Exam 1 – Math 215

1 True/False

Problem 1.1 (60 points; 6 points each)**.** Decide if each of the following are true or false. You do not need to justify your choice here.

(a) True The proposition $\neg(p \vee \neg(q \vee p))$ is equivalent to $\neg p \wedge q$.

Method 1: This is easy to see using equivalences:

$$\begin{aligned} \neg(p \vee \neg(q \vee p)) &\equiv \neg p \wedge (q \vee p) \\ &\equiv (\neg p \wedge q) \vee (\neg p \wedge p) \\ &\equiv (\neg p \wedge q) \vee F \\ &\equiv \neg p \wedge q \end{aligned}$$

Method 2: Alternatively, $\neg p \wedge q$ can only be true if p is false and q is true. $\neg(p \vee \neg(q \vee p))$ is true when $p \vee \neg(q \vee p)$ is false, so p must be false and $\neg(q \vee p)$ must also be false. Hence, $q \vee p$ must be true. But as p is false, it must be that q is true. So both are true exactly when p is false and q is true. Hence, they are equivalent.

Method 3: Use a truth table

p	q	$\neg(p \vee \neg(q \vee p))$	$\neg p \wedge q$
T	T	F	F
T	F	F	F
F	T	T	T
F	F	F	F

(b) False Up to propositional equivalence, there are only 2^3 compound propositions using the atomic propositional variables, p, q, and r.

For any compound proposition, the truth table has 2^3 rows, namely, the 2^3 distributions of T's and F's to the three variables, p, q, and r. But for each of the 2^3 distributions of T's and F's we can choose a resultant T or F. So there are $2^{(2^3)}$ distinct propositions, or if you like, distinct truth tables.

(c) True $\neg\exists x(P(x) \rightarrow Q(x)) \equiv \forall x P(x) \wedge \neg\exists x Q(x))$

Using rules:

$$\begin{aligned}\neg\exists x(P(x) \to Q(x)) &\equiv \forall x \neg(P(x) \to Q(x)) \\ &\equiv \forall x \neg(\neg P(x) \lor Q(x)) \\ &\equiv \forall x(P(x) \land \neg Q(x))) \\ &\equiv \forall x P(x) \land \forall x \neg Q(x) \\ &\equiv \forall x P(x) \land \neg\exists x Q(x)\end{aligned}$$

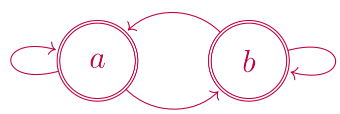
Using models: The only way $\neg\exists x(P(x) \to Q(x))$ can be true is if for all x, $P(x) \to Q(x)$ is false. This would mean that for all x, $P(x)$ is true, but $Q(x)$ is false. But this is exactly what is required for $\forall x P(x) \land \neg\exists Q(x)$ to hold.

(d) False $\exists x P(x) \land \exists x Q(x) \equiv \exists x(P(x) \land Q(x))$.

Consider any model with at least two elements and where P and Q are such that P, and Q are non-empty, but disjoint. For example, $U = \mathbb{N}$, $Q =$ even numbers, $P =$ odd numbers.

(e) True There is a finite model of $\exists x(x = x)$ (something exists), $\forall x \exists y(N(x,y) \land x \neq y)$ (for everything there is a different thing *next to* it), $\forall x, y, z(N(x,y) \land N(y,z) \to N(x,z))$ (being *next to* is transitive).

Sure, there can be just two things $U = \{a, b\}$ and $N = \{(a,a),(b,b),(a,b),(b,a)\}$. You can picture this model as:



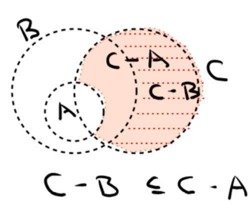
(f) True $A \subseteq B \implies C - B \subseteq C - A$

Suppose $A \subseteq B$ and $x \in C - B$, then $x \in C$ and $x \notin B$. But then $x \in C$ and $x \notin A$, since $x \in A \to x \in B$. So $x \in C - A$. We have shown

$$x \in C - B \to x \in C - A$$

So $C - B \subseteq C - A$.

Here is a Venn-Diagram proof:



(g) False For all sets A, $A \subseteq \mathcal{P}(A)$ (the powerset of A), For $A \subseteq \mathcal{P}(A)$ to be true we require that for all $a \in A$, $a \in \mathcal{P}(A)$, that is $a \subseteq A$. But there is no reason this should be the case. For example, $A = \{1\}$, where we don't view 1 as a set, but just as an integer. Then it is not the case that $a \subseteq A$, in fact, it does not even make sense to say $1 \subseteq A$. 1 is not a set.

So for $A = \{1\}$, it is true that $A = \{1\} \in \mathcal{P}(A)$, but it is not true that $1 \in \mathcal{P}(A)$, so $A \not\subseteq \mathcal{P}(A)$.

(h) True The set of all polynomials with rational coefficients, denoted $\mathbb{Q}[x]$, is countable.

A polynomial is an expression of the form $q^n x^n + \cdots + q_1 x + q_0$. This polynomial can be identified with $(q_n, \ldots, q_0) \in \mathbb{Q}^{n+1}$. Since $\mathbb{Q}$ is countable, so is $\mathbb{Q}^n$ for each n and hence $\bigcup_{n \in \mathbb{N}} \mathbb{Q}^n$. (See Problem 2.4)

(i) True The set of irrationals is uncountable.

The irrationals are $\mathbb{R} - \mathbb{Q}$ and $\mathbb{R} = (\mathbb{R} - \mathbb{Q}) \cup \mathbb{Q}$. If $\mathbb{R} - \mathbb{Q}$ were countable, then $\mathbb{R}$ would be countable.

(j) False In the inductive step of a proof by induction of $\forall n \geq 0 P(n)$, one may assume that for all $n \geq 0$, $P(n) \to P(n+1)$.

In the inductive step one assumes $P(n)$ and then must prove $P(n+1)$.

2 Free Response

100 points total, 20 points each

Problem 2.1. Prove or give a counter-example to the following statement:

The product of two irrational numbers is irrational.

Just consider $\sqrt{2}$, $\sqrt{2} \cdot \sqrt{2} = 2$. So, two irrationals can multiply to give a rational. For an example with two distinct numbers, we can use $a = \sqrt{3} + \sqrt{2}$ and $b = \sqrt{3} - \sqrt{2}$, so that $a \cdot b = (\sqrt{3})^2 - (\sqrt{2})^2 = 3 - 2 = 1$.

Problem 2.2. Prove that the sum of a rational number and an irrational number is irrational.

Suppose r is irrational, s is rational, and $r + s = t$ is rational. Then $r = t - s$ is rational, contrary to the assumption on r.

Problem 2.3. Prove:

$$p \wedge (q_1 \vee q_2 \vee \cdots \vee q_n) \equiv (p \wedge q_1) \vee (p \wedge q_2) \vee \cdots \vee (p \wedge q_n)$$

You may use induction (strong or weak), the well-ordering principle, or infinite decent. Make clear what you are doing, for example, if you use induction, make the base case(s) and inductive step clear.

The thing we are trying to prove is $P(n)$ for $n \geq 1$ where $P(n)$ is exactly as given

$$P(n) : p \wedge (q_1 \vee q_2 \vee \cdots \vee q_n) \equiv (p \wedge q_1) \vee (p \wedge q_2) \vee \cdots \vee (q \wedge q_n)$$

Base Case ($n = 1$):

$$P(1) : p \wedge q_1 = p \wedge q_1$$

is trivially true.

Inductive Step: Suppose $P(n)$. We must prove $P(n+1)$:

$$\begin{aligned}
&p \wedge (q_1 \vee q_2 \vee \cdots \vee q_n \vee q_{n+1}) \\
&\quad \equiv p \wedge ((q_1 \vee \cdots \vee q_n) \vee q_{n+1}) && \text{associativity} \\
&\quad \equiv p \wedge (q_1 \vee \cdots \vee q_n) \vee (p \wedge q_{n+1}) && \text{distributivity} \\
&\quad \equiv ((p \wedge q_1) \vee (p \wedge q_2) \vee \cdots \vee (q \wedge q_n)) \vee (p \wedge q_{n+1}) && \text{by the induction hypothesis } P(n) \\
&\quad \equiv (p \wedge q_1) \vee (p \wedge q_2) \vee \cdots \vee (q \wedge q_n) \vee (p \wedge q_{n+1}) && P(n+1)
\end{aligned}$$

Thus $P(n+1)$ holds.

By induction, $\forall n \geq 1 P(n)$.

Problem 2.4. Explain why the set of finite sequences of rational numbers, $\mathbb{Q}^{<\infty} = \{(q_1, q_2, \ldots, q_n) \mid n \in \mathbb{N}\}$, is countable.

Use facts you know about countability, including the countability of $\mathbb{Q}$ and facts about which set operations preserve countability, e.g., the countable union of countable sets is countable, etc. Make clear which facts you are using.

You might recognize that this is related to one of the T/F questions.

$\mathbb{Q}$ is countable and hence $\mathbb{Q}^n$ is countable. You can just use this or you can argue that $\mathbb{Q} \times \mathbb{Q}$ is countable, $(\mathbb{Q} \times \mathbb{Q}) \times \mathbb{Q}$ is countable, etc. (This is really an induction.)

Now $\mathbb{Q}^{<\infty} = \bigcup_{i=1}^{\infty} \mathbb{Q}^i$ is a countable union of countable sets and hence is countable.

Finally, and not part of this problem, the map taking a tuple, $(q_0, \ldots, q_n) \mapsto q_0 + q_1 x + q_2 x^2 + \cdots + q_n x^n$ is *clearly bijection between $Q^{<\infty}$ and* $\mathbb{Q}[x]$.

Problem 2.5. Show that if $a_1, a_2, \ldots, a_n$ are real numbers and $a = \frac{1}{n}\sum_{i=1}^{n} a_i$ (the average), then for some i, $a_i \geq a$.

There are many ways to proceed. Induction is an option, or a direct argument. Here is one way. Suppose $a_i < a$ for all i, then $\sum_{i=1}^{n} a_i < \sum_{i=1}^{n} a = n \cdot a$, but then $\frac{1}{n}\sum_{i=1}^{n} a_i < a$ contradicting the definition of a.