Name: ___________________________ Exam 2 - MAT345

Part I: True/False

Each problem is points for a total of 60 points. (10 problems 6 points each; 3 points for correct T/F; 3 points for correct explanation.)

Problem 1. Decide if each of the following is true or false. For each, provide an example or counter-example or an argument as required. You may refer to a theorem if that applies.

a) False Let V be a vector space and $S = \{\boldsymbol{v}_1, \dots, \boldsymbol{v}_n\}$ such that $\operatorname{span}(S) = V$. S can be extended to a basis for V.

Any spanning set can be restricted to a basis. Any expansion $T \supsetneq S$ would definitely not be linearly independent and hence not a basis.

b) True Suppose $\mathcal{B}$ is a basis for V, then for any vector $\boldsymbol{v} \notin \mathcal{B}$, $\mathcal{B} \cup \{\boldsymbol{v}\}$ is dependent.

$\boldsymbol{v} \in \operatorname{span}(\mathcal{B})$ since $\mathcal{B}$ is a basis, but then $\boldsymbol{v} = \alpha_1 \boldsymbol{v}_1 + \cdots + \alpha_k \boldsymbol{v}_k$ for some scalars $\alpha_i \neq 0$ and $\boldsymbol{v}_i \in \mathcal{B}$. But then

$$\boldsymbol{v} - (\alpha_1 \boldsymbol{v}_1 + \cdots + \alpha_k \boldsymbol{v}_k) = \boldsymbol{0}$$

is a non-trivial linear combination of vectors, so $\mathcal{B} \cup \{\boldsymbol{v}\}$ is dependent.

c) False $U = \{(x, y) \in \mathbb{R}^2 \mid x \text{ and } y \text{ have the same sign}\}$ is a subspace of $\mathbb{R}^2$.

Not closed under addition: $(2, 2) + (-3, -1) = (-1, 1)$.

d) False The map $L : \mathbb{R}^2 \to \mathbb{R}$ given by $L(x_1, x_2) = |x_1 - x_2|$ is linear.

There are many ways to see that this is not linear. one is just that $L((-1)(x_1, x_2)) = L(-x_1, -x_2) = |(-x_1) - (-x_2)| = |x_2 - x_1| = x_1 - x_2| \neq (-1)L(x_1, x_2)$.

e) True The evaluation map at c, $e_c : P \to \mathbb{R}$ given by $e_c(p(x)) = p(c)$ is linear where P is the vector space of all polynomials with real coefficients.

This is true. Clearly

$$e_c(\alpha_1 p_1(x) + \alpha_2 p_2(x)) = \alpha_1 p_1(c) + \alpha_2 p_2(c) = \alpha_1 e_c(p_1(x)) + \alpha_2 e_c(p_2(x)).$$

f) False There are subspaces $V_0 = \mathbb{R}^4 \supsetneq V_1 \supsetneq V_2 \supsetneq V_3 \supsetneq V_4 \supsetneq V_5 = \{\boldsymbol{0}\}$ where each V_i is a proper subspace of V_{i-1}.

Since we know $\dim(V_0) = 4 > \dim(V_1) > \dim(V_2) > \dim(V_3) > \dim(V_4) > \dim(V_5) = 0$, which is impossible.

g) True Given any three linearly independent vectors $\{\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3\}$,from $\mathbb{R}^3$ and any three vectors $\{p_1(x), p_2(x), p_3(x)\}$ from P_6 (polynomials of degree 6), there is a unique linear function $L : \mathbb{R}^3 \to P_6$ satisfying $L(\boldsymbol{v}_i) = p_i(x)$, for $i = 1, 2, 3$.

This is true, any linear map $L : V \to W$ is completely determined by where the basis vectors are mapped

h) True Suppose $L : \mathbb{R}^{2\times 3} \to \mathbb{R}^4$ is linear and onto, that is, $\operatorname{Img}(L) = \mathbb{R}^4$. Then $\dim(\ker(L)) = 2$.

Recall $\mathbb{R}^{2\times 3}$ is the space of 2×3 matrices.

$\dim(\mathbb{R}^{2\times 3}) = 6$ and so $\dim(\ker(L)) + \dim(\operatorname{Img}(L)) = \dim(\ker(L)) + 4 = 6$.

i) True Let $\mathcal{B} = \{\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3\}$ be a basis for V and suppose $\boldsymbol{v} = \alpha_1\boldsymbol{v}_1 + \alpha_2\boldsymbol{v}_2 + \alpha_3\boldsymbol{v}_3$. Then

$$[\boldsymbol{v}]_{\mathcal{B}} = \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{bmatrix}$$

This is trivially true as this is the definition of $[\boldsymbol{v}]_{\mathcal{B}}$.

j) False $L : \mathbb{R}^{3\times 3} \to \mathbb{R}^{3\times 3}$ is given by $L(A) = BA$ for a 3×3 matrix B. If $\mathcal{B}$ is a basis for $\mathbb{R}^{3\times 3}$, then $[L]_{\mathcal{B}} = B$.

$[L]_{\mathcal{B}}$ acts on representations of matrices wrt $\mathcal{B}$, it is a 9×9 matrix, not a 3×3 matrix.

Part II: Computational (80 points)

Show all computations so that you make clear what your thought processes are.

Problem 2 (20 pts)**.** Consider A given by

$$A = \begin{bmatrix} 1 & 2 & -4 & 3 & 2 \\ -3 & -6 & 14 & -13 & -3 \\ 0 & 0 & 3 & -6 & 4 \\ 2 & 4 & -7 & 4 & 5 \end{bmatrix}$$

Find a basis for each of NS(A), CS(A), and RS(A).

Hint: This should require exactly one (not two or three) reduction of a matrix to echelon form.

Gauss-Jordan elimination to get echelon form:

$$\begin{bmatrix} 1 & 2 & -4 & 3 & 2 \\ -3 & -6 & 14 & -13 & -3 \\ 0 & 0 & 2 & -4 & 3 \\ 2 & 4 & -7 & 4 & 5 \end{bmatrix} \xrightarrow[\substack{R_2+3R_1\to R_2 \\ R_4-2R_1\to R_4}]{} \begin{bmatrix} 1 & 2 & -4 & 3 & 2 \\ 0 & 0 & 2 & -4 & 3 \\ 0 & 0 & 2 & -4 & 3 \\ 0 & 0 & 1 & -2 & 1 \end{bmatrix}$$

$$\xrightarrow[R_2\leftrightarrow R_4]{} \begin{bmatrix} 1 & 2 & -4 & 3 & 2 \\ 0 & 0 & 1 & -2 & 1 \\ 0 & 0 & 2 & -4 & 3 \\ 0 & 0 & 2 & -4 & 3 \end{bmatrix}$$

$$\xrightarrow[\substack{R_3-3R_2\to R_3 \\ R_4-2R_2\to R_4}]{} \begin{bmatrix} 1 & 2 & -4 & 3 & 2 \\ 0 & 0 & 1 & -2 & 1 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\xrightarrow[R_4-R_3\to R_4]{} \begin{bmatrix} 1 & 2 & -4 & 3 & 2 \\ 0 & 0 & 1 & -2 & 1 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

From this we know:

$$\begin{aligned} \mathrm{CS}(A) &= \mathrm{span}\{(1,2,-3),(-4,-7,14),(2,5,-3)\} \\ \mathrm{RS}(A) &= \mathrm{span}\{(1,2,-4,3,2),(0,0,1,-2,1),(0,0,0,0,1)\} \end{aligned}$$

Note: RS(A) is not the span of the first three rows of A.

To find a basis for NS(A) we are looking for solutions to $A\boldsymbol{x} = \mathbf{0}$. First we

have back-substitution: x_2 and x_4 are free, let $x_2 = s$ and $x_4 = t$, then

$$\boxed{x_5 = 0}$$
$$\boxed{x_4 = t}$$
$$x_3 - 2t = 0 \to \boxed{x_3 = 2t}$$
$$\boxed{x_2 = s}$$
$$x_1 + 2s - 4(2t) + 3t = 0 \to \boxed{x_1 = -2s + 5t}$$

Any vector $\boldsymbol{x}$ satisfying, $A\boldsymbol{x} = \boldsymbol{0}$ can be written as:

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} -2s + 5t \\ s \\ 2t \\ t \\ 0 \end{bmatrix} = s \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + t \begin{bmatrix} 5 \\ 0 \\ 2 \\ 1 \\ 0 \end{bmatrix}$$

So $\{(-2, 1, 0, 0, 0), (5, 0, 2, 1, 0)\}$ is a basis for $\mathrm{NS}(A)$, that is,

$$\mathrm{NS}(A) = \mathrm{span}\{(-2, 1, 0, 0, 0), (5, 0, 2, 1, 0)\}$$

Problem 3 (20 pts)**.** Let $L : \mathbb{R}^{3\times 2} \to \mathbb{R}^{2\times 2}$ given by $L(A) = DA$ where

$$D = \begin{bmatrix} 1 & 2 & 0 \\ 2 & -1 & 1 \end{bmatrix}$$

a) (8 points) Show that L is a linear map.

b) (12 points) Give the matrix $[L]_{\mathcal{B},\mathcal{C}}$ in terms of the basis $\mathcal{B}$ for $\mathbb{R}^{3\times 2}$ and $\mathcal{C}$ for $\mathbb{R}^{2\times 2}$ given by:

$$\mathcal{B} = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}$$
$$\mathcal{C} = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}$$

$$\begin{aligned} [L]_{\mathcal{B},C} &= \Big[[DB^{1,1}]_{\mathcal{C}} \, [DB^{1,2}]_{\mathcal{C}} \cdots [DB^{3,2}]_{\mathcal{C}} \Big] \\ &= \Big[\begin{bmatrix} 1 & 0 \\ 2 & 0 \end{bmatrix}_{\mathcal{C}} \begin{bmatrix} 0 & 1 \\ 0 & 2 \end{bmatrix}_{\mathcal{C}} \begin{bmatrix} 2 & 0 \\ -1 & 0 \end{bmatrix}_{\mathcal{C}} \begin{bmatrix} 0 & 2 \\ 0 & -1 \end{bmatrix}_{\mathcal{C}} \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}_{\mathcal{C}} \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}_{\mathcal{C}} \Big] \\ &= \begin{bmatrix} 1 & 0 & 2 & 0 & 0 & 0 \\ 0 & 1 & 0 & 2 & 0 & 0 \\ 2 & 0 & -1 & 0 & 1 & 0 \\ 0 & 2 & 0 & -1 & 0 & 1 \end{bmatrix} \end{aligned}$$

Problem 4 (20 pts)**.** Consider the map $L : \mathbb{R}^3 \to \mathbb{R}^3$ that maps any point in $\mathbb{R}^3$ onto the plane spanned by $(1, -2, 1)$ and $(2, 0, -2)$ in such a way that points in the plane are fixed and which maps $(1, 1, 1)$ to $(0, 0, 0)$.

a) (7 points) Find $[L]_{\mathcal{B}}$ for $\mathcal{B} = \{(1, -2, 1), (2, 0, -2), (1, 1, 1)\}$.

b) (5 points) Find the change of basis matrix $[\mathrm{id}]_{\mathcal{B},E}$ (from the basis $\mathcal{B}$ to the standard basis.)

c) (8 points) Find the matrix for L wrt the standard basis using the first two parts. (Give me the decomposition: $[\mathrm{id}]_{\mathcal{B},\mathcal{E}}[L]_{\mathcal{B}}[\mathrm{id}]_{\mathcal{E},\mathcal{B}}$ as well as the resulting matrix.

$L((1, -2, 1)) = (1, -2, 1)$ so $[L(1, -2, 1)]_{\mathcal{B}} = (1, 0, 0)$, similarly $[L(2, 0, -2)]_{\mathcal{B}} = (0, 1, 0)$, and $[L(1, 1, 1)]_{\mathcal{B}} = (0, 0, 0)$. So

$$[L]_{\mathcal{B}} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

The change of basis matrix from $\mathcal{B}$ to $\mathcal{E}$ is

$$[\mathrm{id}]_{\mathcal{B},\mathcal{E}} = B = \begin{bmatrix} 1 & 2 & 1 \\ -2 & 0 & 1 \\ 1 & -2 & 1 \end{bmatrix}$$

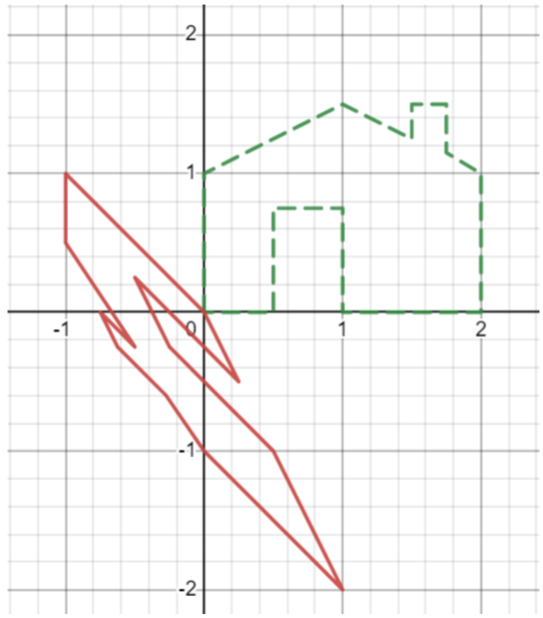
and

$$[\mathrm{id}]_{\mathcal{E},\mathcal{B}} = B^{-1}$$

So

$$\begin{aligned} [L] = B[L]_{\mathcal{B}}B^{-1} &= \begin{bmatrix} 1 & 2 & 1 \\ -2 & 0 & 1 \\ 1 & -2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1/6 & -1/3 & 1/6 \\ 1/4 & 0 & -1/4 \\ 1/3 & 1/3 & 1/3 \end{bmatrix} \\ &= \begin{bmatrix} 2/3 & -1/3 & -1/3 \\ -1/3 & 2/3 & -1/3 \\ -1/3 & -1/3 & 2/3 \end{bmatrix} \end{aligned}$$

Problem 5 (20 pts)**.** The green (dashed) house has been transformed to the red (solid) house by a linear transformation $L : \mathbb{R}^2 \to \mathbb{R}^2$.



Desmos

a) What is $L(\boldsymbol{e}_1)$?

b) What is $L(\boldsymbol{e}_2)$?

c) What is $[L]$?

Clearly

$$L(\boldsymbol{e}_1) = \begin{bmatrix} 1/2 \\ -2 \end{bmatrix} \qquad L(\boldsymbol{e}_2) = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

So

$$\Big[L\Big] = \Big[L(\boldsymbol{e}_1)\, L(\boldsymbol{e}_2)\Big] = \begin{bmatrix} 1/2 & -1 \\ -2 & 1 \end{bmatrix}$$

Part III: Theory and Proofs (30 points; 10 points each)

Choose three of the five options. If you try all five, I will grade the first three, not the best three. You must decide what should be graded.

This part is take-home. You should complete this work on your own without consulting websites, friends, the Math Center, etc.

Problem 6 (10 points)**.** Suppose S is an independent set of vectors from a vector space V, then

$$S \cup \{\boldsymbol{v}\} \text{ is dependent} \quad \Longleftrightarrow \quad \boldsymbol{v} \in \operatorname{span}(S).$$

($\Longleftarrow$) $\boldsymbol{v} \in \operatorname{span}(S)$ means that $\boldsymbol{v} = \alpha_1\boldsymbol{v}_1 + \cdots + \alpha_k\boldsymbol{v}_k$ for some scalars α_i and $\boldsymbol{v}_i \in S$. Clearly then

$$\boldsymbol{v} - (\alpha_1\boldsymbol{v}_1 + \cdots + \alpha_k\boldsymbol{v}_k) = \boldsymbol{0}$$

so $S \cup \{\boldsymbol{v}\}$ is dependent since we have written $\boldsymbol{0}$ as a non-trivial linear combination of vectors from $S \cup \{\boldsymbol{v}\}$.

($\Longrightarrow$) $S \cup \{\boldsymbol{v}\}$ is dependent so $\boldsymbol{v} = \alpha_1\boldsymbol{v}_1 + \cdots + \alpha_k\boldsymbol{v}_k = \boldsymbol{0}$ for some scalars $\alpha_i \neq 0$ and $\boldsymbol{v}_i \in S \cup \{\boldsymbol{v}\}$. Since S is independent, it must be that $\boldsymbol{v}$ is one of the $\boldsymbol{v}_i$'s. WLOG suppose $v = \boldsymbol{v}_1$, then

$$\boldsymbol{v} = -\frac{1}{\alpha_1}(\alpha_2\boldsymbol{v}_2 + \cdots + \alpha_k\boldsymbol{v}_k)$$

and so $\boldsymbol{v} \in \operatorname{span}(S)$.

Problem 7 (10 points)**.** Show that if $L : V \to W$ is linear and $\{L(\boldsymbol{v}_1), L(\boldsymbol{v}_2), L(\boldsymbol{v}_3)\}$ is linearly independent, then $\{\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3\}$ is linearly independent.

More generally, if $L : V \to W$ is linear, then the pre-image of S, $L^{-1}(S) = \{\boldsymbol{v} \mid L(\boldsymbol{v}) \in S\}$ is linearly independent for any linearly independent set S.

Let

$$\alpha_1\boldsymbol{v}_1 + \alpha_2\boldsymbol{v}_2 + \alpha_3\boldsymbol{v}_3 = \boldsymbol{0},$$

then

$$L(\alpha_1\boldsymbol{v}_1 + \alpha_2\boldsymbol{v}_2 + \alpha_3\boldsymbol{v}_3) = \alpha_1 L(\boldsymbol{v}_1) + \alpha_2 L(\boldsymbol{v}_2) + \alpha_3 L(\boldsymbol{v}_3) = L(\boldsymbol{0}) = \boldsymbol{0}$$

so by the independence of $\{L(\boldsymbol{v}_1), L(\boldsymbol{v}_2), L(\boldsymbol{v}_3)\}$ we have $\alpha_1 = \alpha_2 = \alpha_3 = 0$ and thus $\{\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3\}$ is linearly independent.

Problem 8 (10 points)**.** Suppose $A = \begin{bmatrix}\boldsymbol{a}_1\, \boldsymbol{a}_2\, \boldsymbol{a}_3\, \boldsymbol{a}_4\, \boldsymbol{a}_5\end{bmatrix}$ is a 4×5 matrix and

$$\operatorname{NS}(A) = \operatorname{span}\{(-2, 1, 0, 0, 0), (5, 0, 2, 1, 0)\}$$

Find rref(A) and explain how you know that what you have found is rref(A).

We know a typical element of NS(A) is of the form $(x_1, x_2, x_3, x_4, x_5) = (-2s + 5t, s, 2t, t, 0)$ and since $A\boldsymbol{x} = \mathbf{0}$ can be written as a linear combination of columns of A we know

$$(-2s+5t)\boldsymbol{a}_1 + s\boldsymbol{a}_2 + 2t\boldsymbol{a}_3 + t\boldsymbol{a}_4 + 0\boldsymbol{a}_5 = \mathbf{0}$$

Letting $s = 1$ and $t = 0$ we get $-2\boldsymbol{a}_1 + \boldsymbol{a}_2 = 0$ and letting $s = 0$ and $t = 1$ we get $5\boldsymbol{a}_1 + 2\boldsymbol{a}_3 + \boldsymbol{a}_4 = 0$. Thus we have

$$\boldsymbol{a}_2 = 3\boldsymbol{a}_1 \quad \text{and} \quad \boldsymbol{a}_4 = -5\boldsymbol{a}_2 - 2\boldsymbol{a}_3$$

Thus we get

$$\begin{bmatrix}\boldsymbol{a}_1\,\boldsymbol{a}_3\,\boldsymbol{a}_5\end{bmatrix}\begin{bmatrix}1 & 2 & 0 & -5 & 0\\ 0 & 0 & 1 & -2 & 0\\ 0 & 0 & 0 & 0 & 1\end{bmatrix} = \begin{bmatrix}\boldsymbol{a}_1\,\boldsymbol{a}_2\,\boldsymbol{a}_3\,\boldsymbol{a}_4\,\boldsymbol{a}_5\end{bmatrix} = A$$

We know rank(A) $= 3 = 5 - \dim(\text{NS}(A))$ so $\{\boldsymbol{a}_1, \boldsymbol{a}_3, \boldsymbol{a}_5\}$ are linearly independent vectors in $\mathbb{R}^4$. Let $\boldsymbol{b} \in \mathbb{R}^4$ be so that $\{\boldsymbol{a_1}, \boldsymbol{a_2}, \boldsymbol{a_3}, \boldsymbol{b}\}$ is a basis and lat $M = \begin{bmatrix}\boldsymbol{a}_1\,\boldsymbol{a}_3\,\boldsymbol{a}_5\,\boldsymbol{b}\end{bmatrix}$, then M is invertible and

$$M\begin{bmatrix}1 & 2 & 0 & -5 & 0\\ 0 & 0 & 1 & -2 & 0\\ 0 & 0 & 0 & 0 & 1\\ 0 & 0 & 0 & 0 & 0\end{bmatrix} = MR = A$$

So A is equivalent to R. But R is in RREF form so $R = \text{rref}(A)$, since there is only one RREF matrix equivalent to A.

Note: Recall A and B are equivalent if B can be formed from a sequence of elementary row operations applied to A; equivalently, A and B are equivalent iff $B = MA$ for some invertible M. We know

$$A \text{ and } B \text{ are equivalent} \implies \text{NS}(A) = \text{NS}(B).$$

It turns out that for matrices of the same size

$$A \text{ is equivalent to B} \iff \text{NS}(A) = \text{NS}(B)$$

To see this it suffices to show that

$$\text{NS}(A) = \text{NS}(B) \implies \text{rref}(A) = \text{rref}(B).$$

The above basically does this argument by showing that rref(A) can be computed from a basis for NS(A).

Problem 9 (10 points)**.** Suppose A is a 5×5 matrix and $A^n = O$ for some n, then $A^5 = O$.

There are several ways to proceed. Here is one. Note that $\mathrm{NS}(A^{m+1}) \supseteq \mathrm{NS}(A^m)$ for all m since $A^m\boldsymbol{x} = \mathbf{0} \implies A^{m+1}\boldsymbol{x} = A(A^m)\boldsymbol{x} = \mathbf{0}$.

If $\mathrm{NS}(A^{m+1}) = \mathrm{NS}(A^m)$, then $\mathrm{NS}(A^{m+k}) = \mathrm{NS}(A^m)$ for all k. To see this, suppose $\mathrm{NS}(A^{m+k}) = \mathrm{NS}(A^m)$, then

$$\begin{aligned} A^{m+k+1}\boldsymbol{x} = \mathbf{0} &\iff A^{m+k}(A\boldsymbol{x}) = \mathbf{0} && \text{(by assumption)} \\ &\iff A^m(A\boldsymbol{x}) = \mathbf{0} && (1) \\ &\iff A^{m+1}\boldsymbol{x} = \mathbf{0} && (2) \\ &\iff A^m\boldsymbol{x} = \mathbf{0} && (3) \end{aligned}$$

This means that we have the following situation

$$\mathrm{NS}(A) \subsetneqq \mathrm{NS}(A^2) \subsetneqq \cdots \mathrm{NS}(A^{m-1}) \subsetneqq \mathrm{NS}(A^m) = \mathrm{NS}(A^n) \text{ for all } n \geq m$$

Since $0 < \dim(\mathrm{NS}(A)) < \dim(\mathrm{NS}(A^2)) < \cdots < \dim(\mathrm{NS}(A^m)) \leq 5$ we know $m \leq 5$.

If $A^n = O$ for any n, then $\mathrm{NS}(A^n) = \mathbb{R}^5$. But the first place where $\mathrm{NS}(A^n) = \mathbb{R}^5$ will be for $n \leq 5$ and so $A^5 = O$.

Problem 10 (10 points)**.** For A and B are $n \times n$ matrices. Show that

$$AB \text{ is invertible} \iff \text{both } A \text{ and } B \text{ are invertible}$$

($\Longleftarrow$) case: If A and B are invertible, then AB is invertible, since $(AB)^{-1} = B^{-1}A^{-1}$.

($\implies$) case (Proof 1 using NS) If B is not invertible, then $\mathrm{NS}(B) \neq \{\mathbf{0}\}$, but $B\boldsymbol{x} = \mathbf{0} \implies A(B\boldsymbol{x}) = (AB)\boldsymbol{x} = \mathbf{0}$, so $\mathrm{NS}(AB) \neq \{\mathbf{0}\}$ and hence AB is not invertible.

If B is invertible, but A is not, then again let $\boldsymbol{x} \in \mathrm{NS}(A)$, since B is invertible, $\boldsymbol{x} = B\boldsymbol{y}$ for some $\boldsymbol{y}$, in fact, $\boldsymbol{y} = B^{-1}\boldsymbol{x}$. But then, $A(B\boldsymbol{y}) = (AB)\boldsymbol{y} = \mathbf{0}$ and so $\mathrm{NS}(AB) \neq \{\mathbf{0}\}$, so again AB is not invertible.

So if either A or B is not invertible, then neither is AB, and hence if AB is invertible, then both A and B must be invertible.

($\implies$) case (Proof 2 using det) Suppose AB is invertible, then $0 \neq \det(AB) = \det(A)\det(B)$ so $\det(A) \neq 0 \neq \det(B)$ and so A and B are invertible.

($\implies$) case (Proof 3 using algebra.) Suppose AB is invertible, then $A(B(AB)^{-1}) = (AB)(AB)^{-1} = I$ so $A^{-1} = B(AB)^{-1}$ and $B^{-1} = (AB)^{-1}A$ for a similar reason.

Note: This actually uses that $E = F^{-1}$ iff $EF = I$ **or** $FE = I$, whereas the actual definition has "**and**" not "or." To prove this, one usually uses one of the above arguments.