Math 571 - Homework 6

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Problem 6.1 (R:5:8)**.** Suppose f' is continuous on $[a, b]$ and $\epsilon > 0$. Show that there is $\delta > 0$ so that for all t such that $0 < |t - x| < \delta$ and all $a \leq x \leq b$

$$\left| \frac{f(t) - f(x)}{t - x} - f'(x) \right| < \epsilon$$

This could be stated as f' is *uniform continuity* on $[a, b]$ provided f' is continuous on $[a, b]$. Does this hold for vector-valued functions?

Problem 6.2 (R:5:9)**.** Suppose f is continuous on $\mathbb{R}$, and it is known that $f'(x)$ exists for all $x \neq 0$ and $f'(x) \to 3$ as $x \to 0$. Must $f'(0)$ exist?

Problem 6.3 (R:5:11)**.** Suppose f is defined in a nbhd of x and $f''(x)$ exists. Show that

$$\lim_{h \to 0} \frac{f(x + h) + f(x - h) - 2f(x)}{h^2} = f''(x)$$

Show, by example, that the above limit can exist even if $f''(x)$ does not.

Problem 6.4 (R:5:16)**.** Suppose f is twice differentiable on $(0, \infty)$ and f'' is bounded on $(0, \infty)$, and $f(x) \to 0$ as $x \to \infty$. Show that $f'(x) \to 0$ as $x \to \infty$.

Problem 6.5 (R:5:22)**.** Let $f : [a, b] \to [A, B]$ be differentiable on (a, b) and continuous on $[a, b]$. Here a, b, A, or B could be infinite, in which case we just identify something like $[-\infty, 2]$ with the more usual notation $(-\infty, 2]$. A point x is a **fixed** point of f iff $f(x) = x$.

(a) Show that if $f'(t) \neq 1$ for all $t \in (a, b)$, then f can have at most one fixed point.

(b) Show that $f(t) = t + (1 + e^t)^{-1}$ satisfies $|f'(t)| < 1$ and f has no fixed points.

(c) Show that if there is $A < 1$ so that $|f'(t)| \leq A$ for all $t \in (a, b)$, then f has a fixed point and moreover given any $x_0 \in (a, b)$ and taking $x_{n+1} = f(x_n)$ it turns out that $x_n \to x$ and $f(x) = x$ is the unique fixed point of f.

Problem 6.6. Show that $f(x, y) = \sqrt{|xy|}$ is not differentiable at $(0, 0)$, but both partials $f_x(0, 0)$ and $f_y(0, 0)$ exist.