Quiz 2

Problem 1 (7 points)**.** Use the following three facts about determinants to compute the determinant of a matrix using row operations.

a. If B is diagonal, then $\det(B) = b_{11} \cdot b_{22} \cdots b_{nn}$.

b. If B arises from A by a type I row operation, i.e., interchanging two rows, then $\det(B) = -\det(A)$.

c. If B arises from A by a type III row operation, i.e., $r_i + ar_j \to r_i$, that is, row i is replaced by row i plus a scalar multiple of row j, where $i \neq j$. Then $\det(A) = \det(B)$.

Compute $\det(A)$ by:

1. Reducing A to a diagonal matrix B using only type I and III operations.
2. Keep track of how many row swaps were made.
3. Compute $\det(B)$ by multiplying the diagonal elements of B.

$$A = \begin{bmatrix} 8 & 6 & -3 & 20 \\ 4 & 2 & -5 & -7 \\ 8 & 2 & 7 & 20 \\ 4 & 2 & -11 & -4 \end{bmatrix}$$

Show the work for the above computation here.

On your own, don't include this in the quiz, try computing this determinant by expanding on a row or column.

Discuss which method, "expansion along a row or column" or "using elementary row operations" is, in general, a faster method of computing a determinant.

I will only write this out using row reduction, the other way ... expansion along a row/column would be too painful.

$$\begin{bmatrix} 8 & 6 & -3 & 20 \\ 4 & 2 & -5 & -7 \\ 8 & 2 & 7 & 20 \\ 4 & 2 & -11 & -4 \end{bmatrix} \xrightarrow[r_4 \leftrightarrow r_1]{} \begin{bmatrix} 4 & 2 & -11 & -4 \\ 4 & 2 & -5 & -7 \\ 8 & 2 & 7 & 20 \\ 8 & 6 & -3 & 20 \end{bmatrix}$$

$$\xrightarrow[\substack{r_2 - r_1 \to r_2 \\ r_3 - 2r_1 \to r_3 \\ r_4 - 2r_1 \to r_4}]{} \begin{bmatrix} 4 & 2 & -11 & -4 \\ 0 & 0 & 6 & -3 \\ 0 & -2 & 29 & 28 \\ 0 & 2 & 19 & 28 \end{bmatrix}$$

$$\xrightarrow[r_2 \leftrightarrow r_3]{} \begin{bmatrix} 4 & 2 & -11 & -4 \\ 0 & -2 & 29 & 28 \\ 0 & 0 & 6 & -3 \\ 0 & 2 & 19 & 28 \end{bmatrix}$$

$$\xrightarrow[r_4 + r_2 \longrightarrow r_4]{} \begin{bmatrix} 4 & 2 & -11 & -4 \\ 0 & -2 & 29 & 28 \\ 0 & 0 & 6 & -3 \\ 0 & 0 & 48 & 56 \end{bmatrix}$$

$$\xrightarrow[r_4 - 8 * r_3 \longrightarrow r_4]{} \begin{bmatrix} 4 & 2 & -11 & -4 \\ 0 & -2 & 29 & 28 \\ 0 & 0 & 6 & -3 \\ 0 & 0 & 0 & 80 \end{bmatrix}$$

We had two row swaps so $\det(A) = (-1)^2(4)(-2)(6)(80) = -3840$.

The expansion would start as (I'll expand along first column)

$$\begin{aligned}
\det \begin{bmatrix} 8 & 6 & -3 & 20 \\ 4 & 2 & -5 & -7 \\ 8 & 2 & 7 & 20 \\ 4 & 2 & -11 & -4 \end{bmatrix} &= 8 \det \begin{bmatrix} 2 & -5 & -7 \\ 2 & 7 & 20 \\ 2 & -11 & -4 \end{bmatrix} - 4 \det \begin{bmatrix} 6 & -3 & 20 \\ 2 & 7 & 20 \\ 2 & -11 & -4 \end{bmatrix} \\
&+ 8 \det \begin{bmatrix} 6 & -3 & 20 \\ 2 & -5 & -7 \\ 2 & -11 & -4 \end{bmatrix} - 4 \det \begin{bmatrix} 6 & -3 & 20 \\ 2 & -5 & -7 \\ 2 & 7 & 20 \end{bmatrix} \\
&= 8 \left(2 \det \begin{bmatrix} 7 & 20 \\ -11 & -4 \end{bmatrix} - 2 \det \begin{bmatrix} -5 & -7 \\ -11 & 4 \end{bmatrix} + 2 \det \begin{bmatrix} -5 & -7 \\ 7 & 20 \end{bmatrix} \right) \\
&- 4 \left(6 \det \begin{bmatrix} 7 & 20 \\ -11 & 4 \end{bmatrix} - 2 \det \begin{bmatrix} -3 & 20 \\ -11 & -4 \end{bmatrix} + 2 \det \begin{bmatrix} -3 & 20 \\ 7 & 20 \end{bmatrix} \right) \\
&+ 8 \left(6 \det \begin{bmatrix} -5 & -7 \\ -11 & -4 \end{bmatrix} - 2 \det \begin{bmatrix} -3 & 20 \\ -11 & -4 \end{bmatrix} + 2 \det \begin{bmatrix} -3 & 20 \\ -5 & -7 \end{bmatrix} \right) \\
&- 4 \left(6 \det \begin{bmatrix} -5 & -7 \\ 7 & 20 \end{bmatrix} - 2 \det \begin{bmatrix} -3 & 20 \\ 7 & 20 \end{bmatrix} + 2 \det \begin{bmatrix} -3 & 20 \\ -5 & -7 \end{bmatrix} \right)
\end{aligned}$$

I will let the reader finish this, the result will be as above -3840 and the expansion could be done on any row/column. I used the first row for each expansion.

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Problem 2 (7 points)**.** Let A be as above, consider $A\boldsymbol{x} = \boldsymbol{b}$ where $\boldsymbol{b} = (31, 2, 21, 11)$. Find x_1 using Cramer's rule. (You may use MATLAB/Octave to compute the determinants, but write out what you are computing.)

$$x_1 = \frac{\begin{vmatrix} 31 & 6 & -3 & 20 \\ 2 & 2 & -5 & -7 \\ 21 & 2 & 7 & 20 \\ 11 & 2 & -11 & -4 \end{vmatrix}}{\begin{vmatrix} 8 & 6 & -3 & 20 \\ 4 & 2 & -5 & -7 \\ 8 & 2 & 7 & 20 \\ 4 & 2 & -11 & -4 \end{vmatrix}} = 1$$

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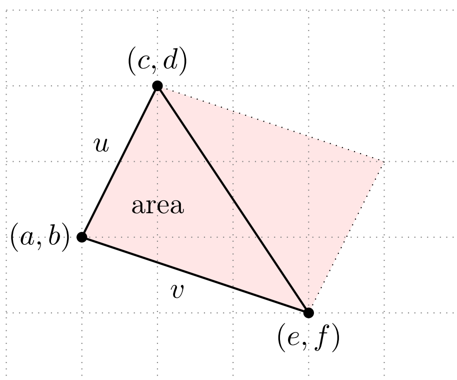
Problem 3 (7 points)**.** Given vectors $v = \begin{bmatrix} v_0 \\ v_1 \end{bmatrix}$ and $u = \begin{bmatrix} u_0 \\ u_1 \end{bmatrix}$ it is true that $|\det(u, v)| = \left|\det \begin{bmatrix} u_0 & v_0 \\ u_1 & v_1 \end{bmatrix}\right|$ is the area of the parallelogram formed by u and v in the natural way. Thus we know:



$$\text{area} = \frac{1}{2}\det(u, v) = \frac{1}{2}\det \begin{bmatrix} c-a & e-a \\ d-b & f-b \end{bmatrix}$$

Show that the area of a triangle with vertices (a, b), (c, d), and (e, f) is given by

$$\text{area} = \frac{1}{2}\det \begin{bmatrix} 1 & 1 & 1 \\ a & c & e \\ b & d & f \end{bmatrix}$$

To get started show:

$$\det \begin{bmatrix} c-a & e-a \\ d-b & f-b \end{bmatrix} = \det \begin{bmatrix} 1 & 1 & 1 \\ 0 & c-a & e-a \\ 0 & d-b & f-b \end{bmatrix}$$

For this just expand along the first column. For the rest use what you know about row operations and determinants. (Only type III row operations are require.)

$$\begin{aligned}
\text{area} &= \frac{1}{2}\det\begin{bmatrix} c-a & e-a \\ d-b & f-b \end{bmatrix} \\
&= \frac{1}{2}\det\begin{bmatrix} 1 & 1 & 1 \\ 0 & c-a & e-a \\ 0 & d-b & f-b \end{bmatrix} && \text{(expand along 1}^{\text{st}}\text{ column)} \\
&= \frac{1}{2}\det\begin{bmatrix} 1 & 1 & 1 \\ a & c & e \\ 0 & d-b & f-b \end{bmatrix} && (R_2 \leftarrow R_2 + aR_1 \text{ (Type III)}) \\
&= \frac{1}{2}\det\begin{bmatrix} 1 & 1 & 1 \\ a & c & e \\ b & d & f \end{bmatrix} && (R_3 \leftarrow R_3 + bR_1 \text{ (Type III)})
\end{aligned}$$

This last is what was desired.

You could do this by expansion

$$\det\begin{bmatrix} 1 & 1 & 1 \\ a & c & e \\ b & d & f \end{bmatrix} = (df - de) - (af - be) + (ad - bc)$$

and

$$\det\begin{bmatrix} c-a & e-a \\ d-b & f-b \end{bmatrix} = (c-a)(f-b) - (d-b)(e-a)$$

Some work will show these to be the same.

Problem 4 (14 points)**.** submit the completion certificate for the OnRamp tutorial from MATLAB.