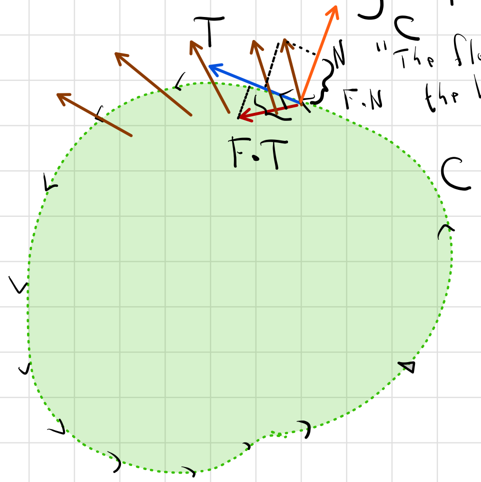
Green's Theorem and 2D versions of div/curl

Review: Given a vector field $F = (P, Q)$ with P, Q being continuously differentiable and a closed simple curve C that is piecewise smooth and positively oriented



$$\oint_C F \cdot dr = \iint_R \det\begin{bmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial y} \\ P & Q \end{bmatrix} dA$$

"The flow around the boundary"

$$= \iint_R \det\begin{bmatrix} \nabla \\ F \end{bmatrix} dA$$

$$T = \frac{r'}{|r'|} = \frac{\langle x', y' \rangle}{|r'|} \qquad N = \frac{\langle y', -x' \rangle}{|r'|}$$

$$T\,ds = dr = \langle dx, dy \rangle \qquad N\,ds = \langle dy, -dx \rangle$$

$$F \cdot dr = \langle P, Q \rangle \cdot \langle dx, dy \rangle \qquad F \cdot N\,ds = \langle P, Q \rangle \cdot \langle dy, -dx \rangle$$

$$= P\,dx + Q\,dy \qquad\qquad = -Q\,dx + P\,dy$$

$$= G \cdot dr$$

where $G = (-Q, P)$

$$\underbrace{\oint_C F \cdot N\,ds}_{\text{"The amount of flow out of R through the boundary."}} = \oint_C G \cdot dr = \iint_R \det\begin{pmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial y} \\ -Q & P \end{pmatrix} dA$$

$$= \iint_R \frac{\partial}{\partial x} P + \frac{\partial}{\partial y} Q \; dA$$

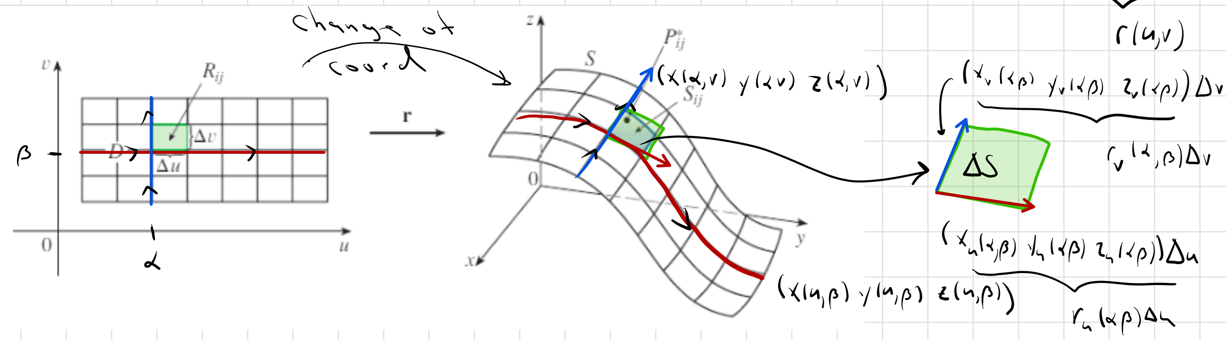
$$= \iint_R \operatorname{div}(F)\, dA$$

$\boxed{\text{Practice: Q5, Q8, Q9}}$

Surface Integrals

$$\mathbb{R}^2 \xrightarrow{r} \mathbb{R}^3$$

map the (u,v) plane to a surface $S: (u,v) \mapsto \underbrace{(x(u,v), y(u,v), z(u,v))}_{r(u,v)}$



$$\Delta S = |r_u \Delta u \times r_v \Delta v| = |r_u \times r_v| \Delta u \Delta v$$

$$\boxed{dS = |r_u \times r_v|\, du\, dv = |r_u \times r_v|\, dA}$$

$$\boxed{\iint_S f(x,y,z)\, dS = \iint_R f(x(u,v)\ y(u,v)\ z(u,v))\, |r_u \times r_v|\, dA}$$

$$R = r^{-1} S = \{ (u,v) \mid r(u,v) \in S \}$$

Practice: Q11

Special Case $\quad r: x = x,\ y = y \quad z = z(x,y) \qquad r_x \times r_y = \det\begin{bmatrix} i & j & k \\ 1 & 0 & z_x \\ 0 & 1 & z_y \end{bmatrix}$

$$r_x = (1, 0, z_x) \qquad r_y = (0, 1, z_y) \qquad = -z_x \hat{i} - z_y \hat{j} + \hat{k}$$

$$|r_x \times r_y| = (z_x^2 + z_y^2 + 1)^{1/2}$$

$$\boxed{\iint_S f(x,y,z)\, dS = \iint_R f(x,y,z(x,y)) \left(\left(\frac{dz}{dx}\right)^2 + \left(\frac{dz}{dy}\right)^2 + 1 \right)^{1/2} dA}$$

Orientable Surfaces

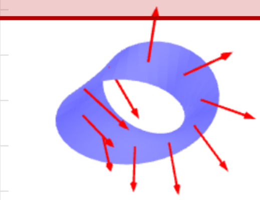
A surface is orientable if there is a continuous $N: S \to \mathbb{R}^3$ s.t. $N(x,y,z)$ is a unit normal to S

If S is given by $r: \underset{\subseteq \mathbb{R}^2}{R} \to S$ then $\vec{n} = \dfrac{r_u \times r_v}{|r_u \times r_v|}$ is a good choice

if S is orientable. (Must be checked.)

See example $(u,v) \mapsto (\cos(u)(1 + v\sin(u/2)),\ \sin(u)(1 + v\sin(u/2)),\ v\cos(u/2))$

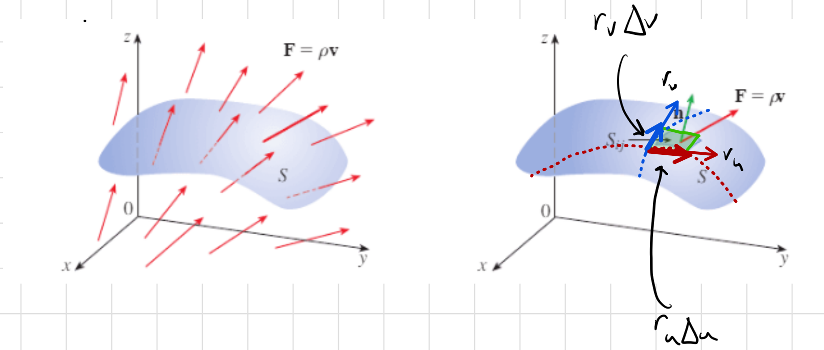
on $[0, 2\pi] \times [-1, 1] = R$ gives S (a möbius strip)



$\boxed{\text{Flux (Surface integral over vector field)}}$

P = amount of F passing through S

ΔP = amount passing ΔS



$$\Delta P = F \cdot \underbrace{\frac{r_u \times r_v}{|r_u \times r_v|}}_{\vec{n}} \underbrace{|r_u \times r_v|\,\Delta u\,\Delta v}_{dS}$$

$$= F \cdot (\underbrace{\vec{n}\,\Delta S}_{\Delta \vec{S}})$$

$$\boxed{d\vec{S} = \vec{n} \cdot dS = (r_u \times r_v)\, dA}$$

$$\boxed{dP = F \cdot d\vec{S}}$$

$$\boxed{P = \iint_S dP = \iint_S F \cdot d\vec{S} = \iint_R F \cdot (r_u \times r_v)\, dA}$$

<u>Example</u> Find the flux of $F(x\,y\,z) = \langle x, y, z\rangle$ through the surface of the solid bounded above by the upper hemisphere of a unit sphere centered at $(0,0,0)$ and below by $x^2+y^2=1$.

S_1 = upper hemisphere use spherical coord $(x(\theta,\varphi), y(\theta,\varphi), z(\theta,\varphi)) = r(\theta,\varphi)$

$= (\cos(\theta)\sin(\varphi), \sin(\theta)\sin(\varphi), \cos(\varphi))$ $\quad 0 \le \phi \le \pi/2, \quad 0 \le \theta \le 2\pi$

$$r_\theta = \langle -\sin(\theta)\sin(\varphi), \cos(\theta)\sin(\varphi), 0\rangle$$

$$r_\varphi = \langle \cos\theta\cos\varphi, \sin(\theta)\cos(\varphi), -\sin(\varphi)\rangle$$

$$r_\theta \times r_\varphi = \det\begin{bmatrix} \vec{i} & \vec{j} & \vec{k} \\ -\sin(\theta)\sin(\varphi) & \cos(\theta)\sin(\varphi) & 0 \\ \cos\theta\cos\varphi & \sin\theta\cos\varphi & -\sin\varphi \end{bmatrix} = (-\sin^2\varphi\cos\theta)\vec{i} - (\sin^2\varphi\sin\theta)\vec{j}$$
$$+(-\sin^2(\theta)\sin\phi\cos\phi - \cos^2\theta\sin\phi\cos\phi)\vec{k}$$

$$= \langle -\sin^2\varphi\cos\theta, -\sin^2\varphi\sin\theta, -\sin\phi\cos\phi\rangle$$

This orientation is inward - I should use $r(\varphi,\theta)$ instead of $r(\theta,\varphi)$ - or - just introduce a "−" here

$$= -\sin\varphi\langle \sin\varphi\cos\theta, \sin\varphi\sin\theta, \cos\phi\rangle$$

$$= -\sin\varphi\,\langle x, y, z\rangle$$

$$\iint_{S_1} F \cdot d\vec{S} = \iint_{R_1} \langle x,y,z\rangle \cdot \langle x,y,z\rangle \sin(\varphi)\, dA = \iint_{R_1} (x^2+y^2+z^2)\sin\varphi\, dA = \iint_{R_1} \sin\varphi\, dA$$

$R_1 = [0,2\pi] \times [0, \pi/2]$

So $\iint_{S_1} F \cdot d\vec{S} = \int_0^{2\pi}\int_0^{\pi/2} \sin\varphi \, d\varphi \, d\theta = \left(\int_0^{2\pi} d\theta\right)\left(\int_0^{\pi/2} \sin\varphi \, d\varphi\right)$

$= 2\pi\left(-\cos\varphi\Big|_0^{\pi/2}\right) = 2\pi\left(-\cos(\pi/2) - (-\cos 0\right) = 2\pi$

S_2 is best with cylindrical coord: $r(\theta,\rho) = (\rho\cos\theta, \rho\sin\theta, \rho^2-1)$

$r_\theta = \langle -\rho\sin\theta, \rho\cos\theta, 0 \rangle$

$r_\rho = \langle \cos\theta, \sin\theta, 2\rho \rangle$

$r_\theta \times r_\rho = \det\begin{bmatrix} i & j & k \\ -\rho\sin\theta & \rho\cos\theta & 0 \\ \cos\theta & \sin\theta & 2\rho \end{bmatrix} = \langle 2\rho^2\cos\theta, -(-2\rho^2\sin\theta), -\rho^2 \rangle$

$R_2 = [0,2\pi]\times[0,1]$

$= 2\rho^2 \langle 2\cos\theta, 2\sin\theta, -1 \rangle$

$\iint_{S_2} F \cdot d\vec{S} = \iint_{R_2} 2\rho^2\left(2\rho\cos^2\theta + 2\rho\sin^2\theta - (\rho^2-1)\right) dA$

$= \int_0^{2\pi}\int_0^1 2\rho^2(2\rho - \rho^2 + 1)\, dA = \int_0^{2\pi} d\theta \int_0^1 (4\rho^3 - 2\rho^4 + 2\rho^2)\, d\rho$

$= 2\pi\left(1 - \frac{2}{5} + \frac{2}{3}\right) = 2\pi\left(\frac{19}{15}\right)$

So $\iint_S F \cdot d\vec{S} = 2\pi + 2\pi\left(\frac{19}{15}\right) = 2\pi\left(\frac{34}{15}\right)$

Practice! Q 14, Q 15