SVD Example: Shear

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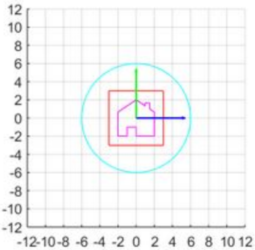
Example: Shear

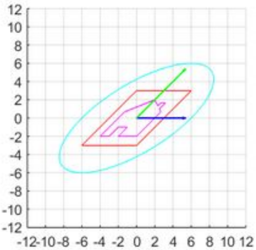
Action

Consider the matrix

$$A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$$

The induced mapping $S : \mathbb{R}^2 \to \mathbb{R}^2$ is called a *shear*. The image shows how shear acts on the standard basis as well as simple shapes.





Eigenvalues/Eigenvectors

$$\det \begin{bmatrix} 1-\lambda & 1 \\ 0 & 1-\lambda \end{bmatrix} = (1-\lambda)^2$$

This has one eigenvalue $\lambda_1 = 1$.

To find eigenvectors compute $\mathrm{NS}(A - \lambda_1 I)$:

$$\mathrm{NS}\left(\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}\right) = \mathrm{span}\left\{\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right\}$$

So there is no basis of eigenvectors, the matrix is called *deficient*, hence A is not diagonalizable.

SVD Calculation: Eigenvalues for $A^T A$

$$A^T A = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}$$

So

$$\det(A^T A - \lambda I) = \det \begin{bmatrix} 1-\lambda & 1 \\ 1 & 2-\lambda \end{bmatrix} = \lambda^2 - 3\lambda + 1$$

This has roots (eigenvalues):

$$\lambda_1 = \frac{3+\sqrt{5}}{2} > \lambda_2 = \frac{3-\sqrt{5}}{2}$$

SVD Calculation: Eigenvectors for A^TA

For λ_1 we must find $\mathrm{NS}(A^TA - \lambda_1 I)$

$$\mathrm{NS}\begin{bmatrix} 1-\lambda_1 & 1 \\ 1 & 2-\lambda_1 \end{bmatrix} = \mathrm{NS}\begin{bmatrix} 1-\lambda_1 & 1 \\ 0 & 0 \end{bmatrix} = \mathrm{span}\left\{\begin{bmatrix} 1 \\ \lambda_1 - 1 \end{bmatrix}\right\} = \mathrm{span}\left\{\begin{bmatrix} 1 \\ \frac{1+\sqrt{5}}{2} \end{bmatrix}\right\}$$

The calculation for λ_2 is the same and we have:

$$\mathrm{NS}\begin{bmatrix} 1-\lambda_2 & 1 \\ 1 & 2-\lambda_2 \end{bmatrix} = \mathrm{NS}\begin{bmatrix} 1-\lambda_2 & 1 \\ 0 & 0 \end{bmatrix} = \mathrm{span}\left\{\begin{bmatrix} 1 \\ \lambda_2 - 1 \end{bmatrix}\right\} = \mathrm{span}\left\{\begin{bmatrix} 1 \\ \frac{1-\sqrt{5}}{2} \end{bmatrix}\right\}$$

SVD Calculation: Eigenvectors for $A^T A$ = Right Singular Vectors for A

For SVD the eigenvectors must be normalized:

$$\left\| \begin{bmatrix} 1 \\ \frac{1\pm\sqrt{5}}{2} \end{bmatrix} \right\|^2 = 1 + \left(\frac{1 \pm \sqrt{5}}{2} \right)^2 = 1 + \frac{1 + 5 \pm 2\sqrt{5}}{4} = \frac{5 \pm \sqrt{5}}{2}$$

So take normalized eigenvectors $\boldsymbol{v}_1$ and $\boldsymbol{v}_2$ for λ_1 and λ_2 respectively to be:

$$\boldsymbol{v}_1 = \left(\frac{5 + \sqrt{5}}{2} \right)^{-1/2} \begin{bmatrix} 1 \\ \frac{1+\sqrt{5}}{2} \end{bmatrix} \text{ and } \boldsymbol{v}_2 = \left(\frac{5 - \sqrt{5}}{2} \right)^{-1/2} \begin{bmatrix} 1 \\ \frac{1-\sqrt{5}}{2} \end{bmatrix}$$

SVD Calculation: Singular values.

The singular values $\sigma_1 = \sqrt{\lambda_1} > \sigma_2 = \sqrt{\lambda_2}$, a little work finds nice roots of the λ_i:

$$\sigma_1 = \frac{1+\sqrt{5}}{2} \text{ and } \sigma_2 = \frac{\sqrt{5}-1}{2}$$

It is easy enough to check:

$$\sigma_1^2 = \left(\frac{1+\sqrt{5}}{2}\right)^2 = \frac{3+\sqrt{5}}{2} = \lambda_1 \text{ and similarly } \sigma_2^2 = \lambda_2.$$

It is a good exercise to compute $\sqrt{\lambda_i}$ to get σ_i.

SVD Calculation: Left Singular Vectors for A

The left singular vectors are simply: $\boldsymbol{u}_i = \frac{1}{\sigma_i} A\boldsymbol{v}_i$

$$\boldsymbol{u}_1 = \left(\frac{2}{1+\sqrt{5}}\right)\left(\frac{5+\sqrt{5}}{2}\right)^{-1/2}\begin{bmatrix}\frac{3+\sqrt{5}}{2}\\ \frac{1+\sqrt{5}}{2}\end{bmatrix} = \left(\frac{5+\sqrt{5}}{2}\right)^{-1/2}\begin{bmatrix}\frac{1+\sqrt{5}}{2}\\ 1\end{bmatrix}$$

and

$$\boldsymbol{u}_2 = \left(\frac{2}{1-\sqrt{5}}\right)\left(\frac{5-\sqrt{5}}{2}\right)^{-1/2}\begin{bmatrix}\frac{3-\sqrt{5}}{2}\\ \frac{1-\sqrt{5}}{2}\end{bmatrix} = \left(\frac{5-\sqrt{5}}{2}\right)^{-1/2}\begin{bmatrix}\frac{1-\sqrt{5}}{2}\\ 1\end{bmatrix}$$

SVD Calculation: Left Singular Vectors for A

A little inspection in this case shows

$$\boldsymbol{v}_1 = (1+\sigma_1^2)^{-1/2} \begin{bmatrix} 1 \\ \sigma_1 \end{bmatrix} \qquad \boldsymbol{u}_1 = (1+\sigma_1^2)^{-1/2} \begin{bmatrix} \sigma_1 \\ 1 \end{bmatrix}$$

$$\boldsymbol{v}_2 = (1+\sigma_2^2)^{-1/2} \begin{bmatrix} 1 \\ -\sigma_2 \end{bmatrix} \qquad \boldsymbol{u}_2 = (1+\sigma_2^2)^{-1/2} \begin{bmatrix} -\sigma_2 \\ 1 \end{bmatrix}$$

SVD Calculation: The Decomposition $A = USV^T$

We have

$$V = \begin{bmatrix} (1+\sigma_1^2)^{-1/2} & (1+\sigma_2^2)^{-1/2} \\ \sigma_1\,(1+\sigma_1^2)^{-1/2} & \sigma_2\,(1+\sigma_2^2)^{-1/2} \end{bmatrix}$$

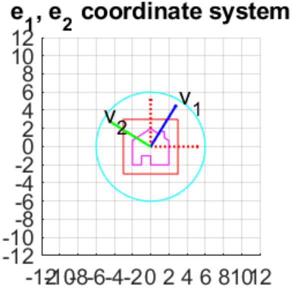
$$\Sigma = \begin{bmatrix} \sigma_1 & 0 \\ 0 & \sigma_1 \end{bmatrix}$$

$$U = \begin{bmatrix} \sigma_1\,(1+\sigma_1^2)^{-1/2} & \sigma_2\,(1+\sigma_2^2)^{-1/2} \\ (1+\sigma_1^2)^{-1/2} & (1+\sigma_2^2)^{-1/2} \end{bmatrix}$$

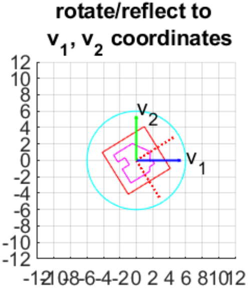
Example: Shear

SVD Geometry: (Rotate) Change to $\mathcal{V} = \{\mathbf{v}_1, \mathbf{v}_2\}$ coordinates

Start with the standard basis $\mathcal{E} = \{\mathbf{e}_1, \mathbf{e}_2\}$



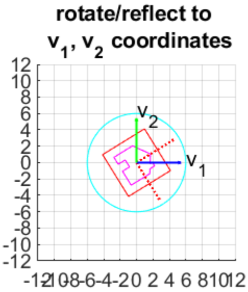
id : $\mathbb{R}^2 \to \mathbb{R}^2$



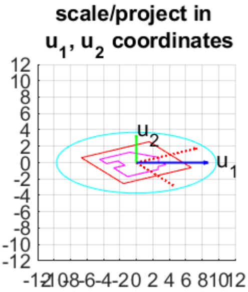
Apply $[\mathrm{id}]_{\mathcal{E},\mathcal{V}} = V^{-1}$ to transition to the $\mathcal{V}$ basis. Since V is unitary, $V^{-1} = V^T$ and this is a rigid motion of the plane, a rotation in this case. Notice $V^T\mathbf{e}_i$ is the $\mathcal{V}$ representation of $\mathbf{e}_i$.

SVD Geometry: (Scale) Apply map in the $\mathcal{V}, \mathcal{U} = \{\boldsymbol{u}_1, \boldsymbol{u}_2\}$ coordinates

Apply S in the $\mathcal{V}, \mathcal{U}$ bases, $[S]_{\mathcal{V},\mathcal{U}} = \Sigma$.



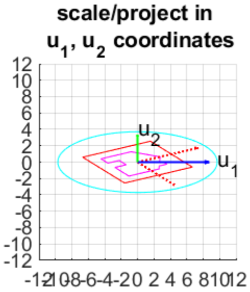
$S : \mathbb{R}^2 \to \mathbb{R}^2$



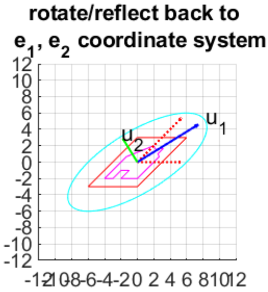
Applying S maps $\boldsymbol{v}_1 \mapsto \sigma_1 \boldsymbol{u}_1$ and $\boldsymbol{v}_2 \mapsto \sigma_2 \boldsymbol{u}_2$.

SVD Geometry: (Rotate) Change back to standard coordinates

Now change to the $\mathcal{E}$ coordinates. This uses $[\mathrm{id}]_{\mathcal{U},\mathcal{E}} = U$.



$\mathrm{id} : \mathbb{R}^2 \to \mathbb{R}^2$



Applying U applies a second rigid motion to the plane, in this case another rotation, transitioning back to standard coordinates.