Quiz 3

Problem 1 (20 points; 5 points each)**.** Decide if each of the following are true or false and provide a small proof or counterexample in each case. All vector spaces are assumed to be finite-dimensional here.

(a) ______ Given a basis $\mathcal{B} = \{\boldsymbol{v}_1, \dots, \boldsymbol{v}_n\}$ for a vector space V and U a subspace of V, then there is $\mathcal{C} \subseteq \mathcal{B}$ that is a basis for U.

This is false. For example, take $U = \operatorname{span}\{(1,1)\}$, $V = \mathbb{R}^2$, and $\mathcal{B}$ be the standard basis for $\mathbb{R}^2$.

(b) ______ Given a basis $\mathcal{C}$ for a subspace U of a vector space V, $\mathcal{C}$ can be extended to a basis $\mathcal{B}$ for V.

This is true. This is part of Theorem 3.4.4.

(c) ______ If $\{\boldsymbol{v}_1, \dots, \boldsymbol{v}_n\}$ is linearly independent and $\boldsymbol{v} \in \operatorname{span}(\{\boldsymbol{v}_1, \dots, \boldsymbol{v}_n\})$, then it is possible that there are distinct $\boldsymbol{c}, \boldsymbol{b} \in \mathbb{R}^n$ such that $\boldsymbol{v} = \sum_{i=1}^n c_i \boldsymbol{v}_i = \sum_{i=1}^n b_i \boldsymbol{v}_i$.

This is false. If $\boldsymbol{v} = \sum_{i=1}^n c_i \boldsymbol{v}_i = \sum_{i=1}^n b_i \boldsymbol{v}_i$, then $\boldsymbol{0} = \sum_{i=1}^n (c_i - b_i)\boldsymbol{v}_i$. By independence, $c_i - b_i = 0$ for all i, so $c_i = b_i$ for all i.

(d) ______ If $\{\boldsymbol{v}_1, \dots, \boldsymbol{v}_n\}$ is linearly independent and $V = \operatorname{span}(\{\boldsymbol{v}_1, \dots, \boldsymbol{v}_n\}) = \operatorname{span}(\{\boldsymbol{u}_1, \dots, \boldsymbol{u}_n\})$, then $\{\boldsymbol{u}_1, \dots, \boldsymbol{u}_n\}$ is linearly independent.

This is true. Again, Theorem 3.4.4.

Problem 2 (8 pts)**.** Find a basis for $\operatorname{span}\{\boldsymbol{u}_1, \dots, \boldsymbol{u}_5\}$ from among the vectors $\boldsymbol{u}_1, \dots, \boldsymbol{u}_5$, where

$$\boldsymbol{u}_1 = \begin{bmatrix}1\\2\\2\end{bmatrix} \qquad \boldsymbol{u}_2 = \begin{bmatrix}2\\5\\4\end{bmatrix} \qquad \boldsymbol{u}_3 = \begin{bmatrix}1\\3\\2\end{bmatrix} \qquad \boldsymbol{u}_4 = \begin{bmatrix}2\\7\\4\end{bmatrix} \qquad \boldsymbol{u}_5 = \begin{bmatrix}1\\1\\0\end{bmatrix}$$

Make sure to show all work and explain your reasoning.

Again using the notation $A \sim B$ to mean A and B are related by performing elementary row operations, or $B = EA$ where E is invertible. Let $A = \begin{bmatrix}\boldsymbol{u}_1 & \cdots & \boldsymbol{u}_5\end{bmatrix}$, then

$$A \sim \begin{bmatrix}1 & 2 & 1 & 2 & 1\\0 & 1 & 1 & 3 & -1\\0 & 0 & 0 & 0 & -1\end{bmatrix}$$

B is an echelon form of A with pivots in columns 1, 2, and 5. Thus $\{\boldsymbol{u}_1, \boldsymbol{u}_2, \boldsymbol{u}_5\}$ is a basis for $\operatorname{span}\{\boldsymbol{u}_1, \dots, \boldsymbol{u}_5\} = \operatorname{CS}(A)$.

This was all that was asked here. The rest is just something like you have on the exam.

We also know that $\boldsymbol{v}_1 = (1,2,1,2,1)$, $\boldsymbol{v}_2 = (0,1,1,3,-1)$, and $\boldsymbol{v}_3 = (0,0,0,0,-1)$ is a basis for $\operatorname{RS}(A)$.

By back substitution, letting $x_3 = s$ and $x_4 = t$ we have $x_5 = 0$, $x_2 = -s - 3t$, and $x_1 = -2x_2 - s - 2t = 2(s+3t) - s - 2t = s + 4t$, so $\operatorname{NS}(A)$ consists of vectors of the form $(s+4t, -s-3t, s, t, 0) =$

$s(1,-1,1,0,0) + t(4,-3,0,1,0)$, so $\mathrm{NS}(A) = \mathrm{span}\{\boldsymbol{v}+4, \boldsymbol{v}_5\}$, where $\boldsymbol{v}_4 = (1,-1,1,0,0)$ and $\boldsymbol{v}_5 = (4,-3,0,1,0)$.

Quick check: Check that $\boldsymbol{v}_4$ and $\boldsymbol{v}_5$ are orthogonal to all rows of A.

Problem 3 (7 pts)**.** Let $c_1, c_2, \ldots, c_n$ be n distinct real numbers. Let $p_i = \prod_{\substack{j=1 \\ j\neq i}}^{n}(x-c_j)/(c_i-c_j)$. Show that $\mathcal{B} = \{p_1, p_2, \ldots, p_n\}$ is a basis for P_{n-1}.

Hint: Consider $p_i(c_j)$.

I'll do the n-dimensional case. Suppose $c_1, \cdots, c_n$ are distinct reals then define $p_i \in P_{n-1}$

$$p_i = \prod_{\substack{j=1 \\ j\neq i}}^{n}(x-c_j)/(c_i-c_j)$$

It is trivial to see that

$$p_i(c_j) = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}$$

This shows independence since if $p = \sum_{i=1}^{n} \alpha_i p_i$, then $p_i(c_j) = \alpha_j$ so if $p = 0$, then $\alpha_j = 0$ for all j.

This all you were asked to do.

Just for fun, for the basis, $\mathcal{B} = \{p_1, p_2, \ldots, p_n\}$, $p = \sum_{i=1}^{n} p(c_i)p_i$, that is,

$$[p]_{\mathcal{B}} = \begin{bmatrix} p(c_1) \\ p(c_2) \\ \vdots \\ p(c_n) \end{bmatrix}$$

So given n points (x_i, y_i) with the x_i's distinct. The unique degree $(n-1)$-polynomial through these points is $p(x) = \sum_{i=1}^{n} y_i \cdot p_i(x)$.