Math 571 - Exam 1 (50 points)

Richard Ketchersid

Notation/Definition: Let (X, d) be a metric space for $A, B \subset X$ define $d(A, B) = \inf\{d(a, b) \mid a \in A \text{ and } b \in B\}$ and set $d(a, B) = d(\{a\}, B)$.

Question 1 (10 points). Let (X, d) be a metric space, prove that

a) For any closed set F and $x \notin F$, $d(x, F) > 0$.

Proof 1: Suppose $d(x, F) = 0$, then there is $x_i \in F$ such that $\lim_i d(x, x_i) = 0$, but then, $\lim_i x_i \to x$ so $x \in F$, which is a contradiction.

Proof 2: Let $\varepsilon > 0$ and $N_\varepsilon(x) \cap F = \emptyset$. $\varepsilon exists$ since $x \notin F$ and F is closed. Now for $y \in F$ we have $y \notin N_\varepsilon(x)$ so $d(x, y) \geq \varepsilon$ and hence $d(x, F) \geq \varepsilon$.

b) For any compact K and closed F with $K \cap F = \emptyset$, $d(K, F) > 0$.

Proof 1: For $x \notin F$ there is $\varepsilon_x > 0$ so that $N_{2\varepsilon_x}(x) \cap F = \emptyset$. The set of open sets $\mathcal{O} = \{N_{\varepsilon_x}(x) \mid x \in K\}$ is an open cover of K and hence has a finite subcover, $\mathcal{O}' = \{N_{\varepsilon_{x_1}}(x_1), \ldots, N_{\varepsilon_{x_k}}(x_k)\}$. Let $\varepsilon = \min\{\varepsilon_{x_1}, \ldots, \varepsilon_{x_k}\} > 0$. This is only true because we have a finite collection here, and this follows from ***compactness***.

Let $x \in K$, then $x \in N_{\varepsilon_{x_i}}(x_i)$ for some i and so for $y \in F$, $d(x, y) \leq d(x, x_i) + d(x_i, y)$ is $d(x_i, y) \geq d(x, y) - d(x_i, x) \geq 2\varepsilon_{x_i} - \varepsilon_{x_i} = \varepsilon_{x_i} \geq \varepsilon$. So $d(x, y) \geq \varepsilon$ for all $x \in K$ and $y \in F$ and hence $d(K, F) \geq \varepsilon > 0$.

Proof 2: Suppose $d(K, F) = 0$, then there is $(x_i, y_i) \in K \times F$ so that $\lim_i d(x_i, y_i) = 0$. Since K is compact there is an $x \in K$ and subsequence x_{i_j} so that $\lim_j x_{i_j} = x$. But then $\lim_j y_{i_j} = x$, so $x \in F$. You must use ***sequential compactness*** here.

c) Can the assumption that K is compact be replaced by K closed in (b)? That is, is there a metric space (X, d) and closed sets A, B so that $A \cap B = \emptyset$ and yet $d(A, B) = 0$?

It is simple to see that compactness is required here.

Example 1: Consider $A = \{(x, 1/x) \mid x > 0\}$ and $B = \{(x, -1/x) \mid x > 0\}$. Clearly, $d(A, B) = 0$ and as $x \mapsto 1/x$ is continuous, A and B are closed.

Example 2: K closed and bounded also does not suffice, but to see this, we must look into a space where closed and bounded does not imply compact. We don't have to look far. Consider $X = (0, 1)$, the open unit interval. Here X is closed (in X) and bounded but not compact. Consider $F = \{1/i \mid i > 0, i \in \mathbb{N}, \text{ and even}\}$ and $K = \{1/i \mid i \in \mathbb{N} \text{ and odd}\}$. Clearly, $K \cap H = \emptyset$ yet $d(1/i, 1/i + 1) \to 0$ so $d(F, K) = 0$.

Note: It is however true that for A, B closed with $A \cap B = \emptyset$, there are U, V open so that $A \supseteq U$, $B \supseteq V$, and $U \cap V = \emptyset$. This is the ***normality*** property.

RECALL: In a metric space (X, d), $\operatorname{diam}(A) = \sup\{d(a, b) \mid a, b \in A\}$.

Question 2 (10 pts)**.** Let (X, d) be a metric space prove or disprove each of the following:

a) $\operatorname{diam}(A) = \operatorname{diam}(\operatorname{Cl}(A))$.

Take $\varepsilon > 0$, let $a, a' \in \operatorname{Cl}(A)$. There is $b, b' \in A$ so that $d(b, a) < \varepsilon$ and $d(b', a') < \varepsilon$. By the triangle inequality $d(a, a') \leq d(b, b') + d(a, b) + d(a', b') < \operatorname{diam}(A) + 2\varepsilon$. So $\operatorname{diam}(\operatorname{Cl}(A)) \leq \operatorname{diam}(A) + 2\varepsilon$. Since ε is arbitrary, $\operatorname{diam}(\operatorname{Cl}(A)) \leq \operatorname{diam}(A)$.

b) $\operatorname{diam}(A) = \operatorname{diam}(\operatorname{Int}(A))$.

This is trivially false. For example in $\mathbb{R}$ let $A = \{a, b\}$, then $\operatorname{diam}(A) = |b - a|$, but $\operatorname{Int}(A) = \emptyset$, so $\operatorname{diam}(\operatorname{Int}(A)) = 0$.

Question 3 (10 pts)**.** Let (X, d) be a metric space and $(x_i)_{i \in \mathbb{N}}$ and $(y_i)_{i \in \mathbb{N}}$ be two Cauchy sequences. Show that $(d(x_i, y_i))_{i \in \mathbb{N}}$ converges.

$d(x_i, y_i) \leq d(x_i, x_j) + d(x_j, y_j) + d(y_j, y_j')$ so that $d(x_i, y_i) - d(x_j, y_j) \leq d(x_i, x_j) + d(y_i, y_j)$. Swapping the rolls of i and j gives $d(x_j, y_j) - d(x_i, y_i) \leq d(x_i, x_j) + d(y_i, y_j)$ so we get

$$|d(x_j, y_j) - d(x_i, y_i)| \leq d(x_i, x_j) + d(y_i, y_j)$$

Now for $\varepsilon > 0$ take N so that $d(x_i, x_j) < \varepsilon/2$ and $d(y_i, y_j) < \varepsilon/2$ for $i, j > N$, then for $i, j > N$

$$|d(x_j, y_j) - d(x_i, y_i)| \leq d(x_i, x_j) + d(y_i, y_j) < \varepsilon.$$

so $(d(x_i, y_i))$ is a Cauchy sequence.

Question 4 (Is supremum "linear"; 10 pts)**.** For $A, B \subseteq \mathbb{R}$, is it true that

i) $\sup(\alpha A) = \alpha \sup(A)$ for $\alpha \geq 0$, and

This is true. This is clear if $\alpha = 0$, so assume $\alpha > 0$. There are two things to show, namely, (1) $\sup(\alpha A) \leq \alpha \sup(A)$ and (2) $\sup(\alpha A) \geq \alpha \sup(A)$. This means that we must show $(1')$ $\alpha \sup(A)$ is an upper bound of αA and $(2')$ $\frac{1}{\alpha} \sup(\alpha A)$ is an upper bound of A. $(2')$ is equivalent to $\sup(\alpha A)$ is an upper bound of αA, but this is clear.

For (1'), let $a \in A$, then $a \leq \sup(A)$ and so $\alpha a \leq \alpha \sup(A)$. Thus $\alpha A \leq \alpha \sup(A)$ and we get that $\alpha \sup(A)$ is an upper bound of αA.

ii) $\sup(A + B) = \sup(A) + \sup(B)$.

Again there are two things to show. (1) $\sup(A+B) \geq \sup(A) + \sup(B)$ and (2) $\sup(A + B) \leq \sup(A) + \sup(B)$. As before, (2) is equivalent to $(2')$ $\sup(A) + \sup(B)$ is an upper bound on $A + B$ and this is clear since if $a \in A$ and $b \in B$, then $\sup(A) + \sup(B) \geq a + b$.

For (1), suppose $\sup(A) + \sup(B) > \sup(A + B)$, then $\sup(A) + b > \sup(A + B)$ for some $b \in B$. Applying this logic a second time we get $a \in A$ such that $a + b > \sup(A + B)$. this is absurd, so it must be that $\sup(A) + \sup(B) \leq \sup(A + B)$.

Question 5 (Compact sets get crowded; 10 pts)**.** Show that if X is compact, then for any $\varepsilon > 0$, there is $N > 0$ so that for all $S \subset X$ with $|S| \geq N$, there are two points in S whose distance is $< \varepsilon$.

Proof 1: Consider the open cover $\mathcal{O} = \{N_{\frac{\varepsilon}{2}}(x) \bigm| x \in X\}$ of X. Let $\mathcal{O}' = \{N_{\frac{\varepsilon}{2}}(x_i) \bigm| i = 1, \ldots, N\}$ be a finite open subcover. Let $S \subset X$ with $|S| > N$. By the pigeonhole principle, there are at least two elements $s, s' \in S$ which must fall in the same nbhd $N_{\frac{\varepsilon}{2}}(x_i)$ for some i, so that $d(s, s') \leq d(s, x_i) + d(x_i, s') < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$.

Proof 2: Fix $\varepsilon > 0$. Take a maximal set $S = \{x_1, \ldots, x_m\}$ so that $d(x_i, x_j) \geq \varepsilon/2$. Note that there cannot be an infinite set of such points all $\varepsilon/2$ apart, as then you would have an infinite sequence with no convergent subsequence, contradicting sequential compactness. The maximality of S means that $X \subseteq \bigcup_{i=1}^m N_{\varepsilon/2}(x_i)$ else there is $x \in X$ with $d(x, x_i) \geq \varepsilon/2$ for all $i = 1, \ldots, n$, contradicting the maximality of S.

Let S' be any set of size $m+1$. Since each $x \in S' \in N_{\varepsilon/2}(x_i)$ for some $i = 1, \ldots, m$ by the pigeon-hole property,there is $x, x' \in S'$ so that $x, x' \in N_{\varepsilon/2}(x_i)$, but the $d(x, x') \leq \varepsilon$. Since S' was arbitrary, this is what we wanted to prove.

Note here that one set of size m maximal for $\varepsilon/2$ means that all sets of $n > m$ have points within ε. A maximally $\varepsilon/2$-*uncrowded* set of size m, implies all sets of size $n > m$ are ε-crowded.