Homework 2 Partial Solutions

Homework 2 Problems:

Section 2.1

3.

(f)

$$\begin{aligned}
\det \begin{bmatrix} 2 & -1 & 2 \\ 1 & 3 & 2 \\ 5 & 1 & 6 \end{bmatrix} &= (2)\det \begin{bmatrix} 3 & 2 \\ 1 & 6 \end{bmatrix} - (-1)\det \begin{bmatrix} 1 & 2 \\ 5 & 6 \end{bmatrix} + 2\det \begin{bmatrix} 1 & 3 \\ 5 & 1 \end{bmatrix} \\
&= (2)((3)(6) - (1)(2)) + ((1)(6) - (5)(2)) + (2)((1)(1) - (5)(3)) \\
&= (2)(16) + (-4) + (2)(-14) = 0
\end{aligned}$$

Here is an alternate method

$$\begin{bmatrix} 2 & -1 & 2 \\ 1 & 3 & 2 \\ 5 & 1 & 6 \end{bmatrix} \xrightarrow[-R_2+R_3\to R_3]{} \begin{bmatrix} 2 & -1 & 2 \\ 1 & 3 & 2 \\ 4 & -2 & 4 \end{bmatrix}$$

The determinant of the right matrix is 0 since the rows are not independent and since a type III row operation was used the determinant of the left and right matrices are the same.

(g)

$$\begin{aligned}
\det \begin{bmatrix} 2 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 1 & 6 & 2 & 0 \\ 1 & 1 & -2 & 3 \end{bmatrix} &= 2\det \begin{bmatrix} 1 & 0 & 0 \\ 6 & 2 & 0 \\ 1 & -2 & 3 \end{bmatrix} - \det \begin{bmatrix} 0 & 1 & 0 \\ 1 & 6 & 2 \\ 1 & 1 & -2 \end{bmatrix} \\
&= (2)(1)(2)(3) - (-1)\det \begin{bmatrix} 1 & 2 \\ 1 & -2 \end{bmatrix} \\
&= 12 + ((1)(-2) - (1)(2)) \\
&= 12 - 4 = 8
\end{aligned}$$

(h) Here I will just use row operations and modify the determinant without specifying the row operations, it should be clear.

$$
\begin{aligned}
\det \begin{bmatrix} 2 & 1 & 2 & 1 \\ 3 & 0 & 1 & 1 \\ -1 & 2 & -2 & 1 \\ -3 & 2 & 3 & 1 \end{bmatrix} &= -\det \begin{bmatrix} -1 & 2 & -2 & 1 \\ 3 & 0 & 1 & 1 \\ 2 & 1 & 2 & 1 \\ -3 & 2 & 3 & 1 \end{bmatrix} \\
&= -\det \begin{bmatrix} -1 & 2 & -2 & 1 \\ 0 & 6 & -5 & 4 \\ 0 & 5 & -2 & 3 \\ 0 & -4 & 9 & -2 \end{bmatrix} \\
&= \det \begin{bmatrix} 6 & -5 & 4 \\ 5 & -2 & 3 \\ -4 & 9 & -2 \end{bmatrix} \\
&= (6)(4-27) + (-5)(10-36) + (-4)(-15+8) \\
&= -(6)(23) + (5)(26) + (4)(7) \\
&= 20
\end{aligned}
$$

4.

(a) $\det \begin{bmatrix} 3 & 5 \\ 2 & 4 \end{bmatrix} = (3)(4) - (2)(5) = 2.$

(b) This is a diagonal matrix so the determinant is simply the product of the diagonal elements:

$$\det \begin{bmatrix} 2 & 0 & 0 \\ 4 & 1 & 0 \\ 7 & 3 & -2 \end{bmatrix} = (2)(1)(-2) = -4$$

(c)

$$\det \begin{bmatrix} 3 & 0 & 0 \\ 2 & 1 & 1 \\ 1 & 2 & 2 \end{bmatrix} = 0$$

since two columns are the same.

(d) any matrix with a column/row of 0's has 0 determinant.

5.

$$
\begin{aligned}
\det \begin{bmatrix} a-x & b & c \\ 1 & -x & 0 \\ 0 & 1 & -x \end{bmatrix} &= (a-x)\det \begin{bmatrix} -x & 0 \\ 1 & -x \end{bmatrix} - \det \begin{bmatrix} b & c \\ 1 & -x \end{bmatrix} \\
&= (a-x)x^2 - (-bx-c) \\
&= -x^3 + ax^2 + bx + c
\end{aligned}
$$

6.

$$
\begin{aligned}
\det \begin{bmatrix} 2-\lambda & 4 \\ 3 & 3-\lambda \end{bmatrix} &= (2-\lambda)(3-\lambda) - 12 \\
&= 6 - 5\lambda + \lambda^2 - 12 \\
&= \lambda^2 - 5\lambda - 6 \\
&= (\lambda - 6)(\lambda + 1)
\end{aligned}
$$

So $\lambda_1 = -1$ and $\lambda_2 = 5$ are the two zeros.

11.

(a) This is false and this can be shown by an example:

$$0 = \det\begin{bmatrix}0 & 0\\0 & 0\end{bmatrix} = \det\left(\begin{bmatrix}1 & 0\\0 & 1\end{bmatrix} + \begin{bmatrix}-1 & 0\\0 & -1\end{bmatrix}\right) \neq \det\begin{bmatrix}1 & 0\\0 & 1\end{bmatrix} + \det\begin{bmatrix}-1 & 0\\0 & -1\end{bmatrix} = 1 + 1 = 2$$

(b) This can not be shown by example, you must use an arbitrary matrix!

$$\begin{aligned}
\det\left(\begin{bmatrix}a & b\\c & d\end{bmatrix}\begin{bmatrix}A & B\\C & D\end{bmatrix}\right) &= \det\begin{bmatrix}aA+bC & aB+bD\\cA+dC & cB+dD\end{bmatrix}\\
&= (aA+bC)(cB+dD) - (cA+dC)(aB+bD)\\
&= aAcB + aAdD + bCcB + bCdD - cAaB - cAbD - dCaB - dCbD\\
&= adAD - bcAD + bcBC - adBC\\
&= ad(AD-BC) - bc(AD-BC)\\
&= (ad-bc)(AD-BC)\\
&= \det\begin{bmatrix}a & b\\c & d\end{bmatrix}\det\begin{bmatrix}A & B\\C & D\end{bmatrix}
\end{aligned}$$

(c) This follows from (b) since

$$\det(AB) = \det(A)\det(B) = \det(B)\det(A) = \det(BA)$$

13.

$$\begin{aligned}
\det\begin{bmatrix}a_{11} & a_{12} & 0 & 0\cdots & \\ a_{21} & a_{22} & a_{23} & 0 & \cdots\\ 0 & a_{32} & a_{33} & a_{34} & \cdots\\ 0 & 0 & a_{43} & a_{44} & \ddots\\ \vdots & \vdots & 0 & \ddots & \ddots\end{bmatrix} &= a_{11}\det M_{11} - a_{12}\det M_{12}\\
&= a_{11}\det M_{11} - a_{12}a_{21}\det B\\
&= a_{11}\det M_{11} - a_{12}^2\det B
\end{aligned}$$

since $a_{12} = a_{21}$ by symmetry of A.

Section 2.2

3. Determine if the following are singular or non-singular.

(b) $A = \begin{bmatrix}3 & 1\\4 & 2\end{bmatrix}$ is clearly non-singular, for example $\det(A) = 6 - 4 \neq 0$.

(e) Let $A \sim B$ mean that A and B are similar, that is there is a sequence of elementary row operations leading from A to B, or equivalently $B = EA$ for some invertible E. We know

$$A \sim B \implies (A \text{ is singular} \iff B \text{ is singular})$$

Now

$$\begin{bmatrix}2 & -1 & 3\\-1 & 2 & -2\\1 & 4 & 0\end{bmatrix} \sim \begin{bmatrix}1 & 4 & 0\\0 & 6 & -2\\0 & 3 & -1\end{bmatrix}$$

It is clear that the RHS is singular, so the original matrix is also singular.

9. From the "Summary" at the top of page 97 in your text, or by using the properties of determinants as describes in the Class Notes, you know the effect of a row operation on a determinant. The point here was to compute the determinant using this fact without appealing to $\det(AB) = \det(A)\det(B)$.

E_1, E_2, and E_3 are elementary matrices corresponding to Type I, II, and III row operations respectively. E_2 comes from multiplying a row by 3. If $\det(A) = 6$ compute the following:

(a) Since E_1A is the result of a Type I operation on A:

$$\det(E_1A) = -\det(A) = -6$$

The above was all that was asked of you. It is however important that you note that

$$\det(E_1) = \det(E_1I) = -\det(I) = -1$$

so

$$\det(E_1A) = -\det(A) = \det(E_1)\det(A)$$

In this way you are proving a specific instance of $\det(EA) = \det(E)\det(A)$.

(b) Since E_2A is the result of a Type II operation on A:

$$\det(E_2A) = 3 \cdot \det(A) = 18$$

Note that

$$\det(E_2) = \det(E_2I) = 3 \cdot \det(I) = -3$$

so

$$\det(E_2A) = 3 \cdot \det(A) = \det(E_2)\det(A)$$

(c) Since E_1A is the result of a Type III operation on A:

$$\det(E_3A) = \det(A) = 6$$

Note that

$$\det(E_3) = \det(E_3I) = \det(I) = 1$$

so

$$\det(E_3A) = \det(A) = \det(E_3)\det(A)$$

(d)

$$\det(AE_1) = \det((AE_1)^T) = \det(E_1)^T A^T) - = \det(A^T)$$

Since $E_1^T = E_1$ and (a).

(e) By (a)

$$\det(E_1E_1) = -\det(E_1) = -\det(E_1I) = -(-\det(I)) = -(-1) = 1$$

(f) By (a)

$$\begin{aligned}
\det(E_1E_2E_3) &= -\det(E_2E_3) && \text{(by (a))} \\
&= -(3 \cdot \det(E_3)) && \text{(by (b))} \\
&= -(3 \cdot \det(E_3I)) \\
&= -(3 \cdot \det(I)) && \text{(by (c))} \\
&= -(3 \cdot 1) = -3
\end{aligned}$$

12.

$$\det\begin{bmatrix}1 & x_1 & x_1^2\\ 1 & x_2 & x_2^2\\ 1 & x_3 & x_3^2\end{bmatrix} = \det\begin{bmatrix}1 & x_1 & x_1^2\\ 0 & x_2-x_1 & x_2^2-x_1^2\\ 0 & x_3-x_1 & x_3^2-x_1^2\end{bmatrix}$$
$$= \det\begin{bmatrix}x_2-x_1 & x_2^2-x_1^2\\ x_3-x_1 & x_3^2-x_1^2\end{bmatrix}$$
$$= (x_2-x_1)(x_3-x_1)\det\begin{bmatrix}1 & x_2+x_1\\ 1 & x_3+x_1\end{bmatrix}$$
$$= (x_2-x_1)(x_3-x_1)((x_3-x_1)-(x_2-x_1))$$
$$= (x_2-x_1)(x_3-x_1)(x_3-x_2)$$

Clearly, V is non-singular whenever x_1, x_2, and x_3 are all distinct.

Section 2.3

3. and 4. If $A^{-1} = [b_1|b_2|b_3]$, then $AA^{-1} = [Ab_1|Ab_2|Ab_3] = I$. So $Ab_3 = \begin{bmatrix}0\\0\\1\end{bmatrix}$ and from Cramer's rule we have

$$b_3(1) = (A^{-1})_{13} = \frac{\det\begin{bmatrix}0 & 2 & 1\\ 0 & 4 & 3\\ 1 & 2 & 2\end{bmatrix}}{\det A} = 1/2 \qquad b_3(2) = (A^{-1})_{23} = \frac{\det\begin{bmatrix}1 & 0 & 1\\ 0 & 0 & 3\\ 1 & 1 & 1\end{bmatrix}}{\det A} = -3/4$$

$$b_3(3) = (A^{-1})_{33} = \frac{\det\begin{bmatrix}1 & 2 & 0\\ 0 & 4 & 0\\ 1 & 2 & 1\end{bmatrix}}{\det A} = 1$$

So $b_3 = \begin{bmatrix}1/2\\ -3/4\\ 1\end{bmatrix}$. You can check this with MATLAB and see

$$A^{-1} = \begin{bmatrix}1/2 & -1/2 & 1/2\\ 3/4 & 1/4 & -3/4\\ -1 & 0 & 1\end{bmatrix}$$

6. $(\det A)I = A \operatorname{adj} A = \mathbf{0}$.

7. Consider solving $I\mathbf{x} = \mathbf{b}$, then you get $x_i = \det B_i / \det I = \det B_i$. But $I\mathbf{x} = \mathbf{x}$ for all $\mathbf{x}$ so $\mathbf{x} = \mathbf{b}$ and $b_i = \det B_i$.

13. Assume $Q^T = Q^{-1}$. Then

$$Q(i,j) = Q^T(j,i) = Q^{-1}(j,i) = \frac{1}{\det(Q)} \cdot Q_{i,j}$$

2.3 #16 This is just a computation, $\mathbf{x} \times \mathbf{y} = A_x\mathbf{y}$ is as follows:

Recall (one definition of the cross-product):

$$
\begin{aligned}
\boldsymbol{x} \times \boldsymbol{y} &= \det \begin{bmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \end{bmatrix} \\
&= (x_2 y_3 - y_2 x_3)\boldsymbol{i} - (x_1 y_3 - y_1 x_3)\boldsymbol{j} + (x_1 y_2 - y_1 x_2)\boldsymbol{k} \\
&= \begin{bmatrix} -x_3 y_2 + x_2 y_3 \\ x_3 y_1 - x_1 y_3 \\ -x_2 y_1 + x_1 y_2 \end{bmatrix}
\end{aligned}
$$

$$
\begin{bmatrix} 0 & -x_3 & x_2 \\ x_3 & 0 & -x_1 \\ -x_2 & x_1 & 0 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} -x_3 y_2 + x_2 y_3 \\ x_3 y_1 - x_1 y_3 \\ -x_2 y_1 + x_1 y_2 \end{bmatrix} = \mathbf{x} \times \mathbf{y}
$$

It is clear that $A_x^T = -A_x$ so that

$$
A_x^T \boldsymbol{y} = -A_x \boldsymbol{y} = -(\boldsymbol{x} \times \boldsymbol{y}) = \boldsymbol{y} \times \boldsymbol{x}
$$