Name: ______________________________ Quiz 2 - MAT345

Problem 1 (20 points; 4 points each)**.** Decide if each of the following are true or false and provide a small proof or counterexample in each case. All vector spaces are assumed to be finite-dimensional here.

(a) False Given a basis $\mathcal{B} = \{\boldsymbol{v}_1, \dots, \boldsymbol{v}_n\}$ for a vector space V and U a subspace of V, then there is $\mathcal{C} \subseteq \mathcal{B}$ that is a basis for U.

FALSE: $\mathcal{B} = \{\boldsymbol{e}_1, \boldsymbol{e}_2\}$ is a basis for $\mathbb{R}^2$ and $U = \text{span}\{(1,1)\}$ is a subspace, namely, the line with slope 1 through the origin. You cannot throw away one of $\boldsymbol{e}_1$ or $\boldsymbol{e}_2$ to get a basis for U.

(b) True Given a basis $\mathcal{C}$ for a subspace U of a vector space V, $\mathcal{C}$ can be extended to a basis $\mathcal{B}$ for V.

This is one of the theorems that you have, any linearly independent set can be expanded to a basis.

(c) False If $\{\boldsymbol{v}_1, \dots, \boldsymbol{v}_n\}$ is linearly independent and $\boldsymbol{v} \in \text{span}(\{\boldsymbol{v}_1, \dots, \boldsymbol{v}_n\})$, then it is possible that there are distinct $\boldsymbol{c}, \boldsymbol{b} \in \mathbb{R}^n$ such that $\boldsymbol{v} = \sum_{i=1}^n c_i \boldsymbol{v}_i = \sum_{i=1}^n b_i \boldsymbol{v}_i$.

If such $\boldsymbol{c}$ and $\boldsymbol{b}$ exists, then $\boldsymbol{0} = \boldsymbol{v} - \boldsymbol{v} = (\sum b_i \boldsymbol{v}_i) - (\sum c_i \boldsymbol{v}_i) = \sum (b_i - c_i)\boldsymbol{v}_i$. Since $\boldsymbol{v}_i$'s are independent, $b_i - c_i = 0$ for all i, so $\boldsymbol{c} = \boldsymbol{b}$.

(d) True If $\{\boldsymbol{v}_1, \dots, \boldsymbol{v}_n\}$ is linearly independent and $V = \text{span}(\{\boldsymbol{v}_1, \dots, \boldsymbol{v}_n\}) = \text{span}(\{\boldsymbol{u}_1, \dots, \boldsymbol{u}_n\})$, then $\{\boldsymbol{u}_1, \dots, \boldsymbol{u}_n\}$ is linearly independent.

This too, is a theorem. Since $V = \text{span}\{\boldsymbol{v}_1, \dots, \boldsymbol{v}_n\}$ and $\boldsymbol{v}_i$ are independent, you know $\{\boldsymbol{v}_1, \dots, \boldsymbol{v}_n\}$ is a basis for V and so $\dim(V) = n$. since $\text{span}\{\boldsymbol{u}_1, \dots, \boldsymbol{u}_n\}$ span V you know this set can be reduced to a basis, but any basis must have n elements, so $\{\boldsymbol{u}_1, \dots, \boldsymbol{u}_n\}$ must already be a basis, and hence is linearly independent.

(e) False Suppose V is a vector space and $U \subseteq V$ is a subspace. For any $\boldsymbol{v} \in V$, there is a **unique** $\boldsymbol{u} \in U$ so that $\boldsymbol{v} = \boldsymbol{u} + (\boldsymbol{v} - \boldsymbol{u})$, that is, there is a unique "projection" of V into U.

Again take $U = \text{span}\{(1,1)\} \subset \mathbb{R}^2 = V$ and let $\boldsymbol{v} = (2,3)$, then $\boldsymbol{v} = (1,1) + (1,2) = (2,2) + (0,1)$.

Note: If we fixed W so that $V = U \oplus W$, then there would be for every $\boldsymbol{v} \in V$ a unique $\boldsymbol{u} \in U, \boldsymbol{w} \in W$ so that $\boldsymbol{v} = \boldsymbol{u} + \boldsymbol{w}$. For example, take U as above and $W = \text{span}\{(0,1)\}$, then $(2,3) = (2,2) + (0,1)$ is the unique decomposition of $(2,3)$ into something from U and something from W.

Problem 2 (10 pts)**.** Show that the collection, U, of upper triangular 3×3 matrices is a subspace of $\mathbb{R}^{3\times 3}$ (the space of all 3×3 matrices). Give a basis $\mathcal{B}$ for U and for $\boldsymbol{v} = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 4 & 5 \\ 0 & 0 & 6 \end{bmatrix}$, give $[\boldsymbol{v}]_{\mathcal{B}}$.

To show that U is a subspace we need only show that $\alpha\boldsymbol{v}+\boldsymbol{u} \in U$ for $\boldsymbol{v},\boldsymbol{u} \in U$. So let $\boldsymbol{u} = \begin{bmatrix} u_{11} & u_{12} & u_{13} \\ 0 & u_{22} & u_{23} \\ 0 & 0 & u_{33} \end{bmatrix}$ and let $\boldsymbol{v} = \begin{bmatrix} v_{11} & v_{12} & v_{13} \\ 0 & v_{22} & v_{23} \\ 0 & 0 & v_{33} \end{bmatrix}$, then $\alpha\boldsymbol{u}+\boldsymbol{v} = \begin{bmatrix} \alpha u_{11}+v_{11} & \alpha u_{12}+v_{12} & \alpha u_{13}+v_{13} \\ 0 & \alpha u_{22}+v_{22} & \alpha u_{23}+v_{23} \\ 0 & 0 & \alpha u_{33}+v_{33} \end{bmatrix} \in U$.

A basis is clearly given by $E^{ij}_{lk} = 1$ if $i=j$ and $l=k$ and $j \le i$ and 0 otherwise. So $E^{11} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$, $E^{12} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$, etc. This basis has six elements, so $\dim(U) = 6$.

With this basis, clearly $\boldsymbol{v} = E^{11} + 2E^{12} + 3E^{13} + 4E^{22} + 5E^{23} + 6E^{33}$.

Problem 3. Find a basis for $\operatorname{span}\{\boldsymbol{v}_1,\boldsymbol{v}_2,\boldsymbol{v}_3,\boldsymbol{v}_4,\boldsymbol{v}_5\}$ where

$$\boldsymbol{v}_1 = \begin{bmatrix} 1 \\ 3 \\ -3 \\ 2 \end{bmatrix}, \boldsymbol{v}_2 = \begin{bmatrix} 1 \\ 4 \\ 0 \\ 4 \end{bmatrix}, \boldsymbol{v}_3 = \begin{bmatrix} 2 \\ 4 \\ -12 \\ 0 \end{bmatrix}, \boldsymbol{v}_4 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 2 \end{bmatrix}, \boldsymbol{v}_5 = \begin{bmatrix} -1 \\ -2 \\ 8 \\ 4 \end{bmatrix}$$

Method 1: Let

$$A = \begin{bmatrix} 1 & 1 & 2 & 0 & -1 \\ 3 & 4 & 4 & 0 & -2 \\ -3 & 0 & -12 & 1 & 8 \\ 2 & 4 & 0 & 2 & 4 \end{bmatrix}$$

$$A \underset{\substack{R_2-3R_1\to R_2 \\ R_3+3R_1\to R_3 \\ R_4-2R_1\to R_4}}{\Longrightarrow} \begin{bmatrix} 1 & 1 & 2 & 0 & -1 \\ 0 & 1 & -2 & 0 & 1 \\ 0 & 3 & -6 & 1 & 5 \\ 0 & 2 & -4 & 2 & 6 \end{bmatrix} \underset{\substack{R_3-3R_2\to R_3 \\ R_4-2R_2\to R_4}}{\Longrightarrow} \begin{bmatrix} 1 & 1 & 2 & 0 & -1 \\ 0 & 1 & -2 & 0 & 1 \\ 0 & 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 2 & 4 \end{bmatrix} \underset{R_4-2R_3\to R_4}{\Longrightarrow} \begin{bmatrix} \boxed{1} & 1 & 2 & 0 & -1 \\ 0 & \boxed{1} & -2 & 0 & 1 \\ 0 & 0 & 0 & \boxed{1} & 2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

So $\mathcal{B} = \{\boldsymbol{v}_1,\boldsymbol{v}_2,\boldsymbol{v}_4\}$ is a basis. (This is all you need.)

In fact, from our CR decomposition, we know

$$A = \begin{bmatrix} 1 & 1 & 0 \\ 3 & 4 & 0 \\ -3 & 0 & 1 \\ 2 & 2 & 2 \end{bmatrix} \begin{bmatrix} \boxed{1} & 1 & 2 & 0 & -1 \\ 0 & \boxed{1} & -2 & 0 & 1 \\ 0 & 0 & 0 & \boxed{1} & 2 \end{bmatrix}$$

So we know $\boldsymbol{v}_3 = 2\boldsymbol{v}_2 - 2\boldsymbol{v}_2$ and $\boldsymbol{v}_5 = -\boldsymbol{v}_1 + \boldsymbol{v}_2 + 2\boldsymbol{v}_3$.

Method 2: Let

$$B = \begin{bmatrix} 1 & 3 & -3 & 2 \\ -1 & -2 & 8 & 4 \\ 0 & 0 & 1 & 2 \\ 1 & 4 & 0 & 4 \\ 2 & 4 & -12 & 0 \end{bmatrix}$$

Then eliminate:

$$B \underset{\substack{R_2+R_1\to R_2 \\ R_4-R_1\to R_4 \\ R_5-2R_1\to R_5}}{\Longrightarrow} \begin{bmatrix} 1 & 3 & -3 & 2 \\ 0 & 1 & 5 & 6 \\ 0 & 0 & 1 & 2 \\ 0 & 1 & 3 & 2 \\ 0 & -2 & -6 & -4 \end{bmatrix} \underset{\substack{R_4-R_2\to R_4 \\ R_5+R_2\to R_5}}{\Longrightarrow} \begin{bmatrix} 1 & 3 & -3 & 2 \\ 0 & 1 & 5 & 6 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & -2 & -4 \\ 0 & 0 & 4 & 8 \end{bmatrix} \underset{\substack{R_4+2R_3\to R_4 \\ R_5-4R_3\to R_5}}{\Longrightarrow} \begin{bmatrix} 1 & 3 & -3 & 2 \\ 0 & 1 & 5 & 6 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} = B'$$

So $\mathrm{span}\{\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3, \boldsymbol{v}_4, \boldsymbol{v}_5\} = \mathrm{RS}(B) = \mathrm{RS}(B') = \mathrm{span}\{(1, 3, -3, 2), (0, 1, 5, 6), (0, 0, 1, 2)\}$. So a basis for $\mathrm{span}\{\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3, \boldsymbol{v}_4, \boldsymbol{v}_5\}$ is $\mathcal{B}' = \{\boldsymbol{u}_1, \boldsymbol{u}_2, \boldsymbol{u}_3\}$ where

$$\boldsymbol{u}_1 = \begin{bmatrix} 1 \\ 3 \\ -3 \\ 2 \end{bmatrix}, \boldsymbol{u}_2 = \begin{bmatrix} 0 \\ 1 \\ 5 \\ 6 \end{bmatrix}, \boldsymbol{u}_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 2 \end{bmatrix}$$