

# Math 571 - Homework 4

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**Problem 0.1** (R:3:8). Suppose  $\sum_n a_n$  converges and  $(b_n)$  is monotonic and bounded, show that  $\sum_n a_n b_n$  converges.

**Problem 0.2** (R:3:9). Find the radius of convergence of the following power series.

a)  $\sum_n n^3 z^n$

b)  $\sum_n \frac{2^n}{n!} z^n$

c)  $\sum_n \frac{2^n}{n^2} z^n$

d)  $\sum_n \frac{n^3}{3^n} z^n$

**Problem 0.3** (R:3:11). Suppose  $a_n > 0$ ,  $s_n = \sum_{i=1}^n a_i$ , and  $\sum_i a_i = \lim_i s_i$  diverges.

a) Show that  $\sum_i \frac{a_i}{a_i+1}$  diverges.

b) Show that  $\frac{a_N}{s_N} + \frac{a_{N+1}}{s_{N+1}} + \dots + \frac{a_{N+k}}{s_{N+k}} \geq 1 - \frac{s_N}{s_{N+k}}$  and deduce that  $\sum_i \frac{a_i}{s_i}$  diverges.

c) Show that  $\frac{a_N}{s_N^2} \leq \frac{1}{s_{N-1}} - \frac{1}{s_N}$  and deduce that  $\sum_i \frac{a_i}{s_i^2}$  converges.

**Problem 0.4** (R:3:16(18)\*). Fix  $\alpha > 1$  and  $x_1 > \sqrt{\alpha}$ , define

$$x_{n+1} = \frac{1}{2} \left( x_n + \frac{\alpha}{x_n} \right)$$

a) Prove that  $(x_n)$  decreases monotonically and  $\lim_{n \rightarrow \infty} x_n = \sqrt{\alpha}$ .

b) Let  $\varepsilon_n = x_n - \sqrt{\alpha}$  be the error at the  $n^{\text{th}}$  term, show that

$$\varepsilon_{n+1} = \frac{\varepsilon_n^2}{2x_n} < \frac{\varepsilon_n^2}{2\sqrt{\alpha}}$$

Setting  $\beta = 2\sqrt{\alpha}$ , gives

$$\varepsilon_{n+1} < \beta \left( \frac{\varepsilon_1}{\beta} \right)^{2^n}$$

c) Choose a number  $\alpha > 3$  find a bound for how many terms are need to compute  $\sqrt{\alpha}$  correct to 20 decimal places where  $x_1$  is chosen minimally so that  $x_1^2 > \alpha$ . Prove your answer and do the computation. You might use [Python](#) or MATLAB.

d) Replace the recursion above by

$$x_{n+1} = \frac{p-1}{p} x_n + \frac{\alpha}{p} x_n^{-p+1}$$

Discuss the behavior of  $(x_n)$  under suitable conditions. Don't bother trying to compute a recursive expression for  $\varepsilon_n$  in this case, but do prove your claims.

**Problem 0.5.** Show that a normed vector space  $(X, \|\cdot\|)$  is complete iff every absolutely summable series is summable.

**Problem 0.6** (R:3:21\*). Let  $E_n$  be a descending sequence of closed subsets in a complete metric space, i.e.,  $E_{n+1} \subseteq E_n$ . Notice that  $\text{diam}(E_{n+1}) \leq \text{diam}(E_n)$  so  $\lim_n \text{diam}(E_n) = \delta$  exists in  $[0, \infty]$ . In each of the following cases what are all of the possibilities for  $\bigcap_n E_n$ .

a)  $\delta = \infty$ .

b)  $0 < \delta < \infty$ .

c)  $\delta = 0$ .