

Name: _____

Exam 2 - MAT345

1 True/False (50 points; 5 points each)

Recall: A and B are equivalent if there is a sequence of elementary row operations leading from A to B , or equivalently, $B = MA$ for some invertible matrix M . This is different from $A \sim B$ (A and B are *similar* which means $B = S^{-1}AS$ for some invertible S).

Problem 1.1. In class, you need only provide a T/F (make it clear!) As usual, you may earn back up to 50% of the lost points by supplying justifications afterward.

False The collection of 3×4 echelon matrices is a subspace of $\mathbb{R}^{3 \times 4}$.

True The set of $n \times n$ matrices with all diagonal elements being 0 is a subspace of $\mathbb{R}^{n \times n}$.

True Consider the map $L : \mathbb{R}^{m \times n} \rightarrow \mathbb{R}^m$ defined by $L(A)_i = \text{ave}(A_{i,*})$, that is, the i^{th} entry of $L(A)$ is the average of the i^{th} row of A . L is a linear map.

False For all linear $L : V \rightarrow W$, if $\{\mathbf{v}_1, \dots, \mathbf{v}_k\}$ is independent, then $\{L(\mathbf{v}_1), \dots, L(\mathbf{v}_k)\}$ is independent.

True For all linear $L : V \rightarrow W$, if $\{L(\mathbf{v}_1), \dots, L(\mathbf{v}_k)\}$ is independent, then $\{\mathbf{v}_1, \dots, \mathbf{v}_k\}$ is independent.

False There are subspaces $V_0 = P_3 \supsetneq V_1 \supsetneq V_2 \supsetneq V_3 \supsetneq V_4 \supsetneq V_5 = \{\mathbf{0}\}$ where each V_i is a proper subspace of V_{i-1} .

True Given any basis $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$, from \mathbb{R}^4 and any four matrices $M_1, M_2, M_3, M_4 \in \mathbb{R}^{2 \times 3}$ there is a unique linear transformation $L : \mathbb{R}^4 \rightarrow \mathbb{R}^{2 \times 3}$ where $L(\mathbf{v}_i) = M_i$.

True Suppose $L : P_5 \rightarrow \mathbb{R}^4$ is linear and onto, that is, $\text{Im}(L) = \mathbb{R}^4$. Then $\dim(\ker(L)) = 2$.

Recall P_5 is the space of polynomials of degree ≤ 5 .

True Let

$$\mathcal{B} = \left\{ \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ 2 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} \right\}$$

be a basis for \mathbb{R}^3 . Then for $\mathbf{v} = \begin{bmatrix} 3 \\ 3 \\ 4 \end{bmatrix}$

$$[\mathbf{v}]_{\mathcal{B}} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

False $L : \mathbb{R}^{3 \times 3} \rightarrow \mathbb{R}^3$ is given by $L(A) = A\mathbf{b}$ for a $\mathbf{b} \in \mathbb{R}^3$. If \mathcal{B} is a basis for $\mathbb{R}^{3 \times 3}$, then $[L]_{\mathcal{B}} = \mathbf{b}$.

2 Multiple Choice (30 points; 10 points each)

Each correct box counts for two points.

Problem 2.1 (10 points). Which of the following are equivalent to “ A is *equivalent* to B ”? Mark ‘Y’ if equivalent and ‘N’ if not.

☐ Y B results from a series of row operations from A .

☐ N $B = AM$ for some invertible matrix M .

☐ Y $B = MA$ for some invertible matrix M .

☐ N $\text{CS}(A) = \text{CS}(B)$.

☐ Y $\text{RS}(A) = \text{RS}(B)$.

Problem 2.2 (10 points). Which of the following are equivalent to A is invertible for an $n \times n$ matrix A . Mark ‘Y’ if equivalent and ‘N’ if not.

☐ Y A is equivalent to I .

☐ N $\dim(\text{RS}(A)) = \dim(\text{CS}(A))$.

☐ Y $\text{NS}(A) = \{\mathbf{0}\}$.

☐ Y $A\mathbf{x} = \mathbf{b}$ has at least one solution for all $\mathbf{b} \in \mathbb{R}^n$.

☐ Y $A\mathbf{x} = \mathbf{b}$ has a unique solution for some \mathbf{b} .

Problem 2.3. Which of the following implies that $\mathcal{B} = \{\mathbf{v}_1, \dots, \mathbf{v}_n\}$ is linearly independent. Mark ‘Y’ if the property implies \mathcal{B} is independent, ‘N’ otherwise.

☐ N For every \mathbf{v} in V , \mathbf{v} can be written as a linear combination of vectors in \mathcal{B} , i.e., there is $\alpha_i \in \mathbb{R}$ so that $\mathbf{v} = \sum_{i=1}^n \alpha_i \mathbf{b}_i$.

☐ Y If $\sum_{i=1}^n \alpha_i \mathbf{b}_i = \sum_{i=1}^n \beta_i \mathbf{b}_i$, then $\alpha_i = \beta_i$ for all i .

☐ Y $\mathbf{b}_i \notin \text{span}(\mathcal{B} - \{\mathbf{b}_i\})$, that is, \mathbf{b}_i is not a linear combination of the other vectors in \mathcal{B} .

☐ Y There is a linearly independent set $\mathcal{C} = \{\mathbf{c}_1, \dots, \mathbf{c}_n\}$ so that $\mathcal{C} \subset \text{span}(\mathcal{B})$.

☐ N There is a linearly independent set $\mathcal{C} = \{\mathbf{c}_1, \dots, \mathbf{c}_n\}$ so that $\mathcal{B} \subset \text{span}(\mathcal{C})$.

3 Computational (80 points; 20 points each)

Show all computations so that you make clear what your thought processes are.

Problem 3.1 (20 pts). Consider A given by

$$A = \begin{bmatrix} -2 & 4 & -4 & -4 & 4 \\ -8 & 16 & -15 & -18 & 18 \\ -8 & 16 & -11 & -26 & 27 \\ -4 & 8 & -8 & -8 & 4 \end{bmatrix}$$

Find a basis for each of $\text{NS}(A)$, $\text{CS}(A)$, and $\text{RS}(A)$.

$$\text{rref}(A) = \begin{bmatrix} 1 & -2 & 0 & 6 & 0 \\ 0 & 0 & 1 & -2 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

From this we know:

$$\text{CS}(A) = \text{span}\{(-2, -8, -8, -4), (-4, -15, -11, -8), (4, 18, 27, 4)\}$$

$$\text{RS}(A) = \text{span}\{(1, -2, 0, 6, 0), (0, 0, 1, -2, 0), (0, 0, 0, 0, 1)\}$$

Note: $\text{RS}(A)$ is not the span of the first three rows of A .

To find a basis for $\text{NS}(A)$ we are looking for solutions to $Ax = 0$. First, we have back-substitution: x_2 and x_4 are free, let $x_2 = s$ and $x_4 = t$, then

$$\boxed{x_5 = 0}$$

$$\boxed{x_4 = t}$$

$$x_3 - 2t = 0 \rightarrow \boxed{x_3 = 2t}$$

$$\boxed{x_2 = s}$$

$$x_1 - 2s + 6t = 0 \rightarrow \boxed{x_1 = 2s - 6t}$$

Any vector x satisfying, $Ax = 0$ can be written as:

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} 2s - 6t \\ s \\ 2t \\ t \\ 0 \end{bmatrix} = s \begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + t \begin{bmatrix} -6 \\ 0 \\ 2 \\ 1 \\ 0 \end{bmatrix}$$

So $\{(2, 1, 0, 0, 0), (-6, 0, 2, 1, 0)\}$ is a basis for $\text{NS}(A)$, that is,

$$\text{NS}(A) = \text{span}\{(2, 1, 0, 0, 0), (-6, 0, 2, 1, 0)\}$$

Problem 3.2 (20 pts). Consider $L : P_3 \rightarrow P_6$ given by $L(p(x)) = q(x)p(x)$ where $q(x) = a_0 + a_1x + a_2x^2 + a_3x^3$.

- a) (8 points) Show that L is a linear map.
- b) (8 points) Give the matrix $[L]$ where the standard basis is used for both P_3 and P_6 . Just to be definite, the standard basis for P_k is $\mathcal{E} = \{1, x, x^2, \dots, x^k\}$.
- c) (4 points) With $p(x) = b_0 + b_1x + b_2x^2 + b_3x^3$ compute $[q(x)p(x)]$ using $[L]$ and $[p]$.

To see that L is linear we just note that for $p(x), h(x) \in P_3$ and $c, d \in \mathbb{R}$ we have

$$\begin{aligned} L(c \cdot p(x) + d \cdot h(x)) &= q(x)(c \cdot p(x) + d \cdot h(x)) \\ &= c \cdot (q(x)p(x)) + d \cdot (q(x)h(x)) \\ &= c \cdot L(p(x)) + d \cdot L(h(x)) \end{aligned}$$

so linearity of L is shown.

To compute $[L]$, first notice that $\dim(P_3) = 4$ and $\dim(P_6) = 7$ so $[L]$ is 7×4 (a good sanity check on our solution).

$$[L] = \begin{bmatrix} [q(x)] & [q(x) \cdot x] & [q(x) \cdot x^2] & [q(x) \cdot x^3] \end{bmatrix} = \begin{bmatrix} a_0 & 0 & 0 & 0 \\ a_1 & a_0 & 0 & 0 \\ a_2 & a_1 & a_0 & 0 \\ a_3 & a_2 & a_1 & a_0 \\ 0 & a_3 & a_2 & a_1 \\ 0 & 0 & a_3 & a_2 \\ 0 & 0 & 0 & a_3 \end{bmatrix}$$

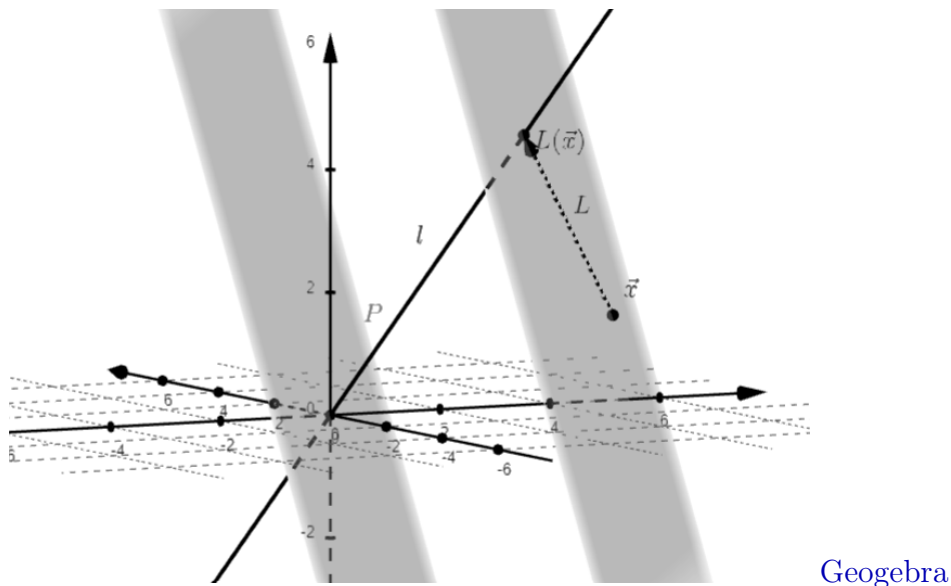
Finally,

$$[q(x)p(x)] = [L(p(x))] = [L][p(x)] = \begin{bmatrix} a_0 & 0 & 0 & 0 \\ a_1 & a_0 & 0 & 0 \\ a_2 & a_1 & a_0 & 0 \\ a_3 & a_2 & a_1 & a_0 \\ 0 & a_3 & a_2 & a_1 \\ 0 & 0 & a_3 & a_2 \\ 0 & 0 & 0 & a_3 \end{bmatrix} \begin{bmatrix} b_0 \\ b_1 \\ b_2 \\ b_3 \end{bmatrix} = \begin{bmatrix} a_0b_0 \\ a_1b_0 + a_0b_1 \\ a_2b_0 + a_1b_1 + a_0b_2 \\ a_3b_0 + a_2b_1 + a_1b_2 + a_0b_3 \\ a_3b_1 + a_2b_2 + a_1b_3 \\ a_3b_2 + a_2b_3 \\ a_3b_3 \end{bmatrix}$$

Pretty, no?

Note $[q(x)p(x)]_i = \sum_{l=0}^i a_l b_{i-l} = \sum_{l+k=i} a_l b_k$, which you might know from studying polynomials in an algebra class.

Problem 3.3 (20 pts). Consider the map $L : \mathbb{R}^3 \rightarrow \mathbb{R}^3$ that projects a point in \mathbb{R}^3 onto the line $l : \left\{ t \begin{bmatrix} -1 \\ -1 \\ 2 \end{bmatrix} \mid t \in \mathbb{R} \right\}$ along the plane $P : 3x - 2y + z = 0$.



Find a basis \mathcal{B} for \mathbb{R}^3 so that $[L]_{\mathcal{B}}$ is simple. Give both \mathcal{B} and $[L]_{\mathcal{B}}$. (9 points for this.) Next, find $[L]$ using some change of basis and the $[L]_{\mathcal{B}}$ that you found. (9 points for this part.) Finally, find $L((4, -4, 0))$. (2 points)

Note: Points on P are mapped to $\mathbf{0}$, that is, $\ker(L) = P$, while points in l are fixed.

There are many choices for \mathcal{B} , I will use the two vectors $\mathbf{v}_1 = (1, 1, -1)$ and $\mathbf{v}_2 = (0, 1, 2)$ in P and $\mathbf{v}_3 = (1, -1, 2)$ in l . So

$$\mathcal{B} = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\} = \left\{ \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix} \right\}$$

and

$$[L]_{\mathcal{B}} = [[L(\mathbf{v}_1)]_{\mathcal{B}} \ [L(\mathbf{v}_2)]_{\mathcal{B}} \ [L(\mathbf{v}_3)]_{\mathcal{B}}] = [\mathbf{0}]_{\mathcal{B}} \ [\mathbf{0}]_{\mathcal{B}} \ [\mathbf{v}_3]_{\mathcal{B}}] = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

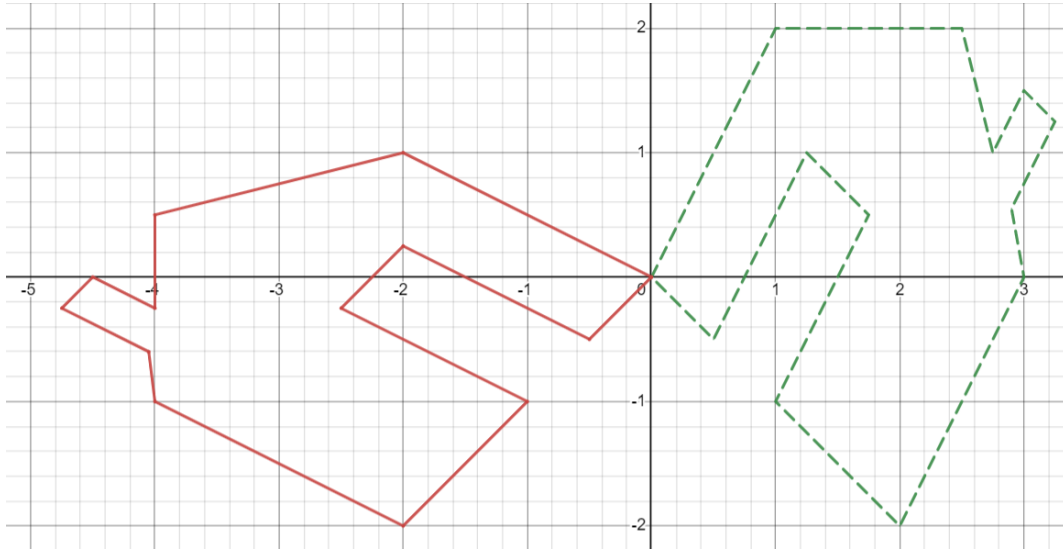
Finding $[L]$ is now trivial.

$$[L] = B[L]_{\mathcal{B}}B^{-1} = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & -1 \\ -1 & 2 & 2 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & -1 \\ -1 & 2 & 2 \end{bmatrix}^{-1} = \frac{1}{7} \begin{bmatrix} 3 & -2 & 1 \\ -3 & 2 & -1 \\ 6 & -4 & 2 \end{bmatrix}$$

and

$$L\left(\begin{bmatrix} 4 \\ -4 \\ 0 \end{bmatrix}\right) = \frac{20}{7} \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}$$

Problem 3.4 (20 pts). The green (dashed) house has been transformed to the red (solid) house by a linear transformation $L : \mathbb{R}^2 \rightarrow \mathbb{R}^2$.



Desmos

Find $[L]$ by first choosing basis \mathcal{G} (for the green house) and basis \mathcal{R} (for the red house) and find $[L]_{\mathcal{G}, \mathcal{R}}$, then use change of basis matrices to find $[L]$.

There are many options here; I will take

$$\mathcal{G} = \{\mathbf{v}_1, \mathbf{v}_2\} = \left\{ \begin{bmatrix} 2 \\ -2 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \end{bmatrix} \right\}$$

and

$$\mathcal{R} = \{\mathbf{u}_1, \mathbf{u}_2\} = \left\{ \begin{bmatrix} -2 \\ -2 \end{bmatrix}, \begin{bmatrix} -2 \\ 1 \end{bmatrix} \right\}$$

Then

$$[L]_{\mathcal{G}, \mathcal{R}} = [[L(\mathbf{v}_1)]_{\mathcal{R}} \ [L(\mathbf{v}_2)]_{\mathcal{R}}] = [[\mathbf{u}_1]_{\mathcal{R}} \ [\mathbf{u}_2]_{\mathcal{R}}] = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

and

$$[L] = R[L]_{\mathcal{G}, \mathcal{R}}G^{-1} = RG^{-1} = \begin{bmatrix} -2 & -2 \\ -2 & 1 \end{bmatrix} \begin{bmatrix} 2 & 1 \\ -2 & 2 \end{bmatrix}^{-1} = \frac{1}{3} \begin{bmatrix} 4 & -1 \\ -1 & 2 \end{bmatrix}$$

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4 Theory and Proofs (30 points; 10 points each)

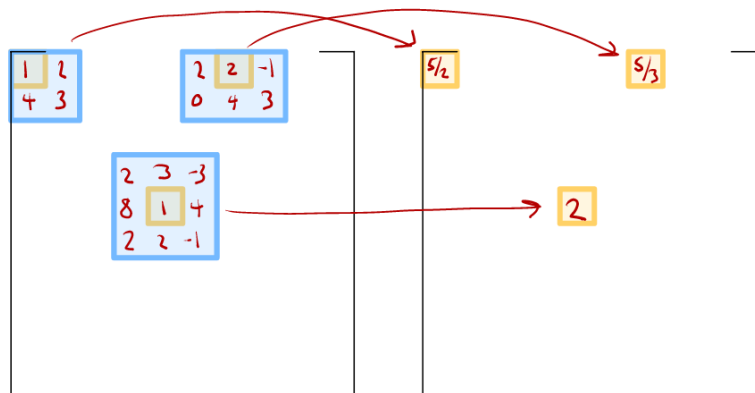
Choose three of the four options. If you try more than three, I will grade only the first three, not the best three. You must decide what should be graded. These will be due on 3/9 in class. Make sure your work is complete and clear. Explain your work; a proof is not just a collection of math symbols, it is an explanation of why something is true.

This part is take-home. You should complete this work on your own without consulting websites, friends, the Math Center, etc.

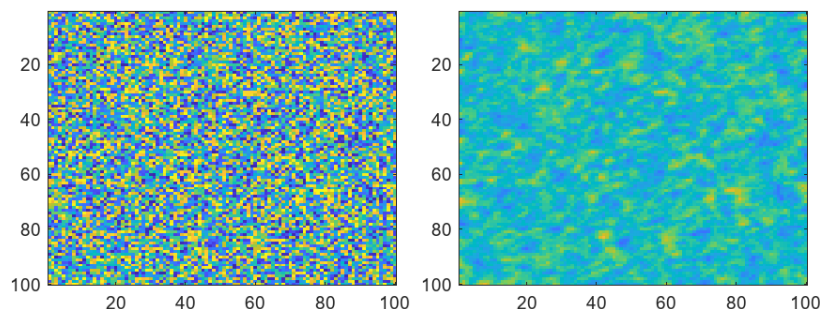
Problem 4.1 (10 points). Let V be a vector space with $\dim(V) = n$ and $U \subseteq V$ a subspace with $\dim(U) = k$, show that there is a subspace $W \subseteq V$ with $\dim(W) = l$ so that $l + k = n$ and $V = U \oplus W$.

Problem 4.2 (10 points). Show that if $L : V \rightarrow W$ is linear and $\ker(L) = \{\mathbf{0}\}$, then for any linearly independent set $\{\mathbf{v}_1, \dots, \mathbf{v}_k\}$ from V , $\{L(\mathbf{v}_1), \dots, L(\mathbf{v}_k)\}$ is independent.

Problem 4.3 (10 points). Consider the following operation. Given an $m \times n$ matrix A , $S(A)$ will be the $m \times n$ matrix where each entry has been replaced by the average of the entry with its neighbors. S for “smear” (often called “blur”).



Example applied to random noise (numeric value represented by color):



[MATLAB code](#)

Main Question (15 points): Show that $S : \mathbb{R}^{m \times n} \rightarrow \mathbb{R}^{m \times n}$ is linear.

Thought Question (3 points): What do you think would happen if you repeatedly applied smearing? That is, consider $A_1 = A$, $A_2 = S(A)$, $A_3 = S(S(A)) = S^2(A)$, etc. What do you think $S^k(A)$ would look like for large k (as an image)? What would the limiting value be?

How might you verify your conjecture? (2 points): Since S is linear

$$A_k = S^k(A) = \sum_{i,j} A_{i,j} S^k(E_{i,j})$$

How might you use this to verify your conjecture?

Problem 4.4 (10 points). Suppose A is a $n \times n$ matrix and $A^{m+1} = A^m$ for some m , then already $A^{n+1} = A^n$.

Hint: This is similar, but different, to one from the exam you had for practice. You can use the same ideas. Note that $A^{m+1} = A^m$ can be written as $A^m(A - I) = O$, do remember that $AB = O$ does not mean that $A = O$ or $B = O$.