Name: ____________________ Quiz 3 - MAT345

Problem 1 (15 points; 3 points each)**.** Decide if each of the following are true or false and provide a justification or counterexample in each case. A justification could consist of a theorem from the text. All vector spaces are assumed to be finite-dimensional here.

(a) False There is a unique least squares solution to $A\boldsymbol{x} = \boldsymbol{b}$.

This was in your homework. You showed that the set of least square solutions to $A\boldsymbol{x} = \boldsymbol{b}$ is exactly $\hat{\boldsymbol{x}} + \mathrm{NS}(A)$, where $\hat{\boldsymbol{x}}$ is any fixed least squares solution, that is, $\hat{\boldsymbol{x}}$ satisfies, $A^T A\hat{\boldsymbol{x}} = A^T\boldsymbol{b}$.

In general, there is no reason for $A^T A$ to be invertible, so $(A^T A)^{-1}$ need not even exist. If it does exist, then $\hat{x} = (A^T A)^{-1}A^T\boldsymbol{b}$ is unique.

What you do know is that there is a unique $\hat{\boldsymbol{b}}$ so that $\hat{\boldsymbol{b}}$ is the closest thing of the form $A\boldsymbol{x}$ to $\boldsymbol{b}$, in other words, $\|\hat{\boldsymbol{b}} - \boldsymbol{b}\|_2^2 = \min\{\|A\boldsymbol{x} - \boldsymbol{b}\|_2^2 \;\big|\; \boldsymbol{x} \in \mathbb{R}^n\}$ and a least-square solution is a a solution to $A\boldsymbol{x} = \hat{\boldsymbol{b}}$.

(b) True There is a unique $\boldsymbol{y}$ so that $\|\boldsymbol{y} - \boldsymbol{b}\|$ is minimal and $A\boldsymbol{x} = \boldsymbol{y}$.

This is is the conclusion of the main theorem about the existence of "least-square" solutions. This is covered in the notes and in the book.

(c) True If $\{\boldsymbol{u}_1, \ldots, \boldsymbol{u}_n\}$ is an orthonormal basis for V with respect to an inner product $\langle\cdot,\cdot\rangle : V \times V \to \mathbb{C}$ and $\boldsymbol{v} = \sum_{i=1}^n \alpha_i\boldsymbol{u}_i$, then $\|\boldsymbol{v}\|_2^2 = \sum_{i=1}^n |\alpha_i|^2$.

This is true and is essentially the Pythagorean Theorem. This is a computation

$$\begin{aligned}
\|\boldsymbol{v}\|_2^2 = \langle\boldsymbol{v}, \boldsymbol{v}\rangle &= \left\langle \sum_{i=1}^n \alpha_i\boldsymbol{u}_i, \sum_{i=1}^n \alpha_i\boldsymbol{u}_i \right\rangle \\
&= \sum_{i=1}^n \alpha_i \left\langle \boldsymbol{u}_i, \sum_{i=1}^n \alpha_i\boldsymbol{u}_i \right\rangle \\
&= \sum_{i=1}^n \alpha_i \left(\sum_{j=1}^n \bar{\alpha}_j \langle\boldsymbol{u}_i, \boldsymbol{u}_j\rangle \right) \\
&= \sum_{i=1}^n \sum_{j=1}^n \alpha_i\bar{\alpha}_j\delta_{i,j} \\
&= \sum_{i=1}^n \alpha_i\bar{\alpha}_i = \sum_{i=1}^n |\alpha_i|^2
\end{aligned}$$

Here

$$\delta_{i,j} = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}$$

(d) False All norms $\|\cdot\| : \mathbb{R}^n \to [0, \infty)$ on $\mathbb{R}^n$ come from an inner product.

This is false. The book provides several norms. For a norm $\|\cdots\|$ to be given by an inner product it must satisfy the parallelogram law $\|\boldsymbol{u} - \boldsymbol{v}\|^2 + \|\boldsymbol{u} + \boldsymbol{v}\|^2 = 2\|\boldsymbol{u}\|^2 + 2\|\boldsymbol{v}\|^2$.

Of all of the norms $\|\cdot\|_p$ for $1 \le p \le \infty$, the only one that satisfies the parallelogram law is $p = 2$, this is the only one given by an inner product.

For example, $\|(a,b)\|_\infty = \max\{|a|,|b|\}$ and clearly we can choose a, b, c, and d so that

$$\max\{|a-c|,|b-d|\} + \max\{|a+c|,|b+d|\} \neq 2\max\{|a|,|b|\} + 2\max\{|c|,|d|\}$$

Let $(a,b) = (1,3)$ and $(c,d) = (2,1)$, then

$$\begin{aligned} \max\{|1-2|,|3-1|\} + & \max\{|1+2|,|3+1|\} = 2+4 \\ & \neq 2\max\{|1|,|3|\} + 2\max\{|2|,|1|\} = 6+4 \end{aligned}$$

(e) If $\mathcal{C} = \{\boldsymbol{u}_1, \dots, \boldsymbol{u}_n\}$ is an orthonormal basis for V with respect to an inner product $\langle\cdot,\cdot\rangle : V \times V \to \mathbb{C}$ and $\boldsymbol{v} \in V$, then for any $(c_1, \dots, c_n) = [\boldsymbol{v}]_{\mathcal{C}}$, $c_i = \langle v, u_i\rangle$.

This is another computation. Say $(c_1,\dots,c_n) = [\boldsymbol{v}]_{\mathcal{C}}$, then $\boldsymbol{v} = \sum_{i=1}^n c_i \boldsymbol{u}_i$. Now just compute

$$\langle \boldsymbol{v}, \boldsymbol{u}_j\rangle = \left\langle \sum_{i=1}^n c_i \boldsymbol{u}_i, \boldsymbol{u}_j \right\rangle = \sum_{i=1}^n c_i \langle \boldsymbol{u}_i, \boldsymbol{u}_j\rangle = \sum_{i=1}^n c_i \delta_{i,j} = c_j$$

Problem 2 (25 points)**.** You are given some data points $\{(x_i, y_i) \mid i = 1, \dots, N\}$ and want to model the data by a function of the form $f(x) = a + bx + c\cos(x) + d\sin(x)$. This involves setting up a matrix A and finding a least-squares solution to $A\mathbf{x} = \mathbf{b}$.

a) (5 points) What is $\mathbf{b}$? (In terms of the data.)

$\boldsymbol{b}$ is the vector of y_i's

$$\boldsymbol{b} = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_N \end{bmatrix}$$

b) (8 points) What is A? (Again, in terms of the data.)

Let $\boldsymbol{X}$ be the vector of x_i's.

$$\boldsymbol{X} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_N \end{bmatrix}$$

A is the matrix given my columns is

$$A = \begin{bmatrix} \mathbf{1} \; \boldsymbol{X} \; \cos(\boldsymbol{X}) \; \sin(\boldsymbol{X}) \end{bmatrix} = \begin{bmatrix} 1 & x_1 & \cos(x_1) & \sin(x_1) \\ 1 & x_2 & \cos(x_2) & \sin(x_2) \\ \vdots & \vdots & \vdots & \vdots \\ 1 & x_N & \cos(x_N) & \sin(x_N) \end{bmatrix}$$

c) (7 points) Suppose you have the least-squares solution $\hat{\mathbf{x}}$. What is $f(x)$? (In terms of $\hat{\mathbf{x}}$)

Let

$$\hat{\boldsymbol{x}} = \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix}$$

then $A\hat{\boldsymbol{x}} = a\mathbf{1} + b\boldsymbol{X} + c\cos(\boldsymbol{X}) + d\sin(\boldsymbol{X}) = \hat{\boldsymbol{b}}$ is the vector closest to $\boldsymbol{b}$ that is of the form $A\boldsymbol{x}$ for some $\boldsymbol{x}$. The model is

$$f(x) = a + bx + c\cos(x) + d\sin(x)$$

So $f(\boldsymbol{X}) = \hat{\boldsymbol{b}}$.

d) (5 points) What is the relationship between $\hat{\mathbf{b}} = A\hat{\mathbf{x}}$ and $\mathbf{b}$?

This is described above, $\hat{\boldsymbol{b}}$ is the vector in $\mathbb{R}^N$ closest to $\boldsymbol{b}$ where $\hat{\boldsymbol{b}}$ is of the form $A\boldsymbol{x}$ for some $\boldsymbol{x} \in \mathbb{R}^4$.