

Homework 7 Partial Solutions

Section 6.4

14. Write $A = B^H B$ where

$$A = \begin{bmatrix} 4 & 0 & 0 \\ 0 & 1 & i \\ 0 & -i & 1 \end{bmatrix}$$

clearly A is Hermitian, so we know A is orthogonally diagonalized.

$A - tI = \begin{bmatrix} 4-t & 0 & 0 \\ 0 & 1-t & i \\ 0 & -i & 1-t \end{bmatrix}$ so $\det(A - tI) = (4 - t)((1 - t)^2 + i^2) = (4 - t)(t^2 - 2t + 1 - 1) = -(t - 4)(t - 2)t$ so the eigenvalues are $4 > 2 > 0$. Note all eigenspaces are dimension 1 so we need only find a single eigenvector in each eigenspace to get a basis.

$$\text{NS}(A - 4I) = \text{NS} \begin{bmatrix} 0 & 0 & 0 \\ 0 & -3 & i \\ 0 & -i & -3 \end{bmatrix} = \text{span}\{(1, 0, 0)\}$$

$$\text{NS}(A - 2I) = \text{NS} \begin{bmatrix} 2 & 0 & 0 \\ 0 & -1 & i \\ 0 & -i & -1 \end{bmatrix} = \text{span}\{(0, i, 1)\}$$

$$\text{NS}(A - 0I) = \text{NS} \begin{bmatrix} 4 & 0 & 0 \\ 0 & 1 & i \\ 0 & -i & 1 \end{bmatrix} = \text{span}\{(0, 1, i)\}$$

Our orthonormal basis will be

$$\mathbf{u}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \quad \mathbf{u}_2 = \begin{bmatrix} 0 \\ i/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix} \quad \mathbf{u}_3 = \begin{bmatrix} 0 \\ 1/\sqrt{2} \\ i/\sqrt{2} \end{bmatrix}$$

So

$$A = UDU^H = \begin{bmatrix} 1 & 0 & 0 \\ 0 & i/\sqrt{2} & 1/\sqrt{2} \\ 0 & 1/\sqrt{2} & i/\sqrt{2} \end{bmatrix} \begin{bmatrix} 4 & & \\ & 2 & \\ & & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & -i/\sqrt{2} & 1/\sqrt{2} \\ 0 & 1/\sqrt{2} & -i/\sqrt{2} \end{bmatrix}$$

Let

$$B = D^{1/2}U^H = \begin{bmatrix} 2 & & \\ & \sqrt{2} & \\ & & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & -i/\sqrt{2} & 1/\sqrt{2} \\ 0 & 1/\sqrt{2} & -i/\sqrt{2} \end{bmatrix} = \begin{bmatrix} 2 & 0 & 0 \\ 0 & -i & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

then

$$A = B^H B$$

as desired.

15. Let U be unitary.

(a) $U^H U = U U^H = I$ so U is normal.

(b) Clearly $\langle U\mathbf{x}, U\mathbf{y} \rangle = (U\mathbf{y})^H (U\mathbf{x}) = \mathbf{y}^H U^H U \mathbf{x} = \mathbf{y}^H \mathbf{x} = \langle \mathbf{x}, \mathbf{y} \rangle$. So $\|x\|_2^2 = \langle \mathbf{x}, \mathbf{x} \rangle = \langle U\mathbf{x}, U\mathbf{x} \rangle = \|U\mathbf{x}\|_2^2$. So $\|\mathbf{x}\|_2 = \|U\mathbf{x}\|_2$.

(c) If λ is an eigenvalue, then from (b)

$$1 = \frac{\langle U\mathbf{x}, U\mathbf{x} \rangle}{\langle \mathbf{x}, \mathbf{x} \rangle} = \frac{\lambda \bar{\lambda} \langle \mathbf{x}, \mathbf{x} \rangle}{\langle \mathbf{x}, \mathbf{x} \rangle} = |\lambda|^2$$

So $|\lambda| = 1$.

26. If you are at site i , the probability of jumping to site j is a_{ij} and is determined as follows. There is an 85% chance that the user will choose on of the m_i links on the page and there is a 15% chance the user will just choose a random link from the i possible links. (This is just a heuristic.)

$$a_{ij} = \begin{cases} (.85)\frac{1}{m_i} + (.15)\frac{1}{n} & \text{if there is a link from } i \text{ to } j + (.15)\frac{1}{n} \\ (.15)\frac{1}{n} & \text{otherwise} \end{cases}$$

So for this problem the matrix is:

$$A = \begin{bmatrix} .15/4 & .85/2 + .15/4 & .15/4 & 1/4 \\ .85/3 + .15/4 & .15/4 & .15/4 & 1/4 \\ .85/3 + .15/4 & .85/2 + .15/4 & .15/4 & 1/4 \\ .85/3 + .15/4 & .15/4 & .85 + .15/4 & 1/4 \end{bmatrix} = \begin{bmatrix} 0.037500 & 0.4625 & 0.0375 & 0.25 \\ 0.320833 & 0.0375 & 0.0375 & 0.25 \\ 0.320833 & 0.4625 & 0.0375 & 0.25 \\ 0.320833 & 0.0375 & 0.8875 & 0.25 \end{bmatrix}$$

This is a Markov matrix and we are interested in the steady state eigenvector, that is \mathbf{x} so that $A\mathbf{x} = \mathbf{x}$ and $x_i \geq 0$ and $\sum x_i = 1$. This vector is $\mathbf{x} = (0.19322, 0.17401, 0.24797, 0.38479)$. To get this with MATLAB:

```
1 [V,D] = eig(A);
2 x = V(:,1)/sum(V(:,1))
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27. $A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ so $\det(A - tI) = \det \begin{bmatrix} -t & 1 \\ 1 & -t \end{bmatrix} = t^2 - 1 = (t-1)(t+1)$. The eigenvalues are $1 > -1$.

$$\text{NS}(A - 1 \cdot I) = \text{NS} \begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix} = \text{span}\{(1, 1)\}.$$

$$\text{NS}(A + 1 \cdot I) = \text{NS} \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} = \text{span}\{(-1, 1)\}.$$

Let

$$\mathbf{u}_1 = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \end{bmatrix} \quad \mathbf{u}_2 = \frac{1}{\sqrt{2}} \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

and

$$A = [\mathbf{u}_1 \quad \mathbf{u}_2] \begin{bmatrix} \lambda_1 & \\ & \lambda_2 \end{bmatrix} \begin{bmatrix} \mathbf{u}_1^H \\ \mathbf{u}_2^H \end{bmatrix} = \lambda_1 \mathbf{u}_1 \mathbf{u}_1^H + \lambda_2 \mathbf{u}_2 \mathbf{u}_2^H$$

28. Let $\mathbf{u}_1, \dots, \mathbf{u}_n$ be an orthonormal basis of eigenvectors for Hermitian A with eigenvalues $\lambda_1, \dots, \lambda_n$. Let $\mathbf{x} = \sum_{i=1}^n c_i \mathbf{u}_i$.

(a) $A\mathbf{x} = \sum_{i=1}^n \lambda_i c_i \mathbf{u}_i$, so $\mathbf{x}^H A \mathbf{x} = \left(\sum_{j=1}^n \bar{c}_j \mathbf{u}_j^H \right) \left(\sum_{i=1}^n c_i \mathbf{u}_i \right) = \sum_{j=1}^n \sum_{i=1}^n \bar{c}_j c_i \mathbf{u}_j^H \mathbf{u}_i = \sum_{i=1}^n |c_i|^2 \lambda_i$.

Similarly, $\mathbf{x}^H \mathbf{x} = \sum_{i=1}^n |c_i|^2$ so

$$\rho(\mathbf{x}) = \frac{\mathbf{x}^H A \mathbf{x}}{\mathbf{x}^H \mathbf{x}} = \frac{\sum_{i=1}^n |c_i|^2 \lambda_i}{\sum_{i=1}^n |c_i|^2}$$

(b) If $\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_n$, then clearly

$$\lambda_1 = \frac{\sum_{i=1}^n |c_i|^2 \lambda_i}{\sum_{i=1}^n |c_i|^2} \geq \rho(\mathbf{x}) = \frac{\sum_{i=1}^n |c_i|^2 \lambda_i}{\sum_{i=1}^n |c_i|^2} \geq \frac{\sum_{i=1}^n |c_i|^2 \lambda_n}{\sum_{i=1}^n |c_i|^2} = \lambda_n$$

Section 6.5

4. We are given an SVD of A as

$$A = \begin{bmatrix} 3/5 & -4/5 & 0 \\ 4/5 & 3/5 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 30 & & \\ & 15 & \\ & & 3 \end{bmatrix} \begin{bmatrix} 3/5 & 4/5 & 0 \\ -4/5 & 3/5 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

A rank 2 matrix of minimal $\|\cdot\|_F$ distance is

$$\begin{aligned} A_2 &= \begin{bmatrix} 3/5 & -4/5 & 0 \\ 4/5 & 3/5 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 30 & & \\ & 15 & \\ & & 0 \end{bmatrix} \begin{bmatrix} 3/5 & 4/5 & 0 \\ -4/5 & 3/5 & 0 \\ 0 & 0 & 1 \end{bmatrix} \\ &= \begin{bmatrix} 3/5 & -4/5 \\ 4/5 & 3/5 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 30 & \\ & 15 \end{bmatrix} \begin{bmatrix} 3/5 & 4/5 & 0 \\ -4/5 & 3/5 & 0 \end{bmatrix} \end{aligned}$$

A rank 1 matrix of minimal $\|\cdot\|_F$ distance is

$$\begin{aligned} A_1 &= \begin{bmatrix} 3/5 & -4/5 & 0 \\ 4/5 & 3/5 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 30 & & \\ & 0 & \\ & & 0 \end{bmatrix} \begin{bmatrix} 3/5 & 4/5 & 0 \\ -4/5 & 3/5 & 0 \\ 0 & 0 & 1 \end{bmatrix} \\ &= \begin{bmatrix} 3/5 \\ 4/5 \\ 0 \end{bmatrix} [30] \begin{bmatrix} 3/5 & 4/5 & 0 \end{bmatrix} \\ &= \sigma_1 \mathbf{u}_1 \mathbf{u}_1^T \end{aligned}$$

7. $A = U \Sigma V^T$, if $\text{rank}(A) = r$, then

$$A = [\mathbf{u}_1 \quad \dots \quad \mathbf{u}_r] \begin{bmatrix} \sigma_1 & & \\ & \ddots & \\ & & \sigma_r \end{bmatrix} \begin{bmatrix} \mathbf{v}_1^T \\ \vdots \\ \mathbf{v}_r^T \end{bmatrix} = B \begin{bmatrix} \mathbf{v}_1^T \\ \vdots \\ \mathbf{v}_r^T \end{bmatrix}$$

for some B so all rows of A are linear combinations of the vectors $\{\mathbf{v}_1, \dots, \mathbf{v}_r\}$ and so $\text{rng}(A^T) = \text{RS}(A) = \text{span}\{\mathbf{v}_1, \dots, \mathbf{v}_r\}$. Trivially, $\{\mathbf{v}_1, \dots, \mathbf{v}_r\}$ is independent so it is a basis for $\text{rng}(A^T)$.

9. If A is $n \times n$ and $A = U\Sigma V^T$, then $\det(A) = \det(U) \det(\Sigma) \det(V^T) = \det(\Sigma) = \prod_{i=1}^n \sigma_i$.

Since U and V are unitary $\det(U), \det(V) \in \{1, -1\}$ and so

$$\det(A) = \pm \prod_{i=1}^n \sigma_i = \prod_{i=1}^n \lambda_i$$

Since all $\sigma_i \geq 0$, it follows that

$$\prod_{i=1}^n \sigma_i = \left| \prod_{i=1}^n \lambda_i \right|$$

11. Let σ be a singular value for A and \mathbf{v} be a corresponding right singular vector. $A\mathbf{v} = \sigma\mathbf{u}$ where \mathbf{u} is the associated left singular vector. Then

$$\frac{\langle A\mathbf{v}, A\mathbf{v} \rangle}{\langle \mathbf{v}, \mathbf{v} \rangle} = \frac{\langle \sigma\mathbf{u}, \sigma\mathbf{u} \rangle}{\langle \mathbf{v}, \mathbf{v} \rangle} = \frac{\sigma^2 \|\mathbf{u}\|_2^2}{\|\mathbf{v}\|_2^2} = \sigma^2$$

12. This defines what is called the psuedo-inverse of a matrix A . This is very important, say in least squares. Recall, that in least squares we have a matrix A and we want to "solve" $A\mathbf{x} = \mathbf{b}$, but this might not have an exact solution, so instead we search for \mathbf{b} that is as close as possible to $\text{CS}(A)$.

In general, given a subspace $S \subset V$ (in an inner product space), if we want the $\hat{\mathbf{b}} \in S$ closest to some $\mathbf{b} \in V$, then what we are looking for is the *orthogonal projection of \mathbf{b} on S* . This means that we want $\mathbf{b} - \hat{\mathbf{b}} \perp S$ and $\hat{\mathbf{b}} \in S$. To see why this yields $\hat{\mathbf{b}}$ is the closest point in S to \mathbf{b} take any other $\mathbf{v} \in S$, then we have

$$\|\mathbf{b} - \mathbf{v}\|^2 = \|\mathbf{b} - \hat{\mathbf{b}} + \hat{\mathbf{b}} - \mathbf{v}\|^2 = \|\mathbf{b} - \hat{\mathbf{b}}\|^2 + \|\hat{\mathbf{b}} - \mathbf{v}\|^2 \geq \|\mathbf{b} - \hat{\mathbf{b}}\|^2$$

This is by Pythagorean theorem (**this is a good exercise!**) using the fact that $\mathbf{b} - \hat{\mathbf{b}} \perp \mathbf{v} - \hat{\mathbf{b}}$, since $\mathbf{v} - \hat{\mathbf{b}} \in S$ and $\mathbf{b} - \hat{\mathbf{b}} \perp S$.

If we want to find the least square solution to $A\mathbf{x} = \mathbf{b}$, then we want to find a point $\hat{\mathbf{b}} \in \text{CS}(A)$ so that $\mathbf{b} - \hat{\mathbf{b}} \perp \text{CS}(A)$. Just take S from above to be $\text{CS}(A)$. This means we want two things $\hat{\mathbf{b}} = A\hat{\mathbf{x}}$ for some $\hat{\mathbf{x}}$ and $A^T(\mathbf{b} - \hat{\mathbf{b}}) = \mathbf{0}$. Putting these together, this means we want

$$A^T\mathbf{b} = A^T A\hat{\mathbf{x}} \tag{1}$$

Let $A^\dagger = V\Sigma^\dagger U^T$ where $A = U\Sigma V^T$ is the svd decomposition of A and

$$\Sigma^\dagger = \begin{bmatrix} 1/\sigma_1 & 0 & 0 & \dots \\ 0 & 1/\sigma_2 & 0 & \dots \\ 0 & 0 & 1/\sigma_3 & \dots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix}$$

is $n \times m$, if A is $m \times n$. Notice

$$\Sigma\Sigma^\dagger = \begin{bmatrix} I_k & O \\ O & O \end{bmatrix}$$

is $m \times m$ and

$$\Sigma^T \Sigma \Sigma^\dagger = \Sigma^T \quad (2)$$

Now what makes A^\dagger the psuedo-inverse is exactly that $A^\dagger \mathbf{b}$ is a least squares solution to $A\mathbf{x} = \mathbf{b}$. To show this we need to show $\hat{\mathbf{x}} = A^\dagger \mathbf{b}$ satisfies equation (1), that is:

$$A^T \mathbf{b} = A^T A \hat{\mathbf{x}} = A^T A A^\dagger \mathbf{b}$$

For this, just compute and use equation (2).

$$A^T A A^\dagger = V \Sigma^T U^T U \Sigma V^T V \Sigma^\dagger U^T = V \Sigma^T I \Sigma I \Sigma^\dagger U^T = V \Sigma^T (\Sigma \Sigma^\dagger) U^T = V \Sigma^T U^T = A^T$$

There are two ways to compute the psuedo-inverse in MATLAB and three ways to compute a least-squares solution to $A\mathbf{x} = \mathbf{b}$.

```

1 [U,S,V] = svd(A);
2 % Compute the psuedo-inverse using the definition
3 Adag1 = V*diag(diag(S(1:rank(A),1:rank(A))).^-1)*U';
4 % Compute the psuedo-inverse using the builtin function
5 Adag2 = pinv(A);
6 % Note Adag1 = Adag2
7
8 % Compute the least squares solution to Ax = b using "\"
9 xhat1 = A\b;
10 % Compute the least squares solution using the psuedo-inverse
11 xhat2 = Adag*b;
12 % or
13 xhat3 = Ainv*b;
14 % xhat1 = xhat2 = xhat3

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There are other solutions to the least squares problem, namely, any element of $\hat{\mathbf{x}} + \ker(A)$ is a solution.

Section 7.6

5. Let

$$A = \begin{bmatrix} 5 & 2 & 2 \\ -2 & 1 & -1 \\ -3 & -4 & 2 \end{bmatrix}$$

a. Verify that $(4, (2, -2, 1))$ is an "eigenpair".

This is trivial, just verify that

$$\begin{bmatrix} 5 & 2 & 2 \\ -2 & 1 & -1 \\ -3 & -4 & 2 \end{bmatrix} \begin{bmatrix} 2 \\ -2 \\ 1 \end{bmatrix} = 4 \begin{bmatrix} 2 \\ -2 \\ 1 \end{bmatrix}$$

b. Find a Householder transformation H so that

$$HAH = HAH^{-1} = \begin{bmatrix} 4 & * & * \\ 0 & * & * \\ 0 & 0 & * \end{bmatrix} \quad (3)$$

The point is that if A and B are similar, then A and B have the same eigenvalues, since they have the same characteristic equation. (Quick proof: $\chi_A(\lambda) \det(A - \lambda I) = \det(SBS^{-1} - \lambda SIS^{-1}) = \det(S(B - \lambda I)S^{-1}) = \det(S) \det(S - \lambda I) \det(S)^{-1} = \det(S - \lambda I) = \chi_B(\lambda)$.)

Since

$$HAH^{-1} = \begin{bmatrix} 4 & B \\ \mathbf{0} & C \end{bmatrix}$$

we have $\det(HAH^{-1}) = (\lambda - 4) \det(C - \lambda I)$ so we have reduced our problem to finding the eigenvalues of a smaller matrix C .

Recall a Householder transformation is of the form $H_{\mathbf{x}} = I - 2\mathbf{u}\mathbf{u}^T$ and is defined for a specific vector \mathbf{x} so that $H\mathbf{x} = \|\mathbf{x}\|\mathbf{e}_1$. Some things to remember:

- H is symmetric. (Proof: $H^T = (I^T - 2(\mathbf{u}^T)^T \mathbf{u}^T) = (I - \mathbf{u}\mathbf{u}^T)$.)
- H is orthogonal. (Proof: $H^T H = H H = H^2 = (I - 2\mathbf{u}\mathbf{u}^T)(I - 2\mathbf{u}\mathbf{u}^T) = (I^2 - 2I\mathbf{u}\mathbf{u}^T - 2\mathbf{u}\mathbf{u}^T I + 4(\mathbf{u}\mathbf{u}^T)(\mathbf{u}\mathbf{u}^T)) = I - 4\mathbf{u}\mathbf{u}^T + 4\mathbf{u}(\mathbf{u}^T \mathbf{u})\mathbf{u}^T = I - 4\mathbf{u}\mathbf{u}^T + 4\mathbf{u}I\mathbf{u}^T = I$.)
- So $H = H^{-1}$, that is, H is **idempotent**.
- $H\mathbf{x} = \mathbf{v}$ iff $H\mathbf{v} = \mathbf{x}$. (Proof: $H\mathbf{x} = \mathbf{v}$ implies $HH\mathbf{x} = \mathbf{x} = H\mathbf{v}$. Similarly in the other direction.)

For this problem, let $\mathbf{x} = 4/3(-2, 2, -1)$ so that $\|\mathbf{x}\| = 4$ and \mathbf{x} is an eigenvector for 4 for A . This way $H = H_{\mathbf{x}}$ satisfies $H\mathbf{x} = 4\mathbf{e}_1$ and thus

$$HAH\mathbf{e}_1 = HA(1/4\mathbf{x}) = 1/4HA\mathbf{x} = 1/4H(4\mathbf{x}) = (1/4)(4)H\mathbf{x} = 4\mathbf{e}_1$$

so \mathbf{e}_1 is an eigenvector for 4 in HAH . Thus equation (3) holds.

All that is left is to provide \mathbf{u} to define H , this is done as follows:

$$\mathbf{u} = \frac{1}{\sqrt{2\beta}}(x_1 - \alpha, x_2, x_3),$$

where $\beta = \alpha(\alpha - x_1)$ and

$$\alpha = \begin{cases} -\|\mathbf{x}\| & \text{if } x_1 > 0 \\ \|\mathbf{x}\| & \text{if } x_1 \leq 0 \end{cases}$$

Note that $\beta > 0$. (α is chosen this way to optimize numeric stability of the calculation.) It is a simple exercise to show $\|\mathbf{u}\| = 1$.

Let $\mathbf{v} = (x_1 - \alpha, x_2, x_3)$, then

$$2\mathbf{u}\mathbf{u}^T = \frac{1}{\beta}\mathbf{v}\mathbf{v}^T \quad \text{so} \quad H = I - 2\mathbf{u}\mathbf{u}^T = I - \frac{1}{\beta}\mathbf{v}\mathbf{v}^T$$

and

$$H\mathbf{x} = \mathbf{x} - \frac{1}{\beta}\mathbf{v}(\mathbf{v}^T \mathbf{x}) = \mathbf{x} - \frac{1}{\beta}\mathbf{v}(x_1^2 - \alpha x_1 + x_2 + x_3^2) = \mathbf{x} - \frac{1}{\beta}\mathbf{v}(\alpha^2 - \alpha x_1) = \mathbf{x} - \mathbf{v} = \alpha\mathbf{e}_1$$

as desired.

So $\mathbf{x} = 4/3(-2, 2, -1)$ and thus $\alpha = 4$ and $\beta = 4(4 + 8/3) = 80/3$ and $\mathbf{v} = (-8/3 - 4, 8/3, -4/3) = 4/3(-5, 2, -1)$ and

$$H = I - \frac{1}{\beta} \mathbf{v} \mathbf{v}^T = I - \begin{bmatrix} 5/3 & -2/3 & 1/3 \\ -2/3 & 4/15 & -2/15 \\ 1/3 & -2/15 & 1/15 \end{bmatrix} = \begin{bmatrix} -2/3 & 2/3 & -1/3 \\ 2/3 & 11/15 & 2/15 \\ -1/3 & 2/15 & 14/15 \end{bmatrix}$$

c. Compute $H A H$ and find the remaining two eigenvalues.

$$H A H = \begin{bmatrix} 4 & -12/5 & -9/5 \\ 0 & 53/25 & -4/25 \\ 0 & -154/25 & 47/25 \end{bmatrix}$$

So

$$C = \frac{1}{25} \begin{bmatrix} 53 & -4 \\ -154 & 47 \end{bmatrix}$$

and

$$\det(C - \lambda I) = (\lambda - 3)(\lambda - 1)$$

so the remaining two eigenvalues of A are 3 and 1.