## Part IV: Proofs (15 points each; 60 points)

a) Let $A$ be an $n \times n$ matrix. Prove that

$$AB = BA \text{ for all } B \text{ iff } A = \alpha I_n$$

where $I_n$ is the $n \times n$ identity matrix and $\alpha \in \mathbb{R}$.

As always with "if and only if" statements there are two directions to do

**If case** ($\implies$): This is the harder direction. We must somehow show that $A$ has a certain form. First let's show that $A$ is diagonal. Let $a_{ij}$ be the entries of $A$ and suppose that $a_{ij} \neq 0$ for some $i \neq j$. Let $B$ be the matrix $b_{ii} = b$ and $b_{jj} = b'$ and $b_{lm} = 0$ for all other $l$ and $m$. Then

$$(AB)_{ij} = \sum_{k=1}^{n} a_{ik} b_{kj} = b_{jj} A_{ij} = b a_{ij}$$

$$(BA)_{ij} = \sum_{k=1}^{n} b_{ik} a_{kj} = b_{ii} a_{ij} = b' a_{ij}$$

So $(BA)_{ij} \neq (AB)_{ij}$. So $AB \neq BA$.

This shows that $A = \operatorname{diag}(a_{11}, \ldots, a_{nn})$. Suppose $a_{ii} \neq a_{jj}$ let $B$ be a matrix with a single 1 in the $ij$ position, then $(AB)_{ij} = a_{ii} b_{ij} = a_{ii}$ and $(BA)_{ij} = b_{ij} a_{jj} = a_{jj}$ so $AB \neq BA$.

**Only if case** ($\Longleftarrow$): This is trivial $(\alpha I)B = \alpha(IB) = \alpha B = (BI)\alpha = B(\alpha I)$.

b) **Prove:** For an invertible $n \times n$ matrix. Show that $A^{-1}$ is symmetric if $A$ is symmetric.

This is a simple computation:

$$(A^{-1})^T A = (A^{-1})^T A^T = (AA^{-1})^T = I^T = I.$$

Since $(A^{-1})^T A = I$ we know that $A^{-1} = (A^{-1})^T$.

c) **Prove:** If $V = U + W$, then

$$\begin{aligned} & U \cap W = \{\mathbf{0}\} \\ \iff & \text{ every } \boldsymbol{v} \in V \text{ can be written uniquely as } \boldsymbol{v} = \boldsymbol{u} + \boldsymbol{w} \text{ for some } \boldsymbol{u} \in U \text{ and } \boldsymbol{v} \in V \\ \iff & (\forall \boldsymbol{v} \in V)(\forall \boldsymbol{u}, \boldsymbol{u}' \in U)(\forall \boldsymbol{w}, \boldsymbol{w}' \in W)(\boldsymbol{v} = \boldsymbol{u} + \boldsymbol{w} = \boldsymbol{u}' + \boldsymbol{w}' \implies \boldsymbol{u} = \boldsymbol{u}' \text{ and } \boldsymbol{w} = \boldsymbol{w}') \end{aligned}$$

Remark: $V = U \oplus W$ is defined as $V = U + W$ and $U \cap W = \{\mathbf{0}\}$. This result means that $V = U \oplus W$ iff every $\boldsymbol{v} \in V$ is uniquely decomposed as $\boldsymbol{v} = \boldsymbol{u} + \boldsymbol{w}$ where $\boldsymbol{u} \in U$ and $\boldsymbol{w} \in W$. For this reason, this is often taken as the definition of $V = U \oplus W$.

d) **Prove:** Suppose $\mathbb{R}^n = U \oplus W$ and $A$ and $B$ are matrices so that

- $\operatorname{rng}(A) = U$, $A^2 = A$, and $\boldsymbol{x} - A\boldsymbol{x} \in W$ for all $\boldsymbol{x} \in \mathbb{R}^n$,

- $\text{rng}(B) = U$, $B^2 = B$, and $\boldsymbol{x} - B\boldsymbol{x} \in W$ for all $\boldsymbol{x} \in \mathbb{R}^n$.

Show that $A = B$.

You know that for all $\boldsymbol{x}$, $\boldsymbol{x} = \boldsymbol{x}_U + \boldsymbol{x}_W$ for unique $\boldsymbol{x}_U \in U$ and $\boldsymbol{x}_W \in W$. Since $\boldsymbol{x} = A\boldsymbol{x} + (\boldsymbol{x} - A\boldsymbol{x})$ we know that $A\boldsymbol{x} = \boldsymbol{x}_U$ and $\boldsymbol{x} - A\boldsymbol{x} = \boldsymbol{x}_W$. Similarly for $B$ so that $B\boldsymbol{x} = \boldsymbol{x}_U = A\boldsymbol{x}$ for all $\boldsymbol{x}$.

Now we need to see that if for all $\boldsymbol{x}$, $A\boldsymbol{x} = B\boldsymbol{x}$, then $A = B$. It is clear that for all $\boldsymbol{x}$, $(A - B)\boldsymbol{x} = \mathbf{0}$.

It suffices to show that if for all $\boldsymbol{x}$, $C\boldsymbol{x} = \mathbf{0}$, then $C = \mathbf{0}$. This is easy, $C\boldsymbol{e}_i = c_{\cdot i} = \mathbf{0}$, where $\boldsymbol{c}_{\cdot i}$ is the i$^{\text{th}}$ column of $C$. Clearly if ll columns of $C$ are the $\mathbf{0}$ column vector, then $C = \mathbf{0}$.