Quiz 2

Problem 1 (10 points)**.** Use the following three facts about determinants to compute the determinant of a matrix using row operations.

a. If B is diagonal, then $\det(B) = b_{11} \cdot b_{22} \cdots b_{nn}$.

b. If B arises from A by a type I row operation, i.e., interchanging two rows, then $\det(B) = -\det(A)$.

c. If B arises from A by a type III row operation, i.e., $r_i + ar_j \to r_i$, that is, row i is replaced by row i plus a scalar multiple of row j, where $i \neq j$. Then $\det(A) = \det(B)$.

Compute $\det(A)$ by:

1. Reducing A to a triangular matrix B using only type I and III operations. (I would say echelon form, except for the issue with pivots being 1).
2. Keep track of how many row swaps were made.
3. Compute $\det(B)$ by multiplying the diagonal elements of B.

$$A = \begin{bmatrix} 2 & 6 & 3 & 2 \\ 4 & 2 & 3 & 2 \\ 2 & 2 & 2 & 1 \\ 4 & 2 & 1 & 5 \end{bmatrix}$$

Show the work for the above computation here.

On your own, don't include this in the quiz, try computing this determinant by expanding on a row or column.

Discuss which method, "expansion along a row or column" or "using elementary row operations" is, in general, a faster method of computing a determinant.

I will only write this out using row reduction, the other way ... expansion along a row/column would be too painful.

$$
\begin{aligned}
\begin{bmatrix} 2 & 6 & 3 & 2 \\ 4 & 2 & 3 & 2 \\ 2 & 2 & 2 & 1 \\ 4 & 2 & 1 & 5 \end{bmatrix}
&\xrightarrow[\substack{r_2-2r_1\to r_2 \\ r_3-r_1\to r_3 \\ r_4-2r_1\to r_4}]{}
\begin{bmatrix} 2 & 6 & 3 & 2 \\ 0 & -10 & -3 & -2 \\ 0 & -4 & -1 & -1 \\ 0 & -10 & -5 & 1 \end{bmatrix} \\
&\xrightarrow[\substack{r_3-4/10r_2\to r_3 \\ r_4-10r_2\to r_4}]{}
\begin{bmatrix} 2 & 6 & 3 & 2 \\ 0 & -10 & -3 & -2 \\ 0 & 0 & 1/5 & -1/5 \\ 0 & 0 & -2 & 3 \end{bmatrix} \\
&\xrightarrow[r_4+10r_3\longrightarrow r_3]{}
\begin{bmatrix} 2 & 6 & 3 & 2 \\ 0 & -10 & -3 & -2 \\ 0 & 0 & 1/5 & -1/5 \\ 0 & 0 & 0 & -1 \end{bmatrix}
\end{aligned}
$$

We had two row swaps so $\det(A) = (2)(-10)(1/5)(1) = -4$.

I leave the expansion along a column to the reader. You should note that it is a much longer and more involved process. For finding $\det(A)$ it is much quicker to use the method of applying elimination and keeping track of how the determinant changes.

Problem 2 (5 points)**.** Let A be as above, consider $A\boldsymbol{x} = \boldsymbol{b}$ where $\boldsymbol{b} = (-3, -3, -2, 1)$. Find x_1 using Cramer's rule. (You may use MATLAB/Octave to compute the determinants, but write out what you are computing.)

$$
x_1 = \frac{\det \begin{bmatrix} -3 & 6 & 3 & 2 \\ -3 & 2 & 3 & 2 \\ -2 & 2 & 2 & 1 \\ 1 & 2 & 1 & 5 \end{bmatrix}}{\det \begin{bmatrix} 2 & 6 & 3 & 2 \\ 4 & 2 & 3 & 2 \\ 2 & 2 & 2 & 1 \\ 4 & 2 & 1 & 5 \end{bmatrix}} = -2
$$

Problem 3 (10 points; 2 points each)**.** Decide if each of the following are true or false and provide a small proof or counterexample in each case.

(a) ______ If A is an $n \times n$ matrix all of whose entries are integers and $\det(A) = \pm 1$, then A^{-1} also has only integer entries.

This is true as can be seen by considering the adjoint.

(b) ______ If A and B are similar, then $\det(A) = \det(B)$.

Here two $n \times n$ matrices A and B are called *similar* iff $A = SBS^{-1}$ for some invertible S.

This is clearly true since

$$
\det(A) = \det(SBS^{-1}) = \det(S)\det(B)\det(S^{-1}) = \det(S)\det(S)^{-1}\det(B) = \det(B)
$$

(c) ______ Three vectors in $\mathbb{R}^3$, $\boldsymbol{v}_1$, $\boldsymbol{v}_2$, and $\boldsymbol{v}_3$ are co-planar iff $\det(\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3) = 0$.

This is true. The three vectors are co-planar iff they don't span $\mathbb{R}^3$ iff they are linearly dependent iff the determinant is 0.

(d) ______ $\det(A^2 + B^2) \geq 0$ for all $n \times n$ matrices A and B with real entries that commute.

This is also true. Here there is a nice trick, consider the complex matrix $A + Bi$. Since $A^2+B^2 = (A+Bi)(A-Bi)$ we have $\det(A^2+B^2) = \det(A+Bi)\det(A-Bi) = \det(A+Bi)\overline{\det(A+Bi)} = |\det(A+Bi)|^2$.

(e) ______ The determinant can be viewed as a multilinear function $\det : (\mathbb{R}^n)^n \to \mathbb{R}$ with the properties that $\det(\boldsymbol{e}_1, \ldots, \boldsymbol{e}_n) = 1$ and $\det(\boldsymbol{v}_1, \ldots, \boldsymbol{v}_n) = \det(\boldsymbol{v}_1', \ldots, \boldsymbol{v}_n')$, where $(\boldsymbol{v}_1', \ldots, \boldsymbol{v}_n)'$ is the result of swapping two of the vectors in $(\boldsymbol{v}_1, \ldots, \boldsymbol{v}_n)$.

True. This is just a check to see that you are looking at the notes that I post. (See here.)

Problem 4 (10 points)**.** Submit the completion certificate for the OnRamp tutorial from MATLAB in the MATLAB shared drive.