Exam 2 – Math 215

1 True/False

Problem 1.1 (60 points; 6 points each)**.** Decide if each of the following are true or false. You do not need to justify your choice here.

(a) TRUE $(x+a)^2 \equiv x^2 + a^2 \pmod 2$

$(x+a)^2 = x^2 + 2ax + a^2 \equiv x^2 + a^2 \pmod 2$.

(b) FALSE $a^n \equiv b^n \pmod m$ and $n^a \equiv n^b \pmod m$ whenever $a \equiv b \pmod m$.

For example, $1 = 2^0 \not\equiv 2^3 = 8 \pmod 3$ but $3 \equiv 0 \pmod 3$.

(c) TRUE Suppose $ax + by = 1$, then $x = a^{-1} \pmod b$ and $x = a^{-1} \pmod y$.

From $ax + by = 1$ we have $ax \equiv 1 \pmod b$ so $x \equiv a^{-1} \pmod b$.

(d) TRUE $G(x) = \frac{1}{(1-x^5)}\frac{1}{(1-x^{10})}$ is the generating function for the number of solutions (a, b) of $5a + 10b = n$.

We have here

$$\begin{aligned} G(x) &= \frac{1}{(1-x^5)}\frac{1}{(1-x^{10})} \\ &= (1 + x^5 + (x^5)^2 + \cdots)(1 + x^{10} + (x^{10})^2 + \cdots) \end{aligned}$$

So the coefficient on any x^n is the number of ways to get n as some sum of 5's and 10's.

(e) TRUE For all a, a has a multiplicative inverse modulo m iff $ab + mn = 1$ for some b and n.

Well (c) is part of this, the if part. Conversely, if $b \equiv a^{-1} \pmod m$, then $ab \equiv 1 \pmod m$ so $ab = qm + 1$ and so $ab + (-q)m = 1$ so letting $n = -q$ does it.

(f) FALSE For any integers a and b, $ax + by = 1$ is a line so there are infinitely many integers x and y satisfying $ax + by = 1$, namely, all integer pairs (x, y) that fall on this line.

This is not true. First, $ax+by=1$ iff $\gcd(a,b)=1$, that is, a and b are relatively prime. So, there might be no solutions.

It is possible that you took as an assumption that $ax+by=1$ has a solution. If there is a solution, that is, $\gcd(a,b)=1$, then x and y here are not unique. Suppose $ax'+by'=1$, then $a(x-x')+b(y-y')=0$ so $a(x-x')=b(y-y')$. Since $\gcd(a,b)=1$, $a \mid y-y'$ and $b \mid x-x'$ so $x \equiv x' \pmod{b}$ and $y = y' \pmod{a}$. This would mean $y-y'=ma$ and $x-x'=nb$ so $y=ma+y'$ and $x=nb+x'$ so

$$1 = ax+by = a(nb+x')+b(ma+y') = nab+mab+ax'+by' = ab(n+m)+1$$

So $ab(n+m)=0$ and hence $n=-m$. Thus we have $x=nb+x'$ and $y=-nb+y'$.

Seeing this we see that given a solution $ax+by=1$, then $a(nb+x)+b(-na+y)=1$ as well, so we have infinitely many $\{(nb+x,-na+y) \mid n \in \mathbb{Z}\}$ is an infinite number of solutions. (If you work this out, then I will give credit.)

(g) FALSE The characteristic function for the recurrence relation $a_n = 3 \cdot a_{n-1} + 2 \cdot a_{n-3}$ is x^2-3x+2.

The characteristic equation here is x^3-3x^2-2. This a 3[rd] order, not second order, linear recurrence relation.

(h) TRUE If $f(n)$ and $g(n)$ are solutions to $a_n = 3a_n - 2a_{n-1} + a_{n-3}$, then $c_1 \cdot f(n) + c_2 \cdot g(n)$ where c_1 and c_2 are scalars, is also a solution.

Linear combinations of solutions are solutions.

(i) TRUE There are two solutions to $x^2 \equiv 2 \pmod{7}$.

You can just compute $3^2 = 9 \equiv 2 \pmod{7}$ and $4^2 = 16 \equiv 2 \pmod{7}$. You can further check, $2^2 \equiv 5^2 \equiv 4 \pmod{7}$ and $6^2 \equiv 1 \pmod{7}$.

(j) TRUE $\sum_{i=0}^{n} \binom{n}{i} 4^i = \sum_{i-0}^{n} \binom{n}{i} (-1)^{n-i} 6^i$.

This is seen by expanding $5^n = (4+1)^n = (6-1)^n$ using the binomial theorem.

2 Free Response

100 points total, 15 points each

Problem 2.1. Find all solutions to $13x - 8 \equiv 23 \pmod{5}$.

$13 \equiv 3 \pmod{5}$, $-8 \equiv 2 \pmod{5}$, and $23 \equiv 3 \pmod{5}$, so our original equation is the same as $3x+2 \equiv 3 \pmod{5}$ or $3x \equiv 1 \pmod{5}$. $3^{-1} \equiv 2 \pmod{5}$ so $x \equiv 3^{-1}3x \equiv 3^{-1} \cdot 1 \equiv 2 \pmod{5}$. So any x such that $x \equiv 2 \pmod{5}$ is a solution, namely, $x = 5 \cdot k + 2$ for $k \in \mathbb{Z}$.

Problem 2.2. Find s and t so that $s \cdot 954 + t \cdot 858 = \gcd(956, 858)$ using the extended Euclidean algorithm. Give the table (trace) of all intermediate values obtained along the way.

The rule is $[s_i, t_i] = [s_{i-2}, t_{i-2}] - q_{i-1}[s_{i-1}, t_{i-1}]$

i	0	1	2	3	4	5	
r	956	858	98	74	24	2	0
s_i	1	0	1	-8	9	-35	
t_i	0	1	-1	9	-10	39	
q_i		1	8	1	3	2	
check: $s_1 \cdot 956 + t_i \cdot 858$	956	858	98	74	24	2	

So $(-35)(956) + (39)(858) = 2 = \gcd(956, 858)$.

Problem 2.3. Find a closed form solution to the recurrence relation $a_n = 5a_{n-1} - 6a_{n-2}$ given $a_0 = 2$ and $a_1 = 1$.

The characteristic function is $x^2 - 5x + 6 = (x-2)(x-3)$ so the general solution is $f(n) = c_0(2^n)^n + c_1(3^n)$. We need to solve

$$\begin{aligned} 2 &= c_0 + c_1 \\ 1 &= 2c_0 + 3c_1 \end{aligned}$$

$c_0 = 2 - c_1$ so $1 = 2(2 - c_1) + 3c_1 = 4 + c_1$ and so $c_1 = -3$ and $c_0 = 5$. So we have $a_n = 5(2^n) - 3(3^n)$.

Problem 2.4. How many non-negative integer solutions are there to $x_1 + x_2 + x_3 \leq 20$ if $x_1 \leq 10$?

To count this, we find the number of solutions to $x_1 + x_2 + x_3 \leq 20$ with no restriction and subtract the number of solutions where $x_1 > 10$.

The number of solutions where there are no restrictions (except x_i is a non-negative integer) is given by adding a *slack* variable x_4 and solving $x_1 + x_2 + x_3 + x_4 = 20$. This number is $\binom{(20-1)+4}{4}$.

The number of solutions where $x_1 > 10$ (except x_i is a non-negative integer) is given by adding a *slack* variable x_4 and solving $x_1 + x_2 + x_3 + x_4 = 9$. This number is $\binom{(9-1)+4}{4}$.

So, the number we want is

$$\binom{(20-1)+4}{4} - \binom{(9-1)+4}{4}$$

Problem 2.5. A lottery places 50 balls numbered 1 through 50 into a machine and randomly selects 8 of these. You write down 8 numbers between 1 and 50, without repetitions, on 8 slips of paper. You win if you correctly choose at least 6 of the numbers selected by the lottery. How many ways are there for you to win?

In case it is not clear in the setup, the order does not matter in your numbers or those chosen by the lottery.

This involve three mutually exclusive possibilities, you get exactly 6 number correct, you get exactly 7 number correct, or you get exactly 8 numbers correct.

Exactly 6 correct: $\binom{8}{6}\binom{42}{2}$. Six numbers from the chosen 8 and 2 from those not chosen.

Exactly 7 correct: $\binom{8}{7}\binom{42}{1}$.

Exactly 8 correct: $\binom{8}{8}\binom{42}{0}$, i.e., 1.

So the answer is

$$\binom{8}{6}\binom{42}{2} + \binom{8}{7}\binom{42}{1} + \binom{8}{8}\binom{42}{0}$$

Problem 2.6. Compute $1005^{51} \pmod 7$ by hand using the fast exponentiation algorithm.

$51 = 32 + 16 + 2 + 1 = (110011)_2$ and $1001 \equiv 4 \pmod 7$ so you need to calculate $4^{51} \pmod 7$

n	2^n	$4^{2^n} \pmod 7$	a
0	1	4	4
1	2	2	1
2	4	4	1
3	8	2	1
4	16	4	4
5	32	2	1

So $1005^{51} \equiv 1 \pmod 7$.

Problem 2.7. Give a combinatorial argument for

$$\binom{n+1}{m} = \binom{n}{m} + \binom{n}{m-1}$$

$\binom{n+1}{m}$ is the number of m-element subsets $A \subset \{1, \dots, n+1\}$. Such a set comes in one of two flavors, (Type 1) $A \subset \{1, \dots, n\}$ or (Type 2) $A = A' \cup \{n+1\}$ for $A' \subset \{1, \dots n\}$. There are $\binom{n}{m}$ Type 1 sets and $\binom{n}{m-1}$ Type 2. The two types are mutually exclusive, so there are $\binom{n}{m} + \binom{n}{m-1}$ total.