

Name: \_\_\_\_\_

Quiz 2 - MAT345

**Problem 1** (20 points; 4 points each). Decide if each of the following are true or false and provide a small proof or counterexample in each case. All vector spaces are assumed to be finite-dimensional here.

- (a) False Given a basis  $\mathcal{B} = \{\mathbf{v}_1, \dots, \mathbf{v}_n\}$  for a vector space  $V$  and  $U$  a subspace of  $V$ , then there is  $\mathcal{C} \subseteq \mathcal{B}$  that is a basis for  $U$ .

FALSE:  $\mathcal{B} = \{\mathbf{e}_1, \mathbf{e}_2\}$  is a basis for  $\mathbb{R}^2$  and  $U = \text{span}\{(1, 1)\}$  is a subspace, namely, the line with slope 1 through the origin. You cannot throw away one of  $\mathbf{e}_1$  or  $\mathbf{e}_2$  to get a basis for  $U$ .

- (b) True Given a basis  $\mathcal{C}$  for a subspace  $U$  of a vector space  $V$ ,  $\mathcal{C}$  can be extended to a basis  $\mathcal{B}$  for  $V$ .

This is one of the theorems that you have, any linearly independent set can be expanded to a basis.

- (c) False If  $\{\mathbf{v}_1, \dots, \mathbf{v}_n\}$  is linearly independent and  $\mathbf{v} \in \text{span}(\{\mathbf{v}_1, \dots, \mathbf{v}_n\})$ , then it is possible that there are distinct  $\mathbf{c}, \mathbf{b} \in \mathbb{R}^n$  such that  $\mathbf{v} = \sum_{i=1}^n c_i \mathbf{v}_i = \sum_{i=1}^n b_i \mathbf{v}_i$ .

If such  $\mathbf{c}$  and  $\mathbf{b}$  exists, then  $\mathbf{0} = \mathbf{v} - \mathbf{v} = (\sum b_i \mathbf{v}_i) - (\sum c_i \mathbf{v}_i) = \sum (b_i - c_i) \mathbf{v}_i$ . Since  $\mathbf{v}_i$ 's are independent,  $b_i - c_i = 0$  for all  $i$ , so  $\mathbf{c} = \mathbf{b}$ .

- (d) True If  $\{\mathbf{v}_1, \dots, \mathbf{v}_n\}$  is linearly independent and  $V = \text{span}(\{\mathbf{v}_1, \dots, \mathbf{v}_n\}) = \text{span}(\{\mathbf{u}_1, \dots, \mathbf{u}_n\})$ , then  $\{\mathbf{u}_1, \dots, \mathbf{u}_n\}$  is linearly independent.

This too, is a theorem. Since  $V = \text{span}\{\mathbf{v}_1, \dots, \mathbf{v}_n\}$  and  $\mathbf{v}_i$  are independent, you know  $\{\mathbf{v}_1, \dots, \mathbf{v}_n\}$  is a basis for  $V$  and so  $\dim(V) = n$ . since  $\text{span}\{\mathbf{u}_1, \dots, \mathbf{u}_n\} \text{ span } V$  you know this set can be reduced to a basis, but any basis must have  $n$  elements, so  $\{\mathbf{u}_1, \dots, \mathbf{u}_n\}$  must already be a basis, and hence is linearly independent.

- (e) False Suppose  $V$  is a vector space and  $U \subseteq V$  is a subspace. For any  $\mathbf{v} \in V$ , there is a **unique**  $\mathbf{u} \in U$  so that  $\mathbf{v} = \mathbf{u} + (\mathbf{v} - \mathbf{u})$ , that is, there is a unique "projection" of  $V$  into  $U$ .

Again take  $U = \text{span}\{(1, 1)\} \subset \mathbb{R}^2 = V$  and let  $\mathbf{v} = (2, 3)$ , then  $\mathbf{v} = (1, 1) + (1, 2) = (2, 2) + (0, 1)$ .

Note: If we fixed  $W$  so that  $V = U \oplus W$ , then there would be for every  $\mathbf{v} \in V$  a unique  $\mathbf{u} \in U, \mathbf{w} \in W$  so that  $\mathbf{v} = \mathbf{u} + \mathbf{w}$ . For example, take  $U$  as above and  $W = \text{span}\{(0, 1)\}$ , then  $(2, 3) = (2, 2) + (0, 1)$  is the unique decomposition of  $(2, 3)$  into something from  $U$  and something from  $W$ .

**Problem 2** (10 pts). Show that the collection,  $U$ , of upper triangular  $3 \times 3$  matrices is a subspace of  $\mathbb{R}^{3 \times 3}$  (the space of all  $3 \times 3$  matrices). Give a basis  $\mathcal{B}$  for  $U$  and for  $\mathbf{v} = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 4 & 5 \\ 0 & 0 & 6 \end{bmatrix}$ , give  $[\mathbf{v}]_{\mathcal{B}}$ .

To show that  $U$  is a subspace we need only show that  $\alpha\mathbf{v} + \mathbf{u} \in U$  for  $\mathbf{v}, \mathbf{u} \in U$ . So let  $\mathbf{u} = \begin{bmatrix} u_{11} & u_{12} & u_{13} \\ 0 & u_{22} & u_{23} \\ 0 & 0 & u_{33} \end{bmatrix}$  and let  $\mathbf{v} = \begin{bmatrix} v_{11} & v_{12} & v_{13} \\ 0 & v_{22} & v_{23} \\ 0 & 0 & v_{33} \end{bmatrix}$ , then  $\alpha\mathbf{u} + \mathbf{v} = \begin{bmatrix} \alpha u_{11} + v_{11} & \alpha u_{12} + v_{12} & \alpha u_{13} + v_{13} \\ 0 & \alpha u_{22} + v_{22} & \alpha u_{23} + v_{23} \\ 0 & 0 & \alpha u_{33} + v_{33} \end{bmatrix} \in U$ .

A basis is clearly given by  $E_{lk}^{ij} = 1$  if  $i = j$  and  $l = k$  and  $j \leq i$  and 0 otherwise. So  $E^{11} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ ,  $E^{12} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ , etc. This basis has six elements, so  $\dim(U) = 6$ .

With this basis, clearly  $\mathbf{v} = E^{11} + 2E^{12} + 3E^{13} + 4E^{22} + 5E^{23} + 6E^{33}$ .

**Problem 3.** Find a basis for  $\text{span}\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4, \mathbf{v}_5\}$  where

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ 3 \\ -3 \\ 2 \end{bmatrix}, \mathbf{v}_2 = \begin{bmatrix} 1 \\ 4 \\ 0 \\ 4 \end{bmatrix}, \mathbf{v}_3 = \begin{bmatrix} 2 \\ 4 \\ -12 \\ 0 \end{bmatrix}, \mathbf{v}_4 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 2 \end{bmatrix}, \mathbf{v}_5 = \begin{bmatrix} -1 \\ -2 \\ 8 \\ 4 \end{bmatrix}$$

**Method 1:** Let

$$A = \begin{bmatrix} 1 & 1 & 2 & 0 & -1 \\ 3 & 4 & 4 & 0 & -2 \\ -3 & 0 & -12 & 1 & 8 \\ 2 & 4 & 0 & 2 & 4 \end{bmatrix}$$

$$A \xRightarrow{\substack{R_2 - 3R_1 \rightarrow R_2 \\ R_3 + 3R_1 \rightarrow R_3 \\ R_4 - 2R_1 \rightarrow R_4}} \begin{bmatrix} 1 & 1 & 2 & 0 & -1 \\ 0 & 1 & -2 & 0 & 1 \\ 0 & 3 & -6 & 1 & 5 \\ 0 & 2 & -4 & 2 & 6 \end{bmatrix} \xRightarrow{\substack{R_3 - 3R_2 \rightarrow R_3 \\ R_4 - 2R_2 \rightarrow R_4}} \begin{bmatrix} 1 & 1 & 2 & 0 & -1 \\ 0 & 1 & -2 & 0 & 1 \\ 0 & 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 2 & 4 \end{bmatrix} \xRightarrow{R_4 - 2R_3 \rightarrow R_4} \begin{bmatrix} \boxed{1} & 1 & 2 & 0 & -1 \\ 0 & \boxed{1} & -2 & 0 & 1 \\ 0 & 0 & 0 & \boxed{1} & 2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

So  $\mathcal{B} = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_4\}$  is a basis. (This is all you need.)

In fact, from our CR decomposition, we know

$$A = \begin{bmatrix} 1 & 1 & 0 \\ 3 & 4 & 0 \\ -3 & 0 & 1 \\ 2 & 2 & 2 \end{bmatrix} \begin{bmatrix} \boxed{1} & 1 & 2 & 0 & -1 \\ 0 & \boxed{1} & -2 & 0 & 1 \\ 0 & 0 & 0 & \boxed{1} & 2 \end{bmatrix}$$

So we know  $\mathbf{v}_3 = 2\mathbf{v}_2 - 2\mathbf{v}_2$  and  $\mathbf{v}_5 = -\mathbf{v}_1 + \mathbf{v}_2 + 2\mathbf{v}_3$ .

**Method 2:** Let

$$B = \begin{bmatrix} 1 & 3 & -3 & 2 \\ -1 & -2 & 8 & 4 \\ 0 & 0 & 1 & 2 \\ 1 & 4 & 0 & 4 \\ 2 & 4 & -12 & 0 \end{bmatrix}$$

Then eliminate:

$$B \xRightarrow{\substack{R_2 + R_1 \rightarrow R_2 \\ R_4 - R_1 \rightarrow R_4 \\ R_5 - 2R_1 \rightarrow R_5}} \begin{bmatrix} 1 & 3 & -3 & 2 \\ 0 & 1 & 5 & 6 \\ 0 & 0 & 1 & 2 \\ 0 & 1 & 3 & 2 \\ 0 & -2 & -6 & -4 \end{bmatrix} \xRightarrow{\substack{R_4 - R_2 \rightarrow R_4 \\ R_5 + R_2 \rightarrow R_5}} \begin{bmatrix} 1 & 3 & -3 & 2 \\ 0 & 1 & 5 & 6 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & -2 & -4 \\ 0 & 0 & 4 & 8 \end{bmatrix} \xRightarrow{\substack{R_4 + 2R_3 \rightarrow R_4 \\ R_5 - 4R_3 \rightarrow R_5}} \begin{bmatrix} 1 & 3 & -3 & 2 \\ 0 & 1 & 5 & 6 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} = B'$$

So  $\text{span}\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4, \mathbf{v}_5\} = \text{RS}(B) = \text{RS}(B') = \text{span}\{(1, 3, -3, 2), (0, 1, 5, 6), (0, 0, 1, 2)\}$ . So a basis for  $\text{span}\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4, \mathbf{v}_5\}$  is  $\mathcal{B}' = \{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$  where

$$\mathbf{u}_1 = \begin{bmatrix} 1 \\ 3 \\ -3 \\ 2 \end{bmatrix}, \mathbf{u}_2 = \begin{bmatrix} 0 \\ 1 \\ 5 \\ 6 \end{bmatrix}, \mathbf{u}_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 2 \end{bmatrix}$$