Question 1 (20 points; 4 points each). No justification required for the this quiz, just the T/F responses.

(a) <u>True</u> If all eigenvalues of A are 0, and A is diagonalizable, then $A = O$.

(b) <u>False</u> The following matrix is diagonalizable

$$\begin{bmatrix} 0 & 1 & 2 & 3 \\ 0 & 0 & 4 & 5 \\ 0 & 0 & 0 & 6 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

(c) <u>True</u> If v is an eigenvector for both A and B, then $\mathbf{v}$ is an eigenvector for AB.

(d) <u>False</u> If A and B are similar matrices and λ, v is an eigenvalue/eigenvector pair for A, then the same is true for B.

(e) <u>True</u> If $\lambda_1 = \frac{1}{2}$ and $\lambda_2 = -2$ are the eigenvalues for $L : \mathbb{R}^2 \to \mathbb{R}^2$ with corresponding eigenvectors $v_1 = (1, -1)$ and $v_2 = (1, 1)$. Then for any $x \in \mathbb{R}^2$, $L^n(x)$ approaches the line $E_{\lambda_2} = \text{span}\{v_2\}$ as $n \to \infty$, specifically $\text{dist}(L^n(x), E_{\lambda_2}) = \| \text{proj}_{E_{\lambda_2}}^{\perp}(L^n(x)) \| \to 0$ as $n \to \infty$.

Question 2 (20 points). Let $A = \begin{bmatrix} 1/2 & 1 \\ 1/2 & 0 \end{bmatrix}$. Diagonalize A, i.e. write $A = S\Lambda S^{-1}$ where Λ is the diagonal matrix of eigenvalues for A.

i) What is $p_A(t)$?

$$p_A(t) = \det \begin{bmatrix} 1/2 - t & 1 \\ 1/2 & -t \end{bmatrix} = (1/2 - t)(-t) - 1/2 = t^2 - 1/2t - 1/2$$

ii) What are the eigenvalues?

$$p_A(t) = 0 \iff 2t^2 - t - 1 = (2t+1)(t-1) = 0 \iff t = -1/2 \text{ or } t = 1$$

So the eigenvalues are $\lambda_1 = -1/2$ and $\lambda_2 = 1$.

iii) Find eigenvectors for each eigenvalue.

$$E_{-1/2} = \text{NS} \begin{bmatrix} 1 & 1 \\ 1/2 & 1/2 \end{bmatrix} = \text{span}\left\{ \begin{bmatrix} -1 \\ 1 \end{bmatrix} \right\}$$

and

$$E_1 = \text{NS} \begin{bmatrix} -1/2 & 1 \\ 1/2 & -1 \end{bmatrix} = \text{span}\left\{ \begin{bmatrix} 1 \\ 1/2 \end{bmatrix} \right\}$$

iv) Diagonalize A, i.e., write $A = BDB^{-1}$ where D is diagonal.

$$A = \begin{bmatrix} -1 & 1 \\ 1 & 1/2 \end{bmatrix} \begin{bmatrix} -1/2 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} -1 & 1 \\ 1 & 1/2 \end{bmatrix}^{-1} = \begin{bmatrix} -1 & 1 \\ 1 & 1/2 \end{bmatrix} \begin{bmatrix} -1/2 & 0 \\ 0 & 1 \end{bmatrix} \left(-\tfrac{2}{3} \right) \begin{bmatrix} 1/2 & -1 \\ -1 & -1 \end{bmatrix}$$

Bonus (5pts - Can complete this at home.) Give a formula for $A^n \mathbf{v}$ where $\mathbf{v} = \begin{bmatrix} a \\ b \end{bmatrix}$ and use this to find $\lim_{n \to \infty} A^n \mathbf{v}$.