Exam 1

• This exam covers Topics 1 - 3, Topic 4 will not be covered here.

• I will write $(a_1, a_2, \ldots, a_n)$ in place of $\begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix}$ to save space on occasion. The book writes $\left[\begin{smallmatrix} a_1 & a_2 & \cdots & a_n \end{smallmatrix} \right]^T$ for the same purpose.

• I use $\mathrm{NS}(A)$ for the null space of A, $\mathrm{RS}(A)$ for the row space of A, $\mathrm{CS}(A)$ for the column space of A. Note that $\mathrm{CS}(A) = \mathrm{rng}(A)$ is the range of A.

• When I say you can use some fact from another part of the exam, this means that you can use the fact whether or not you have completed that part of the exam correctly.

• If you have questions ask via Remind or email.

• Use only arguments that you fully understand. I am aware that you can find solutions to some of these online. I am also aware that some of these solutions use concepts and theorems far past what we have covered in Ch 1 - 3. If you use such ideas, then I will ask you to verbally explain your solution so that I can verify that you understand what you have submitted as your own work. In short, **this is an exam** and the usual expectations of academic honesty apply.

Part I: True/False (5 points each; 25 points)

For each of the following mark as (T) true or (F) false.

For the exam you do not need to provide justification.

a) _____ $A\boldsymbol{x} = \mathbf{0} \iff \operatorname{rref}(A)\boldsymbol{x} = \mathbf{0}$

b) _____ $\operatorname{tr}(AB) = \operatorname{tr}(BA)$ for an $n \times n$ matrices A and B, where

$$\operatorname{tr}(C) = \sum_{i=1}^{n} C_{ii} = \text{the sum of the diagonal elements of } C.$$

c) _____ $\operatorname{tr}(AB) = \operatorname{tr}(A)\operatorname{tr}(B)$ for an $n \times n$ matrices A, B, and C.

d) _____ If W is a subspace of a vector space V, then there is a subspace U so that $V = W \oplus U$.

This notation is a little hard to find in your text: $V = U + W$ means that for all $\boldsymbol{v} \in V$, there is $\boldsymbol{u} \in U$ and $\boldsymbol{w} \in W$ so that $\boldsymbol{v} = \boldsymbol{u} + \boldsymbol{w}$. $V = U \oplus V$ means $V = U + V$ **and** $U \cap W = \{\mathbf{0}\}$, equivalently, for every $\boldsymbol{v} \in V$, there is a **unique** $\boldsymbol{u} \in U$ and $\boldsymbol{w} \in W$ so that $\boldsymbol{v} = \boldsymbol{u} + \boldsymbol{w}$.

e) _____ For any $m \times n$ matrices A and B,

$$B = EA \text{ for some invertible } E \iff \operatorname{NS}(A) = \operatorname{NS}(B).$$

Part II: Definitions and short answer (5 points each; 25 points)

a) Define what it means for a set of vectors $\mathcal{B} = \{\boldsymbol{v}_1, \ldots, \boldsymbol{v}_n\}$ from a real vector space V to span V.

b) Define what it means for a set of vectors $\{\boldsymbol{v}_1, \ldots, \boldsymbol{v}_n\}$ from a real vector space V to be linearly independent.

c) What does it mean to say $(c_1, \ldots, c_n) \in \mathbb{R}^n$ represents a vector $\boldsymbol{v} \in V$ with respect to the basis $\mathcal{B} = \{\boldsymbol{v}_1, \ldots, \boldsymbol{v}_n\}$?

d) Define $\dim(V)$.

e) What conditions must be checked to verify that $W \subseteq V$ is a subspace of a vector space. V

Part III: Computational (15 points each; 45 point)

a) Given that A is a 3×4 matrix and

$$\mathrm{NS}(A) = \mathrm{span}\left(\left\{\begin{bmatrix} 1 \\ -2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 3 \\ 0 \\ 1 \end{bmatrix}\right\}\right)$$

compute $\mathrm{rref}(A)$. You may use the fact that

$$\mathrm{rref}(A) = \mathrm{rref}(B) \iff \mathrm{NS}(A) = \mathrm{NS}(B).$$

b) For the same (unknown) A used in (a) for each of RS(A) and CS(A) find a basis if possible and explain how you know that you have found a basis; if it is not possible to find a basis, then explain why it is not.

c) Consider the transformation $D : P_4 \to P_3$ defined by $D(p) = \frac{d}{dx}p$, so specifically, $D(c_4x^4 + c_3x^3 + c_2x^2 + c_1x^1 + c_0) = 4c_4x^3 + 3c_3x^2 + 2c_2x + c_1$. Find the matrix $[L] = [L]_{\mathcal{B},\mathcal{C}}$ for D with respect to the standard basis $\mathcal{B} = \{x^4, x^3, x^2, x, 1\}$ for P_4 and the standard basis $\mathcal{C} = \{x^3, x^2, x, 1\}$ for P_3. Represent $p = 2x^4 - x^2 + 5x - 5$ with respect to $\mathcal{B}$ and then use $[L]$ to get the representation of $D(p)$ with respect to $\mathcal{C}$.

Part IV: Proofs (15 points each; 60 points)

Provide complete arguments/proofs for the following.

a) Prove that $\det(A) = \det(B)$ if A and B are similar.

b) Let $\boldsymbol{V} = (0, \infty) \subset \mathbb{R}$ and define vector addition by $\boldsymbol{v} \oplus \boldsymbol{u} = \boldsymbol{u}\boldsymbol{v}$ and for $\alpha \in \mathbb{R}$ define scalar multiplication by $\alpha \odot \boldsymbol{v} = \boldsymbol{v}^{\alpha}$. Show that $(V, \oplus, \odot)$ is a vector space over $\mathbb{R}$. Make sure to clearly verify all the axioms for being a vector space.

Just to be clear, if $\boldsymbol{u} = 3$ and $\boldsymbol{v} = 2$, then $\boldsymbol{u} \oplus \boldsymbol{v} = (3)(2) = 6$ and $\sqrt{2} \odot \boldsymbol{u} = 3^{\sqrt{2}}$.

c) Let A be a fixed $n \times n$ matrix. Show that

$$AB = BA \text{ for all } n \times n \text{ matrices } B \iff A = \alpha I \text{ for some } \alpha$$