Name: ____________________ Exam 3 - MAT345

1 Part I: True/False

Each problem is points for a total of 80 points. (10 problems worth 8 points each.)

Problem 1.1. For the exam, you need only indicate True or False. No justification is required. If you want to earn back some points, you can supply full justifications for **all** of the problems. You may earn back 50% of lost points.

a) True There is a 3×3 matrix with eigenvalues $\lambda_1 = 1/2$ and $\lambda_2 = -1/3$ with

$$E_{\lambda_1} = \text{span}\left\{\begin{bmatrix}1\\1\\1\end{bmatrix}, \begin{bmatrix}1\\0\\2\end{bmatrix}\right\} \text{ and } E_{\lambda_2} = \text{span}\left\{\begin{bmatrix}0\\0\\1\end{bmatrix}\right\}$$

This is true, and we can write down the matrix quite easily

$$A = \begin{bmatrix}1 & 1 & 0\\1 & 0 & 0\\1 & 2 & 1\end{bmatrix}\begin{bmatrix}1/2 & 0 & 0\\0 & 1/2 & 0\\0 & 0 & -1/3\end{bmatrix}\begin{bmatrix}1 & 1 & 0\\1 & 0 & 0\\1 & 2 & 1\end{bmatrix}^{-1}$$

b) False There is a 3×3 matrix with eigenvalues $\lambda_1 = 1/2$ and $\lambda_2 = -1/3$ with

$$E_{\lambda_1} = \text{span}\left\{\begin{bmatrix}1\\1\\1\end{bmatrix}, \begin{bmatrix}1\\0\\2\end{bmatrix}\right\} \text{ and } E_{\lambda_2} = \text{span}\left\{\begin{bmatrix}2\\1\\3\end{bmatrix}\right\}$$

If this were true, then $E_{\lambda_1} \cap E_{\lambda_2} = \{\mathbf{0}\}$ and as

$$\begin{bmatrix}2\\1\\3\end{bmatrix} = \begin{bmatrix}1\\1\\1\end{bmatrix} + \begin{bmatrix}1\\0\\2\end{bmatrix}$$

we see that this is false.

c) False If a 3×3 matrix A has eigenvalues $\lambda_1 = 1/2$ and $\lambda_2 = -1/3$, and no others, and

$$E_{\lambda_1} = \text{span}\left\{\begin{bmatrix}1\\1\\1\end{bmatrix}\right\} \text{ and } E_{\lambda_2} = \text{span}\left\{\begin{bmatrix}2\\1\\3\end{bmatrix}\right\}$$

Then A is diagonalizable.

In this case one of the eigenvalues has algebraic multiplicity 2, but geometric multiplicity 1, so A is deficient and hence not diagonalizable.

d) <u>True</u> If a 3×3 matrix has eigenvalues 1, $1/2$, and $-1/4$, then A is diagonalizable.

If an $n \times n$ matrix has n distinct eigenvalues, then it is diagonalizable.

e) <u>False</u> There is a 3×3 symmetric matrix with eigenvalues $\lambda_1 = 1/2$ and $\lambda_2 = -1/3$ with

$$E_{\lambda_1} = \text{span}\left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix} \right\} \text{ and } E_{\lambda_2} = \text{span}\left\{ \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}$$

This is false since $E_{\lambda_1} \perp E_{\lambda_2}$ if A is symmetric. But this is false in the present case.

f) <u>True</u> There is a 3×3 symmetric matrix with eigenvalues $\lambda_1 = 1/2$ and $\lambda_2 = -1/3$ with

$$E_{\lambda_1} = \text{span}\left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix} \right\} \text{ and } E_{\lambda_2} = \text{span}\left\{ \begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix} \right\}$$

This is true. To start with, notice $E_{\lambda_1} \perp E_{\lambda_2}$ and so we can find an orthonormal basis and produce the needed matrix as $A = U \Lambda U^T$. (This is fine for an explanation.)

Details follow:

$$\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} - \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}$$

and so

$$E_{\lambda_1} = \text{span}\left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} \right\} = \text{span}\left\{ \begin{bmatrix} 1/\sqrt{3} \\ 1/\sqrt{3} \\ 1/\sqrt{3} \end{bmatrix}, \begin{bmatrix} 0 \\ 1/\sqrt{2} \\ -1/\sqrt{2} \end{bmatrix} \right\}$$

$$E_{\lambda_2} = \text{span}\left\{ \begin{bmatrix} -2/\sqrt{6} \\ 1/\sqrt{6} \\ 1/\sqrt{6} \end{bmatrix} \right\}$$

So we have an orthonormal basis of eigenvectors for $\mathbb{R}^3$, and we have

$$A = \begin{bmatrix} 1/\sqrt{3} & 0 & -2/\sqrt{6} \\ 1/\sqrt{3} & 1/\sqrt{2} & 1/\sqrt{6} \\ 1/\sqrt{3} & -1/\sqrt{2} & 1/\sqrt{6} \end{bmatrix} \begin{bmatrix} 1/2 & 0 & 0 \\ 0 & 1/2 & 0 \\ 0 & 0 & -1/3 \end{bmatrix} \begin{bmatrix} 1/\sqrt{3} & 1/\sqrt{3} & 1/\sqrt{3} \\ 0 & 1/\sqrt{2} & -1/\sqrt{2} \\ -2/\sqrt{6} & 1/\sqrt{6} & 1/\sqrt{6} \end{bmatrix}$$

This turns out to be:

$$A = \frac{1}{36} \begin{bmatrix} -2 & 10 & 10 \\ 10 & 13 & -5 \\ 10 & -5 & 13 \end{bmatrix}$$

g) <u>True</u> For any $m \times n$ matrix A with real entries, $A^T A$ is diagonalizable with all eigenvalues being real and non-negative.

This is true and is the basis of the first step of the SVD decomposition.

h) False If A is an $n \times n$ real matrix, then there is an orthogonal (unitary and real) $n \times n$ real matrix U and real $n \times n$ diagonal Λ so that $A = U\Lambda U^T$.

This is, in general, false. This sort of looks like the Spectral Theorem, but that only applies to symmetric A, and this sort of looks like SVD, but that gives $A = U\Lambda V^T$ where $U \neq V$, is often true. As a counter-example, consider the rotation by 45° counter-clockwise matrix

$$A = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$$

This has complex eigenvalues $\lambda = \pm i$ since $p_A(t) = t^2 + 1 = (t - i)(t + i)$. So A cannot have the form stated.

i) True If A is an $n \times n$ real matrix, then there are orthogonal (unitary and real) $n \times n$ real matrices U and V and real $n \times n$ diagonal Λ so that $A = U\Lambda V^T$.

This is the statement of the SVD Theorem.

j) True For any symmetric matrix A with non-negative eigenvalues, we can find a symmetric matrix B, also with non-negative eigenvalues, so that $A = B^2$.

By the Spectral Theorem, $A = U\Lambda U^T$ and we can let $B = U\Lambda^{1/2}U^T$ so that

$$\begin{aligned} B^2 = (U\Lambda^{1/2}U^T)(U\Lambda^{1/2}U^T) &= U\Lambda^{1/2}(U^T U)\Lambda^{1/2}U^T \\ &= U\Lambda^{1/2}I\Lambda^{1/2}U^T = U\Lambda^{1/2}\Lambda^{1/2}U^T = U\Lambda U^T = A \end{aligned}$$

2 Part II: Computational (120 points; 5 problems, each worth 25 points.)

Show all computations so that you make clear what your thought processes are.

Problem 2.1 (20 pts)**.** Diagonalize A if possible. If not diagonalizable, explain why.

$$A = \begin{bmatrix} 8 & 3 & -3 \\ -16 & -7 & 7 \\ -4 & -3 & 3 \end{bmatrix}$$

The eigenvalues are $\lambda_1 = 2$ and $\lambda_2 = 0$.

$$E_2 = \mathrm{NS} \begin{bmatrix} 6 & 3 & -3 \\ -16 & -9 & 7 \\ -4 & -3 & 1 \end{bmatrix} = \mathrm{NS} \begin{bmatrix} 2 & 1 & -1 \\ 0 & -1 & -1 \\ 0 & -1 & -1 \end{bmatrix} == \mathrm{NS} \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix} = \mathrm{span} \left\{ \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} \right\}$$

and

$$E_0 = \mathrm{NS} \begin{bmatrix} 8 & 3 & -3 \\ -16 & -7 & 7 \\ -4 & -3 & 3 \end{bmatrix} = \mathrm{NS} \begin{bmatrix} 8 & 3 & 3 \\ 0 & -1 & 1 \\ 0 & -3/2 & 3/2 \end{bmatrix} = \mathrm{NS} \begin{bmatrix} 8 & 0 & 0 \\ 0 & -1 & 1 \\ 0 & 0 & 0 \end{bmatrix} = \mathrm{span} \left\{ \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \right\}$$

The matrix A is deficient.

Problem 2.2 (20 pts)**.** Let P be the plane through the origin and the points $(1, 1, 0)$ and $(1, -1, 0)$. Let $p : \mathbb{R}^3 \to P$ be the orthogonal projection onto P. That is, $p(x, y, z)$ is the point (a, b, c) in the plane P so that

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} - \begin{bmatrix} a \\ b \\ c \end{bmatrix} \perp P$$

p is a linear map and hence given by a matrix A.

a) What are the eigenvalues of A?

b) For each eigenvalue what is the associated eigenspace?

c) Given the answer to the first two questions, write $A = SDS^{-1}$.

Note: There is really nothing that you need to calculate here, this is just checking that you understand what eigenvalues and eigenvectors are, at least geometrically.

If a point, vector, $\boldsymbol{v}$, is in the plane P, then $p(\boldsymbol{v}) = \boldsymbol{v}$, i.e., the point does not move. That is $p(\boldsymbol{v}) = 1 \cdot \boldsymbol{v}$, and we see that 1 is an eigenvalue with eigenspace $E_1 = P$.

If a point, $\boldsymbol{v}$, is on the line, L, through the origin perpendicular to the plane, then $P(\boldsymbol{v}) = \boldsymbol{0}$, and so $\ker(P) = E_0 = L$ is the line L to the plane through the origin.

By inspection, $(0, 0, 1)$ is a point on the normal line and

$$E_0 = L = \text{span}\left\{ \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}$$

Similarly,

$$E_1 = P = \text{span}\left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} \right\}$$

and letting

$$S = \begin{bmatrix} 1 & 1 & 0 \\ 1 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \text{ and } D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

then we have

$$P(\boldsymbol{v}) = SDS^{-1}\boldsymbol{v}$$

Problem 2.3 (20 pts)**.** Suppose

$$A = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1/2 & 0 & 0 \\ 0 & -1/3 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 0 \end{bmatrix}^{-1}$$

Describe the "long-term" behavior of $A^n\boldsymbol{v}$ for an arbitrary point $\boldsymbol{v} \in \mathbb{R}^3$. More specifically, in the limit as $n \to \infty$ what happens to $A^n\boldsymbol{v}$.

Note: Depending on where $\boldsymbol{v}$ is in $\mathbb{R}^3$, there might be different long-term behavior.

For any $\boldsymbol{v}$, we may write $\boldsymbol{v} = a\boldsymbol{v}_1 + b\boldsymbol{v}_2 + c\boldsymbol{v}_3$ where

$$\boldsymbol{v}_1 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \quad \boldsymbol{v}_2 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \quad \boldsymbol{v}_3 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$$

and we see that

$$A^n\boldsymbol{v} = (1/2)^n a\boldsymbol{v}_1 + (-1/3)^n b\boldsymbol{v}_2 + (1)^n c\boldsymbol{v}_3$$

and since both $(1/2)^n$ and $(-1/3)^n$ approach 0 as n gets large, $A^n\boldsymbol{v}$ approaches $c\boldsymbol{v}$. All points are "attracted" to the line L.

More specifically,

$$\begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 0 \end{bmatrix}^{-1} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \frac{1}{2} \begin{bmatrix} 1 & -1 & 1 \\ -1 & 1 & 1 \\ 1 & 1 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

So $c = \frac{1}{2}(x+y-z)$ and so

$$A^n \begin{bmatrix} x \\ y \\ z \end{bmatrix} \to \frac{1}{2}(x+y-z) \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$$

For the next two problems, let

$$A = \begin{bmatrix} 3/2 & 3/2 \\ 3/2 & 3/2 \\ -1 & 1 \\ -1 & 1 \end{bmatrix}.$$

Problem 2.4 (20 pts)**.** Unitarily diagonalize the 2×2 matrix $A^T A$. That is, find V a unitary (real) matrix and diagonal Λ so that $A = V \Lambda V^T$. Make sure that $\Lambda = \left[\begin{smallmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{smallmatrix}\right]$ where $\lambda_1 \geq \lambda_2$.

Check your answer! Make sure that your V is unitary and check that $A^T A = V \Lambda V^T$. You need these in the next step so you want to double-check here to make sure that they are correct.

$$A^T A = \begin{bmatrix} 13/2 & 5/2 \\ 5/2 & 13/2 \end{bmatrix}$$

So $p_A(t) = (13/2 - t)^2 - (5/2)^2$ and

$$\begin{aligned} p_A(t) = 0 &\iff (13-2t)^2 - 5^2 = 0 \\ &\iff 4t^2 - 52t + (13^2 - 5^2) = 0 \\ &\iff 4t^2 - 52t + 12^2 = 0 \\ &\iff t^2 - 13t + 36 = (t-9)(t-4) = 0 \end{aligned}$$

So the eigenvalues are 9 and 4. So

$$\Lambda = \begin{bmatrix} 9 & 0 \\ 0 & 4 \end{bmatrix}$$

Now find the basis for the eigenspaces

$$E_9 = \mathrm{NS}(A - 9I) = \mathrm{NS} \begin{bmatrix} -5/2 & 5/2 \\ -5/2 & 5/2 \end{bmatrix} = \mathrm{span}\left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right\}$$

and

$$E_4 = \mathrm{NS}(A - 4I) = \mathrm{NS} \begin{bmatrix} 5/2 & 5/2 \\ 5/2 & 5/2 \end{bmatrix} = \mathrm{span}\left\{ \begin{bmatrix} -1 \\ 1 \end{bmatrix} \right\}$$

So

$$V = \begin{bmatrix} 1/\sqrt{2} & -1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}$$

Problem 2.5 (20 pts)**.** Find the SVD for A (same A as in the previous problem.) Explain how you get the singular values and the left singular vectors.

Check! When done, you should have unitary 4×4 matrix U, a diagonal 4×2 matrix Σ, and the unitary 2×2 matrix V (from above) so that $A = U\Sigma V^T$.

The singular values are just $\sigma_1 = \sqrt{9}$ and $\sigma_2 = \sqrt{4} = 2$ and so we know from the previous problem that

$$\Sigma = \begin{bmatrix} 3 & 0 \\ 0 & 2 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \text{ and } V = \begin{bmatrix} 1/\sqrt{2} & -1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}$$

We have

$$\boldsymbol{u}_1 = \frac{A\boldsymbol{v}_1}{\sigma_1} = \frac{1}{3}\begin{bmatrix} 3/\sqrt{2} \\ 3/\sqrt{2} \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \\ 0 \\ 0 \end{bmatrix}$$

and

$$\boldsymbol{u}_2 = \frac{A\boldsymbol{v}_1}{\sigma_2} = \frac{1}{2}\begin{bmatrix} 0 \\ 0 \\ 2/\sqrt{2} \\ 2/\sqrt{2} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix}$$

For $\boldsymbol{u}_3$ and $\boldsymbol{u}_4$ we need an orthonormal basis for NS(A)

$$\begin{aligned} \mathrm{NS}(A^T) &= \mathrm{NS}\begin{bmatrix} 3/2 & 3/2 & -1 & -1 \\ 3/2 & 3/2 & 1 & 1 \end{bmatrix} = \mathrm{NS}\begin{bmatrix} 3/2 & 3/2 & -1 & -1 \\ 0 & 0 & 2 & 2 \end{bmatrix} \\ &= \mathrm{NS}\begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix} = \mathrm{span}\left\{ \begin{bmatrix} 1 \\ -1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ -1 \\ 1 \end{bmatrix} \right\} \end{aligned}$$

So

$$\boldsymbol{u}_3 = \begin{bmatrix} 1/\sqrt{2} \\ -1/\sqrt{2} \\ 0 \\ 0 \end{bmatrix} \text{ and } \boldsymbol{u}_4 = \begin{bmatrix} 0 \\ 0 \\ 1/\sqrt{2} \\ -1/\sqrt{2} \end{bmatrix}$$

and so

$$U = \begin{bmatrix} 1/\sqrt{2} & 0 & 1/\sqrt{2} & 0 \\ 1/\sqrt{2} & 0 & -1/\sqrt{2} & 0 \\ 0 & 1/\sqrt{2} & 0 & 1/\sqrt{2} \\ 0 & 1/\sqrt{2} & 0 & -1/\sqrt{2} \end{bmatrix}$$

So

$$A = \begin{bmatrix} 1/\sqrt{2} & 0 & 1/\sqrt{2} & 0 \\ 1/\sqrt{2} & 0 & -1/\sqrt{2} & 0 \\ 0 & 1/\sqrt{2} & 0 & 1/\sqrt{2} \\ 0 & 1/\sqrt{2} & 0 & -1/\sqrt{2} \end{bmatrix} \begin{bmatrix} 3 & 0 \\ 0 & 2 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ -1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}$$

$$= \frac{1}{2} \begin{bmatrix} 1 & 0 & 1 & 0 \\ 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & -1 \end{bmatrix} \begin{bmatrix} 3 & 0 \\ 0 & 2 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}$$

$$= \frac{1}{2} \begin{bmatrix} 3 & 0 \\ 3 & 0 \\ 0 & 2 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}$$

$$= \frac{1}{2} \begin{bmatrix} 3 & 3 \\ 3 & 3 \\ -2 & 2 \\ -2 & 2 \end{bmatrix} \text{ Correct!}$$