Homework 4 Partial Solutions

Notation: To keep notation simpler lets agree that

$$(x_1, x_2, \ldots, x_n) = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$$

and agree to write $L(x_1, x_2, \ldots, x_n)$ in place of the more correct $L((x_1, x_2, \ldots, x_n))$. This way we can write things like:

$$L(x_1, x_2, x_3) = (x_1 + x_2, x_3)$$

instead of the more cumbersome:

$$L([x_1, x_2, x_3]^T) = [x_1 + x_2, x_3]^T \quad \text{or} \quad L\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}\right) = \begin{bmatrix} x_1 + x_2 \\ x_3 \end{bmatrix}$$

Section 4.1

5. Determine if the following maps : $\mathbb{R}^3 \to \mathbb{R}^2$ are linear.

(a) (Linear) : $L(\boldsymbol{x}) = (x_2, x_3)$ (projection onto the last two coordinates).

Clearly $L(\boldsymbol{x} + \alpha\boldsymbol{y}) = (x_2, x_3) + (\alpha y_2, \alpha y_3) = (x_2, x_3) + \alpha(y_2, y_3) = L(\boldsymbol{x}) + \alpha L(\boldsymbol{y})$.

(b) (Linear) $L(\boldsymbol{x}) = (0, 0)$ (constant $\mathbf{0}$ map)

$L(\boldsymbol{x} + \alpha\boldsymbol{y}) = \mathbf{0} = \mathbf{0} + \alpha\mathbf{0} = L(\boldsymbol{x}) + \alpha L(\boldsymbol{y})$.

(c) (Non-Linear) $L(\boldsymbol{x}) = (1 + x_1, x_2)$.

$L(\mathbf{0}) = (1, 0) \neq \mathbf{0}$ so L is non-linear.

(d) (Linear) $L(\boldsymbol{x}) = (x_3, x_1 + x_2)$.

$L(\boldsymbol{x}+\alpha\boldsymbol{y}) = (x_3+\alpha y_3, (x_1+\alpha y_1)+(x_2+\alpha y_2)) = (x_3, x_1+x_2)+\alpha(y_3, y_1+y_2) = L(\boldsymbol{x})+\alpha L(\boldsymbol{y})$.

6. Determine if $L : \mathbb{R}^2 \to \mathbb{R}^3$ is linear.

(a) $L(x_1, x_2) = (x_1, x_2, 1)$

If L is linear, then $L(\mathbf{0}) = \mathbf{0}$, since $L(0\boldsymbol{x}) = 0L(\boldsymbol{x}) = \mathbf{0}$. For the given transformation, this fails, so the given L is not linear.

(b) $L(x_1, x_2) = (x_1, x_2, x_1 + 2x_2)$

Let $r \in \mathbb{R}$ and $(x_1, x_2), (y_1, y_2) \in \mathbb{R}^2$, then:

$$\begin{aligned} L((x_1, x_2) + r(y_1, y_2)) &= L(x_1 + ry_1, x_2 + ry_2) = (x_1 + ry_1, x_2 + ry_2, x_1 + ry_1 + 2(x_2 + ry_2) \\ &= (x_1, x_2, x_1 + 2x_2) + r(y_1, y_2, y_1 + 2y_2) = L(x_1, x_2) + rL(y_1, y_2). \end{aligned}$$

So L is linear.

(c) $L(x_1, x_2) = (x_1, 0, 0)$

Let $r \in \mathbb{R}$ and $(x_1, x_2), (y_1, y_2) \in \mathbb{R}^2$, then:

$$\begin{aligned} L((x_1, x_2) + r(y_1, y_2)) &= L(x_1 + ry_1, x_2 + ry_2) = (x_1 + ry_1, 0, 0) \\ &= (x_1, 0, 0) + r(y_1, 0, 0) = L(x_1, x_2) + rL(y_1, y_2). \end{aligned}$$

So L is linear.

(d) $L(x_1, x_2) = (x_1, x_2, x_1^2 + x_2^2)$

$L((0, 1) + (0, 1)) = L(0, 2) = (0, 2, 4)$ whereas $L(0, 1) + L(0, 1) = (0, 1, 1) + (0, 1, 1) = (0, 2, 2)$ so clearly $L(\boldsymbol{x} + \boldsymbol{y}) \neq L(\boldsymbol{x}) + L(\boldsymbol{y})$ for $\boldsymbol{x} = \boldsymbol{y} = (0, 1)$. Hence L is not linear.

13. Let $\boldsymbol{x} \in V$, then there are unique $a_i \in \mathbb{R}$ so that $\boldsymbol{x} = \sum_{i=1}^n a_i \boldsymbol{v}_i$. By linearity of L_1 and L_2 we have:

$$L_1(\boldsymbol{x}) = \sum_{i=1}^n a_i L_1(\boldsymbol{v}_i) = \sum_{i=1}^n a_i L_2(\boldsymbol{v}_i) = L_2(\boldsymbol{x})$$

So for all $\boldsymbol{x} \in V$, $L_1(\boldsymbol{x}) = L_2(\boldsymbol{x})$. Thus $L_1 = L_2$.

17.

(a) Clearly $\ker(L) = \{\mathbf{0}\}$ and $\mathrm{Img}(L) = \mathbb{R}^3$.

(b) $\ker(L) = \{(0, 0, x_3) \mid x_3 \in \mathbb{R}\}$ and $\mathrm{Img}(L) = \{(x_1, x_2, 0) \mid x_1, x_2 \in \mathbb{R}\}$.

(c) $\ker(L) = \{(0, x_2, x_3) \mid x_2, x_3 \in \mathbb{R}\}$ and $\mathrm{Img}(L) = \{(x_1, x_1, x_1) \mid x_1 \in \mathbb{R}\}$.

19 Find $\ker(L)$ for each linear $L : P_3 \to P_3$.

(a) $L(f) = x \cdot f'$.

Clearly $x \cdot f' = 0$ iff $f' = 0$ for $x \neq 0$. But this means f is constant for $x > 0$ and being a polynomial, f must just be constant. So $\ker(L)$ is the set of all constant maps, and hence essentially, $\ker(L) = P_0 = \mathbb{R}$.

(b) $L(p) = p - p'$. Since $p - p' = 0$ iff $p = p'$ we see this is equivalent to $\frac{p'}{p} = 1$ or $\frac{d}{dx} \ln(|p|) = 1$, so $\ln(|p|) = x + c$ or $|p| = e^{x+c} = Ke^x$. No polynomial satisfies this except when $K = 0$ and so $p = 0$. Thus $\ker(L) = \{0\}$.

(c) $L(p) = p(0)x + p(1)$

$L(p) = 0$ iff $p(0)x + p(1) = 0$ so $p(1) = p(0) = 0$. Now $p(x) = a_3x^3 + a_2x^2 + a_1x + a_0$ and $p(0) = a_0 = 0$ and $p(1) = a_3 + a_2 + a_1 = 0$, so $a_1 = -(a_3 + a_2)$. Thus $p = a_3x^3 + a_2x^2 - (a_3 + a_2)x = a_3(x^3 - x) + a_2(x^2 - x)$. So $\ker(L) = \mathrm{span}\{x^3 - x, x^2 - x\}$.

Section 4.2

2. For each linear $L : \mathbb{R}^3 \to \mathbb{R}^2$ find A so that $L(\boldsymbol{x}) = A\boldsymbol{x}$.

A couple of things to note. The standard basis for $\mathbb{R}^n$ will be $\mathcal{E} = \{\boldsymbol{e}_1, \dots, \boldsymbol{e}_n\}$ where n is clear from the context. When working in $\mathbb{R}^n$ in the standard basis we have $[\boldsymbol{v}]_{\mathcal{E}} = \boldsymbol{v}$, this helps reduce notation. For example, $[L(\boldsymbol{e}_i)]_{\mathcal{E}} = L(\boldsymbol{e}_i)$ so long as everything is wrt the standard basis.

(b) $L((x_1, x_2, x_3)) = (x_1, x_2)$.

$$A = \big[[L(\boldsymbol{e}_1) \quad L(\boldsymbol{e}_2) \quad L(\boldsymbol{e}_3)\big] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

(c) $L((x_1, x_2, x_3)) = (x_2 - x_1, x_3 - x_2)$.

$$A = \big[[L(\boldsymbol{e}_1) \quad L(\boldsymbol{e}_2) \quad L(\boldsymbol{e}_3)\big] = \begin{bmatrix} -1 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix}$$

3. For each $L : \mathbb{R}^3 \to \mathbb{R}^3$ find A so that $A\boldsymbol{x} = L(\boldsymbol{x})$. (See (2) above for notation.)

(b) $L((x_1, x_2, x_3)) = (x_1, x_1 + x_2, x_1 + x_2 + x_3)$

$$A = \big[L(\boldsymbol{e}_1) \quad L(\boldsymbol{e}_2) \quad L(\boldsymbol{e}_3)\big] = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix}$$

(c) $L((x_1, x_2, x_3)) = (2x_3, x_2 + 3x_1, 2x_1 - x_3)$

$$A = \big[L(\boldsymbol{e}_1) \quad L(\boldsymbol{e}_2) \quad L(\boldsymbol{e}_3)\big] = \begin{bmatrix} 0 & 0 & 2 \\ 3 & 1 & 0 \\ 2 & 0 & -1 \end{bmatrix}$$

5. In each case $[L] = \big[L(\boldsymbol{e}_1) \quad L(\boldsymbol{e}_2)\big]$

(a) $L(1, 0) = (\sqrt{2}/2, -\sqrt{2}/2)$ and $L(0, 1) = (\sqrt{2}/2, \sqrt{2}/2)$ so

$$[L] = \frac{\sqrt{2}}{2} \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}$$

(b) $(1, 0) \mapsto (1, 0) \mapsto (0, 1)$ and $(0, 1) \mapsto (0, -1) \mapsto (1, 0)$ so $L(1, 0) = (0, 1)$ and $L(0, 1) = (1, 0)$

$$[L] = [L(\boldsymbol{e}_1)\, L(\boldsymbol{e}_2)] = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

(c) The counter clockwise rotation is

$$[R_{30^\circ}] = \begin{bmatrix} \sqrt{3}/2 & -1/2 \\ 1/2 & \sqrt{3}/2 \end{bmatrix}$$

Stretching by 2 is

$$[T_2] = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} = 2I_2$$

so

$$[L] = [R_{30^\circ} \circ T_2] = \begin{bmatrix} \sqrt{3}/2 & -1/2 \\ 1/2 & \sqrt{3}/2 \end{bmatrix} \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} = \begin{bmatrix} \sqrt{3} & -1 \\ 1 & \sqrt{3} \end{bmatrix}$$

(d) Projection about $x_1 = x_2$ is

$(1,0) \mapsto (0,1) \mapsto (0,0)$ and $(0,1) \mapsto (1,0) \mapsto (1,0)$ so the matrix is

$$[L] = [L(\boldsymbol{e}_1)\, L(\boldsymbol{e}_2)] = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$$

8. Let

$$\boldsymbol{y}_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \quad \boldsymbol{y}_2 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \quad \boldsymbol{y}_3 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

$L : \mathbb{R}^3 \to \mathbb{R}^3$ is given by

$$L(c_1\boldsymbol{y}_1 + c_2\boldsymbol{y}_2 + c_3\boldsymbol{y}_3) = (c_1 + c_2 + c_3)\boldsymbol{y}_1 + (2c_1 + c_3)\boldsymbol{y}_2 - (2c_2 + c_3)\boldsymbol{y}_3$$

Start by finding the matrix for L wrt $\mathcal{B} = \{\boldsymbol{y}_1, \boldsymbol{y}_2, \boldsymbol{y}_3\}$. For this notice that $L(\boldsymbol{y}_1) = \boldsymbol{y}_1 + 2\boldsymbol{y}_2$, $L(\boldsymbol{y}_2) = \boldsymbol{y}_1 - 2\boldsymbol{y}_3$, and $L(\boldsymbol{y}_3) = \boldsymbol{y}_1 + \boldsymbol{y}_2 - \boldsymbol{y}_3$. So

$$[L(\boldsymbol{y}_1)]_{\mathcal{B}} = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} \quad [L(\boldsymbol{y}_2)]_{\mathcal{B}} = \begin{bmatrix} 1 \\ 0 \\ -2 \end{bmatrix} \quad [L(\boldsymbol{y}_3)]_{\mathcal{B}} = \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}$$

So

$$[L]_{\mathcal{B},\mathcal{B}} = \begin{bmatrix} 1 & 1 & 1 \\ 2 & 0 & 1 \\ 0 & -2 & -1 \end{bmatrix}$$

Now if we want the matrix wrt the standard basis we need to do the change of basis

$$[\mathrm{id}]_{\mathcal{B},\mathcal{E}} = B = \begin{bmatrix} \boldsymbol{y}_1 & \boldsymbol{y}_2 & \boldsymbol{y}_3 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

and

$$[\mathrm{id}]_{\mathcal{E},\mathcal{B}} = ([\mathrm{id}]_{\mathcal{B},\mathcal{E}})^{-1} = B^{-1} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & -1 \\ 1 & -1 & 0 \end{bmatrix}$$

(a) $[L]_{\mathcal{B},\mathcal{B}}$ (above)

(b) Here we ask for $[L]_{\mathcal{E},\mathcal{E}}$

$$[L]_{\mathcal{E},\mathcal{E}} = [\mathrm{id}]_{\mathcal{B},\mathcal{E}}[L]_{\mathcal{B},\mathcal{B}}[\mathrm{id}]_{\mathcal{E},\mathcal{B}} = B[L]_{\mathcal{B},\mathcal{B}}B^{-1} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 \\ 2 & 0 & 1 \\ 0 & -2 & -1 \end{bmatrix} \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & -1 \\ 1 & -1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & -2 & 4 \\ 2 & -1 & 2 \\ 1 & 0 & 0 \end{bmatrix}$$

(i) $L((7,5,2)) = \begin{bmatrix} 1 & -2 & 4 \\ 2 & -1 & 2 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 7 \\ 5 \\ 2 \end{bmatrix} = \begin{bmatrix} 5 \\ 13 \\ 7 \end{bmatrix}$.

You could also argue this way: $(7,5,2) = 2\boldsymbol{y}_1 + 3\boldsymbol{y}_2 + 2\boldsymbol{y}_3$, so

$$\begin{aligned} L((7,5,2)) &= L(2\boldsymbol{y}_1 + 3\boldsymbol{y}_2 + 2\boldsymbol{y}_3) \\ &= (2+3+2)\boldsymbol{y}_1 + (2(2)+2)\boldsymbol{y}_2 - (2(3)+2)\boldsymbol{y}_3 = 7\boldsymbol{y}_1 + 6\boldsymbol{y}_2 - 8\boldsymbol{y}_3 \\ &= (7,7,7) + (6,6,0) - (8,0,0) = (5,13,7) \end{aligned}$$

(ii) $L((3,2,1)) = \begin{bmatrix} 1 & -2 & 4 \\ 2 & -1 & 2 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 3 \\ 6 \\ 3 \end{bmatrix}$.

(iii) $L((1,2,3)) \begin{bmatrix} 1 & -2 & 4 \\ 2 & -1 & 2 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 9 \\ 6 \\ 1 \end{bmatrix}$.

9.

(a) R is a unit square.(One vertex at origin, one side along x_1 axis and one along x_2 axis.)

(b)

(i) This is the unit square shrunk by a factor of 1/2.

(ii) This is the unit square rotated counter clockwise by 45°.

(iii) This is the unit square shifted 2 units in the x_1 direction and -3 units in the x_2 direction.

Section 4.3

4. Given a basis $\mathcal{B} = \{\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3\}$ in $\mathbb{R}^3$, the change of basis matrix from $\mathcal{B}$ to the standard basis is just $B = \begin{bmatrix} \boldsymbol{v}_1 & \boldsymbol{v}_2 & \boldsymbol{v}_3 \end{bmatrix}$, that is just write down the elements of $\mathcal{B}$ as the columns. Then the change of basis matrix from the standard basis $\mathcal{E}$ to $\mathcal{B}$ is just B^{-1}. So $[L]_{\mathcal{B}} = B^{-1}[L]B$ and thus

$$[L]_{\mathcal{B}} = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 2 & -2 \\ 1 & 0 & 1 \end{bmatrix}^{-1} \begin{bmatrix} 3 & -1 & -2 \\ 2 & 0 & -2 \\ 2 & -1 & -1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 0 \\ 1 & 2 & -2 \\ 1 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

It is clear now that with respect to the basis $\mathcal{B}$, L simply fixes $\boldsymbol{v}_2$ and $\boldsymbol{v}_3$ and kills $\boldsymbol{v}_1$.

5. Let : $P_3 \to P_3$ be $L(p) = xp' + p''$

(a) Let $\mathcal{B} = \{1, x, x^2\}$, then

$$[L(1)]_{\mathcal{B}} = [0]_{\mathcal{B}} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \qquad [L(x)]_{\mathcal{B}} = [x]_{\mathcal{B}} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \qquad [L(x^2)]_{\mathcal{B}} = [2x^2+2]_{\mathcal{B}} = \begin{bmatrix} 2 \\ 0 \\ 2 \end{bmatrix}$$

So

$$A = [L]_{\mathcal{B},\mathcal{B}} = \begin{bmatrix} 0 & 0 & 2 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

(b) Let $\mathcal{C} = \{1, x, x^2+1\}$, then

$$[L(1)]_{\mathcal{C}} = [0]_{\mathcal{C}} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \qquad [L(x)]_{\mathcal{C}} = [x]_{\mathcal{C}} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \qquad [L(x^2+1)]_{\mathcal{C}} = [2x^2+2]_{\mathcal{C}} = \begin{bmatrix} 0 \\ 0 \\ 2 \end{bmatrix}$$

So

$$B = [L]_{\mathcal{C},\mathcal{C}} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

(c) The matrix from $\mathcal{C}$ to $\mathcal{B}$ is

$$S = \begin{bmatrix} [1]_{\mathcal{B}} & [x]_{\mathcal{B}} & [x^2+1]_{\mathcal{B}} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

S^{-1} transforms from $\mathcal{B}$ to $\mathcal{C}$

$$S^{-1} = \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

So

$$B = [L]_{\mathcal{C},\mathcal{C}} = [\mathrm{id}]_{\mathcal{B},\mathcal{C}}[L]_{\mathcal{B},\mathcal{B}}[\mathrm{id}]_{\mathcal{C},\mathcal{B}} = S^{-1}AS$$

(d) Compute $L^n(p)$ for $p = a_0 + a_1 x + a_2(x^2+1)$, so $[p]_{\mathcal{C}} = (a_0, a_1, a_2)$

$$[L^n(p)]_{\mathcal{C}} = ([L]_{\mathcal{C},\mathcal{C}})^n [p]_{\mathcal{C}} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1^n & 0 \\ 0 & 0 & 2^n \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ a_2 \end{bmatrix} = \begin{bmatrix} 0 \\ a_1 \\ 2^n a_3 \end{bmatrix}$$

So $L^n(p) = a_1 x + 2^n a_3(x^2+1)$

8. Suppose $A = S\Lambda S^{-1}$ and $\boldsymbol{s}_i$ is the i^{th} column of S. Then

(a) $AS = \Lambda S$. Since Λ is diagonal we have

$$AS = \begin{bmatrix} A\boldsymbol{s}_1 & \cdots & A\boldsymbol{s}_n \end{bmatrix} = \Lambda S = \begin{bmatrix} \Lambda\boldsymbol{s}_1 & \cdots & \Lambda\boldsymbol{s}_n \end{bmatrix} = \begin{bmatrix} \lambda_1\boldsymbol{s}_1 & \cdots & \lambda_n\boldsymbol{s}_n \end{bmatrix}$$

Thus, clearly $A\boldsymbol{s}_i = \lambda_i \boldsymbol{s}_i$.

(b) If $\boldsymbol{x} = \alpha_1\boldsymbol{s}_1 + \cdots + \alpha_n\boldsymbol{s}_n$, then $L(\boldsymbol{x}) = \sum_{i=1}^n \alpha_i L(\boldsymbol{s}_i) = \sum_{i=1}^n \lambda_i\alpha_i\boldsymbol{s}_i$. So by a simple inductiom, $L^m(\boldsymbol{x}) = \sum_{i=1}^n \lambda_i^m \alpha_i \boldsymbol{s}_i$.

(c) Clearly if $|\lambda_i| < 1$, then $\lambda_i^m \to 0$ as $m \to \infty$, so $L^m(\boldsymbol{x}) \to 0$ as $m \to \infty$.