Exam 2 – Math 215

1 True/False

Problem 1.1 (60 points; 6 points each)**.** Decide if each of the following are true or false. You do not need to justify your choice here.

(a) TRUE $(x+a)^2 \equiv x^2 + a^2 \pmod 2$

(b) FALSE $a^n \equiv b^n \pmod m$ and $n^a \equiv n^b \pmod m$ whenever $a \equiv b \pmod m$.

(c) TRUE Suppose $ax + by = 1$, then $x = a^{-1} \pmod b$ and $x = a^{-1} \pmod y$.

(d) TRUE $G(x) = \frac{1}{(1-x^5)}\frac{1}{(1-x^{10})}$ is the generating function for the number of solutions (a,b) of $5a + 10b = n$.

(e) TRUE For all a, a has a multiplicative inverse modulo m iff $ab + mn = 1$ for some b and n.

(f) FALSE For any integers a and b, $ax + by = 1$ is a line so there are infinitely many integers x and y satisfying $ax + by = 1$, namely, all integer pairs (x, y) that fall on this line.

(g) FALSE The characteristic function for the recurrence relation $a_n = 3 \cdot a_{n-1} + 2 \cdot a_{n-3}$ is $x^2 - 3x + 2$.

(h) TRUE If $f(n)$ and $g(n)$ are solutions to $a_n = 3a_n - 2a_{n-1} + a_{n-3}$, then $c_1 \cdot f(n) + c_2 \cdot g(n)$ where c_1 and c_2 are scalars, is also a solution.

(i) TRUE There are two solutions to $x^2 \equiv 2 \pmod 7$.

(j) TRUE $\sum_{i=0}^{n} \binom{n}{i} 4^i = \sum_{i-0}^{n} \binom{n}{i} (-1)^{n-i} 6^i$.

2 Free Response

100 points total, 15 points each

Problem 2.1. Find all solutions to $13x + 9 \equiv 1 \pmod 7$.

$$13x + 9 \equiv 6x + 2 \equiv 1 \pmod 7$$
$$6x \equiv -1 \equiv 6 \pmod 7$$
$$x \equiv 1 \pmod 7$$

So the solutions are all numbers $x = 7k + 1$.

Problem 2.2. Find s and t so that $s \cdot 953 + t \cdot 859 = 1$ using the extended Euclidean algorithm. Give the table (trace) of all intermediate values obtained along the way.

The rule is $[s_i, t_i] = [s_{i-2}, t_{i-2}] - q_{i-1}[s_{i-1}, t_{i-1}]$

i	0	1	2	3	4	5	
r	953	859	94	13	3	1	0
s_i	1	0	1	-9	64	-265	
t_i	0	1	-1	10	-71	294	
q_i		1	9	7	4	3	
check: $s_1 \cdot 954 + t_i \cdot 859$	953	859	94	13	3	1	

So $(-265)(953) + (294)(859) = 1$.

Problem 2.3. Find a closed form solution to the recurrence relation $a_n = a_{n-1} + 2a_{n-2}$ given $a_0 = 2$ and $a_1 = 1$.

The characteristic function is $x^2 - x - 2 = (x-2)(x+1)$ so the general solution is $f(n) = c_0(-1)^n + c_1(2^n)$. We need to solve

$$2 = c_0 + c_1$$
$$1 = -c_0 + 2c_1$$

This can be seen by inspection, $c_0 = c_1 = 1$ so we have $a_n = (-1)^n + 2^n$.

Problem 2.4. How many non-negative integer solutions are there to $x_1 + x_2 + x_3 \leq 20$ if $x_1, x_2 > 1$ and $x_3 > 2$?

There are two slight complications here, first $x_1, x_2 > 1$ and $x_3 > 2$. We can instead look at counting the number of solutions to $y_1 + y_2 + y_3 \leq 13$. For each such solution, we get a solution to the original problem by letting $x_1 = y_1 + 2$, $x_2 = y_2 + 2$, and $x_3 = y_3 + 3$. The second issue is the $\leq$, for this we introduce a y_4 and find the number of non-negative integer solutions to $y_1 + y_2 + y_3 + y_4 = 13$. This is the number of ways to distribute 13 balls into 4 bins and that is given by

$$\binom{13+4-1}{13} \text{ or } \binom{13+4-1}{4-1}$$

Problem 2.5. How many codes can be formed from the letters and numbers in 100221-XMNMXXN if every code must have all numbers preceding all letters? For example, 001122-MMNNXXX is a second code in this set.

The number of rearrangements of 100221 and XMNMXXN are $\frac{6!}{2!2!2!}$ and $\frac{7!}{2!2!3!}$ respectively. The total number of codes is

$$\left(\frac{6!}{2!2!2!}\right) \cdot \left(\frac{7!}{2!2!3!}\right)$$

Problem 2.6 (*fixed*)**.** Show that $\gcd(4n+2, 3n+1) = 1$ for all $n \geq 0$.

$$\gcd(4n+1, 3n+1) = \gcd(3n+1, n) = \gcd(n, 1) = 1$$

The point here is $4n+1 = (3n+1)+n$ or $4n+1 \pmod{3n+1} = n$ and then similarly, $\gcd(3n+1, n) = 1$.

Problem 2.7. Give a combinatorial argument for

$$\sum_{i \leq n, i \text{ even}} \binom{n}{i} = \sum_{i \leq n, i \text{ odd}} \binom{n}{i}$$

Proof 1: On the left-hand side, you have the number subsets of $\{1, \ldots, n\}$ of even size, and on the right, those of odd size. So, the equality just follows from knowing that there are as many even-sized subsets as odd-sized subsets.

There are many ways to accomplish this. Induction on n is a good method. Here, I will just define a one-one and onto the map, although you can see the obvious hint of the inductive argument. Let

$$E^+ = \{S \subseteq \{1, \ldots, n\} \mid n \in S \wedge |S| \text{ is even}\}$$
$$E^- = \{S \subseteq \{1, \ldots, n\} \mid n \notin S \wedge |S| \text{ is even}\}$$

$E = E^+ \cup E^-$ is the set of all even-sized subsets of $\{1, \ldots, n\}$.

Similarly define O, O^+, and O^-. Clearly, $|O^-| = |E^+|$ as every $S \in E^+ = S' \cup \{n\}$ for a unique $S' \in O^-$ and conversely, for each $S' \in O^-$, $S = S' \cup \{n\} \in E^+$. Similarly, $|O^+| = |E^-|$ and as $O^+ \cap O^- = \emptyset = E^+ \cap E^-$ we have

$$|E| = |E^+| + |E^-| = |O^-| + |O^+| = |O|$$

Proof 2: By induction. (Actually, this made me realize that I needed $n \geq 1$ as for $n = 0$, there is one even -ized subset of $\emptyset$, namely, $\emptyset$, but no odd-sized.

Base Case: For $n = 1$, let $S = \{1\}$, then there is one even sized subset, namely, $\emptyset$ and one odd sized subset, $\{1\}$. The equation itself reduces to just $\binom{1}{0} = \binom{1}{1}$ which is true as both sides are just 1.

Induction step: Suppose the claim is true for n. Then consider subsets of $\{1, \ldots, n, n+1\}$. There are those subsets $A \subseteq \{1, \ldots, n\}$, and by induction, there is the same number of these that are even as odd. Then there are those $A = A' \cup \{n + 1\}$ where $A' \subseteq \{1, \ldots, n\}$. Clearly, $|A|$ is odd/even iff A' is even/odd respectively, and again by induction, there are the same number of each. So we actually see that if $E_n = |\{A \subseteq \{1, \ldots, n\} \mid |A| \text{ is even}\}$ and likewise O_n, then $|E_n| = |O_n|$ by the induction hypothesis and $E_{n+1} = |E_n| + |O_n| = 2|E_n|$ and similarly $|O_{n+1}| = 2|O_n|$.

Proof 3: (From a student.) *There are two issues here: (1) This proof is not combinatorial, and that was part of the instructions, but still, it is a cute proof. (2) This only works for n odd.*

For any $n \geq 1$ odd, for $k < n$ even, $n - k$ is odd, and we have:

$$\sum_{i \leq n, i \text{ even}} \binom{n}{i} = \sum_{i \leq n, i \text{ even}} \binom{n}{n-i} = \sum_{k \leq n, k \text{ odd}} \binom{n}{k}$$

This is just because $\binom{n}{i} = \binom{n}{n-i}$.