Homework 5 Solutions

Ch 16: 25, 27, 35, 37, 57, 58, 63, 64 - 66 (these are all related), 67, 68

25. If $x - 2$ is a factor of $p(x) = x^4 - 2x - 2$, then $p(2) = 0$, $p(2) = 10 \bmod p = 0$ so $p = 2$ and $p = 5$.

27. (Used hint from the book here.) $U(p)$ is abelian of order $p - 1$, if $U(p)$ were not cyclic, then by the fundamental theorem of abelian groups, for some q prime, $q \mid p - 1$, there is $H \simeq \mathbb{Z}_q \times \mathbb{Z}_q < (U(p), \cdot, 1)$ (the multiplicative group). Let $\phi : \mathbb{Z}_q \times \mathbb{Z}_q \simeq H$ and let $x_{a,b} = \phi(a, b) \in U(p)$, then $x_{a,b}^q = 1$ and so $p(x) = x^q - 1$ has q^2 many solutions, which we know is impossible.

35. Show that $p(x) = x^3 - 2x^2 - 9$ has a root in every field. $p(3) = 3^3 - 2(3^2) - 9 = 3(3^2) - 2(3^2) - 3^2 = (3 - 2 - 1)(3^2) = 0$. So 3 is a root in any field. In $\mathbb{Z}_2$, $3 = 1$ and in $\mathbb{Z}_3$, $3 = 0$, but the argument still holds.

37. Let F be a field and $I = \{f(x) \in F[x] \mid f(1) = 0 \text{ and } f(2) = 0\}$. Find $g(x) \in F[x]$ so that $I = (g(x))$.

Let $g(x) = (x - 1)(x - 2) = x^2 - 3x + 2$, then $(g(x)) = \{f(x)(x - 1)(x - 2) \mid f(x) \in F[x]\}$. Clearly, $(g) \subseteq I$, conversely, the division algorithm shows that if $f(x) \in I$, then $f(x) = f'(x)(x - 1)(x - 2)$ for some $f'(x)$.

57. Show that in $\mathbb{Z}_p[x]$, $x^{p-1} - 1 = \prod_{a=1}^{p-1}(x - a)$.

This is because $a^{p-1} = 1$ in $\mathbb{Z}_p$ for all $a \in U(p) = \{1, \cdots, p - 1\}$. Thus each element is a root of $x^{p-1} - 1$, and so the factorization follows.

58. (Wilson's Theorem) For every integer $n > 1$, $(n - 1)! \bmod n = n - 1$ iff n is prime.

If n is prime, then

$$x^{n-1} - 1 = (x - 1)(x^{n-2} + x^{n-3} + \cdots + 1) = (x - 1)(x - 2) \cdots (x - (n - 1))$$

So

$$x^{n-2} + x^{n-3} + \cdots + 1 = (x - 2)(x - 3) \cdots (x - (n - 1)) \bmod n$$

Evaluating both sides at $x = 1$ gives

$$n - 1 = (-1)(-2) \cdots (-(n - 1)) = (n - 1)(n - 2) \cdots (1) = (n - 1)! \bmod n$$

Conversely, if $n = s \cdot t$ is not prime, then $n \mid (n - 1)!$ so $(n - 1)! = 0 \bmod n$.

63. For a field that properly contains the field of complex numbers, the first thing that comes to mind is the quotient field of $\mathbb{C}[x]$. That is the field of rational functions over $\mathbb{C}$.

64. If I is an ideal of R show that $I[x]$ is an ideal of $R[x]$. It is clear that $I[x]$ is closed under addition. For the multiplicative closure a little effort is required, consider $p(x) \in I[x]$ with coefficients $a_i \in I$ and $q(x) \in R[x]$ with coefficients $b_i \in R$, then the coefficient of x^i in $p(x)q(x)$ is

$$c_i = \sum_{j=0}^{i} a_j b_{i-j} \in I$$

So $p(x)q(x) \in I[x]$.

65. $2\mathbb{Z}$ is a maximal ideal in $\mathbb{Z}$, since $\mathbb{Z}/2\mathbb{Z} \simeq \mathbb{Z}_2$ is a field. But, $\mathbb{Z}[x]/2\mathbb{Z}[x] \simeq \mathbb{Z}_2[x]$ is an integral domain, but not a field.

66. Show that if I is a prime ideal of R (commutative and unitary), then $I[x]$ is a prime ideal of $R[x]$.

If I is prime, then R/I is an integral domain. Now $R[x]/I[x] \simeq (R/I)[x]$ and since R/I is an integral domain, so is $R/I[x]$.

Note To prove $R[x]/I[x] \simeq (R/I)[x]$ define the map $\phi : R[x] \to (R/I)[x]$ by $\sum_{i=1}^{n} r_i x^i \mapsto \sum_{i=1}^{n} (r_i/I)x^i$. It is easy to see that this is a homomorphism and is surjective. Now show that $\ker(\phi) = I[x]$.

67. Show that $x = 1$ is the only solution to $x^{25} - 1$ in $\mathbb{Z}_{37}$.

For $x^{25} = 1$ in $U(37)$ we know that $|x| \mid 25 = 5^2$, on the other hand, $|x| \mid |U(37)| = 36 = 6^2$. Only $\gcd(36, 25) = 1$ so $|x| = 1$ and hence $x = 1$.

68. Show that $\mathbb{Q}[x]/(x^2 - 2) \simeq \mathbb{Q}[\sqrt{2}]$.

There are several ways to do this. Here is one. Define $\phi : \mathbb{Q}[x] \to \mathbb{Q}[\sqrt{2}]$ by $x \mapsto \sqrt{2}$ and everything else maps as must be. A little effort verifies this to be a homomorphism and onto. So suppose $\phi(p(x)) = 0$, then $\sqrt{2}$ is a root of $p(x)$. We know $\overline{p(\sqrt{2})} = \bar{p}(\overline{\sqrt{2}}) = p(-\sqrt{2}) = 0$ as well, so $x^2 - 2 \mid p(x)$ and thus $\ker(\phi) = (x^2 - 2)$.

Note Here as usual $\overline{a + b\sqrt{2}} = a - b\sqrt{2}$.

Ch 17: 7, 12, 14, 15, 19, 28, 38, 39, 40

7. Suppose $r + 1/r$ is an odd integer, show that r is irrational.

Let n be an integer and consider $2n + 1 = x + 1/x$ or $x^2 - (2n + 1)x + 1 = 0$. If r is rational, then this must factor over $\mathbb{Q}$. But if this factors over $\mathbb{Q}$, then it factors over $\mathbb{Z}$ as $(x - p)(x - q)$ with $p, q \in \mathbb{Z}$ so that either $p = q = 1$ or $p = q = -1$ and $2n + 1 = p + q = \pm 2$.

12. Construct a field of order 27.

Consider $x^3 + 2x + 1$. This has no root in $\mathbb{Z}_3$, so it is irreducible in $\mathbb{Z}_3[x]$ and hence $\mathbb{Z}_3[x]/(x^3 + x + 1)$ is a field, since $\mathbb{Z}_3[x]$ is a PID. The classes of $\mathbb{Z}_3[x]$ are given by $ax^2 + bx + c$ with $a, b, c \in \mathbb{Z}[3]$ so there are $3^3 = 27$ elements.

14. Which of the following are irreducible over $\mathbb{Q}$?

a. $x^5 + 9x^4 + 12x^2 + 6$: This is irreducible over $\mathbb{Q}$ since since $3 \nmid 1$, $3 \mid 9, 12, 6$, and $3^2 \nmid 6$.

b. x^4+x+1: x^4+x+1 has no linear factors since the only possible roots are ± 1. If it factors into quadratics, then we must have $x^4+x+1=(x^2+ax+1)(x^2+bx+1)=x^4+(a+b)x^3+(ab+2)x^2+(a+b)x+1$. But then $a+b=1$ and $a+b=0$, so this can't happen either.

c. x^4+3x^2+3: This is like (a.). $3\nmid 1$, $3\mid 0,3,3$, and $3^2\nmid 3$.

d. x^5+5x^2+1: Let's see if this is irreducible in $\mathbb{Z}_2[x]$. There are no linear factors, no roots in $\mathbb{Z}_2$. If there is a quadratic factor, it must be one of $x^2, x^2+1, x^2+x, x^2+x+1$. Each of x^2, x^2+1, x^2+x has roots in $\mathbb{Z}_2$, so these can't be a factor. So x^2+x+1 is the only option. You can check that x^2+x+1 does not divide x^5+5x^2+1 in $\mathbb{Z}_2[x]$, so x^5+5x^2+1 is irreducible in $\mathbb{Z}_2[x]$ and by the mod2 test, x^5+5x^2+1 is irreducible in $\mathbb{Q}$.

e. $(5/2)x^5+(9/2)x^4+15x^2+6x+3/14$: $(1/14)(35x^5+63x^4+210x^2+84x+3)$. Again, as above $3\nmid 35$, $3\mid 63,210,84,3$, and $3^2\nmid 3$. So the polynomial is irreducible.

15. Consider $\mathbb{Z}_2[x]/(x^3+x+1)$.

$$(x^2+x)^2=x^2(x+1)^2=x^2(x^2+2x+1)=x^2(x^2+1)=x(x^3+x)=x(-1)=-x \bmod (x^3+x+1)$$

and noting that $1=-1=x^3+x$ we can divide x^3+x by x^2+x and get $x+1$.

$$(x^2+x)(x+1)=x^3+2x^2+x=x^3+x=-1=1$$

So $(x^2+x)^{-1}=x+1$.

19. Consider $F=\mathbb{Z}_7[x]/(x^2+2)$. x^2+2 has no roots in $\mathbb{Z}_7$ so x^2+2 is irreducible and $\mathbb{Z}_7[x]/(x^2+2)$ is a field.

$$\begin{aligned}
x^1 &= x, & x^2 &= -2 = 5,\\
x^3 &= 5x, & x^4 &= 5^2 = 25 = 4,\\
x^5 &= 4x, & x^6 &= 4x^2 = 20 = 6,\\
x^7 &= 6x, & x^8 &= 6x^2 = 30 = 2,\\
x^9 &= 2x, & x^{10} &= 2x^2 = 10 = 3,\\
x^{11} &= 3x, & x^{12} &= 15 = 1
\end{aligned}$$

So $|x|=12$

$$\begin{aligned}
(x+1) &= x+1, & (x+1)^2 &= 2x+6,\\
(x+1)^3 &= x+2, & (x+1)^4 &= 3x,\\
(x+1)^5 &= 3x+1, & (x+1)^6 &= 4x+2,\\
(x+1)^7 &= 6x+1, & (x+1)^8 &= 3\\
(x+1)^9 &= 3x+3, & (x+1)^{10} &= 6x+4\\
(x+1)^{11} &= 3x+6, & (x+1)^{12} &= 2x\\
&\vdots & &\vdots
\end{aligned}$$

I got tired of this one so I made python do it for me. We see here that $U(F)$ is cyclic, and $(x+1)$ is a primitive 48^{th} root of unity.

A better solution: $(x+1)^m = (x+1)^{8k}(x+1)^m$ for $m = 0, 2, \ldots, 7$. $(x+1)^{8k} = ((x+1)^8)^k = 3^k \pmod 7$. Now in $\mathbb{Z}_7$:

$$3^1 = 3, 3^2 = 2, 3^3 = 6, 3^4 = 2^2 = 4, 3^5 = 2 \cdot 6 = 5, 3^6 = 6^2 = 1$$

Now check that $(x+1)^{8k}(x+1)^m \neq 1$ for $k = 1, \ldots, 5$ and $m = 0, \ldots, 7$. For example

$$(x+1)^{8\cdot 4}(x+1) = 4x+4, \qquad (x+1)^{8\cdot 4}(x+1)^2 = 4(2x+6) = x+3,$$
$$(x+1)^{8\cdot 4}(x+1)^3 = 4(x+2) = 4x+1, \qquad (x+1)^{8\cdot 4}(x+1)^4 = 4(3x) = 5x,$$
$$(x+1)^{8\cdot 4}(x+1)^5 = 4(3x+1) = 5x+4, \qquad (x+1)^{8\cdot 4}(x+1)^6 = 4(4x+2) = 2x+1,$$

28. (a) and (b) seem to be asking the same thing as the quadratic monic polynomials are just those polynomials of the form $x^2 + ax + b$. These are irreducible so long as they have no root in $\mathbb{Z}_p$. That is $x^2 + ax + b \neq (x-m)(x-n) = x^2 - (m+n)x + mn$ for any $m, n \in \mathbb{Z}_p$. There are $p(p-1)/2$ of the form $(x-m)(x-n)$ where $m \neq n$ and p where $m = n$ for a total of $p(p-1)/2 + p$ many reducible monomial quadratics and thus $p^2 - (p(p-1)/2 + p) = p^2 - (p^2 - p)/2 + p = p^2/2 + p/2 = (p)(p+1)/2$ irreducible.

38. If $x^{p-1} - x^{p-2} + \cdots - x + 1 = p(-x) = (-x)^{p-1} + (-x)^{p-2} + \cdots + (-x)^1 + 1 = f(x)g(x)$ with $\deg(g), deg(f) > 0$. Then $p(x) = p(--x) = x^{p-1} + x^{p-2} + \cdots + x + 1 = f(-x)g(-x) = f_1(x)g_2(x)$. But this contradicts the irreducibility of the cyclotomic polynomial.

39. The evaluation map is obviously a homomorphism. Let $f(x) \in \ker(\phi)$. If $p(x) \nmid f(x)$, then as $p(x)$ is irreducible, we know $\gcd(f(x), p(x)) = 1$ (constant polynomial). We can use the Euclidean algorithm to find $q(x)$ and $r(x)$ so that $q(x)p(x) + r(x)f(x) = 1$. This is a contradiction since $q(a)p(a) + r(a)f(a) = q(a) \cdot 0 + r(a) \cdot 0 = 0 \neq 1$. So $p(x) \mid f(x)$.

40. We have seen before that $\mathbb{Z}[x]/(x^2+1) \simeq \mathbb{Z}[i]$ is an integral domain, but not a field, so (x^2+1) is prime and not maximal.