Homework 7 Partial Solutions

Section 6.4

14. Write $A = B^H B$ where

$$A = \begin{bmatrix} 4 & 0 & 0 \\ 0 & 1 & i \\ 0 & -i & 1 \end{bmatrix}$$

clearly A is Hermitian,so we know A is orthogonally diagonalized.

$A - tI = \left[\begin{smallmatrix} 4-t & 0 & 0 \\ 0 & 1-t & i \\ 0 & -i & 1-t \end{smallmatrix}\right]$ so $\det(A - tI) = (4-t)((1-t)^2 + i^2) = (4-t)(t^2 - 2t + 1 - 1) = -(t-4)(t-2)t$ so the eigenvalues are $4 > 2 > 0$. Note all eigenspaces are dimension 1 so we need only find a single eigenvector in each eigenspace to get a basis.

$\mathrm{NS}(A - 4I) = \mathrm{NS}\left[\begin{smallmatrix} 0 & 0 & 0 \\ 0 & -3 & i \\ 0 & -i & -3 \end{smallmatrix}\right] = \mathrm{span}\{(1,0,0)\}$

$\mathrm{NS}(A - 2I) = \mathrm{NS}\left[\begin{smallmatrix} 2 & 0 & 0 \\ 0 & -1 & i \\ 0 & -i & -1 \end{smallmatrix}\right] = \mathrm{span}\{(0,i,1)\}$

$\mathrm{NS}(A - 0I) = \mathrm{NS}\left[\begin{smallmatrix} 4 & 0 & 0 \\ 0 & 1 & i \\ 0 & -i & 1 \end{smallmatrix}\right] = \mathrm{span}\{(0,1,i)\}$

Our orthonormal basis will be

$$\boldsymbol{u}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \qquad \boldsymbol{u}_2 = \begin{bmatrix} 0 \\ i/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix} \qquad \boldsymbol{u}_3 = \begin{bmatrix} 0 \\ 1/\sqrt{2} \\ i/\sqrt{2} \end{bmatrix}$$

So

$$A = UDU^H = \begin{bmatrix} 1 & 0 & 0 \\ 0 & i/\sqrt{2} & 1/\sqrt{2} \\ 0 & 1/\sqrt{2} & i/\sqrt{2} \end{bmatrix} \begin{bmatrix} 4 & & \\ & 2 & \\ & & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & -i/\sqrt{2} & 1/\sqrt{2} \\ 0 & 1/\sqrt{2} & -i/\sqrt{2} \end{bmatrix}$$

Let

$$B = D^{1/2}U^H = \begin{bmatrix} 2 & & \\ & \sqrt{2} & \\ & & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & -i/\sqrt{2} & 1/\sqrt{2} \\ 0 & 1/\sqrt{2} & -i/\sqrt{2} \end{bmatrix} = \begin{bmatrix} 2 & 0 & 0 \\ 0 & -i & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

then

$$A = B^H B$$

as desired.

15. Let U be unitary.

(a) $U^H U = UU^H = I$ so U is normal.

(b) Clearly $\langle U\boldsymbol{x}, U\boldsymbol{y}\rangle = (U\boldsymbol{y})^H(U\boldsymbol{x}) = \boldsymbol{y}^H U^H U\boldsymbol{x} = \boldsymbol{y}^H\boldsymbol{x} = \langle \boldsymbol{x}, \boldsymbol{y}\rangle$. So $\|x\|_2^2 = \langle \boldsymbol{x}, \boldsymbol{x}\rangle = \langle U\boldsymbol{x}, U\boldsymbol{x}\rangle = \|U\boldsymbol{x}\|_2^2$. So $\|\boldsymbol{x}\|_2 = \|U\boldsymbol{x}\|_2$.

(c) If λ is an eigenvalue, then from (b)

$$1 = \frac{\langle U\boldsymbol{x}, U\boldsymbol{x}\rangle}{\langle \boldsymbol{x}, \boldsymbol{x}\rangle} = \frac{\lambda\bar{\lambda}\langle \boldsymbol{x}, \boldsymbol{x}\rangle}{\langle \boldsymbol{x}, \boldsymbol{x}\rangle} = |\lambda|^2$$

So $|\lambda| = 1$.

26. If you are at site i, the probability of jumping to site j is a_{ij} and is determined as follows. There is an 85% chance that the user will choose on of the m_i links on the page and there is a 15% chance the user will just choose a random link from the i possible links. (This is just a heuristic.)

$$a_{ij} = \begin{cases} (.85)\frac{1}{m_i} + (.15)\frac{1}{n} & \text{if there is a link from } i \text{ to } j + (.15)\frac{1}{n} \\ (.15)\frac{1}{n} & \text{otherwise} \end{cases}$$

So for this problem the matrix is:

$$A = \begin{bmatrix} .15/4 & .85/2 + .15/4 & .15/4 & 1/4 \\ .85/3 + .15/4 & .15/4 & .15/4 & 1/4 \\ .85/3 + .15/4 & .85/2 + .15/4 & .15/4 & 1/4 \\ .85/3 + .15/4 & .15/4 & .85 + .15/4 & 1/4 \end{bmatrix} = \begin{bmatrix} 0.037500 & 0.4625 & 0.0375 & 0.25 \\ 0.320833 & 0.0375 & 0.0375 & 0.25 \\ 0.320833 & 0.4625 & 0.0375 & 0.25 \\ 0.320833 & 0.0375 & 0.8875 & 0.25 \end{bmatrix}$$

This is a Markov matrix and we are interested in the steady state eigenvector, that is $\boldsymbol{x}$ so that $A\boldsymbol{x} = \boldsymbol{x}$ and $x_i \geq 0$ and $\sum x_i = 1$. This vector is $\boldsymbol{x} = (0.19322, 0.17401, 0.24797, 0.38479)$. To get this with MATLAB:

```
[V,D] = eig(A);
x = V(:,1)/sum(V(:,1))
```

27. $A = \left[\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right]$ so $\det(A - tI) = \det\left[\begin{smallmatrix} -t & 1 \\ 1 & -t \end{smallmatrix}\right] = t^2 - 1 = (t-1)(t+1)$. The eigenvalues are $1 > -1$.

$\mathrm{NS}(A - 1\cdot I) = \mathrm{NS}\left[\begin{smallmatrix} -1 & 1 \\ 1 & -1 \end{smallmatrix}\right] = \mathrm{span}\{(1,1)\}$.

$\mathrm{NS}(A + 1\cdot I) = \mathrm{NS}\left[\begin{smallmatrix} 1 & 1 \\ 1 & 1 \end{smallmatrix}\right] = \mathrm{span}\{(-1,1)\}$.

Let

$$\boldsymbol{u}_1 = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ 1 \end{bmatrix} \qquad \boldsymbol{u}_2 = \frac{1}{\sqrt{2}}\begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

and

$$A = \begin{bmatrix} \boldsymbol{u}_1 & \boldsymbol{u}_2 \end{bmatrix}\begin{bmatrix} \lambda_1 & \\ & \lambda_2 \end{bmatrix}\begin{bmatrix} \boldsymbol{u}_1^H \\ \boldsymbol{u}_2^H \end{bmatrix} = \lambda_1\boldsymbol{u}_1\boldsymbol{u}_1^H + \lambda_2\boldsymbol{u}_2\boldsymbol{u}_2^H$$

28. Let $\boldsymbol{u}_1, \dots, \boldsymbol{u}_n$ be an orthonormal basis of eigenvectors for Hermitian A with eigenvalues $\lambda_1, \dots, \lambda_n$S. Let $\boldsymbol{x} = \sum_{i=1}^n c_i \boldsymbol{u}_i$.

(a) $A\boldsymbol{x} = \sum_{i=1}^n \lambda_i c_i \boldsymbol{u}_i$, so $\boldsymbol{x}^H A\boldsymbol{x} = \left(\sum_{j=1}^n \bar{c}_j \boldsymbol{u}_j^H\right)\left(\sum_{i=1}^n c_i \boldsymbol{u}_i\right) = \sum_{j=1}^n \sum_{i=1}^n \bar{c}_j c_i \boldsymbol{u}_j^H \boldsymbol{u}_i = \sum_{i=1}^n |c_i|^2 \lambda_i$.

Similarly, $\boldsymbol{x}^H \boldsymbol{x} = \sum_{i=1}^n |c_i|^2$ so

$$\rho(\boldsymbol{x}) = \frac{\boldsymbol{x}^H A\boldsymbol{x}}{\boldsymbol{x}^H \boldsymbol{x}} = \frac{\sum_{i=1}^n |c_i|^2 \lambda_i}{\sum_{i=1}^n |c_i|^2}$$

(b) If $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$, then clearly

$$\lambda_1 = \frac{\sum_{i=1}^n |c_i|^2 \lambda_1}{\sum_{i=1}^n |c_i|^2} \geq \rho(\boldsymbol{x}) = \frac{\sum_{i=1}^n |c_i|^2 \lambda_i}{\sum_{i=1}^n |c_i|^2} \geq \frac{\sum_{i=1}^n |c_i|^2 \lambda_n}{\sum_{i=1}^n |c_i|^2} = \lambda_n$$

Section 6.5

4. We are given an SVD of A as

$$A = \begin{bmatrix} 3/5 & -4/5 & 0 \\ 4/5 & 3/5 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 30 & & \\ & 15 & \\ & & 3 \end{bmatrix} \begin{bmatrix} 3/5 & 4/5 & 0 \\ -4/5 & 3/5 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

A rank 2 matrix of minimal $\|\cdot\|_F$ distance is

$$\begin{aligned} A_2 &= \begin{bmatrix} 3/5 & -4/5 & 0 \\ 4/5 & 3/5 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 30 & & \\ & 15 & \\ & & 0 \end{bmatrix} \begin{bmatrix} 3/5 & 4/5 & 0 \\ -4/5 & 3/5 & 0 \\ 0 & 0 & 1 \end{bmatrix} \\ &= \begin{bmatrix} 3/5 & -4/5 \\ 4/5 & 3/5 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 30 & \\ & 15 \end{bmatrix} \begin{bmatrix} 3/5 & 4/5 & 0 \\ -4/5 & 3/5 & 0 \end{bmatrix} \end{aligned}$$

A rank 1 matrix of minimal $\|\cdot\|_F$ distance is

$$\begin{aligned} A_1 &= \begin{bmatrix} 3/5 & -4/5 & 0 \\ 4/5 & 3/5 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 30 & & \\ & 0 & \\ & & 0 \end{bmatrix} \begin{bmatrix} 3/5 & 4/5 & 0 \\ -4/5 & 3/5 & 0 \\ 0 & 0 & 1 \end{bmatrix} \\ &= \begin{bmatrix} 3/5 \\ 4/5 \\ 0 \end{bmatrix} \begin{bmatrix} 30 \end{bmatrix} \begin{bmatrix} 3/5 & 4/5 & 0 \end{bmatrix} \\ &= \sigma_1 \boldsymbol{u}_1 \boldsymbol{u}_1^T \end{aligned}$$

7. $A = U\Sigma V^T$, if $\mathrm{rank}(A) = r$, then

$$A = \begin{bmatrix} \boldsymbol{u}_1 & \cdots & \boldsymbol{u}_r \end{bmatrix} \begin{bmatrix} \sigma_1 & & \\ & \ddots & \\ & & \sigma_r \end{bmatrix} \begin{bmatrix} \boldsymbol{v}_1^T \\ \vdots \\ \boldsymbol{v}_r^T \end{bmatrix} = B \begin{bmatrix} \boldsymbol{v}_1^T \\ \vdots \\ \boldsymbol{v}_r^T \end{bmatrix}$$

for some B so all rows of A are liner combinations of the vectors $\{\boldsymbol{v}_1, \dots, \boldsymbol{v}_r\}$ and so $\text{rng}(A^T) = \text{RS}(A) = \text{span}\{\boldsymbol{v}_1, \dots, \boldsymbol{v}_r\}$. Trivially, $\{\boldsymbol{v}_1, \dots, \boldsymbol{v}_r\}$ is independent so it is a basis for $\text{rng}(A^T)$.

9. If A is $n \times n$ and $A = U\Sigma V^T$, then $\det(A) = \det(U)\det(\Sigma)\det(V^T) = \det(\Sigma) = \prod_{i=1}^n \sigma_n$. Since U and V are unitary $\det(U), \det(V) \in \{1, -1\}$ and so

$$\det(A) = \pm \prod_{i=1}^{n} \sigma_i = \prod_{i=1}^{n} \lambda_i$$

Since all $\sigma_i \geq 0$, it follows that

$$\prod_{i=1}^{n} \sigma_i = \left| \prod_{i=1}^{n} \lambda_i \right|$$

11. Let σ be a singular value for A and $\boldsymbol{v}$ be a corresponding right singular vector. $A\boldsymbol{v} = \sigma \boldsymbol{u}$ where $\boldsymbol{u}$ is the associated left singular vector. Then

$$\frac{\langle A\boldsymbol{v}, A\boldsymbol{v} \rangle}{\langle \boldsymbol{v}, \boldsymbol{v} \rangle} = \frac{\langle \sigma\boldsymbol{u}, \sigma\boldsymbol{u} \rangle}{\langle \boldsymbol{v}, \boldsymbol{v} \rangle} = \frac{\sigma^2 \|\boldsymbol{u}\|_2^2}{\|\boldsymbol{v}\|_2^2} = \sigma^2$$

12. This defines what is called the psuedo-inverse of a matrix A. This is very important, say in least squares. Recall, that in least squares we have a matrix A and we want to "solve" $A\boldsymbol{x} = \boldsymbol{b}$, but this might not have an exact solution, so instead we search for $\boldsymbol{b}$ that is as close as possible to $\text{CS}(A)$.

In general, given a subspace $S \subset V$ (in an inner product space), if we want the $\hat{\boldsymbol{b}} \in S$ closest to some $\boldsymbol{b} \in V$, then what we are looking for is the *orthogonal projection of* $\boldsymbol{b}$ *on* S. This means that we want $\boldsymbol{b} - \hat{\boldsymbol{b}} \perp S$ and $\hat{\boldsymbol{b}} \in S$. To see why this yields $\hat{\boldsymbol{b}}$ is the closest point in S to $\boldsymbol{b}$ take any other $\boldsymbol{v} \in S$, then we have

$$||\boldsymbol{b} - \boldsymbol{v}||^2 = ||\boldsymbol{b} - \hat{\boldsymbol{b}} + \hat{\boldsymbol{b}} - \boldsymbol{v}||^2 = ||\boldsymbol{b} - \hat{\boldsymbol{b}}||^2 + ||\hat{\boldsymbol{b}} - \boldsymbol{v}||^2 \geq ||\boldsymbol{b} - \hat{\boldsymbol{b}}||^2$$

This is by Pythagorean theorem **(this is a good exercise!)** using the fact that $\boldsymbol{b} - \hat{\boldsymbol{b}} \perp \boldsymbol{v} - \hat{\boldsymbol{b}}$, since $\boldsymbol{v} - \hat{\boldsymbol{b}} \in S$ and $\boldsymbol{b} - \hat{\boldsymbol{b}} \perp S$.

If we want to find the least square solution to $A\boldsymbol{x} = \boldsymbol{b}$, then we want to find a point $\hat{b} \in \text{CS}(A)$ so that $\boldsymbol{b} - \hat{\boldsymbol{b}} \perp \text{CS}(A)$. Just take S from above to be $\text{CS}(A)$. This means we want two things $\hat{\boldsymbol{b}} = A\hat{\boldsymbol{x}}$ for some $\hat{\boldsymbol{x}}$ and $A^T(\boldsymbol{b} - \hat{\boldsymbol{b}}) = \boldsymbol{0}$. Putting these together, this means we want

$$A^T\boldsymbol{b} = A^T A\hat{\boldsymbol{x}} \tag{1}$$

Let $A^\dagger = V\Sigma^\dagger U^T$ where $A = U\Sigma V^T$ is the svd decomposition of A and

$$\Sigma^\dagger = \begin{bmatrix} 1/\sigma_1 & 0 & 0 & \dots \\ 0 & 1/\sigma_2 & 0 & \cdots \\ 0 & 0 & 1/\sigma_3 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix}$$

is $n \times m$, if A is $m \times n$. Notice

$$\Sigma\Sigma^\dagger = \begin{bmatrix} I_k & O \\ O & O \end{bmatrix}$$

is $m \times m$ and

$$\Sigma^T \Sigma \Sigma^\dagger = \Sigma^T \tag{2}$$

Now what makes $A^\dagger$ the psuedo-inverse is exactly that $A^\dagger \boldsymbol{b}$ is a least squares solution to $A\boldsymbol{x} = \boldsymbol{b}$. To show this we need to show $\hat{\boldsymbol{x}} = A^\dagger \boldsymbol{b}$ satisfies equation (1), that is:

$$A^T \boldsymbol{b} = A^T A \hat{\boldsymbol{x}} = A^T A A^\dagger \boldsymbol{b}$$

For this, just compute and use equation (2).

$$A^T A A^\dagger = V \Sigma^T U^T U \Sigma V^T V \Sigma^\dagger U^T = V \Sigma^T I \Sigma I \Sigma^\dagger U^T = V \Sigma^T (\Sigma \Sigma^\dagger) U^T = V \Sigma^T U^T = A^T$$

There are two ways to compute the psuedo-inverse in MATLAB and three ways to compute a least-squares solution to $A\boldsymbol{x} = \boldsymbol{b}$.

```
[U,S,V] = svd(A);
% Compute the psuedo-inverse using the definition
Adag1 = V*diag(diag(S(1:rank(A),1:rank(A))).^-1)*U';
% Compute the psuedo -nverse using the builtin function
Adag2 = pinv(A);
% Note Adag1 = Adag2

% Compute the least squares solution to Ax = b using "\"
xhat1 = A\b;
% Compute the least squares solution using the psuedo-inverse
xhat2 = Adag*b;
% or
xhat3 = Ainv*b;
% xhat1 = xhat2 = xhat3
```

There are other solutions to the least squares problem, namely, any element of $\hat{\boldsymbol{x}} + \ker(A)$ is a solution.

Section 7.6

5. Let

$$A = \begin{bmatrix} 5 & 2 & 2 \\ -2 & 1 & -1 \\ -3 & -4 & 2 \end{bmatrix}$$

a. Verify that $(4, (2, -2, 1))$ is an "eigenpair".

This is trivial, just verify that

$$\begin{bmatrix} 5 & 2 & 2 \\ -2 & 1 & -2 \\ -3 & -4 & 2 \end{bmatrix} \begin{bmatrix} 2 \\ -2 \\ 1 \end{bmatrix} = 4 \begin{bmatrix} 2 \\ -2 \\ 1 \end{bmatrix}$$

b. Find a Householder transformation H so that

$$HAH = HAH^{-1} = \begin{bmatrix} 4 & * & * \\ 0 & * & * \\ 0 & 0 & * \end{bmatrix} \tag{3}$$

The point is that if A and B are similar, then A and B have the same eigenvalues, since they have the same characteristic equation. (Quick proof: $\chi_A(\lambda) \det(A - \lambda I) = \det(SBS^{-1} - \lambda SIS^{-1}) = \det(S(B - \lambda I)S^{-1}) = \det(S)\det(S - \lambda I)\det(S)^{-1} = \det(S - \lambda I) = \chi_B(\lambda)$.)

Since

$$HAH^{-1} = \begin{bmatrix} 4 & B \\ \mathbf{0} & C \end{bmatrix}$$

we have $\det(HAH^{-1}) = (\lambda - 4)\det(C - \lambda I)$ so we have reduced out problem to finding the eigenvalues of a smaller matrix C.

Recall a Householder transformation is of the form $H_{\boldsymbol{x}} = I - 2\boldsymbol{u}\boldsymbol{u}^T$ and is defined for a specific vector $\boldsymbol{x}$ so that $H\boldsymbol{x} = ||\boldsymbol{x}||\boldsymbol{e}_1$. Some things to remember:

- H is symmetric. (Proof: $H^T = (I^T - 2(\boldsymbol{u}^T)^T\boldsymbol{u}^T) = (I - \boldsymbol{u}\boldsymbol{u}^T)$.)
- H is orthogonal. (Proof: $H^TH = HH = H^2 = (I - 2\boldsymbol{u}\boldsymbol{u}^T)(I - 2\boldsymbol{u}\boldsymbol{u}^T) = (I^2 - 2I\boldsymbol{u}\boldsymbol{u}^T - 2\boldsymbol{u}\boldsymbol{u}^TI + 4(\boldsymbol{u}\boldsymbol{u}^T)(\boldsymbol{u}\boldsymbol{u}^T)) = I - 4\boldsymbol{u}\boldsymbol{u}^T + 4\boldsymbol{u}(\boldsymbol{u}^T\boldsymbol{u})\boldsymbol{u}^T = I - 4\boldsymbol{u}\boldsymbol{u}^T + 4\boldsymbol{u}I\boldsymbol{u}^T = I$.)
- So $H = H^{-1}$, that is, H is **idempotent**.
- $H\boldsymbol{x} = \boldsymbol{v}$ iff $H\boldsymbol{v} = \boldsymbol{x}$. (Proof: $H\boldsymbol{x} = \boldsymbol{v}$ implies $HH\boldsymbol{x} = \boldsymbol{x} = H\boldsymbol{v}$. Similarly in the other direction.)

For this problem, let $\boldsymbol{x} = 4/3(-2, 2, -1)$ so that $||\boldsymbol{x}|| = 4$ and $\boldsymbol{x}$ is an eigenvector for 4 for A. This way $H = H_{\boldsymbol{x}}$ satisfies $H\boldsymbol{x} = 4\boldsymbol{e}_1$ and thus

$$HAH\boldsymbol{e}_1 = HA(1/4\boldsymbol{x}) = 1/4HA\boldsymbol{x} = 1/4H(4\boldsymbol{x}) = (1/4)(4)H\boldsymbol{x} = 4\boldsymbol{e}_1$$

so $\boldsymbol{e}_1$ is an eigenvector for 4 in HAH. Thus equation (3) holds.

All that is left is to provide $\boldsymbol{u}$ to define H, this is done as follows:

$$\boldsymbol{u} = \frac{1}{\sqrt{2\beta}}(x_1 - \alpha, x_2, x_3),$$

where $\beta = \alpha(\alpha - x_1)$ and

$$\alpha = \begin{cases} -||x|| & \text{if } x_1 > 0 \\ ||x|| & \text{if } x_1 \leq 0 \end{cases}$$

Note that $\beta > 0$. (α is chosen this way to optimize numeric stability of the calculation.) It is a simple exercise to show $||u|| = 1$.

Let $\boldsymbol{v} = (x_1 - \alpha, x_2, x_3)$, then

$$2\boldsymbol{u}\boldsymbol{u}^T = \frac{1}{\beta}\boldsymbol{v}\boldsymbol{v}^T \quad \text{so} \quad H = I - 2\boldsymbol{u}\boldsymbol{u}^T = I - \frac{1}{\beta}\boldsymbol{v}\boldsymbol{v}^T$$

and

$$H\boldsymbol{x} = \boldsymbol{x} - \frac{1}{\beta}\boldsymbol{v}(\boldsymbol{v}^T\boldsymbol{x}) = \boldsymbol{x} - \frac{1}{\beta}\boldsymbol{v}(x_1^2 - \alpha x_1 + x_2 + x_3^2) = \boldsymbol{x} - \frac{1}{\beta}\boldsymbol{v}(\alpha^2 - \alpha x_1) = \boldsymbol{x} - \boldsymbol{v} = \alpha\boldsymbol{e}_1$$

as desired.

So $\boldsymbol{x} = 4/3(-2, 2, -1)$ and thus $\alpha = 4$ and $\beta = 4(4 + 8/3) = 80/3$ and $\boldsymbol{v} = (-8/3 - 4, 8/3, -4/3) = 4/3(-5, 2, -1)$ and

$$H = I - \frac{1}{\beta}\boldsymbol{v}\boldsymbol{v}^T = I - \begin{bmatrix} 5/3 & -2/3 & 1/3 \\ -2/3 & 4/15 & -2/15 \\ 1/3 & -2/15 & 1/15 \end{bmatrix} = \begin{bmatrix} -2/3 & 2/3 & -1/3 \\ 2/3 & 11/15 & 2/15 \\ -1/3 & 2/15 & 14/15 \end{bmatrix}$$

c. Compute HAH and find the remaining two eigenvalues.

$$HAH = \begin{bmatrix} 4 & -12/5 & -9/5 \\ 0 & 53/25 & -4/25 \\ 0 & -154/25 & 47/25 \end{bmatrix}$$

So

$$C = \frac{1}{25}\begin{bmatrix} 53 & -4 \\ -154 & 47 \end{bmatrix}$$

and

$$\det(C - \lambda I) = (\lambda - 3)(\lambda - 1)$$

so the remaining two eigenvalues of A are 3 and 1.