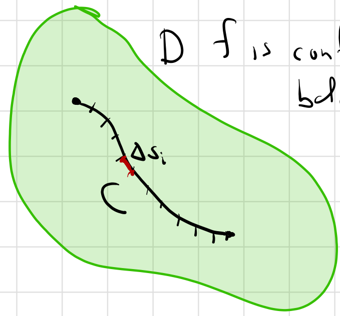
Line integrals

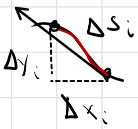
Let C be a smooth curve: C is given by $r(t)$ $a \le t \le b$

$r'(t)$ is continuous and $r'(t) \neq 0$.



D f is continuous & bdd on D

$$\int_C f \, ds = \lim_{\Delta s_i \to 0} \sum_{i=1}^{n} f(x_i^*, y_i^*) \Delta s_i$$



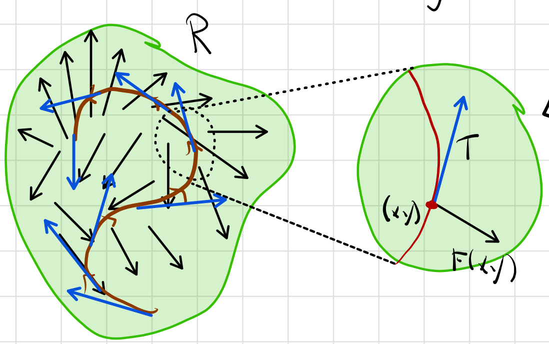
$$\Delta s_i \simeq (\Delta x_i^2 + \Delta y_i^2)^{1/2} \qquad ds = \underbrace{(dx^2 + dy^2)^{1/2}}$$

$$= \int_a^b f(r(t)) \overbrace{|r'(t)| \, dt}$$

Work

Let $F: \mathbb{R}^2 \to \mathbb{R}^2$ or $F: \mathbb{R}^3 \to \mathbb{R}^3$ be a continuous map (vector field).

The work done along C is defined as $\int_C F \cdot T \, ds$



$$\Delta W_i = \underbrace{F \cdot T}_{\text{scalar proj of } F \text{ on } T} \Delta s_i \qquad W = \int_C F \cdot T \, ds$$

$$T = \frac{r'}{|r'|} \quad ds = |r'|$$

$$\boxed{W = \int_a^b F \cdot r' \, dt = \int_a^b F \cdot dr}$$

$$F \cdot r' = \langle P, Q \rangle \cdot \left\langle \frac{dx}{dt}, \frac{dy}{dt} \right\rangle$$
$$= P \frac{dx}{dt} + Q \frac{dy}{dt}$$

$$= \int_a^b P \frac{dx}{dt} dt + Q \frac{dy}{dt} dt$$

$$= \int_C P \, dx + Q \, dy$$

FTC for line integrals

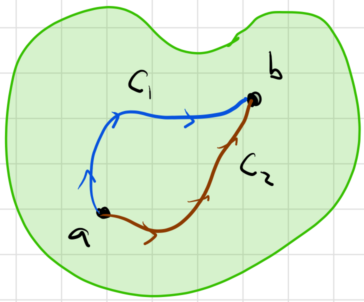
Theorem: Let C be smooth and $f: \mathbb{R}^2 \to \mathbb{R}$ with ∇f cont on C, then $\int_C \nabla f \cdot dr = f(\vec{r}(b)) - f(\vec{r}(a))$

<u>Proof</u>. $\nabla f \cdot \vec{r}' = \frac{d}{dt} f(\vec{r}(t))$ and $\int_a^b \nabla f \cdot \vec{r}' \, dt = \int_a^b \frac{d}{dt} f(\vec{r}(t)) \, dt$

$$= f(\vec{r}(b)) - f(\vec{r}(a))$$

Independence of Path

$$\int_{C_1} f\,ds = \int_{C_2} f\,ds$$

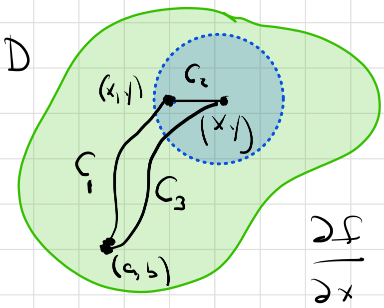


If $F = \nabla f$, then $\int_{C_1} F\cdot T_1\,ds = f(b)-f(a) = \int_{C_2} F\cdot T_2\,ds$

So if F is a potential field the line integral is independent of path.
$F = \nabla f$

Path independence $\Leftrightarrow \oint_C F\cdot T\,ds = 0$

Theorem If $\int_C F\cdot T\,ds$ is independent of C on an open and ~~simply~~ connected region D, then $F = \nabla f$ for some potential function f.



$$\int_{C_1} F\cdot dr + \int_{C_2} F\cdot dr = \int_{C_3} F\cdot dr = f(x,y)$$

$$F(x,y) = (P(x,y), Q(x,y))$$

$$\frac{\partial f}{\partial x}(x,y) = \lim_{x_1\to x}\frac{f(x_1,y)-f(x,y)}{x_1-x} = \lim_{t_1\to t}\frac{\int_{t_1}^{t_0} F\cdot dr}{t_1-t}$$

$$\begin{cases} (x,y) = (x(t_0), y(t_0)) \\ (x_1,y) = (x(t_1), y(t_1)) \\ \frac{dx}{dt} = 1 \end{cases}$$

$$= \lim_{t_1\to t_0}\frac{\int_{t_1}^{t_0} P(x,y)\frac{dx}{dt}\,dt}{t_1-t_0}$$

$$= P(x,y)$$

Similarly, $\frac{\partial f}{\partial y} = Q(x,y)$ ∎

Since $\frac{\partial}{\partial x}\frac{\partial}{\partial y} f = \frac{\partial}{\partial y}\frac{\partial}{\partial x} f$ we have $\frac{\partial}{\partial x} Q = \frac{\partial}{\partial y} P$

Theorem: If $F = P\vec{i} + Q\vec{j}$ has continuous first partials on a **simply connected open region** D and $\frac{\partial Q}{\partial x} = \frac{\partial P}{\partial y}$, then $F = \nabla f$

Definition. A vector field F that is of the form $F = \nabla f$ is called a **conservative vector field**

Do #9, #12 from Week 12

Green's Theorem

Theorem Suppose C is a **simple closed** **positively oriented** **piecewise smooth** **plane curve**. Suppose further that $F: \mathbb{R}^2 \to \mathbb{R}^2$ is a vector field $F(x,y) = \langle P(x,y), Q(x,y) \rangle$ with **continuous first partials**, then

$$\oint_C P\,dx + Q\,dy = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dA$$

D = the simply connected region bounded by C, as $C = \partial D$ this is often written

$$\int_{\partial D} P\,dx + Q\,dy = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dA$$

with the assumption being that ∂D has the required properties.

Recalling that $dr = \langle dx, dy \rangle$, if you think of $\nabla = \langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y} \rangle$ as an operator, then $\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = -\det\begin{pmatrix} P & Q \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} \end{pmatrix} = -\det(F, \nabla)$

Divergence and Curl

$F: \mathbb{R}^3 \to \mathbb{R}^3 \quad F = P\vec{i} + Q\vec{j} + R\vec{k}$

$$\text{Curl } F = \nabla \times F = \det \begin{bmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ P & Q & R \end{bmatrix}$$

$$\text{div } F = F \cdot \nabla = \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z}$$

If F is conservative $\text{curl } F = \vec{0}$

$$\nabla \times \nabla f = \det \begin{bmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} & \frac{\partial f}{\partial z} \end{bmatrix} = (f_{yz} - f_{zy})\vec{i} - (f_{xz} - f_{zx})\vec{j} + (f_{yx} - f_{xy})\vec{k} = \vec{0}$$

(Alt) $\nabla \times \nabla f = (\nabla \times \nabla) f = \vec{0} f = \vec{0}$

There is a converse to this

Theorem If $\text{curl } F = \vec{0}$ in a "nice" region D of $\mathbb{R}^3$ and if the components of F have continuous partials on D, then $F = \nabla f$.

$$\text{div curl } F = \nabla \cdot (\nabla \times F) = \det \begin{bmatrix} \nabla \\ \nabla \\ F \end{bmatrix} = 0$$