

## Math 571 - Homework 7 (05.22)

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**Problem 1** (R:5:26). Suppose  $f(x)$  is diff'ble on  $[a, b]$ ,  $f(a) = 0$ , and there is a fixed  $A$  such that  $|f'(x)| \leq A|f(x)|$  for all  $x$  in  $[a, b]$ . Show that  $f(x) = 0$  on  $[a, b]$ .

**Proof 1:** Let  $Z = \{x \in [a, b] \mid \forall y \leq x (f(y) = 0)\}$ . If  $Z = [a, b]$ , then we are done. Else let  $c \in [a, b] - Z$ , then there is  $y \leq c$  with  $f(y) \neq 0$ . Say  $f(y) > 0$ , then  $f(y) > \delta > 0$  for some  $\delta$  and we get an open nbhd of  $y$ , say,  $N_r(y)$  so that  $f(N_r(y)) \subseteq (\delta, \infty)$ . This shows  $c \in (y, b]$  and  $(y, b] \cap Z = \emptyset$ . So  $c$  is in an open nbhd of  $[a, b]$  disjoint from  $Z$ . We have shown that for  $c \notin Z$ , there is an open nbhd of  $c$  disjoint from  $Z$ , thus  $Z$  is closed.

Let  $z^* = \sup(Z) < b$  and choose  $c > z^*$  so that  $c - z^* < 1/A$ . Say  $|f(c)| = M$  look at  $C = \{c \mid |f(c)| \geq M\}$ , this set is closed and clearly  $c \geq c^* = \inf(C) > z^*$  and  $|f(c^*)| = M$ . Assume  $f(c^*) = M$ , alternatively,  $f(c^*) = -M$  and the same argument works with the obvious modifications.

By MVT there is  $d \in (z^*, c^*)$  so that

$$f(c^*) - f(z^*) = f(c^*) - 0 = M = f'(d)(c^* - z^*) \leq A(c^* - z^*)f(d) < f(d)$$

But this means we have  $d \in (z^*, c^*)$  with  $f(d) > M$  contradicting the choice of  $c^*$ . What leads to this contradiction? It was the assumption that  $Z \neq [a, b]$ , we conclude that  $Z = [a, b]$ .

**Proof 2: (As hint in text using a student solution.)** Fix  $N$  so that  $A(b-1)/N < 1$  and let  $x_i = a + i(b-a)/N$ . So  $x_0 = a$  and  $x_n = b$ . Suppose  $f$  is 0 on  $[x_0, x_i]$  we will see that  $f$  must then be 0 on  $[x_0, x_{i+1}]$ .

Let  $M_0 = \sup(f([x_i, x_{i+1}]))$  and  $M_1 = \sup(f'([x_i, x_{i+1}]))$ . Notice that  $M_1 \leq AM_0$  by assumption. We know for  $x \in [x_i, x_{i+1}]$  that  $f(x) - f(x_i) = f(x) = f'(c)(x_{i+1} - x_i)$  by MVT. So  $f(x) \leq M_1(x_{i+1} - x_i) \leq AM_0(x_{i+1} - x_i)$ , but this means  $M_0 \leq M_0A(x_{i+1} - x_i)$  and this is non-sense as  $M_0A(x_{i+1} - x_i) < M_0$  unless  $M_0 = 0$ . So  $M_0 = 0$  and thus  $f$  is 0 on  $[x_i, x_{i+1}]$ .

**Problem 2** (R:5:27). Let  $\phi : [a, b] \times [\alpha, \beta] \rightarrow \mathbb{R}$ . A *solution to the initial-value problem* (IVP)

$$y' = \phi(x, y), \quad y(a) = c \text{ for } a \leq c \leq b$$

is a function  $f : [a, b] \rightarrow [\alpha, \beta]$  satisfying

$$f(a) = c, \quad f'(x) = \phi(x, f(x)) \text{ for all } a \leq x \leq b$$

Show that if there is a constant  $A \geq 0$  so that

$$|\phi(x, y_1) - \phi(x, y_2)| \leq A|y_1 - y_2| \text{ for all } x \in [a, b] \text{ and } y_1, y_2 \in [\alpha, \beta]$$

then there is at most one solution to any such IVP.

Suppose  $f_1$  and  $f_2$  are two such solutions, then note that by assumption

$$|f_1'(x) - f_2'(x)| = |\phi(x, f_1(x)) - \phi(x, f_2(x))| \leq A|f_1(x) - f_2(x)| \text{ for } x \in [a, b]$$

Letting  $h(x) = f_1(x) - f_2(x)$  we have  $h(a) = 0$ ,  $h$  is diff'ble on  $[a, b]$ , and  $|h'(x)| \leq A|h(x)|$  for  $x \in [a, b]$ . Thus by Problem 1,  $h = 0$  on  $[a, b]$  and thus  $f_1 = f_2$ .

The book points out an example  $y(0) = 0$  and  $y' = y^{1/2}$  on  $[0, 1]$ . Note that this fails the hypotheses since there is no  $A \geq 0$  with  $|\sqrt{y}| < A|y|$  on  $[0, 1]$ , in particular,  $\lim_{y \rightarrow 0^+} \frac{\sqrt{y}}{y} = \infty$ . The book gives two solutions  $y = 0$  and  $y = x^2/4$ . To find all solutions note

$$\begin{aligned} \frac{y'}{y^{1/2}} &= 1 \\ y^{-1/2} \frac{dy}{dx} &= 1 \\ y^{-1/2} dy &= dx \\ \int y^{-1/2} dy &= \int dx \\ \frac{y^{1/2}}{1/2} + d &= x + c && (d \text{ and } c \text{ arbitrary constants}) \\ y^{1/2} &= \frac{x}{2} + C && (C \text{ an arbitrary constant}) \\ y &= \frac{x^2}{4} + 2Cx + C^2 \end{aligned}$$

If  $y(0) = 0$ , then  $C^2 = 0$  so  $C = 0$  and thus the to given solutions are all.

**Problem 3.** Show that the following are equivalent for a bounded function  $f$  on  $[a, b]$ :

- i)  $f \in \mathcal{R}$ , i.e.,  $f$  is Riemann integrable,
- ii) For all  $\epsilon > 0$ , there is a  $\delta > 0$  such that

$$\|P\| < \delta \implies U(f, P) - L(f, P) < \epsilon$$

First show (i) implies (ii). Let  $f \in \mathcal{R}$  and  $\epsilon > 0$ . We have a partition  $P$  so that  $U(f, P) - L(f, P) < \epsilon/2$ . Take  $\delta > 0$  so that  $\Delta x_i > 2\delta$  for all  $i$  and so that  $\delta < \frac{\epsilon}{12MN}$  where  $M = \sup\{|f(x)| \mid x \in [a, b]\}$  and  $N = |P|$ .

Let  $P'$  be a partition with  $\|P'\| < \delta$  and let  $P'' = P \cup P'$ , then  $L(P') \leq L(P'') \leq U(P'') \leq U(P')$  and  $L(P) \leq L(P'') \leq U(P'') \leq U(P)$ . So  $U(P'') - L(P'') \leq U(P) - L(P) < \epsilon/2$ . We want to show that  $U(P') - U(P'') < \epsilon/4$  and  $L(P'') - L(P') < \epsilon/4$ , then

$$\begin{aligned} U(P') - L(P') &= (U(P'') + (U(P') - U(P''))) - (L(P'') - (L(P'') - L(P'))) \\ &< (U(P'') + \epsilon/4) - (L(P'') - \epsilon/4) = (U(P'') - L(P'')) + \epsilon/2 \\ &< \epsilon/2 + \epsilon/2 = \epsilon \end{aligned}$$

All that needs be proved here is

$$U(P') - U(P'') < \epsilon/4, \quad L(P'') - L(P') < \epsilon/4$$

Let  $P' = a = y_0 < y_1 < \dots < y_m = b$  and  $P = a = x_0 < x_1 < \dots < x_N = b$ . For each  $i = 1, 2, \dots, N-1$  there is  $y_{k_i}$  so that  $x_i \in [y_{k_i-1}, y_{k_i}]$ . If  $x_i \in \{y_{k_i-1}, y_{k_i}\}$ , then adding  $x_i$  to  $P'$  adds nothing new, so in the worst case  $x_i \in (y_{k_i-1}, y_{k_i})$ . Let us assume this always occurs (since this is the worst case). In this case we have

$$\begin{aligned} U(P') - U(P'') &= \sum_{i=1}^{N-1} \sup(f([y_{k_i-1}, y_{k_i}]))(y_{k_i} - y_{k_i-1}) \\ &\quad - (\sup(f([y_{k_i-1}, x_i]))(x_i - y_{k_i-1}) + \sup(f([x_i, y_{k_i}]))(y_{k_i} - x_i)) \\ &\leq \sum_{i=1}^{N-1} 3M\|P'\| = 3(N-1)M\|P\| < \epsilon/4 \end{aligned}$$

The other direction (ii) implies (i) is trivial since all that is required for  $f \in \mathcal{R}$  is that for all  $\epsilon > 0$ , there is  $P$  so that  $U(f, P) - L(f, P) < \epsilon$ .

**Problem 4** (R:6:1). Suppose  $\alpha : [a, b] \rightarrow \mathbb{R}$  is monotonic increasing and continuous at  $x_0 \in [a, b]$ . consider  $f : [a, b] \rightarrow 0, 1$  given by  $f(x_0) = 1$  and  $f(x) = 0$  for  $x \neq x_0$ . Show that  $f \in \mathcal{R}(\alpha)$  and  $\int_a^b f d\alpha = 0$ .

Pick  $\epsilon > 0$ . Since  $\alpha$  is continuous at  $x_0$  take  $\delta$  so that  $\alpha(N_\delta(x_0)) \subseteq N_{\epsilon/2}(\alpha(x_0))$ . Let  $P = y_0 = a < y_1 < y_2 < y_3 = b$  where  $[y_1, y_2] \subset (x_0 - \delta, x_0 + \delta)$ , so that  $\Delta\alpha_2 = \alpha(y_2) - \alpha(y_1) < \epsilon$ . Then  $M_i^{f,P} = m_i^{f,P}$  for  $i \neq 2$  and  $M_2^{f,P} = \sup(f([y_1, y_2])) = 1$  while  $m_i^{f,P} = \inf(f([y_1, y_2])) = 0$  so that

$$U(f, P) - L(f, P) = (1 - 0)\Delta\alpha_2 < \epsilon$$

**Problem 5** (R:6:2). Suppose  $f : [a, b] \rightarrow \mathbb{R}$  is continuous,  $f \geq 0$ , and  $\int_a^b f dx = 0$ , then  $f = 0$ .

Note that where Rudin asks you to compare with (1) there might be the thought that these do not compare since (1) is about  $\mathcal{R}(\alpha)$  while (2) is about  $\mathcal{R}$ , but taking  $\alpha = \text{id}$  in (1) allows you to make the comparison.

This is really almost trivial. If  $f \neq 0$ , then  $f(x) > 0$  for some  $x \in [a, b]$ , but then  $f(x) > \delta > 0$  and so there is an open nbhd of  $x$ ,  $N_\delta(x) = (x - \delta, x + \delta)$  so that  $f((x - \delta, x + \delta) \cap [a, b]) \subset (\delta, \infty)$ . Say  $(c, d) \subseteq (x - \delta, x + \delta) \cap [a, b]$ , then clearly  $\int_a^b f dx \geq \delta(d - c) > 0$ .

The difference in the example from R:6:1 and R:6:2 is clearly that in R:6:1 the function is not continuous. In fact  $\int_a^b f dx = 0$  whenever  $\{x \in [a, b] \mid f(x) \neq 0\}$  has **measure 0**. A set  $Z$  has measure 0 whenever

$$0 = \inf \left\{ \sum_{i=1}^{\infty} (b_i - a_i) \mid Z \subseteq \bigcup_{i=1}^{\infty} (a_i, b_i) \right\}$$

**Problem 6** (R:6:3). Define  $\beta_i : [-1, 1] \rightarrow [0, 1]$  by  $\beta_i = 0$  for  $x < 0$  and  $\beta_i = 1$  for  $x > 0$ , then  $\beta_1(0) = 0$ ,  $\beta_2(0) = 1$ , and  $\beta_3(0) = 1/2$ . In particular  $\beta_i$  has a simple discontinuity at 0 with  $\beta_1(0-) = \beta_1(0) = 0$  (continuous from the left),  $\beta_2(0+) = \beta_2(0) = 1$  (continuous from the right), while  $\beta_3$  is neither continuous from the left or right. Let  $f : [-1, 1] \rightarrow \mathbb{R}$  be bounded. show that

- i)  $f \in \mathcal{R}(\beta_1)$  iff  $f(0+) = f(0)$ , that is,  $f$  is continuous from the right at 0.
- ii)  $f \in \mathcal{R}(\beta_2)$  iff  $f(0-) = f(0)$ , that is,  $f$  is continuous from the left at 0.
- iii)  $f \in \mathcal{R}(\beta_3)$  iff  $f$  is continuous at 0.

These are all very similar. It suffices to consider partitions that include 0 so that  $P = -1 = x_0 < x_1 < \dots < x_n = 1$  and where  $x_k = 0$ . For  $\beta_i$  we have

$$(\Delta\beta_i)_k = \beta_i(0) - \beta_i(x_{k-1}) = \begin{cases} 0 & i = 1 \\ 1 & i = 2 \\ 1/2 & i = 3 \end{cases}$$

and

$$(\Delta\beta_i)_{k+1} = \beta_i(k+1) - \beta_i(0) = \begin{cases} 1 & i = 1 \\ 0 & i = 2 \\ 1/2 & i = 3 \end{cases}$$

All other  $(\Delta\beta_i)_j = 0$  and thus we see

$$\begin{aligned} U(f, P) - L(f, P) &= (M_k - m_k)(\Delta\beta_i)_k + (M_{k+1} - m_{k+1})(\Delta\beta_i)_{k+1} \\ &= \begin{cases} M_{k+1} - m_{k+1} & i = 1 \\ M_k - m_k & i = 2 \\ 1/2((M_k - m_k) + (M_{k+1} - m_{k+1})) & i = 3 \end{cases} \end{aligned}$$

Now  $f \in \mathcal{R}(\beta_i)$  iff for all  $\epsilon > 0$  there is a  $P$  so that

$$U(f, P) - L(f, P) < \epsilon \iff \begin{cases} M_{k+1} - m_{k+1} < \epsilon & i = 1 \\ M_k - m_k < \epsilon & i = 2 \\ 1/2((M_k - m_k) + (M_{k+1} - m_{k+1})) < \epsilon & i = 3 \end{cases}$$

Take the  $i = 1$  case, this says that for all  $\epsilon > 0$  there is  $x_{k+1} = h > 0$  so that  $\sup(f([0, h]) - \inf(f([0, h]))) < \epsilon$  which says exactly that  $f(0+) = f(0)$ . Similarly for  $i = 2$  and  $i = 3$ .

**Problem 7** (R:6:6). Let  $f : [0, 1] \rightarrow \mathbb{R}$  be bounded and continuous off of the Cantor set  $\mathcal{C}$ . Show that  $f \in \mathcal{R}$ .

Recall the construction of the Cantor set.  $C_0 = [0, 1]$   $C_1 = [0, 1] - (1/3, 2/3)$  (removing middle third).  $C_2 = C_1 - (1/9, 2/9) - (7/9, 8/9)$ , again remove middle thirds from what was left.

Notice the lengths of what is removed:  $1/3, 1/3 + 2/9, 1/3 + 2/9 + 4/27$ , etc. Consider

$$\sum_{i=0}^{\infty} \frac{2^i}{3^{i+1}} = \frac{1}{3} \sum_{i=0}^{\infty} \left(\frac{2}{3}\right)^i = \frac{1}{3} \left(\frac{1}{1 - 2/3}\right) = 1$$

We can cover  $C_i$  by  $2^i$  many disjoint intervals of length  $(1/3)^i + \epsilon$  for any  $\epsilon$ . Since  $\mathcal{C} = \bigcap C_i$  we see that  $\mathcal{C}$  has measure 0 as defined above.

Suppose  $f$  is continuous off of a measure 0 set  $Z \subset [a, b]$ . Let  $\mathcal{O}$  be an open cover of  $Z$  by intervals  $(a_i, b_i)$  so that  $\sum_i (b_i - a_i) < \epsilon$ . For each  $x \notin Z$  take  $\delta_x$  so that  $f(N_{\delta_x}(x)) \subset$

$N_{\epsilon/2}(f(x))$ . As  $[a, b]$  is compact we can find a finite subcover  $\{(u_i, v_i) \mid i = 1, \dots, n\}$  so that  $u_1 < a < u_2 < v_1 < u_3 < v_2 < \dots < u_n < v_{n-1} < b < v_n$  where each  $(u_i, v_i)$  is from our cover of  $Z$  or else is one of the  $N_{\delta_x}(x)$ .

Use  $x_0 = a$ ,  $x_i = (u_{i+1} + v_i)/2$  for  $i < n$ , and  $x_n = b$  as the partition:  $P = a = x_0 < x_1 < \dots < b = x_n$ . Let  $M$  be a bound on  $|f|$  on  $[a, b]$ . Let  $T = \{i \mid (x_{i-1}, x_i) \subseteq (a_j, b_j) \text{ for some } j\}$ . Then we have

$$\begin{aligned} U(f, P) - L(f, P) &= \sum_{i \in T} (M_i - m_i) \Delta x_i + \sum_{i \notin T} (M_i - m_i) \Delta x_i \\ &< \sum_{i \in T} 2M \Delta x_i + \sum_{i \notin T} \epsilon \Delta x_i \\ &\leq 2M\epsilon + \epsilon(b - a) = \epsilon(2M + (b - a)) \end{aligned}$$

**Problem 8** (R:6:10). See text. This is mostly done in [the notes](#).

In particular whenever  $p_i \geq 0$  and  $\sum_{i=1}^n p_i = 1$ , then  $\prod_{i=1}^n a_i^{p_i} \leq \sum_{i=1}^n p_i a_i$ . In particular, if  $\frac{1}{p} + \frac{1}{q} = 1$ , then

$$uv = (u^p)^{1/p} (v^q)^{1/q} \leq \frac{u^p}{p} + \frac{v^q}{q}$$

This basically completes (a). For (b) notice

$$fg \leq \frac{f^p}{p} + \frac{g^q}{q}$$

so

$$\int_a^b fg \, d\alpha \leq \int_a^b \frac{f^p}{p} \, d\alpha + \int_a^b \frac{g^q}{q} \, d\alpha = \frac{1}{p} + \frac{1}{q} = 1$$

For (c) the proof is exactly as for Hölder's inequality in the notes already mentioned above. Define  $\|f\|_p = \left( \int_a^b |f|^p \, d\alpha \right)^{1/p}$  provided that  $|f|^p \in \mathcal{R}(\alpha)$ . Let  $L^p(\alpha)$  be all those bounded  $f[a, b] \rightarrow \mathbb{R}$  with  $\|f\|_p < \infty$ . The spaces of function  $L^p(\alpha)$  are normed vector spaces with norm  $\|\cdot\|_p$ . We want to see if  $f \in L^p(\alpha)$  and  $g \in L^q(\alpha)$  with  $\frac{1}{p} + \frac{1}{q} = 1$  we have  $fg \in L^1(\alpha)$  and

$$\|fg\|_1 \leq \|f\|_p \|g\|_q \tag{†}$$

We can replace  $f$  with  $\hat{f} = \frac{f}{\|f\|_p}$  and  $g$  with  $\hat{g} = \frac{g}{\|g\|_q}$ , then we have  $\|\hat{f}\|_p = 1 = \|\hat{g}\|_q$  and from above

$$\|\hat{f}\hat{g}\|_1 \leq 1 = \frac{\|\hat{f}\|_p^p}{p} + \frac{\|\hat{g}\|_q^q}{q}$$

But from this we have

$$\left\| \frac{f}{\|f\|_p} \frac{g}{\|g\|_q} \right\| = \frac{1}{\|f\|_p} \frac{1}{\|g\|_q} \|fg\|_1 \leq 1$$

From this (†) follows immediately.

**Problem 9** (Functions with only countable many discontinuities are integrable.). Let  $f$  be bounded on  $[a, b]$  with at most countable many discontinuities on  $[a, b]$ . Let  $\alpha : [a, b] \rightarrow \mathbb{R}$  is monotonic increasing and  $\alpha$  is continuous at every discontinuity of  $f$ . Show that  $f \in \mathcal{R}(\alpha)$ .

Hint: Fix an enumeration  $S = \{s_i \mid i \in \mathbb{N}\}$  of the discontinuities of  $f$ . Fix  $\epsilon > 0$  and  $\epsilon_i > 0$  so that  $\sum_i \epsilon_i \leq \epsilon$ . Since  $\alpha$  is continuous at  $s_i$  fix  $\delta_i$  so that  $\alpha(N_{\delta_i}(s_i)) \subset N_{\epsilon_i}(\alpha(s_i))$ . For  $x \notin S$ , fix  $\delta_x$  so that  $f(N_{\delta_x}(x)) \subset N_\epsilon(f(x))$ . Now  $\mathcal{O} = \{N_{\delta_i}(s_i) \mid i \in \mathbb{N}\} \cup \{N_{\delta_x}(x) \mid x \notin S\}$  is an open cover of  $[a, b]$ . Apply compactness to get a finite subcover and then do something *similar* (not the same) as in the proof of 6.10.

**Proof 1:** Fix  $\epsilon > 0$  and  $\epsilon_i > 0$  so that  $\sum_i \epsilon_i < \epsilon$ . Let  $S = \{s_i \mid i \in \mathbb{N}\}$  be the discontinuities of  $f$ . Since  $\alpha$  is continuous at  $s_i$  let  $\delta_i$  be so that  $\alpha(N_{\delta_i}(s_i)) \subset N_{\epsilon_i}(\alpha(s_i))$ . For  $x \notin S$  let  $\delta_x$  be chosen so that  $f(N_{\delta_x}(x)) \subset N_\epsilon(f(x))$ . Let  $\mathcal{O} = \{N_{\delta_i}(s_i) \mid i \in \mathbb{N}\} \cup \{N_{\delta_x}(x) \mid x \notin S\}$  be the associated open cover of  $[a, b]$ . Let  $\mathcal{O}' \subseteq \mathcal{O}$  be a finite subcover of  $[a, b]$ . Notice that  $\mathcal{O}'$  consists of intervals  $(u_i, v_i)$  and we may assume that  $a = u_0 < u_1 < v_0 < u_2 < v_1 < u_3 < v_2 \cdots$  (a “chain”). Thus we define  $x_0 = a < x_1 = (u_1 + v_0)/2 < x_2 = (u_2 + v_1)/2 < \cdots < x_{n-1} = (u_{n-1} + v_{n-2})/2 < x_n = v_n = b$ . Thus  $[x_{i-1}, x_i] \subset N_{\delta_j}(s_j)$  for some  $j$  or  $[x_{i-1}, x_i] \subset N_{\delta_x}(x)$  for some  $x \notin S$ .

Let  $T = \{i \mid [x_{i-1}, x_i] \subset N_{\delta_i}(s_i) \text{ for some } s_i \in S\}$ . Then letting  $|f(x)| \leq M$  and  $\alpha(b) - \alpha(a) = N$ :

$$\begin{aligned} \sum_{i=1}^n |M_i - m_i|(\alpha(x_i) - \alpha(x_{i-1})) &= \sum_{i \in T} |M_i - m_i|(\alpha(x_i) - \alpha(x_{i-1})) + \sum_{i \notin T} |M_i - m_i|(\alpha(x_i) - \alpha(x_{i-1})) \\ &< \sum_{i \in T} 2M\epsilon_i + \sum_{i \notin T} \epsilon \alpha(x_i) - \alpha(x_{i-1}) \\ &\leq 2M\epsilon + N\epsilon = \epsilon(2M + N) \end{aligned}$$

**Proof 2:** (This proof was attempted by everyone in one of my classes. I thought I would write it out correctly so that everyone in that class could see what should have been presented.)

Start like the above. Since  $\alpha$  is continuous at  $s_i$  pick  $(a_i, b_i)$  satisfying:

- $(a_i, b_i) \cap (a_j, b_j) = \emptyset$ . (mutually disjoint)
- $s_i \in (a_i, b_i)$ .
- $\alpha((a_i, b_i)) \subseteq S_{\epsilon/2}(s_i)$  so that if  $t, t' \in (a_i, b_i)$ , then  $|\alpha(t') - \alpha(t)| < \epsilon_i$ . Where  $\sum_i \epsilon_i = \epsilon$  and  $\epsilon_i$  will be chosen at the end.

Let  $K = [a, b] - \bigcup_i (b_i, a_i)$ .  $K$  is closed and bounded, hence compact. Since  $f$  is continuous on  $K$  it is uniformly continuous and thus we can pick  $\delta > 0$  so that  $|x - y| < \delta \implies |f(x) - f(y)| < \epsilon$  for all  $x, y \in K$ .

$\mathcal{O} = \{(x - \delta/2, x + \delta/2) \mid x \in K\} \cup \{(a_i, b_i) \mid i \in \mathbb{N}\}$  is an open cover of  $[a, b]$  and hence has a finite subcover  $\mathcal{O}'$ . Let  $\mathcal{O}' = \{(u_i, v_i) \mid i < m\}$  we may assume that for no  $i \neq j$  do we have  $(u_i, v_i) \subset (u_j, v_j)$ , as we could just toss out  $(u_i, v_i)$  in this case. So  $u_0 < a < u_1 < v_0 < u_2 < v_1 < \cdots < u_{m-1} < v_{m-2} < b < v_{m-1}$ . For  $i = 1, \dots, m-2$  let  $y_i = (u_i + v_{i-1})/2$  and set  $y_0 = a$  and  $y_{m-1} = b$  and let  $P = \{y_i \mid i = 0, \dots, m-1\}$ . Then we know for each  $i = 1, \dots, m-1$  that either  $[u_{i-1}, u_i] \subset (a_j, b_j)$  for some  $j$  or else  $[u_{i-1}, u_i] \subset (x - \delta/2, x + \delta/2)$  for some  $x \in K$ .

Let  $A = \{i \mid [u_{i-1}, u_i] \subset (a_{j_i}, b_{j_i}) \text{ for some } j_i\}$ , then for  $i \in A$  we have  $\Delta\alpha_i = \alpha(u_i) - \alpha(u_{i-1}) <$

$\epsilon_{j_i}$  and for  $i \notin A$ ,  $|M_i^{f,P} - m_i^{f,P}| < \epsilon$ . Thus

$$\begin{aligned} U(f, P, \alpha) - L(f, P, \alpha) &= \sum_{i=1}^{m-1} |M_i^{f,P} - m_i^{f,P}| \Delta \alpha_i \\ &\leq \sum_{i \in A} |M_i^{f,P} - m_i^{f,P}| \epsilon_{j_i} + \sum_{i \notin A} \epsilon \Delta \alpha_i \leq 2M\epsilon + \epsilon(\alpha(b) - \alpha(a)) \end{aligned}$$

where  $M = \sup |f(x)|$  on  $[a, b]$ .

Since  $M$  and  $\alpha(b) - \alpha(a)$  are fixed constants we can make the  $\epsilon(2M + \alpha(b) - \alpha(a))$  arbitrarily small. Thus  $f \in \mathcal{R}(\alpha)$ .

**Problem 10** (An integrable function with uncountable many discontinuities.). Let  $\mathcal{C}$  be the Cantor set and  $f$  be defined by

$$f(x) = \begin{cases} 1 & x \in \mathcal{C} \\ 0 & x \notin \mathcal{C} \end{cases}$$

Show that  $f \in \mathcal{R}$ , namely,  $\int_0^1 f dx = 0$ . That  $f$  has uncountably many points of discontinuity is clear since each point of  $\mathcal{C}$  is a discontinuity of  $f$  and  $\mathcal{C}$  is perfect, hence uncountable.

**Proof 1:** The argument from Problem 7 works here. Basically, that argument showed that if  $g = f$  off of a measure zero set, then  $f \in \mathcal{R} \iff g \in \mathcal{R}$  and  $\int_a^b f dx = \int_a^b g dx$ . So here take  $g = 0$  on  $[0, 1]$ .

**Proof 2:** (From a student.) Let  $P_i$  be the partition that breaks  $[0, 1]$  into  $3^i$  many pieces. Let  $C_i$  be the  $i^{\text{th}}$  approximation of the Cantor set, so  $C_0 = [0, 1]$ ,  $C_1 = [0, 1/3] \cup [2/3, 1]$ , etc. In the picture, the black part represents the closed sets  $C_i$ .



Notice that on each of these sections  $M_j - m_j = 1$  since there are always points outside  $\mathcal{C}$  in any interval, yet the endpoints are always in  $\mathcal{C}$ . The same argument holds for the orange segments. In  $P_i$ ,  $2^i$  pieces of the partition are black. The orange segments satisfy a recurrence, namely, if there are  $a_i$  in  $P_i$ , then there are  $2 \cdot a_i + 2^i$  in  $P_{i+1}$  since each black gives one orange and each orange splits into two orange. So  $a_{i+1} = 2a_i + 2^i$ . You can check that  $a_i = i2^{i-1}$ . In particular,  $(0)(2^{-1}) = 0$  and  $(i+1)2^i = i2^i + 2^i = 2(i2^{i-1}) + 2^i = 2a_i + 2^i$  as required. So there are  $i2^{i-1} + 2^i = (i+2)2^{i-1}$  many elements of the partition where  $M_i - m_i = 1$  so that

$$U(f, P) - L(f, P) = \frac{(i+2)}{3} \left(\frac{2}{3}\right)^{i-1} \rightarrow 0 \text{ as } i \rightarrow \infty$$

The following is for a future class, but it came up here so I wanted to record it. Let  $\alpha : [a, b] \rightarrow \mathbb{R}$  and say  $Z \subseteq [a, b]$  has  $\alpha$ -measure 0 iff for all  $\epsilon > 0$  there is  $(a_i, b_i)$  so that  $Z \subseteq \bigcup_{i=0}^{\infty} (a_i, b_i)$

and  $\sum_{i=0}^{\infty} \alpha(b_i) - \alpha(a_i) < \epsilon$ . The argument above works for  $\mathcal{R}(\alpha)$  with  $\alpha$ -measure zero replacing measure 0.

Notice that if  $Z$  has  $\alpha$ -measure zero and  $z \in Z$ , then  $\alpha$  is continuous at  $z$ . To see this let  $\epsilon > 0$  and take  $\{(a_i, b_i) \mid i \in \mathbb{N}\}$  covering  $Z$  with  $\sum_i \alpha(b_i) - \alpha(a_i) < \epsilon$ . Then  $z \in (a_i, b_i)$  and clearly  $\alpha((a_i, b_i)) \subset N_\epsilon(\alpha(z))$ , since  $\alpha(b_i) - \alpha(a_i) < \epsilon$ . So if  $Z$  is the set of discontinuities of  $f$ , then  $\alpha$  must be continuous at each  $z \in Z$ .

**Problem 11.** Show that if  $f[a, b] \rightarrow \mathbb{R}$  is bounded and  $Z = \{x \mid f \text{ is discontinuous at } x\}$  is countable  $\alpha$  is continuous at each in  $Z$ , then  $Z$  has  $\alpha$ -measure zero.

**Problem 12** (Generalization of Problem 7). Show that if  $f : [a, b] \rightarrow \mathbb{R}$  is bounded and  $\alpha : [a, b] \rightarrow \mathbb{R}$  is monotonic increasing with  $f$  discontinuous on a set  $Z$  of  $\alpha$ -measure zero with  $\alpha$  continuous at each point in  $Z$ , then  $f \in \mathcal{R}(\alpha)$ .

**Problem 13.** Let  $f, g : [a, b] \rightarrow \mathbb{R}$  and let  $Z = \{z \mid g(z) \neq f(z)\}$ . If  $Z$  has  $\alpha$ -measure zero show that

- i)  $f \in \mathcal{R}(\alpha) \iff g \in \mathcal{R}(\alpha)$
- ii) If  $f \in \mathcal{R}(\alpha)$ , then  $\int_a^b f d\alpha = \int_a^b g d\alpha$