High Level Overview

Solving Equations	Everything Else
	Structure / Factor
$Ax = b$	$A = LU$
$Ax = \lambda x$	$A = S \Lambda S^{-1}$
$Av = \sigma u$	$A = U \Sigma V^T$
	Geometry
$A\hat{x} = \hat{b}$	inner products
	length
	$A = QR$
Simple	General
$\vec{v} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$	vector space - Fourier - Functions

Simple	General
matrices	linear maps
$A = \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \\ a_{m1} & \cdots & a_{mn} \end{bmatrix}$	$\int$, $\frac{d}{dx}$

Quick intro to vectors and matrices

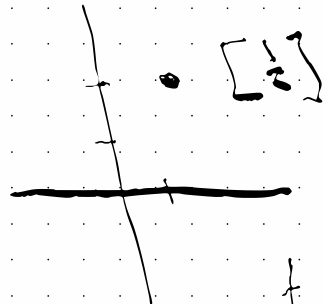
A matrix is an array of scalars

$$A = \begin{array}{c} \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \\ a_{m1} & \cdots & a_{mn} \end{bmatrix} \Big\} m \text{ rows} \\ \underbrace{\hphantom{a_{m1} \cdots a_{mn}}}_{n \text{ columns}} \end{array}$$

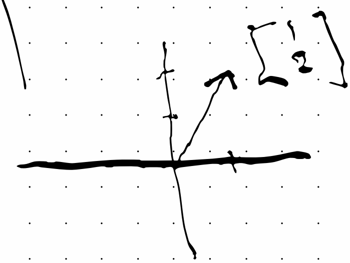
For now a vector is just a 1-column matrix $v = \begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix} \in \mathbb{R}^n (\mathbb{C}^n)$

Geometry and algebra of vectors

$$\begin{bmatrix}1\\2\end{bmatrix}$$

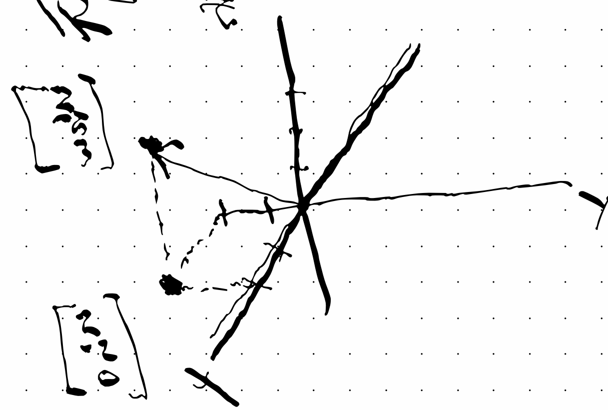


or



Same idea in $\mathbb{R}^3$.

$$v = \begin{bmatrix}2\\-2\\3\end{bmatrix}$$



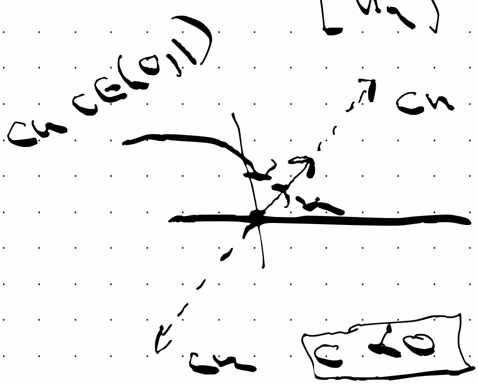
Hard to draw in $\mathbb{R}^3$, impossible for $\mathbb{R}^n$ $n>3$ or $\mathbb{C}^2$. Pics will be in $\mathbb{R}^2$, but thought must be used.

adding vectors

$$v+u = \begin{bmatrix} v_1 \\ \vdots \\ v_n \end{bmatrix} + \begin{bmatrix} u_1 \\ \vdots \\ u_n \end{bmatrix} = \begin{bmatrix} u_1+v_1 \\ \vdots \\ u_n+v_n \end{bmatrix}$$

Scalar Mult

$$cu = c\begin{bmatrix} u_1 \\ \vdots \\ u_n \end{bmatrix} = \begin{bmatrix} cu_1 \\ \vdots \\ cu_n \end{bmatrix}$$

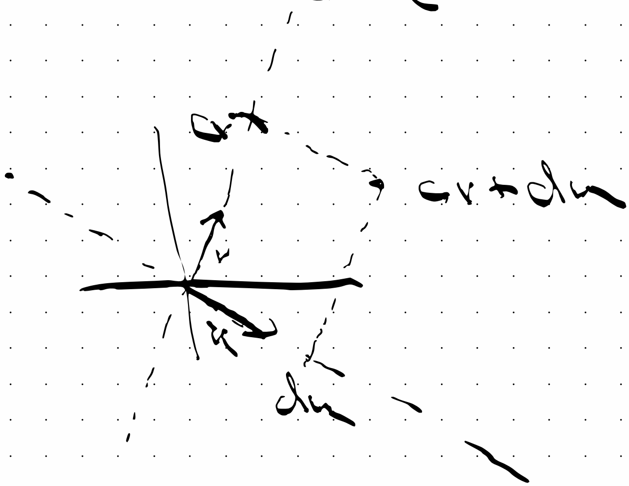


$0 = 0u$

Linear Combinations

(Most import concept)

$$c_1 u_1 + c_2 u_2 + \cdots + c_n u_n = b$$



Clearly if u, v are not on same line then lin comb give a plane.

Ax as a lin comb of vectors

$$\begin{array}{c} \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \vdots \\ a_{m1} & & a_{mn} \end{bmatrix} \\ v_1 \quad\quad\quad v_n \end{array} \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} = x_1 v_1 + \cdots + x_n v_n \in \mathbb{R}^m$$

$x \in \mathbb{R}^n$

In this way Ax represents a linear combination of the columns of A an

$$\{Ax \mid x \in \mathbb{R}^n\} = \text{all lin comb of columns of } A$$
$$= CS(A)$$

So if $u \nparallel v$ in $\mathbb{R}^2$, $A = [u\,v]$

then $CS(A) =$ all lin comb u,v
$$= \mathbb{R}^2$$

We can view $m \times n$ matrix A as a function $\underset{\mathbb{R}^n}{x} \mapsto \underset{\mathbb{R}^m}{Ax}$

$$L_A : \mathbb{R}^n \to \mathbb{R}^m$$

Solving systems of Eqn

$$\boxed{\begin{aligned} 3x + 4y &= -5 \\ -2x - y &= 0 \end{aligned}}$$

View #1:
$$\begin{bmatrix} 3 \\ -2 \end{bmatrix} x + \begin{bmatrix} 4 \\ -1 \end{bmatrix} y = \begin{bmatrix} -5 \\ 0 \end{bmatrix}$$

~~Desmos~~ Geogebra

View #2. Intersection of 2 lines

~~Desmos~~ Geogebra

Solving Systems

Idea convert system into simple system to solve - a triangular system

$$x_1 + 2x_2 - 3x_3 = 0$$
$$4x_2 + x_3 = 5$$
$$2x_3 = 2$$

Back substitution

$$x_3 = 1$$

$$4x_2 + 1 = 5 \Rightarrow 4x_2 = 4$$
$$\Rightarrow x_2 = 1$$

$$x_1 + 2 - 3 = 0 \Rightarrow x_1 = 1$$

Solution is $(1, 1, 1)$

system - - - → Δ-system → back sub
transform
to equivalent
systems

Def. 2 systems are equiv if same set of solutions

Operations that transform to eq systems

I. swap 2 eqns
II. mult an eq by non-zero
III. add eqn to mult of another and replace the first

Consider the 2/3 D systems from geogebra

$3x - 2y = 7$ $\qquad\qquad \begin{pmatrix} 1 \\ -2 \end{pmatrix}$

$-2x - y = 0$

Matrix Representation

$$Ax = b$$

$$\Big\downarrow$$

$$(A \mid b)$$

Elementary OPS → Elementary Row ops

I. Swap two rows $R_i \leftrightarrow R_j$

II. Multiply Row by Non-zero const.

$R_i \to cR_i$

III. $R_i + cR_j \to R_i$

$$x_1 + 2x_2 + x_3 = 3$$

$$3x_1 - x_2 - 3x_3 = -1$$

$$2x_1 + 3x_2 + x_3 = 4$$

Pivot Elem

$$\begin{array}{cc} \text{Pivot Row} \rightarrow & \left[\begin{array}{ccc|c} \boxed{1} & 2 & 1 & 3 \\ 3 & -1 & -3 & -1 \\ 2 & 3 & 1 & 4 \end{array}\right] \end{array} \xrightarrow{\substack{R_2 - 3R_1 \rightarrow R_2 \\ R_3 - 2R_1 \rightarrow R_3}}$$

$$\rightarrow \left[\begin{array}{ccc|c} 1 & 2 & 1 & 3 \\ 0 & -7 & -6 & -10 \\ 0 & -1 & -1 & -2 \end{array}\right] \xrightarrow{R_2 \leftrightarrow R_3}$$

$$\left[\begin{array}{ccc|c} 1 & 2 & 1 & 3 \\ 0 & -1 & -1 & -2 \\ 0 & -7 & -6 & -10 \end{array}\right] \xrightarrow{R_3 - 7R_2 \rightarrow R_3}$$

$$\left[\begin{array}{ccc|c} 1 & 2 & 1 & 3 \\ 0 & -1 & -1 & -2 \\ 0 & -7 & -6 & -10 \end{array}\right] \xrightarrow{R_3 - 7R_2 \to R_3}$$

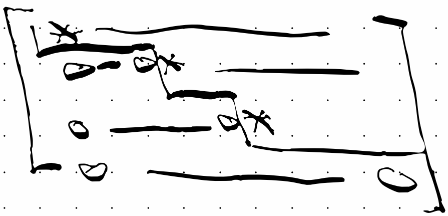
$$\left[\begin{array}{ccc|c} 1 & 2 & 1 & 3 \\ 0 & -1 & -1 & -2 \\ 0 & 0 & 1 & 4 \end{array}\right]$$

Back Subst

$$\begin{cases} x_3 = 4 \\ -x_2 - 4 = -2 \Rightarrow x_2 = 2 \\ \qquad \Rightarrow x_2 = -2 \\ x_1 + (-4) + 4 = 3 \Rightarrow x_1 = 3 \end{cases}$$

More general case

Echelon Form



← stair step

$$x_1 + x_2 + x_3 + x_4 + x_5 = 2$$
$$x_1 + x_2 + x_3 + 2x_4 + 2x_5 = 3$$
$$x_1 + x_2 + x_3 + 2x_4 + 3x_5 = 2$$

$$\left[\begin{array}{ccccc|c} \boxed{1} & 1 & 1 & 1 & 1 & 2 \\ 1 & 1 & 1 & 2 & 2 & 3 \\ 1 & 1 & 1 & 2 & 3 & 2 \end{array}\right] \quad \begin{array}{l} R_2 - R_1 \to R_2 \\ R_3 - R_1 \to R_3 \end{array} \longrightarrow$$

$$\left[\begin{array}{ccccc|c} 1 & 1 & 1 & 1 & 1 & 2 \\ 0 & 0 & 0 & \boxed{1} & 1 & 1 \\ 0 & 0 & 0 & 1 & 2 & 0 \end{array}\right] \quad R_3 - R_2 \to R_2 \longrightarrow$$

$$\left[\begin{array}{ccccc|c} 1 & 1 & 1 & 1 & 1 & 2 \\ 0 & 0 & 0 & \boxed{1} & 1 & 1 \\ 0 & 0 & 0 & 1 & 2 & 0 \end{array}\right] \xrightarrow{R_3 - R_2 \to R_2}$$

$$\begin{array}{ccccc|c} \boxed{1} & 1 & 1 & 1 & 1 & 2 \\ 0 & 0 & 0 & \boxed{1} & 1 & 1 \\ 0 & 0 & 0 & 0 & \boxed{1} & -1 \end{array}$$

Pivots = dep var

non-pivots = ind var

$x_5 = -1$

$x_4 + -1 = 1 \to x_4 = 0$

$x_1 + x_2 + x_3 + 0 - 1 = 2$

$x_2 = s, \; x_3 = t$ Plane in $\mathbb{R}^5$

$x_1 = 3 - s - t$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} 3-s-t \\ s \\ t \\ 0 \\ -1 \end{bmatrix} = \begin{bmatrix} 3 \\ 0 \\ 0 \\ 0 \\ -1 \end{bmatrix} + s\begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + t\begin{bmatrix} -1 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}$$

Note. In this 3×5 situation at least 2 variable are free