Name: ____________________ Quiz 1 - MAT345

Problem 1.1 (15 points; 3 points each)**.** Decide if each of the following is true or false. You do not need to provide reasons.

(a) True Suppose a matrix B comes from a sequence of elementary row operations applied to a matrix A. Then $A\boldsymbol{x} = \mathbf{0} \iff B\boldsymbol{x} = \mathbf{0}$.

This was essentially the point of elementary row operations. They came from elementary operations on systems of equations that do not change the solution set.

(b) False $(A+B)(A-B) = A^2 - B^2$ for all $n \times n$ matrices A and B.

Essentially choosing random 2×2 matrices will provide a counterexample. For a specific example, take $A = \left[\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right]$ and $B = \left[\begin{smallmatrix} 0 & 0 \\ 1 & 0 \end{smallmatrix}\right]$.

(c) True Given that

$$\operatorname{rref}(A) = \begin{bmatrix} 1 & -2 & 0 & 2 & 0 \\ 0 & 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

The solution to $A\boldsymbol{x} = \mathbf{0}$ is the set of $\boldsymbol{x}$ such that

$$\boldsymbol{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = s \begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + t \begin{bmatrix} -2 \\ 0 \\ -2 \\ 1 \\ 0 \end{bmatrix} \quad \text{for } s, t \in \mathbb{R}$$

x_2 and x_4 are free, since these are not columns of pivots. So set $x_2 = s$ and $x_4 = t$, then what is left is

$$\begin{aligned} x_5 &= 0 \\ x_4 &= t \\ x_3 + 2t = 0 \rightarrow x_3 &= -2t \\ x_2 &= s \\ x_1 - 2s + 2t = 0 \rightarrow x_1 &= 2s - 2t \end{aligned}$$

(d) False A is in row echelon form

$$A = \begin{bmatrix} 2 & -2 & 1 & 2 & 0 \\ 0 & 0 & 4 & 2 & 3 \\ 0 & 0 & 0 & 0 & 3 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

The $(3,5)$ entry is a pivot, so to be in echelon form the 1 in the $(4,5)$ entry should be 0.

(e) True Let A a 3×6 matrix, and let B be the result of performing the elementary row operation $R_2 - 3R_1 \to R_2$. Then $B = EA$ where

$$E = \begin{bmatrix} 1 & 0 & 0 \\ -3 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Letting the i^{th} row of A be $A_{i,:}$ we have

$$EA = \begin{bmatrix} 1 & 0 & 0 \\ -3 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} A = \begin{bmatrix} \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} A \\ \begin{bmatrix} -3 & 1 & 0 \end{bmatrix} A \\ \begin{bmatrix} 0 & 0 & 1 \end{bmatrix} A \end{bmatrix}$$

So the 2^{nd} row of EA is $\begin{bmatrix} -3 & 1 & 0 \end{bmatrix} A$ which is

$$\begin{bmatrix} -3 & 1 & 0 \end{bmatrix} \begin{bmatrix} A_{1,:} \\ A_{2,:} \\ A_{3,:} \end{bmatrix} = (-3)A_{1,:} + (1)A_{2,:} + (0)A_{3,:} \qquad \text{(I.e. } R_2 - 3R_1\text{)}$$

Problem 1.2 (25 points)**.** Solve $A\boldsymbol{x} = \mathbf{0}$ for

$$A = \begin{bmatrix} 2 & -2 & 2 & 2 & 1 \\ -4 & 2 & -5 & -3 & -4 \\ -2 & 4 & -1 & -5 & 1 \\ -4 & 6 & -3 & -7 & 0 \end{bmatrix}$$

Follow the procedure discussed in class

- (10 points) Use elementary row operations to reduce to an echelon matrix.
- (10 points) Write down the resulting triangular system and use **back-substitution** to solve.
- (5 points) Write out your solution as a linear combination of vectors.

Reduction to echelon form requires six row manipulations:

This part is 10/25 points.

$$\begin{bmatrix} 2 & -2 & 2 & 2 & 1 \\ -4 & 2 & -5 & -3 & -4 \\ -2 & 4 & -1 & -5 & 1 \\ -4 & 6 & -3 & -7 & 0 \end{bmatrix} \underset{\substack{R_2+2R_1\to R_2 \\ R_3+R_1\to R_3 \\ R_4+2R_2\to R_4}}{\longrightarrow} \begin{bmatrix} 2 & -2 & 2 & 2 & 1 \\ 0 & -2 & -1 & 1 & -2 \\ 0 & 2 & 1 & -3 & 2 \\ 0 & 2 & 1 & -3 & 2 \end{bmatrix}$$

$$\underset{\substack{R_3+R_2\to R_3 \\ R_4+R_2\to R_4}}{\longrightarrow} \begin{bmatrix} 2 & -2 & 2 & 2 & 1 \\ 0 & -2 & -1 & 1 & -2 \\ 0 & 0 & 0 & -2 & 0 \\ 0 & 0 & 0 & -2 & 0 \end{bmatrix}$$

$$\underset{R_4-R_3\to R_4}{\longrightarrow} \begin{bmatrix} 2 & -2 & 2 & 2 & 1 \\ 0 & -2 & -1 & 1 & -2 \\ 0 & 0 & 0 & -2 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Back-substitution: So x_3 and x_5 are the free variables. Let $x_3 = s$ and $x_5 = t$. Back substitution gives

$$\begin{aligned} x_5 &= t \\ x_4 &= 0 \\ x_3 &= t \\ -2x_2 - s + 0 - 2t = 0 \to x_2 &= -\frac{1}{2}s - t \\ 2x_1 - 2\big(-\frac{1}{2}s - t\big) + 2s + t = 2x_1 + 3s + 3t = 0 \to x_1 &= -\frac{3}{2}s - \frac{3}{2}t \end{aligned}$$

Final answer as linear combination of vectors:

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_4 \end{bmatrix} = \begin{bmatrix} -\frac{3}{2}s - \frac{3}{2}t \\ -\frac{1}{2}s - t \\ s \\ 0 \\ t \end{bmatrix} = s \begin{bmatrix} -\frac{3}{2} \\ -\frac{1}{2} \\ 1 \\ 0 \\ 0 \end{bmatrix} + t \begin{bmatrix} -\frac{3}{2} \\ -1 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

This final answer is the remaining 5/25 points.