Math 571 - Homework 3

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Problem 1. Define a metric on $\mathbb{Z}$ for each integer $n > 1$ as follows. Let $s \in \mathbb{Z}$ and define $e_n(s) = \max\{a \in \mathbb{N} \mid n^a | s\}$. Set $d_n(s,t) = n^{-e_n(s-t)}$ if $s \neq t$ and $d_n(s,s) = 0$.

a) Show that $d_n : \mathbb{Z} \times \mathbb{Z} \to [0,1)$ is a metric. (Look at (c) before proving the triangle inequality.)

   Symmetry and reflexivity are trivial.The triangle inequality is dealt with below.

b) Interpret the metric, for example what does it mean to say $d_n(s,t) < \delta$ (s and t are within δ of each other.)

   $d_n(s,t) < d_n(s,t')$ iff there are more copies of n in the factorization of $s-t$ than of $s-t'$. So the closer t is to s the more factors of n there are in $s-t$.

   Suppose $n^a \mid s,t$ and $s = n^a s'$ and $t = n^a t'$, then $e_n(s,t) = a + e_n(s',t')$.

c) $d_n(s,t) \leq \max\{d_n(s,r), d_n(r,t)\}$. ($d_n$ is an **ultrametric**.)

   Suppose $\max\{d_n(s,r), d_n(s,t)\} = d_n(s,r)$. Let $a = e_n(s,r)$ so $n^a|s-r$ and $n^a|s-t$. Now $s-t = (s-r)+(r-t)$ so $n^a|s-t$ and hence $d_n(s,t) \leq \max\{d_n(s,r), d_n(r,t)\} = n^{-a}$.

Problem 2 (R:2:17)**.** Consider all reals in $[0,1]$ whose decimal expansion requires only the digits 3 and 5. Call this set Y. Is Y

It is important to think a bit about the representation of reals in $[0,1]$ in decimal form before jumping into this. clearly there is a map $\phi : 10^{\mathbb{N}} \to [0,1]$ given by $\phi(x) = \sum_{i \in N} x(i)10^{-(i+1)}$. So for example

$$(3,1,4,1,5,9,\cdots) \underset{\phi}{\mapsto} \frac{3}{10} + \frac{1}{10^2} + \frac{4}{10^3} + \frac{1}{10^5} + \cdots = 0.314159$$

There is a very natural topology and metric on $10^{\mathbb{N}}$ the basic open sets are $[s] = \{x \mid x \supset s\}$ for $s \in 10^{<\mathbb{N}}$. The metric can be expressed as $d(x,y) = \frac{1}{10^{i+1}}$ where i is least so that $x|i \neq y|i$. (This is an ultrametric again). So $[s] = N_{10^{(-\operatorname{len}(s))}}(x)$ for any $x \in [s]$. Note that this is related to but different from the metric on $[0,1]$ given by $\rho(x,y) = \left|\sum_{i \in \mathbb{N}} (y(i) - x(i))10^{-(i+1)}\right|$ (the usual metric). In particular, $\rho(0.1\overline{00}, 0.0\overline{99}) = 0$ so ρ is not a metric on $10^{\mathbb{N}}$. We could use $d'(x,y) = \sum_{i \in \mathbb{N}} |y(i) - x(i)| 10^{-(i+1)}$ which looks more like ρ, but is still different since $d'(0.1\overline{00}, 0.0\overline{99}) = \frac{1}{10} + \sum_{i=2}^{\infty} \frac{9}{10^i} = 0.1 + 0.1 = .2$. Anyway, d' records more info than we need in $10^{\mathbb{N}}$ where all we really care about is the first digit on which x and y disagree.

Trivially, each $[s]$ is clopen, since $[s]$ is open and

$$[s]^c = \bigcup\{t \mid t|n-2 = s|n-2 \wedge t(n-1) \neq s(n-1) \wedge n = \operatorname{len}(s)\}$$

so $[s]^c$ is open. This means that $10^{\mathbb{N}}$ is **totally disconnected**, namely, $\{[s] \mid s \in 10^{<\mathbb{N}}\}$ is a base of clopen sets, so the only connected sets are singletons and the empty set.

Notice that $\phi|Y' : Y' \to Y$ where $Y' = \{3,5\}^{\mathbb{N}} \subseteq 10^{\mathbb{N}}$ is bijective and continuous. In fact, Y and Y' are homeomorphic, so any topological property that Y has is shared by Y' and vice-versa. Note by the way that Y has no finite length decimals, for example, $0.33 = 0.33\overline{0} = 0.32\overline{9} \notin Y$.

a) dense in $[0,1]$?

It is clear that Y' is not dense in $10^{\mathbb{N}}$ since for example $[1] \cap Y' = \emptyset$. This can easily be turned into an argument that Y is not dense on $[0,1]$, namely, $N_{10^{-1}}(0) \cap Y = \emptyset$, since the closest element in Y to 0 is $0.33333\cdots$.

b) nowhere dense in $[0,1]$?

Again, it is trivial to see that $Y' = \{3,5\}^{\mathbb{N}}$ is nowhere-dense in $10^{\mathbb{N}}$. Take any open set O and $[s] \subset O$. Then $[s0] \subset [s]$ and $[s0] \cap Y' = \emptyset$.

Again a variant of this works in $[0,1]$. Let O be open, take $N_\delta(x) \subset O$. Say $x = 0.d_0 d_1 \cdots d_i \cdots$ get $y = 0.d_0 \cdots d_i 0 \cdots$ where $10^{-(i+2)} < \delta/2$. Now consider $N_{10^{-(i+3)}}(y) \subset O$ and disjoint from Y.

c) countable?

The standard diagonalization argument shows that Y (Y') is uncountable.

d) closed?

Again it is trivial to see that Y' is closed in $10^{\mathbb{N}}$. We can see this in $[0,1]$ with essentially the same argument. Suppose $x_i \in Y$ and $x_i \to x$. Then it is clear that for all n, $x \mid i \subset x \mid n$ for all sufficiently large i. Thus $x \in Y$.

e) compact?

This is easy to show directly, but Y is a closed subset of a compact space, hence compact.

f) perfect?

Again this is trivial looking at Y' inside of $10^{\mathbb{N}}$ and the same argument works in $[0,1]$.

Problem 3 (R:2:20*)**.** If E is connected is $\mathrm{Cl}(E)$ and/or $\mathrm{Int}(E)$ necessarily connected?

Of course, give a proof or a counterexample.

The easiest here is $\mathrm{Int}(E)$. Take $E \subseteq \mathbb{R}^2$ to be $N_1((-1,0)) \cup N_1((0,1))$. So E consists of unit discs centered at $(-1,0)$ and $(0,1)$ that just "kiss" at $(0,0)$. $\mathrm{Int}(E)$ is the union of the interiors of the two discs, and these form their own separating sets. So if E is connected, $\mathrm{Int}(E)$ **need not be** connected.

Suppose A, B witness that $\mathrm{Cl}(E)$ is disconnected. Then

i) $A' = A \cap E \neq \emptyset \neq B \cap E = B'$,

ii) $A' \cup B' = E$, and

iii) $A' \cap B' = \emptyset$.

So E must also be disconnected. (i) uses the following:

For any open O:

$$O \cap E \neq \emptyset \iff O \cap \mathrm{Cl}(E) \neq \emptyset$$

$(\implies)$ is trivial. For the other direction argue the contrapositive

$$O \cap E = \emptyset \implies O \cap \mathrm{Cl}(E) = \emptyset$$

This is true since

$$O \cap E = \emptyset \implies O \subseteq \mathrm{Int}(E^c) \implies O \cap (\mathrm{Int}(E^c))^c = O \cap \mathrm{Cl}(E) = \emptyset$$

So if E is connected, then $\mathrm{Cl}(E)$ **is** connected.

Problem 4. Show that E is connected iff for all $p, q \in E$ there is a connected open relative to E set $A \subseteq E$ with $p, q \in A$.

The "only if" part is trivial since E is open in E, E is connected, and $p, q \in E$.

For the "if" part show the contrapositive. Suppose E is not connected, then there are open (relative to E) $A, B \subset E$ such that $A, B \neq \emptyset$ and $A \cap B = \emptyset$, and $A \cup B = E$. Let $p \in A$ and $q \in B$. Let $O \subset E$ be open in E with $p, q \in O$. Then $A' = A \cap O$ and $B' = B \cap O$ witness that O is not connected. So there is no open connected subset of E containing p and q.

Problem 5 (R:2:21*)**.** Prove that every convex subset of $\mathbb{R}^k$ is connected.

The original problem in Rudin is a four part problem with this being the last part. You might use the original problem as a hint/guide here.

Suppose G is our convex set and let $A, B \subset G$ be non-empty and open in G with $A \cup B = G$. Let $a \in A$ and $b \in B$. Then $\{ta + (1-t)b \mid t \in [0,1]\} \subseteq G$. Loot at $\{t \in [0,1] \mid ta + (1-t)b \in A\} = A'$ and $B' = \{t \mid ta + (1-t)b \in B\}$.

Claim: A', B' witness that $[0,1]$ is not connected.

The only thing that requires argument is that A' and B' are open in $[0,1]$. Let $c = ta + (1-t)b \in A$. Then $B_\epsilon(c) \subset A$ for some $\epsilon > 0$. Consider, $(t+h)a + (1-(t+h))b = ta + (1-t)b + h(a-b) = c + h(a-b)$. If $|h| < \epsilon/|a-b| = \delta$, then $t + h \in A'$. This means that $(t-\delta, t+\delta) \cap [0,1] \subseteq A'$ and so A' is open in $[0,1]$.

Fact: $[0,1]$ is connected. (Rudin 2.47)

Problem 6 (R:2:26)**.** Let X be a metric space in which every infinite set has a limit. Show that X is compact.

I prove this in the notes. It is an important and very useful characterization of compactness in a metric space, namely, **sequential compactness**. I do not want you to reproduce the proof I give. Use the hint from Rudin and try it the way he suggests. This builds on some problems you did last week.

Let $\mathcal{O}$ be an open cover of X. Our goal is to produce a finite subcover of X. Problems 6 from homework 2 gives us that X has a countable base. Thus we can easily get a countable subcover $\mathcal{O}' \subseteq \mathcal{O}$, simply assign to each x a base set U_x and O_{U_x} so that $x \in U_x \subseteq O_{U_x} \in \mathcal{O}$. Since there are only countably many U_x, there are also only countable many $O_{U_x} \in \mathcal{O}$ required. So we may assume $\mathcal{O} = \{G_i \mid i \in \mathbb{N}\}$.

Assume there is no finite subcover from $\mathcal{O}$, then $\bigcup_{i=0}^{n} G_i$ fails to cover X and so $F_n = X - \bigcup_{i=0}^{n} G_i$ is closed and non-empty. Further, $F_{n+1} \subseteq F_n$ and by assumption $\bigcap_{i \in \mathbb{N}} F_i = \emptyset$. Let $x_i \in F_i$ for each i. By assumption there is $x \in \mathrm{Lim}(\{x_i\}_{i \in \mathbb{N}})$, but then $x \in F_i$ for all i since $\{x_k\}_{k \geq i} \subseteq F_i$ and F_i is closed. This is a contradiction so the assumption that there is no finite subcover must be false.

Problem 7 (R:2:28)**.** Show that every closed set, F, in a separable metric space can be written as $P = P \cup C$ where P is perfect (perhaps empty) and C is countable.

A different hint from Rudin's: I gave you a sort of hint in class, define $F' = F - \mathrm{Iso}(F)$, recall $\mathrm{Iso}(F)$ is the set of isolated points of F. F' is called the derivative of the set F. Argue that $\mathrm{Iso}(F)$ is countable, in some natural sense F' is *closer to perfection*, since we have removed some isolated points. Notice that F' is closed. If you haven't reached perfection repeat the process. In this way you build a sequence of closed sets $F \supset F_1 \supset F_2 \cdots$ and countable sets C_i so that $F = \bigcap F_i \cup \bigcup C_i$. If $\bigcap F_i = F_\omega$ still has isolated points, continue! A transfinite recursion!

Proof 1: Continue as suggested in the hint. There is a strictly descending sequence $F_\alpha \supset F_\beta$ for $\alpha < \beta < \gamma$ with the additional property that $C_\beta = \mathrm{Iso}\left(\bigcap_{\alpha<\beta} F_\alpha\right) \neq \emptyset$ and $F_\beta \cup C_\beta = \bigcup_{\alpha<\beta} F_\alpha$. But let $c_\beta \in C_\beta$, then there is open O_β with $c_\beta \in O_\beta$ and $s_\beta \in O_\beta \cap S$, where S is the separable set. Clearly, for $\alpha < \beta$, $s_\alpha \neq s_\beta$ so this must halt after countably many steps. That is we reach γ so that $\bigcap_{\alpha<\gamma} F_\alpha = F$ and $\mathrm{Iso}(F) = \emptyset$, for F is perfect and $C = \bigcup_{\alpha<\gamma} C_\alpha$ is countable.

Proof 2: (Follow the text.) Let P be the set of condensation points of F. If $x \notin P$, then $x \in O$ for some open O with $O \cap F$ countable. We can choose this O from a countable base, $\mathcal{B}$ and thus $C = \bigcup\{O \cap F \in \mathcal{B} \mid |O \cap P| \leq \aleph_0\}$. C is countable and $F = C \cup P$.

If $x \in P$, then x is a condensation point of P, not just a condensation point of F. For suppose there is O with $x \in O$ and $|O \cap P| \leq \aleph_0$, then $|O \cap F| = |(O \cap P) \cup (O \cap C)| \leq \aleph_0$. So clearly, $\mathrm{Iso}(P) = \emptyset$ and $P \subset \mathrm{Lim}(P)$, so P is closed.