Name: ____________________________ Quiz 4 - MAT345

Question 1 (20 points; 4 points each)**.** No justification required for the this quiz, just the T/F responses.

(a) False If all eigenvalues of A are 0, then $A = 0$.

(b) False All $n \times n$ matrices are diagonalizable.

(c) True If A and B are $n \times n$ matrices and each are diagonalizable using the same set of eigenvectors, that is, $A = S\Lambda_A S^{-1}$ and $B = S\Lambda_B S^{-1}$ (**same** S), then A and B commute, i.e., $AB = BA$.

(d) True If $p(x)$ is a polynomial and $A = S^{-1}DS$ where $D = \operatorname{diag}(d_1, \ldots, d_n)$, then

$$p(A) = S^{-1} \operatorname{diag}(p(d_1), \ldots, p(d_n))S$$

Here

$$\operatorname{diag}(d_1, \ldots, d_n) = \begin{bmatrix} d_1 & 0 & \cdots & 0 \\ 0 & d_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & d_n \end{bmatrix}$$

(e) True If A is upper-triangular, then the eigenvalues of A are exactly the diagonal elements of A.

Question 2 (10 points)**.** Let $A = \begin{bmatrix} 5/4 & -3/4 \\ -3/4 & 5/4 \end{bmatrix}$. Diagonalize A, i.e. write $A = S\Lambda S^{-1}$ where Λ is the diagonal matrix of eigenvalues for A.

Find the eigenvalues: $\det(A - tI) = (5/4 - t)^2 - (3/4)^2 = (5/4 - t - 3/4)(5/4 - t + 3/4) = (1/2 - t)(2 - t)$ so $\lambda_1 = 2 > \lambda_2 = 1/2$ and

$$\Lambda = \begin{bmatrix} 2 & 0 \\ 0 & 1/2 \end{bmatrix}$$

Find the eigenspace for λ_1:

$$\mathrm{NS}\left(\begin{bmatrix} -3/4 & -3/4 \\ -3/4 & -3/4 \end{bmatrix}\right) = \mathrm{NS}\left(\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}\right)$$

So a basis is given by $\boldsymbol{v_1} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$.

Similarly a basis for the eigenspace of λ_2 is $\boldsymbol{v_2} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$.

$$S = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}; \quad \Lambda = \begin{bmatrix} 2 & 0 \\ 0 & 1/2 \end{bmatrix}; \quad S^{-1} = \frac{1}{2}\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$$

So

$$A = \begin{bmatrix} 5/4 & -3/4 \\ -3/4 & 5/4 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} 2 & 0 \\ 0 & 1/2 \end{bmatrix}\begin{bmatrix} 1/2 & -1/2 \\ 1/2 & 1/2 \end{bmatrix} = S\Lambda S^{-1}$$

Question 3 (10 points)**.** Suppose the matrix A from Question 2 is used to transform points in the plane iteratively. What is $A^n\left[\begin{smallmatrix} 3 \\ 5 \end{smallmatrix}\right]$?

Bonus (5 points) What is $A^n\left[\begin{smallmatrix} a \\ b \end{smallmatrix}\right]$?

$$\begin{bmatrix} 3 \\ 5 \end{bmatrix} = (-1)\begin{bmatrix} 1 \\ -1 \end{bmatrix} + (4)\begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

So

$$A^n\begin{bmatrix} 3 \\ 5 \end{bmatrix} = (-1)A^n\begin{bmatrix} 1 \\ -1 \end{bmatrix} + (4)A^n\begin{bmatrix} 1 \\ 1 \end{bmatrix} = (-1)2^n\begin{bmatrix} 1 \\ -1 \end{bmatrix} + (4)(1/2)^n\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} -2^n + 4/2^n \\ 2^n + 4/2^n \end{bmatrix}$$

Bonus

$$\left[\begin{bmatrix} a \\ b \end{bmatrix}\right]_{\{\boldsymbol{v}_1, \boldsymbol{v}_2\}} = A^{-1}\begin{bmatrix} a \\ b \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} (a-b)/2 \\ (a+b)/2 \end{bmatrix}$$

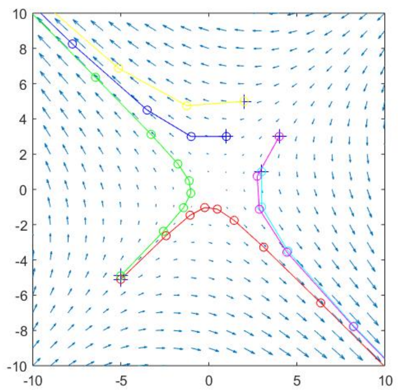
So

$$\begin{bmatrix} a \\ b \end{bmatrix} = \frac{a-b}{2}\begin{bmatrix} 1 \\ -1 \end{bmatrix} + \frac{a+b}{2}\begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

and so

$$A^n \begin{bmatrix} a \\ b \end{bmatrix} = \left(\frac{a-b}{2}\right) \cdot 2^n \begin{bmatrix} 1 \\ -1 \end{bmatrix} + \left(\frac{a+b}{2}\right) \cdot 2^{-n} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} (a-b)2^{n-1} + (a+b)2^{-(n+1)} \\ (b-a)2^{n-1} + (a+b)2^{-(n+1)} \end{bmatrix}$$

This is all I expected. For a visual, here is an image of a few points traced out by this action and the code that generated the image.



Here is the code that generated the image, just for fun. (download code)

```
clear;
close all;
clc;

A = 1/4*[5 -3; -3 5];
x = -10:1:10;
y = x;
[X,Y] = meshgrid(x,y);
Z = A*[X(:)'; Y(:)'];
u = Z(1,:);
v = Z(2,:);
fig = figure();
quiver(X(:)',Y(:)',u-X(:)',v-Y(:)')
axis([-10 10 -10 10]);
daspect([1 1 1]);
hold on;


x = [   -5    -5    1  3  2  4];
y = [-5.1  -4.9    3  1  5  3];

plot(x,y,"b+",'MarkerSize',10);

[M N] = size(x);

for i = 1:N
    p(:,:,i) = [x(i);y(i)];
```

```
end

colors =["red","green","blue","cyan","yellow","magenta"]

for i=2:10
    for j=1:N
        R(:,:,j) = A*p(:,i-1,j);
    end
    p = cat(2,p,R);
    for k=1:N
        plot(p(1,i-1:i,k),p(2,i-1:i,k),'Color',colors(k),...
            'Marker','o');
    end
    drawnow;
    pause(.5);
end
```