Part III: Theory and Proofs (30 points; 10 points each)

Choose three of the five options. If you try all five, I will grade the first three, not the best three. You must decide what should be graded.

This part is take-home. You should complete this work on your own without consulting websites, friends, the Math Center, etc.

Problem 6 (10 points)**.** Suppose S is an independent set of vectors from a vector space V, then

$$S \cup \{\boldsymbol{v}\} \text{ is dependent } \iff \boldsymbol{v} \in \operatorname{span}(S).$$

($\Longleftarrow$) $\boldsymbol{v} \in \operatorname{span}(S)$ means that $\boldsymbol{v} = \alpha_1 \boldsymbol{v}_1 + \cdots + \alpha_k \boldsymbol{v}_k$ for some scalars α_i and $\boldsymbol{v}_i \in S$. Clearly then

$$\boldsymbol{v} - (\alpha_1 \boldsymbol{v}_1 + \cdots + \alpha_k \boldsymbol{v}_k) = \boldsymbol{0}$$

so $S \cup \{\boldsymbol{v}\}$ is dependent since we have written $\boldsymbol{0}$ as a non-trivial linear combination of vectors from $S \cup \{\boldsymbol{v}\}$.

($\Longrightarrow$) $S \cup \{\boldsymbol{v}\}$ is dependent so $\boldsymbol{v} = \alpha_1 \boldsymbol{v}_1 + \cdots + \alpha_k \boldsymbol{v}_k = \boldsymbol{0}$ for some scalars $\alpha_i \neq 0$ and $\boldsymbol{v}_i \in S \cup \{\boldsymbol{v}\}$. Since S is independent, it must be that $\boldsymbol{v}$ is one of the $\boldsymbol{v}_i$'s. WLOG suppose $v = \boldsymbol{v}_1$, then

$$\boldsymbol{v} = -\frac{1}{\alpha_1}(\alpha_2 \boldsymbol{v}_2 + \cdots + \alpha_k \boldsymbol{v}_k)$$

and so $\boldsymbol{v} \in \operatorname{span}(S)$.

Problem 7 (10 points)**.** Show that if $L : V \to W$ is linear and $\{L(\boldsymbol{v}_1), L(\boldsymbol{v}_2), L(\boldsymbol{v}_3)\}$ is linearly independent, then $\{\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3\}$ is linearly independent.

More generally, if $L : V \to W$ is linear, then the pre-image of S, $L^{-1}(S) = \{\boldsymbol{v} \mid L(\boldsymbol{v}) \in S\}$ is linearly independent for any linearly independent set S.

Let

$$\alpha_1 \boldsymbol{v}_1 + \alpha_2 \boldsymbol{v}_2 + \alpha_3 \boldsymbol{v}_3 = \boldsymbol{0},$$

then

$$L(\alpha_1 \boldsymbol{v}_1 + \alpha_2 \boldsymbol{v}_2 + \alpha_3 \boldsymbol{v}_3) = \alpha_1 L(\boldsymbol{v}_1) + \alpha_2 L(\boldsymbol{v}_2) + \alpha_3 L(\boldsymbol{v}_3) = L(\boldsymbol{0}) = \boldsymbol{0}$$

so by the independence of $\{L(\boldsymbol{v}_1), L(\boldsymbol{v}_2), L(\boldsymbol{v}_3)\}$ we have $\alpha_1 = \alpha_2 = \alpha_3 = 0$ and thus $\{\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3\}$ is linearly independent.

Problem 8 (10 points)**.** Suppose $A = \begin{bmatrix} \boldsymbol{a}_1 \, \boldsymbol{a}_2 \, \boldsymbol{a}_3 \, \boldsymbol{a}_4 \, \boldsymbol{a}_5 \end{bmatrix}$ is a 4×5 matrix and

$$\operatorname{NS}(A) = \operatorname{span}\{(-2, 1, 0, 0, 0), (5, 0, 2, 1, 0)\}$$

Find rref(A) and explain how you know that what you have found is rref(A).

We know a typical element of NS(A) is of the form $(x_1, x_2, x_3, x_4, x_5) = (-2s + 5t, s, 2t, t, 0)$ and since $A\boldsymbol{x} = \boldsymbol{0}$ can be written as a linear combination of columns of A we know

$$(-2s + 5t)\boldsymbol{a}_1 + s\boldsymbol{a}_2 + 2t\boldsymbol{a}_3 + t\boldsymbol{a}_4 + 0\boldsymbol{a}_5 = \boldsymbol{0}$$

Letting $s = 1$ and $t = 0$ we get $-2\boldsymbol{a}_1 + \boldsymbol{a}_2 = 0$ and letting $s = 0$ and $t = 1$ we get $5\boldsymbol{a}_1 + 2\boldsymbol{a}_3 + \boldsymbol{a}_4 = 0$. Thus we have

$$\boldsymbol{a}_2 = 3\boldsymbol{a}_1 \quad \text{and} \quad \boldsymbol{a}_4 = -5\boldsymbol{a}_2 - 2\boldsymbol{a}_3$$

Thus we get

$$\begin{bmatrix}\boldsymbol{a}_1 \, \boldsymbol{a}_3 \, \boldsymbol{a}_5\end{bmatrix} \begin{bmatrix} 1 & 2 & 0 & -5 & 0 \\ 0 & 0 & 1 & -2 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix}\boldsymbol{a}_1 \, \boldsymbol{a}_2 \, \boldsymbol{a}_3 \, \boldsymbol{a}_4 \, \boldsymbol{a}_5\end{bmatrix} = A$$

We know $\text{rank}(A) = 3 = 5 - \dim(\text{NS}(A))$ so $\{\boldsymbol{a}_1, \boldsymbol{a}_3, \boldsymbol{a}_5\}$ are linearly independent vectors in $\mathbb{R}^4$. Let $\boldsymbol{b} \in \mathbb{R}^4$ be so that $\{\boldsymbol{a_1}, \boldsymbol{a_2}, \boldsymbol{a_3}, \boldsymbol{b}\}$ is a basis and lat $M = \begin{bmatrix}\boldsymbol{a}_1 \, \boldsymbol{a}_3 \, \boldsymbol{a}_5 \, \boldsymbol{b}\end{bmatrix}$, then M is invertible and

$$M \begin{bmatrix} 1 & 2 & 0 & -5 & 0 \\ 0 & 0 & 1 & -2 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} = MR = A$$

So A is equivalent to R. But R is in RREF form so $R = \text{rref}(A)$, since there is only one RREF matrix equivalent to A.

Note: Recall A and B are equivalent if B can be formed from a sequence of elementary row operations applied to A; equivalently, A and B are equivalent iff $B = MA$ for some invertible M. We know

$$A \text{ and } B \text{ are equivalent} \implies \text{NS}(A) = \text{NS}(B).$$

It turns out that for matrices of the same size

$$A \text{ is equivalent to B} \iff \text{NS}(A) = \text{NS}(B)$$

To see this it suffices to show that

$$\text{NS}(A) = \text{NS}(B) \implies \text{rref}(A) = \text{rref}(B).$$

The above basically does this argument by showing that rref(A) can be computed from a basis for NS(A).

Problem 9 (10 points)**.** Suppose A is a 5×5 matrix and $A^n = O$ for some n, then $A^5 = O$.

There are several ways to proceed. Here is one. Note that $\mathrm{NS}(A^{m+1}) \supseteq \mathrm{NS}(A^m)$ for all m since $A^m\boldsymbol{x} = \mathbf{0} \implies A^{m+1}\boldsymbol{x} = A(A^m)\boldsymbol{x} = \mathbf{0}$.

If $\mathrm{NS}(A^{m+1}) = \mathrm{NS}(A^m)$, then $\mathrm{NS}(A^{m+k}) = \mathrm{NS}(A^m)$ for all k. To see this, suppose $\mathrm{NS}(A^{m+k}) = \mathrm{NS}(A^m)$, then

$$\begin{aligned} A^{m+k+1}\boldsymbol{x} = \mathbf{0} &\iff A^{m+k}(A\boldsymbol{x}) = \mathbf{0} && \text{(by assumption)} \\ &\iff A^m(A\boldsymbol{x}) = \mathbf{0} && (1) \\ &\iff A^{m+1}\boldsymbol{x} = \mathbf{0} && (2) \\ &\iff A^m\boldsymbol{x} = \mathbf{0} && (3) \end{aligned}$$

This means that we have the following situation

$$\mathrm{NS}(A) \subsetneqq \mathrm{NS}(A^2) \subsetneqq \cdots \mathrm{NS}(A^{m-1}) \subsetneqq \mathrm{NS}(A^m) = \mathrm{NS}(A^n) \text{ for all } n \geq m$$

Since $0 < \dim(\mathrm{NS}(A)) < \dim(\mathrm{NS}(A^2)) < \cdots < \dim(\mathrm{NS}(A^m)) \leq 5$ we know $m \leq 5$.

If $A^n = O$ for any n, then $\mathrm{NS}(A^n) = \mathbb{R}^5$. But the first place where $\mathrm{NS}(A^n) = \mathbb{R}^5$ will be for $n \leq 5$ and so $A^5 = O$.

Problem 10 (10 points)**.** For A and B are $n \times n$ matrices. Show that

$$AB \text{ is invertible} \iff \text{both } A \text{ and } B \text{ are invertible}$$

($\Longleftarrow$) case: If A and B are invertible, then AB is invertible, since $(AB)^{-1} = B^{-1}A^{-1}$.

($\implies$) case (Proof 1 using NS) If B is not invertible, then $\mathrm{NS}(B) \neq \{\mathbf{0}\}$, but $B\boldsymbol{x} = \mathbf{0} \implies A(B\boldsymbol{x}) = (AB)\boldsymbol{x} = \mathbf{0}$, so $\mathrm{NS}(AB) \neq \{\mathbf{0}\}$ and hence AB is not invertible.

If B is invertible, but A is not, then again let $\boldsymbol{x} \in \mathrm{NS}(A)$, since B is invertible, $\boldsymbol{x} = B\boldsymbol{y}$ for some $\boldsymbol{y}$, in fact, $\boldsymbol{y} = B^{-1}\boldsymbol{x}$. But then, $A(B\boldsymbol{y}) = (AB)\boldsymbol{y} = \mathbf{0}$ and so $\mathrm{NS}(AB) \neq \{\mathbf{0}\}$, so again AB is not invertible.

So if either A or B is not invertible, then neither is AB, and hence if AB is invertible, then both A and B must be invertible.

($\implies$) case (Proof 2 using det) Suppose AB is invertible, then $0 \neq \det(AB) = \det(A)\det(B)$ so $\det(A) \neq 0 \neq \det(B)$ and so A and B are invertible.

($\implies$) case (Proof 3 using algebra.) Suppose AB is invertible, then $A(B(AB)^{-1}) = (AB)(AB)^{-1} = I$ so $A^{-1} = B(AB)^{-1}$ and $B^{-1} = (AB)^{-1}A$ for a similar reason.

Note: This actually uses that $E = F^{-1}$ iff $EF = I$ **or** $FE = I$, whereas the actual definition has "**and**" not "or." To prove this, one usually uses one of the above arguments.