Exam 1

- This exam covers Topics 1 - 3, Topic 4 will not be covered here.
- I will write $(a_1, a_2, \ldots, a_n)$ in place of $\begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix}$ to save space on occasion. The book writes $\begin{bmatrix} a_1 & a_2 & \cdots & a_n \end{bmatrix}^T$ for the same purpose.
- I use NS(A) for the null space of A, RS(A) for the row space of A, CS(A) for the column space of A. Note that CS$(A) = \text{rng}(A)$ is the range of A.
- When I say you can use some fact from another part of the exam, this means that you can use the fact whether or not you have completed that part of the exam correctly.
- If you have questions ask via Remind or email.
- Use only arguments that you fully understand. I am aware that you can find solutions to some of these online. I am also aware that some of these solutions use concepts and theorems far past what we have covered in Ch 1 - 3. If you use such ideas, then I will ask you to verbally explain your solution so that I can verify that you understand what you have submitted as your own work. In short, **this is an exam** and the usual expectations of academic honesty apply.

Part I: True/False (5 points each; 25 points)

For each of the following mark as true or false.

a) _____ $\mathrm{tr}(AB) = \mathrm{tr}(BA)$ for an $n \times n$ matrices A and B, where

$$\mathrm{tr}(C) = \sum_{i=1}^{n} C_{ii} = \text{the sum of the diagonal elements of } C.$$

b) _____ $\mathrm{tr}(AB) = \mathrm{tr}(A)\,\mathrm{tr}(B)$ for an $n \times n$ matrices A, B, and C.

c) _____ If W is a subspace of a vector space V and $\mathcal{B}$ is a basis for V, then B can be restricted to a basis for W.

d) _____ If W is a subspace of a vector space V, then there is a subspace U so that $V = W \oplus U$. Here, $V = U \oplus W$ means $V = U + W$ and $U \cap W = \{\mathbf{0}\}$, equivalently, for every $\boldsymbol{v} \in V$, there is a **unique** $\boldsymbol{u} \in U$ and $\boldsymbol{w} \in W$ so that $\boldsymbol{v} = \boldsymbol{u} + \boldsymbol{w}$.

e) _____ For any $m \times n$ matrices A and B,

$$B = EA \text{ for some invertible } E \iff \mathrm{NS}(A) = \mathrm{NS}(B).$$

Part II: Definitions and Theorems (5 points each; 25 points)

a) Define what it means for a set of vectors $\mathcal{B} = \{\boldsymbol{v}_1, \ldots, \boldsymbol{v}_n\}$ from a real vector space V to span V.

b) Define what it means for a set of vectors $\{\boldsymbol{v}_1, \ldots, \boldsymbol{v}_n\}$ from a real vector space V to be linearly independent.

c) Define what it means for a set of vectors $\mathcal{B} = \{\boldsymbol{v}_1, \ldots, \boldsymbol{v}_n\}$ to be a basis for a vector space V.

d) State the Rank-Nullity Theorem.

e) What conditions must be checked to verify that $W \subseteq V$ is a subspace of a vector space. V

Part III: Computational (15 points each; 45 point)

a) Given that A is a 3×4 matrix and

$$\mathrm{NS}(A) = \mathrm{span}\left(\left\{\begin{bmatrix} 1 \\ -2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 3 \\ 0 \\ 1 \end{bmatrix}\right\}\right)$$

compute $\mathrm{rref}(A)$. Make sure to explain how you arrive at your result. You may use (a) from the "Proofs" part below.

b) For the same (unknown) A used in (a) for each of $\mathrm{RS}(A)$ and $\mathrm{CS}(A)$ find a basis if possible and explain how you know that you have found a basis; if it is not possible to find a basis, then explain why it is not.

c) Show that the upper-triangular $n \times n$ matrices form a subspace of all $n \times n$ matrices and find a basis for this subspace.

Part IV: Proofs (15 points each; 60 points)

Provide complete arguments/proofs for the following.

a) Show that if A and B are 3×4 rref matrices, then

$$\boxed{A = B \iff \mathrm{NS}(A) = \mathrm{NS}(B).}$$

Note: the 3×4 is a red-herring, this holds for arbitrary $m \times n$ matrices. If this helps you , then just prove this more general result. Also notice that this actually gives

$$\mathrm{rref}(A) = \mathrm{rref}(B) \iff \mathrm{NS}(A) = \mathrm{NS}(B)$$

since $\mathrm{NS}(\mathrm{rref}(A)) = \mathrm{NS}(A)$, trivially.

b) **Prove:** For $m \times n$ matrices A and B define $A \underset{\text{row}}{\sim} B$ to mean that you can get from A to B by a series of elementary row operations. Use the $m \times n$ version of (a), namely: $\text{rref}(A) = \text{rref}(B) \iff \text{NS}(A) = \text{NS}(B)$ to show that

$$\boxed{A \underset{\text{row}}{\sim} B \iff \text{rref}(A) = \text{rref}(B)}$$

Remark: Using elementary matrices one can show

$$A \underset{\text{row}}{\sim} B \iff A = EB \text{ for some invertible matrix } E$$

This is done in the text and in my notes. So you get here

$$A \underset{\text{row}}{\sim} B \iff \text{rref}(A) = \text{rref}(B) \iff A = EB \text{ for some invertible matrix } E$$

c) **Prove:** Let A be an $m \times n$ matrix, $\mathbb{R}^n = \mathrm{NS}(A) \oplus \mathrm{RS}(A)$.

Recall: $V = U \oplus W$ means $V = U + W = \{\boldsymbol{u} + \boldsymbol{w} \mid \boldsymbol{u} \in U \text{ and } \boldsymbol{w} \in W\}$ and $U \cap W = \{\mathbf{0}\}$.

d) **Prove:** If A is an $n \times n$ matrix and $A^k = \mathbf{0}$ for any k, then $A^n = \mathbf{0}$.