

Math 571 - Homework 3

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Problem 3.1. Define a metric on \mathbb{Z} for each integer $n > 1$ as follows. Let $s \in \mathbb{Z}$ and define $e_n(s) = \max\{a \in \mathbb{N} \mid n^a \mid s\}$. Set $d_n(s, t) = n^{-e_n(s-t)}$ if $s \neq t$ and $d_n(s, s) = 0$.

- a) Show that $d_n : \mathbb{Z} \times \mathbb{Z} \rightarrow [0, 1]$ is a metric. (Look at (c) before proving the triangle inequality.)

Symmetry and reflexivity are trivial. The triangle inequality is dealt with below.

- b) Interpret the metric, for example what does it mean to say $d_n(s, t) < \delta$ (s and t are within δ of each other.)

$d_n(s, t) < d_n(s, t')$ iff there are more copies of n in the factorization of $s - t$ than of $s - t'$. So the closer t is to s the more factors of n there are in $s - t$.

Suppose $n^a \mid s, t$ and $s = n^a s'$ and $t = n^a t'$, then $e_n(s, t) = a + e_n(s', t')$.

- c) $d_n(s, t) \leq \max\{d_n(s, r), d_n(r, t)\}$. (d_n is an **ultrametric**.)

Suppose $\max\{d_n(s, r), d_n(s, t)\} = d_n(s, r)$. Let $a = e_n(s, r)$ so $n^a \mid s - r$ and $n^a \mid s - t$. Now $s - t = (s - r) + (r - t)$ so $n^a \mid s - t$ and hence $d_n(s, t) \leq \max\{d_n(s, r), d_n(r, t)\} = n^{-a}$.

Problem 3.2 (R:2:17). Consider all reals in $[0, 1]$ whose decimal expansion requires only the digits 3 and 5, no 0's, so there are infinitely many 3's and 5's. Call this set Y .

Notice that Y has no finite length decimals. For example, $0.33 = 0.33\bar{0} = 0.32\bar{9} \notin Y$.

Prove or disprove each of the following:

It is important to think a bit about the representation of reals in $[0, 1]$ in decimal form before jumping into this. clearly there is a map $\phi : 10^{\mathbb{N}} \rightarrow [0, 1]$ given by $\phi(x) = \sum_{i \in \mathbb{N}} x(i)10^{-(i+1)}$. So for example

$$(3, 1, 4, 1, 5, 9, \dots) \mapsto_{\phi} \frac{3}{10} + \frac{1}{10^2} + \frac{4}{10^3} + \frac{1}{10^5} + \dots = 0.314159$$

There is a very natural topology and metric on $10^{\mathbb{N}}$ the basic open sets are $[s] = \{x \mid x \supset s\}$ for $s \in 10^{<\mathbb{N}}$. The metric can be expressed as $d(x, y) = \frac{1}{10^{i+1}}$ where i is least so that $x(i) \neq y(i)$. (This is an ultrametric again). So $[s] = N_{10^{(-\text{len}(s))}}(x)$ for any $x \in [s]$. Note that this is related to but different from the metric on $[0, 1]$ given by $\rho(x, y) = |\sum_{i \in \mathbb{N}} (y(i) - x(i))10^{-(i+1)}|$ (the usual metric). In particular, $\rho(0.\overline{100}, 0.099) = 0$ so ρ is not a metric on $10^{\mathbb{N}}$. We could use $d'(x, y) = \sum_{i \in \mathbb{N}} |y(i) - x(i)|10^{-(i+1)}$ which looks more like ρ , but is still different since $d'(0.\overline{100}, 0.099) = \frac{1}{10} + \sum_{i=2}^{\infty} \frac{9}{10^i} = 0.1 + 0.1 = .2$. Anyway, d' records more info than we need in $10^{\mathbb{N}}$ where all we really care about is the first digit on which x and y disagree.

Trivially, each $[s]$ is clopen, since $[s]$ is open and

$$[s]^c = \bigcup \{t \mid t|n-2 = s|n-2 \wedge t(n-1) \neq s(n-1) \wedge n = \text{len}(s)\}$$

so $[s]^c$ is open. This means that $10^{\mathbb{N}}$ is **totally disconnected**, namely, $\{[s] \mid s \in 10^{<\mathbb{N}}\}$ is a base of clopen sets, so the only connected sets are singletons and the empty set.

Notice that $\phi|Y' : Y' \rightarrow Y$ where $Y' = \{3, 5\}^{\mathbb{N}} \subseteq 10^{\mathbb{N}}$ is bijective and continuous. In fact, Y and Y' are homeomorphic, so any topological property that Y has is shared by Y' and vice-versa.

a) Y is dense in $[0, 1]$?

It is clear that Y' is not dense in $10^{\mathbb{N}}$ since for example $[1] \cap Y' = \emptyset$. This can easily be turned into an argument that Y is not dense on $[0, 1]$, namely, $N_{10^{-1}}(0) \cap Y = \emptyset$, since the closest element in Y to 0 is $0.33333\ldots$.

b) Y nowhere dense in $[0, 1]$?

Again, it is trivial to see that $Y' = \{3, 5\}^{\mathbb{N}}$ is nowhere-dense in $10^{\mathbb{N}}$. Take any open set O and $[s] \subset O$. Then $[s0] \subset [s]$ and $[s0] \cap Y' = \emptyset$.

Again a variant of this works in $[0, 1]$. Let O be open, take $N_{\delta}(x) \subset O$. Say $x = 0.d_0d_1\cdots d_i\cdots$ get $y = 0.d_0\cdots d_i0\cdots$ where $10^{-(i+2)} < \delta/2$. Now consider $N_{10^{-(i+3)}}(y) \subset O$ and disjoint from Y .

c) Y is countable?

The standard diagonalization argument shows that Y (Y') is uncountable.

d) Y is closed?

Again it is trivial to see that Y' is closed in $10^{\mathbb{N}}$. We can see this in $[0, 1]$ with essentially the same argument. Suppose $x_i \in Y$ and $x_i \rightarrow x$. Then it is clear that for all n , $x \upharpoonright i \subset x \upharpoonright n$ for all sufficiently large i . Thus $x \in Y$.

e) Y is compact?

This is easy to show directly, but Y is a closed subset of a compact space, hence compact.

f) Y is perfect?

We have that Y is closed, so we just need to see that there are no isolated points. This is clear, let $x \in Y$ and let $x_i \in Y$ with $x_i \upharpoonright i = x \upharpoonright i$, then $x_i \rightarrow x$ so $x \in \text{Lim}(Y)$.

Problem 3.3 (R:2:20*). If E is connected is $\text{Cl}(E)$ and/or $\text{Int}(E)$ necessarily connected?

Of course, give a proof or a counterexample.

The easiest here is $\text{Int}(E)$. Take $E \subseteq \mathbb{R}^2$ to be $\text{Cl}(N_1((-1, 0))) \cup \text{Cl}(N_1((0, 1)))$. So E consists of two closed unit discs centered at $(-1, 0)$ and $(0, 1)$ that just “kiss” at $(0, 0)$. $\text{Int}(E)$ is the union of the interiors of the two discs, and these form their own separating sets. So if E is connected, $\text{Int}(E)$ **need not be** connected.

Suppose A, B witness that $\text{Cl}(E)$ is disconnected. Then

- i) $A' = A \cap E \neq \emptyset \neq B \cap E = B'$,
- ii) $A' \cup B' = E$, and
- iii) $A' \cap B' = \emptyset$.

So E must also be disconnected. (i) uses the following:

For any open O :

$$O \cap E \neq \emptyset \iff O \cap \text{Cl}(E) \neq \emptyset$$

(\implies) is trivial. For the other direction argue the contrapositive

$$O \cap E = \emptyset \implies O \cap \text{Cl}(E) = \emptyset$$

This is true since

$$O \cap E = \emptyset \implies O \subseteq \text{Int}(E^c) \implies O \cap (\text{Int}(E^c))^c = O \cap \text{Cl}(E) = \emptyset$$

So if E is connected, then $\text{Cl}(E)$ is connected.

Problem 3.4. Show that E is connected iff for all $p, q \in E$ there is a connected open relative to E set $A \subseteq E$ with $p, q \in A$.

The “only if” part is trivial since E is open in E , E is connected, and $p, q \in E$.

For the “if” part show the contrapositive. Suppose E is not connected, then there are open (relative to E) $A, B \subset E$ such that $A, B \neq \emptyset$ and $A \cap B = \emptyset$, and $A \cup B = E$. Let $p \in A$ and $q \in B$. Let $O \subset E$ be open in E with $p, q \in O$. Then $A' = A \cap O$ and $B' = B \cap O$ witness that O is not connected. So there is no open connected subset of E containing p and q .

Problem 3.5 (R:2:21*). Prove that every convex subset of \mathbb{R}^k is connected.

The original problem in Rudin is a four part problem with this being the last part. You might use the original problem as a hint/guide here.

Suppose G is our convex set and let $A, B \subset G$ be non-empty and open in G with $A \cup B = G$. Let $a \in A$ and $b \in B$. Then $\{ta + (1-t)b \mid t \in [0, 1]\} \subseteq G$. Look at $\{t \in [0, 1] \mid ta + (1-t)b \in A\} = A'$ and $B' = \{t \mid ta + (1-t)b \in B\}$.

Claim: A', B' witness that $[0, 1]$ is not connected.

The only thing that requires argument is that A' and B' are open in $[0, 1]$. Let $c = ta + (1-t)b \in A$. Then $B_\epsilon(c) \subset A$ for some $\epsilon > 0$. Consider, $(t+h)a + (1-(t+h))b = ta + (1-t)b + h(a-b) = c + h(a-b)$. If $|h| < \epsilon/|a-b| = \delta$, then $t+h \in A'$. This means that $(t-\delta, t+\delta) \cap [0, 1] \subseteq A'$ and so A' is open in $[0, 1]$.

Fact: $[0, 1]$ is connected. (Rudin 2.47)

Problem 3.6 (R:2:26). Let X be a metric space in which every infinite set has a limit. Show that X is compact.

I prove this in the notes. It is an important and very useful characterization of compactness in a metric space, namely, **sequential compactness**. I do not want you to reproduce the

proof I give. Use the hint from Rudin and try it the way he suggests. This builds on some problems you did last week.

Let \mathcal{O} be an open cover of X . Our goal is to produce a finite subcover of X . Problems 6 from homework 2 gives us that X has a countable base. Thus we can easily get a countable subcover $\mathcal{O}' \subseteq \mathcal{O}$, simply assign to each x a base set U_x and O_{U_x} so that $x \in U_x \subseteq O_{U_x} \in \mathcal{O}$. Since there are only countably many U_x , there are also only countably many $O_{U_x} \in \mathcal{O}$ required. So we may assume $\mathcal{O} = \{G_i \mid i \in \mathbb{N}\}$.

Assume there is no finite subcover from \mathcal{O} , then $\bigcup_{i=0}^n G_i$ fails to cover X and so $F_n = X - \bigcup_{i=0}^n G_i$ is closed and non-empty. Further, $F_{n+1} \subseteq F_n$ and by assumption $\bigcap_{i \in \mathbb{N}} F_i = \emptyset$. Let $x_i \in F_i$ for each i . By assumption there is $x \in \text{Lim}(\{x_i\}_{i \in \mathbb{N}})$, but then $x \in F_i$ for all i since $\{x_k\}_{k \geq i} \subseteq F_i$ and F_i is closed. This is a contradiction so the assumption that there is no finite subcover must be false.

Problem 3.7 (R:2:28). Show that every closed set, F , in a separable metric space can be written as $P = P \cup C$ where P is perfect (perhaps empty) and C is countable.

A different hint from Rudin's: I gave you a sort of hint in class, define $F' = F - \text{Iso}(F)$, recall $\text{Iso}(F)$ is the set of isolated points of F . F' is called the derivative of the set F . Argue that $\text{Iso}(F)$ is countable, in some natural sense F' is *closer to perfection*, since we have removed some isolated points. Notice that F' is closed. If you haven't reached perfection repeat the process. In this way you build a sequence of closed sets $F \supset F_1 \supset F_2 \cdots$ and countable sets C_i so that $F = \bigcap F_i \cup \bigcup C_i$. If $\bigcap F_i = F_\omega$ still has isolated points, continue! A transfinite recursion!

Proof 1: Continue as suggested in the hint. There is a strictly descending sequence $F_\alpha \supset F_\beta$ for $\alpha < \beta$ with the additional properties:

- For λ a limit, $F_\lambda = \bigcap_{\alpha < \lambda} F_\alpha$.
- $C_\alpha = \text{Iso}(F_\alpha)$ and $F_{\alpha+1} = F'_\alpha = F_\alpha - C_\alpha$.

Note that $O_\alpha = X - F_\alpha$ is a strictly increasing sequence of open sets and thus must be countable since $S \cap (O_{\alpha+1} - O_\alpha) \neq \emptyset$ for all α . So there is a countable β where $\text{Iso}(F_\beta) = \emptyset$. Similarly, since X is separable, $\text{Iso}(A)$ is countable for any $A \subseteq X$ so $C = \bigcup_{\alpha < \beta} C_\alpha$ is countable. Let $P = F_\beta$. By construction, this is closed and has no isolated points.

Proof 2: (Follow the text.) Let P be the set of condensation points of F . If $x \notin P$, then $x \in O$ for some open O with $O \cap F$ countable. We can choose this O from a countable base, \mathcal{B} and thus $C = \bigcup \{O \cap F \in \mathcal{B} \mid |O \cap P| \leq \aleph_0\}$. C is countable and $F = C \cup P$.

If $x \in P$, then x is a condensation point of P , not just a condensation point of F . So clearly there are no isolated points of P . If $x \in \text{Cl}(P)$, then $x \in F$ so either $x \in C$ or $x \in P$. If $x \in C$, then by construction, there is open nbhd O of x so that $O \cap P = \emptyset$, so it must be that $x \in P$ and thus P is closed.

Note: It is possible that C is dense in F so that $\text{Cl}(C) = F$. For example, consider Y_1 to be the subset of $[0, 1]$ consisting of all $\sum_{i=0}^{\infty} a_i 10^{-(i+1)}$ where $a_i \in \{2, 3\}$. This is different from Y in problem 2 in that this allows certain finite decimals, namely, $\sum_{i=0}^n a_i 10^{-(i+1)} = 0.a_1 a_2 \cdots a_n$. These are isolated and dense in Y_1 , so $Y_1 = Y \cup C$ and $\text{Cl}(C) = Y_1$.