Math 571 - Exam 2 (05.22)

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Definition: $\prod_n b_n$ converges to P iff $P_n = \prod_{i=1}^n b_i \to P$ as $n \to \infty$ for $P \neq 0$. (If $P = 0$, then we say the product **diverges to** 0.)

Problem 1 (Convergent Products)**.** (ii) and (iii) below follow easily from (i).

i) Show that for sequences (a_n) and (b_n) where $a_n \geq 0$ and $b_n > 0$, if $\lim_n \frac{a_n}{b_n} = c > 0$, then

$$\sum a_n \text{ converges} \iff \sum b_n \text{ converges}$$

ii) Show that $\sum \ln(1 + a_n)$ converges iff $\sum a_n$ converges, for $a_n > 0$.

iii) If $a_n > 0$ for all n, show that $\prod_{i=1}^{\infty}(1 + a_n)$ converges iff $\sum_{i=1}^{\infty} a_n$ converges.

Recall: $f : X \to Y$ is an *open map* iff $f(O)$ is open in Y for every open $O \subseteq X$. So if f is a bijection, then f^{-1} is continuous iff f is open.

Problem 2. Let $f : \mathbb{R} \to \mathbb{R}$ be open and continuous. Show that f must be monotonic.

Definition: Fix an interval $[a, b]$ and $\alpha : [a, b] \to \mathbb{R}$ monotonic increasing. For f bounded on $[a, b]$ define $\|f\|_1 = \int_a^b |f| \, d\alpha$. Let $L^1(\alpha)$ be the set of all bounded f on $[a, b]$ so that $\|f\|_1 < \infty$ where f and g are considered the "same" if $\|f - g\|_1 = 0$. $L^1(\alpha)$ is a vector space with norm $\|\cdot\|_1$. Thus $d_1(f, g) = \|f - g\|_1$ is a metric which turns out to be both complete and separable. Here we condider two dense sets of functions in $L^1(\alpha)$.

Problem 3. Let α be monotonic increasing on $[a, b]$ and f bounded on $[a, b]$ with $f \in \mathcal{R}(\alpha)$. Show that for any $\delta > 0$ there is a step function f_1 so that $\|f_1 - f\|_1 < \delta$.

As a consequence, in the space $L^1(\alpha)$, the set of step functions is a dense subset.

Note: A step function on $[a, b]$ is given as follows: There is a partition $a = x_0 < \cdots < x_n = b$ so that on $[x_i, x_{i-1}]$ the function just takes a constant value, that is, $f|_{(x_{i-1}, x_i)} = a_i$ for some a_i. (I do not specify exactly what f does at the x_i's, but this does not really matter. Can you explain why? – Not part of the exam, just a thought question.)

Hint: Fix a partition P so that $U(f, P) - L(f, P) < \delta$ and use this to define a step function, s, so that $\|f - s\|_1 = \int_a^b |f - s| \, d\alpha < \delta$. You must argue here that $|f - s| \in \mathcal{R}(\alpha)$.

Problem 4 (Generalization of FTC)**.** Suppose that F is a differentiable function on $[a, b]$ with $F' = f$, show that

$$\underline{\int_a^b} f\,dx \leq F(b) - F(a) \leq \overline{\int_a^b} f\,dx$$