Exam 2

To avoid any confusion, unless specified otherwise, vector spaces are complex vector spaces, inner-products are complex inner-products, and matrices are complex matrices. The standard inner product is $\langle \boldsymbol{u}, \boldsymbol{v} \rangle = \boldsymbol{v}^H \boldsymbol{u} = \sum_{i=1}^{n} \bar{\boldsymbol{v}}_i \boldsymbol{u}_i$. Keep in mind that $A^H = A^T$ for real matrices and symmetric = Hermitian for real matrices.

Part I: True/False

Each problem is points for a total of 50 points. (5 points each.)

You do not need to justify the answers here, this is unlike the quizzes.

1	2	3	4	5	6	7	8	9	10
T	F	F	T	T	T	T	F	F	F

1. TRUE If U is unitary, then U is itself unitarily diagonalizable. This means there is a unitary V so that $U = VDV^H$ where D is diagonal.

   $U^H U = UU^H = I$, so U is normal, hence unitarily diagonalizable.

2. FALSE For any diagonalizable matrix A, one can use Gram-Schmidt to find an orthogonal basis consisting of eigenvectors.

   You must first have that the eigenspaces for different eigenvalues are orthogonal.

3. FALSE The collection of rank k $n \times n$ matrices is a subspace of $\mathbb{R}^{n \times n}$, for $k < n$.

   In fact SVD shows how to write any matrix as a sum of rank 1 matrices.

4. TRUE If A is unitary, then $|\lambda| = 1$ for all eigenvalues λ of A.

   Let λ be an eigenvalue, with unit eigenvector $\boldsymbol{v}$. then $\langle A\boldsymbol{v}, A\boldsymbol{v} \rangle = \langle \lambda\boldsymbol{v}, \lambda\boldsymbol{v} \rangle = \bar{\lambda}\lambda \|\boldsymbol{v}\|_2^2 = |\lambda|^2 = (A\boldsymbol{v})^H (A\boldsymbol{v}) = \boldsymbol{v}^H (A^H A)\boldsymbol{v} = \boldsymbol{v}^H I \boldsymbol{v} = \|\boldsymbol{v}\|_2^2 = 1$. So $|\lambda|^2 = 1$.

5. TRUE If $p(t)$ is a polynomial and $\boldsymbol{v}$ is an eigenvector of A with associated eigenvalue λ, then $p(A)\boldsymbol{v} = p(\lambda)\boldsymbol{v}$.

   $p(x) = \sum_{i=1}^{k} a_i x^i$, so $p(A)\boldsymbol{v} = \sum_{i=1}^{k} a_i A^i \boldsymbol{v} = \sum_{i=1}^{k} a_i \lambda^i \boldsymbol{v} = p(\lambda)\boldsymbol{v}$

6. TRUE If A and B are both $n \times n$ and $\mathcal{B}$ is a basis for $\mathbb{C}^n$ consisting of eigenvectors for both A and B, then A and B commute.

   $AB = (SD_A S^{-1})(SB_B S^{-1}) = AD_A D_B S^{-1} = SD_B D_A S^{-1} = (SD_B S^{-1})(SD_A S^{-1}) = BA$.

7. TRUE Any matrix A can be written as a weighted sum of rank 1 matrices.

   This is essentially one of the statements of the SVD. $U\Sigma V^T = \sum_{i=1}^{r} \sigma_i u_i v_i^T$ where $r = \text{rank}(A)$. Each $u_i v_i^T$ is an $m \times n$ rank-1 matrix.

8. FALSE For all Hermitian matrices A, there is a matrix B so that $B^H B = A$.

   A variant that is true is given in the first problem in part III. The point is that $B^H B$ is not only Hermitian, but also positive.

9. TRUE If A is an $m \times n$ matrix, then $\mathrm{rng}(A) \oplus \mathrm{NS}(A^T) = \mathbb{R}^m$

   You have previously proved that $\mathrm{RS}(A) \oplus \mathrm{NS}(A) = \mathbb{R}^n$. You just apply this result to A^T noting that $\mathrm{RS}(A^T) = \mathrm{CS}(A) = \mathrm{rng}(A)$.

10. FALSE If A is an invertible $n \times n$ matrix, then $ABA^{-1} = B$ for all $n \times n$ matrices B.

    You have shown that the only $n \times n$ matrices that commute with all other $n \times n$ matrices are the diagonal matrices.

Part II: Computational (60 points)

P1. (15 points) Find B so that $B^2 = A$ where

$$A = \begin{bmatrix} 13 & -5 & 5 \\ -8 & 10 & -8 \\ -3 & -3 & 5 \end{bmatrix}$$

This is like 6.3 #4.

First diagonalize A.

Find the eigenvalues:

$$\begin{aligned}
\det\left(\begin{bmatrix} 13-\lambda & -5 & 5 \\ -8 & 10-\lambda & -8 \\ -3 & -3 & 5-\lambda \end{bmatrix}\right) &= (13-\lambda)((10-\lambda)(5-\lambda)-24) - (-5)((-8)(5-\lambda)-24) + (5)((24+(3)(10-\lambda)) \\
&= (13-\lambda)(26-15\lambda+\lambda^2) + (5)(-64+8\lambda) + (5)(54-3\lambda) \\
&= (13-\lambda)(\lambda-13)(\lambda-2) + 5(-10+5\lambda) \\
&= (13-\lambda)(\lambda-13)(\lambda-2) + 25(-2+\lambda) \\
&= (\lambda-2)[(13-\lambda)(\lambda-13)+25] \\
&= (\lambda-2)(5-(13-\lambda))(5+(13-\lambda)) \\
&= -(\lambda-2)(\lambda-8)(\lambda-18)
\end{aligned}$$

So the eigenvalues are $\lambda_1 = 18 > \lambda_2 = 8 > \lambda_3 = 2$.

This means $A = S\begin{bmatrix} 18 & & \\ & 8 & \\ & & 2 \end{bmatrix}S^{-1}$ and so $B = S\begin{bmatrix} \sqrt{18} & & \\ & \sqrt{8} & \\ & & \sqrt{2} \end{bmatrix}S^{-1}$ will be our matrix, where $S = \begin{bmatrix} \boldsymbol{v}_1 & \boldsymbol{v}_2 & \boldsymbol{v}_3 \end{bmatrix}$ where $\boldsymbol{v}_i$ is an eigenvector for λ_i.

Find eigenspaces:

$E_{18} = \text{NS}\left(\begin{bmatrix} -5 & -5 & 5 \\ -8 & -8 & -8 \\ -3 & -3 & -13 \end{bmatrix}\right) = \text{NS}\left(\begin{bmatrix} -1 & -1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}\right) = \text{span}\left(\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}\right)$

$E_8 = \text{NS}\left(\begin{bmatrix} 5 & -5 & 5 \\ -8 & 2 & -8 \\ -3 & -3 & -3 \end{bmatrix}\right) = \text{NS}\left(\begin{bmatrix} 1 & -1 & 1 \\ -4 & 1 & -4 \\ -1 & -1 & -1 \end{bmatrix}\right) = \text{NS}\left(\begin{bmatrix} 1 & -1 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}\right) = \text{span}\left(\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}\right)$

$E_2 = \text{NS}\left(\begin{bmatrix} 11 & -5 & 5 \\ -8 & 8 & -8 \\ -3 & -3 & 3 \end{bmatrix}\right) = \text{NS}\left(\begin{bmatrix} 11 & -5 & 5 \\ -1 & 1 & -1 \\ -1 & -1 & 1 \end{bmatrix}\right) \text{NS}\left(\begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}\right) = \text{span}\left(\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}\right)$

So here we could use $S = \begin{bmatrix} 1 & 1 & 0 \\ -1 & 0 & 1 \\ 0 & -1 & 1 \end{bmatrix}$

$$B = SDS^{-1} = \begin{bmatrix} 1 & 1 & 0 \\ -1 & 0 & 1 \\ 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} 3\sqrt{2} & & \\ & 2\sqrt{2} & \\ & & \sqrt{2} \end{bmatrix} \begin{bmatrix} 1/2 & -1/2 & 1/2 \\ 1/2 & 1/2 & -1/2 \\ 1/2 & 1/2 & 1/2 \end{bmatrix} = \frac{\sqrt{2}}{2}\begin{bmatrix} 5 & -1 & 1 \\ -2 & 4 & -2 \\ -1 & 1 & 3 \end{bmatrix}$$

P2. (15 points) Find B so that $B^H B = A$ where

$$A = \begin{bmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{bmatrix}$$

This is like 6.4 #14.

First diagonalize A.

Find the eigenvalues:

$\det\left(\begin{bmatrix} 1-\lambda & -1 & 0 \\ -1 & 2-\lambda & -1 \\ 0 & -1 & 1-\lambda \end{bmatrix}\right) = (1-\lambda)((2-\lambda)(1-\lambda)-1)-(-1)((-1)(1-\lambda)-0) = (1-\lambda)(1-3\lambda+\lambda^2)-(1-\lambda) = (1-\lambda)((1-3\lambda-\lambda^2-1) = (1-\lambda)(-3\lambda+\lambda^2) = (1-\lambda)(\lambda)(-3+\lambda)$. So the eigenvalues are $\lambda_1 = 3 > \lambda_2 = 1 > \lambda_3 = 0$.

This means $A = S\begin{bmatrix} 3 & & \\ & 1 & \\ & & 0 \end{bmatrix}S^{-1}$ and so $B = S\begin{bmatrix} \sqrt{3} & & \\ & 1 & \\ & & 0 \end{bmatrix}S^{-1}$ will be our matrix, where $S = \begin{bmatrix} \boldsymbol{v}_1 & \boldsymbol{v}_2 & \boldsymbol{v}_3 \end{bmatrix}$ where $\boldsymbol{v}_i$ is an eigenvector for λ_i.

Find eigenspaces:

$E_3 = \mathrm{NS}\left(\begin{bmatrix} -2 & -1 & 0 \\ -1 & -1 & -1 \\ 0 & -1 & -2 \end{bmatrix}\right) = \mathrm{NS}\left(\begin{bmatrix} -1 & -1 & -1 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix}\right) = \mathrm{span}\left(\begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}\right)$

$E_1 = \mathrm{NS}\left(\begin{bmatrix} 0 & -1 & 0 \\ -1 & 1 & -1 \\ 0 & -1 & 0 \end{bmatrix}\right) = \mathrm{NS}\left(\begin{bmatrix} -1 & 2 & -1 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix}\right) = \mathrm{span}\left(\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}\right)$

$E_0 = \mathrm{NS}(A) = \mathrm{NS}\left(\begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}\right) = \mathrm{span}\left(\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}\right)$

$$S = \begin{bmatrix} 1/\sqrt{6} & 1/\sqrt{2} & 1/\sqrt{3} \\ -2/\sqrt{6} & 0 & 1/\sqrt{3} \\ 1/\sqrt{6} & -1/\sqrt{2} & 1/\sqrt{3} \end{bmatrix}$$

this is unitary so $S^{-1} = S^T$ and finally

Let $B = D^{1/2}S^H$ where $A = B^H B = SDS^H$ just as above. So

$$B = \begin{bmatrix} \sqrt{3} & & \\ & 1 & \\ & & 0 \end{bmatrix}\begin{bmatrix} 1/\sqrt{6} & -2/\sqrt{6} & 1/\sqrt{6} \\ 1/\sqrt{2} & 0 & 1/\sqrt{2} \\ 1/\sqrt{3} & 1/\sqrt{3} & 1/\sqrt{3} \end{bmatrix} = \frac{\sqrt{2}}{2}\begin{bmatrix} 1 & -2 & 1 \\ 1 & 0 & -1 \\ 0 & 0 & 0 \end{bmatrix}$$

P3. (15 points) Find the best rank 2 approximation to A from (2) with respect to $\|\cdot\|_F$.

This is like 6.5 #4.

You know $\mathrm{rank}(A) = 2$ so the best rank 2 approximation of A is A, but if you just plug into the computation, you get the following:

You already have the SVD of $A = U\Sigma V^T = SDS^T$, so $U = V$ in this case and $D = \Sigma$. Now the best rank-2 approximation of A is thus (using MATLAB type notation)

$$\begin{aligned} C &= S(:,1:2)D(1:2,1:2)S^T(1:2,:) \\ &= \begin{bmatrix} 1/\sqrt{6} & 1/\sqrt{2} \\ -2/\sqrt{6} & 0 \\ 1/\sqrt{6} & -1/\sqrt{2} \end{bmatrix}\begin{bmatrix} 3 & \\ & 1 \end{bmatrix}\begin{bmatrix} 1/\sqrt{6} & -2/\sqrt{6} & 1/\sqrt{6} \\ 1/\sqrt{2} & 0 & 1/\sqrt{2} \end{bmatrix} \\ &= \begin{bmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{bmatrix} = A \end{aligned}$$

Note: Actually, you didn't need to do anything, my bad! $\mathrm{rank}(A) = 2$, so it was clear before doing anything that A is its own best rank 2 approximation.

P4. (15 points) Let

$$A = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix},$$

find the characteristic polynomial and all eigenvalues, both real and complex. Explain why A is diagonalizable and compute A^{2020}. Note, I do not ask you to diagonalize A.

Find eigenvalues:

$\det\left(\begin{bmatrix} -\lambda & 0 & 1 \\ 1 & -\lambda & 0 \\ 0 & 1 & -\lambda \end{bmatrix}\right) = -\lambda(-\lambda)^2 - (-1) = -\lambda^3 + 1$, so the roots are 1, $e^{i\frac{2\pi}{3}} = -\frac{1}{2} + i\frac{\sqrt{3}}{2}$, and $e^{i\frac{4\pi}{3}} = -\frac{1}{2} - i\frac{\sqrt{3}}{2}$.

Compute A^{2020}:

We see $2020 = 673{\cdot}3{+}1$, so $\lambda_i^{2020} = (\lambda_i^3)^{673} \cdot \lambda_i = \lambda_i$. So $S^{2020} = SD^{2020}S^{-1} = S\begin{bmatrix} \lambda_1^{2020} & & \\ & \lambda_2^{2020} & \\ & & \lambda_3^{2020} \end{bmatrix}S^{-1} = S\begin{bmatrix} \lambda_1 & & \\ & \lambda_2 & \\ & & \lambda_3 \end{bmatrix}S^{-1} = A$.

Note we actually don't need to know the eigenvalues, just that $\lambda^3 = 1$.

Alternatively, you might just compute that $A^3 = I$, so $A^{2020} = I^{637}A = A$.

Part III: Theory and Proofs (45 points; 15 points each)

Pick three of the following four options. If you try all four, I will grade the first three, so if this is not what you intend, then just do three, or at least make it clear which I should grade.

P1. Let $L : V \to V$ be a linear transformation and let $\mathcal{B} = \{\boldsymbol{v}_1, \ldots, \boldsymbol{v}_n\}$ be an ordered basis. Show that $[L]_{\mathcal{B}}$ is upper triangular iff $L(\boldsymbol{v}_i) \in \operatorname{span}\{\boldsymbol{v}_1, \ldots, \boldsymbol{v}_i\}$ for all i.

This is an "if and only if" so there are two things to do.

($\Longrightarrow$) Assume $[L]_{\mathcal{B}}$ is upper-triangular. To make notation simpler suppose $[L]_{\mathcal{B}} = A$ and $_{ij}$ is the ij^{th} entry in A. Then $[L]_{\mathcal{B}} = \left[\, [L(v_1)]_{\mathcal{B}} \cdots [L(v_n)]_{\mathcal{B}} \,\right]$ and since A is upper-triangular

$$[L(v_i)]_{\mathcal{B}} = \begin{bmatrix} a_{1i} \\ \vdots \\ a_{ii} \\ 0 \\ \vdots \\ 0 \end{bmatrix} \tag{$\dagger$}$$

and so $L(\boldsymbol{v}_i) = \sum_{j=1}^{i} a_{ji} \boldsymbol{v}_j \in \operatorname{span}\{\boldsymbol{v}_1, \ldots, \boldsymbol{v}_i\}$.

($\Longleftarrow$) Suppose $L(\boldsymbol{v}_i) \in \operatorname{span}\{\boldsymbol{v}_1, \ldots, \boldsymbol{v}_i\}$ for all i, then $L(\boldsymbol{v}_i) = \sum_{j=1}^{i} a_{ji} \boldsymbol{v}_j$ and thus ($\dagger$) holds here too, so $[L]_{\mathcal{B}}$ is upper-triangular.

P2. Let P be an $n \times n$ symmetric matrix that satisfies $P^2 = P$.

(a) Let $U = \operatorname{rng}(P) = \operatorname{CS}(P)$. Show that $\boldsymbol{v} - P\boldsymbol{v} \perp U$.

Take an $\boldsymbol{u} \in U$, then $\boldsymbol{u} = P\boldsymbol{w}$, we need to show

$$\langle \boldsymbol{v} - P\boldsymbol{v}, P\boldsymbol{w} \rangle = 0$$

This is just a computation:

$$\begin{aligned} \langle \boldsymbol{v} - P\boldsymbol{v}, P\boldsymbol{w} \rangle &= (P\boldsymbol{w})^T(\boldsymbol{v} - P\boldsymbol{v}) \\ &= \boldsymbol{w}^T P^T (\boldsymbol{v} - P\boldsymbol{v}) \\ &= \boldsymbol{w}^T P (\boldsymbol{v} - P\boldsymbol{v}) && (P^T = P) \\ &= \boldsymbol{w}^T (P\boldsymbol{v} - P^2\boldsymbol{v}) \\ &= \boldsymbol{w}^T \boldsymbol{0} = 0 && (P = P^2) \end{aligned}$$

(b) Use (a) to see that $\|\boldsymbol{v} - P\boldsymbol{v}\|_2^2 = \min\{\|\boldsymbol{v} - \boldsymbol{u}\|_2^2 \mid \boldsymbol{u} \in U\}$. That is $P\boldsymbol{v}$ is the vector in U closest to $\boldsymbol{v}$.

Again, take $\boldsymbol{u} \in U$, we want to see that $\|\boldsymbol{v} - P\boldsymbol{v}\|_2^2 \le \|\boldsymbol{v} - \boldsymbol{u}\|_2^2$. Here we have

$$\begin{aligned} \|\boldsymbol{v} - \boldsymbol{u}\|_2^2 &= \|(\boldsymbol{v} - P\boldsymbol{v}) + (P\boldsymbol{v} - \boldsymbol{u})\|_2^2 \\ &= \|\boldsymbol{v} - P\boldsymbol{v}\|_2^2 + \|P\boldsymbol{v} - \boldsymbol{u}\|_2^2 && (\dagger) \text{ since } \boldsymbol{v} - P\boldsymbol{v} \perp P\boldsymbol{v} - \boldsymbol{u} \\ &\ge \|\boldsymbol{v} - P\boldsymbol{v}\|_2^2 \end{aligned}$$

($\dagger$) is true since (1) $P\boldsymbol{v} - \boldsymbol{u} \in U$ and $\boldsymbol{v} - P\boldsymbol{v} \perp U$ and (2) (Pythagorean Theorem) when $\boldsymbol{w} \perp \boldsymbol{w}'$, then $\|\boldsymbol{w} + \boldsymbol{w}'\|_2^2 = \|\boldsymbol{w}\|_2^2 + \|\boldsymbol{w}'\|_2^2$.

P3. Let $\lambda_1, \lambda_2, \ldots, \lambda_n$ be the complete set of n eigenvalues of an $n \times n$ matrix A. Show that

$$\det(A) = \prod_{i=1}^{n} \lambda_i = \lambda_1 \cdot \lambda_2 \cdot \cdots \cdot \lambda_n.$$

Note: You do not know that A is diagonalizable, there might fail to be a basis of eigenvectors.

The characteristic polynomial $p(t) = \det(A - tI) = (\lambda_1 - t)(\lambda_2 - t)\cdots(\lambda_n - t)$. By setting $t = 0$ and evaluating we get $\det(A) = \lambda_1 \cdot \lambda_2 \cdots \lambda_n$.

P4. Use the SVD to show that any square matrix A can be written as $A = UP$ where U is unitary and P is Hermitian.

Let $A = V\Sigma W^H$ as in SVD and let $U = VW^H$, this is unitary since both V and W are unitary. So

$$A = (VW^H(W\Sigma W^H)) = UP$$

where $P = W\Sigma W^H$. This P is clearly Hermitian.

Note: If we think of matrices like generalizations of complex numbers, then Hermitian matrices are like the reals, $P^H = P$ is like $\bar{z} = z$ for $z \in \mathbb{C}$. A unitary is "like" a rotation, so here we represent A as a rotation followed by a "real." this is like writing $z = e^{i\theta}r$, the polar form of a complex number.