

## Quiz 6

**Question 1** (15 points; 3 points each). Decide if each of the following are true or false and provide a justification or counterexample in each case. A justification could consist of a theorem from the text. All vector spaces are assumed to be finite-dimensional here. All vector spaces are now over  $\mathbb{C}$  unless otherwise stated.

- (a) \_\_\_\_\_ If the characteristic polynomial of a  $4 \times 4$  matrix is  $p(t) = (t - 1)^2 t^2$ , then there must be an invertible matrix  $S$  so that  $A = SDS^{-1}$  where

$$D = \text{diag}(1, 1, 0, 0) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

This is false. For example,

$$A = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

- (b) \_\_\_\_\_ If  $A$  and  $B$  are  $n \times n$  matrices and  $\lambda_A$  and  $\lambda_B$  are eigenvalues for  $A$  and  $B$  respectively with respect to the same eigenvector  $\mathbf{v}$ , then  $AB\mathbf{v} = BA\mathbf{v}$ .

This is true. This is a trivial computation

$$AB\mathbf{v} = A\lambda_B\mathbf{v} = \lambda_B A\mathbf{v} = \lambda_B \lambda_A \mathbf{v}.$$

Similarly,  $BA\mathbf{v} = \lambda_A \lambda_B \mathbf{v}$  and since  $\lambda_A \lambda_B = \lambda_B \lambda_A$  it is clear that the assertion is true.

- (c) \_\_\_\_\_ If  $A$  and  $B$  are  $n \times n$  matrices and  $\lambda_A$  is an eigenvalue of  $A$  and  $\lambda_B$  is an eigenvalue of  $B$ , then  $\lambda_A \lambda_B$  is an eigenvalue of  $AB$ .

This is false. This would be true if there is a  $\mathbf{v}$  that is simultaneously an eigenvector for  $A$  and  $\lambda_A$  and  $B$  and  $\lambda_B$ . A counterexample can be found even for the simplest of matrices:

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}$$

Then 2 is an eigenvalue for both  $A$  and  $B$ , but

$$AB = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$$

so clearly  $4 = 2 \cdot 2$  is not an eigenvalue for  $AB$ .

- (d) \_\_\_\_\_ If  $A$  and  $B$  are diagonalizable  $n \times n$  matrices, then  $AB$  is diagonalizable.

This is false, a counterexample suffices

$$\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$$

The two on the LHS are diagonalizable since they each have two distinct eigenvalues, the RGS is the typical example of a non-diagonalizable matrix. The only eigenvalue is 1, it has algebraic degree 2, but  $E_1 = \text{span}\{(0, 1)\}$  so the geometric degree is 1.

- (e) \_\_\_\_\_ Suppose  $A$  is diagonalizable, then  $e^A$  is diagonalizable and  $e^\lambda$  is an eigenvalue of  $e^A$  iff  $\lambda$  is an eigenvalue of  $A$ .

One direction is trivial, if  $\lambda$  is an eigenvalue of  $A$ , then this is a simple calculation. If  $A = SDS^{-1}$ , then  $e^A = Se^DS^{-1}$  and  $e^{\text{diag}(d_1, \dots, d_n)} = \text{diag}(e^{d_1}, \dots, e^{d_n})$ . So  $e^\lambda$  is an eigenvalue of  $e^A$ .

For the converse we use the fact that we have a full basis of eigenvectors for  $e^A$ . That is if  $\{\lambda_1, \dots, \lambda_m\}$  are eigenvalues of  $A$  and  $E_{\lambda_i}$  is the eigenspace associated to  $\lambda_i$ , then we know that  $E_{\lambda_i} \subseteq E_{e^{\lambda_i}}$  (the eigenspace for  $e^A$  w.r.t.  $e^{\lambda_i}$ ). But  $\mathbb{R}^n = E_{\lambda_1} \oplus \dots \oplus E_{\lambda_m}$  and thus  $\mathbb{R}^n = E_{e^{\lambda_1}} \oplus \dots \oplus E_{e^{\lambda_m}}$ , thus the  $e^{\lambda_i}$  are ALL of the eigenvalues for  $e^A$ . Thus if  $\mu$  is an eigenvalue for  $e^A$ , then  $\mu = e^{\lambda_i}$  for some  $i$ .

**Question 2** (10 points). Let  $A = \begin{bmatrix} 0 & 0 & 2 \\ 0 & 3 & 0 \\ 2 & 0 & 0 \end{bmatrix}$ , write  $A = U\Lambda U^{-1}$  where  $U$  is unitary, columns are orthonormal basis for  $\mathbb{R}^3$  and  $\Lambda = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix}$  with  $\lambda_1 > \lambda_2 > \lambda_3$ .

Recall:  $U^{-1} = U^T$  for unitary  $U$ .

Find the eigenvalues:  $\det(A - tI) = -(t - 3)(t - 2)(t + 2)$  so the eigenvalues are  $\lambda_1 = 3 > \lambda_2 = 2 > \lambda_3 = -2$ .

Find the eigenspace for  $\lambda_1 = 3$ :

$$\text{NS}\left(\begin{bmatrix} -3 & 0 & 2 \\ 0 & 0 & 0 \\ 2 & 0 & -3 \end{bmatrix}\right) = \text{span}\left(\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}\right).$$

Find the eigenspace for  $\lambda_2 = 2$ :

$$\text{NS}\left(\begin{bmatrix} -2 & 0 & 2 \\ 0 & 1 & 0 \\ 2 & 0 & -2 \end{bmatrix}\right) = \text{span}\left(\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}\right).$$

Find the eigenspace for  $\lambda_3 = -2$ :

$$\text{NS}\left(\begin{bmatrix} 2 & 0 & 2 \\ 0 & 1 & 0 \\ 2 & 0 & 2 \end{bmatrix}\right) = \text{span}\left(\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}\right).$$

$$U = \begin{bmatrix} 0 & 1/\sqrt{2} & 1/\sqrt{2} \\ 1 & 0 & 0 \\ 0 & 1/\sqrt{2} & -1/\sqrt{2} \end{bmatrix}$$

and

$$\begin{bmatrix} 0 & 0 & 2 \\ 0 & 3 & 0 \\ 2 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1/\sqrt{2} & 1/\sqrt{2} \\ 1 & 0 & 0 \\ 0 & 1/\sqrt{2} & -1/\sqrt{2} \end{bmatrix} \begin{bmatrix} 3 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & -2 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \\ 1/\sqrt{2} & 0 & 1/\sqrt{2} \\ 1/\sqrt{2} & 0 & -1/\sqrt{2} \end{bmatrix}$$

**Question 3** (10 points). Suppose the matrix  $A = \frac{1}{12} \begin{bmatrix} 5 & -1 \\ -1 & 5 \end{bmatrix}$  is used to transform points in the plane iteratively. That is, given a point  $\mathbf{v}$ , consider the sequence  $\mathbf{v}_n = A^n \mathbf{v}$ . Letting  $U = [\mathbf{u}_1 \quad \mathbf{u}_2]$  so that  $\mathbf{u}_i$  is an eigenvector associated to  $\lambda_i$  and letting  $\mathbf{v} = c_1 \mathbf{u}_1 + c_2 \mathbf{u}_2$  what is a simple expressions for  $a_n$  and  $b_n$  so that  $\mathbf{v}_n = A^n \mathbf{v} = a_n \mathbf{u}_1 + b_n \mathbf{u}_2$ .

After a little work, you have  $A = UDU^{-1}$  where

$$U = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} \quad \text{and} \quad D = \begin{bmatrix} 1/3 & 0 \\ 0 & 1/2 \end{bmatrix}$$

$$A(c_1 \mathbf{u}_1 + c_2 \mathbf{u}_2) = c_1 A \mathbf{u}_1 + c_2 A \mathbf{u}_2 = \lambda_1 c_1 \mathbf{u}_1 + \lambda_2 c_2 \mathbf{u}_2$$

$$A^2(c_1 \mathbf{u}_1 + c_2 \mathbf{u}_2) = A(\lambda_1 c_1 \mathbf{u}_1 + \lambda_2 c_2 \mathbf{u}_2) = \lambda_1 c_1 A(\mathbf{u}_1) + \lambda_2 c_2 A(\mathbf{u}_2) = \lambda_1^2 c_1 \mathbf{u}_1 + \lambda_2^2 c_2 \mathbf{u}_2$$

$\vdots$

$$A^n(c_1 \mathbf{u}_1 + c_2 \mathbf{u}_2) = \lambda_1^n c_1 \mathbf{u}_1 + \lambda_2^n c_2 \mathbf{u}_2 = (1/3)^n c_1 \mathbf{u}_1 + (1/2)^n c_2 \mathbf{u}_2$$

So  $a_n = (1/3)^n c_1$  and  $b_n = (1/2)^n c_2$ .