Name: ______________________ Exam 2 - MAT513

Here are nine problems, 20 points each, straight from the text, for a total of 180 points. You may assume all rings are commutative and unital.

Problem 16.70. Let F be a field. Let $I = \{f(x) \in F[x] \mid f(a) = 0 \text{ for all } a \in F\}$. Show that I is an ideal of $F[x]$ and that I is infinite when F is finite and $I = \{0\}$ when F is infinite.

Problem 16.72. Let R be a ring, prove that $R[x]$ and $R[x^2]$ are isomorphic.

Problem 17.10. Let $f(x) \in \mathbb{Z}_p[x]$ be irreducible and $\deg(f(x)) = n$, show that $\mathbb{Z}_p[x]/\langle f(x)\rangle$ is a field of size p^n.

Problem 17.30. Show that for every integer $n > 1$ there are infinitely many monic irreducible $f(x) \in \mathbb{Q}[x]$ with $\deg(f(x)) = n$.

Problem 18.26. Show that every element of the form $(3 + 2\sqrt{2})^n$ is a unit in $\mathbb{Z}[\sqrt{2}]$.

Problem 18.34. Show that in $\mathbb{Z}_5[x]$, $3x^2+4x+3 = (3x+2)(x+4) = (4x+1)(2x+3)$. Why does this not violate Theorem 18.3?

Problem 18.44. Let F be a field and R be the subring of $F[x]$ generated by x^2 and x^3 and so that $F \subset R$, that is,

$$R = \bigcap \Big\{ D \subset F[x] \;\Big|\; F \cup \{x^2, x^3\} \subset D \text{ and } D \text{ is a ring} \Big\}.$$

Show that R is not a UFD.

Hint: Clearly, R is an integral domain since any subring of an integral domain is an integral domain. So the problem is unique factorization. Start by showing that x^2 and x^3 are irreducible in R. (You do need to **show** this!)

Problem 19.34. Show that $f(x) = x^{19} + x^8 + 1$ has at least one repeated root in some extension of $\mathbb{Z}_3$.

Problem 19.38. Find the splitting field of $f(x) = x^4 - x^2 - 2$ over $\mathbb{Z}_3$.