Math 571 - Homework 4

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Problem 0.1 (R:3:8)**.** Suppose $\sum_n a_n$ converges and (b_n) is monotonic and bounded, show that $\sum_n a_n b_n$ converges.

Use the partial sum trick. Define $A_i = \sum_{j\leq i} a_j$ and define $A_{-1} = 0$ so that $a_i = A_i - A_{i-1}$. Then

$$\begin{aligned}
\sum_{i=m}^{n} a_i b_i &= \sum_{i=m}^{n} (A_i - A_{i-1}) b_i = \sum_{i=m}^{n} A_i b_i - \sum_{i=m}^{n} A_{i-1} b_i \\
&= \sum_{i=m}^{n} A_i b_i - \sum_{i=m}^{n} A_{i-1} b_i = \sum_{i=m}^{n} A_i b_i - \sum_{i=m-1}^{n-1} A_i b_{i+1} \\
&= \sum_{i=m}^{n-1} A_i (b_i - b_{i+1}) + A_n b_n - A_{m-1} b_m
\end{aligned}$$

The last bit we can play with:

$$A_n b_n - A_{m-1} b_m = A_n b_n - A_{m-1} b_m + A_n b_m - A_n b_m = A_n (b_n - b_m) + (A_n - A_{m-1}) b_m$$

So we get

$$\left|\sum_{i=m}^{n} a_i b_i\right| \leq \sum_{i=m}^{n-1} |A_i||b_i - b_{i+1}| + |A_n||b_n - b_m| + |A_n - A_{m-1}||b_m|$$

We know $|A_i| < A$ and $|b_i| < B$ for all i for some fixed bounds A and B so

$$\begin{aligned}
\left|\sum_{i=m}^{n} a_i b_i\right| &\leq \sum_{i=m}^{n-1} A|b_i - b_{i+1}| + A|b_n - b_m| + |A_n - A_{m-1}|B \\
&= A\sum_{i=m}^{n-1} |b_i - b_{i+1}| + A|b_n - b_m| + |A_n - A_{m-1}|B \\
&= A\left|\sum_{i=m}^{n-1} b_i - b_{i+1}\right| + A|b_n - b_m| + |A_n - A_{m-1}|B \qquad \text{(since } b_i \text{ is monotonic.)} \\
&= A|b_n - b_m| + A|b_n - b_m| + |A_n - A_{m-1}|B
\end{aligned}$$

Since $\{A_i\}$ and $\{b_i\}$ are Cauchy sequences, it follow that $\{\sum_{i=m}^{n} a_i b_i\}$ is a Cauchy sequence and so $\sum_{i=0}^{\infty}$ converges.

Problem 0.2 (R:3:9)**.** Find the radius of convergence of the following power series.

a) $\sum_n n^3 z^n$

$\limsup \frac{(n+1)^3|z|^{n+1}}{n^3|z|^n} = |z|$ so this converges absolutely for $|z| < 1$, that is, $R = 1$.

b) $\sum_n \frac{2^n}{n!} z^n$

$\frac{2^{n+1}|z|^{n+1}}{(n+1)!} \Big/ \frac{2^n|z|^n}{n!} = \frac{2^{n+1}}{2^n} \cdot \frac{n!}{(n+1)!} \cdot |z| = \frac{2}{n+1}|z| \to 0$. So the radius of convergence is $R = \infty$.

c) $\sum_n \frac{2^n}{n^2} z^n$

$\frac{2^{n+1}|z|^{n+1}}{(n+1)^2} \Big/ \frac{2^n|z|^n}{n^2} = \frac{2^{n+1}}{2^n} \cdot \frac{n^2}{(n+1)^2} \cdot |z| = 2 \cdot \frac{n^2}{(n+1)^2} \cdot |z| \to 2|z|$. This converges absolutely for $|z| < \frac{1}{2}$.

d) $\sum_n \frac{n^3}{3^n} z^n$

$\frac{(n+1)^3|z|^{n+1}}{3^{n+1}} \Big/ \frac{n^3|z|^n}{3^n} = \frac{(n+1)^3}{n^3} \cdot \frac{1}{3} \cdot |z| = \frac{1}{3} \cdot \frac{(n+1)^3}{n^3} \cdot |z| \to \frac{1}{3}|z|$. This converges absolutely for $|z| < 3$.

Problem 0.3 (R:3:11)**.** Suppose $a_n > 0$, $s_n = \sum_{i=1}^n a_i$, and $\sum_i a_i = \lim_i s_i$ diverges.

a) Show that $\sum_i \frac{a_i}{a_i+1}$ diverges.

Let $t_n = \sum_{i=1}^n \frac{a_i}{a_i+1}$. Suppose $\sum_i \frac{a_i}{a_i+1}$ does converge, then $\lim_i \frac{a_i}{a_i+1} = 0$ and from this it is easy to see that $\lim_i a_i = 0$, for example, $\lim_i \frac{a_i}{a_i+1} = \lim_i \frac{1}{1+1/a_i} = 0$ so $\lim_i \frac{1}{a_i} = \infty$. This means that $r_n = \sum_{i=1}^n \frac{a_i^2}{a_i+1}$ also converges.

Clearly, $a_i - \frac{a_i}{a_i+1} = \frac{a_i^2}{a_i+1}$, so that $s_n - t_n = r_n$ and thus $s_n = t_n + r_n$. But this would mean that $\lim_i s_i = \lim_i (t_i + r_i) = \lim_i t_i + \lim_i r_i$ exists, and is finite. This contradicts the divergence of $\sum_i a_i$.

b) Show that $\frac{a_N}{s_N} + \frac{a_{N+1}}{s_{N+1}} + \cdots \frac{a_{N+k}}{s_{N+k}} \geq 1 - \frac{s_N}{s_{N+k}}$ and deduce that $\sum_i \frac{a_i}{s_i}$ diverges.

$$\begin{aligned}\frac{a_N}{s_N} + \frac{a_{N+1}}{s_{N+1}} + \cdots \frac{a_{N+k}}{s_{N+k}} &\geq \frac{a_N}{s_{N+k}} + \frac{a_{N+1}}{s_{N+k}} + \cdots \frac{a_{N+k}}{s_{N+k}} \\ &= \frac{s_{N+k} - s_N}{s_{N+k}} = 1 - \frac{s_N}{s_{N+k}}\end{aligned}$$

Let $t_n = \sum_{i=1}^n \frac{a_i}{s_i}$, then we have here that $(t_i)_{i\in\mathbb{N}}$ fails to be a Cauchy sequence and hence fails to converge. So $\sum_i \frac{a_i}{s_i}$ diverges.

c) Show that $\frac{a_N}{s_N^2} \leq \frac{1}{s_{N-1}} - \frac{1}{s_N}$ and deduce that $\sum_i \frac{a_i}{s_i^2}$ converges.

$\frac{a_N}{s_N^2} \leq \frac{a_N}{s_{N-1}s_N} = \frac{a}{s_{N-1}} - \frac{1}{s_N}$. But then $\sum_2^n \frac{a_i}{s_i^2} \leq \sum_{i=2}^n \left(\frac{1}{s_{i-1}} - \frac{1}{s_i}\right) = \frac{1}{s_1} - \frac{1}{s_N} \to \frac{1}{s_1}$. So $\sum_i \frac{a_i}{s_i^2}$ converges.

Problem 0.4 (R:3:16(18)*)**.** Fix $\alpha > 1$ and $x_1 > \sqrt{\alpha}$, define

$$x_{n+1} = \frac{1}{2}\left(x_n + \frac{\alpha}{x_n}\right)$$

a) Prove that (x_n) decreases monotonically and $\lim_{n\to\infty} x_n = \sqrt{\alpha}$.

A little argument and trick shows that $x_n^2 \geq \alpha$. The "trick" is to see that

$$\frac{x_n^2}{\alpha} + \frac{\alpha}{x_n^2} \geq 2$$

For this we need $x_n^4 + \alpha^2 > 2\alpha x_n^2$, but this is just $x_n^4 + \alpha^2 - 2\alpha x_n^2 = (x_n^2 - \alpha)^2 \geq 0$.

Now we can use this as follows:

$$x_{n+1}^2 = \frac{1}{4}\left(x_n^2 + 2\alpha + \frac{\alpha^2}{x_n^2}\right) = \frac{1}{4}\left(\frac{x_n^2}{\alpha} + 2 + \frac{\alpha}{x_n^2}\right) \cdot \alpha \geq \frac{1}{4}(2+2) \cdot \alpha = \alpha$$

Let's see now that $x_{n+1} \leq x_n$, this will use the above:

$$x_{n+1} = \frac{1}{2}x_n + \frac{1}{2}\left(\frac{\alpha}{x_n}\right) = \frac{1}{2}x_n + \frac{1}{2}\left(\frac{\alpha}{x_n^2}\right) x_n \leq \frac{1}{2}x_n + \frac{1}{2}x_n = x_n$$

We know $\gamma = \lim_{n\to\infty} x_n$ and by the standard limit laws

$$\gamma = \lim_{n\to\infty} x_{n+1} = \lim_{n\to\infty} \frac{1}{2}\left(x_n + \frac{\alpha}{x_n}\right) = \frac{1}{2}\left(\gamma + \frac{\alpha}{\gamma}\right)$$

Now we have:

$$2\gamma = \gamma + \frac{\alpha}{\gamma}$$

and hence $\gamma^2 = \alpha$, as desired.

b) Let $\varepsilon_n = x_n - \sqrt{\alpha}$ be the error at the n^{th} term, show that

$$\varepsilon_{n+1} = \frac{\varepsilon_n^2}{2x_n} < \frac{\varepsilon_n^2}{2\sqrt{\alpha}}$$

Setting $\beta = 2\sqrt{\alpha}$, gives

$$\varepsilon_{n+1} < \beta\left(\frac{\varepsilon_1}{\beta}\right)^{2^n}$$

$$\begin{aligned}
\varepsilon_{n+1} &= x_{n+1} - \sqrt{\alpha} = \frac{1}{2}\left(x_n + \frac{\alpha}{x_n}\right) - \sqrt{\alpha} \\
&= \frac{x_n^2 + \alpha - 2x_n\sqrt{\alpha}}{2x_n} = \frac{(\varepsilon_n + \sqrt{\alpha})^2 + \alpha - 2(\varepsilon_n + \sqrt{\alpha})\sqrt{\alpha}}{2x_n} \\
&= \frac{\varepsilon_n^2 + 2\varepsilon_n\sqrt{\alpha} + \alpha + \alpha - 2(\varepsilon_n + \sqrt{\alpha})\sqrt{\alpha}}{2x_n} = \frac{\varepsilon_n^2}{2x_n}
\end{aligned}$$

Since $x_n > \sqrt{\alpha}$ we do have

$$\varepsilon_{n+1} < \frac{\varepsilon_n^2}{2\sqrt{\alpha}} = \frac{\varepsilon_n^2}{\beta} = \beta\left(\frac{\varepsilon_n}{\beta}\right)^2 < \beta\left(\frac{\varepsilon_{n-1}}{\beta}\right)^{2^2} \cdots < \beta\left(\frac{\varepsilon_{n-(n-1)}}{\beta}\right)^{2^n} = \beta\left(\frac{\varepsilon_1}{\beta}\right)^{2^n}$$

c) Choose a number $\alpha > 3$ find a bound for how many terms are need to compute $\sqrt{\alpha}$ correct to 20 decimal places where x_1 is chosen minimally so that $x_1^2 > \alpha$. Prove your answer and do the computation. You might use Python or MATLAB.

This will vary, I'll take $\alpha = 145$ and hence $x_1 = 13$. $\beta = 2\sqrt{145} < 26$ and $\beta > 24$ and $\epsilon_1 = 13 - \sqrt{145} < 1$ and so

$$\varepsilon_{n+1} < \beta \left(\frac{\varepsilon_1}{\beta}\right)^{2^n} < (26)\left(\frac{1}{24}\right)^{2^n}$$

So it would suffice to have n such that $(26)(24)^{-2^n} < 10^{-21}$, this is the same as $(1/26)(24)^{2^n} > 10^{21}$, a little playing shows $n = 4$ suffices, so take $n + 1 = 5$ terms. Testing with Python get

$$\begin{aligned}
x_1 &= 13 \\
\varepsilon_1 &= 0.958 \\
x_2 &= \mathbf{12.07}6923076923076923076923076923076923076923 \\
\varepsilon_2 &= 0.0353 \\
x_3 &= \mathbf{12.041}646251837334639882410583047525722684958353748 \\
\varepsilon_3 &= 0.0000517 \\
x_4 &= \mathbf{12.041594578}903165021043862247214042390984013945153 \\
\varepsilon_4 &= 1.11e - 10 \\
x_5 &= \mathbf{12.04159\,45787\,92295\,48012}\,8751430185805358673288917241 \\
\varepsilon_5 &= 5.10e - 22
\end{aligned}$$

The actual value is:

$$\sqrt{145} = \mathbf{12.04159\,45787\,92295\,48012}\,8241030378608052425352405054...$$

d) Replace the recursion above by

$$x_{n+1} = \frac{p-1}{p}\,x_n + \frac{\alpha}{p}\,x_n^{-p+1}$$

Discuss the behavior of (x_n) under suitable conditions. Don't bother trying to compute a recursive expression for ε_n in this case, but do prove your claims.

Assume $x_n^p > \alpha$, then

$$x_{n+1} = \left(\frac{p-1}{p}\right)x_n + \left(\frac{1}{p}\right)\left(\frac{\alpha}{x_n^p}\right)x_n \le x_n$$

Now we must show that $x_{n+1} > \alpha^{1/p}$ given that $x_n > \alpha$. A little algebra gives:

$$\begin{aligned}
x_{n+1} &= \frac{\alpha^{1/p}}{p}\left((p-1)\left(\frac{x_n}{\alpha^{1/p}}\right) + \left(\frac{\alpha^{1/p}}{x_n}\right)^{p-1}\right) \\
&= \frac{\alpha^{1/p}}{p}\left((p-1)z^{-1} + z^{p-1}\right)
\end{aligned}$$

It suffices to show that

$$(p-1)z^{-1} + z^{p-1} > p \tag{†}$$

for then we get $x_{n+1} = \alpha^{1/p}(1/p)((p-1)z^{-1} + z^{p-1}) > \alpha^{1/p}$. In order to verify (†) it suffices to show $(p-1) + z^p > zp$ which is equivalent to $z^p - 1 > (z-1)p$ so we may show

$$\frac{z^p - 1}{z - 1} = z^{p-1} + z^{p-1} + \cdots + 1 > p \tag{‡}$$

But as $z > 1$ we have $z^i > 1$ and so we have the sum of p-terms all ≥ 1 and where $p-1$ of them are > 1, so (‡) holds.

So the series x_n is monotonic decreasing and bounded below by $\alpha^{1/p}$. Let $\gamma = \lim_{n\to\infty} x_n$ we have

$$\gamma = \lim_{n\to\infty} x_n = \lim_{n\to\infty}\left(\frac{p-1}{p}x_n + \frac{1}{p}\frac{\alpha}{x_n^{p-1}}\right) = \frac{p-1}{p}\gamma + \frac{1}{p}\frac{\alpha}{\gamma^{p-1}}$$

This gives $p\gamma^p = (p-1)\gamma^p + \alpha$ which is the same as $\gamma^p = \alpha$.

Problem 0.5. Show that a normed vector space $(X, \|\cdot\|)$ is complete iff every absolutely summable series is summable.

The first thing to notice is that if $\{u_i\}$ is absolutely summable, then the *partial sums* $v_m = \sum_{i=0}^m$ form a C-seq, since for $m < j$

$$\|v_m - v_j\| = \|u_{m+1} + \cdots + u_j\| \leq \sum_{i=m+1}^{j} \|u_i\| \leq \sum_{i=m+1}^{\infty} \|u_i\| \to 0 \text{ as } m \to 0$$

Thus if X is complete, then $v_i \to v$ for some v and hence $\{u_i\}$ is summable.

Suppose now that absolutely summable $\implies$ summable. Let (u_i) be a C-seq. Choose a subsequence (u_{i_j}) where for $l \geq k$, $\|u_{i_l} - u_{i_k}\| < 2^{-(k+1)}$. Define $u'_k = u_{i_k} - u_{i_{k-1}}$ so that $v'_k = \sum_{l=0}^k u'_k = u_{i_k} - u_{i_0}$ where $\|v'_k\| \leq \sum_{l=0}^k \|u_{i_l} - u_{i_{l-1}}\| \leq \sum_{l=0}^k 2^{-(l+1)} < 1$. So the sequence (u'_k) is absolutely summable and summable, so $v'_k = u_{i_k} - u_{i_0} \to v = u - u_{i_0}$ and so $u = v + u_{i_0}$ is the limit of the sequence (u_{i_k}) and since (u_i) was a C-seq, $u_i \to u$. So we have X is complete.

It might seem that we almost proved that it $\lim_i \|u_i\|$ exists, then $\lim_i u_i$ exists. Did we? Is this true?

Problem 0.6 (R:3:21*)**.** Let E_n be a descending sequence of closed subsets in a complete metric space, i.e., $E_{n+1} \subseteq E_n$. Notice that $\operatorname{diam}(E_{n+1}) \leq \operatorname{diam}(E_n)$ so $\lim_n \operatorname{diam}(E_n) = \delta$ exists in $[0, \infty]$. In each of the following cases what are all of the possibilities for $\bigcap_n E_n$.

a) $\delta = \infty$.

It is easy to produce examples where $\bigcap E_n = \emptyset$, one I like was $E_n = \{i \in \mathbb{Z} \mid |i| \geq n\}$. It is also easy to create examples where $\bigcap_n E_n$ is non-empty, in particular let S be any closed set and let $F_n = E_n \cup S$, then F_n is closed, $\operatorname{diam}(F_n) = \infty$, and $\bigcap_n F_n = S$. Since $\bigcap_n F_n$ is closed this is all of the possibilities.

b) $0 < \delta < \infty$.

This is just as in the preceding case, in fact the examples from (a) can be directly mapped to examples on $[a, b]$ where $b - a = \delta$. To be more explicit take $E_n = \{a\} \cup \{a + 1/m \mid m \geq n\} \cup \{b - 1/m \mid m \geq n\} \cup \{b\}$, then set $F_n = E_n \cup S$ for any closed S. We get $\operatorname{diam}(F_n) = \delta$, F_n is closed, $\bigcap_n F_n = S$.

c) $\delta = 0$.

In this case $\bigcap E_n$ has exactly one element. Just take $x_i \in E_i$ for each i, then $i, j > n \implies d(x_i, x_j) < \operatorname{diam}(E_n)$. So (x_i) is a Cauchy sequence. This proves existence.

For uniqueness, suppose $x, y \in \bigcap E_n$. Then $d(x, y) < \operatorname{diam}(E_n)$ for all n, so $d(x, y) = 0$.

Problem 0.7 (R:3:23)**.** Let (X, d) be a metric space and $(x_i)_{i\in\mathbb{N}}$ and $(y_i)_{i\in\mathbb{N}}$ be two Cauchy sequences. Show that $(d(x_i, y_i))_{i\in\mathbb{N}}$ converges.

$$d(x_i, y_i) \leq d_(x_i, x_j) + d(x_j, y_j) + d(y_j, y_i)$$

So $d(x_i, y_i) - d(x_j, y_j) \leq d(x_i, x_j) + d(y_i, y_j)$. Similarly, $d(x_j, y_j) - d(x_i, y_i) \leq d(x_i, x_j) + d(y_i, y_j)$, so

$$|d(x_j, y_j) - d(x_i, y_i)| \leq d(x_i, x_j) + d(y_i, y_j)$$

But for any $\epsilon > 0$, there is $N > 0$ so that for all $i, j > N$, $d(x_i, x_j) < \epsilon/2$ and $d(y_i, y_j) < \epsilon/2$. So for all $\epsilon > 0$ there is $N > 0$ so that for all $i, j > N$,

$$|d(x_j, y_j) - d(x_i, y_i)| \leq d(x_i, x_j) + d(y_i, y_j) < \epsilon$$

Thus $(d(x_i, y_i))$ is a Cauchy sequence in $\mathbb{R}$.