Exam 2

To avoid any confusion, unless specified otherwise, vector spaces are complex vector spaces, inner-products are complex inner-products, and matrices are complex matrices. The standard inner product is $\langle \boldsymbol{u}, \boldsymbol{v} \rangle = \boldsymbol{v}^H \boldsymbol{u} = \sum_{i=1}^{n} \bar{\boldsymbol{v}}_i \boldsymbol{u}_i$. Keep in mind that $A^H = A^T$ for real matrices and symmetric = Hermitian for real matrices.

Part I: True/False

Each problem is points for a total of 50 points. (5 points each.)

You do not need to justify the answers here, this is unlike the quizzes.

1. ______ If U is unitary, then U is itself unitarily diagonalizable. This means there is a unitary V so that $U = VDV^H$ where D is diagonal.

2. ______ For any diagonalizable matrix A, one can use Gram-Schmidt to find an orthogonal basis consisting of eigenvectors.

3. ______ The collection of rank k $n \times n$ matrices is a subspace of $\mathbb{R}^{n \times n}$, for $k < n$.

4. ______ If A is unitary, then $|\lambda| = 1$ for all eigenvalues λ of A.

5. ______ If $p(t)$ is a polynomial and $\boldsymbol{v}$ is an eigenvector of A with associated eigenvalue λ, then $p(A)\boldsymbol{v} = p(\lambda)\boldsymbol{v}$.

6. ______ If A and B are both $n \times n$ and $\mathcal{B}$ is a basis for $\mathbb{C}^n$ consisting of eigenvectors for both A and B, then A and B commute.

7. ______ Any matrix A can be written as a weighted sum of rank 1 matrices.

8. ______ For all Hermitian matrices A, there is a matrix B so that $B^H B = A$.

9. ______ If A is an $m \times n$ matrix, then $\text{rng}(A) \oplus \text{NS}(A^T) = \mathbb{R}^m$

10. ______ If A is an invertible $n \times n$ matrix, then $ABA^{-1} = B$ for all $n \times n$ matrices B.

Part II: Computational (60 points)

P1. (15 points) Find B so that $B^2 = A$ where

$$A = \begin{bmatrix} 13 & -5 & 5 \\ -8 & 10 & -8 \\ -3 & -3 & 5 \end{bmatrix}$$

P2. (15 points) Find B so that $B^H B = A$ where

$$A = \begin{bmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{bmatrix}$$

P3. (15 points) Find the best rank 2 approximation to A from (2) with respect to $\|\cdot\|_F$.

P4. (15 points) Let

$$A = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix},$$

find the characteristic polynomial and all eigenvalues, both real and complex. Explain why A is diagonalizable and compute A^{2020}. Note, I do not ask you to diagonalize A.

Part III: Theory and Proofs (45 points; 15 points each)

Pick three of the following four options. If you try all four, I will grade the first three, so if this is not what you intend, then just do three, or at least make it clear which I should grade.

P1. Let $L : V \to V$ be a linear transformation and let $\mathcal{B} = \{\boldsymbol{v}_1, \ldots, \boldsymbol{v}_n\}$ be an ordered basis. Show that $[L]_{\mathcal{B}}$ is upper triangular iff $L(\boldsymbol{v}_i) \in \operatorname{span}\{\boldsymbol{v}_1, \ldots, \boldsymbol{v}_i\}$ for all i.

P2. Let P be an $n \times n$ symmetric matrix that satisfies $P^2 = P$.

(a) Let $U = \mathrm{rng}(P) = \mathrm{CS}(P)$. Show that $\boldsymbol{v} - P\boldsymbol{v} \perp U$.

(b) Use (a) to see that $\|\boldsymbol{v} - P\boldsymbol{v}\|_2^2 = \min\{\|\boldsymbol{v} - \boldsymbol{u}\|_2^2 \mid \boldsymbol{u} \in U\}$. That is $P\boldsymbol{v}$ is the vector in U closest to $\boldsymbol{v}$.

P3. Let $\lambda_1, \lambda_2, \ldots, \lambda_n$ be the complete set of n eigenvalues of an $n \times n$ matrix A. Show that

$$\det(A) = \prod_{i=1}^{n} \lambda_i = \lambda_1 \cdot \lambda_2 \cdot \cdots \cdot \lambda_n.$$

Note: You do not know that A is diagonalizable, there might fail to be a basis of eigenvectors.

P4. Use the SVD to show that any square matrix A can be written as $A = UP$ where U is unitary and P is Hermitian.