Name: ______________________ Quiz 2 - MAT345

Problem 2.1 (20 points; 4 points each)**.** Decide if each of the following are true or false and provide a small proof or counterexample in each case. All vector spaces are assumed to be finite-dimensional here.

(a) False Given a basis $\mathcal{B} = \{\boldsymbol{v}_1, \ldots, \boldsymbol{v}_n\}$ for a vector space V and U a subspace of V, then there is $\mathcal{C} \subseteq \mathcal{B}$ that is a basis for U.

$\mathcal{B} = \{\boldsymbol{e}_1, \boldsymbol{e}_2\}$ is a basis for $\mathbb{R}^2$ and $U = \operatorname{span}\{(1,1)\}$ is a subspace, namely, the line with slope 1 through the origin. You cannot throw away one of $\boldsymbol{e}_1$ or $\boldsymbol{e}_2$ to get a basis for U.

(b) True Given a basis $\mathcal{C}$ for a subspace U of a vector space V, $\mathcal{C}$ can be extended to a basis $\mathcal{B}$ for V.

This is one of the theorems that you have, any linearly independent set can be expanded to a basis.

(c) False If $\boldsymbol{v} \in \operatorname{span}\{\boldsymbol{v}_1, \ldots, \boldsymbol{v}_n\}$, then it is guaranteed that there are unique scalars $\alpha_1, \ldots, \alpha_n$ so that $\boldsymbol{v} = \sum \alpha_i \boldsymbol{v}_i$.

It is not assumed that the vectors are linearly independent. If they are linearly dependent, then there would be infinitely many n-tuples of scalars $\alpha_1, \ldots, \alpha_n$ so that $\boldsymbol{v} = \sum \alpha_i \boldsymbol{v}_i$.

(d) True If $\{\boldsymbol{v}_1, \ldots, \boldsymbol{v}_n\}$ span V and $\{\boldsymbol{u}_1, \ldots, \boldsymbol{u}_n\} \subseteq V$ is linearly independent, then $\{\boldsymbol{u}_1, \ldots, \boldsymbol{u}_n\}$ span V.

Since $V = \operatorname{span}\{\boldsymbol{v}_1, \ldots, \boldsymbol{v}_n\}$ we know $\dim(V) \leq n$, but given that $\{\boldsymbol{u}_1, \ldots, \boldsymbol{u}_n\} \subset V$ is linearly independent, then $\dim(V) \geq n$. Thus $\dim(V) = n$ so $\{\boldsymbol{u}_1, \ldots, \boldsymbol{u}_n\}$ must be a basis

(e) False Suppose V is a vector space and $U \subseteq V$ is a subspace. For any $\boldsymbol{v} \in V$, there is a **unique** $\boldsymbol{u} \in U$ so that $\boldsymbol{v} = \boldsymbol{u} + (\boldsymbol{v} - \boldsymbol{u})$, that is, there is a unique "projection" of V into U.

Again take $U = \operatorname{span}\{(1,1)\} \subset \mathbb{R}^2 = V$ and let $\boldsymbol{v} = (2,3)$, then $\boldsymbol{v} = (1,1) + (1,2) = (2,2) + (0,1)$.

Note: If we fixed W so that $V = U \oplus W$, then there would be for every $\boldsymbol{v} \in V$ a unique $\boldsymbol{u} \in U, \boldsymbol{w} \in W$ so that $\boldsymbol{v} = \boldsymbol{u} + \boldsymbol{w}$. For example, take U as above and $W = \operatorname{span}\{(0,1)\}$, then $(2,3) = (2,2) + (0,1)$ is the unique decomposition of $(2,3)$ into something from U and something from W.

Problem 2.2 (10 pts)**.** A square matrix A is called **horizontally-symmetric** if $\text{flip}(A) = A$ where $\text{flip}(A)$ is the matrix you obtain from A by flipping it horizontally, for example,

$$\text{flip}\begin{bmatrix}1 & 2 & 3\\4 & 5 & 6\\7 & 8 & 9\end{bmatrix} = \begin{bmatrix}3 & 2 & 1\\6 & 5 & 4\\9 & 8 & 7\end{bmatrix}$$

a) Show that the flip-symmetric 3×3 matrices form a subspace of all 3×3 matrices.

b) Give a basis, $\mathcal{B}$, for the 3×3 flip-symmetric matrices.

c) Give representation $[\boldsymbol{v}]_{\mathcal{B}}$ for $\boldsymbol{v} = \begin{bmatrix}0 & 1 & 0\\1 & 0 & 1\\-2 & -3 & -2\end{bmatrix}$ with respect to the basis that you gave.

To see that the flip-symmetric matrices for a subspace note that the set is non-empty and that

$$\alpha\begin{bmatrix}a & b & a\\c & d & c\\e & f & e\end{bmatrix} + \beta\begin{bmatrix}A & B & A\\C & D & C\\E & F & E\end{bmatrix} = \begin{bmatrix}\alpha a+\beta A & \alpha b+\beta B & \alpha a+\beta A\\\alpha c+\beta C & \alpha d+\beta D & \alpha c+\beta C\\\alpha e+\beta E & \alpha f+\beta F & \alpha e+\beta E\end{bmatrix}$$

A basis $\mathcal{B}$ would be:

$$\begin{bmatrix}1 & 0 & 1\\0 & 0 & 0\\0 & 0 & 0\end{bmatrix}\quad\begin{bmatrix}0 & 1 & 0\\0 & 0 & 0\\0 & 0 & 0\end{bmatrix}\quad\begin{bmatrix}0 & 0 & 0\\1 & 0 & 1\\0 & 0 & 0\end{bmatrix}\quad\begin{bmatrix}0 & 0 & 0\\0 & 1 & 0\\0 & 0 & 0\end{bmatrix}\quad\begin{bmatrix}0 & 0 & 0\\0 & 0 & 0\\1 & 0 & 1\end{bmatrix}\quad\begin{bmatrix}0 & 0 & 0\\0 & 0 & 0\\0 & 1 & 0\end{bmatrix}$$

$$[\boldsymbol{v}]_{\mathcal{B}} = \begin{bmatrix}0\\1\\1\\0\\-2\\-3\end{bmatrix}$$

Problem 2.3. Suppose U and W subspaces of a vector space V such that

$$U + W = V, \text{ and } U \cap W = \{\mathbf{0}\}.$$

Then for every $\boldsymbol{v} \in V$, there is a **unique pair** $\boldsymbol{u} \in U, \boldsymbol{w} \in W$ so that $u + w = v$.

Recall: $U + W = \{\boldsymbol{u} + \boldsymbol{w} \mid \boldsymbol{u} \in U \text{ and } \boldsymbol{w} \in W\}$.

The only issue here is uniqueness since by assumption every $\boldsymbol{v} \in V$ can be written as $\boldsymbol{u} + \boldsymbol{w}$ for some pair $(\boldsymbol{u}, \boldsymbol{w})$. Suppose $\boldsymbol{v} = \boldsymbol{u} + \boldsymbol{w} = \boldsymbol{u}' + \boldsymbol{w}'$, then

$$\mathbf{0} = \boldsymbol{v} - \boldsymbol{v} = (\boldsymbol{u} + \boldsymbol{w}) - (\boldsymbol{u}' + \boldsymbol{w}') = (\boldsymbol{u} - \boldsymbol{u}') - (\boldsymbol{w}' - \boldsymbol{w})$$

so

$$\boldsymbol{w}' - \boldsymbol{w} = \boldsymbol{u} - \boldsymbol{u}' \in U \cap W$$

hence $\boldsymbol{w}' - \boldsymbol{w} = \mathbf{0} = \boldsymbol{u} - \boldsymbol{u}'$ and so $\boldsymbol{u} = \boldsymbol{u}'$ and $\boldsymbol{w} = \boldsymbol{w}'$.