Math 571 - Homework 7

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Problem 7.1 (R:5:26)**.** Suppose $f(x)$ is differentiable on $[a, b]$, $f(a) = 0$, and there is a fixed A such that $|f'(x)| \leq A|f(x)|$ for all x in $[a, b]$. Show that $f(x) = 0$ on $[a, b]$.

Problem 7.2 (R:5:27)**.** Let $\phi : [a, b] \times [\alpha, \beta] \to \mathbb{R}$. A *solution to the initial–value problem* (IVP)

$$y' = \phi(x, y), \quad y(a) = c \text{ for } a \leq c \leq b$$

is a function $f : [a, b] \to [\alpha, \beta]$ satisfying

$$f(a) = c, \quad f'(x) = \phi(x, f(x)) \text{ for all } a \leq x \leq b$$

Show that if there is a constant $A \geq 0$ so that

$$|\phi(x, y_1) - \phi(x, y_2)| \leq A|y_1 - y_2| \text{ for all } x \in [a, b] \text{ and } y_1, y_2 \in [\alpha, \beta],$$

then there is at most one solution to any such IVP.

Problem 7.3. Show that the following are equivalent for a bounded function f on $[a, b]$:

i) $f \in \mathcal{R}$, i.e., f is Riemann integrable,

ii) For all $\epsilon > 0$, there is a $\delta > 0$ such that

$$\|P\| < \delta \implies U(f, P) - L(f, P) < \epsilon$$

Problem 7.4 (R:6:1)**.** Suppose $\alpha : [a, b] \to \mathbb{R}$ is monotonic increasing and continuous at $x_0 \in [a, b]$. consider $f : [a, b] \to \{0, 1\}$ given by $f(x_0) = 1$ and $f(x) = 0$ for $x \neq x_0$. Show that $f \in \mathcal{R}(\alpha)$ and $\int_a^b f \, d\alpha = 0$.

Problem 7.5 (R:6:2)**.** Suppose $f : [a, b] \to \mathbb{R}$ is continuous, $f \geq 0$, and $\int_a^b f \, dx = 0$, then $f = 0$.

Note that where Rudin asks you to compare with (1), there might be the thought that these do not compare since (1) is about $\mathcal{R}(\alpha)$ while (2) is about $\mathcal{R}$, but taking $\alpha = \text{id}$ in (1) allows you to make the comparison.

Problem 7.6 (R:6:3)**.** Define $\beta_i : [-1,1] \to [0,1]$ by $\beta_i = 0$ for $x < 0$ and $\beta_i = 1$ for $x > 0$, then $\beta_1(0) = 0$, $\beta_2(0) = 1$, and $\beta_3(0) = 1/2$. In particular β_i has a simple discontinuity at 0 with $\beta_1(0-) = \beta_1(0) = 0$ (continuous from the left), $\beta_2(0+) = \beta_2(0) = 1$ (continuous from the right), while β_3 is neither continuous from the left or right. Let $f : [-1,1] \to \mathbb{R}$ be bounded. show that

i) $f \in \mathcal{R}(\beta_1)$ iff $f(0+) = f(0)$, that is, f is continuous from the right at 0.

ii) $f \in \mathcal{R}(\beta_2)$ iff $f(0-) = f(0)$, that is, f is continuous from the left at 0.

iii) $f \in \mathcal{R}(\beta_3)$ iff f is continuous at 0.

Problem 7.7 (R:6:6)**.** Let $f : [0,1] \to \mathbb{R}$ be bounded and continuous off of the Cantor set $\mathcal{C}$. Show that $f \in \mathcal{R}$.

Problem 7.8 (R:6:10)**.** See text. This is mostly done in the notes.

Problem 7.9 (Functions with only countable many discontinuities are integrable.)**.** Let f be bounded on $[a,b]$ with at most countable many discontinuities on $[a,b]$. Let $\alpha : [a,b] \to \mathbb{R}$ is monotonic increasing and α is continuous at every discontinuity of f. Show that $f \in \mathcal{R}(\alpha)$.

Hint: Fix an enumeration $S = \{s_i \mid i \in \mathbb{N}\}$ of the discontinuities of f. Fix $\epsilon > 0$ and $\epsilon_i > 0$ so that $\sum_i \epsilon_i \leq \epsilon$. Since α is continuous at s_i fix δ_i so that $\alpha(N_{\delta_i}(s_i)) \subset N_{\epsilon_i}(\alpha(s_i))$. For $x \notin S$, fix δ_x so that $f(N_{\delta_x}(x)) \subset N_\epsilon(f(x))$. Now $\mathcal{O} = \{N_{\delta_i}(s_i) \mid i \in \mathbb{N}\} \cup \{N_{\delta_x}(x) \mid x \notin S\}$ is an open cover of $[a,b]$. Apply compactness to get a finite subcover and then do something ***similar*** (not the same) as in the proof of 6.10.

Problem 7.10 (An integrable function with uncountable many discontinuities.)**.** Let $\mathcal{C}$ be the Cantor set and f be defined by

$$f(x) = \begin{cases} 1 & x \in \mathcal{C} \\ 0 & x \notin \mathcal{C} \end{cases}$$

Show that $f \in \mathcal{R}$, namely, $\int_0^1 f\, dx = 0$. That f has uncountably many points of discontinuity is clear since each point of $\mathcal{C}$ is a discontinuity of f and $\mathcal{C}$ is perfect, hence uncountable.

The following is for a future class, but it came up here so I wanted to record it. Let $\alpha : [a,b] \to \mathbb{R}$ and say $Z \subseteq [a,b]$ has α-measure 0 iff for all $\epsilon > 0$ there is (a_i, b_i) so that $Z \subseteq \bigcup_{i=0}^{\infty} (a_i, b_i)$ and $\sum_{i=0}^{\infty} \alpha(b_i) - \alpha(a_i) < \epsilon$. The argument above works for $\mathcal{R}(\alpha)$ with α-measure zero replacing measure 0.

Notice that if Z has α-measure zero and $z \in Z$, then α is continuous at z. To see this let $\epsilon > 0$ and take $\{(a_i, b_i) \mid i \in \mathbb{N}\}$ covering Z with $\sum_i \alpha(b_i) - \alpha(a_i) < \epsilon$. Then $z \in (a_i, b_i)$ and clearly $\alpha((a_i, b_i)) \subset N_\epsilon(\alpha(z))$, since $\alpha(b_i) - \alpha(a_i) < \epsilon$. So if Z is the set of discontinuities of f, then α must be continuous at each $z \in Z$.

Problem 7.11. Show that if $f[a,b] \to \mathbb{R}$ is bounded and $Z = \{x \mid f \text{ is discontinuous at } x\}$ is countable α is continuous at each in Z, then Z has α-measure zero.

Problem 7.12 (Generalization of Problem 7)**.** Show that if $f : [a, b] \to \mathbb{R}$ is bounded and $\alpha : [a, b] \to \mathbb{R}$ is monotonic increasing with f discontinuous on a set Z of α-measure zero with α continuous at each point in Z, then $f \in \mathcal{R}(\alpha)$.

Problem 7.13. Let $f, g : [a, b] \to \mathbb{R}$ and let $Z = \{z \mid g(z) \neq f(z)\}$. If Z has α-measure zero show that

i) $f \in \mathcal{R}(\alpha) \iff g \in \mathcal{R}(\alpha)$

ii) If $f \in \mathcal{R}(\alpha)$, then $\int_a^b f \, d\alpha = \int_a^b g \, d\alpha$