Math 571 - Homework 2

Richard Ketchersid

Definition 1. A set $S \subseteq X$ is **discrete** iff every point in S is isolated.

Problem 2.1 (R:2:5*)**.** Prove the following for discrete $S \subset \mathbb{R}$:

a) $\mathrm{Lim}(S) \cap S = \emptyset$ and S is countable.

b) There is discrete set $A \subset \mathbb{R}$ so that $\mathrm{Lim}(A) = \mathrm{Cl}(S)$.

c) Give an example of a discrete set S where there is no set A such that $\mathrm{Lim}(A) = S$.

For the following use the definition that I provided for $\mathrm{Cl}(E)$, namely, $\mathrm{Cl}(E) = \bigcap\{F \mid F \text{ is closed and } E \subseteq F\}$.

Problem 2.2 (R:2:6)**.** For X a metric space and $E \subseteq X$, show that

a) $\mathrm{Lim}(\mathrm{Lim}(E)) \subseteq \mathrm{Lim}(E)$ and equality need not obtain.

b) $\mathrm{Lim}(A \cup B) = \mathrm{Lim}(A) \cup \mathrm{Lim}(B)$.

c) $E \cup \mathrm{Lim}(E)$ is closed and $E \cup \mathrm{Lim}(E) = \mathrm{Cl}(E)$.

d) $\mathrm{Lim}(E)$ is closed and $\mathrm{Lim}(E) = \mathrm{Lim}(\mathrm{Cl}(E))$.

Problem 2.3 (R:2:9*)**.** Let X be a metric space, or just any topological space. Are the following true for all $E \subseteq X$? For each either prove the statement true or give a counterexample. For a counterexample you must provide both X and E.

a) $\mathrm{Int}(E)^c = \mathrm{Cl}(E^c)$.

b) $\mathrm{Cl}(E) = \mathrm{Int}(E^c)^c$?

c) $\mathrm{Cl}(E) = \mathrm{Cl}(\mathrm{Int}(E))$?

d) $\mathrm{Int}(E) = \mathrm{Int}(\mathrm{Cl}(E))$

Problem 2.4. Let $\tau = \{(a, \infty) \mid a \in \mathbb{R} \cup \{\infty, -\infty\}\}$. This might be called *the (right) ray topology.*

(a) Show that $(\mathbb{R}, \tau)$ is a topological space.

(b) Compute $\mathrm{Int}((0, 1))$.

(c) Compute $\mathrm{Ext}((0, 1))$.

(d) Compute $\partial(0, 1)$.

(e) Compute $\mathrm{Cl}((0, 1))$.

(f) Compute $\mathrm{Lim}((0, 1))$. (The derived set of $(0, 1)$.)

Definition 2. A metric space X is **separable** iff there is a countable $E \subseteq X$ with E dense in X.

Problem 2.5 (R:2:22)**.** Show the $\mathbb{R}^k$ is separable.

Definition 3. A set $\mathcal{B}$ of open sets is called a **base** for X iff for all $x \in X$ and open set U with $x \in U$, there is $V \in \mathcal{B}$ so that $x \in V \subset U$.

Problem 2.6 (R:2:23*)**.** Prove that if a topological space has a countable base, i.e., is second countable, then it is separable. Prove that a metric space is separable iff it has a countable base.

Problem 2.7 (R:2:24)**.** Prove that if X is a metric space and every infinite sequence has a limit point, then X is separable. (See the hint in the text.)