

Math 571 - Exam 1

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NOTATION/DEFINITION: Let (X, d) be a metric space for $A, B \subset X$ define $d(A, B) = \sup\{d(a, b) \mid a \in A \text{ and } b \in B\}$. Set $d(a, B) = d(\{a\}, B)$.

Question 1 (12 points). Let (X, d) be a metric space, prove that

- a) For any closed set F and $x \notin F$, $d(x, F) > 0$.
- b) For any compact K and closed F with $K \cap F = \emptyset$, $d(K, F) > 0$.
- c) Can the assumption that K is compact be replaced by K closed in (b)? That is, is there a metric space (X, d) and closed sets A, B so that $A \cap B = \emptyset$ and yet $d(A, B) = 0$?

RECALL: In a metric space (X, d) , $\text{diam}(A) = \inf\{d(a, b) \mid a, b \in A\}$.

Question 2 (12 pts). Let (X, d) be a metric space prove or disprove each of the following:

- a) $\text{diam}(A) = \text{diam}(\text{Cl}(A))$.
- b) $\text{diam}(A) = \text{diam}(\text{Int}(A))$.

Question 3 (12 pts). Let (X, d) be a metric space and $(x_i)_{i \in \mathbb{N}}$ and $(y_i)_{i \in \mathbb{N}}$ be two Cauchy sequences. Show that $(d(x_i, y_i))_{i \in \mathbb{N}}$ converges.

For the next problem, $(x_{i_k})_{k=0}^\infty$ is a **subsequence** of $(x_i)_{i=0}^\infty$ means $i_0 < i_1 < \dots$. A sequence $(x_i)_{i=0}^\infty$ is **monotone increasing** iff $x_0 \leq x_1 \leq x_2 \dots$. Similarly define **monotone decreasing**. A sequence is **monotone** iff it is either monotone increasing or monotone decreasing.

Question 4 (12 pts). Show that every infinite sequence of real numbers has a monotone subsequence that converges to $\limsup_i x_i$.

NOTE: The same is true for $\liminf_i x_i$.

Question 5 (Is supremum “linear”; 12 pts). For $A, B \subseteq \mathbb{R}$, is it true that

- i) $\sup(\alpha A) = \alpha \sup(A)$ for $\alpha \geq 0$, and
- ii) $\sup(A + B) = \sup(A) + \sup(B)$.

Question 6 (Compact sets get crowded; 15 pts). Show that if X is compact, then for any $\epsilon > 0$, there is $N > 0$ so that for all $S \subset X$ with $|S| \geq N$, there are two points in S whose distance is $< \epsilon$.