Quiz 3

Problem 1 (15 points; 3 points each)**.** Decide if each of the following are true or false and provide a small proof or counterexample in each case. All vector spaces are assumed to be finite-dimensional here.

(a) ______ Given a basis $\mathcal{B} = \{\boldsymbol{v}_1, \ldots, \boldsymbol{v}_n\}$ for a vector space V and U a subspace of V, then there is $\mathcal{C} \subseteq \mathcal{B}$ that is a basis for U.

(b) ______ Given a basis $\mathcal{C}$ for a subspace U of a vector space V, $\mathcal{C}$ can be extended to a basis $\mathcal{B}$ for V.

(c) ______ If $\{\boldsymbol{v}_1, \ldots, \boldsymbol{v}_n\}$ is linearly independent and $\boldsymbol{v} \in \operatorname{span}(\{\boldsymbol{v}_1, \ldots, \boldsymbol{v}_n\})$, then it is possible that there are distinct $\boldsymbol{c}, \boldsymbol{b} \in \mathbb{R}^n$ such that $\boldsymbol{v} = \sum_{i=1}^n c_i \boldsymbol{v}_i = \sum_{i=1}^n b_i \boldsymbol{v}_i$.

(d) ______ If $\{\boldsymbol{v}_1, \ldots, \boldsymbol{v}_n\}$ is linearly independent and $V = \operatorname{span}(\{\boldsymbol{v}_1, \ldots, \boldsymbol{v}_n\}) = \operatorname{span}(\{\boldsymbol{u}_1, \ldots, \boldsymbol{u}_n\})$, then $\{\boldsymbol{u}_1, \ldots, \boldsymbol{u}_n\}$ is linearly independent.

(e) ______ Suppose V is a vector space and $U \subseteq V$ is a subspace. For any $\boldsymbol{v} \in V$, there is a **unique** $\boldsymbol{u} \in U$ so that $\boldsymbol{v} = \boldsymbol{u} + (\boldsymbol{v} - \boldsymbol{u})$, that is, there is a unique "projection" of V into U.

Problem 2 (10 pts)**.** Show that the collection, U, of upper triangular 3×3 matrices is a subspace of $\mathbb{R}^{3\times 3}$ (the space of all 3×3 matrices). Give a basis $\mathcal{B}$ for U and for $\boldsymbol{v} = \left[\begin{smallmatrix} 1 & 2 & 3 \\ 0 & 4 & 5 \\ 0 & 0 & 6 \end{smallmatrix}\right]$, give $[\boldsymbol{v}]_{\mathcal{B}}$.

Problem 3 (10 pts)**.** Let $c_1, c_2, \ldots, c_n$ be n distinct real numbers. Let $p_i = \prod_{\substack{j=1 \\ j \neq i}}^{n} (x - c_j)/(c_i - c_j)$. Show that $\mathcal{B} = \{p_1, p_2, \ldots, p_n\}$ is a basis for P_{n-1}.

Hint: Compute $p_i(c_j)$ and look at what happens when $i = j$ and when $i \neq j$. Use this to argue the independence of $\mathcal{B}$.