Name: ____________________ Exam 3 - MAT345

1 True/False

Each problem is worth 10 points for a total of 100 points. Here, you need only provide the "T" and "F" you may submit a set of justifications to earn back 50% of lost points. I can't post the "answers" until after the exam on Friday. These justifications will be due by Sunday at midnight.

Problem 1.1. For the exam, you need only indicate True or False. No justification is required. If you want to earn back some points, you can supply full justifications for **all** of the problems. You may earn back 50% of lost points.

a) False There is a 3×3 matrix with eigenvalues $\lambda_1 = 1/2$ and $\lambda_2 = -1/3$ with

$$E_{\lambda_1} = \text{span}\left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix} \right\} \text{ and } E_{\lambda_2} = \text{span}\left\{ \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix} \right\}$$

b) True There is a 3×3 matrix with eigenvalues $\lambda_1 = 1/2$ and $\lambda_2 = -1/3$ with

$$E_{\lambda_1} = \text{span}\left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix} \right\} \text{ and } E_{\lambda_2} = \text{span}\left\{ \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}$$

c) True If a 3×3 matrix A has eigenvalues $\lambda_1 = 1/2$ and $\lambda_2 = -1/3$, and no others, and

$$E_{\lambda_1} = \text{span}\left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \right\} \text{ and } E_{\lambda_2} = \text{span}\left\{ \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix} \right\}$$

Then there is no diagonal matrix D similar to A.

d) True If a 3×3 matrix has eigenvalues 1, $1/2$, and $-1/4$, then A is diagonalizable.

e) True There is a 3×3 symmetric matrix with eigenvalues $\lambda_1 = 1/2$ and $\lambda_2 = -1/3$ with

$$E_{\lambda_1} = \text{span}\left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix} \right\} \text{ and } E_{\lambda_2} = \text{span}\left\{ \begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix} \right\}$$

f) False For A a 2×3 matrix, there is always a unique $\hat{\boldsymbol{x}} \in \mathbb{R}^3$ so that $\|\boldsymbol{b} - A\hat{\boldsymbol{x}}\| \leq \|\boldsymbol{b} - A\boldsymbol{x}\|$ for all $\boldsymbol{x} \in \mathbb{R}^3$.

g) True For A a 2×3 matrix, there is always a unique $\hat{\boldsymbol{b}} \in \mathbb{R}^2$ so that $\hat{\boldsymbol{b}} = A\hat{\boldsymbol{x}}$ for some $\hat{\boldsymbol{x}} \in \mathbb{R}^3$, and $\|\boldsymbol{b} - A\hat{\boldsymbol{x}}\| \leq \|\boldsymbol{b} - A\boldsymbol{x}\|$ for all $\boldsymbol{x} \in \mathbb{R}^3$.

h) True The sheer matrix $A = \left[\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right]$ can be written as

$$A = UDV$$

where $U^{-1} = U^T$ and $V^{-1} = V^T$ and D is diagonal. (All matrices are real.)

i) False The sheer matrix $A = \left[\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right]$ can be written

$$A = UDU^{-1}$$

where D is diagonal. (All matrices are real.)

j) True Every finite dimensional vector space has an orthonormal basis.

2 Computational (100 points; 5 problems, each worth 20 points.)

Show all computations so that you make clear what your thought processes are.

Problem 2.1 (20 pts)**.** Let P be the plane through the origin and the points $(1,1,0)$ and $(1,-1,0)$. Let $p : \mathbb{R}^3 \to P$ be the orthogonal projection onto P. Let L be the line through the origin perpendicular to P. p is a linear map and hence given by a matrix A.

a) What are the eigenvalues of A? Explain without calculation.

   All points on the plane P are fixed, so 1 is an eigenvalue with $E_1 = P$. All points on L are mapped to 0, so 0 is an eigenvalue with $E_0 = \ker(p) = L$.

b) For each eigenvalue, what is the associated eigenspace in terms of P and L?

   already answered in the answer to (a)

c) Given the answer to the first two questions, write $A = SDS^{-1}$. (Read the next item first, you might kill two birds with one stone here.)

   This could just be answered in (d), but given the answer to (a) and (b) we have the following. $P = \text{span}\{(1,1,0),(1,-1,0)\}$ and $L = \text{span}\{(0,0,1)\}$. So

$$A = \begin{bmatrix} 1 & 1 & 0 \\ 1 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 1 & 0 \\ 1 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}^{-1}$$

d) If possible, write $A = UDU^T$ for some unitary U.

   Given the preceding, we can just normalize the two vectors we chose for the basis to P to get

$$A = \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} & 0 \\ 1/\sqrt{2} & -1/\sqrt{2} & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} & 0 \\ 1/\sqrt{2} & -1/\sqrt{2} & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

   Note here that $U^T = U$.

Note: There is almost nothing that you need to calculate here. This is checking that you understand what eigenvalues and eigenvectors are, at least geometrically.

Problem 2.2 (20 pts)**.** Suppose

$$A = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1/2 & 0 & 0 \\ 0 & -1/3 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 0 \end{bmatrix}^{-1}$$

Describe the "long-term" behavior of $A^n \boldsymbol{v}$ for an arbitrary point $\boldsymbol{v} \in \mathbb{R}^3$. More specifically, in the limit as $n \to \infty$ what happens to $A^n \boldsymbol{v}$.

Note: Depending on where $\boldsymbol{v}$ is in $\mathbb{R}^3$, there might be different long-term behavior.

For any $\boldsymbol{v}$, we may write $\boldsymbol{v} = a\boldsymbol{v}_1 + b\boldsymbol{v}_2 + c\boldsymbol{v}_3$ where

$$\boldsymbol{v}_1 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \quad \boldsymbol{v}_2 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \quad \boldsymbol{v}_3 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$$

and we see that

$$A^n \boldsymbol{v} = (1/2)^n a\boldsymbol{v}_1 + (-1/3)^n b\boldsymbol{v}_2 + (1)^n c\boldsymbol{v}_3$$

and since both $(1/2)^n$ and $(-1/3)^n$ approach 0 as n gets large, $A^n \boldsymbol{v}$ approaches $c\boldsymbol{v}$. All points are "attracted" to the line L, in particular $\lim_{n\to\infty} A^n \boldsymbol{v} = c\boldsymbol{v}_3 = \left[\begin{smallmatrix} c \\ c \\ 0 \end{smallmatrix}\right]$.

You can calculate c to get

$$\begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 0 \end{bmatrix}^{-1} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \frac{1}{2} \begin{bmatrix} 1 & -1 & 1 \\ -1 & 1 & 1 \\ 1 & 1 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

So $c = \frac{1}{2}(x + y - z)$ and so

$$A^n \begin{bmatrix} x \\ y \\ z \end{bmatrix} \to \frac{1}{2}(x + y - z) \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$$

For the next two problems, let

$$A = \begin{bmatrix} 3/2 & 3/2 \\ 3/2 & 3/2 \\ -1 & 1 \\ -1 & 1 \end{bmatrix}.$$

Problem 2.3 (20 pts)**.** Diagonalize the 2×2 matrix $A^T A$.

Note: Make sure that you give the characteristic polynomial $p(t)$ and its roots. Explain how you find eigenvectors. Check the answer; you will be using it below.

$$A^T A = \begin{bmatrix} 13/2 & 5/2 \\ 5/2 & 13/2 \end{bmatrix}$$

So $p_A(t) = (13/2 - t)^2 - (5/2)^2$ and

$$\begin{aligned} p_A(t) = 0 &\iff (13 - 2t)^2 - 5^2 = 0 \\ &\iff 4t^2 - 52t + (13^2 - 5^2) = 0 \\ &\iff 4t^2 - 52t + 12^2 = 0 \\ &\iff t^2 - 13t + 36 = (t-9)(t-4) = 0 \end{aligned}$$

So the eigenvalues are 9 and 4. So

$$\Lambda = \begin{bmatrix} 9 & 0 \\ 0 & 4 \end{bmatrix}$$

Now find the basis for the eigenspaces

$$E_9 = \mathrm{NS}(A - 9I) = \mathrm{NS}\begin{bmatrix} -5/2 & 5/2 \\ -5/2 & 5/2 \end{bmatrix} = \mathrm{span}\left\{\begin{bmatrix} 1 \\ 1 \end{bmatrix}\right\}$$

and

$$E_4 = \mathrm{NS}(A - 4I) = \mathrm{NS}\begin{bmatrix} 5/2 & 5/2 \\ 5/2 & 5/2 \end{bmatrix} = \mathrm{span}\left\{\begin{bmatrix} -1 \\ 1 \end{bmatrix}\right\}$$

So

$$V = \begin{bmatrix} 1/\sqrt{2} & -1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}$$

Problem 2.4 (20 pts)**.** Find the SVD for A (same A as in the previous problem.) Explain how you get the singular values and the left singular vectors.

Check! When done, you should have unitary 4×4 matrix U, a diagonal 4×2 matrix Σ, and the unitary 2×2 matrix V (from above) so that $A = U\Sigma V^T$.

The singular values are just $\sigma_1 = \sqrt{9}$ and $\sigma_2 = \sqrt{4} = 2$ and so we know from the previous problem that

$$\Sigma = \begin{bmatrix} 3 & 0 \\ 0 & 2 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \text{ and } V = \begin{bmatrix} 1/\sqrt{2} & -1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}$$

We have

$$\boldsymbol{u}_1 = \frac{A\boldsymbol{v}_1}{\sigma_1} = \frac{1}{3}\begin{bmatrix} 3/\sqrt{2} \\ 3/\sqrt{2} \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \\ 0 \\ 0 \end{bmatrix}$$

and

$$\boldsymbol{u}_2 = \frac{A\boldsymbol{v}_1}{\sigma_2} = \frac{1}{2}\begin{bmatrix} 0 \\ 0 \\ 2/\sqrt{2} \\ 2/\sqrt{2} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix}$$

For $\boldsymbol{u}_3$ and $\boldsymbol{u}_4$ we need an orthonormal basis for NS(A)

$$\mathrm{NS}(A^T) = \mathrm{NS}\begin{bmatrix} 3/2 & 3/2 & -1 & -1 \\ 3/2 & 3/2 & 1 & 1 \end{bmatrix} = \mathrm{NS}\begin{bmatrix} 3/2 & 3/2 & -1 & -1 \\ 0 & 0 & 2 & 2 \end{bmatrix}$$
$$= \mathrm{NS}\begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix} = \mathrm{span}\left\{ \begin{bmatrix} 1 \\ -1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ -1 \\ 1 \end{bmatrix} \right\}$$

So

$$\boldsymbol{u}_3 = \begin{bmatrix} 1/\sqrt{2} \\ -1/\sqrt{2} \\ 0 \\ 0 \end{bmatrix} \text{ and } \boldsymbol{u}_4 = \begin{bmatrix} 0 \\ 0 \\ 1/\sqrt{2} \\ -1/\sqrt{2} \end{bmatrix}$$

and so

$$U = \begin{bmatrix} 1/\sqrt{2} & 0 & 1/\sqrt{2} & 0 \\ 1/\sqrt{2} & 0 & -1/\sqrt{2} & 0 \\ 0 & 1/\sqrt{2} & 0 & 1/\sqrt{2} \\ 0 & 1/\sqrt{2} & 0 & -1/\sqrt{2} \end{bmatrix}$$

So

$$A = \begin{bmatrix} 1/\sqrt{2} & 0 & 1/\sqrt{2} & 0 \\ 1/\sqrt{2} & 0 & -1/\sqrt{2} & 0 \\ 0 & 1/\sqrt{2} & 0 & 1/\sqrt{2} \\ 0 & 1/\sqrt{2} & 0 & -1/\sqrt{2} \end{bmatrix} \begin{bmatrix} 3 & 0 \\ 0 & 2 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ -1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}$$
$$= \frac{1}{2}\begin{bmatrix} 1 & 0 & 1 & 0 \\ 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & -1 \end{bmatrix} \begin{bmatrix} 3 & 0 \\ 0 & 2 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}$$
$$= \frac{1}{2}\begin{bmatrix} 3 & 0 \\ 3 & 0 \\ 0 & 2 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}$$
$$= \frac{1}{2}\begin{bmatrix} 3 & 3 \\ 3 & 3 \\ -2 & 2 \\ -2 & 2 \end{bmatrix} \text{ Correct!}$$