Name: ______________________________ Exam 1 - MAT513

Warning! I can (eh ... do) make mistakes, if you think I have something wrong here, please ask.

Part I: True/False

Each problem is points for a total of 40 points. (10 problems 4 points each; 2 points for correct T/F; 2 points for correct, but brief, explanation.)

Problem 1. Decide if each of the following is true or false. For each, provide an example, counter-example, or argument as required. You may refer to a theorem if one applies.

a) False If $H < G$, the set of cosets $G/H = \{gH \mid g \in G\}$ form a group under the operation $(aH) \cdot (bH) = (ab)H$.

This is a basic fact that we have learned. For G/H to be a group as described, H must be normal. As an example, consider $H = \{e, f\}$ where f is a reflection in D_n and r a non-trivial rotation, then $rHrH = \{r, rf\}\{r, rf\} = \{r^2, r^2f, rfr, rfrf\} = \{e, f, r^2, r^2f\} \neq r^2H = \{r^2, r^2f\}$.

b) True $Z(G) = \bigcap_{g\in G} C(g)$.

This is just unpacking the definitions

$$\begin{aligned} x \in Z(G) &\iff \text{for all } g \in G, xg = gx \\ &\iff \text{for all } g \in G, x \in G(g) \\ &\iff x \in \bigcap_{g\in G} C(g) \end{aligned}$$

c) True For $c \in \mathbb{R}$ and $c \neq 0$, $\phi : \mathbb{R} \to \mathbb{R}$ given by $\phi(x) = cx$ is an automorphism of $(\mathbb{R}, +)$.

This is homomorphism since $\phi(a+b) = c(a+b) = ca+cb = \phi(a)+\phi(b)$ and $\phi(0) = 0$. $\ker(\phi) = \{0\}$, that is, $ca = 0 \iff a = 0$. So ϕ is 1-1. Since $\phi(a/c) = a$ we see ϕ is onto.

d) True Let $\phi : G \to H$ be a homomorphism,

$$\ker(\phi) = \{e_G\} \iff \phi \text{ is 1-1}$$

If $\phi(x) = \phi(y)$ and $\ker(\phi) = \{e_G\}$, then $e_H = \phi(x)\phi(y)^{-1} = \phi(x)\phi(y^{-1}) = \phi(xy^{-1})$ so $xy^{-1} = e_G$ and thus $x = y$, so ϕ is 1-1.

Conversely, suppose ϕ is 1-1 and $\phi(x) = e_H$, then since $\phi(e_G) = e_H$ we have $x = e_G$, so $\ker(\phi) = \{e\}$.

e) False S_9 has an element of order 11.

$|S_9| = 9!$ and $11 \nmid 9!$ so no element of order 11.

f) False There are finite groups that are not isomorphic to a subgroup of S_n for some n.

Cayley's theorem.

g) False There is a finite group of order n and a prime p such that $p \mid n$, but no element of G has order p.

Cauchy's Theorem.

h) False There is a group G and non-normal subgroup $H < G$ so that $|H| \nmid |G|$.

Langrange's Theorem.

i) True In S_8, $(135)(456)(567)$ is even.

$$(135)(456)(567) = (15)(13)(46)(45)(57)(56)$$

So 6 transpositions, hence even. You might also recall that 3-cycles are even, so 3 even is even.

j) True $[S_4 : D_4] = 3$.

$|S_4|/|D_4| = 4!/8 = 3$.

Part II: Short Answer

Each problem is 8 points for a total of 40 points. (5 problems, 8 points each)

Problem 2 (8 points)**.** Let ϕ be a homomorphism from G to H. What is the relationship between G, $\ker(\phi)$, $\mathrm{Img}(\phi) = \phi(G)$, and H. If G and H are finite, what is the relationship between $|G|$, $|\ker(\phi)|$, $\mathrm{Img}(\phi)$, and $|H|$?

You might mention $\ker(\phi) \leq G$ and $\mathrm{Img}(\phi) \leq H$, but primarily I need to see

$$G/\ker(\phi) \simeq \mathrm{Img}(\phi) < H$$

and hence

$$|G| = |\ker(\phi)||\,\mathrm{Img}(\phi)| \text{ and } |\,\mathrm{Img}(\phi)| \mid |H|.$$

Problem 3 (8 points)**.** List the abelian groups of order 12 up to isomorphism.

By the fundamental theorem of abelian groups we just need to consider all ways of factoring 12 (order irrelevant). We have

$$12 = (4)(3) = (2)(2)(3) = (2)(6)$$

Since 3 and 4 are relatively prime, $\mathbb{Z}_{12} \simeq \mathbb{Z}_4 \times \mathbb{Z}_3$; similarly $\mathbb{Z}_2 \times \mathbb{Z}_3 \simeq \mathbb{Z}_6$. So there are just two possible isomorphism types of abelian groups of order 12:

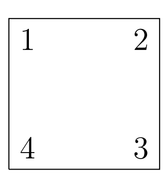
$$\mathbb{Z}_4 \times \mathbb{Z}_3 \simeq \mathbb{Z}_{12} \text{ and } \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_3 \simeq \mathbb{Z}_2 \times \mathbb{Z}_6.$$

Problem 4 (8 points)**.** What is $\mathrm{Aut}(\mathbb{Z}_{45})$ up to isomorphism, in terms of products of $\mathbb{Z}_n$'s. (Explain or show "computation.")

$$\mathrm{Aut}(\mathbb{Z}_{45}) = U(45) = U(3^3 \cdot 5) = U(3^2) \times U(5) = \mathbb{Z}_{3(2)} \times \mathbb{Z}_4 = \mathbb{Z}_6 \times \mathbb{Z}_4$$

Problem 5 (8 points)**.** Show that D_4 is not a normal subgroup of S_4.

Use

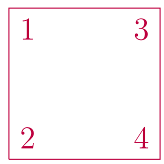


So that D_4 is generated by the rotation $R = (1234)$ and the horizontal reflection $H = (12)(34)$.

We just need to see that $\sigma D_4 \sigma^{-1} \neq D_4$ for some σ. Take $\sigma = (12)$ (or (123) or basically any $\sigma \in S_4 - D_4$, then

$$(12)(1234)(12) = (1342)$$

this would correspond to labeling like



and this cannot be achieved in D_8 since adjacent labels must stay adjacent.

Note This is a good example to keep in mind. We know that if $[G : H] = 2$, then H is normal in G. We might guess that if $[G : H]$ is prime, the same holds. This example shows that this is false. It is true that if p is the smallest prime such that $p \mid G$ and $[G : H] = p$, then H is normal.

Problem 6 (8 points)**.** What is the largest cyclic subgroup of $G = \mathbb{Z}_6 \times \mathbb{Z}_{20} \times \mathbb{Z}_{24} \times \mathbb{Z}_{45}$?

$6 = 2 \cdot 3$, $20 = 2^2 \cdot 5$, $24 = 2^3 \cdot 3$, and $45 = 3^2 \cdot 5$. So $\mathrm{lcm}(6, 20, 24, 45) = 2^3 3^2 5 = 360$. In fact $(1, 1, 1, 1)$ has this order and for any $g \in G$, $|g| = \mathrm{lcm}(|g_1|, |g_2|, |g_3|, |g_4|) \mid \mid \mathrm{lcm}(6, 20, 24, 45)$ since $|g_1| \mid 6$, $|g_2| \mid 20$, $|g_3| \mid 24$, and $|g_4| \mid 45$.

Part III: Theory and Proofs (100 points; 20 points each)

This part is to be completed in the comfort of your home, cafe, or wherever you prefer to ponder math. You should complete this work on your own without consulting websites, friends, classmates, etc.

Problem 7 (20 points)**.** Show that $\phi(x) = x^2$ is an automorphism of G iff G abelian group and for all $g \in G$, $|g|$ is finite and odd.

($\implies$) If $\phi(x) = x^2$ is an automorphism, then $\phi(ab) = (ab)^2 = \phi(a)\phi(b) = a^2b^2$ so $abab = aabb$. Multiply both sides on the right by b^{-1} and left by a^{-1} to get $ba = ab$. So G is commutative.

If $\langle g \rangle$ is infinite, then $\phi : \langle g \rangle \to \langle g^2 \rangle$ and so ϕ is not onto. Thus for all $g \in G$, g has finite order.

If $|g| = 2m$ were even, then $\phi(g^m) = e$ where $g^m \neq e$ so ϕ is not 1-1. Thus $|g|$ is odd for all $g \in G$.

($\impliedby$) If G is abelian, then $\phi(x) = x^2$ is clearly a homomorphism, since $\phi(ab) = (ab)^2 = a^2b^2 = \phi(a)\phi(b)$ and $\phi(e) = e^2 = e$.

If all elements of G (except e) have odd order, then ϕ is 1-1 since $a^2 = b^2 \implies a^2b^{-2} = (ab^{-1})^2 = e$ so as ab^{-1} has odd order so $ab^{-1} = e$ and hence $a = b$.

Since $|g|$ is finite and $\phi : \langle g \rangle \to \langle g \rangle$ is 1-1, it must also be onto. Thus ϕ must also be onto.

Problem 8 (20 points)**.** Prove that $(\mathbb{Q}, +)$ (rationals under addition) is not isomorphic to any **proper** subgroup of itself.

Suppose $\pi : (\mathbb{Q}, +) \to (\mathbb{Q}, +)$. Let $x = p/q$, then $qx = p$, so

$$\underbrace{x + x + \cdots + x}_{q\text{-times}} = \underbrace{1 + 1 + \cdots + 1}_{p\text{-times}}$$

so

$$\pi(qx) = q \cdot \pi(x) = p \cdot \pi(1) = \pi(p)$$

Note π is only guaranteed to preserve the additive structure; we are not using anything like $\pi(px) = \pi(p)\pi(x)$. We have

$$\pi(x) = \frac{p}{q}\pi(1)$$

So either $\pi(1) = 0$ and π maps all of $(\mathbb{Q}, +)$ to 0, or else π is 1-1 and onto, i.e., an automorphism.

So we have shown that every endomorphism except for the 0 map is an automorphism of the form $x \mapsto \alpha \cdot x$ where $\alpha = \pi(1)$.

Problem 9 (20 points)**.** Suppose $|G| = 2n+1$ and $(ab)^2 = (ba)^2$ for all $a, b \in G$. Show that G is abelian.

Since $x^{2n+1} = e$ for all $x \in G$ we know $(ab)^{2n+1} = (ba)^{2n+1} = e$ and so

$$(ab)^{2n+1} = ((ab)^2)^n(ab) = \boxed{((ba)^2)^n(ab) = ((ba)^2)^n(ba)} = (ba)^{2n+1}$$

So $ab = ba$.

Problem 10 (20 points)**.** Show that if G is abelian and $|G| = mn$ where m and n are relatively prime, then $G = G^m G^n$ (internal direct product) where $G^k = \{x \in G \mid x^k = e\}$.

Clearly, $G^n \cap G^m = \{e\}$. Let $x \in G$, then $|x| = rs$ where $r \mid m$ and $s \mid n$. We know $|x^r| = s$ and $|x^s| = r$ so that $x^s \in G^m$ and $x^r \in G^n$. Also, $x = x^{as+br} = (x^s)^a(x^r)^b$ for some $a, b \in \mathbb{Z}$. This shows that $G^m G^n = G$.

Problem 11 (20 points)**.** Suppose $H, K \lhd G$ and $H \cap K = \{e\}$. Show that G is isomorphic to a subgroup of $G/H \times G/K$.

Define $\phi : G \to G/H \times G/K$ by $\phi(g) = (gH, gK)$. Then $g \in \ker(\phi)$ iff $gH = eH$ and $gK = eK$, that is $g \in H$ and $g \in K$. So $g = e$ and thus $\ker(\phi) = \{e\}$ is trivial. So

$$G \simeq G/\{e\} = G/\ker(\phi) \simeq \text{Img}(\phi) < G/H \times G/K$$

Actually, you can drop the $H \cap K = \{e\}$ assumption and the same argument gives

$$G/(H \cap K) < G/H \times G/K$$