Quiz 3

Problem 1 (15 points; 3 points each)**.** Decide if each of the following are true or false and provide a small proof or counterexample in each case. All vector spaces are assumed to be finite-dimensional here.

(a) ______ Given a basis $\mathcal{B} = \{\boldsymbol{v}_1, \ldots, \boldsymbol{v}_n\}$ for a vector space V and U a subspace of V, then there is $\mathcal{C} \subseteq \mathcal{B}$ that is a basis for U.

FALSE: $\mathcal{B} = \{\boldsymbol{e}_1, \boldsymbol{e}_2\}$ is a basis for $\mathbb{R}^2$ and $U = \operatorname{span}\{(1,1)\}$ is a subspace, namely, the line with slope 1 through the origin. You cannot throw away one of $\boldsymbol{e}_1$ or $\boldsymbol{e}_2$ to get a basis for U.

(b) ______ Given a basis $\mathcal{C}$ for a subspace U of a vector space V, $\mathcal{C}$ can be extended to a basis $\mathcal{B}$ for V.

TRUE: This is one of the theorems that you have, any linearly independent set can be expanded to a basis.

(c) ______ If $\{\boldsymbol{v}_1, \ldots, \boldsymbol{v}_n\}$ is linearly independent and $\boldsymbol{v} \in \operatorname{span}(\{\boldsymbol{v}_1, \ldots, \boldsymbol{v}_n\})$, then it is possible that there are distinct $\boldsymbol{c}, \boldsymbol{b} \in \mathbb{R}^n$ such that $\boldsymbol{v} = \sum_{i=1}^n c_i \boldsymbol{v}_i = \sum_{i=1}^n b_i \boldsymbol{v}_i$.

FALSE: If such $\boldsymbol{c}$ and $\boldsymbol{b}$ exists, then $\mathbf{0} = \boldsymbol{v} - \boldsymbol{v} = (\sum b_i \boldsymbol{v}_i) - (\sum c_i \boldsymbol{v}_i) = \sum (b_i - c_i)\boldsymbol{v}_i$. Since $\boldsymbol{v}_i$'s are independent, $b_i - c_i = 0$ for all i, so $\boldsymbol{c} = \boldsymbol{b}$.

(d) ______ If $\{\boldsymbol{v}_1, \ldots, \boldsymbol{v}_n\}$ is linearly independent and $V = \operatorname{span}(\{\boldsymbol{v}_1, \ldots, \boldsymbol{v}_n\}) = \operatorname{span}(\{\boldsymbol{u}_1, \ldots, \boldsymbol{u}_n\})$, then $\{\boldsymbol{u}_1, \ldots, \boldsymbol{u}_n\}$ is linearly independent.

TRUE: This too is a theorem. Since $V = \operatorname{span}\{\boldsymbol{v}_1, \ldots, \boldsymbol{v}_n\}$ and $\boldsymbol{v}_i$ are independent, you know $\{\boldsymbol{v}_1, \ldots, \boldsymbol{v}_n\}$ is a basis for V and so $\dim(V) = n$. since $\operatorname{span}\{\boldsymbol{u}_1, \ldots, \boldsymbol{u}_n\}$ span V you know this set can be reduced to a basis, but any basis must have n elements, so $\{\boldsymbol{u}_1, \ldots, \boldsymbol{u}_n\}$ must already be a basis, and hence is linearly independent.

(e) ______ Suppose V is a vector space and $U \subseteq V$ is a subspace. For any $\boldsymbol{v} \in V$, there is a **unique** $\boldsymbol{u} \in U$ so that $\boldsymbol{v} = \boldsymbol{u} + (\boldsymbol{v} - \boldsymbol{u})$, that is, there is a unique "projection" of V into U.

FALSE: Again take $U = \operatorname{span}\{(1,1)\} \subset \mathbb{R}^2 = V$ and let $\boldsymbol{v} = (2,3)$, then $\boldsymbol{v} = (1,1) + (1,2) = (2,2) + (0,1)$.

Note: If we fixed W so that $V = U \oplus W$, then there would be for every $\boldsymbol{v} \in V$ a unique $\boldsymbol{u} \in U, \boldsymbol{w} \in W$ so that $\boldsymbol{v} = \boldsymbol{u} + \boldsymbol{w}$. For example, take U as above and $W = \operatorname{span}\{(0,1)\}$, then $(2,3) = (2,2) + (0,1)$ is the unique decomposition of $(2,3)$ into something from U and something from W.

Problem 2 (10 pts)**.** Show that the collection, U, of upper triangular 3×3 matrices is a subspace of $\mathbb{R}^{3\times 3}$ (the space of all 3×3 matrices). Give a basis $\mathcal{B}$ for U and for $\boldsymbol{v} = \left[\begin{smallmatrix} 1 & 2 & 3 \\ 0 & 4 & 5 \\ 0 & 0 & 6 \end{smallmatrix}\right]$, give $[\boldsymbol{v}]_{\mathcal{B}}$.

To show that U is a subspace we need only show that $\alpha\boldsymbol{v}+\boldsymbol{u} \in U$ for $\boldsymbol{v}, \boldsymbol{u} \in U$. So let $\boldsymbol{u} = \left[\begin{smallmatrix} u_{11} & u_{12} & u_{13} \\ 0 & u_{22} & u_{23} \\ 0 & 0 & u_{33} \end{smallmatrix}\right]$ and let $\boldsymbol{v} = \left[\begin{smallmatrix} v_{11} & v_{12} & v_{13} \\ 0 & v_{22} & v_{23} \\ 0 & 0 & v_{33} \end{smallmatrix}\right]$, then $\alpha\boldsymbol{u} + \boldsymbol{v} = \left[\begin{smallmatrix} \alpha u_{11}+v_{11} & \alpha u_{12}+v_{12} & \alpha u_{13}+v_{13} \\ 0 & \alpha u_{22}+v_{22} & \alpha u_{23}+v_{23} \\ 0 & 0 & \alpha u_{33}+v_{33} \end{smallmatrix}\right] \in U$.

A basis is clearly given by $E^{ij}_{lk} = 1$ if $i = j$ and $l = k$ and $j \leq i$ and 0 otherwise. So $E^{11} = \left[\begin{smallmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{smallmatrix}\right]$, $E^{12} = \left[\begin{smallmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{smallmatrix}\right]$, etc. This basis has six elements, so $\dim(U) = 6$.

With this basis, clearly $\boldsymbol{v} = E^{11} + 2E^{12} + 3E^{13} + 4E^{22} + 5E^{23} + 6E^{33}$.

Problem 3 (10 pts)**.** Let $c_1, c_2, \ldots, c_n$ be n distinct real numbers. Let $p_i = \prod_{\substack{j=1 \\ j\neq i}}^{n}(x - c_j)/(c_i - c_j)$. Show that $\mathcal{B} = \{p_1, p_2, \ldots, p_n\}$ is a basis for P_{n-1}.

Hint: Compute $p_i(c_j)$ and look at what happens when $i = j$ and when $i \neq j$. Use this to argue the independence of $\mathcal{B}$.

There are two ways to proceed. We know $\dim(P_{n-1}) = n$ so it would suffice to show either that $\mathcal{B} = \{p_1, \ldots, p_n\}$ spans or is linearly independent, since either implies other for any set of n vectors in P_{n-1}.

Proof 1: (linear independence) It is trivial to see that

$$p_i(c_j) = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}$$

This shows independence since if $p = \sum_{i=1}^{n} \alpha_i p_i$, then $p_i(c_j) = \alpha_j$ so if $p = 0$, then $\alpha_j = 0$ for all j.

Proof 2: (spanning) Let $q \in P_{n-1}$ and let $\alpha_i = q(c_i)$, then Exactly as above, if $p = \sum_{i=1}^{n} \alpha_i p_i$ we see that $p(c_i) = \alpha_i = q(c_i)$.

We just need to see that $p = q$ and we have the desired spanning. Note that $r = p - q$ has roots at each c_i, but this is n distinct roots for an $n-1$-degree polynomial, thus $r = 0$ and hence $p = q$ as required.