

Homework 1 Partial Solutions

Homework 1 problems:

Section 1.1

8. Use elimination and back substitution to solve the given system:

$$x_1 + 2x_2 - 2x_3 = 1$$

$$2x_1 + 5x_2 + x_3 = 9$$

$$x_1 + 3x_2 + 4x_3 = 9$$

Elimination on the augmented system looks like:

$$\left[\begin{array}{ccc|c} 1 & 2 & -2 & 1 \\ 2 & 5 & 1 & 9 \\ 1 & 3 & 4 & 9 \end{array} \right] \xrightarrow[r_3 - r_1 \rightarrow r_3]{r_2 - 2r_1 \rightarrow r_2} \left[\begin{array}{ccc|c} 1 & 2 & -2 & 1 \\ 0 & 1 & 5 & 7 \\ 0 & 1 & 6 & 8 \end{array} \right] \xrightarrow{r_3 - r_2 \rightarrow r_3} \left[\begin{array}{ccc|c} 1 & 2 & -2 & 1 \\ 0 & 1 & 5 & 7 \\ 0 & 0 & 1 & 1 \end{array} \right]$$

This gives the triangular system

$$x_1 + 2x_2 - 2x_3 = 1$$

$$x_2 + 5x_3 = 7$$

$$x_3 = 1$$

Back substitution gives:

$$x_2 = 7 - 5(1) = 2$$

$$x_1 = 1 + 2(1) - 2(2) = -1$$

So we have the solution $(-1, 2, 1)$. (Check this in the initial system!)

9.

(a) Suppose $m_1 \neq m_2$. Assume $m_1 \neq 0$. (If $m_1 = 0$, then $m_2 \neq 0$ by our assumption so we could just swap the rolls below.) Multiply the first equation by $-m_2/m_1$ add to the second

and replace the second. This yields the new system:

$$\begin{aligned} -m_1x_1 + x_2 &= b_1 \\ (1 - m_2/m_1)x_2 &= b_2 - (m_2/m_1)b_1 \end{aligned}$$

Now you can use back substitution:

$$\begin{aligned} x_2 &= \frac{b_2 - (m_2/m_1)b_1}{1 - m_2/m_1} = \frac{m_1b_2 - m_2b_1}{m_1 - m_2} && (\text{ok since } m_1 \neq m_2) \\ x_1 &= -\frac{b_1 - x_2}{m_1} = \frac{2m_2b_1 - m_1(b_1 + b_2)}{m_1(m_1 - m_2)} \end{aligned}$$

So we have a unique solution.

(b) If $m_1 = m_2$, then the system is just

$$\begin{aligned} -m_1x_1 + x_2 &= b_1 \\ -m_1x_1 + x_2 &= b_2 \end{aligned}$$

This is equivalent to the single equation $-m_1x_1 + x_2 = b_1$ if $b_1 = b_2$, else you get $0 = b_2 - b_1 \neq 0$ which cannot happen.

(c) For (a) saying $m_1 \neq m_2$ is equivalent to saying that the two lines have different slopes and hence have a unique point of intersection. For (b), if $m_1 = m_2$, the lines are parallel, they are the same line if $b_1 = b_2$, else they are distinct parallel lines and hence never intersect.

Section 1.2

5.

(c)

$$\begin{aligned} x_1 + x_2 &= 0 \\ 2x_1 + 3x_2 &= 0 \\ 3x_1 - 2x_2 &= 0 \end{aligned}$$

Gaussian Elimination looks like

$$\left[\begin{array}{cc|c} 1 & 1 & 0 \\ 2 & 3 & 0 \\ 3 & -2 & 0 \end{array} \right] \xrightarrow[r_3 - 3r_1 \rightarrow r_3]{r_2 - 2r_1 \rightarrow r_2} \left[\begin{array}{cc|c} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & -5 & 0 \end{array} \right] \xrightarrow{r_3 + 5r_2 \rightarrow r_3} \left[\begin{array}{cc|c} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{array} \right]$$

This results in the triangular system

$$\begin{aligned} x_1 + x_2 &= 0 \\ x_2 &= 0 \end{aligned}$$

Which is solved by back substitution to give: $x_2 = 0$, and $x_1 = -x_2 = 0$.

(d)

$$\begin{aligned} 3x_1 + 2x_2 - x_3 &= 4 \\ x_1 - 2x_2 + 2x_3 &= 1 \\ 11x_1 + 2x_2 + x_3 &= 14 \end{aligned}$$

Gaussian Elimination looks like

$$\begin{aligned} \left[\begin{array}{ccc|c} 3 & 2 & -1 & 4 \\ 1 & -2 & 2 & 1 \\ 11 & 2 & 1 & 14 \end{array} \right] &\xrightarrow{r_1 \leftrightarrow r_2} \left[\begin{array}{ccc|c} 1 & -2 & 2 & 1 \\ 3 & 2 & -1 & 4 \\ 11 & 2 & 1 & 14 \end{array} \right] \xrightarrow{\substack{r_2 - 3r_1 \rightarrow r_2 \\ r_3 - 11r_1 \rightarrow r_3}} \\ \left[\begin{array}{ccc|c} 1 & -2 & 2 & 1 \\ 0 & 8 & -7 & 1 \\ 0 & 24 & -21 & 3 \end{array} \right] &\xrightarrow{r_3 - 3r_2 \rightarrow r_3} \left[\begin{array}{ccc|c} 1 & -2 & 2 & 1 \\ 0 & 8 & -7 & 1 \\ 0 & 0 & 0 & 0 \end{array} \right] \end{aligned}$$

This is equivalent to the "triangular system"

$$\begin{aligned} x_1 - 2x_2 + 2x_3 &= 1 \\ 8x_2 - 7x_3 &= 1 \end{aligned}$$

So x_1 and x_2 are the two "pivot variables" and x_3 is the "free variable." Let $x_3 = t$ be any value, then we use back substitution to get:

$$\begin{aligned} x_2 &= 1/8 + 7/8t \\ x_1 - 2(1/8 + 7/8t) + 2t &= 1 \\ x_1 &= 5/4 - 1/4t \end{aligned}$$

So

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 5/4 - 1/4t \\ 1/8 + 7/8t \\ t \end{bmatrix} = \begin{bmatrix} 5/4 \\ 1/8 \\ 0 \end{bmatrix} + t \begin{bmatrix} -1/4 \\ 7/8 \\ 1 \end{bmatrix}$$

(e)

$$\begin{aligned} 2x_1 + 3x_2 + x_3 &= 1 \\ x_1 + x_2 + x_3 &= 3 \\ 3x_1 + 4x_2 + 2x_3 &= 4 \end{aligned}$$

Elimination on the augmented system looks like:

$$\left[\begin{array}{ccc|c} 2 & 3 & 1 & 1 \\ 1 & 1 & 1 & 3 \\ 3 & 4 & 2 & 4 \end{array} \right] \xrightarrow{r_1 \leftrightarrow r_2} \left[\begin{array}{ccc|c} 1 & 1 & 1 & 3 \\ 2 & 3 & 1 & 1 \\ 3 & 4 & 2 & 4 \end{array} \right] \xrightarrow{\substack{r_2 - 2r_1 \rightarrow r_2 \\ r_3 - 3r_1 \rightarrow r_3}} \left[\begin{array}{ccc|c} 1 & 1 & 1 & 3 \\ 0 & 1 & -1 & -5 \\ 0 & 1 & -1 & -5 \end{array} \right] \xrightarrow{r_3 - r_2 \rightarrow r_3} \left[\begin{array}{ccc|c} 1 & 1 & 1 & 3 \\ 0 & 1 & -1 & -5 \\ 0 & 0 & 0 & 0 \end{array} \right]$$

The echelon form of the matrix coefficient matrix is now shown.

This system is consistent. if we let $x_3 = t$, then by back substitution get

$$\begin{aligned}x_3 &= t \\x_2 &= -5 + t \\x_1 &= 3 - t - (-5 + t) = 8 - 2t\end{aligned}$$

so the solution set is

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 8 \\ -5 \\ 0 \end{bmatrix} + t \begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix}$$

8. For what values of a does the following have a unique solution?

$$\left[\begin{array}{ccc|c} 1 & 2 & 1 & 1 \\ -1 & 4 & 3 & 2 \\ 2 & -2 & a & 3 \end{array} \right]$$

$$\begin{aligned} \left[\begin{array}{ccc|c} 1 & 2 & 1 & 1 \\ -1 & 4 & 3 & 2 \\ 2 & -2 & a & 3 \end{array} \right] & \xrightarrow[r_3 - 2r_1 \rightarrow r_3]{r_2 + r_1 \rightarrow r_2} \left[\begin{array}{ccc|c} 1 & 2 & 1 & 1 \\ 0 & 6 & 4 & 3 \\ 0 & -6 & a - 2 & 1 \end{array} \right] \\ & \xrightarrow{r_3 + r_2 \rightarrow r_3} \left[\begin{array}{ccc|c} 1 & 2 & 1 & 3 \\ 0 & 6 & 4 & -5 \\ 0 & 0 & a + 2 & 4 \end{array} \right] \end{aligned}$$

All that is required to get a unique solution is that $a + 2 \neq 0$ or $a \neq -2$.

11. Solve

$$\begin{aligned}x_1 + 2x_2 &= 2 & x_1 + 2x_2 &= 1 \\ 3x_1 + 7x_2 &= 8 & 3x_1 + 7x_2 &= 7\end{aligned}$$

Set this up as:

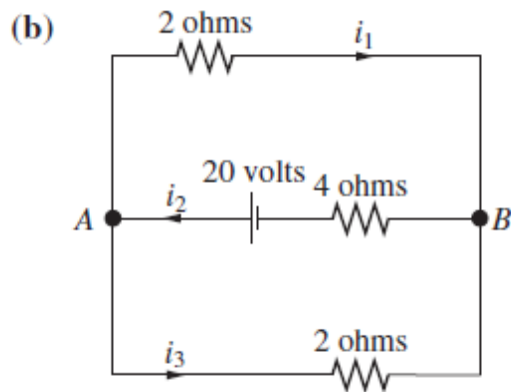
$$\left[\begin{array}{cc|cc} 1 & 2 & 2 & 1 \\ 3 & 7 & 8 & 7 \end{array} \right] \xrightarrow{r_2 - 3r_1 \rightarrow r_2} \left[\begin{array}{cc|cc} 1 & 2 & 2 & 1 \\ 0 & 1 & 2 & 4 \end{array} \right] \xrightarrow{r_1 - 2r_2 \rightarrow r_1} \left[\begin{array}{cc|cc} 1 & 0 & -2 & -7 \\ 0 & 1 & 2 & 4 \end{array} \right]$$

So the solutions are

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -2 \\ 2 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -7 \\ 4 \end{bmatrix}$$

respectively. As usual, you should check this.

(b)



From this we have

$$\begin{aligned}
 i_1 - i_2 + i_3 &= 0 && \text{(node B)} \\
 -i_1 + i_2 - i_3 &= 0 && \text{(node A)} \\
 2i_1 + 4i_2 &= 20 && \text{(top loop)} \\
 4i_2 + 2i_3 &= 20 && \text{(bottom loop)}
 \end{aligned}$$

This gives

$$\begin{aligned}
 \left[\begin{array}{ccc|c} 1 & -1 & 1 & 0 \\ -1 & 1 & -1 & 0 \\ 2 & 4 & 0 & 20 \\ 0 & 4 & 2 & 20 \end{array} \right] &\xrightarrow[r_3 - 2r_1 \rightarrow r_3]{r_2 + r_1 \rightarrow r_2} \left[\begin{array}{ccc|c} 1 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 6 & -2 & 20 \\ 0 & 4 & 2 & 20 \end{array} \right] \\
 &\xrightarrow[1/2r_2 \rightarrow r_2]{r_2 \leftrightarrow r_4} \left[\begin{array}{ccc|c} 1 & -1 & 1 & 0 \\ 0 & 2 & 1 & 10 \\ 0 & 3 & -1 & 10 \\ 0 & 0 & 0 & 0 \end{array} \right] \\
 &\xrightarrow[r_3 - 3/2r_2 \rightarrow r_3]{} \left[\begin{array}{ccc|c} 1 & -1 & 1 & 0 \\ 0 & 2 & 1 & 10 \\ 0 & 0 & -5/2 & -5 \\ 0 & 0 & 0 & 0 \end{array} \right]
 \end{aligned}$$

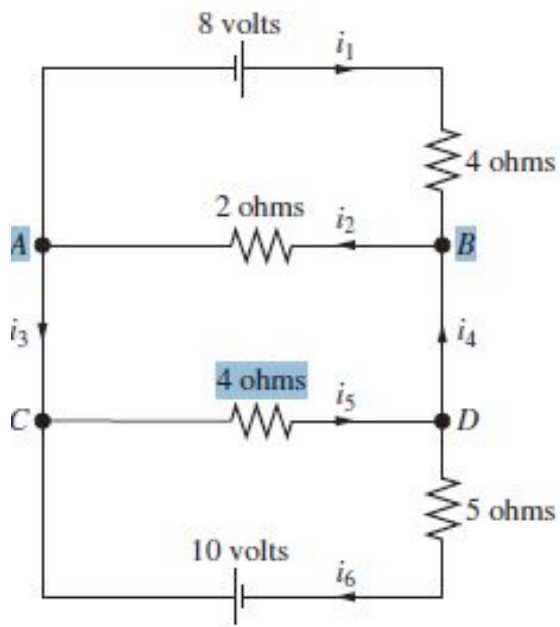
This gives the triangular system

$$\begin{aligned}
 i_1 - i_2 + i_3 &= 0 \\
 2i_2 + i_3 &= 10 \\
 -5/2i_3 &= -5
 \end{aligned}$$

So

$$\begin{aligned}
 i_3 = 2 &\implies i_3 = 2 \\
 2i_2 + 2 &= 10 \implies i_2 = 4 \\
 i_1 - 4 + 2 &= 0 \implies i_1 = 2
 \end{aligned}$$

(c)



The equations here are

$$-i_1 + i_2 - i_3 = 0 \quad (\text{Node A})$$

$$i_1 - i_2 + i_4 = 0 \quad (\text{Node B})$$

$$i_3 - i_5 + i_6 = 0 \quad (\text{Node C})$$

$$-i_4 + i_5 - i_6 = 0 \quad (\text{Node D})$$

$$4i_1 + 2i_2 = 8 \quad (\text{Top Loop})$$

$$4i_5 + 5i_6 = 10 \quad (\text{Bottom Node})$$

The augmented matrix is

$$\left[\begin{array}{cccccc|c} -1 & 1 & -1 & 0 & 0 & 0 & 0 \\ 1 & -1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & -1 & 1 & -1 & 0 \\ 4 & 2 & 0 & 0 & 0 & 0 & 8 \\ 0 & 0 & 0 & 0 & 4 & 5 & 10 \end{array} \right]$$

solving gives

$$\begin{bmatrix} 1/2 \\ 3 \\ 5/2 \\ 5/2 \\ 0 \end{bmatrix}$$

Section 1.3

6. This is just computational. Do this by hand, but this is also good to verify in MATLAB.

7. This is just computational. Do this by hand, but this is also good to verify in MATLAB.

13.

(a) Say the variables are x_1, x_2, x_3, x_4, x_5 the variables x_2, x_4 and x_5 will be independent since the columns 2, 4, and 5 do not contain pivot elements. The others can be solved in terms of those. Let s, t, u be arbitrary real numbers and set $x_2 = s$, $x_4 = t$, and $x_5 = u$, then

$$x_1 = -2 - 2s - 3t - u$$

$$x_2 = s$$

$$x_3 = 5 - 2t - 4u$$

$$x_4 = t$$

$$x_5 = u$$

We can write this nicely as

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} -2 \\ 0 \\ 5 \\ 0 \\ 0 \end{bmatrix} + s \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + t \begin{bmatrix} -3 \\ 0 \\ -2 \\ 1 \\ 0 \end{bmatrix} + u \begin{bmatrix} -1 \\ 0 \\ -4 \\ 0 \\ 1 \end{bmatrix} \quad (1)$$

(b)

We know

$$A\mathbf{x} = \mathbf{b} \iff \begin{bmatrix} 1 & 2 & 0 & 3 & 1 \\ 0 & 0 & 1 & 2 & 4 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \mathbf{x} = \begin{bmatrix} -2 \\ 5 \\ 0 \\ 0 \end{bmatrix}$$

We also know that

$$\mathbf{x} = \begin{bmatrix} -2 \\ 0 \\ 5 \\ 0 \\ 0 \end{bmatrix}$$

is a solution, namely the solution where $s = t = u = 0$

$$\text{So } A\mathbf{x} = -2\mathbf{a}_1 + 5\mathbf{a}_3 = -2 \begin{bmatrix} 1 \\ 1 \\ 3 \\ 4 \end{bmatrix} + 5 \begin{bmatrix} 2 \\ -1 \\ 1 \\ 3 \end{bmatrix} = \begin{bmatrix} 8 \\ -7 \\ -1 \\ 7 \end{bmatrix} = \mathbf{b}$$

16. $A^T = -A$ implies $a_{ii} = -a_{ii}$ so $a_{ii} = 0$.

Section 1.4

8. Check that $A^2 = I$, so $A^{2n+1} = A$ and $A^{2n} = I$.

1.4 #10 Assume $A^T = A$ and $B^T = B$.

(a) Symmetric since: $(A + B)^T = A^T + B^T = A + B$

(b) Symmetric since: $(A^2)^T = A^T A^T = AA = A^2$

(c) Not necessarily transitive since: $(AB)^T = B^T A^T = BA$. Symmetry would only happen if $AB = BA$, i.e., A and B commute.

Example: Let $A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & -1 \\ -1 & -1 \end{bmatrix}$, notice both A and B are symmetric. $AB = \begin{bmatrix} 0 & -2 \\ 0 & -2 \end{bmatrix}$ is not symmetric so $(AB)^T \neq AB$.

(d) Symmetric since: $(ABA)^T = A^T B^T A^T = ABA$

(e) Symmetric since: $(AB + BA)^T = (AB)^T + (BA)^T = B^T A^T + A^T B^T = BA + AB =$

$$AB + BA$$

(f) Not symmetric since: $(AB - BA)^T = (AB)^T - (BA)^T = B^T A^T - A^T B^T = BA - AB = -(AB - BA)$. So $AB - BA$ is symmetric iff $AB - BA = 0$, that is iff A and B commute.

The same example as in (c) works here.

22.

$$\begin{aligned} RR^T &= \begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix} \begin{bmatrix} \cos(\theta) & \sin(\theta) \\ -\sin(\theta) & \cos(\theta) \end{bmatrix} \\ &= \begin{bmatrix} \cos^2(\theta) + \sin^2(\theta) & \cos(\theta)\sin(\theta) - \sin(\theta)\cos(\theta) \\ \sin(\theta)\cos(\theta) - \cos(\theta)\sin(\theta) & \sin^2(\theta) + \cos^2(\theta) \end{bmatrix} \\ &= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \end{aligned}$$

23. $H = I - 2\mathbf{u}\mathbf{u}^T$, so

$$\begin{aligned} H^2 &= (I - 2\mathbf{u}\mathbf{u}^T)(I - 2\mathbf{u}\mathbf{u}^T) \\ &= I^2 - 2\mathbf{u}\mathbf{u}^T I - 2I\mathbf{u}\mathbf{u}^T + 4\mathbf{u}\mathbf{u}^T \mathbf{u}\mathbf{u}^T \\ &= I^2 - 4\mathbf{u}\mathbf{u}^T + 4\mathbf{u}\mathbf{u}^T \mathbf{u}\mathbf{u}^T \\ &= I^2 - 4\mathbf{u}\mathbf{u}^T + 4\mathbf{u}(1)\mathbf{u}^T \\ &= I^2 - 4\mathbf{u}\mathbf{u}^T + 4\mathbf{u}\mathbf{u}^T \\ &= I^2 \\ &= I \end{aligned}$$

27. Assume $A^2 = I$ and let $B = \frac{1}{2}(I + A)$ and $C = \frac{1}{2}(I - A)$. Note

$$B^2 = \frac{1}{4}(I^2 + IA + AI + A^2) = \frac{1}{4}(I + 2A + I) = \frac{1}{4}(2)(I + A) = B$$

Similarly, $C^2 = C$.

30.

$$(A + A^T)^T = A^T + (A^T)^T = A^T + A = A + A^T$$

So $A + A^T$ is symmetric. Similarly,

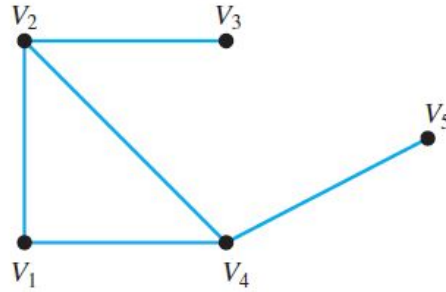
$$(A - A^T)^T = A^T - (A^T)^T = A^T - A = -(A - A^T)$$

So $A - A^T$ is skew-symmetric.

Since $A = \frac{1}{2}(A + A^T) + \frac{1}{2}(A - A^T)$ we see A can be written as symmetric + skew-symmetric.

32. This is false, for example let $A = B = I$ and $C = -I$, then clearly A is row equivalent to B and C , but $B + C = \mathbf{0}$ (the all 0 matrix). It is not true that A is row equivalent to $\mathbf{0}$, else $I = \mathbf{0}$ (a version of $1 = 0$).

33.



(a) Adjacency Matrix, A :

$$A = \begin{bmatrix} 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$

(b)

$$A^2 = \begin{bmatrix} 2 & 1 & 1 & 1 & 1 \\ 1 & 3 & 0 & 1 & 1 \\ 1 & 0 & 1 & 1 & 0 \\ 1 & 1 & 1 & 3 & 0 \\ 1 & 1 & 0 & 0 & 1 \end{bmatrix}$$

There are 6 walks starting at V_1 and that is the sum of column 1 or row 1 (the matrix is symmetric). $V_1V_2V_1$, $V_1V_4V_1$, V_1, V_2V_3 , $V_1V_2V_4$, $V_1V_4V_2$, and $V_1V_4V_5$.

(c)

$$A^2 = \begin{bmatrix} 2 & 4 & 1 & 4 & 1 \\ 4 & 2 & 3 & 5 & 1 \\ 1 & 3 & 0 & 1 & 1 \\ 4 & 5 & 1 & 2 & 3 \\ 1 & 1 & 1 & 3 & 0 \end{bmatrix}$$

There are 5 walks of length 3 from V_2 to V_4 and 1 of length 2 so 6 altogether.