Name: ________________________ Quiz 2 - MAT345

Problem Q2.1 (20 points; 4 points each). Decide if each of the following is true or false. As usual, you may supply reasons for some points back.

Let $V = \operatorname{span}\{\boldsymbol{v_1}, \boldsymbol{v_2}, \boldsymbol{v_3}, \boldsymbol{v_4}, \boldsymbol{v_5}\}$ where

$$\boldsymbol{v}_1 = \begin{bmatrix} -1 \\ -5 \\ 5 \\ 3 \end{bmatrix} \quad \boldsymbol{v}_2 = \begin{bmatrix} -3 \\ -16 \\ 20 \\ 8 \end{bmatrix} \quad \boldsymbol{v}_3 = \begin{bmatrix} 2 \\ 7 \\ 5 \\ -9 \end{bmatrix} \quad \boldsymbol{v}_4 = \begin{bmatrix} -2 \\ -12 \\ 19 \\ 7 \end{bmatrix} \quad \boldsymbol{v}_5 = \begin{bmatrix} 1 \\ 3 \\ 7 \\ -9 \end{bmatrix}$$

and we have

$$\operatorname{rref}\begin{bmatrix} -1 & -3 & 2 & -2 & 1 \\ -5 & -16 & 7 & -12 & 3 \\ 5 & 20 & 5 & 19 & 7 \\ 3 & 8 & -9 & 6 & -9 \end{bmatrix} = \begin{bmatrix} 1 & 0 & -11 & 0 & -15 \\ 0 & 1 & 3 & 0 & 6 \\ 0 & 0 & 0 & 1 & -2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}; \operatorname{rref}\begin{bmatrix} -1 & -5 & 5 & 3 \\ -3 & -16 & 20 & 8 \\ 2 & 7 & 5 & -9 \\ -2 & -12 & 19 & 7 \\ 1 & 3 & 7 & -9 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 32 \\ 0 & 1 & 0 & -9 \\ 0 & 0 & 1 & -2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

(a) False $\mathcal{B}$ is a basis for V where

$$\mathcal{B} = \left\{ \begin{bmatrix} -1 \\ -5 \\ 5 \\ 3 \end{bmatrix}, \begin{bmatrix} -3 \\ -16 \\ 20 \\ 8 \end{bmatrix}, \begin{bmatrix} 2 \\ 7 \\ 5 \\ -9 \end{bmatrix} \right\}$$

We know from the matrix where the $\boldsymbol{v}_i$'s are the columns that $\{\boldsymbol{v_1}, \boldsymbol{v_2}, \boldsymbol{v_4}\}$ is a basis and in particular that $\boldsymbol{v}_3 \in \operatorname{span}\{\boldsymbol{v_1}, \boldsymbol{v_2}\}$.

(b) True $\mathcal{B}'$ is a basis for V where

$$\mathcal{B}' = \left\{ \begin{bmatrix} -1 \\ -5 \\ 5 \\ 3 \end{bmatrix}, \begin{bmatrix} -3 \\ -16 \\ 20 \\ 8 \end{bmatrix}, \begin{bmatrix} 1 \\ 3 \\ 7 \\ -9 \end{bmatrix} \right\}$$

Note: This is true, but there is a mistake in intent. I should have had columns 1, 2, and 4, not 5. It is still true that columns 1, 2, and 5 of $\operatorname{rref}(A)$ are linearly independent, and so do form a basis for $\operatorname{CS}(\operatorname{rref}(A))$ and hence 1, 2, and 5 are a basis for $\operatorname{CS}(A)$. However, this was not my intent. See answer to (a).

(c) True $\mathcal{B}''$ is a basis for V where

$$\mathcal{B}'' = \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \\ 32 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ -9 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ -2 \end{bmatrix} \right\}$$

We see this by looking at the matrix where the $\boldsymbol{v}_i$'s are the rows.

(d) True If $\{\boldsymbol{v}_1, \dots, \boldsymbol{v}_n\}$ spans a vector space V and $\{\boldsymbol{u}_1, \dots, \boldsymbol{u}_n\} \subseteq V$ is independent. Then $\{\boldsymbol{u}_1, \dots, \boldsymbol{u}_n\}$ spans V.

That the $\boldsymbol{v}_i$'s span V tells us that $\dim(V) \leq n$. That the $\boldsymbol{u}_i$'s are independent tells us that $\dim(V) \geq n$. Together we know that $\dim(V) = n$, and hence both the given sets of vectors must be a basis.

(e) True Suppose U and W subspaces of a vector space V such that

$$U + W = V, \text{ and } U \cap W = \{\mathbf{0}\}.$$

Then for every $\boldsymbol{v} \in V$, there is a **unique pair** $\boldsymbol{u} \in U, \boldsymbol{w} \in W$ so that $u + w = v$.

Recall: $U + W = \{\boldsymbol{u} + \boldsymbol{w} \mid \boldsymbol{u} \in U \text{ and } \boldsymbol{w} \in W\}$.

The only issue here is uniqueness since by assumption every $\boldsymbol{v} \in V$ can be written as $\boldsymbol{u} + \boldsymbol{w}$ for some pair $(\boldsymbol{u}, \boldsymbol{w})$. Suppose $\boldsymbol{v} = \boldsymbol{u} + \boldsymbol{w} = \boldsymbol{u}' + \boldsymbol{w}'$, then

$$\mathbf{0} = \boldsymbol{v} - \boldsymbol{v} = (\boldsymbol{u} + \boldsymbol{w}) - (\boldsymbol{u}' + \boldsymbol{w}') = (\boldsymbol{u} - \boldsymbol{u}') - (\boldsymbol{w}' - \boldsymbol{w})$$

so

$$\boldsymbol{w}' - \boldsymbol{w} = \boldsymbol{u} - \boldsymbol{u}' \in U \cap W$$

hence $\boldsymbol{w}' - \boldsymbol{w} = \mathbf{0} = \boldsymbol{u} - \boldsymbol{u}'$ and so $\boldsymbol{u} = \boldsymbol{u}'$ and $\boldsymbol{w} = \boldsymbol{w}'$.

Problem Q2.2 (10 pts)**.** A square matrix A is called **anti-symmetric** if $A^T = -A$.

a) Show that the anti-symmetric 3×3 matrices form a subspace of all 3×3 matrices.

b) Give a basis, $\mathcal{B}$, for the 3×3 anti-symmetric matrices.

c) Give representation $[\boldsymbol{v}]_{\mathcal{B}}$ for $\boldsymbol{v} = \begin{bmatrix} 0 & 1 & 2 \\ -1 & 0 & 3 \\ -2 & -3 & 0 \end{bmatrix}$ with respect to the basis that you gave.

Denote by U the set of 3×3 anti-symmetric matrices.

Proof 1 that U is a subspace: Clearly, $O = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \in U$. Next, we need to show that U is closed under scalar multiplication and addition. This is done by just taking arbitrary elements of U and computing:

$$\begin{bmatrix} 0 & a & b \\ -a & 0 & c \\ -b & -c & 0 \end{bmatrix} + \begin{bmatrix} 0 & A & B \\ -A & 0 & C \\ -B & -C & 0 \end{bmatrix} = \begin{bmatrix} 0 & a+A & b+B \\ -(a+A) & 0 & c+C \\ -(b+B) & -(c+C) & 0 \end{bmatrix}$$

and

$$\alpha \begin{bmatrix} 0 & a & b \\ -a & 0 & c \\ -b & -c & 0 \end{bmatrix} = \begin{bmatrix} 0 & \alpha a & \alpha b \\ -\alpha a & 0 & \alpha c \\ -\alpha b & -\alpha c & 0 \end{bmatrix}$$

(Easier) Proof 2 that U is a subspace: Let $A, B \in U$, then

$$(aA + bB)^T = aA^T + bB^T = a(-A) + b(-B) = -(aA + bB)$$

A basis is given by

$$\mathcal{B} = \left\{ \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix} \right\}$$

With this basis, clearly

$$\boldsymbol{v} = \begin{bmatrix} 0 & 1 & 2 \\ -1 & 0 & 3 \\ -2 & -3 & 0 \end{bmatrix} = (1)\begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} + (2)\begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{bmatrix} + (3)\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix}$$

So $[\boldsymbol{v}]_{\mathcal{B}} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$

Problem Q2.3. Find a basis for $\operatorname{span}\{\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3, \boldsymbol{v}_4, \boldsymbol{v}_5\}$ from the given vectors

$$\boldsymbol{v}_1 = \begin{bmatrix} 1 \\ 2 \\ -2 \\ -3 \end{bmatrix}, \boldsymbol{v}_2 = \begin{bmatrix} 2 \\ 4 \\ -4 \\ -6 \end{bmatrix}, \boldsymbol{v}_3 = \begin{bmatrix} 0 \\ 1 \\ -2 \\ 4 \end{bmatrix}, \boldsymbol{v}_4 = \begin{bmatrix} -3 \\ -4 \\ 2 \\ 17 \end{bmatrix}, \boldsymbol{v}_5 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ -3 \end{bmatrix}$$

Let

$$A = \begin{bmatrix} 1 & 2 & 0 & -3 & 0 \\ 2 & 4 & 1 & -4 & 0 \\ -2 & -4 & -2 & 2 & 1 \\ -3 & -6 & 4 & 17 & -3 \end{bmatrix}$$

$$A \underset{\substack{R_2 - 2R_1 \to R_2 \\ R_3 + 2R_1 \to R_3 \\ R_4 + 3R_1 \to R_4}}{\Longrightarrow} \begin{bmatrix} 1 & 2 & 0 & -3 & 0 \\ 0 & 0 & 1 & 2 & 0 \\ 0 & 0 & -2 & -4 & 1 \\ 0 & 0 & 4 & 8 & -3 \end{bmatrix} \underset{\substack{R_3 + 2R_2 \to R_3 \\ R_4 - 4R_2 \to R_4}}{\Longrightarrow} \begin{bmatrix} 1 & 2 & 0 & -3 & 0 \\ 0 & 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & -3 \end{bmatrix} \underset{R_4 + 3R_3 \to R_4}{\Longrightarrow} \begin{bmatrix} \boxed{1} & 2 & 0 & -3 & 0 \\ 0 & 0 & \boxed{1} & 2 & 0 \\ 0 & 0 & 0 & 0 & \boxed{1} \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

So $\mathcal{B} = \{\boldsymbol{v}_1, \boldsymbol{v}_3, \boldsymbol{v}_5\}$ is a basis. (This is all you need.)

In fact, from our CR decomposition, we know

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ -2 & -2 & 1 \\ -3 & 4 & -3 \end{bmatrix} \begin{bmatrix} \boxed{1} & 2 & 0 & -3 & 0 \\ 0 & 0 & \boxed{1} & 2 & 0 \\ 0 & 0 & 0 & 0 & \boxed{1} \end{bmatrix}$$

So we know $\boldsymbol{v}_2 = 2\boldsymbol{v}_1$ and $\boldsymbol{v}_4 = -3\boldsymbol{v}_1 + 2\boldsymbol{v}_3$.