Math 571 - Homework 1

Problem 1.1 (R:1:2*). Show that for any positive integer n, if n is not a perfect square, then $\sqrt{n}$ is irrational.

Problem 1.2 (R:1:4*). Let E be a non-empty subset of an ordered set $(S, <)$; suppose that α is a lower bound for E in S and β is an upper bound for E in S. Show that $\alpha \leq \beta$. Can $\alpha = \beta$? What happens if $E = \emptyset$?

Problem 1.3 (R:1:5). Let A be a non-empty set of real numbers bounded below. Let $-A = \{-a \mid a \in A\}$. Show that

$$\inf(A) = -\sup(-A)$$

Problem 1.4 (R:1:6). Fix $b > 1$.

(a) If n, m, p, q are integers, $n, q > 0$, and $r = m/n = p/q$, prove that

$$(b^m)^{1/n} = (b^p)^{1/q}.$$

Because of this defining b^r and $b^{m/n}$ for any $m/n = r$ provided $n > 0$ is well-defined.

(b) Prove that $b^{r+s} = b^r b^s$ if r and s are rational.

(c) If $x \in \mathbb{R}$, define $B(x) = \{b^t \bigm| t \in \mathbb{Q} \wedge t \leq x\}$. Prove that

$$b^r = \sup(B(r))$$

whenever r is rational.

Explain why it makes sense to define

$$b^x = \sup(B(x))$$

for every real x.

(d) Prove that $b^{x+y} = b^x b^y$ for every real x and y.

Problem 1.5 (R:1:7). Fix $b > 1$, $y > 1$, and show that there is a unique x so that $b^x = y$. This defines $\log_b(y) = x$.

(a) For any positive integer n, $b^n - 1 \geq n(b-1)$.

(b) $b - 1 \geq n(b^{1/n} - 1)$.

(c) If $t > 1$ and $n(t-1) > (b-1)$, then $b^{1/n} < t$.

(d) If w is such that $b^w < y$, then $b^{w+1/n} < y$ for large enough n. To see this, apply (c) to $t = y \cdot b^{-w}$.

(e) If $b^w > y$, then $b^{w-1/n} > y$ for large enough n.

(f) Let $A(y) = \{w \mid b^w < y\}$ and $x = \sup(A(y))$. Show that $y = b^x$.

(g) Show that x is unique.

Problem 1.6 (R:1:8)**.** Show that $\mathbb{C}$ can not be made into an ordered field.

Problem 1.7 (R:1:14*)**.** Show that for $w, z \in \mathbb{C}$

$$|w+z|^2 + |w-z|^2 = 2|w|^2 + 2|z|^2.$$

Use this to compute $|1+z|^2 + |1-z|^2$ given that $|z| = 1$.

Problem 1.8 (R:1:17)**.** Show that for $x, y \in \mathbb{R}^k(\mathbb{C}^k)$,

$$\|x+y\|_2^2 + \|x-y\|_2^2 = 2\|x\|_2^2 + 2\|y\|_2^2. \qquad \text{(Parallelogram Law)}$$

How does this generalize the Pythagorean theorem?

Problem 1.9 (R:1:18)**.** Show that if $k \geq 2$ and $x \in \mathbb{R}^k(\mathbb{C}^k)$, there is $y \in \mathbb{R}^k(\mathbb{C}^k)$, $y \neq 0$ such that $\langle x, y \rangle = 0$.