Quiz 1

Problem 1 (15 points; 3 points each)**.** Decide if each of the following are true or false and provide a justification or counterexample in each case. A justification could consist of a theorem from the text. All vector spaces are assumed to be finite-dimensional here.

(a) _____ The inverse of a Type III elementary matrix need not be elementary.

This is false. In fact, the inverse of any elementary matrix is an elementary matrix of exactly the same type. In particular the elementary matrix for $R_i + cR_j \to R_i$ is just the elementary matrix for $R_i - cR_j \to R_i$.

This is a theorem in the text, so you can refer to that Theorem 1.5.1.

Note: An example does not suffice for an argument here.

(b) _____ An echelon form of a matrix is unique.

This is false. There is a unique reduced echelon form. Both

$$E_1 = \begin{bmatrix} 2 & 3 & 1 \\ 0 & 3 & 2 \end{bmatrix}$$

and

$$E_2 = \begin{bmatrix} 2 & 0 & -1 \\ 0 & 3 & 2 \end{bmatrix}$$

are both echelon and each can be got from the other by an elementary row operation.

(c) _____ A matrix is invertible if and only if it can be written as the product of elementary matrices.

This is true. If $A = E_1 \cdot E_2 \cdots E_n$ where E_i are elementary, then $A^{-1} = E_n^{-1} \cdot E_{n-1}^{-1} \cdots E_1^{-1}$, so the "if" part is proved. Conversely, if A is invertible, then by 1.5.2 there is a sequence of elementary matrices so that $E_1 \cdots E_n A = I$ and thus $A = E_n^{-1} \cdot E_{n-1}^{-1} \cdots E_1^{-1}$.

I would accept Theorem 1.5.2 as a full justification, although literally the bit of work above is required.

(d) _____ Right cancellation holds for matrix multiplication: If $AC = BC$, then $A = B$.

This is false. We have

$$AC = BC \iff AC - BC = 0 \iff (A-B)C = 0$$

So basically the claim is equivalent to $AB = 0 \iff A = 0$ or $B = 0$ and this is false. Easy example:

$$\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ 1 & -1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

(e) _____ Suppose for all matrices C, $AC = BC$, then $A = B$.

This is true. In particular take $C = \boldsymbol{e}_i$, then $AC = A_{*,i} = B_{*,i} = BC$. In particular, A and B have the same columns, so $A = B$.

Problem 2 (10 points)**.** Write down the augmented matrix associated to this system and then use the elementary row operations I and III to find an equivalent echelon, upper-triangular, form of this matrix. (See the note below.) Show every step of the reduction and indicate what operation you use.

Note: The book requires the leading entries in non-zero rows be 1's. Many, including myself, drop this and just describe the pivot as the leading non-zero entry. If you like, call this more general form, "*upper-triangular*."

Eliminate all the non-pivot leading coefficients in column 1 first, then work on column 2, etc. You can combine all operations for one column in a single step, so this should require 2 or 3 steps depending on how you write things.

$$\begin{aligned} 4x_1 + x_2 - x_3 + 3x_4 &= 5 \\ -4x_1 - 4x_2 + x_3 &= 4 \\ -2x_1 + 4x_2 + 4x_3 - 4x_4 &= -12 \end{aligned}$$

The augmented matrix is

$$\left[\begin{array}{cccc|c} 4 & 1 & -1 & 3 & 5 \\ -4 & -4 & 1 & 0 & 4 \\ -2 & 4 & 4 & -4 & -12 \end{array}\right]$$

Reduction to echelon form requires 3 row manipulations:

$$\begin{aligned} \left[\begin{array}{cccc|c} 4 & 1 & -1 & 3 & 5 \\ -4 & -4 & 1 & 0 & 4 \\ -2 & 4 & 4 & -4 & -12 \end{array}\right] &\xrightarrow[R_3 \leftarrow 2R_3]{} \left[\begin{array}{cccc|c} 4 & 1 & -1 & 3 & 5 \\ -4 & -4 & 1 & 0 & 4 \\ -4 & 8 & 8 & -8 & -24 \end{array}\right] \\ &\xrightarrow[\substack{R_2 \leftarrow R_1 + R_2 \\ R_3 \leftarrow R_1 + R_3}]{} \left[\begin{array}{cccc|c} 4 & 1 & -1 & 3 & 5 \\ 0 & -3 & 0 & 3 & 9 \\ 0 & 9 & 7 & -5 & -19 \end{array}\right] \\ &\xrightarrow[R_3 \leftarrow R_3 + 3R_2]{} \left[\begin{array}{cccc|c} 4 & 1 & -1 & 3 & 5 \\ 0 & -3 & 0 & 3 & 9 \\ 0 & 0 & 7 & 4 & 8 \end{array}\right] \end{aligned}$$

Problem 3 (10 points)**.** Using your echelon form above re-write the initial system as an equivalent triangular system. Let the variable that correspond to non-pivot element be the independent variable and solve for the remaining three variables in terms of this one. This is the "**back substitution**" step. Finally, write the solution set as $\{t\boldsymbol{v} + \boldsymbol{u} \,|\, t \in \mathbb{R}\}$ for some $\boldsymbol{v}, \boldsymbol{u} \in \mathbb{R}^4$. This way it is clear that the solution set is a line in $\mathbb{R}^4$.

The pivots are 4, -3, and 7 with corresponding pivot (dependent) variables x_1, x_2, and x_3, the only independent variable being x_4.

So let $t \in \mathbb{R}$ be arbitrary and set $x_4 = t$ to give the diagonal system

$$\begin{aligned} 4x_1 + x_2 - x_3 &= 5 - 3t \\ -3x_2 &= 9 - 3t \\ 7x_3 &= 8 - 4t \end{aligned}$$

Back substitution gives

$$\begin{aligned} x_2 &= t - 3 \\ x_3 &= \frac{1}{7}(8 - 4t) \end{aligned}$$

and

$$\begin{aligned} 4x_1 + (t-3) - \frac{1}{7}(8-4t) &= 5 - 3t \\ 28x_1 &= 35 - 21t - 7t + 21 - 4t + 8 = 64 - 32t \\ x_1 &= \frac{16}{7} - \frac{8}{7}t \end{aligned}$$

This gives

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 16/7 \\ -3 \\ 8/7 \\ 0 \end{bmatrix} + t \begin{bmatrix} -8/7 \\ 1 \\ -4/7 \\ 1 \end{bmatrix}$$

This shows that the solution set is a straight line in $\mathbb{R}^4$.