Math 571 - Homework 1 (Due 5/17/23)

Problem 1 (R:1:2*). Show that for any positive integer n, if n is not a perfect square, then $\sqrt{n}$ is irrational.

Problem 2 (R:1:4*). Let E be a non-empty subset of an ordered set $(S, <)$; suppose that α is a lower bound for E in S and β is an upper-bound for E in S. Show that $\alpha \leq \beta$. Can $\alpha = \beta$? Is this still true if $E = \emptyset$?

Problem 3 (R:1:5). Let A be a non-empty set of real numbers bounded below. Let $-A = \{-a \mid a \in A\}$. Show that

$$\inf(A) = -\sup(-A)$$

Problem 4 (R:1:6). Fix $b > 1$.

(a) If n, m, p, q are integers, $n, q > 0$, and $r = m/n = p/q$, prove that

$$(b^m)^{1/n} = (b^p)^{1/q}.$$

Explain why it makes sense to define $b^r = (b^m)^{1/n}$.

(b) Prove that $b^{r+s} = b^r b^s$ if r and s are rational.

(c) If $x \in \mathbb{R}$, define $B(x) = \{b^t \mid t \in \mathbb{Q} \wedge t \leq x\}$. Prove that

$$b^r = \sup(B(r))$$

when r is rational. Explain why it makes sense to define

$$b^x = \sup(B(x))$$

for every real x.

(d) Prove that $b^{x+y} = b^x b^y$ for every real x and y.

Problem 5 (R:1:8). Show that $\mathbb{C}$ can not be made into an ordered field.

Problem 6 (R:1:14*). Show that for $w, z \in \mathbb{C}$

$$|w + z|^2 + |w - z|^2 = 2|w|^2 + 2|z|^2.$$

Use this to compute $|1 + z|^2 + |1 - z|^2$ given that $|z| = 1$.

Problem 7 (R:1:17)**.** Show that for $x, y \in \mathbb{R}^k$,

$$\|x+y\|_2^2 + \|x-y\|_2^2 = 2\|x\|_2^2 + 2\|y\|_2^2. \qquad \text{(Parallelogram Law)}$$

How does this generalize the Pythagorean theorem?

Problem 8 (R:1:18)**.** Show that if $k \geq 2$ and $x \in \mathbb{R}^k$, there is $y \in \mathbb{R}^k$, $y \neq 0$ such that $x \bullet y = 0$.

If you recall how this goes, drop the $k \geq 2$ and show that given any non-zero pairwise orthogonal $x_1, x_2, \dots, x_l$ ($l \leq k$) in $\mathbb{R}^k$, you can find $x_{l+1}, \dots, x_k$ so that $x_1, x_2, \dots, x_k$ are pairwise orthogonal.