Exam 1 – Math 215

Problem 1 (30 points; 3 points each)**.** Decide if each of the following are true or false. You do not need to justify your choice here.

(a) _____ $(p \to q) \leftrightarrow (\neg p \wedge q)$ is a tautology.

(b) _____ $(p \to q) \leftrightarrow (q \to p)$ is a tautology.

(c) _____ $\exists x P(x) \wedge \exists x Q(x) \equiv \exists x\,(P(x) \wedge Q(x))$.

(d) _____ $\exists x P(x) \wedge \exists x Q(x) \equiv \exists x \exists y\,(P(x) \wedge Q(y))$.

(e) _____ $\neg(\forall x P(x) \wedge \exists x Q(x)) \equiv \exists x \neg P(x) \vee \forall x \neg Q(x)$

(f) _____ $A \subseteq B \implies C - A \subseteq C - B$

(g) _____ $P(\{\emptyset, \{\emptyset\}\}) = \{\emptyset, \{\emptyset\}, \{\emptyset, \{\emptyset\}\}\}$

(h) _____ $f : \mathbb{Z} \times \mathbb{Z}^+ \to \mathbb{Q}$ given by $f(n, m) = n/m$ is onto or surjective.

(i) _____ $g : \mathbb{Q} \to \mathbb{Z} \times \mathbb{Z}^+$ given by $g(q) = (n, m)$ iff $q = n/m$ where n and m have no common factors is 1-1 or injective.

(j) _____ With f and g as in (h) and (i), $f \circ g : \mathbb{Q} \to \mathbb{Q}$ is the identity on $\mathbb{Q}$, so $f = g^{-1}$.

Problem 2 (Multiple Choice; 40 points; 4 points each). You may select any number of choices, 0 – 4. You get one point per each correct item, meaning if the item should be selected you get a point, if it should not be selected you get a point.

(a) Which of the following are propositions?

- ◯ The rat was a spy.
- ◯ Wait, wait, don't tell me.
- ◯ This sentence is false.
- ◯ $x + 2 = 4$.

(b) Which of the conditionals are true?

- ◯ $5 > 3 \to 5$ is prime.
- ◯ $5 > 3 \to 5$ is not prime
- ◯ $5 < 3 \to 5$ is prime
- ◯ $5 < 3 \to 5$ is not prime.

(c) Which biconditionals are true for arbitrary integer n?

- ◯ $5 > n \leftrightarrow 25 > n^2$.
- ◯ $5 > n \leftrightarrow 5n > n^2$.
- ◯ $5 > n \leftrightarrow 25 > 5n$.
- ◯ $5 > n \leftrightarrow 1/5 < 1/n$.

(d) Which are equivalent to $p \to q$?

- ◯ $\neg q \to \neg p$.
- ◯ $\neg p \vee q$.
- ◯ $(p \wedge \neg q) \to F$.
- ◯ $T \to (p \to q)$.

(e) Which of the following are contingencies?

- ◯ $(p \wedge q) \vee (\neg p \wedge q) \vee (p \wedge \neg q) \vee (\neg p \wedge \neg q)$
- ◯ $(p \to q) \wedge (q \to r) \wedge (r \to \neg p)$
- ◯ $(p \wedge q \wedge \neg r) \vee (\neg p \wedge q \wedge r) \vee (p \wedge \neg q \wedge r)$
- ◯ $(p \vee q \vee \neg r) \wedge (\neg p \vee \neg q \vee r) \wedge (p \vee q \vee r)$.

(f) Which of the following say that there are exactly two things satisfying P(x):

- ◯ $\exists x P(x) \wedge \exists y P(y) \wedge \neg \exists z P(z)$.
- ◯ $\exists x \exists y (P(x) \wedge P(y) \wedge x \neq y) \wedge \neg \exists x \exists y \exists z (P(x) \wedge P(y) \wedge P(z) \wedge x \neq y \wedge x \neq z \wedge y \neq z)$.
- ◯ $\exists x \exists y \big(P(x) \wedge P(y) \wedge x \neq y \wedge \forall z \big(P(z) \to (z = x \vee z = y)\big)\big)$
- ◯ $\forall x \forall y \forall z \big(P(x) \wedge P(y) \wedge P(z) \to (x = y \vee x = z \vee y = z)\big)$.

(g) f is continuous at a means $\lim_{x\to a} f(x) = f(a)$ and this is defined as

$$\forall\epsilon > 0 \exists\delta > 0 \forall x \big(0 < |x-a| < \delta \to |f(x) - f(a)| < \epsilon\big)$$

Which of the following imply that f is not continuous at a?

○ $\exists\epsilon > 0 \exists\delta > 0 \exists x \big(0 < |x-a| < \delta \wedge |f(x) - f(a)| \geq \epsilon\big)$
○ $\exists\epsilon > 0 \forall\delta > 0 \exists x \big(0 < |x-a| < \delta \to |f(x) - f(a)| \geq \epsilon\big)$
○ $\exists\epsilon > 0 \forall\delta > 0 \exists x \big(0 < |x-a| < \delta \wedge |f(x) - f(a)| \geq \epsilon\big)$
○ $\exists\epsilon > 0 \forall\delta > 0 \exists x \big(0 < |x-a| < \delta \to |f(x) - f(a)| < \epsilon\big)$.

(h) Which of the following imply $A = B$?

○ $A \subseteq B \wedge B \subseteq A$.
○ $A - B = \emptyset \wedge A \cup B = A$.
○ $A \cap B = B \wedge A \cup B = B$.
○ $A - B = \emptyset = B - A$.

(i) Which of the following are always true where all sets are subsets of a universal set U?

○ $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$.
○ $\overline{A \cup (B \cap C)} = (\overline{A} \cup \overline{B}) \cap (\overline{A} \cup \overline{C})$.
○ $\overline{A \cup (B \cap C)} = \overline{A} \cup (\overline{B} \cap \overline{C})$.
○ $\overline{A \cup B} = \overline{A} \cap \overline{B}$.

(j) Which of the following are countably infinite?

○ The set of points on the unit interval $(0, 1)$.
○ The set of all polynomials with integer coefficients.
○ The set of all grains of sand on the Earth.
○ The set of finite strings of letters in the standard alphabet.

Problem 3 (Short answer; 40 points; 10 points each). Choose four of the five problems, I will grade the first four chosen, so if you do all five and get 1, 2, 3, and 5 correct but 4 wrong, you will score 30/40, since I will have graded 1 - 4. It is your job to decide which four I grade.

(a) Show by any method that the following is a tautology.

$$((p \wedge \neg q) \to F) \leftrightarrow (p \to q)$$

(b) Write down a sentence using quantifiers and logical connectives which asserts that $P(x)$ has at most one items satisfying it.

(c) Give a compound proposition in p, q, and r that is true when exactly two of p, q, or r are true.

(d) Explain why proof by contradiction is valid. That is, you want to prove $p \to q$ and to do this you prove $(p \wedge \neg q) \to F$, that is, assuming p and $\neg q$ you derive a contradiction.

(e) Explain why $|\mathbb{Z} \times \mathbb{Z}| \leq |\mathbb{Z}|$ by providing an injection $f : \mathbb{Z} \times \mathbb{Z} \to \mathbb{Z}$.

Problem 4 (Free response; 60 points; 20 points each)**.** Select three of the following four to complete. As above, you must make clear which three you choose.

(a) Either prove or disprove the following: For finite sets A and B,

$$\mathcal{P}(A \times B) \neq \{C \times D \mid C \in \mathcal{P}(A) \wedge D \in \mathcal{P}(B)\}.$$

(b) Use the rules of inference to provide an argument that from the premises $\forall x(P(x) \to Q(x))$ and $\forall x(Q(x) \to R(x))$ the conclusion $\forall x(P(x) \to R(x))$ follows. Make sure to indicate the rules of inference used and to what they are applied.

$$
\begin{array}{ll}
\forall x(P(x) \to Q(x)) & \text{Given} \\
\forall x(Q(x) \to R(x)) & \text{Given} \\
\vdots & \vdots \\
\hline
\therefore \forall x(P(x) \to R(x)) &
\end{array}
$$

(c) Prove that there are 100 consecutive integers that are not perfect squares. Is your proof direct/indirect? Is it constructive/nonconstructive?

(d) Prove the triangle inequality for real numbers, $|x| + |y| \geq |x + y|$. What methods do you use? Indirect/direct proof? Proof by cases? Etc.