

Math 571 - Homework 6

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Problem 6.1 (R:5:8). Suppose f' is continuous on $[a, b]$ and $\varepsilon > 0$. Show that there is $\delta > 0$ so that for all t such that $0 < |t - x| < \delta$ and all $a \leq x \leq b$

$$\left| \frac{f(t) - f(x)}{t - x} - f'(x) \right| < \varepsilon$$

This could be stated as f' is *uniform continuity* on $[a, b]$ provided f' is continuous on $[a, b]$. Does this hold for vector-valued functions?

f' is continuous on $[a, b]$ and hence uniformly continuous there since $[a, b]$ is compact. Fix $\varepsilon > 0$ and $\delta > 0$ so that $|f'(x) - f'(x')| < \varepsilon$ whenever $|x - x'| < \delta$. Let $t \in N_\delta(x)$, then MVT gives $c \in N_\delta(x)$ so that

$$\left| \frac{f(t) - f(x)}{t - x} - f'(x) \right| = |f'(c) - f'(x)| < \varepsilon$$

If $f : [a, b] \rightarrow \mathbb{R}^n$ (or \mathbb{C}^n) then this is still true as all of the component functions satisfy the conclusion. That is $f(x) = (f_1(x), \dots, f_n(x))$ in the real case and $f_i(x) = u_i(x) + iv_i(x)$ in the complex case.

Problem 6.2 (R:5:9). Suppose f is continuous on \mathbb{R} , and it is known that $f'(x)$ exists for all $x \neq 0$ and $f'(x) \rightarrow 3$ as $x \rightarrow 0$. Must $f'(0)$ exist?

By MVT $\frac{f(0+h)-f(0)}{h} = f'(c)$ for c between 0 and h and so $\lim_{h \rightarrow 0} \frac{f(0+h)-f(0)}{h} = \lim_{c \rightarrow 0} f'(c) = 3$.

Problem 6.3 (R:5:11). Suppose f is defined in a nbhd of x and $f''(x)$ exists. Show that

$$\lim_{h \rightarrow 0} \frac{f(x+h) + f(x-h) - 2f(x)}{h^2} = f''(x)$$

Show, by example, that the above limit can exist even if $f''(x)$ does not.

Let $F(h) = f(x+h) + f(x-h)$, then $F'(0) = f'(x) - f'(x) = 0$ and $F(h) - F(0) =$

$f(x+h) + f(x-h) - 2f(x)$. Let $G(h) = h^2$, then by MVT

$$\begin{aligned}\frac{f(x+h) + f(x-h) - 2f(x)}{h^2} &= \frac{F(h) - F(0)}{G(h) - G(0)} \\ &= \frac{F'(c)}{G'(c)} \text{ for some } c \in N_h(0) \\ &= \frac{f'(x+c) - f'(x-c) - 2f'(x)}{2c} \\ &= \frac{1}{2} \frac{f'(x+c) - f'(x)}{c} + \frac{1}{2} \frac{f'(x-c) - f'(x)}{-c}\end{aligned}$$

So

$$\begin{aligned}\lim_{h \rightarrow 0} \frac{f(x+h) + f(x-h) - 2f(x)}{h^2} &= \frac{1}{2} \lim_{c \rightarrow 0} \frac{f'(x+c) - f'(x)}{c} + \frac{1}{2} \lim_{d \rightarrow 0} \frac{f'(x+d) - f'(x)}{d} \\ &= \frac{1}{2} f''(x) + \frac{1}{2} f''(x) = f''(x)\end{aligned}$$

The "symmetry" in the initial formulation gives a hint at how to find the desired counterexample. Consider $f(x) = x^2 \sin(\frac{1}{x})$ for $x \neq 0$ and $f(0) = 0$. This function is odd so

$$\frac{f(0+h) + f(0-h) - 2f(0)}{h^2} = 0$$

$f'(x) = 2x \sin(\frac{1}{x}) - \cos(\frac{1}{x})$ for $x \neq 0$. At $x = 0$ we have

$$f'(0) = \lim_{h \rightarrow 0} \frac{f(h) - f(0)}{h} = \lim_{h \rightarrow 0} h \sin\left(\frac{1}{h}\right) = 0$$

Clearly, $f'(x)$ is not even continuous at $x = 0$ so $f''(0)$ DNE.

Problem 6.4 (R:5:16). Suppose f is twice differentiable on $(0, \infty)$ and f'' is bounded on $(0, \infty)$, and $f(x) \rightarrow 0$ as $x \rightarrow \infty$. Show that $f'(x) \rightarrow 0$ as $x \rightarrow \infty$.

We have $f(x) = f(a) + f'(a)(x-a) + \frac{f''(c)}{2}(x-a)^2$ for some c between x and a . So $f'(a) = \frac{f(x)-f(a)}{x-a} - \frac{f''(c)}{2}(x-a)$. Let $x = a+h$ so we get

$$|f'(a)| \leq \left| \frac{f(a+h) - f(a)}{h} \right| + M|h|$$

Pick $\varepsilon > 0$. Fixing h we can make $Mh < \varepsilon/2$ and letting $a \rightarrow \infty$ we can make $|f(a+h)|, |f(a)| < h\varepsilon/4$ and thus

$$|f'(a)| \leq \frac{|f(a+h)|}{h} + \frac{|f(a)|}{h} + Mh < \varepsilon/4 + \varepsilon/4 + \varepsilon/2 = \varepsilon$$

Problem 6.5 (R:5:22). Let $f : [a, b] \rightarrow [A, B]$ be differentiable on (a, b) and continuous on $[a, b]$. Here a, b, A , or B could be infinite, in which case we just identify something like $[-\infty, 2]$ with the more usual notation $(-\infty, 2]$. A point x is a **fixed** point of f iff $f(x) = x$.

- (a) Show that if $f'(t) \neq 1$ for all $t \in (a, b)$, then f can have at most one fixed point.

If there were $x, y \in [a, b]$ such that $x \neq y$, $f(x) = x$, and $f(y) = y$, then from MVT, there is c between x and y so that $f(x) - f(y) = x - y = f'(c)(x - y)$, but then $f'(c) = 1$.

- (b) Show that $f(t) = t + (1 + e^t)^{-1}$ satisfies $|f'(t)| < 1$ and f has no fixed points.

$$f'(t) = 1 - \frac{e^t}{(1+e^t)^2}, \text{ but } 0 < \frac{e^t}{(1+e^t)^2} < 1 \text{ so } 0 < f'(t) < 1.$$

It can't be the case that $f(t) = t$, since $t = t + (1 + e^t)^{-1}$ would imply $(1 + e^t)^{-1} = 0$ which is false.

- (c) Show that if there is $A < 1$ so that $|f'(t)| \leq A$ for all $t \in (a, b)$, then f has a fixed point and moreover given any $x_0 \in (a, b)$ and taking $x_{n+1} = f(x_n)$ it turns out that $x_n \rightarrow x$ and $f(x) = x$ is the unique fixed point of f .

$$|x_n - x_{n-1}| = |f(x_{n-1}) - f(x_{n-2})| = |f'(c)||x_{n-1} - x_{n-2}| \leq A|x_{n-1} - x_{n-2}|$$

Continuing this gives

$$|x_n - x_{n-1}| \leq A^{n-1}|x_1 - x_0|$$

and thus for $n > m$

$$|x_n - x_m| \leq |x_n - x_{n-1}| + \cdots + |x_{m+1} - x_m| \leq (A^{n-2} + \cdots + A^m)|x_1 - x_0|$$

Now $A^{n-1} + \cdots + A^m = A^m(A^{n-m-1} + \cdots + 1) = A^m \left(\frac{1-A^{n-m}}{1-A} \right) < A^m/(1-A)$ So for $\varepsilon > 0$, if N is chosen so that $A^N/(1-A) < \varepsilon$ and $m, n \geq N$, then

$$|x_n - x_m| < A^N/(1-A) < \varepsilon$$

Thus (x_n) is a C-seq and so $\lim_{n \rightarrow \infty} x_n = x$ exists and by continuity of f , $\lim_{n \rightarrow \infty} f(x_n) = f(x)$, but by definition $\lim_{n \rightarrow \infty} f(x_n) = \lim_{n \rightarrow \infty} x_{n+1} = x$ and thus $f(x) = x$. Uniqueness follows from (a).

Problem 6.6. Show that $f(x, y) = \sqrt{|xy|}$ is not differentiable at $(0, 0)$, but both partials $f_x(0, 0)$ and $f_y(0, 0)$ exist.

Compute

$$f_x(0, 0) = \lim_{h \rightarrow 0} \frac{\sqrt{|h \cdot 0|} - \sqrt{|0 \cdot 0|}}{h} = 0$$

$$f_y(0, 0) = \lim_{h \rightarrow 0} \frac{\sqrt{|0 \cdot h|} - \sqrt{|0 \cdot 0|}}{h} = 0$$

If f is differentiable at $(0, 0)$, then $D_f(0, 0)(h, k) = ah + bk$ for some a and b .

$$o_f(0, 0)(h, k) = (f(0 + h, 0 + k) - f(0, 0)) - D_f(0, 0)(h, k) = \sqrt{|hk|} - (ah + bk)$$

for some fixed a and b . This must satisfy

$$0 = \lim_{(h,k) \rightarrow 0} \frac{|(f(0 + h, 0 + k) - f(0, 0)) - D_f(0, 0)(h, k)|}{\|(h, k)\|} = \lim_{(h,k) \rightarrow 0} \frac{\sqrt{|hk|} - (ah + bk)}{\sqrt{h^2 + k^2}}$$

Notice: We know that $a = \frac{\partial f}{\partial x}(0,0) = 0$ and $b = \frac{\partial f}{\partial y}(0,0) = 0$, but we won't use this here and derive this directly from the definition.

If you approach along $t(1,0)$ (the x-axis), then you have

$$\lim_{(h,k) \rightarrow 0} \frac{\sqrt{|hk|} - (ah + bk)}{\sqrt{h^2 + k^2}} = \lim_{t \rightarrow 0} \frac{-at}{\sqrt{t^2}} = -a \operatorname{sgn}(t) = 0$$

so $a = 0$. Similarly along $t(0,1)$ gives $b = 0$.

So we must have $(f(0+h, 0+k) - f(0,0)) - D_f(0,0)(h,k) = \sqrt{|hk|} - (ah + bk) = \sqrt{|hk|}$.

Finally, if you let (h,k) approach $(0,0)$ along $t(1,1)$, then

$$\lim_{(h,k) \rightarrow 0} \frac{\sqrt{|hk|} - (ah + bk)}{\sqrt{h^2 + k^2}} = \lim_{t \rightarrow 0} \frac{\sqrt{t^2}}{\sqrt{2t^2}} = \frac{1}{\sqrt{2}} \neq 0$$

This is a contradiction.