Math 571 - Homework 2 (05.22)

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Problem 0.1 (R:2:2*)**.** A complex number γ is **algebraic** iff γ is a root to a polynomial with integer coefficients. Prove that there are complex numbers that are not algebraic.

The following are two fun facts (not related to the question, but just for intellectual curiosity):

Let $\mathbb{A} \subset \mathbb{C}$ be the set of algebraic numbers.

1. $\mathbb{A}$ is a field.
2. $\mathbb{A}$ is algebraically closed, that is, if α is a root of a polynomial in $\mathbb{A}[x]$, then $\alpha \in \mathbb{A}$. So in the definition of algebraic numbers you can use any ring of coefficients R, $\mathbb{Z} \subseteq R \subseteq \mathbb{A}$.

Here is a write-up of the proofs, you should have the background to read this, but it is not an easy read.

Definition 1. A set $S \subseteq X$ is **discrete** iff every point in S is isolated.

Problem 0.2 (R:2:5*)**.** Prove the following for discrete $S \subset \mathbb{R}$:

a) $\mathrm{Lim}(S) \cap S = \emptyset$ and S is countable.

b) There is countable set $A \subset \mathbb{R}$ so that $\mathrm{Lim}(A) = \mathrm{Cl}(S)$.

c) Give an example of a discrete set S where there is no set A such that $\mathrm{Lim}(A) = S$.

For the following use the definition that I provided for $\mathrm{Cl}(E)$, namely, $\mathrm{Cl}(E) = \bigcap\{F \mid F \text{ is closed and } E \subseteq F\}$.

Problem 0.3 (R:2:6)**.** For X a metric space and $E \subseteq X$, show that

a) $\mathrm{Cl}(E) = E \cup \mathrm{Lim}(E)$.

b) $\mathrm{Lim}(E)$ is closed.

Either show or give a counterexample to $\mathrm{Lim}(E) = \mathrm{Lim}(\mathrm{Cl}(E))$.

$\mathrm{Lim}(E) \subseteq \mathrm{Lim}(\mathrm{Cl}(E))$ simply because $A \subseteq B \implies \mathrm{Lim}(A) \subseteq \mathrm{Lim}(B)$. Let $x \in \mathrm{Lim}(\mathrm{Cl}(E))$ and let O be an open nbhd of x, then $O \cap (E \cup \mathrm{Lim}(E)) \neq \emptyset$. If $O \cap E \neq \emptyset$ then we are done. Else $O \cap \mathrm{Lim}(E) \neq \emptyset$. In this case we argue as we did above. Let $y \in \mathrm{Lim}(E) \cap O$. Let $U \subset O$ be nbhd of y, then $U \cap E \neq \emptyset$, so $O \cap E \neq \emptyset$.

Problem 0.4 (R:2:9*)**.** Let X be a metric space, or just any topological space. Are the following true for all $E \subseteq X$?

a) $\mathrm{Int}(E)^c = \mathrm{Cl}(E^c)$.

b) $\mathrm{Cl}(E) = \mathrm{Int}(E^c)^c$?

c) $\mathrm{Cl}(E) = \mathrm{Cl}(\mathrm{Int}(E))$?

d) $\mathrm{Int}(E) = \mathrm{Int}(\mathrm{Cl}(E))$

For each either prove the statement true or give a counterexample. For a counterexample you must provide both X and E.

Definition 2. A metric space X is **separable** iff there is a countable $E \subseteq X$ with E dense in X.

Problem 0.5 (R:2:22)**.** Show the $\mathbb{R}^k$ is separable.

Definition 3. A set $\mathcal{B}$ of open sets is called a **base** for X iff for all $x \in X$ and open set U with $x \in U$, there is $V \in \mathcal{B}$ so that $x \in V \subset U$.

Problem 0.6 (R:2:23*)**.** Prove that a metric space is separable iff it has a countable base.

Problem 0.7 (R:2:24)**.** Prove that if X is a metric space and every infinite sequence has a limit point, then X is separable. (See the hint in the text.)