Quiz 2

Problem 1 (10 points; 2 points each)**.** Decide if each of the following are true or false and provide a justification or counterexample in each case. A justification could consist of a theorem from the text. All vector spaces are assumed to be finite-dimensional here.

(a) ______ For A and B $n \times n$ matrices, $\det(AB) = \det(A)\det(B)$

This is true and is Theorem 2.2.3.

(b) ______ For A and B $n \times n$ matrices, $\det(AB - BA) = \det(AB) - \det(BA)$

This is false any example works as a sufficient reason. For example

$$\det\left(\begin{bmatrix}0 & 1\\ 0 & 0\end{bmatrix}\begin{bmatrix}1 & 1\\ 1 & 1\end{bmatrix} - \begin{bmatrix}1 & 1\\ 1 & 1\end{bmatrix}\begin{bmatrix}1 & 1\\ 0 & 0\end{bmatrix}\right) = \det\left(\begin{bmatrix}1 & 1\\ 0 & 0\end{bmatrix} - \begin{bmatrix}0 & 1\\ 0 & 1\end{bmatrix}\right) = \det\begin{bmatrix}1 & 0\\ 0 & -1\end{bmatrix} = -1$$

Whereas, from(a), $\det(AB) - \det(BA) = 0$.

To delve a bit deeper, it can be shown (we will later) that $\det(AB - BA) = 0$ iff $AB = BA$, that is, iff A and B commute. In fact, $\det(AB - BA)$ is used as a measure of how badly A and B fail to commute.

(c) ______ Performing type III elementary row operations on a square matrix does not change the value of the determinant.

This is true and is in the summary box on pg 97.

(d) ______ If A is a 4×4 matrix with rows $\boldsymbol{a}_1$, $\boldsymbol{a}_2$, $\boldsymbol{a}_3$, and $\boldsymbol{a}_4$, and $\boldsymbol{a}_1 = 2\boldsymbol{a}_2 - 3\boldsymbol{a}_3 + 4\boldsymbol{a}_4$, then $\det(A) = 2 - 3 + 4 = 3$.

This is false, it one row/column is a linear combination of other rows/columns, then $\det(A) = 0$.

(e) ______ $\det(A)$ has a geometric interpretation.

This is true and was the topic of some of the notes that I provided. $\det(A)$ is the "signed volume" of the n-dimensional parallelepiped determined by the rows/-columns of A.

Problem 2 (10 points)**.** Use the following three facts about determinants to compute the determinant of a matrix using row operations.

a. If B is diagonal, then $\det(B) = b_{11} \cdot b_{22} \cdots b_{nn}$.

b. If B arises from A by a type I row operation, i.e., interchanging two rows, then $\det(B) = -\det(A)$.

c. If B arises from A by a type III row operation, i.e., $r_i + ar_j \to r_i$, that is, row i is replaced by row i plus a scalar multiple of row j, where $i \neq j$. Then $\det(A) = \det(B)$.

Compute $\det(A)$ by:

1. Reducing A to a triangular matrix B using only type I and III operations. (I would say echelon form, except for the issue with pivots being 1).
2. Keep track of how many row swaps were made.
3. Compute $\det(B)$ by multiplying the diagonal elements of B.

$$A = \begin{bmatrix} 2 & 6 & 3 & 2 \\ 4 & 2 & 3 & 2 \\ 2 & 2 & 2 & 1 \\ 4 & 2 & 1 & 5 \end{bmatrix}$$

Show the work for the above computation here.

On your own, don't include this in the quiz, try computing this determinant by expanding on a row or column.

Discuss which method, "expansion along a row or column" or "using elementary row operations" is, in general, a faster method of computing a determinant.

I will only write this out using row reduction, the other way ... expansion along a row/column would be too painful.

$$\begin{bmatrix} 2 & 6 & 3 & 2 \\ 4 & 2 & 3 & 2 \\ 2 & 2 & 2 & 1 \\ 4 & 2 & 1 & 5 \end{bmatrix} \underset{\substack{r_2-2r_1\to r_2 \\ r_3-r_1\to r_3 \\ r_4-2r_1\to r_4}}{\longrightarrow} \begin{bmatrix} 2 & 6 & 3 & 2 \\ 0 & -10 & -3 & -2 \\ 0 & -4 & -1 & -1 \\ 0 & -10 & -5 & 1 \end{bmatrix}$$
$$\underset{\substack{r_3-4/10r_2\to r_3 \\ r_4-10r_2\to r_4}}{\longrightarrow} \begin{bmatrix} 2 & 6 & 3 & 2 \\ 0 & -10 & -3 & -2 \\ 0 & 0 & 1/5 & -1/5 \\ 0 & 0 & -2 & 3 \end{bmatrix}$$
$$\underset{r_4+10r_3\longrightarrow r_3}{\longrightarrow} \begin{bmatrix} 2 & 6 & 3 & 2 \\ 0 & -10 & -3 & -2 \\ 0 & 0 & 1/5 & -1/5 \\ 0 & 0 & 0 & -1 \end{bmatrix}$$

We had two row swaps so $\det(A) = (2)(-10)(1/5)(1) = -4$.

I leave the expansion along a column to the reader. You should note that it is a much longer and more involved process. For finding $\det(A)$ it is much quicker to use the method of applying elimination and keeping track of how the determinant changes.

Formally for an $n \times n$ matrix, reducing to a triangular matrix requires at most $(n-1)+(n-2)+ ... +1 = (n)(n-1)/2 \approx n^2$ many operations. Expending a determinant is a recursive procedure so it is a little harder to compute. For 2×2, let's say 2 operations are used. For 3×3, we have 3×2, since you must do 3 2×2's. For 4×4, you need $4\times 3\times 2$, recognize this! It requires $n!$ operations, that is HUGE! Worse than exponential.

Problem 3 (5 points)**.** Let A be as above, consider $A\boldsymbol{x} = \boldsymbol{b}$ where $\boldsymbol{b} = (-3, -3, -2, 1)$. Find x_1 using Cramer's rule. (You may use MATLAB/Octave to compute the determinants, but write out what you are computing.)

$$x_1 = \frac{\det \begin{bmatrix} -3 & 6 & 3 & 2 \\ -3 & 2 & 3 & 2 \\ -2 & 2 & 2 & 1 \\ 1 & 2 & 1 & 5 \end{bmatrix}}{\det \begin{bmatrix} 2 & 6 & 3 & 2 \\ 4 & 2 & 3 & 2 \\ 2 & 2 & 2 & 1 \\ 4 & 2 & 1 & 5 \end{bmatrix}} = -2$$

Problem 4 (5 points)**.** Submit the completion certificate for the OnRamp tutorial from MATLAB in the MATLAB shared drive.