Math 571 - (Take-Home) Exam 2 (04.24)

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There are 80 points here, so basically, 10 extra points. I'll take the score and use the minimum of 70 and your score as the final grade. Make your answers self-contained. If something here comes straight out of the homework, do not "quote" the homework result as a reason. I am looking for the argument. If in doubt ask!

Question 1 (20 pts)**.** Give a reason for the non-existence of each of the following, or else provide an example. You may use theorems, but you must state the theorem; a theorem number does not suffice. In either case, you must show that your example has the required properties.

(a) A continuous function $f : S_1 \to \mathbb{Q}$ that is not a constant function where $S_1 = \{(x, y) \mid x^2 + y^2 = 1\}$ is the unit circle.

S_1 is connected, $\mathbb{Q}$ is not.

(b) A continuous function $f : \mathbb{Q} \to \mathbb{Z}$ that is not a constant function.

It is simple enough to choose such a function whose range is $\{0, 1\}$, for example, $f(r) = 0$ if $r < \sqrt{2}$ and $f(r) = 1$ if $r > \sqrt{2}$.

For fun, let's find an onto example. Take $(s_i)_{i\in\mathbb{Z}}$ to be irrational numbers such that $i < j \implies s_i < s_j$. Then simply map (s_i, s_{i+1}) to $i \in \mathbb{Z}$. For each i, $f^{-1}(i) = (s_i, s_{i+1}) \cap \mathbb{Q}$ is *clopen* (closed and open) and $f^{-1}(i) \cap f^{-1}(j) = \emptyset$. Let $A \subset Z$ (every subset of $\mathbb{Z}$ is open), then $f^{-1}(A) = \bigcup_{i\in A} f^{-1}(i)$. This is a union of open sets; hence, it is open.

(c) A function $f : \left\{\frac{1}{n} \mid n \in \mathbb{Z} - \{0\}\right\} \to \mathbb{R}$ that fails to be continuous.

$\{\frac{1}{n} \mid n \in \mathbb{Z} - \{0\}\}$ is discrete, so no such map.

(d) A continuous non-constant 1-1 function $f : S_1 \to \mathbb{R}$, where $S_1 = \{(x, y) \mid x^2 + y^2 = 1\}$ is the unit circle.

S_1 is compact and connected, so $f(S_1)$ must also be compact and connected. Thus we would have $f(S_1) = [a, b]$. Say $f(t_a) = a$ and $f(t_b) = b$. If $t_a \neq t_b$, then there are two arcs on S_1 say a_1 and a_2 from t_a to t_b and then from t_b to t_a, going counterclockwise on S_1. Then a_1 and a_2 are compact and connected, so $f(a_1)$ and $f(a_2)$ are closed intervals containing a and b, so $f(a_1) = f(a_2) = [a, b]$. This contradicts f being 1-1.

Definition 1. Let $f_n : [a, b] \to \mathbb{R}$, define $f_n \underset{\text{unif}}{\to} f$ to mean:

$$(\forall \varepsilon > 0)(\exists N > 0)(\forall n > N)(\forall x \in [a, b]) \left(|f_n(x) - f(x)| < \varepsilon\right)$$

Contrast this to the standard *pointwise convergence* where $f_n \to f$ iff

$$(\forall x \in [a,b])(\forall \varepsilon > 0)(\exists N > 0)(\forall n > N)\left(|f_n(x) - f(x)| < \varepsilon\right)$$

The point is that in the uniform case, we get for every $\varepsilon > 0$, a fixed $N_\varepsilon > 0$, which *works* for all x uniformly, whereas in the pointwise case, we have for each $x \in [a,b]$ and $\varepsilon > 0$, a $N_\varepsilon(x)$ that does the job..

Question 2 (15 pts)**.** Let $f_n \in \mathcal{R}([a,b])$ (Riemann integrable functions on $[a,b]$), suppose $f_n \underset{\text{unif}}{\to} f$. Show that $f \in \mathcal{R}([a,b])$ and $\lim_n \int_a^b f(x)\,dx = \int_a^b \lim_n f_n(x)\,dx = \int_a^b f(x)\,dx$.

We must show that for all $\varepsilon > 0$, there is a partition P of $[a,b]$ so that $U(f,P) - L(f,P) < \varepsilon$.

Take $N > 0$ so that for all $n > N$, $\sup(\{|f_n(x) - f(x)| \mid x \in [a,b]\}) < \frac{\varepsilon}{4(b-a)}$. Fix $n > N$ and P a partition of $[a,b]$ so that $U(f_n,P) - L(f_n,P) < \frac{\varepsilon}{2}$. For $x, y \in [x_{i-1}, x_i]$, we see

$$U(f,P) = \sum_i M_i(f)\Delta_i \leq \sum_i \Big(M_i(f_n) + \frac{\varepsilon}{4(b-a)}\Big)\Delta_i = \frac{\varepsilon}{4(b-a)}(b-a) + U(f_n,P)$$

Similarly for $L(f,P)$ and thus we get

$$L(f_n,P) - \frac{\varepsilon}{2} \leq L(f,P) \leq U(f,P) \leq U(f_n,P) + \frac{\varepsilon}{2}$$

This shows at once that for all $\varepsilon > 0$, there is $N > 0$ such that for $n > N$

$$L(f_n) - \frac{\varepsilon}{2} = L(f) \leq U(f) \leq U(f_n) + \frac{\varepsilon}{2}$$

This gives $U(f) - L(f) < \varepsilon$ and $|U(f) - U(f_n)| < \frac{\varepsilon}{2}$ for all $\varepsilon > 0$ and large enough n. So $f \in \mathcal{R}([a,b])$ and $\lim_n U(f_n) = U(f)$ which is what we needed.

Question 3 (15 pts)**.** (1) Let $f : \mathbb{R} \to \mathbb{R}$ be differentiable, suppose f' is bounded. Show that f is uniformly continuous.

(2) Find a uniformly continuous and differentiable function $f : \mathbb{R} \to \mathbb{R}$ whose derivative is not bounded. You must show that your example id uniformly continuous.

Let $M > 0$ be a bound so $|f'(x)| < M$ for all $x \in \mathbb{R}$. By MVT for any $a < b$ we have $f(b) - f(a) = f'(t)(b-a)$ for some $t \in (a,b)$. So $|f(b) - f(a)| < M|b-a|$ and thus for $\epsilon > 0$, let $\delta = \frac{\epsilon}{M}$, then for x, x' with $|x - x'| < \delta$, we have $|f(x) - f(x')| < M|x - x'| < M\delta = \epsilon$.

Example 1: A simple example is:

$$f(x) = \begin{cases} \frac{\sin(x^3)}{x} & x \neq 0 \\ 0 & x = 0 \end{cases}$$

Clearly, this function is continuous given that $\lim_{h\to 0} \frac{\sin(h^3)}{h} = 0$. (Use L'Hospital). Since $\lim_{x\to\infty} f(x)$ and $\lim_{x\to-\infty} f(x)$ both exist, f is is uniformly continuous.

It is also clear that $f(x)$ is differentiable for $x \neq 0$. For $x = 0$ we need to compute:

$$\lim_{h\to 0} \frac{f(h) - f(0)}{h} = \lim_{h\to 0} \frac{\sin(h^2)}{h} \overset{\text{l'H}}{=} \lim_{h\to 0} \frac{3h^2\cos(h^3)}{1} = 0$$

Here is a plot.

For $x \neq 0$ we have $f'(x) = \frac{\cos(x^3)(3x^2)}{x} - \frac{\sin(x^3)}{x^2} = 3x\cos(x^3) - \frac{\sin(x^3)}{x^2}$. As $x \to \infty$. it is clear that $f'(x)$ is unbounded.

Example 2: Another, related, example is

$$g(x) = \begin{cases} x\sin(1/x^3) & x \neq 0 \\ 0 & x = 0 \end{cases}$$

Here is a plot

This function is clearly continuous has limits at $\pm\infty$, hence uniformly continuous. The derivative is clearly not bounded.

Note the two examples are related by $g(x) = f(1/x)$

Example 3: Here is another nice example from a student

$$f(x) = \frac{\sin(e^x)}{1+x^2}$$

Here is a plot

See the fact below for uniform continuity of f.

Note: You must do something to show that your example is uniformly continuous. For example $\sin(x^2)$ is not uniformly continuous. Here is a "proof by picture". Note how the intervals in the shaded area grow smaller. The following is useful for showing a function is uniformly continuous.

Fact: Suppose $f : \mathbb{R} \to \mathbb{R}$ is continuous and $\lim_{x\to\infty} f(x) = L$ and $\lim_{x\to-\infty} f(x) = M$, then f is uniformly continuous. We want to see that f is uniformly continuous. Fix $\epsilon > 0$. There is N so that for all $x > N$, $|f(x) - L| < \epsilon/2$ and $|f(-x) - M| < \epsilon/2$. Notice from this it is clear that for $x > x' > N$ or $x < x' < -N$ we have $|f(x) - f(x')| < \epsilon$. Now f is uniformly continuous on $[-N-1, N+1]$ and so there is $1/2 > \delta > 0$ so that $|x - x'| < \delta \implies |f(x) - f(x')| < \epsilon$. It follows that $|x - x'| < \delta$ either puts x and x' in $(-\infty, N)$, $(-N - 1/2, N + 1/2)$, or (N, ∞) and thus $|x - x'| < \delta \implies |f(x) - f(x')| < \epsilon$ for any x and x'.

Question 4 (15 pts)**.** Let X and Y be metric spaces with X **compact** and let $f : X \to Y$ and $g : X \to Z$ be two functions, with no additional assumptions on these functions.

Suppose that for every $x \in X$, at least one of f or g is continuous at x. Show that for every $\epsilon > 0$, there is a $\delta > 0$ such that for all $x, x' \in X$:

$$d_X(x, x') < \delta \implies d_Y(f(x), f(x')) < \epsilon \text{ or } d_Z(g(x), g(x')) < \epsilon$$

This is sort of an "either/or" version of uniform continuity.

Proof 1: Fix $\epsilon > 0$. For each $x \in X$, fix $\delta_x > 0$, so that either

$$d_X(x, x') < \delta_x \implies d_Y(f(x), f(x')) < \epsilon$$

or

$$d_X(x, x') < \delta_x \implies d_Z(g(x), g(x')) < \epsilon$$

In the first case, say x is f-good, and in the second, say x is g-good. Notice that x can be g-good and f-good simultaneously.

Let $\mathscr{O} = \{N_{\delta_x/2}(x) \mid x \in X\}$ is an open cover of X. Let $\{N_{\delta_{x_i}/2}(x_i) \mid i = 1, 2, \dots, n\}$ be a finite subcover. Let $\delta = \min\{\delta_i \mid i = 1, \dots, n\}/2$. Then for $x, x' \in X$, if $d_X(x, x') < \delta$, we have $x \in N_{\delta_{x_i}/2}(x_i)$ for some i and $d_X(x, x') < \delta_{x_i}/2$ so $d_X(x', x_i) < d_X(x', x) + d_X(x, x_i) < \delta_{x_i}$. If x_i is f-good, then we have $d_Y(f(x), f(x')) < \epsilon$, else $d_Z(g(x), g(x')) < \epsilon$. This is what we wanted to prove.

Proof 2: Suppose for some $\varepsilon > 0$ for all $\delta > 0$ there is x, x' with $d_X(x, x') < \delta$ and $d_Y(f(x), f(x')) \geq \varepsilon$ and $d_Z(g(x), g(x')) \geq \varepsilon$. Choose sequences x_i and x_i' so that $d_X(x_i, x_i') < \delta_i$ where $\delta_i \to 0$ and so that $d_Y(f(x_i), f(x_i')) \geq \varepsilon$ and $d_Z(g(x_i), g(x_i')) \geq \varepsilon$. As X is compact, there is a convergent subsequence x_{i_k} of x_i. Let $x_{i_k} \to x$, clearly as $d(x'_{i_k}, x_{i_k}) \to 0$ we also have $x'_{i_k} \to x$.

By assumption, either f or g is continuous at x. If f is continuous at x, then $f(x_{i_k}) \to f(x)$ and $f(x'_{i_k}) \to f(x)$. But then $d_Y(f(x_{i_k}), f(x'_{i_k})) \leq d_Y(f(x), f(x_{i_k}) + d_Y(f(x), f(x_{i_k}))$ can be made arbitrarily small contradicting our assumption.

Question 5 (15 pts)**.** Let f and α be monotonically increasing bounded functions on $[a, b]$. Suppose that α is continuous at every point where f is discontinuous. Show that $f \in \mathscr{R}(\alpha)$.

Let $\epsilon > 0$. By the Question 4, there is $\delta > 0$ so that:

$$|x - x'| < \delta \implies |f(x) - f(x')| < \epsilon \text{ or } |\alpha(x) - \alpha(x')| < \epsilon$$

Let P be a partition $a = x_0 < x_1 < \cdots < x_n = b$ with $x_i - x_{i-1} < \delta$ for $i = 1, 2, \dots, x_n$, then

$$\begin{aligned}
U(P, \alpha, f) - L(P, \alpha, f) &= \sum_{i=1}^{n} (M_i^{P,f} - m_i^{P,f})(\alpha(x_i) - \alpha(x_{i-1})) \\
&= \sum_{i=1}^{n} (f(x_i) - f(x_{i-1}))(\alpha(x_i) - \alpha(x_{i-1})) \\
&\leq \sum_{i \in A} \epsilon(\alpha(x_i) - \alpha(x_{i-1})) + \sum_{i \in B} (f(x_i) - f(x_{i-1}))\epsilon \\
&\leq \sum_{i=1}^{n} \epsilon(\alpha(x_i) - \alpha(x_{i-1})) + \sum_{i=1}^{n} (f(x_i) - f(x_{i-1}))\epsilon \\
&= \epsilon(\alpha(b) - \alpha(a)) + (f(b) - f(a))\epsilon \qquad \text{(monotonicity used)} \\
&= \epsilon((f(b) - f(a)) + (\alpha(b) - \alpha(a)))
\end{aligned}$$

where A is the set of i so that $|f(x_i) - f(x_{i-1})| < \epsilon$ and B is the rest, so for $i \in B$ we have $|\alpha(x_i) - \alpha(x_{i-1})| < \epsilon$ By replacing the original ϵ by $\frac{\epsilon}{(f(b)-f(a))+(\alpha(b)-\alpha(a))}$, we have what we want.