

Homework 5 Partial Solutions

Homework 5 Problems

5.1

1.

(a)

$\cos(\theta) = \frac{\mathbf{w}^T \mathbf{v}}{\|\mathbf{w}\| \|\mathbf{v}\|} = 1$ so \mathbf{w} and \mathbf{v} are in the same direction.

This is clear since $3(2, 1, 3) = (6, 3, 9)$.

5. $y = 2x$ is the same as $U = \text{span}\{\mathbf{u}\}$, where $\mathbf{u} = (1, 2)$. The point in U closest to $\mathbf{v} = (5, 2)$ is the projection $P_U \mathbf{v}$ where P_U is the projection map onto U , this has matrix $P_U = A(A^T A)^{-1} A^T$, where $A = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ and $U = \text{CS}(A)$.

Notice, $A(A^T A)^{-1} A^T \mathbf{v} = \mathbf{u}(\mathbf{u}^T \mathbf{u})^{-1}(\mathbf{u}^T \mathbf{v}) = \frac{\langle \mathbf{v}, \mathbf{u} \rangle}{\langle \mathbf{u}, \mathbf{u} \rangle} \mathbf{u} = \text{proj}_{\mathbf{u}}(\mathbf{v})$, which is how the text defines the projection of \mathbf{v} onto \mathbf{u} .

$$P_U = \frac{1}{5} \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}$$

so $P_U \mathbf{v} = \frac{1}{5}(9, 18)$ or if you like $\text{proj}_{\mathbf{u}}(\mathbf{v}) = \frac{9}{5}(1, 2)$.

13. Let \mathbf{v} and \mathbf{u} be vectors in any inner product space. We have

$$\begin{aligned} \|\mathbf{v} + \mathbf{u}\|^2 &= \langle \mathbf{v} + \mathbf{u}, \mathbf{v} + \mathbf{u} \rangle \\ &= \langle \mathbf{v}, \mathbf{v} + \mathbf{u} \rangle + \langle \mathbf{u}, \mathbf{v} + \mathbf{u} \rangle \\ &= \langle \mathbf{v}, \mathbf{v} \rangle + \langle \mathbf{v}, \mathbf{u} \rangle + \langle \mathbf{u}, \mathbf{v} \rangle + \langle \mathbf{u}, \mathbf{u} \rangle \\ &= \|\mathbf{v}\|^2 + \|\mathbf{u}\|^2 + 2\langle \mathbf{v}, \mathbf{u} \rangle \\ &\leq \|\mathbf{v}\|^2 + \|\mathbf{u}\|^2 + 2\|\mathbf{v}\|\|\mathbf{u}\| && \text{(Cauchy's Theorem)} \\ &= (\|\mathbf{v}\| + \|\mathbf{u}\|)^2 \end{aligned}$$

So $\|\mathbf{v} + \mathbf{u}\| \leq \|\mathbf{v}\| + \|\mathbf{u}\|$.

Equality will hold when $\langle \mathbf{v}, \mathbf{u} \rangle = 0$, this would mean

$$(\langle \mathbf{u}, \mathbf{v} \rangle)^2 = \langle \mathbf{v}, \mathbf{v} \rangle \langle \mathbf{u}, \mathbf{u} \rangle$$

Assuming $\|\mathbf{u}\| \neq 0 \neq \|\mathbf{v}\|$, then we have

$$\begin{aligned}\langle \|\mathbf{v}\|\mathbf{u} - \|\mathbf{u}\|\mathbf{v}, \|\mathbf{v}\|\mathbf{u} - \|\mathbf{u}\|\mathbf{v} \rangle &= \|\mathbf{v}\|\langle \mathbf{u}, \|\mathbf{v}\|\mathbf{u} - \|\mathbf{u}\|\mathbf{v} \rangle - \|\mathbf{u}\|\langle \mathbf{v}, \|\mathbf{v}\|\mathbf{u} - \|\mathbf{u}\|\mathbf{v} \rangle \\ &= \|\mathbf{v}\|^2\langle \mathbf{u}, \mathbf{u} \rangle - \|\mathbf{v}\|\|\mathbf{u}\|\langle \mathbf{u}, \mathbf{v} \rangle - \|\mathbf{u}\|\|\mathbf{v}\|\langle \mathbf{v}, \mathbf{u} \rangle + \|\mathbf{u}\|^2\langle \mathbf{v}, \mathbf{v} \rangle \\ &= 2\|\mathbf{v}\|^2\|\mathbf{v}\|^2 - 2(\|\mathbf{u}\|\|\mathbf{v}\|)^2 = 0\end{aligned}$$

So $\|\mathbf{v}\|\mathbf{u} - \|\mathbf{u}\|\mathbf{v} = \mathbf{0}$, but this means that \mathbf{u} and \mathbf{v} differ by a scalar multiple.

18.

(a) Show that

$$p_{\mathbf{y}}(\mathbf{x}) = \frac{\langle \mathbf{x}, \mathbf{y} \rangle}{\langle \mathbf{y}, \mathbf{y} \rangle} \mathbf{y}$$

is the orthogonal projection of \mathbf{x} onto \mathbf{y} . That is, $\mathbf{x} - p_{\mathbf{y}}(\mathbf{x}) \perp \mathbf{y}$.

Notice the usual inner product is $\langle \mathbf{x}, \mathbf{y} \rangle = \mathbf{y}^T \mathbf{x}$, so this does solve the problem.

Clearly

$$\langle \mathbf{x} - p_{\mathbf{y}}(\mathbf{x}), \mathbf{y} \rangle = \langle \mathbf{x}, \mathbf{y} \rangle - \frac{\langle \mathbf{x}, \mathbf{y} \rangle}{\langle \mathbf{y}, \mathbf{y} \rangle} \langle \mathbf{y}, \mathbf{y} \rangle = 0$$

(b) In arguing for (13) above we see $\|\mathbf{u} + \mathbf{v}\|^2 = \|\mathbf{u}\|^2 + \|\mathbf{v}\|^2 \iff \langle \mathbf{u}, \mathbf{v} \rangle = 0$, so here we have

$$\|\mathbf{x}\|^2 = \|\mathbf{x} - p_{\mathbf{y}}(\mathbf{x}) + p_{\mathbf{y}}(\mathbf{x})\|^2 = \|\mathbf{x} - p_{\mathbf{y}}(\mathbf{x})\|^2 + \|p_{\mathbf{y}}(\mathbf{x})\|^2 = 8^2 + 6^2 = 10^2$$

So $\|\mathbf{x}\| = 10$.

5.2

1.

(d) Here

$$A = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 1 & 1 & 2 & 2 \end{bmatrix} \quad \text{and} \quad \text{rref}(A) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

So we read off a basis for the four subspaces:

RS(A) = CS(A^T): A basis is $\{(1, 0, 0, 0), (0, 1, 0, 0), (0, 0, 1, 1)\}$ (the non-zero rows of $\text{rref}(A)$)

CS(A): A basis is $\{(1, 0, 0, 1), (0, 1, 0, 1), (0, 1, 1, 2)\}$ (the pivot columns of A).

NS(A): A basis is $\{(0, 0, 0, 1)\}$ since we see that the solutions to $A\mathbf{x} = \mathbf{0}$ are of the form $\mathbf{x} = t(0, 0, -1, 1)$.

Note: NS(A) \perp CS(A^T) as we know must happen.

5. If A is 3×2 and of rank 2, then RS(A) is a 2-dimensional subspace of \mathbb{R}^2 , hence RS(A) = \mathbb{R}^2 . $\mathbb{R}^3 = \text{NS}(A^T) \oplus \text{RS}(A^T)$ and RS(A^T) = CS(A) is a 2-dimensional subset of \mathbb{R}^3 , a plane. So RS(A^T) is a plane in \mathbb{R}^3 and NS(A^T) is the line normal to that plane.

13.

(a) Let $\mathbf{x} \in \text{NS}(A^T A)$. $A\mathbf{x} \in \text{rng}(A)$ by definition of $\text{rng}(A)$, no assumption necessary. Of course $\mathbf{x} \in \text{NS}(A^T A)$ means $(A^T A)\mathbf{x} = \mathbf{0}$, but $(A^T A)\mathbf{x} = A^T(A\mathbf{x}) = \mathbf{0} \implies A\mathbf{x} \in \text{NS}(A^T)$.

(b) For any \mathbf{x} , $\mathbf{x} \in \text{NS}(A) \implies A\mathbf{x} = \mathbf{0} \implies A^T A\mathbf{x} = \mathbf{0}$. So

$$\text{NS}(A) \subseteq \text{NS}(A^T A)$$

So let $\mathbf{x} \in \text{NS}(A^T A)$, then $A\mathbf{x} \in \text{NS}(A^T) \cap \text{rng}(A)$ by (a). But $\text{rng}(A) = \text{CS}(A) = \text{RS}(A^T)$ and so $A\mathbf{x} \in \text{NS}(A^T) \cap \text{RS}(A^T)$. But $\text{NS}(A^T) \perp \text{RS}(A^T)$ so $\text{NS}(A^T) \cap \text{RS}(A^T) = \{\mathbf{0}\}$. Thus $A\mathbf{x} = \mathbf{0}$ and $\mathbf{x} \in \text{NS}(A)$. So we have

$$\text{NS}(A^T A) \subseteq \text{NS}(A)$$

So we have $\text{NS}(A^T A) = \text{NS}(A)$ as desired.

(c) If A is $m \times n$, then $A^T A$ is $n \times n$ and the rank-nullity theorem gives

$$\text{rank}(A) + \dim(\text{NS}(A)) = n = \text{rank}(A^T A) + \dim(\text{NS}(A^T A))$$

By (b), $\dim(\text{NS}(A)) = \dim(\text{NS}(A^T A))$ so $\text{rank}(A) = \text{rank}(A^T A)$.

(d) If A has independent columns, then $\text{rank}(A) = n = \text{rank}(A^T A)$ so $A^T A$ has full rank and is invertible.

14 A is $m \times n$, B is $n \times r$. and $C = AB$.

(a) $\mathbf{x} \in \text{NS}(B) \implies B\mathbf{x} = \mathbf{0} \implies AB\mathbf{x} = \mathbf{0} \implies \mathbf{x} \in \text{NS}(C)$, so $\text{NS}(B) \subseteq \text{NS}(C)$.

(b) $\text{NS}(C) \oplus \text{NS}(C)^\perp = \mathbb{R}^r = \text{NS}(B) \oplus \text{NS}(B)^\perp$, so $\text{rng}(C^T) = \text{CS}(C^T) = \text{NS}(C)^\perp \subseteq \text{NS}(B)^\perp = \text{CS}(B^T) = \text{rng}(B^T)$.

15. By definition $W = U \oplus V$ iff $W = U + V$ for any $\mathbf{w} \in W$, there are unique \mathbf{u} and \mathbf{v} so that $\mathbf{w} = \mathbf{u} + \mathbf{v}$.

If $\mathbf{z} \in U \cap V$ and $\mathbf{z} \neq \mathbf{0}$, then $\mathbf{z} = \mathbf{u} + \mathbf{v}$, where there are two options $\mathbf{u} = \mathbf{z}$ and $\mathbf{v} = \mathbf{0}$ or $\mathbf{u} = \mathbf{0}$ and $\mathbf{v} = \mathbf{z}$.

16. Let $\text{rank}(A) = k$ and $\{\mathbf{x}_1, \dots, \mathbf{x}_k\}$ be a basis for $\text{RS}(A) = \text{rng}(A^T)$. Then to show that $\{A\mathbf{x}_1, \dots, A\mathbf{x}_k\}$ is a basis for $\text{rng}(A) = \text{CS}(A)$ we only need to check independence, since we know $\dim(\text{RS}(A)) = \dim(\text{CS}(A)) = \text{rank}(A)$. Suppose $\sum_{i=1}^k \alpha_i A\mathbf{x}_i = \mathbf{0}$, then $A\left(\sum_{i=1}^k \alpha_i \mathbf{x}_i\right) = \mathbf{0}$, so $\sum_{i=1}^k \alpha_i \mathbf{x}_i \in \text{NS}(A)$. But $\text{NS}(A) \cap \text{rng}(A^T) = \{\mathbf{0}\}$ and thus $\sum_{i=1}^k \alpha_i \mathbf{x}_i = \mathbf{0}$ and so $\alpha_i = 0$ for all i .

5.3

3.

(b) Find first the orthogonal projection of \mathbf{b} onto $\text{CS}(A)$. The first two columns of A are a basis for $\text{CS}(A)$ so we can use those two to find the projection $\hat{\mathbf{b}} = B(B^T B)^{-1} B^T \mathbf{b}$ where

$B = A(:, 1 : 2)$. We find $\hat{\mathbf{b}} = (3, 1, 4)^T$ now we can ask what $\hat{\mathbf{x}}$ satisfy $A\hat{\mathbf{x}} = (3, 1, 4)^T$. Solving this you find solutions have the form

$$\hat{\mathbf{x}} = \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} + t \begin{bmatrix} -2 \\ -1 \\ 1 \end{bmatrix}$$

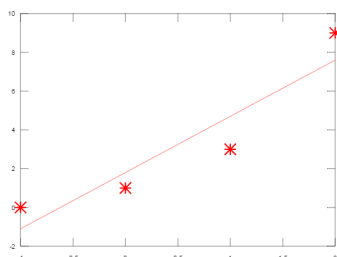
5. Trying to find $(\alpha, \beta)^T$ so that

$$\underbrace{\begin{bmatrix} 1 & -1 \\ 1 & 0 \\ 1 & 1 \\ 1 & 2 \end{bmatrix}}_A \underbrace{\begin{bmatrix} \alpha \\ \beta \end{bmatrix}}_{\hat{\mathbf{x}}} = \underbrace{\begin{bmatrix} 0 \\ 1 \\ 3 \\ 9 \end{bmatrix}}_b$$

For this $\begin{bmatrix} \alpha \\ \beta \end{bmatrix} = (A^T A)^{-1} A^T \mathbf{b}$ and we get:

$$\begin{bmatrix} \alpha \\ \beta \end{bmatrix} = \begin{bmatrix} 1.8 \\ 2.9 \end{bmatrix}$$

So the equation is $y = 2.9x + 1.8$.



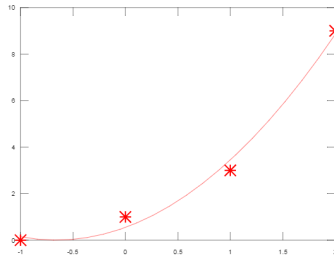
6. Trying to find $(\alpha, \beta)^T$ so that

$$\underbrace{\begin{bmatrix} 1 & -1 & 1 \\ 1 & 0 & 0 \\ 1 & 1 & 1 \\ 1 & 2 & 4 \end{bmatrix}}_A \underbrace{\begin{bmatrix} \alpha \\ \beta \\ \gamma \end{bmatrix}}_{\hat{\mathbf{x}}} = \underbrace{\begin{bmatrix} 0 \\ 1 \\ 3 \\ 9 \end{bmatrix}}_b$$

For this $\begin{bmatrix} \alpha \\ \beta \end{bmatrix} = (A^T A)^{-1} A^T \mathbf{b}$ and we get:

$$\begin{bmatrix} \alpha \\ \beta \\ \gamma \end{bmatrix} = \begin{bmatrix} 0.55 \\ 1.65 \\ 1.25 \end{bmatrix}$$

So the equation is $y = 1.25x^2 + 1.65x + 0.55$.



13. We know that if A is $m \times n$ and $\mathbf{b} \in \mathbb{R}^m$, then there is a unique $\hat{\mathbf{b}} \in \text{rng}(A) = \text{CS}(A)$ so that $\hat{\mathbf{b}} - \mathbf{b} \perp \text{rng}(A)$ and hence $\|\hat{\mathbf{b}} - \mathbf{b}\|$ is minimal among all $\|\mathbf{y} - \mathbf{b}\|$ for all $\mathbf{y} \in \text{rng}(A)$.

You should understand why $\hat{\mathbf{b}} = A(A^T A)^{-1} A^T \mathbf{b}$. So that $\hat{\mathbf{x}} = (A^T A)^{-1} A^T \mathbf{b}$ is such that $\|A\hat{\mathbf{x}} - \mathbf{b}\|$ is minimized and what you know is $A^T A\hat{\mathbf{x}} = A^T \mathbf{b}$.

We are assuming that $\hat{\mathbf{x}}$ is a least-squares solution to $A\mathbf{x} = \mathbf{b}$, so $A^T A\hat{\mathbf{x}} = A^T \mathbf{b}$. There are two things to prove:

(1) If \mathbf{y} is a least-squares solution to $A\mathbf{x} = \mathbf{b}$, then $\mathbf{y} = \hat{\mathbf{x}} + \mathbf{z}$ for $\mathbf{z} \in \text{NS}(A)$.

We know $A^T A\mathbf{y} = A^T \mathbf{b} = A^T A\hat{\mathbf{x}}$ so $A^T A(\mathbf{y} - \hat{\mathbf{x}}) = 0$. Let $\mathbf{z} = \mathbf{y} - \hat{\mathbf{x}}$, then $\mathbf{z} \in \text{NS}(A^T A) = \text{NS}(A)$ and $\mathbf{y} = \hat{\mathbf{x}} + \mathbf{z}$ as desired.

(2) If $\mathbf{y} = \hat{\mathbf{x}} + \mathbf{z}$ for some $\mathbf{z} \in \text{NS}(A)$, then \mathbf{y} is a least square-solution to $A\mathbf{x} = \mathbf{b}$.

$A^T A(\hat{\mathbf{x}} + \mathbf{z}) = A^T A\hat{\mathbf{x}} + A^T A\mathbf{z} = \mathbf{b} + \mathbf{0} = \mathbf{b}$. So $\hat{\mathbf{x}} + \mathbf{z}$ is a least-squares solution to $A\mathbf{x} = \mathbf{b}$.

14. We want to fit a circle to the data $(-1, -2), (0, 2.4), (1.1, -4), (2.4, -1.6)$. An arbitrary circle would satisfy

$$x^2 + y^2 + ax + by + c = 0$$

or

$$ax + by + c = -(x^2 + y^2)$$

So we are trying to solve $A\mathbf{x} = \mathbf{b}$ where

$$\underbrace{\begin{bmatrix} -1 & -2 & 1 \\ 0 & 2.4 & 1 \\ 1.1 & -4 & 1 \\ 2.4 & -1.6 & 1 \end{bmatrix}}_A \underbrace{\begin{bmatrix} a \\ b \\ c \end{bmatrix}}_{\mathbf{x}} = \underbrace{\begin{bmatrix} -((-1)^2 + (-2)^2) \\ -((0)^2 + (2.4)^2) \\ -((1.1)^2 + (-4)^2) \\ -((2.4)^2 + (1.6)^2) \end{bmatrix}}_{\mathbf{b}}$$

Here is [MATLAB](#) code for this and here it is done with [Desmos](#).

5.4

4. Given

$$A = \begin{bmatrix} 1 & 2 & 2 \\ 1 & 0 & 2 \\ 3 & 1 & 1 \end{bmatrix} \quad B = \begin{bmatrix} -4 & 1 & 1 \\ -3 & 3 & 2 \\ 1 & -2 & -2 \end{bmatrix}$$

(a)

$$\begin{aligned}\langle A, B \rangle &= (1)(-4) + (2)(1) + (2)(1) + (1)(-3) + (0)(3) \\ &\quad + (2)(2) + (3)(1) + (1)(-2) + (1)(-2) = 0\end{aligned}$$

So A and B are orthogonal.

(b)

$$\|A\|_F^2 = (1)^2 + (2)^2 + (2)^2 + (1)^2 + (0)^2 + (2)^2 + (3)^2 + (1)^2 + (1)^2 = 25$$

(c)

$$\|A\|_F^2 = (-4)^2 + (1)^2 + (1)^2 + (-3)^2 + (3)^2 + (2)^2 + (1)^2 + (-2)^2 + (-2)^2 = 49$$

(d) $\|A + B\|_F^2 = \|A\|_F^2 + \|B\|_F^2 = 25 + 49 = 74$, since $A \perp_F B$.

8.

(a)

$$\cos(\theta) = \frac{\langle 1, x \rangle}{\|1\| \cdot \|x\|} = \frac{\int_0^1 (1 \cdot x) dx}{\left(\int_0^1 1^2 dx\right)^{1/2} \cdot \left(\int_0^1 x^2 dx\right)^{1/2}} = \frac{1/2}{(1)^{1/2} \cdot (1/3)^{1/2}} = \sqrt{3}/2$$

So $\theta = \pi/6$. Of course, this really doesn't mean anything, the relevant thing here is $\frac{\langle 1, x \rangle}{\|1\| \cdot \|x\|}$.

(b) $p = p_x(1) = \frac{\langle 1, x \rangle}{\langle x, x \rangle} x = \frac{\int_0^1 x dx}{\int_0^1 x^2 dx} = \frac{1/2}{1/3} x = (3/2)x$.

Check that $1 - p \perp p$, $\langle p, 1 - p \rangle = \int_0^1 (3/2)x(1 - (3/2)x) dx = \int_0^1 (3/2)x - (3/2)^2 x^2 dx = (3/2)(1/2) - (1/3)(3/2)^2 = 0$!

(c)

$$\begin{aligned}\|1\|^2 &= \langle 1, 1 \rangle = \int_0^1 1 dx = 1 \\ \|p\|^2 &= \|(3/2)x\|^2 = (3/2)^2 \langle x, x \rangle = \int_0^1 x^2 dx = (3/2)^2 \frac{1}{3} = 3/4 \\ \|1 - p\|^2 &= \int_0^1 (1 - (3/2)x)^2 dx = (2/3) \int_1^{-1/2} u^2 du \\ &= (2/3)(1/3)((-1/2)^3 - (1^3)) = (2/3)(1/3)(-9/8) = 1/4\end{aligned}$$

Now

$$\|p\|^2 + \|1 - p\|^2 = 3/4 + 1/4 = 1 = \|1\|^2 = \|p + (1 - p)\|^2$$

Verifying Pythagoras.

10. Let $x_i = (i - 3)/2$ for $i = 1, \dots, 5$. Define $\langle p, q \rangle = \sum_{i=1}^5 p(x_i)q(x_i)$ for $p, q \in \mathbb{P}_5[x]$.

$$\langle x, x^2 \rangle = \sum_{i=1}^5 (i - 3)/2 \cdot (i - 3)/2^2 = \frac{1}{8} \sum_{i=1}^5 (i - 3)^3 = \frac{1}{8}((-2)^3 + (-1)^3 + (0)^3 + (1)^3 + (2)^3) = 0$$

So $x \perp x^2$ as desired.

5.5

29 Use $\langle f, g \rangle = \int_{-1}^1 f \cdot g \, dx$

(a) $\langle 1, x \rangle = \int_{-1}^1 1 \cdot x \, dx = \frac{1}{2}x^2 \Big|_{-1}^1 = 0$. So $1 \perp x$.

(b)

$$\begin{aligned}\|1\|^2 &= \langle 1, 1 \rangle = \int_{-1}^1 1 \cdot 1 \, dx = x \Big|_{-1}^1 = 2 \\ \|x\|^2 &= \langle x, x \rangle = \int_{-1}^1 x^2 \, dx = \frac{1}{3}x^3 \Big|_{-1}^1 = \frac{2}{3}\end{aligned}$$

(c) To find the best approximation to $x^{1/3}$ we project $x^{1/3}$ onto $\text{span}\{1, x\}$

$$\begin{aligned}p_1(x^{1/3}) &= \frac{\langle x^{1/3}, 1 \rangle}{\langle 1, 1 \rangle} 1 = \frac{\int_{-1}^1 x^{1/3} \, dx}{2} = \frac{(3/4)x^{4/3} \Big|_{-1}^1}{2} = 0 \\ p_x(x^{1/3}) &= \frac{\langle x^{1/3}, x \rangle}{\langle x, x \rangle} x = \frac{\int_{-1}^1 x^{4/3} \, dx}{(2/3)} x = \frac{(3/7)x^{7/3} \Big|_{-1}^1}{(2/3)} x = 2(3/7)(3/2)x = 9/7 x\end{aligned}$$

So the projection of $x^{1/3}$ onto $\text{span}\{1, x\}$ is $0 \cdot 1 + 9/7 \cdot x = 9/7 x$

[Here is a Desmos](#) graph to illustrate this.

30

(a) Let $f_1 = 1$ and $f_2 = 2x - 1$, $\langle f_1, f_2 \rangle = \int_0^1 f_1 \cdot f_2 \, dx = \int_0^1 (2x - 1) \, dx = x^2 - x \Big|_0^1 = 0$, so $f_1 \perp f_2$.

(b) $\|f_1\|_2^2 = \langle f_1, f_1 \rangle = \int_0^1 f_1^2 \, dx = \int_0^1 (1)^2 \, dx = x \Big|_0^1 = 1$ and $\|f_2\|_2^2 = \int_0^1 (2x - 1)^2 \, dx = \frac{1}{2} \int_{-1}^1 u^2 \, du = \frac{1}{2} \frac{u^3}{3} \Big|_{-1}^1 = \frac{1}{3}$. So $\|f_1\| = 1$ and $\|f_2\| = \frac{1}{\sqrt{3}}$, so the unit vector in the direction of f_2 is $\sqrt{3}f_2$.

(c) The projection of $g(x) = \sqrt{x}$ onto $\text{span}\{f_1, f_2\}$ is $\langle g, f_1 \rangle \cdot f_1 + \langle g, \sqrt{3}f_2 \rangle \cdot \sqrt{3}f_2 = \langle g, f_1 \rangle \cdot f_1 + \langle g, f_2 \rangle \cdot 3f_2$. We have

$$\begin{aligned}\langle g, f_1 \rangle &= \int_0^1 x^{1/2} \, dx = \frac{2}{3}x^{3/2} \Big|_0^1 = \frac{2}{3} \\ \langle g, f_2 \rangle &= \int_0^1 x^{1/2}(2x - 1) \, dx \\ &= \int_0^1 2x^{3/2} - x^{1/2} \, dx \\ &= \left(2 \cdot \frac{2}{5}x^{5/2} - \frac{2}{3}x^{3/2} \right) \Big|_0^1 = \frac{2}{15}\end{aligned}$$

So the projection of $g(x) = \sqrt{x}$ onto $\text{span}\{f_1, f_2\}$ is

$$\hat{g} = \frac{2}{3} \cdot f_1 + \frac{2}{15} \cdot 3 \cdot f_2 = \frac{2}{3} + \frac{2}{5}(2x - 1) = \frac{4}{5}x + \frac{4}{15}$$

See demo [here](#)

1 5.6

4. The strategy here is simple:

- Start with $\mathcal{V} = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\} = \{1, x, x^2\}$.
- $\mathbf{u}_1 = \mathbf{v}_1$
- $\mathbf{q}_1 = \mathbf{u}_1 / \|\mathbf{u}_1\|$
- $\mathbf{u}_2 = \mathbf{v}_2 - \langle \mathbf{v}_2, \mathbf{q}_1 \rangle \mathbf{q}_1$
- $\mathbf{q}_2 = \mathbf{u}_2 / \|\mathbf{u}_2\|$
- $\mathbf{u}_3 = \mathbf{v}_3 - \langle \mathbf{v}_3, \mathbf{q}_1 \rangle \mathbf{q}_1 - \langle \mathbf{v}_3, \mathbf{q}_2 \rangle \mathbf{q}_2$
- $\mathbf{q}_3 = \mathbf{u}_3 / \|\mathbf{u}_3\|$

That is the idea, here are the details. This is how all simple Gram-Schmidt proceeds.

Choose $\mathbf{u}_1 = 1$, then $\langle 1, 1 \rangle = \int_0^1 1 \, dx = 1$ so this is already normalized and so set $\mathbf{q}_1 = \mathbf{u}_1$.

Set $\mathbf{u}_2 = x - \langle x, \mathbf{q}_1 \rangle \mathbf{q}_1 = x - \int_0^1 x \, dx = x - \frac{1}{2}$. Now $\|\mathbf{u}_2\|^2 = \int_0^1 \left(x - \frac{1}{2}\right)^2 \, dx = \int_0^1 x^2 - x + \frac{1}{4} \, dx = \frac{1}{3}x^3 - \frac{1}{2}x^2 + \frac{1}{4}x \Big|_{x=0}^{x=1} = \frac{1}{3} - \frac{1}{2} + \frac{1}{4} = \frac{1}{12}$. So $\mathbf{q}_2 = \sqrt{12} \left(x - \frac{1}{2}\right) = \sqrt{3}(2x - 1)$.

Finally, $\mathbf{u}_3 = x^2 - \langle x^2, \mathbf{q}_2 \rangle \mathbf{q}_2 - \langle x^2, \mathbf{q}_1 \rangle \mathbf{q}_1$. We have $\langle x^2, \mathbf{q}_2 \rangle = \int_0^1 \sqrt{3}(2x - 1)x^2 \, dx = \sqrt{3} \left(\frac{1}{2}x^4 - \frac{1}{3}x^3\right) \Big|_{x=0}^{x=1} = \frac{1}{2\sqrt{3}}$. So $\langle x^2, \mathbf{q}_2 \rangle \mathbf{q}_2 = \left(x - \frac{1}{2}\right)$. Also, $\langle x^2, \mathbf{q}_1 \rangle = \int_0^1 x^2 \, dx = \frac{1}{3}$, so $\mathbf{u}_3 = x^2 - \left(x - \frac{1}{2}\right) - \frac{1}{3} = x^2 - x + \frac{1}{6}$.

We have $\|\mathbf{u}_3\|^2 = \int_0^1 \left(x^2 - x + \frac{1}{6}\right)^2 \, dx = \int_0^1 x^4 - 2x^3 + \frac{4}{3}x^2 - \frac{1}{3}x + \frac{1}{36} \, dx = \frac{1}{180}$ and so $\mathbf{q}_3 = \sqrt{5}(6x^2 - 6x + 1)$.

[A SageCell page that does computations](#)

5. Let

(a)

$$A = \begin{bmatrix} 2 & 1 \\ 1 & 1 \\ 2 & 1 \end{bmatrix} = [\mathbf{a}_1 \quad \mathbf{a}_2]$$

Let

$$\begin{aligned} \mathbf{q}_1 &= \mathbf{a}_1 = (2, 1, 2)^T \\ \mathbf{q}_2 &= \mathbf{a}_2 - \frac{\langle \mathbf{a}_1, \mathbf{q}_1 \rangle}{\langle \mathbf{q}_1, \mathbf{q}_1 \rangle} \mathbf{q}_1 \\ &= (1, 1, 1)^T - \frac{\langle (2, 1, 2)^T, (1, 1, 1)^T \rangle}{\langle (2, 1, 2)^T, (2, 1, 2)^T \rangle} (2, 1, 2)^T \\ &= (1/9)(-1, 4, -1)^T \end{aligned}$$

Check that this is orthogonal to $(2, 1, 2)^T$.

Now just normalize these

$$\begin{aligned}\hat{\mathbf{q}}_1 &= (2/3, 1/3, 2/3)^T \\ \hat{\mathbf{q}}_2 &= (1/(3\sqrt{2}))(-1, 4, -1)^T\end{aligned}$$

So

$$Q = \begin{bmatrix} 2/3 & -1/(3\sqrt{2}) \\ 1/3 & 4/(3\sqrt{2}) \\ 2/3 & -1/(3\sqrt{2}) \end{bmatrix}$$

$$R = Q^T A = \begin{bmatrix} \langle \hat{\mathbf{q}}_1, \mathbf{a}_1 \rangle & \langle \hat{\mathbf{q}}_1, \mathbf{a}_2 \rangle \\ 0 & \langle \hat{\mathbf{q}}_2, \mathbf{a}_2 \rangle \end{bmatrix} = \begin{bmatrix} 3 & 5/3 \\ 0 & 2/(3\sqrt{2}) \end{bmatrix}.$$

Check: $A = QR$.

7. We know $\text{CS}(A) \perp \text{NS}(A)$ where $A = \begin{bmatrix} 1 & 1 & 1 & -1 \\ 1 & 1 & 3 & 5 \end{bmatrix}$. To find $\ker(A)$ start with $\text{rref}(A) = \begin{bmatrix} 1 & 1 & 0 & -4 \\ 0 & 0 & 1 & 3 \end{bmatrix}$. So letting $x_2 = s$ and $x_4 = t$ we have

$$\mathbf{x} \in \ker(A) \iff \mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = s \begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \end{bmatrix} + t \begin{bmatrix} 4 \\ 0 \\ -3 \\ 1 \end{bmatrix}$$

so a basis for $\ker(A) = \text{span}\{(-1, 1, 0, 0)^T, (4, 0, -3, 1)^T\}$

Check that these are indeed orthogonal to the given vectors.

Now use GS

$$\begin{aligned}\mathbf{q}_3 &= (-1, 1, 0, 0)^T \\ \mathbf{q}_4 &= (4, 0, -3, 1)^T - \frac{\langle (4, 0, -3, 1)^T, (-1, 1, 0, 0)^T \rangle}{\langle (-1, 1, 0, 0)^T, (-1, 1, 0, 0)^T \rangle} (-1, 1, 0, 0)^T = (2, 2, -3, 1)^T\end{aligned}$$

Now just normalize to make these into unit vectors.

$$\hat{\mathbf{q}}_3 = \frac{1}{\sqrt{2}} \begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \end{bmatrix} \quad \hat{\mathbf{q}}_4 = \frac{1}{3\sqrt{2}} \begin{bmatrix} 2 \\ 2 \\ -3 \\ 1 \end{bmatrix}$$

8. Use Gram-Schmidt to find orthonormal basis for $\text{span}\{\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3\}$ where

$$\mathbf{x}_1 = \begin{bmatrix} 4 \\ 2 \\ 2 \\ 1 \end{bmatrix} \quad \mathbf{x}_2 = \begin{bmatrix} 2 \\ 0 \\ 0 \\ 2 \end{bmatrix} \quad \mathbf{x}_3 = \begin{bmatrix} 1 \\ 1 \\ -1 \\ 1 \end{bmatrix}$$

$$\begin{aligned}
\mathbf{q}_1 &= \mathbf{x}_1 \\
\mathbf{q}_2 &= \mathbf{x}_2 - \frac{\langle \mathbf{q}_1, \mathbf{x}_2 \rangle}{\langle \mathbf{q}_1, \mathbf{q}_1 \rangle} \mathbf{q}_1 \\
&= (2, 0, 0, 2) - \frac{10}{25}(4, 2, 2, 1) \\
&= (2/5, -4/5, -4/5, 8/5) = (2/5)(1, -2, -2, 4) \\
\mathbf{q}_3 &= \mathbf{x}_3 - \frac{\langle \mathbf{x}_3, \mathbf{q}_1 \rangle}{\langle \mathbf{q}_1, \mathbf{q}_1 \rangle} \mathbf{q}_1 - \frac{\langle \mathbf{x}_3, \mathbf{q}_2 \rangle}{\langle \mathbf{q}_2, \mathbf{q}_2 \rangle} \mathbf{q}_2 \\
&= (1, 1, -1, 1) - \frac{5}{25}(4, 2, 2, 1) - \frac{(2/5)(5)}{(2/5)^2(25)}(2/5)(1, -2, -2, 4) \\
&= (1, 1, -1, 1) - (4/5, 2/5, 2/5, 1/5) - (1/5, -2/5, -2/5, 4/5) \\
&= (0, 1, -1, 0) = (0, 1, -1, 0)
\end{aligned}$$

So the final normalized orthonormal basis is

$$\mathbf{q}_1 = \left(\frac{1}{5}\right) \begin{bmatrix} 4 \\ 2 \\ 2 \\ 1 \end{bmatrix} \quad \mathbf{q}_2 = \left(\frac{1}{5}\right) \begin{bmatrix} 1 \\ -2 \\ -2 \\ 4 \end{bmatrix} \quad \mathbf{q}_3 = \left(\frac{1}{2}\right)^{1/2} \begin{bmatrix} 0 \\ 1 \\ -1 \\ 0 \end{bmatrix}$$

9. The modified Gram-Schmidt looks like:

First pass:

$$\begin{aligned}
\mathbf{q}_1 &= \mathbf{x}_1 \\
\mathbf{q}_2 &= \mathbf{x}_2 - \frac{\langle \mathbf{q}_1, \mathbf{x}_2 \rangle}{\langle \mathbf{q}_1, \mathbf{q}_1 \rangle} \mathbf{q}_1 \\
&= (2, 0, 0, 2) - \frac{10}{25}(4, 2, 2, 1) \\
&= (2/5, -4/5, -4/5, 8/5) = (2/5)(1, -2, -2, 4) \\
\mathbf{q}_3 &= \mathbf{x}_3 - \frac{\langle \mathbf{x}_3, \mathbf{q}_1 \rangle}{\langle \mathbf{q}_1, \mathbf{q}_1 \rangle} \mathbf{q}_1 \\
&= (1, 1, -1, 1) - \frac{5}{25}(4, 2, 2, 1) \\
&= (1/5, 3/5, -7/5, 4/5) = (1/5)(1, 3, -7, 4)
\end{aligned}$$

Now \mathbf{q}_2 and \mathbf{q}_3 are orthogonal to \mathbf{q}_1 .

Second pass:

$$\begin{aligned}
\mathbf{q}'_1 &= \mathbf{q}_1 \\
\mathbf{q}'_2 &= \mathbf{q}_2 = (2/5)(1, -2, -2, 4) \\
\mathbf{q}'_3 &= \mathbf{q}_3 - \frac{\langle \mathbf{q}_3, \mathbf{q}_2 \rangle}{\langle \mathbf{q}_2, \mathbf{q}_2 \rangle} \mathbf{q}_2 \\
&= (1/5)(1, 3, -7, 4) - \frac{(1/5)(2/5)25}{(2/5)^2(25)}(2/5)(1, -2, -2, 4) \\
&= (1/5)(1, 3, -7, 4) - (1/5)(1, -2, -2, 4) \\
&= (1/5)(0, 5, -5, 0) = (0, 1, -1, 0)
\end{aligned}$$

Now just normalize and get the same answer as in (8).

14. and 15. Let \mathcal{B} be a basis for $W = U \cap V$ and let $\mathcal{B}_U \supseteq \mathcal{B}$ be a basis for U that extends \mathcal{B}_W . Extend \mathcal{B}_U to \mathcal{B}_{V+U} a basis for $V + U$. For each $\mathbf{z} \in \mathcal{B}_{V+U} - \mathcal{B}_U$, $\mathbf{z} = \mathbf{v}_z + \mathbf{u}_z$ and as $\mathbf{z} \notin U$ we know $\mathbf{v}_z \neq 0$ and if $\mathcal{C} = \langle \mathbf{w}_z \mid \mathbf{z} \in \mathcal{B}_{V+U} - \mathcal{B}_U \rangle$, then clearly $\mathcal{B}_U \cup \mathcal{C}$ is still independent and spans $V + U$. So we may assume $\mathcal{B}_{U+V} - \mathcal{B}_U \subset V$.

Claim: $\mathcal{B}_V \stackrel{\text{def}}{=} \mathcal{B}_W \cup (\mathcal{B}_{U+V} - \mathcal{B}_U)$ is a basis for V .

Independence is for free since \mathcal{B}_{V+U} is independent. So we must show that \mathcal{B}_V spans V . Let $\mathbf{v} \in V$ and $\mathbf{v} = \mathbf{w} + \mathbf{u} + \mathbf{z}$ where $\mathbf{w} \in \text{span}(\mathcal{B}_W)$, $\mathbf{u} \in \text{span}(\mathcal{B}_U - \mathcal{B}_W)$, and $\mathbf{z} \in \text{span}(\mathcal{B}_{U+V} - \mathcal{B}_U) \subseteq V$.

If $\mathbf{u} \neq \mathbf{0}$, then $\mathbf{u} = \mathbf{v} - (\mathbf{w} + \mathbf{z}) \in V$, but then $\mathbf{u} \in U \cap V = W$ which is impossible since then a non-zero linear combination from $\mathcal{B}_U - \mathcal{B}_W$ is also a linear combination from \mathcal{B}_W , contradicting the independence of \mathcal{B}_U .

So $\mathbf{v} = \mathbf{w} + \mathbf{z} \in \text{span}(\mathcal{B}_V)$. The

This shows $|\mathcal{B}_U \cup \mathcal{B}_V| = |\mathcal{B}_U| + |\mathcal{B}_V| - |\mathcal{B}_U \cap \mathcal{B}_V|$ so $\dim(U+V) = \dim(U) + \dim(V) - \dim(U \cap V)$.