$, then as $p(x)$ is irreducible, we know $\gcd(f(x),p(x)) = 1$ (constant polynomial). We can use the Euclidean algorithm to find $q(x)$ and $r(x)$ so that $q(x)p(x)+r(x)f(x) = 1$. This is a contradiction since $q(a)p(a)+r(a)f(a) = q(a)\cdot 0 + r(a)\cdot 0 = 0 \neq 1$. So $p(x) \mid f(x)$.

40. We have seen before that $\mathbb{Z}[x]/(x^2+1) \simeq \mathbb{Z}[i]$ is an integral domain, but not a field, so (x^2+1) is prime and not maximal.

Ch 18: 17, 30, 33, 36, 37, 38, 41, 42

17. Show in $\mathbb{Z}[i]$ that 3 is irreducible, hence prime, since $\mathbb{Z}[i]$ is a PID, and hence UFD, but 2 and 5 are not irreducible.

$$2 = (1-i)(1+i)$$

and

$$5 = (1-2i)(1+4i)$$

Suppose $3 = (a+bi)(c+di)$, then

$$3\bar{3} = 9 = (a+bi)(c+di)\overline{(a+bi)(c+di)} = (a+bi)\overline{(a+bi)}(c+di)\overline{(c+di)} = (a^2+b^2)(c^2+d^2)$$

But then, $3 \mid a^2+b^2$ (or $3 \mid c^2+d^2$). This is the same as $a^2+b^2 = 0 \bmod 3$ and this in turn is the same as

$$(a \bmod 3)^2 + (b \bmod 3)^2 = 0 \bmod 3$$

But we can just check the values for $a \bmod 3$ and $b \bmod 3$. Using the symmetry that we have here, we can just check the pairs (r,s) for (r,s) in $\{(0,0),(1,0),(2,0),(1,1),(2,1),(2,2)\}$ the only one satisfying $r^2+s^2=0$ is for $r=0=s$. So we must $3 \mid a, b$ and hence $3 \mid a+bi$ and so

$$3 = 3(a'+b'i)(c+di)$$

but then $a'+b'i, c+di \in \{1,-1\}$ (a unit) so 3 is irreducible.

29. Show that if $p \mid n$, then p is prime in $\mathbb{Z}_n$.

If $p \mid a\cdot b$ in $\mathbb{Z}_n$, then $a\cdot b = p\cdot m \bmod n$ so $n \mid a\cdot b - p\cdot m$, that is $a\cdot b - p\cdot m = n\cdot q$ and so $p\cdot m = a\cdot b - n\cdot q$ and since $p \mid n$ we must have $p \mid a\cdot b$ so $p \mid a$ or $p \mid b$. It is easy to see that if $p \mid a$, then $p \mid a \bmod n$.

So p is a prime in $\mathbb{Z}_n$.

30. You might think that since all primes are irreducible, we are done from 29. But this was only true in an integral domain. So we must argue the point.

If $p^2 \nmid n$, then n/p and p are relatively prime, so there are s and t such that $sp + t(n/p) = 1$, but then $p = p(sp) + tn$ and thus $p = p(sp) \bmod n$ witnesses that p is decomposible since p and sp are not a units in $\mathbb{Z}_n$.

Conversely, if $p^2 \mid n$ and $p = ab$, then $p - ab = mn$ so $1 - ab/p = m(n/p)$. We know $p \mid b$ or $p \mid a$. Suppose $p \mid b$. We know $q \nmid a$ for any prime factor of n' and so $\gcd(a, n') = \gcd(a, n) = 1$ and so a is a unit in $\mathbb{Z}_n$.

33. This is a trivial induction. Suppose for all $m < n$ is $p \mid a_1 \cdots a_{m-1}$, then $p \mid a_i$ for some $i < m$. Then if $p \mid a_1 \ldots a_{n-1}$ we have $p \mid a_1 \cdots a_{n-2}$ or $p \mid a_{n-1}$. In the latter case, we are done. In the first case, we apply the induction hypothesis to $m = n - 1$.

36. Show that every integral domain with the descending chain condition is a field. First, we may assume $|R|$ is infinite since we already know that any finite integral domain is a field.

If R is not a field, let $r \neq 0$ be a non-unit of R. If $(r^2) = (r)$, then $r = r^2 t$ for some t, but then $r - r^2 t = r(1 - rt) = 0$, so either $r = 0$ or r is a unit. Either is a contradiction. So $(r^2) \subset (r)$. Continuing, we get $(r^3) = (r^2)$ implies $r^2 = r^3 t$ so $r^2(1 - rt) = 0$ and either $r^2 = 0$ or r is a unit. Again, neither can be true so $(r^3) \subset (r^2)$. We can continue thus to get $(r^{n+1}) \subset (r^n)$ for all n. This contradicts the descending chain condition. So it must be that R is a field.

37. Show that R satisfies ACC iff every ideal is finitely generated.

Suppose R satisfies ACC. Fix an ideal I. Take $a_1 \in I$, if $(a_1) \neq I$, then take $a_2 \in I - (a_1)$. If $(a_1, a_2) \neq I$, take $a_3 \in I - (a_1, a_2)$, etc. Since R satisfies ACC, we must reach some k so that $(a_1, a_2, \ldots, a_k) = I$.

Suppose every ideal is finitely generated. Let $I_1 \subset I_2 \subset \cdots$ be proper ideals. Let $I = \bigcup_i I_i$. I is finitely generated so get k such that $(a_1, \ldots, a_k) = I$. Take n so that $a_i \in I_n$ for $i = 1, 2, \ldots, k$. Then $I_n = I$ and we have ACC.

38. It is not true that a subdomain of a Euclidean domain needs be Euclidean as $\mathbb{Z}[x] \subset \mathbb{Q}[x]$ demonstrates. Both are domains, but $\mathbb{Z}[x]$ is not Euclidean.

41. In $\mathbb{Z}[-7]$, clearly $N(6+2\sqrt{-7}) = 6^2+7\cdot 2^2 = 36+28 = 1+63 = 1^2+3^2\cdot 7 = N(1+3\sqrt{-7})$. Also, if $u \in U(\mathbb{Z}[\sqrt{-7}])$, then $N(u) = 1 = a^2 + 7b^2$ where $a, b \in \mathbb{Z}$. The only option here is $u = \pm 1$, that is $U(\mathbb{Z}[\sqrt{-7}]) = \{1, -1\}$. Clearly, $6 + 2\sqrt{-7} \neq \pm(1 + 3\sqrt{-7})$ so $6 + 2\sqrt{-7}$ and $1 + 3\sqrt{-7}$ are not associates.

42. Let $R = \mathbb{Z} \times \mathbb{Z} \times \mathbb{Z} \cdots = \sum_{i \in \mathbb{N}} \mathbb{Z}$. Let $r_i = (1, 1, 1, \cdots, 1, 0, 0, \cdots) \in R$ so that r_i has i many 1's followed by 0's. Clearly $(r_i) \subset (r_{i+1})$, basically,

$$(r_i) = R^i \times \{0\} \times \{0\} \times \cdots \subset R^{i+1} \times \{0\} \times \{0\} \times \cdots = (r_{i+1}).$$