Echelon Form:

$$\begin{bmatrix} * & \cdots & & & & & & \\ 0 & 0 & * & \cdots & & & & \\ 0 & 0 & 0 & 0 & * & \cdots & & \\ \vdots & & & & & & & \\ 0 & 0 & \cdots & & & & & 0 \end{bmatrix}$$

pivots

"stairsteps"

pivots columns = dependent variables

non-pivot columns = independent variables

$$\begin{array}{ccccc} x_1 & x_2 & x_3 & x_4 & x_5 \end{array}$$
$$\begin{bmatrix} \boxed{3} & 2 & 1 & 0 & 1 \\ 0 & 0 & \boxed{2} & 1 & 2 \\ 0 & 0 & 0 & \boxed{2} & 1 \end{bmatrix}$$

Back subs

$x_2 = s,\ x_5 = t$

- $2x_4 + t = 0 \to x_4 = -t/2$
- $2x_3 - t/2 + 2t = 0 \to$
  $2x_3 = -\frac{3}{2}t \to x_3 = -\frac{3}{4}t$
- $3x_1 + 2s - \frac{3}{4}t + t = 0$
  $3x_1 = -2s - \frac{1}{4}t$
  $x_1 = -\frac{2}{3}s - \frac{1}{12}t$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} -\frac{2}{3}s - \frac{1}{2}t \\ s \\ -3/4t \\ -t/2 \\ t \end{bmatrix} = \underbrace{s\begin{bmatrix} -\frac{2}{3} \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + t\begin{bmatrix} -1/2 \\ 0 \\ -3/4 \\ -1/2 \\ 1 \end{bmatrix}}_{\text{a 2-dim plane in } \mathbb{R}^5}$$

Example. Solve

$$\begin{aligned} x_1 + x_2 + x_3 + x_4 &= 0 \\ 2x_1 + x_2 - x_3 + 3x_4 &= 0 \\ x_1 - 2x_2 \quad x_3 \quad x_4 &= 0 \end{aligned} \longrightarrow \begin{bmatrix} \boxed{1} & 1 & 1 & 1 \\ 2 & 1 & -1 & 3 \\ 1 & -2 & 1 & 1 \end{bmatrix}$$

$$\xrightarrow[R_3 - R_1 \to R_3]{R_2 - 2R_1 \to R_2} \begin{bmatrix} \boxed{1} & 1 & 1 & 1 \\ 0 & \boxed{-1} & -3 & 1 \\ 0 & -3 & 0 & 0 \end{bmatrix} \xrightarrow[R_3 - 3R_2 \to R_3]{} \begin{bmatrix} \boxed{1} & 1 & 1 & 1 \\ 0 & \boxed{-1} & -3 & 1 \\ 0 & 0 & \boxed{9} & -3 \end{bmatrix}$$

Free

- $x_4 = t$
- $9x_3 - 3t = 0 \to x_3 = \frac{t}{3}$
- $-x_2 - t + t = 0 \to x_2 = 0$
- $x_1 + \frac{t}{3} + t = 0 \to x_1 = -\frac{4}{3}t$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = t\begin{bmatrix} -4/3 \\ 0 \\ 1/3 \\ 1 \end{bmatrix}$$

set of all solutions: $\underbrace{\left\{ t\begin{bmatrix} -4 \\ 0 \\ 1 \\ 3 \end{bmatrix} \,\middle|\, t \in \mathbb{R} \right\}}_{\text{line in } \mathbb{R}^4}$

Note. Echelon form is not unique!

RREF

Rank

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 3 & -1 \\ 0 & 0 & 1 & -1/3 \end{bmatrix} \xrightarrow[R_1 - R_2 \to R_1]{R_2 - R_3 \to R_2} \begin{bmatrix} 1 & 1 & 0 & 4/3 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1/3 \end{bmatrix} \xrightarrow{R_1 - R_2 \to R_1}$$

RREF

$$\begin{array}{c} \begin{bmatrix} 1 & 0 & 0 & 4/3 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1/3 \end{bmatrix} \begin{array}{c} x_1 \\ x_2 \\ x_3 \end{array} \\ \begin{array}{cccc} x_1 & x_2 & x_3 & t \end{array} \end{array}$$

Notice we can read of the solution to the

System from the RREF:
$$\begin{cases} x_1 + \frac{4}{3}t = 0 \\ x_2 = 0 \\ x_3 - \frac{1}{3}t = 0 \end{cases}$$

Also recall what this means

$$\left(-\tfrac{4}{3}t\right)\begin{bmatrix}1\\2\\1\end{bmatrix} + (0)\begin{bmatrix}1\\1\\-2\end{bmatrix} + \left(\tfrac{1}{3}t\right)\begin{bmatrix}1\\-1\\1\end{bmatrix} + (t)\begin{bmatrix}1\\3\\1\end{bmatrix} = \begin{bmatrix}0\\0\\0\end{bmatrix}$$

$$\begin{bmatrix}1\\3\\1\end{bmatrix} = \left(\tfrac{4}{3}\right)\begin{bmatrix}1\\2\\1\end{bmatrix} + (0)\begin{bmatrix}1\\1\\2\end{bmatrix} + \left(\tfrac{1}{3}\right)\begin{bmatrix}1\\-1\\1\end{bmatrix}$$

$$A = \underbrace{\begin{bmatrix}1 & 1 & 1\\2 & 1 & -1\\1 & 2 & 1\end{bmatrix}}_{\text{independent columns of } A} \underbrace{\begin{bmatrix}1 & 0 & 0 & 4/3\\0 & 1 & 0 & 0\\0 & 0 & 1 & -1/3\end{bmatrix}}_{\text{RREF with rows of 0's removed}}$$ (column-row decomposition)

Suppose
$$\mathrm{RREF}(A) = \begin{array}{c} \begin{array}{ccccc} x_1 & x_2 & x_3 & x_4 & x_5 \end{array} \\ \begin{bmatrix} \boxed{1} & 2 & 0 & 1 & 0 \\ 0 & 0 & \boxed{1} & -1 & 0 \\ 0 & 0 & 0 & 0 & \boxed{1} \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \end{array}$$

$x_2 = s$, $x_4 = t$ (free)

x_1, x_3, x_5 (pivots)

$A = [a_1 \cdots a_5]$

(All columns are lin-comb of C, all rows are lin-comb of R)

$$A = \underset{C}{[a_1\ a_3\ a_5]}\underset{R}{\begin{bmatrix}1 & 2 & 0 & 1 & 0\\0 & 0 & 1 & -1 & 0\\0 & 0 & 0 & 0 & 1\end{bmatrix}} = \begin{bmatrix} a_1 & 2a_1 & a_3 & a_1 - a_3 & a_5 \end{bmatrix}$$

$$\begin{bmatrix}x_1\\x_2\\x_3\\x_4\\x_5\end{bmatrix} = \begin{bmatrix}-2s-t\\s\\t\\t\\0\end{bmatrix}$$

$$a_1(-2s-t) + a_2 s + a_3 t + a_4 t + a_5(0) = \begin{bmatrix}0\\0\\0\\0\\0\end{bmatrix}$$

Set $t = 0$, $s = 1$ and get $a_2 = 2a_1$

$t = 1$, $s = 0$: $a_4 = a_1 - a_3$

$$[a_1\ a_3\ a_5]\begin{bmatrix}2 & 1\\0 & -1\\0 & 0\end{bmatrix} = [a_2\ a_4]$$

Look at MATLAB example.

Note: This shows R is unique and so RREF is unique

<u>Matrix Algebra</u>: cA, $A+B$ (same size)

- A $m \times n$, c is $n \times 1$ we have defined $Ac = a_1c_1 + \cdots + a_nc_n$

(A) • A $m \times n$, C $n \times k = [c_1 \cdots c_k]$ then $\underset{(m \times n)}{AC} = [Ac_1 | Ac_2 | \cdots | Ac_k]$

- $c = [c_1 \cdots c_m]$ $(1 \times m)$; $A = \begin{bmatrix} - a_1 - \\ \vdots \\ - a_m - \end{bmatrix}$ (row vectors)

$$cA = c_1 a_1 + \cdots + c_m a_m \quad 1 \times n$$

(B) - $C = \underset{l \times m}{\begin{bmatrix} - c_1 - \\ \vdots \\ - c_l - \end{bmatrix}}$ $\quad A \quad m \times n \quad$ $CA = \underset{l \times n}{\begin{bmatrix} - c_1 A - \\ - c_2 A - \\ \vdots \\ - c_l A - \end{bmatrix}}$

(C) - $u, v \in \mathbb{R}^n$, $u \cdot v = \langle u, v \rangle = \sum_{i=1}^{n} u_i v_i$ (dot-prod / inner prod)

$$\underset{l \times m}{\overset{C}{\begin{bmatrix} - c_1 - \\ \vdots \\ - c_l - \end{bmatrix}}} \underset{m \times n}{\overset{A}{\begin{bmatrix} | & | & & | \\ a_1 & a_2 & \cdots & a_n \\ | & | & & | \end{bmatrix}}} = \underset{l \times n}{\begin{bmatrix} c_1 \cdot a_1 & \cdots & c_1 \cdot a_n \\ \vdots & & \\ c_l \cdot a_1 & \cdots & c_l \cdot a_n \end{bmatrix}} \quad (CA)_{ij} = c_i \cdot a_j = \sum_{k=1}^{m} C_{ik} A_{kj}$$

- Is this all? No! (Block Op) $\begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & & \end{bmatrix} \begin{bmatrix} B_{11} & B_{12} \\ & \end{bmatrix}$

Example.

$$\underset{2 \times 3}{\begin{bmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{bmatrix}} \underset{3 \times 2}{\begin{bmatrix} 1 & 4 \\ 3 & 2 \\ 2 & -1 \end{bmatrix}} = \text{do this several ways} \quad 2 \times 2$$

(A)
$$\left[\begin{bmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 3 \\ 2 \end{bmatrix} \quad \begin{bmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{bmatrix} \begin{bmatrix} 4 \\ 2 \\ -1 \end{bmatrix} \right] = \left[\begin{bmatrix} 1 \\ 3 \end{bmatrix} + 3 \begin{bmatrix} 2 \\ 2 \end{bmatrix} + 2 \begin{bmatrix} 3 \\ 1 \end{bmatrix} \quad 4 \begin{bmatrix} 1 \\ 3 \end{bmatrix} + 2 \begin{bmatrix} 2 \\ 2 \end{bmatrix} + (-1) \begin{bmatrix} 3 \\ 1 \end{bmatrix} \right]$$

$$= \begin{bmatrix} 13 & 5 \\ 11 & 15 \end{bmatrix}$$

(B)
$$\begin{bmatrix} [1 \ 2 \ 3] \begin{bmatrix} 1 & 4 \\ 3 & 2 \\ 2 & -1 \end{bmatrix} \\ [3 \ 2 \ 1] \begin{bmatrix} 1 & 4 \\ 3 & 2 \\ 2 & -1 \end{bmatrix} \end{bmatrix} = \begin{bmatrix} [1 \ 4] + [6 \ 4] + [6 \ -3] \\ [3 \ 12] + [6 \ 4] + [2 \ -1] \end{bmatrix} = \begin{bmatrix} 13 & 5 \\ 11 & 15 \end{bmatrix}$$

(C)
$$\begin{bmatrix} [1 \ 2 \ 3] \begin{bmatrix} 1 \\ 3 \\ 2 \end{bmatrix} & [1 \ 2 \ 3] \begin{bmatrix} 4 \\ 2 \\ -1 \end{bmatrix} \\ \cdots & \cdots \end{bmatrix} = \begin{bmatrix} 13 & 5 \\ 11 & 15 \end{bmatrix}$$

$(3\ 2\ 1)\ (3\ 2)\quad (3\ 2\ 1)\ (2\ {-1})$