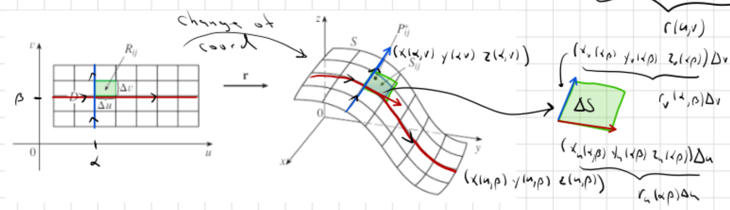
Recall surface integrals - Flux

Surface Integrals

$\mathbb{R}^2 \xrightarrow{r} \mathbb{R}^3$

map the (u,v) plane to a surface $S: (u,v) \mapsto \underbrace{(x(u,v), y(u,v), z(u,v))}_{r(u,v)}$



$\Delta s = |r_u \Delta u \times r_v \Delta v| = |r_u \times r_v| \Delta u \Delta v$

$\boxed{dS = |r_u \times r_v|\, du\, dv = |r_u \times r_v|\, dA}$

$$\boxed{\iint_S f(x,y,z)\, dS = \iint_R f(x(u,v), y(u,v), z(u,v))\, |r_u \times r_v|\, dA}$$

$R = r^{-1} S = \{(u,v) \mid r(u,v) \in S\}$

Key items

$$\boxed{\iint_S F \cdot \vec{n}\, dS = \iint_S F \cdot d\vec{S}}$$

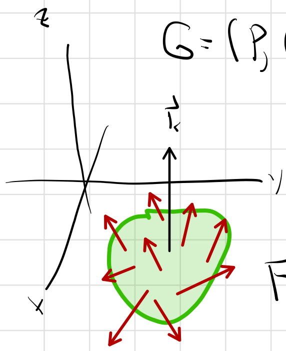
$$\boxed{\vec{n} \cdot dS = d\vec{S} = |r_u \times r_v|\, dA}$$

Stoke's Theorem

Let S be an oriented piecewise smooth surface with ∂S oriented according to S. Let F be a vector field with continuous first partials then

$$\boxed{\int_C F \cdot d\vec{r} = \iint_S \text{curl}\, F \cdot d\vec{S}}$$

Relation to Green's



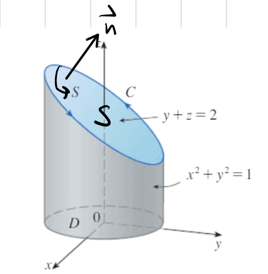
$G = (P, Q, R)$ where $R = 0$

$$\overbrace{\int_C F \cdot dr = \iint_S \text{curl}(G) \cdot d\vec{S}}^{\text{Stoke's}} = \iint_S \text{curl}(G) \cdot \vec{k}\, dA$$

Green's

$F = (P, Q)$

Example. Compute $\int_C F \cdot d\vec{r}$ where $F = \langle -y^2, x, z^2 \rangle$ and C is the intersection of $x^2+y^2=1$ (cylinder) and $y+z=2$ (plane) oriented counter-clockwise



(Shamelessly stolen from the text.)

$\vec{n} = \langle 0,1,1 \rangle \left(\frac{1}{\sqrt{2}}\right)$ (recall what we know about planes)

$$\int_C F \cdot d\vec{r} = \iint_S \text{curl} F \cdot \vec{n}\, dS$$

$$\text{curl} F = \det \begin{bmatrix} i & j & k \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ -y^2 & x & z^2 \end{bmatrix} = 0\vec{i} + 0\vec{j} + (1+2y)\vec{k}$$

$S(x,y) = (x, y, 2-y)$

$dS = |S_x \times S_y| = \left|\det\begin{bmatrix} i & j & k \\ 1 & 0 & 0 \\ 0 & 1 & -1 \end{bmatrix}\right| = |\langle 0,1,1\rangle| = \sqrt{2}$

$$\int_C F \cdot d\vec{r} = \frac{1}{\sqrt{2}} \iint_S \langle 0, 0, 1+2y \rangle \langle 0,1,1 \rangle\, dS$$

$$= \frac{1}{\sqrt{2}} \iint_S (1+2y)\sqrt{2}\, dA$$

$$= \int_0^{2\pi} \int_0^1 (1 + 2r\sin(\theta))\, r\, dr\, d\theta \quad \text{(use cylindrical)}$$

$$= \int_0^{2\pi} \left[\frac{1}{2}r^2 + \frac{2}{3}r^3 \sin\theta \right]_{r=0}^{r=1} d\theta$$

$$= \int_0^{2\pi} \left(\frac{1}{2} + \frac{2}{3}\sin\theta\right) d\theta$$

$$= \left(\frac{1}{2}\theta - \frac{2}{3}\cos(\theta)\right)\Big|_{\theta=0}^{\theta=2\pi}$$

$$= \left(\frac{1}{2} 2\pi\right) = \pi$$

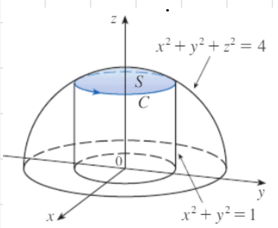
Note: It would have been easier just to use

$$d\vec{S} = (S_x \times S_y)\, dA$$

$$\int_C F \cdot d\vec{r} = \iint_S \text{curl} F \cdot d\vec{S}$$

$$= \iint_R (\text{curl} F) \cdot \langle 0,1,1 \rangle\, dA$$

Example This is sort of the converse of the previous. Use Stoke's to compute the flux of $F = \langle xz, yz, xy \rangle$ through S = part of $x^2+y^2+z^2=4$ lying inside $x^2+y^2=1$ and above the xy-plane.



Here the idea is to compute $\iint_S \text{curl} F \cdot d\vec{S}$ by actually computing $\int_C F \cdot d\vec{r}$

$$r(\theta) = \langle \cos\theta, \sin(\theta), \sqrt{3} \rangle$$

$$\int_0^{2\pi} \langle \sqrt{3}\cos\theta, \sqrt{3}\sin\theta, \cos\theta\sin\theta \rangle \cdot \langle -\sin\theta, \cos\theta, 0 \rangle\, d\theta$$

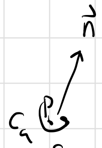
$$= \int_0^{2\pi} -\sqrt{3}\sin\theta\cos\theta + \sqrt{3}\sin\theta\cos\theta \, d\theta = 0$$

$$\text{Stoke's} \Rightarrow \left(\boxed{\text{curl } F = \vec{0} \text{ on } S \Rightarrow \underbrace{F \text{ is conservative}}_{F = \nabla f \Leftrightarrow \underbrace{\int_C F\cdot d\vec{r} = 0}_{\text{path independence}}}} \right)$$

$$\boxed{\int_C F\cdot dr = \iint_S \text{curl } F \cdot d\vec{S} = \iint_S 0 \, d\vec{S} = 0}$$

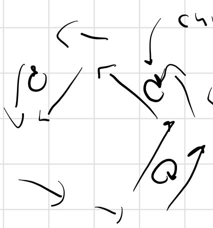
Stokes provides an interpretation of curl F

$$\underbrace{\oint_{C_a} F\cdot d\vec{r}}_{\text{measures rotation}} \approx \iint_{S_a} \text{curl } F(P_0)\cdot\vec{n}(P_0)\, dS = F(P_0)\cdot\vec{n}(P_0)\,\pi a^2$$



S_a disk of radius a curl F on $S_a \approx$ curl $F(P_0)$

So $\text{curl } F\cdot\vec{n} = \lim_{a\to 0} \dfrac{\oint_{C_a} F\cdot d\vec{r}}{\pi a^2}$ ~ measures circulation (at a point)



curl is clockwise

large scale rotation is counter clockwise

Divergence Let E be a "simple" solid region. Let F be a field with continuous first partials, then

$$\boxed{\iint_S F \cdot d\vec{S} = \iiint_E \operatorname{div} \vec{F} \, dV}$$

This provides an interpretation to $\operatorname{div} \vec{F}$ like the one Stoke's provides for $\operatorname{curl} F$. Let B_a be a small radius a ball around P_0, $\operatorname{div} F \simeq \operatorname{div} F(P_0)$ near P_0 so:

$$\iint_{\partial B_a} F \cdot d\vec{S} \approx \iiint_{B_a} \operatorname{div}(F(P_0)) \, dV = \operatorname{div}(F(P_0)) \left(\frac{4}{3}\pi a^3\right)$$

$$\operatorname{div}(F) \propto \text{flux out of small sphere at } P$$

Example Find flux of $F = \langle \cancel{z}, y, x \rangle$ over $S_2 = \{(x,y,z) \mid x^2+y^2+z^2=1\}$

$$\iint_S F \cdot d\vec{S} = \iiint_B \operatorname{div} F \, dV = \iiint_B \left(\frac{\partial z}{\partial x} + \frac{\partial y}{\partial y} + \frac{\partial x}{\partial z}\right) dV$$

$$= \iiint_B 1 \, dV = \frac{4}{3}\pi$$