Part III: Theory and Proofs (30 points; 10 points each)

Choose three of the five options. If you try all five, I will grade the first three, not the best three. You must decide what should be graded.

This part is take-home. You should complete this work on your own without consulting websites, friends, the Math Center, etc.

Problem 6 (10 points)**.** Suppose S is an independent set of vectors from a vector space V, then

$$S \cup \{\boldsymbol{v}\} \text{ is dependent} \iff \boldsymbol{v} \in \operatorname{span}(S).$$

Problem 7 (10 points). Show that if $L : V \to W$ is linear and $\{L(\boldsymbol{v}_1), L(\boldsymbol{v}_2), L(\boldsymbol{v}_3)\}$ is linearly independent, then $\{\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3\}$ is linearly independent.

Problem 8 (10 points)**.** Suppose $A = \begin{bmatrix} \boldsymbol{a}_1 \, \boldsymbol{a}_2 \, \boldsymbol{a}_3 \, \boldsymbol{a}_4 \, \boldsymbol{a}_5 \end{bmatrix}$ is a 4×5 matrix and

$$\mathrm{NS}(A) = \mathrm{span}\{(-2, 1, 0, 0, 0), (5, 0, 2, 1, 0)\}$$

Find $\mathrm{rref}(A)$ and explain how you know that what you have found is $\mathrm{rref}(A)$.

Problem 9 (10 points)**.** Suppose A is a 5×5 matrix and $A^n = O$ for some n, then $A^5 = O$.

Problem 10 (10 points)**.** For A and B are $n \times n$ matrices. Show that

$$AB \text{ is invertible} \iff \text{both } A \text{ and } B \text{ are invertible}$$