Math 571 - (Take-Home) Exam 2 (04.24)

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There are 80 points here, so basically, 10 extra points. I'll take the score and use the minimum of 70 and your score as the final grade. Make your answers self-contained. If something here comes straight out of the homework, do not "quote" the homework result as a reason. I am looking for the argument. If in doubt ask!

Question 1 (20 pts)**.** Give a reason for the non-existence of each of the following, or else provide an example. You may use theorems, but you must state the theorem; a theorem number does not suffice. In either case, you must show that your example has the required properties.

(a) A continuous function $f : S_1 \to \mathbb{Q}$ that is not a constant function where $S_1 = \{(x, y) \mid x^2 + y^2 = 1\}$ is the unit circle.

(b) A continuous function $f : \mathbb{Q} \to \mathbb{Z}$ that is not a constant function.

(c) A function $f : \left\{\frac{1}{n} \mid n \in \mathbb{Z} - \{0\}\right\} \to \mathbb{R}$ that fails to be continuous.

(d) A continuous non-constant 1-1 function $f : S_1 \to \mathbb{R}$, where $S_1 = \{(x, y) \mid x^2 + y^2 = 1\}$ is the unit circle.

Definition 1. Let $f_n : [a, b] \to \mathbb{R}$, define $f_n \underset{\text{unif}}{\to} f$ to mean:

$$(\forall \varepsilon > 0)(\exists N > 0)(\forall n > N)(\forall x \in [a, b]) \left(|f_n(x) - f(x)| < \varepsilon\right)$$

Contrast this to the standard *pointwise convergence* where $f_n \to f$ iff

$$(\forall x \in [a, b])(\forall \varepsilon > 0)(\exists N > 0)(\forall n > N) \left(|f_n(x) - f(x)| < \varepsilon\right)$$

The point is that in the uniform case, we get for every $\varepsilon > 0$, a fixed $N_\varepsilon > 0$, which *works* for all x uniformly, whereas in the pointwise case, we have for each $x \in [a, b]$ and $\varepsilon > 0$, a $N_\varepsilon(x)$ that does the job..

Question 2 (15 pts)**.** Let $f_n \in \mathcal{R}([a, b])$ (Riemann integrable functions on $[a, b]$), suppose $f_n \underset{\text{unif}}{\to} f$. Show that $f \in \mathcal{R}([a, b])$ and $\lim_n \int_a^b f(x)\,dx = \int_a^b \lim_n f_n(x)\,dx = \int_a^b f(x)\,dx$.

Question 3 (15 pts)**.** (1) Let $f : \mathbb{R} \to \mathbb{R}$ be differentiable, suppose f' is bounded. Show that f is uniformly continuous.

(2) Find a uniformly continuous and differentiable function $f : \mathbb{R} \to \mathbb{R}$ whose derivative is not bounded. You must show that your example id uniformly continuous.

Question 4 (15 pts)**.** Let X and Y be metric spaces with X **compact** and let $f : X \to Y$ and $g : X \to Z$ be two functions, with no additional assumptions on these functions.

Suppose that for every $x \in X$, at least one of f or g is continuous at x. Show that for every $\epsilon > 0$, there is a $\delta > 0$ such that for all $x, x' \in X$:

$$d_X(x, x') < \delta \implies d_Y(f(x), f(x')) < \epsilon \text{ or } d_Z(g(x), g(x')) < \epsilon$$

This is sort of an "either/or" version of uniform continuity.

Question 5 (15 pts)**.** Let f and α be monotonically increasing bounded functions on $[a, b]$. Suppose that α is continuous at every point where f is discontinuous. Show that $f \in \mathscr{R}(\alpha)$.