Math 571 - Exam 1

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NOTATION/DEFINITION: Let (X, d) be a metric space for $A, B \subset X$ define $d(A, B) = \inf\{d(a, b) \mid a \in A \text{ and } b \in B\}$. Set $d(a, B) = d(\{a\}, B)$.

Question 1 (12 points)**.** Let (X, d) be a metric space, prove that

a) For any closed set F and $x \notin F$, $d(x, F) > 0$.

Proof 1: Suppose $d(x, F) = 0$, then there is $x_i \in F$ such that $\lim_i d(x, x_i) = 0$, but then, $\lim_i x_i \to x$ so $x \in F$, which is a contradiction.

Proof 2: Let $\varepsilon > 0$ and $N_\varepsilon(x) \cap F = \emptyset$. $\varepsilon exists$ since $x \notin F$ and F is closed. Now for $y \in F$ we have $y \notin N_\varepsilon(x)$ so $d(x, y) \geq \varepsilon$ and hence $d(x, F) \geq \varepsilon$.

b) For any compact K and closed F with $K \cap F = \emptyset$, $d(K, F) > 0$.

Proof 1: For $x \notin F$ there is $\varepsilon_x > 0$ so that $N_{2\varepsilon_x}(x) \cap F = \emptyset$. The set of open sets $\mathcal{O} = \{N_{\varepsilon_x}(x) \mid x \in K\}$ is an open cover of K and hence has a finite subcover, $\mathcal{O}' = \{N_{\varepsilon_{x_1}}(x_1), \ldots, N_{\varepsilon_{x_k}}(x_k)\}$. Let $\varepsilon = \min\{\varepsilon_{x_1}, \ldots, \varepsilon_{x_k}\} > 0$. This is only true because we have a finite collection here, and this follows from ***compactness***.

Let $x \in K$, then $x \in N_{\varepsilon_{x_i}}(x_i)$ for some i and so for $y \in F$, $d(x, y) \leq d(x, x_i) + d(x_i, y)$ is $d(x_i, y) \geq d(x, y) - d(x_i, x) \geq 2\varepsilon_{x_i} - \varepsilon_{x_i} = \varepsilon_{x_i} \geq \varepsilon$. So $d(x, y) \geq \varepsilon$ for all $x \in K$ and $y \in F$ and hence $d(K, F) \geq \varepsilon > 0$.

Proof 2: Suppose $d(K, F) = 0$, then there is $(x_i, y_i) \in K \times F$ so that $\lim_i d(x_i, y_i) = 0$. Since K is compact there is an $x \in K$ and subsequence x_{i_j} so that $\lim_j x_{i_j} = x$. But then $\lim_j y_{i_j} = x$, so $x \in F$. You must use ***sequential compactness*** here.

c) Can the assumption that K is compact be replaced by K closed in (b)? That is, is there a metric space (X, d) and closed sets A, B so that $A \cap B = \emptyset$ and yet $d(A, B) = 0$?

It is simple to see that compactness is required here.

Example 1: Consider $A = \{(x, 1/x) \mid x > 0\}$ and $B = \{(x, -1/x) \mid x > 0\}$. Clearly, $d(A, B) = 0$ and as $x \mapsto 1/x$ is continuous, A and B are closed.

Example 2: K closed and bounded also does not suffice, but to see this, we must look into a space where closed and bounded does not imply compact. We don't have to look far. Consider $X = (0, 1)$, the open unit interval. Here X is closed (in X) and bounded but not compact. Consider $F = \{1/i \mid i > 0, i \in \mathbb{N}, \text{ and even}\}$ and $K = \{1/i \mid i \in \mathbb{N} \text{ and odd}\}$. Clearly, $K \cap H = \emptyset$ yet $d(1/i, 1/i + 1) \to 0$ so $d(F, K) = 0$.

Note: It is however true that for A, B closed with $A \cap B = \emptyset$, there are U, V open so that $A \supseteq U$, $B \supseteq V$, and $U \cap V = \emptyset$. This is the ***normality*** property.

Recall: In a metric space (X,d), $\mathrm{diam}(A) = \sup\{d(a,b) \mid a,b \in A\}$.

Question 2 (12 pts)**.** Let (X,d) be a metric space prove or disprove each of the following:

a) $\mathrm{diam}(A) = \mathrm{diam}(\mathrm{Cl}(A))$.

Take $\varepsilon > 0$, let $a, a' \in \mathrm{Cl}(A)$. There is $b, b' \in A$ so that $d(b,a) < \varepsilon$ and $d(b',a') < \varepsilon$. By the triangle inequality $d(a,a') \leq d(b,b') + d(a,b) + d(a',b') < \mathrm{diam}(A) + 2\varepsilon$. So $\mathrm{diam}(\mathrm{Cl}(A)) \leq \mathrm{diam}(A) + 2\varepsilon$. Since ε is arbitrary, $\mathrm{diam}(\mathrm{Cl}(A)) \leq \mathrm{diam}(A)$.

b) $\mathrm{diam}(A) = \mathrm{diam}(\mathrm{Int}(A))$.

This is trivially false. For example in $\mathbb{R}$ let $A = \{a,b\}$, then $\mathrm{diam}(A) = |b-a|$, but $\mathrm{Int}(A) = \emptyset$, so $\mathrm{diam}(\mathrm{Int}(A)) = 0$.

Question 3 (12 pts)**.** Let (X,d) be a metric space and $(x_i)_{i\in\mathbb{N}}$ and $(y_i)_{i\in\mathbb{N}}$ be two Cauchy sequences. Show that $(d(x_i,y_i))_{i\in\mathbb{N}}$ converges.

$d(x_i,y_i) \leq d(x_i,x_j) + d(x_j,y_j) + d(y_j,y_j')$ so that $d(x_i,y_i) - d(x_j,y_j) \leq d(x_i,x_j) + d(y_i,y_j)$. Swapping the rolls of i and j gives $d(x_j,y_j) - d(x_i,y_i) \leq d(x_i,x_j) + d(y_i,y_j)$ so we get

$$|d(x_j,y_j) - d(x_i,y_i)| \leq d(x_i,x_j) + d(y_i,y_j)$$

Now for $\varepsilon > 0$ take N so that $d(x_i,x_j) < \varepsilon/2$ and $d(y_i,y_j) < \varepsilon/2$ for $i,j > N$, then for $i,j > N$

$$|d(x_j,y_j) - d(x_i,y_i)| \leq d(x_i,x_j) + d(y_i,y_j) < \varepsilon.$$

so $(d(x_i,y_i))$ is a Cauchy sequence.

For the next problem, $(x_{i_k})_{k=0}^{\infty}$ is a **subsequence** of $(x_i)_{i=0}^{\infty}$ means $i_0 < i_1 < \cdots$. A sequence $(x_i)_{i=0}^{\infty}$ is **monotone increasing** iff $x_0 \leq x_1 \leq x_2 \cdots$. Similarly, define **monotone decreasing**. A sequence is **monotone** iff it is either monotone increasing or monotone decreasing.

Question 4 (12 pts)**.** Show that every infinite sequence of real numbers has a monotone subsequence that converges to $\limsup_i x_i$.

Before starting this, you need to recall, or look up, what $\limsup_i x_i$ means; otherwise, there is nothing you can do here.

$$\begin{aligned}\limsup_i x_1 &= \lim_i \sup\{x_j \mid j \geq i\} \\ &= \inf_i \sup\{x_j \mid j \geq i\} = \inf_i \sup_{j\geq i} x_j\end{aligned}$$

Proof 1: Define $\alpha_i = \sup_i\{x_j \mid j \geq i\}$. Clearly, $\alpha_0 \geq \alpha_1 \geq \cdots$, that is (α_i) is a monotonically decreasing sequence. Let $\alpha = \inf_i \alpha_i$, noting that $\alpha = -\infty$ and $\alpha = \infty$ are both possible.

Suppose there is a subsequence (α_{i_j}) that is strictly decreasing, that is $\alpha_{i_j} > \alpha_{i_{j+1}}$. In this case we get $i_j \leq m_j < i_{j+1}$ so that $\alpha_{i_j} \geq x_{m_i} > \alpha_{i_{j+1}}$. In this case (x_{m_i}) is a ***strictly descending sequence*** and $\lim_{x_{m_i}} = \alpha$.

The other case is that $\alpha_i = \alpha$ for all large enough i. It could be that $\alpha \in \{x_j \mid j \geq i\}$ for all large enough i. In this case, there is $x_{j_i} = \alpha$ with $i_0 < i_1 < \cdots$. In this case, the constant sequence (α) is an infinite constant (monotonic) subsequence of (x_i). If this fails to be the case, then for all large enough i, and for all $\varepsilon > 0$, there is $x_j > \alpha - \varepsilon$ for some $j > i$. So now we can build $x_{i_0} < x_{i_1} < \cdots$, a strictly increasing monotonic sequence, so that $\lim_j x_{i_j} = \alpha$.

So there are three possibilities: either there is a strictly increasing subsequence converging to α, a strictly decreasing subsequence converging to α, or else the constant sequence (α) is a subsequence.

Proof 2: (Student solution.) Call x_i a *peak* of $x_i > x_j$ for all $j > i$. If there are infinitely many peaks, (x_{i_j}) then clearly this is a strictly decreasing monotonic subsequence with limit $\limsup_i x_i$.

If there are only finitely many peaks, then let x_N be the last peak. be the last peak and take $\alpha = \sup\{x_i \mid i > N\} \leq x_N$. Clearly, $\alpha = \sup\{x_i \mid i > m\}$ for all $m > N$ and thus $\alpha = \limsup_i x_i$. But clearly if $\alpha = \sup\{x_i \mid i > N\}$, then $\alpha = \lim_j x_{i_j}$ for some $N < i_1 < i_2 < \cdots$ with $x_{i_1} \leq x_{i_2} \leq \cdots$ and this is what we wanted.

Note: There is another way to define $\limsup_i x_i$. Namely, consider, the set E of limits of all subsequences of $(x_i)_i$, then E is closed in the extended reals $\mathbb{R} \cup \{\pm\infty\}$ and $\limsup_i x_i = \max(E) = x^*$. Thus there is a subsequence $(x_{i_j})_j$ of $(x_i)_i$ so that $\lim_j x_{i_j} = \limsup_i x_i$. (This is Theorem 3.7 in the text.) So using this, you could fix such a subsequence of $(x_{i_j})_j$. Then argue that at least one of the following holds:

i) For infinitely many j, $x_{i_j} = x^*$, so that we can extract a further constant (hence monotonic) subsequence that converges to x^*.

ii) For all $\varepsilon > 0$, there is a $N > 0$ such that for infinitely many $j > N$, $x^* - \varepsilon < x_{i_j} < x^*$. In this case, we can construct an infinite monotonic increasing sequence with limit x^* by taking $x^* - \varepsilon_j < x_{i_j} < x^* - \varepsilon_{j+1} < x_{i_{j+1}} < x^*$ and $\varepsilon_j \to 0$.

iii) For all $\varepsilon > 0$, there is a $N > 0$ such that for infinitely many $j > N$, $x^* < x_{i_j} < x^* + \varepsilon$. In this case, we can construct an infinite monotonic decreasing sequence with limit x^* in a fashion similar to the preceding case.

As in the original argument, we have all three possibilities for the requisite monotonic sequence.

NOTE: The same is true for $\liminf_i x_i$.

Question 5 (Is supremum "linear"; 12 pts)**.** For $A, B \subseteq \mathbb{R}$, is it true that

i) $\sup(\alpha A) = \alpha \sup(A)$ for $\alpha \geq 0$, and

   This is true. This is clear if $\alpha = 0$, so assume $\alpha > 0$. There are two things to show, namely, (1) $\sup(\alpha A) \leq \alpha \sup(A)$ and (2) $\sup(\alpha A) \geq \alpha \sup(A)$. This means that we must show ($1'$) $\alpha \sup(A)$ is an upper bound of αA and ($2'$) $\frac{1}{\alpha} \sup(\alpha A)$ is an upper bound of A. ($2'$) is equivalent to $\sup(\alpha A)$ is an upper bound of αA, but this is clear.

   For (1'), let $a \in A$, then $a \leq \sup(A)$ and so $\alpha a \leq \alpha \sup(A)$. Thus $\alpha A \leq \alpha \sup(A)$ and we get that $\alpha \sup(A)$ is an upper bound of αA.

ii) $\sup(A + B) = \sup(A) + \sup(B)$.

Again there are two things to show. (1) $\sup(A+B) \geq \sup(A)+\sup(B)$ and (2) $\sup(A+B) \leq \sup(A)+\sup(B)$. As before, (2) is equivalent to (2$'$) $\sup(A)+\sup(B)$ is an upper bound on $A+B$ and this is clear since if $a \in A$ and $b \in B$, then $\sup(A)+\sup(B) \geq a+b$.

For (1), suppose $\sup(A)+\sup(B) > \sup(A+B)$, then $\sup(A)+b > \sup(A+B)$ for some $b \in B$. Applying this logic a second time we get $a \in A$ such that $a+b > \sup(A+B)$. this is absurd, so it must be that $\sup(A)+\sup(B) \leq \sup(A+B)$.

Question 6 (Compact sets get crowded; 15 pts)**.** Show that if X is compact, then for any $\varepsilon > 0$, there is $N > 0$ so that for all $S \subset X$ with $|S| \geq N$, there are two points in S whose distance is $< \varepsilon$.

Consider the open cover $\mathcal{O} = \{N_{\frac{\varepsilon}{2}}(x) \;\big|\; x \in X\}$ of X. Let $\mathcal{O}' = \{N_{\frac{\varepsilon}{2}}(x_i) \;\big|\; i = 1, \ldots, N\}$ be a finite open subcover. Let $S \subset X$ with $|S| > N$. By the pigeonhole principle, there are at least two elements $s, s' \in S$ which must fall in the same nbhd $N_{\frac{\varepsilon}{2}}(x_i)$ for some i, so that $d(s,s') \leq d(s,x_i) + d(x_i,s') < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$.