Name: ______________________ Exam 2 - MAT345

1 True/False (100 points; 10 points each)

Problem 1.1. In class, you need only provide a T/F (make it clear!) As usual, you may earn back up to 50% of the lost points by supplying justifications afterward.

i) True Let $S \in M_3$ be invertible, then $SM_{3\times 3}S^{-1} = \{SAS^{-1} \mid A \in M_{3\times 3}\}$ is a subspace of $M_{3\times 3}$.

We must show that this set is closed under linear combinations:

$$S(\alpha_1 A_1 + \alpha_2 A_2)S^{-1} = \alpha_1 SA_1S^{-1} + \alpha_2 SA_2A^{-1}$$

ii) True The map mean : $\mathbb{R}^n \to \mathbb{R}$ is linear where $\text{mean}(\boldsymbol{x}) = \frac{1}{n}\sum_{i=1}^n x_i$.

This is clear:

$$\text{mean}(\alpha\boldsymbol{x}+\beta\boldsymbol{y}) = \frac{1}{n}\sum_{i=1}^{n}(\alpha x_i+\beta y_i) = \alpha\frac{1}{n}\sum_i x_i+\beta\frac{1}{n}\sum_i y_i = \alpha\,\text{mean}(\boldsymbol{x})+\beta\,\text{mean}(\boldsymbol{y})$$

iii) False It is clear that for any $\boldsymbol{u} \in \mathbb{R}^n$, the map $l_{\boldsymbol{u}} : \mathbb{R}^n \to \mathbb{R}$ given by $l_{\boldsymbol{u}}(\boldsymbol{v}) = \langle \boldsymbol{v}, \boldsymbol{u}\rangle = \boldsymbol{u}^T\boldsymbol{v}$ is linear. There are however some linear $l : \mathbb{R}^n \to \mathbb{R}$ such that $l \neq l_{\boldsymbol{u}}$ for any $\boldsymbol{u} \in \mathbb{R}^n$.

Let $\boldsymbol{u}^T = [l]_{\mathcal{E}}$, then $l(\boldsymbol{v}) = [l(\boldsymbol{v})]_{\mathcal{E}} = [l]_{\mathcal{E}}[\boldsymbol{v}]_{\mathcal{E}} = \boldsymbol{u}^T\boldsymbol{v}$.

iv) False For all linear $L : V \to W$, if $\{\boldsymbol{v}_1, \ldots, \boldsymbol{v}_k\}$ is independent, then $\{L(\boldsymbol{v}_1), \ldots, L(\boldsymbol{v}_k)\}$ is independent.

This is trivially false. L could just be the $\mathbf{0}$ map, that is, $L(\boldsymbol{v}) = \mathbf{0}$ for all $\boldsymbol{v} \in V$. This would be true if $\ker(L) = \{\mathbf{0}\}$.

v) True For all linear $L : V \to W$, if $\{L(\boldsymbol{v}_1), \ldots, L(\boldsymbol{v}_k)\}$ is independent, then $\{\boldsymbol{v}_1, \ldots, \boldsymbol{v}_k\}$ is independent.

Suppose $\sum \alpha_i \boldsymbol{v}_i = \mathbf{0}$, then $L(\sum \alpha_i\boldsymbol{v}_i) = \sum \alpha_i L(\boldsymbol{v}_i) = L(\mathbf{0}) = \mathbf{0}$. Since $\{L(\boldsymbol{v}_1), \ldots, L(\boldsymbol{v}_k)\}$ is independent it follows that $\alpha_i = 0$ for all i and hence that $\{\boldsymbol{v}_1, \ldots, \boldsymbol{v}_k\}$ is independent.

vi) True Let c_1, c_2, c_3, c_4 be distinct real numbers, then the polynomials $p_1(x) = x-c_1$, $p_2(x) = (x-c_1)(x-c_2)$, $p_3(x) = (x-c_1)(x-c_2)(x-c_3)$, $p_4(x) = (x-c_1)(x-c_2)(x-c_3)(x-c_4)$ is a basis for $S = \{p(x) \in \mathbb{P}_4(x) \mid p(x) = \alpha_1 x + \alpha_2 x^2 + \alpha_3 x^3 + \alpha_4 x^4\}$.

Clearly $\dim(S) = 4$ so we just need to see that $\{p_1, p_2, p_3, p_4\}$ is linearly independent. Suppose $p(x) = \alpha_1 p_1(x) + \alpha_2 p_2(x) + \alpha_3 p_3(x) + \alpha_4 p_4(x) = \mathbf{0}$. then $p(c_2) = \alpha_1(c_2 - c_1) = 0$ so $\alpha_1 = 0$. So $p(x) = \alpha_2 p_2(x) + \alpha_3 p_3(x) + \alpha_4 p_4(x) = \mathbf{0}$. Now $p(c_3) = \alpha_2(c_3-c_1)(c_3-c_2) = 0$ so $\alpha_2 = 0$. Thus $p(x) = \alpha_3 p_3(x)+\alpha_4 p_4(x) = \mathbf{0}$.

Now we have $p(c_4) = \alpha_3(c_4 - c_1)(c_4 - c_2)(c_4 - c_3) = 0$, so $\alpha_3 = 0$. So we have $p(x) = \alpha_4 p_4(x) = 0$. Take $d \notin \{c_1, c_2, c_3, c_4\}$, then $p(d) = \alpha_4(d - c_1)(d - c_2)(d - c_3)(d - c_4) = 0$ and so $\alpha_4 = 0$. So all $\alpha_i = 0$ and hence the set is independent and thus must span.

vii) <u>True</u> Given any basis $\{\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3, \boldsymbol{v}_4\}$, from $\mathbb{R}^4$ and any four matrices $M_1, M_2, M_3, M_4 \in \mathbb{R}^{2\times 3}$ there is a unique linear transformation $L : \mathbb{R}^4 \to \mathbb{R}^{2\times 3}$ where $L(\boldsymbol{v}_i) = M_i$.

Existence: Define $L\left(\sum_{i=1}^4 \alpha_i \boldsymbol{v}_i\right) = \sum_{i=1}^4 \alpha_i M_i$. This is a well-defined function $L : \mathbb{R}^4 \to \mathbb{R}^{2\times 3}$.

Showing that this is linear is just a computation: Let $\boldsymbol{v} = \sum_{i=1}^4 \alpha_i \boldsymbol{v}_i$ and $\boldsymbol{u} = \sum_{i=1}^4 \beta + i\boldsymbol{v}_i$, then

$$\begin{aligned} L(\gamma\boldsymbol{v} + \boldsymbol{u}) &= L\left(\gamma \sum_{i=1}^4 \alpha_i \boldsymbol{v}_i + \sum_{i=1}^4 \beta_i \boldsymbol{v}_i\right) = L\left(\sum_{i=1}^4 (\gamma\alpha_i + \beta_i)\boldsymbol{v}_i\right) \\ &= \sum_{i=1}^4 (\gamma\alpha_i + \beta_i) W_i = \gamma \sum_{i=1}^4 \alpha_i \boldsymbol{W}_i + \sum_{i=1}^4 \beta_i W_i = \gamma \cdot L(\boldsymbol{v}) + L(\boldsymbol{u}) \end{aligned}$$

Uniqueness: Suppose $L' : V \to W$ is linear and sends $\boldsymbol{v}_i$ to $\boldsymbol{W}_i$, then for $\boldsymbol{v} \in V$, $\boldsymbol{v} = \sum_{i=1}^4 \alpha_i \boldsymbol{v}_i$ and $L'(\boldsymbol{v}) = L'\left(\sum_{i=1}^4 \alpha_i \boldsymbol{v}_i\right) = \sum_{i=1}^4 \alpha_i M_i = L(\boldsymbol{v})$ and thus $L = L'$.

viii) <u>False</u> There is an invertible (one-to-one) linear map $L : \mathbb{R}^2 \to \mathbb{R}$.

$\dim(\ker(L)) + \dim(\mathrm{Img}(L)) = 2$, but $\dim(\ker(L)) + \dim(\mathrm{Img}(L)) \leq \dim(\ker(L)) +$

ix) <u>True</u> Let

$$\mathcal{B} = \left\{ \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ 2 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} \right\}$$

be a basis for $\mathbb{R}^3$. If $[\boldsymbol{v}]_{\mathcal{B}} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ then $\boldsymbol{v} = \begin{bmatrix} 3 \\ 3 \\ 4 \end{bmatrix}$

$$\begin{bmatrix} 3 \\ 3 \\ 4 \end{bmatrix} = (1)\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} + (1)\begin{bmatrix} 2 \\ 0 \\ 2 \end{bmatrix} + (1)\begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}$$

x) <u>False</u> Given a linear transformation $L : V \to V$ where $\dim(V) = n$ and $\mathcal{B}$ is a basis for V, the value of $\det([L]_{\mathcal{B}})$ depends on the choice of basis $\mathcal{B}$.

This we discussed in class, given any other basis $\mathcal{B}'$ we have

$$[L]_{\mathcal{B}'} = [\mathrm{id}]_{calB,B'} [L]_{\mathcal{B},\mathcal{C}} [\mathrm{id}]_{\mathcal{B}',\mathcal{B}}$$

Letting $S = [\mathrm{id}]_{calB,B'}$ and so $S^{-1} = [\mathrm{id}]_{calB',B}$. (A common *change of basis* scenario.) So $[L]_{\mathcal{B}'}$ and $[L]_{\mathcal{B}}$ are similar and thus have the same determinant. This way we can define $\det(L)$ for L a linear transformation, not just a matrix.

2 Long Answer (90 points)

Show all computations so that you make clear what your thought processes are.

Problem 2.1 (30 pts)**.** Consider A given by

$$A = \begin{bmatrix} 2 & -2 & 2 & 2 & 1 \\ -4 & 2 & -5 & -3 & -4 \\ -2 & 4 & -1 & -5 & 1 \\ -4 & 6 & -3 & -7 & 0 \end{bmatrix}$$

Find a basis for each of NS(A), CS(A), and RS(A). (10 points each)

$$\text{rref}(A) = \begin{bmatrix} 1 & 0 & \frac{3}{2} & 0 & \frac{3}{2} \\ 0 & 1 & \frac{1}{2} & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

From this we know:

$$\begin{aligned} \text{CS}(A) &= \text{span}\{(-2,-4,-2,-4),(-2,2,4,6),(2,-3,-5,-7)\} \\ \text{RS}(A) &= \text{span}\big\{\big(1,0,\tfrac{3}{2},0,\tfrac{3}{2}\big),\big(0,1,\tfrac{1}{2},0,1\big),(0,0,0,1,0)\big\} \end{aligned}$$

Note: RS(A) is not the span of the first three rows of A.

To find a basis for NS(A) we are looking for solutions to $A\boldsymbol{x} = \mathbf{0}$. First, we have back-substitution: x_3 and x_5 are free, let $x_3 = s$ and $x_5 = t$, then

$$\begin{aligned} x_5 &= t \\ x_4 &= 0 \\ x_3 &= s \\ x_2 &= -\frac{1}{2}s - t \\ x_1 &= -\frac{3}{2}s - \frac{3}{2}t \end{aligned}$$

Any vector $\boldsymbol{x}$ satisfying, $A\boldsymbol{x} = \mathbf{0}$ can be written as:

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} -\frac{3}{2}s - \frac{3}{2}t \\ -\frac{1}{2}s - t \\ s \\ 0 \\ t \end{bmatrix} = s\begin{bmatrix} -\frac{3}{2} \\ -\frac{3}{2} \\ 1 \\ 0 \\ 0 \end{bmatrix} + t\begin{bmatrix} -\frac{3}{2} \\ -1 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

So $\big\{\big(-\frac{3}{2},-\frac{1}{2},1,0,0\big),\big(-\frac{3}{2},-1,0,0,1\big)\big\}$ is a basis for NS(A), that is,

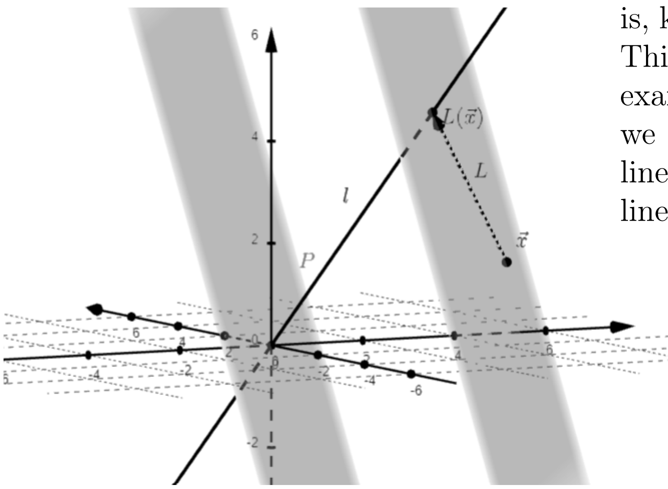
$$\text{NS}(A) = \text{span}\big\{\big(-\tfrac{3}{2},-\tfrac{1}{2},1,0,0\big),\big(-\tfrac{3}{2},-1,0,0,1\big)\big\}$$

Problem 2.2 (20 pts)**.** Consider the map $L : \mathbb{R}^3 \to \mathbb{R}^3$ that projects a point in $\mathbb{R}^3$ onto the line $l : \left\{ t \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix} \;\middle|\; t \in \mathbb{R} \right\}$ along the plane $P : 3x - 2y + z = 0$.

i) (8 points) Find a basis $\mathcal{B}$ for $\mathbb{R}^3$ so that $[L]_{\mathcal{B}}$ is simple. Give both $\mathcal{B}$ and $[L]_{\mathcal{B}}$.

ii) (8 points) Next, find $[L]$ using some change of basis and the $[L]_{\mathcal{B}}$ that you found.

iii) (4 points) Finally, find $L((4, -4, 0))$.



Geogebra

Note: Points on P are mapped to $\mathbf{0}$, that is, $\ker(L) = P$, while points in l are fixed. This is similar to, but different from, the examples done in class. In that example, we were projecting onto a plane along a line, while here, we are projecting onto a line along a plane.

There are many choices for $\mathcal{B}$, I will use the two vectors $\boldsymbol{v}_1 = (1, 1, -1)$ and $\boldsymbol{v}_2 = (0, 1, 2)$ in P and $\boldsymbol{v}_3 = (1, -1, 2)$ in L. So

$$\mathcal{B} = \{\boldsymbol{v_1}, \boldsymbol{v_2}, \boldsymbol{v_3}\} = \left\{ \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix} \right\}$$

and

$$[L]_{\mathcal{B}} = \Big[[L(\boldsymbol{v}_1)]_{\mathcal{B}}\, [L(\boldsymbol{v}_2)]_{\mathcal{B}}\, [L(\boldsymbol{v}_3)]_{\mathcal{B}} \Big] = \Big[[\mathbf{0}]_{\mathcal{B}}\, [\mathbf{0}]_{\mathcal{B}}\, [\boldsymbol{v}_3]_{\mathcal{B}} \Big] = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

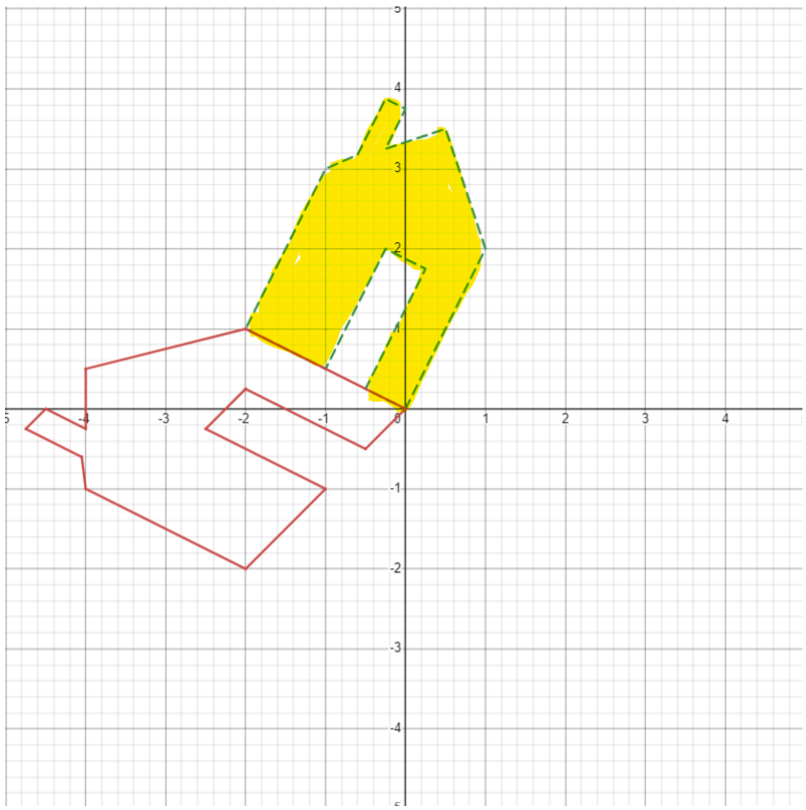
Finding $[L]$ is now trivial.

$$[L] = B[L]_{\mathcal{B}}B^{-1} = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & -1 \\ -1 & 2 & 2 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & -1 \\ -1 & 2 & 2 \end{bmatrix}^{-1} = \frac{1}{7} \begin{bmatrix} 3 & -2 & 1 \\ -3 & 2 & -1 \\ 6 & -4 & 2 \end{bmatrix}$$

and

$$L\left(\begin{bmatrix} 4 \\ -4 \\ 0 \end{bmatrix} \right) = \frac{20}{7} \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}$$

Problem 2.3 (20 pts)**.** The green (dashed/filled) house has been transformed to the red (solid) house by a linear transformation $L : \mathbb{R}^2 \to \mathbb{R}^2$.



Desmos

i) (10 points) Find $[L]$ by first choosing basis $\mathcal{G}$ (for the green house) and basis $\mathcal{R}$ (for the red house) and find $[L]_{\mathcal{G},\mathcal{R}}$.

ii) (10 points) Find $[L]$ by using appropriate change of basis matrices together with $[L]_{\mathcal{G},\mathcal{R}}$

There are many options here. In all cases, you might have chosen a different basis than I did, but the final matrix is the same.

(Exactly as done in class!) Take

$$\mathcal{G} = \{\boldsymbol{v}_1, \boldsymbol{v}_2\} = \left\{ \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \begin{bmatrix} -2 \\ 1 \end{bmatrix} \right\}$$

and

$$\mathcal{R} = \{\boldsymbol{u}_1, \boldsymbol{u}_2\} = \left\{ \begin{bmatrix} -2 \\ 1 \end{bmatrix}, \begin{bmatrix} -2 \\ -2 \end{bmatrix} \right\}$$

Then

$$[L]_{\mathcal{G},\mathcal{R}} = \left[[L(\boldsymbol{v}_1)]_{\mathcal{R}} \, [L(\boldsymbol{v}_2)]_{\mathcal{R}} \right] = \left[[\boldsymbol{u}_1]_{\mathcal{R}} \, [\boldsymbol{u}_2]_{\mathcal{R}} \right] = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

and

$$
\begin{aligned}
[L] = R[L]_{\mathcal{G},\mathcal{R}}G^{-1} = RG^{-1} &= \begin{bmatrix} -2 & -2 \\ 1 & -2 \end{bmatrix} \begin{bmatrix} 1 & -2 \\ 2 & 1 \end{bmatrix}^{-1} \\
&= \begin{bmatrix} -2 & -2 \\ 1 & -2 \end{bmatrix} \frac{1}{5} \begin{bmatrix} 1 & 2 \\ -2 & 1 \end{bmatrix} = \frac{1}{5} \begin{bmatrix} 2 & -6 \\ 5 & 0 \end{bmatrix}
\end{aligned}
$$

Problem 2.4 (20 points)**.** Show that if $L : V \to W$ is linear and $\ker(L) = \{\mathbf{0}\}$, then for any linearly independent set $\{\boldsymbol{v_1, \ldots, v_k}\}$ from V, $\{L(\boldsymbol{v}_1), \ldots, L(\boldsymbol{v}_k)\}$ is independent.

Suppose $\sum \alpha_i L(\boldsymbol{v}_i) = \mathbf{0}$, then we can use linearity to get

$$
L\left(\sum \alpha_i \boldsymbol{v}_i\right) = \mathbf{0}
$$

But since $\ker(L) = \{\mathbf{0}\}$ we have $\sum \alpha_i \boldsymbol{v}_i = \mathbf{0}$ and as the $\boldsymbol{v}_i$'s are independent we know $\alpha_i = 0$ for all i and so the $L(\boldsymbol{v}_i)$'s are shown independent.