Math 571 - Homework 5

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Notation: For $f: X \to Y$ and $E \subseteq X$ set $f(X) = \{f(e) \mid e \in E\}$, this is called the *image of* E under f.

Problem 0.1 (R:4:2*). Let $f: X \to Y$ be continuous. Let $E \subseteq X$, show that $f(\operatorname{Cl}(E)) \subseteq \operatorname{Cl}(f(E))$. By example, show that this containment can be proper, that is $\operatorname{Cl}(f(E)) \nsubseteq f(\operatorname{Cl}(E))$ can hold.

You may take X and Y to be metric if you want, but this is not relevant.

Definition Let $f: E \subseteq X \to Y$, the graph of f is the set $Graph(f) = \{(x, f(x) \mid x \in E\} \subseteq X \times Y.$

Problem 0.2. Let $f: E \subseteq X \to Y$ be continuous where Y is Hausdorff, show that Graph(f) is closed in $E \times Y$.

Problem 0.3 (R:4:6). Suppose $f: E \subseteq X \to Y$ and E is compact. Suppose further that X and Y are Hausdorff (or metric if you prefer). Show that f is continuous on E iff Graph(f) is compact.

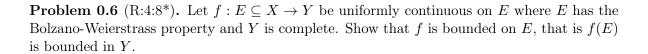
Hint: You may use the fact that if K and H are compact, then $K \times H$ is compact and that If K is compact and $C \subseteq K$ is closed, then C is compact. (Both of these are in notes and book.)

Problem 0.4. Let $f: E \subseteq X \to Y$ where both X and Y are metric spaces with Y complete. suppose f is uniformly continuous on E, show that there is a unique continuous extension $\hat{f}: Cl(E) \to Y$. Moreover, \hat{f} remains uniformly continuous.

Definition: A set $E \subseteq X$ has the *Bolzano-Weierstrass property* iff every sequence in X has a convergent subsequence.

Problem 0.5. Show that if $E \subseteq X$ has the Bolzano-Weierstrass property, then

- a) Cl(E) also has Bolzano-Weierstrass property.
- b) If X is metric, then E is bounded.
- c) For X metric E has the Bolzano-Weierstrass property iff Cl(E) is compact.



Problem 0.7 (R:4:19). Show that if $f: \mathbb{R} \to \mathbb{R}$ satisfies the intermediate value theorem and $f^{-1}(r) = \{x \mid f(x) = r\}$ is closed for $r \in \mathbb{Q}$, then f is continuous. (See the text for a hint. \mathbb{Q} here could be replaced by any dense set.)