Math 571 - Exam 1

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Notation/Definition: Let (X,d) be a metric space for $A, B \subset X$ define $d(A,B) = \inf\{d(a,b) \mid a \in A \text{ and } b \in B\}$. Set $d(a,B) = d(\{a\},B)$.

Question 1 (12 points)**.** Let (X,d) be a metric space, prove that

a) For any closed set F and $x \notin F$, $d(x,F) > 0$.

Suppose $d(x,F) = 0$, then there is $x_i \in F$ such that $\lim_i d(x,x_i) = 0$, but then, $\lim_i x_i \to x$ so $x \in F$, which is a contradiction.

b) For any compact K and closed F with $K \cap F = \emptyset$, $d(K,F) > 0$.

For $x \notin F$ there are open sets U and V with $x \in U$, $F \subseteq V$, and $V \cap U = \emptyset$. Suppose $d(x,F) = a$, then let $U = N_{a/2}(x)$ and $V = \bigcup_{y \in F} N_{a/2}(y)$. Clearly, $x \in U$ and $F \subseteq V$. If $z \in U \cap V$, then $z \in N_{a/2}(x)$ and $z \in N_{a/2}(y)$ for some $y \in F$. But then $d(x,y) \leq d(x,z) + d(z,y) < a$, which is a contradiction.

Now for each $x \in K$ let U_x, V_x be a pair of open sets so that $x \in U_x$, $F \subseteq V_x$, and $U_x \cap V_x = \emptyset$. since K is compact, let $\{U_{x_1}, \ldots, U_{x_n}\}$ cover K. Define $U = \bigcup_{i=1}^n U_{x_i}$ and $V = \bigcap_{i=1}^n V_{x_i}$. Then $K \subseteq U$, $F \subseteq V$, and $K \cap V = \emptyset$.

c) Can the assumption that K is compact be replaced by K closed in (b)? That is, is there a metric space (X,d) and closed sets A, B so that $A \cap B = \emptyset$ and yet $d(A,B) = 0$?

It is simple to see that compactness is required here. **Example 1:** Consider $A = \{(x, 1/x) \mid x > 0\}$ and $B = \{(x, -1/x) \mid x > 0\}$. Clearly, $d(A,B) = 0$ and as $x \mapsto 1/x$ is continuous, A and B are closed.

Example 2: K closed and bounded also does not suffice, but to see this, we must look into a space where closed and bounded does not imply compact. We don't have to look far. Consider $X = (0,1)$, the open unit interval. Here X is closed (in X) and bounded but not compact. Consider $F = \{1/i \mid i > 0, i \in \mathbb{N}, \text{and even}\}$ and $K = \{1/i \mid i \in \mathbb{N} \text{ and odd}\}$. Clearly, $K \cap H = \emptyset$ yet $d(1/i, 1/i+1) \to 0$ so $d(F,K) = 0$.

Note: It is however true that for A, B closed with $A \cap B = \emptyset$, there are U, V open so that $A \supseteq U$, $B \supseteq V$, and $U \cap V = \emptyset$. This is the ***normality*** property.

Recall: In a metric space (X,d), $\text{diam}(A) = \sup\{d(a,b) \mid a, b \in A\}$.

Question 2 (12 pts)**.** Let (X,d) be a metric space prove or disprove each of the following:

a) $\text{diam}(A) = \text{diam}(\text{Cl}(A))$.

Let $x, y \in \mathrm{Cl}(A)$ and $\epsilon > 0$ it is easy to see that $d(x,y) < \mathrm{diam}(A) + \epsilon$. since this is true for all $\epsilon > 0$, $d(x,y) \leq \mathrm{diam}(A)$ and so $\mathrm{diam}(\mathrm{Cl}(A)) \leq \mathrm{diam}(A)$.

b) $\mathrm{diam}(A) = \mathrm{diam}(\mathrm{Int}(A))$.

This is trivially false. For example in $\mathbb{R}$ let $A = \{a, b\}$, then $\mathrm{diam}(A) = |b - a|$, but $\mathrm{Int}(A) = \emptyset$, so $\mathrm{diam}(\mathrm{Int}(A)) = 0$.

Question 3 (12 pts)**.** Let (X, d) be a metric space and $(x_i)_{i \in \mathbb{N}}$ and $(y_i)_{i \in \mathbb{N}}$ be two Cauchy sequences. Show that $(d(x_i, y_i))_{i \in \mathbb{N}}$ converges.

$d(x_i, y_i) \leq d(x_i, x_j) + d(x_j, y_j) + d(y_j, y_j')$ so that $d(x_i, y_i) - d(x_j, y_j) \leq d(x_i, x_j) + d(y_i, y_j)$. Swapping the rolls of i and j gives $d(x_j, y_j) - d(x_i, y_i) \leq d(x_i, x_j) + d(y_i, y_j)$ so we get

$$|d(x_j, y_j) - d(x_i, y_i)| \leq d(x_i, x_j) + d(y_i, y_j)$$

Now for $\epsilon > 0$ take N so that $d(x_i, x_j) < \epsilon/2$ and $d(y_i, y_j) < \epsilon/2$ for $i, j > N$, then for $i, j > N$

$$|d(x_j, y_j) - d(x_i, y_i)| \leq d(x_i, x_j) + d(y_i, y_j) < \epsilon.$$

so $(d(x_i, y_i))$ is a Cauchy sequence.

For the next problem, $(x_{i_k})_{k=0}^{\infty}$ is a **subsequence** of $(x_i)_{i=0}^{\infty}$ means $i_0 < i_1 < \cdots$. A sequence $(x_i)_{i=0}^{\infty}$ is **monotone increasing** iff $x_0 \leq x_1 \leq x_2 \cdots$. Similarly define **monotone decreasing**. A sequence is **monotone** iff it is either monotone increasing or monotone decreasing.

Question 4 (12 pts)**.** Show that every infinite sequence of real numbers has a monotone subsequence that converges to $\limsup_i x_i$.

Define $\alpha_i = \sup_i \{x_j \mid j \geq i\}$. Clearly $\alpha_0 \geq \alpha_1 \geq \cdots$, that is (α_i) is a monotonically decreasing sequence. Let $\alpha = \inf_i \alpha_i$, noting that $\alpha = -\infty$ and $\alpha = \infty$ are both possible.

Suppose there is a subsequence (α_{i_j}) that is strictly decreasing, that is $\alpha_{i_j} > \alpha_{i_{j+1}}$. In this case we get $i_j \leq m_j < i_{j+1}$ so that $\alpha_{i_j} \geq x_{m_i} > \alpha_{i_{j+1}}$. In this case (x_{m_i}) is a strictly descending sequence and $\lim_{x_{m_i}} = \alpha$.

The other case is that $\alpha_i = \alpha$ for all large enough i. It could be that $\alpha \in \{x_j \mid j \geq i\}$ for all large enough i. In this case, there is $x_{j_i} = \alpha$ with $i_0 < i_1 < \cdots$. In this case the constant sequence (α) is an infinite constant (monotonic) subsequence of (x_i). If this fails to be the case, then for all large enough i, and for all $\epsilon > 0$, there is $x_j > \alpha - \epsilon$ for some $j > i$. So now we can build $x_{i_0} < x_{i_1} < \cdots$, a strictly increasing monotonic sequence, so that $\lim_j x_{i_j} = \alpha$.

So there are three main cases, either there is a stictly increasing subsequence converging to α, a strictly decreasing subsequence converging to α, or else the constant sequence (α) is a subsequence.

NOTE: The same is true for $\liminf_i x_i$.

Question 5 (Is supremum "linear"; 12 pts)**.** For $A, B \subseteq \mathbb{R}$, is it true that

i) $\sup(\alpha A) = \alpha \sup(A)$ for $\alpha \geq 0$, and

This is true. This is clear if $\alpha = 0$, so assume $\alpha > 0$. There are two things to show, namely, (1) $\sup(\alpha A) \leq \alpha \sup(A)$ and (2) $\sup(\alpha A) \geq \alpha \sup(A)$. This means that we must show (1′) $\alpha \sup(A)$ is an upper bound of αA and (2′) $\frac{1}{\alpha} \sup(\alpha A)$ is an upper bound of A. (2′) is equivalent to $\sup(\alpha A)$ is an upper bound of αA, but this is clear.

For (1'), let $a \in A$, then $a \leq \sup(A)$ and so $\alpha a \leq \alpha \sup(A)$. Thus $\alpha A \leq \alpha \sup(A)$ and we get that $\alpha \sup(A)$ is an upper bound of αA.

ii) $\sup(A + B) = \sup(A) + \sup(B)$.

Again there are two things to show. (1) $\sup(A+B) \geq \sup(A)+\sup(B)$ and (2) $\sup(A+B) \leq \sup(A) + \sup(B)$. As before, (2) is equivalent to (2′) $\sup(A) + \sup(B)$ is an upper bound on $A+B$ and this is clear since if $a \in A$ and $b \in B$, then $\sup(A)+\sup(B) \geq a+b$.

For (1), suppose $\sup(A)+\sup(B) > \sup(A+B)$, then $\sup(A)+b > \sup(A+B)$ for some $b \in B$. Applying this logic a second time we get $a \in A$ such that $a + b > \sup(A + B)$. this is absurd, so it must be that $\sup(A) + \sup(B) \leq \sup(A + B)$.

Question 6 (Compact sets get crowded; 15 pts)**.** Show that if X is compact, then for any $\epsilon > 0$, there is $N > 0$ so that for all $S \subset X$ with $|S| \geq N$, there are two points in S whose distance is $< \epsilon$.

Consider the open cover $\mathcal{O} = \{N_{\frac{\epsilon}{2}}(x) \mid x \in X\}$ of X. Let $\mathcal{O}' = \{N_{\frac{\epsilon}{2}}(x_i) \mid i = 1, \ldots, N\}$ be a finite open subcover. Let $S \subset X$ with $|S| > N$. By the pigeonhole principle, there are at least two elements $s, s' \in S$ which must fall in the same nbhd $N_{\frac{\epsilon}{2}}(x_i)$ for some i, so that $d(s, s') \leq d(s, x_i) + d(x_i, s') < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$.