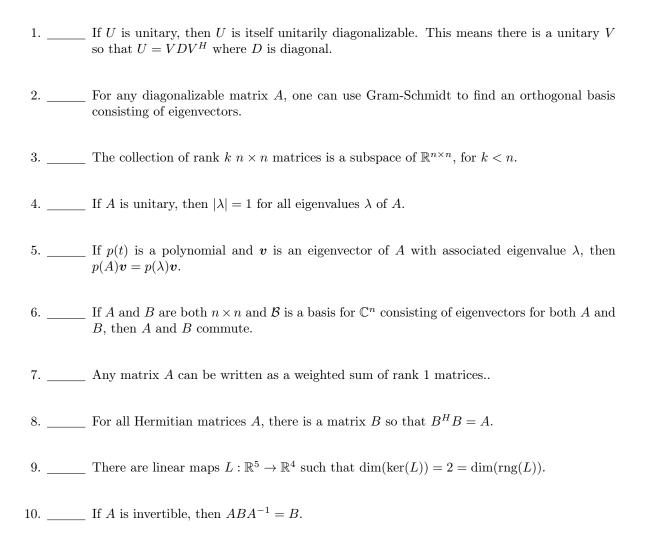
Exam 2

To avoid any confusion, unless specified otherwise, vector spaces are complex vector spaces, inner-products are complex inner-products, and matrices are complex matrices. The standard inner product is $\langle \boldsymbol{u}, \boldsymbol{v} \rangle = \boldsymbol{v}^H \boldsymbol{u} = \sum_{i=1}^n \bar{v}_i \boldsymbol{u}_i$.

Part I: True/False

Each problem is points for a total of 50 points. (5 points each.)

You do not need to justify the answers here, this is unlike the quizzes.



Part II: Computational (60 points)

P1. (15 points) Find B so that $B^2 = A$ where

$$A = \begin{bmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{bmatrix}$$

P2. (15 points) Find B so that $B^HB=A$ where A is from (1).

P3. (15 points) Find the best rank 2 approximation to A from (1) with respect to $\|\cdot\|_F$.

P4. (15 points) Let

$$A = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix},$$

find the characteristic polynomial and all eigenvalues, both real and complex. Explain why A is diagonalizable and compute A^{2020} . Note, I do not ask you to diagonalize A.

Part III: Theory and Proofs (45 points; 15 points each)

Pick three of the following four options. If you try all four, I will grade the first three, so if this is not what you intend, then just do three, or at least make it clear which I should grade.

P1. Let S be a fixed invertible $n \times n$ matrix. Let U be the set of $n \times n$ matrices that are diagonalized by S, that is $A = SD_AS^{-1}$ for some diagonal matrix A. Either prove that that U is a subspace of $\mathbb{C}^{n \times n}$ or show that U is not a subspace of $\mathbb{C}^{n \times n}$.

P2. Let A be a real $m \times n$ matrix and let $A^{\dagger} = V^T \Sigma^{\dagger} U$, where $A = U \Sigma V^T$ where U is $m \times m$, V is $n \times n$, both unitary, Σ is $m \times n$ and Σ^{\dagger} is $n \times m$ have the form

with $\sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_r > 0$.

Show: $\hat{x} = A^{\dagger} b$ is a least-squares solution to Ax = b.

Previously we used $\hat{\boldsymbol{x}} = (A^TA)^{-1}A^T\boldsymbol{b}$ for our least-squares solution, but we had the restriction that the columns of the "data" matrix A were independent, this guarantees that $\mathrm{NS}(A) = \mathrm{NS}(A^TA) = \{\mathbf{0}\}$. It is not hard to see that $A^\dagger = (A^TA)^{-1}A^T$ if A has linear independent columns.

Review the comments about Topic 5 DQ 2 in the Class Notes. Particularly point (2.) concerning what it means to be a least-squares solution to Ax = b.

P3. Prove that any complex inner-product $\langle \cdot, \cdot \rangle_V$ on a complex vector space V, there is a basis $\mathcal{U} = \{u_1, \dots, u_n\}$ so that

$$\langle oldsymbol{x}, oldsymbol{y}
angle_V = [oldsymbol{y}]_{\mathcal{U}}^H [oldsymbol{x}]_{\mathcal{U}}$$

In other words for any finite dimensional inner-product space, there is a choice of basis, so that with respect to that basis, the inner-product is represented by the standard inner-product.

Here, in case you need it, is the definition of an inner-product. All the notation here is as I always use it in my notes.

P4. Use the SVD to show that any square matrix A can be written as A = UP where U is unitary and P is Hermitian.