Math 571 - Homework 5

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Notation: For $f : X \to Y$ and $E \subseteq X$ set $f(X) = \{f(e) \mid e \in E\}$, this is called the *image of E under f*.

Problem 0.1 (R:4:2*)**.** Let $f : X \to Y$ be continuous. Let $E \subseteq X$, show that $f(\mathrm{Cl}(E)) \subseteq \mathrm{Cl}(f(E))$. By example, show that this containment can be proper, that is $f(\mathrm{Cl}(E)) \nsubseteq \mathrm{Cl}(f(E)))$ can hold.

Proof 1: Let $y \in f(\mathrm{Cl}(X))$, so $y = f(x)$ for $x \in \mathrm{Cl}(E)$. Let O be an open nbhd of y and let U be an open nbhd of x so that $f(U) \subset O$. Since $x \in \mathrm{Cl}(E)$ we have $U \cap E \neq \emptyset$. Let $e \in U \cap E$, then $f(e) \in f(U) \cap f(E) \subseteq O \cap f(E)$. So we have shown that for any open nbhd O of y, $y \cap f(E) \neq \emptyset$, thus $y \in \mathrm{Cl}(f(E))$.

Proof 2: Clearly, $f(E) \subseteq \mathrm{Cl}(f(E))$, so $E \subseteq f^{-1}(\mathrm{Cl}(f(E))$, but as $f^{-1}(\mathrm{Cl}(f(E))$ is closed we have $\mathrm{Cl}(E) \subseteq f^{-1}(\mathrm{Cl}(f(E))$ and hence $f(\mathrm{Cl}(E)) \subseteq \mathrm{cl}(f(E))$.

For the example, consider $f : \mathbb{R} \to \mathbb{R}$ given by $f(x) = \frac{1}{1+x^2}$. So $f(\mathbb{R}) = (0, 1] \subsetneq \mathrm{Cl}(f(\mathbb{R})) = \mathrm{Cl}((0, 1]) = [0, 1]$.

Definition Let $f : E \subseteq X \to Y$, the graph of f is the set $\mathrm{Graph}(f) = \{(x, f(x) \mid x \in E\} \subseteq X \times Y$.

Problem 0.2. Let $f : E \subseteq X \to Y$ be continuous where Y is Hausdorff, show that $\mathrm{Graph}(f)$ is closed in $E \times Y$.

(Proof 1) Hausdorff Case: Let $(x, y) \in E \times Y - \mathrm{Graph}(f)$. So $f(x) = y' \neq y$. Let O and O' be open nbhds of y and y' respectively so that $O \cap O' = \emptyset$. (Here we use the Hausdorff property.) Let U be an open nbhd of x so that $f(U \cap E) \subseteq O'$. I claim that $(U \times O) \cap \mathrm{Graph}(f) = \emptyset$. Suppose that $(\tilde{x}, \tilde{y}) \in (U \times O) \cap \mathrm{Graph}(f)$, then $f(\tilde{x}) = \tilde{y}$, so $f(U \cap E) \cap O \neq \emptyset$, contradicting $f(U \cap E) \subseteq O'$ and $O' \cap O = \emptyset$.

(Proof 2) Metric Case: Suppose $((x_i, f(x_i))$ is a convergent sequence in $E \times Y$, that is $((x_i, f(x_i)) \to (x, y)$. In particular, $x_i \to x \in E$ and as f is sequentially continuous $f(x_i) \to f(x)$, thus $y = f(x)$ and we see $\mathrm{Graph}(f)$ is sequentially closed, hence closed.

Problem 0.3 (R:4:6)**.** Suppose $f : E \subseteq X \to Y$ and E is compact. Suppose further that X and Y are Hausdorff (or metric if you prefer). Show that f is continuous on E iff $\mathrm{Graph}(f)$ is compact.

Hint: You may use the fact that if K and H are compact, then $K \times H$ is compact and that If K is compact and $C \subseteq K$ is closed, then C is compact. (Both of these are in notes and book.)

(Proof 1) Hausdorff Case: Assume f is continuous, then $f(E) \subset Y$ is compact and $\text{Graph}(f) \subset E \times f(E)$ is closed, hence $\text{Graph}(f)$ is a closed subset of the compact set $E \times f(E)$ and hence compact.

So now, suppose $\text{Graph}(f)$ is compact. Consider the map $F : E \to \text{Graph}(f)$ given by $F(x) = (x, f(x))$.

Claim: F is continuous iff f is continuous.

Proof of Claim: Suppose F is continuous, then $f(x) = (\pi_2 \circ F)(x)$ where $\pi_2(x, y) = y$ (projection in the second coordinate). $\pi_2 : E \times f(E) \to f(E)$ is continuous and so $f = \pi_2 \circ F$ is a composition of continuous functions and is continuous.

Now suppose f is continuous, then for $U \times V \subset E \times f(E)$ open, $(x, y) \in F^{-1}(U \times V) \iff y = f(x) \wedge x \in U \wedge y \in V \iff x \in U \cap f^{-1}(V)$. So $F^{-1}(U \times V) = u \cap f^{-1}(V)$, which is open in E. (Claim) $\square$

So we need only show now that F is continuous. Since F^{-1} is the projection map $(x, f(x)) \mapsto x$, it is continuous. Since $\text{Graph}(f)$ is compact and X is Hausdorff, F^{-1} is a closed map, and hence $F = (F^{-1})^{-1}$ is continuous. (See here.)

(Proof 2) Metric Case: Suppose $\text{Graph}(f)$ is compact, hence sequentially compact. Suppose $x_i \in E$ and $x_i \to x \in E$. Consider $((x_i, y_i))$ in $\text{Graph}(f)$ we know there is a convergent subsequence $((x_{n_i}, y_{n_i}))_i \to (x, y) \in \text{Graph}(f)$. But then $\lim_i x_{n_i} = \lim_i x_i = x$ and $y = f(x)$ and $y_{n_i} \to y$, so $f(x_{n_i}) \to y$.

Suppose $y_i \not\to y$ as $i \to \infty$, then there is $y' \neq y$ and subsequence $((x_{m_i}, y_{m_i}))_i \to (x, y') \in \text{Graph}(f)$. But then $f(x) = y = y'$ which is a contradiction. so $y_i \to y$, that is $f(x_i) \to f(x)$. Thus f is sequentially continuous and hence continuous.

The other direction is easier. Suppose f is continuous and $((x_i, y_i))$ is a sequence from $\text{Graph}(f)$. then $x_{n_i} \to x \in E$ for some subsequence x_{n_i} since E is sequentially compact. But then $f(x_{n_i}) \to f(x)$ and so $((x_{n_i}, y_{n_i}))_i \to (x, y) = (x, f(x)) \in \text{Graph}(f)$. So $\text{Graph}(f)$ is sequentially compact, hence compact.

Problem 0.4. Let $f : E \subseteq X \to Y$ where both X and Y are metric spaces with Y complete. suppose f is uniformly continuous on E, show that there is a unique continuous extension $\hat{f} : \text{Cl}(E) \to Y$. Moreover, $\hat{f}$ remains uniformly continuous.

Existence: Let $x \in \text{Cl}(E) - E$ so that x is a limit point of E, then $x = \lim_i x_i$ for (x_i) a sequence from E. Since (x_i) is a Cauchy sequence and f is uniformly continuous, $(f(x_i))$ is Cauchy and thus has a limit y. To see that $x \mapsto y$ defines an extension of f we must see that y is independent of the particular sequence (x_i) chosen and that $y = f(x)$ for $x \in E$. The second follows from the first trivially, since letting $x_i = x$ for all i, (x_i) is a Cauchy sequence converging to x. Suppose (x_i') is another sequence from E with $\lim_i x_i' = x$. Then the sequence (z_i) where $z_{2i} = x_i$ and $z_{2i+1} = x_i'$ is a sequence from E converging to x and clearly $(f(x_i))$ and $(f(x_i'))$ are both Cauchy subsequences of the Cauchy sequence $(f(z_i))$, thus all of these must have the same limit y.

To see that $\hat{f}$ is uniform continuous, let $\epsilon > 0$ take δ that witnesses uniform continuity on E, so for all $x, x' \in E$, $d^X(x, x') < \delta \implies d^Y(f(x), f(x')) < \epsilon/2$. Suppose $x, x' \in \text{Cl}(E)$

and $d^X(x, x') < \delta$. Take $u, u' \in E$ with $d^Y(f(u), \hat{f}(x)) < \epsilon/4$, $d^Y(f(u'), \hat{f}(x')) < \epsilon/4$, and $d^X(u, u') < \delta$, then $d^Y(\hat{f}(x), \hat{f}(x')) \leq d^Y(\hat{f}(x), f(u)) + d^Y(f(u), f(u')) + d^Y(\hat{f}(x'), f(u')) < \epsilon$.

Uniqueness: Suppose $g : \mathrm{Cl}(E) \to Y$ is continuous and $f = g|_E$, then we must show that $g = \hat{f}$. This is trivial since if $x \in E$ there is nothing to do. If $x \notin E$, then $x = \lim_i x_i$ for $x_i \in E$, so $g(x) = \lim_i g(x_i) = \lim_i f(x_i) = \hat{f}(x)$.

Definition: A set $E \subseteq X$ has the *Bolzano-Weierstrass property* iff every sequence in X has a convergent subsequence.

Problem 0.5. Show that if $E \subseteq X$ has the Bolzano-Weierstrass property, then

a) $\mathrm{Cl}(E)$ also has Bolzano-Weierstrass property.

Let $x_i \in \mathrm{Cl}(E)$, then for each i there is $x_i' \in E$ so that $d^X(x_i, x_i') < 1/i$. Then x_i' has a convergent subsequence (x_{n_i}') and it is clear that (x_{n_i}) also converges (to the same limit).

b) If X is metric, then E is bounded.

If E is unbounded, then it is simple to choose a sequence $x_i \in E$ so that $d^X(x_i, x_j) > 1$ for all i, j. But then this sequence has no convergent subsequence.

c) For X metric E has the Bolzano-Weierstrass property iff $\mathrm{Cl}(E)$ is compact.

$\mathrm{Cl}(E)$ is sequentially compact, hence compact.

Problem 0.6 (R:4:8*)**.** Let $f : E \subseteq X \to Y$ be uniformly continuous on E where E has the Bolzano-Weierstrass property and Y is complete. Show that f is bounded on E, that is $f(E)$ is bounded in Y.

Proof 1: From problem 4 we can extend f to $\hat{f} : \mathrm{Cl}(E) \to Y$ and from problem 5, $\mathrm{Cl}(E)$ is compact. So $\hat{f}(\mathrm{Cl}(E))$ is compact hence bounded in Y and so $f(E)$ is bounded.

Proof 2: (You don't actually need Problem 5 or the stuff about compactness.) Suppose $f(E)$ is unbounded. Then get $x_i \in E$ so that $d^Y(x_i, x_j) \geq 1$. By assumption there is a convergent and hence Cauchy subsequence of (x_i), say (x_{n_i}). By uniform continuity of f, $(f(x_{n_i}))$ is a Cauchy sequence in Y. But this is a contradiction.

Problem 0.7 (R:4:19)**.** Show that if $f : \mathbb{R} \to \mathbb{R}$ satisfies the intermediate value theorem and $f^{-1}(r) = \{x \mid f(x) = r\}$ is closed for $r \in \mathbb{Q}$, then f is continuous. (See the text for a hint. $\mathbb{Q}$ here could be replaced by any dense set.)

Suppose f fails to be continuous at x. Fix $\epsilon > 0$ such that for all $\delta > 0$, there is some $x' \in (x - \delta, x + \delta)$ so that $f(x') \notin (f(x) - \epsilon, f(x) + \epsilon)$. We can then choose a sequence $x_i \to x$ so the for all i, $f(x_i) \notin N_\epsilon(f(x))$. We may assume WLOG $f(x_i) \leq f(x) - \epsilon < f(x)$ for all i since either infinitely many of the x_i satisfy this or else $f(x) < f(x) + \epsilon \leq f(x_i)$ and the proof would be the same in each case. Fix $r \in (f(x) - \epsilon, f(x)) \cap \mathbb{Q}$. So we can get $t_i \in N_{|x_i - x|}(x)$ so that $f(t_i) = r$ using the IVT property. But then it is clear that $t_i \to x$ and as $t_i \in f^{-1}(r)$ for all i we have $x \in f^{-1}(r)$. But then $f(x) = r$ and this is a contradiction.