Quiz 1

Problem 1 (15 points; 3 points each)**.** Decide if each of the following are true or false and provide a justification or counterexample in each case. A justification could consist of a theorem from the text. All vector spaces are assumed to be finite-dimensional here.

(a) _____ Given a matrix with integer entries, it is possible to use elementary operations I – III with only integer constants in II and III. So that the final echelon matrix and all matrices along the way have only integer entries.

This is true.

(b) _____ There is only one echelon form of a matrix.

This is false. There is a unique reduced echelon form. Both

$$E_1 = \begin{bmatrix} 2 & 3 & 1 \\ 0 & 3 & 2 \end{bmatrix}$$

and

$$E_2 = \begin{bmatrix} 2 & 0 & -1 \\ 0 & 3 & 2 \end{bmatrix}$$

are both echelon and each can be got from the other by an elementary row operation.

(c) _____ The inverse of an elementary row operation is another elementary row operation of exactly the same type.

This is true. Just go through the cases.

Type I The inverse of "swap row i and row j" is just "swap row i and row j." In this case the operation is its own inverse.

Type II The inverse of "multiply row i by c" is "multiply row i by c^{-1}."

Type III The inverse of add "$c\text{row}_i$ to row_j and replace row_j" is "add $-c\text{row}_i$ to row_j and replace row_j"

(d) _____ If $AC = BC$, then $A = B$.

This is false. We have

$$AC = BC \iff AC - BC = 0 \iff (A - B)C = 0$$

so basically the claim is equivalent to $AB = 0 \iff A = 0$ or $B = 0$ and this is false. Easy example:

$$\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ 1 & -1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

(e) _____ If A and B are invertible $n \times n$ matrices, then

$$((AB)^T)^{-1} = ((AB)^{-1})^T = (A^T)^{-1}(B^T)^{-1} = (A^{-1})^T(B^{-1})^T$$

This is true. First off

$$((AB)^T)^{-1} = (B^T A^T)^{-1} = (A^T)^{-1}(B^T)^{-1}$$

Both $(\cdot)^{-1}$ and $(\cdot)^T$ swap the order of multiplication.

For the next bit, we need $(A^T)^{-1} = (A^{-1})^T$, this we argue by

$$I = (A^{-1}A)^T = A^T(A^{-1})^T$$

but we know that $AB = I \implies B = A^{-1}$ for A and B $n \times n$ matrices. This we will discuss more later. In particular

$$A^T(A^{-1})^T = I \implies (A^{-1})^T = (A^T)^{-1}$$

Problem 2 (10 points)**.** Write down the augmented matrix associated to this system and then use the elementary row operations I and III to find an equivalent echelon form of this matrix. Show every step of the reduction and indicate what operation you use. Use only integer constants in II – III.

Eliminate all the non-pivot leading coefficients in column 1 first, then work on column 2, etc. You can combine all operations for one column in a single step, so this should require 2 or 3 steps depending on how you write things.

$$\begin{aligned} 4x_1 + x_2 - x_3 + 3x_4 &= 5 \\ -4x_1 - 4x_2 + x_3 &= 4 \\ -2x_1 + 4x_2 + 4x_3 - 4x_4 &= -12 \end{aligned}$$

The augmented matrix is

$$\left[\begin{array}{cccc|c} 4 & 1 & -1 & 3 & 5 \\ -4 & -4 & 1 & 0 & 4 \\ -2 & 4 & 4 & -4 & -12 \end{array}\right]$$

Reduction to echelon form requires 3 row manipulations:

$$\left[\begin{array}{cccc|c} 4 & 1 & -1 & 3 & 5 \\ -4 & -4 & 1 & 0 & 4 \\ -2 & 4 & 4 & -4 & -12 \end{array}\right] \underset{R_3 \leftarrow 2R_3}{\longrightarrow} \left[\begin{array}{cccc|c} 4 & 1 & -1 & 3 & 5 \\ -4 & -4 & 1 & 0 & 4 \\ -4 & 8 & 8 & -8 & -24 \end{array}\right]$$

$$\underset{\substack{R_2 \leftarrow R_1 + R_2 \\ R_3 \leftarrow R_1 + R_3}}{\longrightarrow} \left[\begin{array}{cccc|c} 4 & 1 & -1 & 3 & 5 \\ 0 & -3 & 0 & 3 & 9 \\ 0 & 9 & 7 & -5 & -19 \end{array}\right]$$

$$\underset{R_3 \leftarrow R_3 + 3R_2}{\longrightarrow} \left[\begin{array}{cccc|c} 4 & 1 & -1 & 3 & 5 \\ 0 & -3 & 0 & 3 & 9 \\ 0 & 0 & 7 & 4 & 8 \end{array}\right]$$

Problem 3 (10 points)**.** Using your echelon form above re-write the initial system as an equivalent triangular system. Let the variable that correspond to non-pivot element be the independent variable and solve for the remaining three variables in terms of this one. This is the "**back substitution**" step. Finally, write the solution set as $\{t\boldsymbol{v} + \boldsymbol{u} \,|\, t \in \mathbb{R}\}$ for some $\boldsymbol{v}, \boldsymbol{u} \in \mathbb{R}^4$. This way it is clear that the solution set is a line in $\mathbb{R}^4$.

The pivots are 4, -3, and 7 with corresponding pivot (dependent) variables x_1, x_2, and x_3, the only independent variable being x_4.

So let $t \in \mathbb{R}$ be arbitrary and set $x_4 = t$ to give the diagonal system

$$\begin{aligned} 4x_1 + x_2 - x_3 &= 5 - 3t \\ -3x_2 &= 9 - 3t \\ 7x_3 &= 8 - 4t \end{aligned}$$

Back substitution gives

$$\begin{aligned} x_2 &= t - 3 \\ x_3 &= \frac{1}{7}(8 - 4t) \end{aligned}$$

and

$$\begin{aligned} 4x_1 + (t-3) - \frac{1}{7}(8 - 4t) &= 5 - 3t \\ 28x_1 = 35 - 21t - 7t + 21 - 4t + 8 &= 64 - 32t \\ x_1 &= \frac{16}{7} - \frac{8}{7}t \end{aligned}$$

This gives

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 16/7 \\ -3 \\ 8/7 \\ 0 \end{bmatrix} + t \begin{bmatrix} -8/7 \\ 1 \\ -4/7 \\ 1 \end{bmatrix}$$

This shows that the solution set is a straight line in $\mathbb{R}^4$.