

Name: _____

Quiz 2 - MAT345

Problem Q2.1 (20 points; 4 points each). Decide if each of the following is true or false. As usual, you may supply reasons for some points back.

Let $V = \text{span}\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4, \mathbf{v}_5\}$ where

$$\mathbf{v}_1 = \begin{bmatrix} -1 \\ -5 \\ 5 \\ 3 \end{bmatrix} \quad \mathbf{v}_2 = \begin{bmatrix} -3 \\ -16 \\ 20 \\ 8 \end{bmatrix} \quad \mathbf{v}_3 = \begin{bmatrix} 2 \\ 7 \\ 5 \\ -9 \end{bmatrix} \quad \mathbf{v}_4 = \begin{bmatrix} -2 \\ -12 \\ 19 \\ 7 \end{bmatrix} \quad \mathbf{v}_5 = \begin{bmatrix} 1 \\ 3 \\ 7 \\ -9 \end{bmatrix}$$

and we have

$$\text{rref} \begin{bmatrix} -1 & -3 & 2 & -2 & 1 \\ -5 & -16 & 7 & -12 & 3 \\ 5 & 20 & 5 & 19 & 7 \\ 3 & 8 & -9 & 6 & -9 \end{bmatrix} = \begin{bmatrix} 1 & 0 & -11 & 0 & -15 \\ 0 & 1 & 3 & 0 & 6 \\ 0 & 0 & 0 & 1 & -2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}; \text{rref} \begin{bmatrix} -1 & -5 & 5 & 3 \\ -3 & -16 & 20 & 8 \\ 2 & 7 & 5 & -9 \\ -2 & -12 & 19 & 7 \\ 1 & 3 & 7 & -9 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 32 \\ 0 & 1 & 0 & -9 \\ 0 & 0 & 1 & -2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

(a) False \mathcal{B} is a basis for V where

$$\mathcal{B} = \left\{ \begin{bmatrix} -1 \\ -5 \\ 5 \\ 3 \end{bmatrix}, \begin{bmatrix} -3 \\ -16 \\ 20 \\ 8 \end{bmatrix}, \begin{bmatrix} 2 \\ 7 \\ 5 \\ -9 \end{bmatrix} \right\}$$

(b) True \mathcal{B}' is a basis for V where

$$\mathcal{B}' = \left\{ \begin{bmatrix} -1 \\ -5 \\ 5 \\ 3 \end{bmatrix}, \begin{bmatrix} -3 \\ -16 \\ 20 \\ 8 \end{bmatrix}, \begin{bmatrix} 1 \\ 3 \\ 7 \\ -9 \end{bmatrix} \right\}$$

Note: This is true, but there is a mistake in intent. I should have had columns 1, 2, and 4, not 5. It is still true that columns 1, 2, and 5 of $\text{rref}(A)$ are linearly independent, and so do form a basis for $\text{CS}(\text{rref}(A))$ and hence 1, 2, and 5 are a basis for $\text{CS}(A)$. However, this was not my intent.

(c) True \mathcal{B}'' is a basis for V where

$$\mathcal{B}'' = \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \\ 32 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ -9 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ -2 \end{bmatrix} \right\}$$

(d) True If $\{\mathbf{v}_1, \dots, \mathbf{v}_n\}$ spans a vector space V and $\{\mathbf{u}_1, \dots, \mathbf{u}_n\} \subseteq V$ is independent. Then $\{\mathbf{u}_1, \dots, \mathbf{u}_n\}$ spans V .

(e) True Suppose U and W subspaces of a vector space V such that

$$U + W = V, \text{ and } U \cap W = \{\mathbf{0}\}.$$

Then for every $\mathbf{v} \in V$, there is a **unique pair** $\mathbf{u} \in U, \mathbf{w} \in W$ so that $\mathbf{u} + \mathbf{w} = \mathbf{v}$.

Recall: $U + W = \{\mathbf{u} + \mathbf{w} \mid \mathbf{u} \in U \text{ and } \mathbf{w} \in W\}$.

Problem Q2.2 (10 pts). A square matrix A is called **anti-symmetric** if $A^T = -A$.

- Show that the anti-symmetric 3×3 matrices form a subspace of all 3×3 matrices.
- Give a basis, \mathcal{B} , for the 3×3 anti-symmetric matrices.
- Give representation $[\mathbf{v}]_{\mathcal{B}}$ for $\mathbf{v} = \begin{bmatrix} 0 & 1 & 2 \\ -1 & 0 & 3 \\ -2 & -3 & 0 \end{bmatrix}$ with respect to the basis that you gave.

Denote by U the set of 3×3 anti-symmetric matrices. Clearly, $O = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \in U$. Next, we need to show that U is closed under scalar multiplication and addition. This is done by just taking arbitrary elements of U and computing:

$$\begin{bmatrix} 0 & a & b \\ -a & 0 & c \\ -b & -c & 0 \end{bmatrix} + \begin{bmatrix} 0 & A & B \\ -A & 0 & C \\ -B & -C & 0 \end{bmatrix} = \begin{bmatrix} 0 & a+A & b+B \\ -(a+A) & 0 & c+C \\ -(b+B) & -(c+C) & 0 \end{bmatrix}$$

and

$$\alpha \begin{bmatrix} 0 & a & b \\ -a & 0 & c \\ -b & -c & 0 \end{bmatrix} = \begin{bmatrix} 0 & \alpha a & \alpha b \\ -\alpha a & 0 & \alpha c \\ -\alpha b & -\alpha c & 0 \end{bmatrix}$$

A basis is clearly given by

$$\mathcal{B} = \left\{ \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & -1 & 0 \end{bmatrix} \right\}$$

With this basis, clearly

$$\mathbf{v} = \begin{bmatrix} 0 & 1 & 2 \\ -1 & 0 & 3 \\ -2 & -3 & 0 \end{bmatrix} = (1) \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} + (2) \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{bmatrix} + (3) \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & -1 & 0 \end{bmatrix}$$

So $[\mathbf{v}]_{\mathcal{B}} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$

Problem Q2.3. Find a basis for $\text{span}\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4, \mathbf{v}_5\}$ from the given vectors

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ 2 \\ -2 \\ -3 \end{bmatrix}, \mathbf{v}_2 = \begin{bmatrix} 2 \\ 4 \\ -4 \\ -6 \end{bmatrix}, \mathbf{v}_3 = \begin{bmatrix} 0 \\ 1 \\ -2 \\ 4 \end{bmatrix}, \mathbf{v}_4 = \begin{bmatrix} -3 \\ -4 \\ 2 \\ 17 \end{bmatrix}, \mathbf{v}_5 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ -3 \end{bmatrix}$$

Let

$$A = \begin{bmatrix} 1 & 2 & 0 & -3 & 0 \\ 2 & 4 & 1 & -4 & 0 \\ -2 & -4 & -2 & 2 & 1 \\ -3 & -6 & 4 & 17 & -3 \end{bmatrix}$$

$$A \xrightarrow{\substack{R_2 - 2R_1 \rightarrow R_2 \\ R_3 + 2R_1 \rightarrow R_3 \\ R_4 + 3R_1 \rightarrow R_4}} \begin{bmatrix} 1 & 2 & 0 & -3 & 0 \\ 0 & 0 & 1 & 2 & 0 \\ 0 & 0 & -2 & -4 & 1 \\ 0 & 0 & 4 & 8 & -3 \end{bmatrix} \xrightarrow{\substack{R_3 + 2R_2 \rightarrow R_3 \\ R_4 - 4R_2 \rightarrow R_4}} \begin{bmatrix} 1 & 2 & 0 & -3 & 0 \\ 0 & 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & -3 \end{bmatrix} \xrightarrow{R_4 + 3R_3 \rightarrow R_4} \begin{bmatrix} \boxed{1} & 2 & 0 & -3 & 0 \\ 0 & 0 & \boxed{1} & 2 & 0 \\ 0 & 0 & 0 & 0 & \boxed{1} \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

So $\mathcal{B} = \{\mathbf{v}_1, \mathbf{v}_3, \mathbf{v}_5\}$ is a basis. (This is all you need.)

In fact, from our CR decomposition, we know

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ -2 & -2 & 1 \\ -3 & 4 & -3 \end{bmatrix} \begin{bmatrix} \boxed{1} & 2 & 0 & -3 & 0 \\ 0 & 0 & \boxed{1} & 2 & 0 \\ 0 & 0 & 0 & 0 & \boxed{1} \end{bmatrix}$$

So we know $\mathbf{v}_2 = 2\mathbf{v}_1$ and $\mathbf{v}_4 = -3\mathbf{v}_1 + 2\mathbf{v}_3$.