Name: ________________________ Exam 1 - MAT345

I True/False (100 points; 10 points each)

Each problem is points for a total of 50 points. (5 points each and one free point.) In class, you only provide the T/F.

Corrections: If you choose to make corrections for 50% back on this section, then you must provide reasons for ALL of these, not just the ones that you miss. A reason might be as simple as, "by Theorem ...," or it might require an example or counterexample. In any case, some correct reason or counterexample must be provided.

Problem I.1 (100 points; 10 points each)**.** Decide if each of the following is true or false.

1. True For A, a 6×5–matrix, and $\boldsymbol{b}, \boldsymbol{c} \in \mathbb{R}^6$, if $A\boldsymbol{x} = \boldsymbol{b}$ has a unique solution, then $A\boldsymbol{x} = \boldsymbol{c}$ has at most one solution.

   If there is a unique solution to $A\boldsymbol{x} = \boldsymbol{b}$, then there are no free variables. This is also the case when trying to solve $A\boldsymbol{x} = \boldsymbol{c}$, and hence, there would be at most one solution.

2. False For A, a 6×5–matrix, and $\boldsymbol{b}, \boldsymbol{c} \in \mathbb{R}^6$, if $A\boldsymbol{x} = \boldsymbol{b}$ has a unique solution, then $A\boldsymbol{x} = \boldsymbol{c}$ has a solution.

   Let A be the 5×5 identity matrix followed by a row of 0's. Then $\text{rng}(A)$ is the set of vectors of the form $\boldsymbol{u} = (u_1, u_2, \ldots, u_5, 0)$.

3. True If A is a 5×6–matrix and $A\boldsymbol{x} = \boldsymbol{b}$ has a solution, then $A\boldsymbol{x} = \boldsymbol{b}$ has infinitely many solutions.

   As with (I.1), if $A\boldsymbol{x} = \boldsymbol{b}$ has a solution, then as $6 > 5$, there must be a free variable, and so there must be infinitely many solutions.

4. True If A is a 5×5–matrix and $A\boldsymbol{x} = \boldsymbol{b}$ has a unique solution for some $\boldsymbol{b} \in \mathbb{R}^5$, then $A\boldsymbol{x} = \boldsymbol{c}$ has a solution for all $\boldsymbol{c} \in \mathbb{R}^5$.

   If $A\boldsymbol{x} = \boldsymbol{b}$ has a unique solution, then A row-reduces to I, and so $A\boldsymbol{x} = \boldsymbol{c}$ has a unique solution.

5. True The following are equivalent:

   (i) A is row reducible to B.

   (ii) $B = MA$ for some invertible matrix M.

   This follows as a matrix M is invertible iff M is reducible to I by a sequence of elementary row operations; equivalently, M is a product of elementary matrices.

6. False If A and B are invertible, then $A+B$ is invertible and $(A+B)^{-1} = A^{-1} + B^{-1}$.

Consider $A = I$ and $B = -I$!

7. True A square upper triangular matrix is invertible exactly when all of its diagonal entries are non-zero.

   There are several ways to see this. You could use that A s invertible iff $\det(A)$ is non-zero and that for upper-triangular A this means that the product of the diagonal elements is not 0. Or you could note that such a matrix is reducible to I.

8. False If A is row reducible to B, then $\det(A) = \det(B)$.

   Consider $A = \left[\begin{smallmatrix} 1 & 1 \\ 1 & 1 \end{smallmatrix}\right]$ and $B = \left[\begin{smallmatrix} 1 & 1 \\ 0 & 0 \end{smallmatrix}\right]$.

9. False Consider the operation flip(A) that "flips" a matrix horizontally, so for example

$$\text{flip}\left(\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}\right) = \begin{bmatrix} 2 & 1 \\ 4 & 3 \end{bmatrix} \text{ while flip}\left(\begin{bmatrix} 1 & 2 & 3 \\ 0 & 4 & 5 \\ 0 & 0 & 6 \end{bmatrix}\right) = \begin{bmatrix} 3 & 2 & 1 \\ 5 & 4 & 0 \\ 6 & 0 & 0 \end{bmatrix}$$

   For any $n \times n$ matrix A, $\det(\text{flip}(A)) = -\det(A)$.

   The point is to consider how many "swaps" of columns are required. If $n = 2k$ or $n = 2k + 1$, then k swaps are required, so for k even $\det(\text{flip}(A)) = \det(A)$, e.g., $n = 1, 4, 5, 8, 9, \ldots$, however, if k is odd, then $\det(\text{flip}(A)) = -\det(A)$, e.g., $2, 3, 6, 7, \ldots$.

10. True $\det(AB) = \det(BA)$

    $\det(AB) = \det(A)\det(B) = \det(B)\det(A) = \det(BA)$.

II Long Answer (90 points)

Show all computations so that you make clear what your thought processes are.

Problem II.1 (35 pts)**.** This is exactly like your quiz, so if you looked at the feedback and the solutions, then you know what I expect here.

- (15 points) Use row operations (show all work and indicate operations) to reduce A to an echelon form. (This should work out very nicely - no fractions required..)
- (15 points) Use back-substitution to solve the resulting system. Make sure to indicate which variables are free. (Or reduce all the way to RREF and read off the solution.)
- (5 points) Write your solution as a linear combination of vectors.

Solve $A\boldsymbol{x} = \mathbf{0}$ where

$$A = \begin{bmatrix} 2 & -2 & 2 & 2 & 1 \\ -4 & 2 & -5 & -3 & -4 \\ -2 & 4 & -1 & -5 & 1 \\ -4 & 6 & -3 & -7 & 0 \end{bmatrix}$$

Gauss-Jordan elimination to get echelon form:

$$\begin{bmatrix} 2 & -2 & 2 & 2 & 1 \\ -4 & 2 & -5 & -3 & -4 \\ -2 & 4 & -1 & -5 & 1 \\ -4 & 6 & -3 & -7 & 0 \end{bmatrix} \underset{\substack{R_2-2R_1\to R_2 \\ R_3+R_1\to R_3 \\ R_4+2R_1\to R_4}}{\longrightarrow} \begin{bmatrix} 2 & -2 & 2 & 2 & 1 \\ 0 & -2 & -1 & 1 & -2 \\ 0 & 2 & 1 & -3 & 2 \\ 0 & 2 & 1 & -3 & 2 \end{bmatrix}$$

$$\underset{\substack{R_3+R_2\to R_3 \\ R_4+R_2\to R_4}}{\longrightarrow} \begin{bmatrix} 2 & -2 & 2 & 2 & 1 \\ 0 & -2 & -1 & 1 & -2 \\ 0 & 0 & 0 & -2 & 0 \\ 0 & 0 & 0 & -2 & 0 \end{bmatrix}$$

$$\underset{R_4-R_3\to R_4}{\longrightarrow} \begin{bmatrix} 2 & -2 & 2 & 2 & 1 \\ 0 & -2 & -1 & 1 & -2 \\ 0 & 0 & 0 & -2 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Back-substitution: x_4 and x_5 are free.

$$\boxed{x_5 = t}$$

$$\boxed{x_4 = 0}$$

$$\boxed{x_3 = s}$$

$$-2x_2 - s + 0 - 2t = 0 \to -2x_2 = s + 2t \to \boxed{x_2 = -\frac{1}{2}s - t}$$

$$2x_1 - 2\left(-\frac{1}{2}s - t\right) + 2(s) + 2(0) + t = 0 \to 2x_1 = -3s - 3t \to \boxed{x_1 = -\frac{3}{2}s - \frac{3}{2}t}$$

Solution as a linear combination of vectors:

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} -\frac{3}{2}s - \frac{3}{2}t \\ -\frac{1}{2}s - t \\ s \\ 0 \\ t \end{bmatrix} = s \begin{bmatrix} -\frac{3}{2} \\ -\frac{1}{2} \\ 1 \\ 0 \\ 0 \end{bmatrix} + t \begin{bmatrix} -\frac{3}{2} \\ -1 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

Problem II.2 (25 pts)**.** For what scalars a and b is A invertible for

$$A = \begin{bmatrix} 1 & 1 & 1 \\ a & 1 & 1 \\ 0 & b & 1 \end{bmatrix}$$

There are many ways to do this, all are pretty simple. Here are two options:

Method 1: Gaussian elimination reduces A to

$$B = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1-a & 1 \\ 0 & & 1-b \end{bmatrix}$$

So A has rank 3, and hence is invertible iff $b \neq 1$ and $a \neq 1$. Recall $\operatorname{rank}(A)$ is the number of pivots and an $n \times n$ matrix is invertible iff $\operatorname{rank}(A) = n$.

Method 2: You could also use that A is invertible iff $\det(A) \neq 0$.

$$\det(A) = 1 \cdot \det\left(\left[\begin{smallmatrix} 1 & 1 \\ b & 1 \end{smallmatrix}\right]\right) - a \cdot \det\left(\left[\begin{smallmatrix} 1 & 1 \\ b & 1 \end{smallmatrix}\right]\right) = (1-b) - a(1-b)$$

So $\det(A) = (1-a)(1-b)$ and thus $\det(A) \neq 0 \iff a \neq 1 \neq b$.

Problem II.3 (30 pts)**.** If A and B are invertible $n \times n$ matrices, show that

$$(AB)^2 = A^2B^2 \iff AB = BA$$

($\Leftarrow$) Assume $AB = BA$, then

$$(AB)^2 = (AB)(AB) = A(BA)B = A(AB)B = (AA)(BB) = A^2B^2$$

So $AB = BA \implies (AB)^2 = A^2B^2$ as desired.

($\Rightarrow$) Suppose $(AB)^2 = A^2B^2$, then

$$\begin{gathered}(AB)(AB) = A^2B^2 = AABB \\ \Downarrow \\ (A^{-1}A)BA(BB^{-1}) = (A^{-1}A)AB(BB^{-1}) \\ \Downarrow \\ IBAI = IABI \\ \Downarrow \\ BA = AB\end{gathered}$$

So $(AB)^2 = A^2B^2 \implies AB = BA$.