Name: ____________________________ Exam 1 - MAT345

Part I: True/False

Each problem is points for a total of 50 points. (7 points each and one free point.)

Problem 1 (50 points; 5 points each). Decide if each of the following is true or false.

(a) ______ If A and B commute, then so do A^T and B^T.

(b) ______ For any invertible matrix A, $(A^T)^{-1} = (A^{-1})^T$.

(c) ______ For all $n \times n$ matrices A and B, $\det(A + B) = \det(A) + \det(B)$

(d) ______ For all $n \times n$ matrices A, $\det(cA) = c \cdot \det(A)$

(e) ______ For all $n \times n$ matrices A and B, $\det(AB) = \det(BA)$.

(f) ______ If

$$\text{rref}(A) = \begin{bmatrix} 1 & 2 & 0 & 1 \\ 0 & 0 & 1 & 2 \end{bmatrix},$$

then the solutions of $A\boldsymbol{x} = \mathbf{0}$ are given by

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = s \begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + t \begin{bmatrix} 1 \\ 0 \\ 2 \\ 1 \end{bmatrix}$$

(g) ______ If A is an $m \times n$ matrix, then in the expression $A\boldsymbol{x} = \boldsymbol{b}$, $\boldsymbol{x}$ represents m variables, or a vector in $\mathbb{R}^m$, and $\boldsymbol{b}$ is a vector in $\mathbb{R}^n$.

Part II: Computational (80 points)

Show all computations so that you make clear what your thought processes are.

Problem 2 (20 pts)**.** Let

$$A = \begin{bmatrix} 4 & 5 & -1 & -3 \\ 2 & -4 & 3 & 0 \\ -1 & 0 & 3 & 0 \end{bmatrix}; \qquad B = \begin{bmatrix} 2 & 0 & 0 \\ 3 & 2 & 5 \\ 3 & -3 & -1 \\ -2 & 0 & 1 \end{bmatrix}$$

1. Express the third row of AB as a linear combination of rows of B.

2. Express the second column of AB as a linear combination of the columns of A.

3. Express $(AB)_{1,2}$ as a product of a row of A and a column of B.

Problem 3 (20 pts)**.** Solve $A\boldsymbol{x} = \boldsymbol{b}$ where

$$A = \begin{bmatrix} 1 & 2 & -4 & 3 & 2 \\ 2 & 4 & -7 & 4 & 5 \\ -3 & -6 & 14 & -13 & -3 \end{bmatrix} \qquad \boldsymbol{b} = \begin{bmatrix} 4 \\ 8 \\ -11 \end{bmatrix}$$

1. (8 points) Use row operations (show all work and indicate operations) to reduce A to an echelon form. (This should work out very nicely - no fractions required..)
2. (7 points) Use back-substitution to solve the resulting system. Make sure to indicate which variables are free.
3. (5 points) Write your solution as a linear combination of vectors.

Workspace

Problem 4 (20 pts)**.** Use Cramer's rule to find x_3, where

$$\begin{bmatrix} 1 & 2 & -4 & 3 \\ 0 & -6 & -4 & 0 \\ 0 & 3 & 0 & 0 \\ 2 & 5 & -3 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \\ 1 \\ 3 \end{bmatrix}$$

Note: These determinants should work out very nicely if you chose how you expand carefully.

Problem 5 (20 pts)**.** Write A in the form LU where L is lower-triangular with 1's on the diagonal, and U is upper-triangular for

$$A = \begin{bmatrix} 2 & -2 & 1 \\ 4 & -1 & 1 \\ -6 & 12 & -4 \end{bmatrix}$$

Part III: Theory and Proofs (60 points; 20 points each)

Choose three of the five options. If you try all five, I will grade the first three, not the best three. You must decide what should be graded.

Problem 6. Show that for any symmetric $n \times n$ matrices A and B that $AB + BA$ is symmetric.

Problem 7 (20 pts)**.** For A and B invertible $n \times n$ matrices, prove

$$((AB)^T)^{-1} = ((AB)^{-1})^T.$$

You may use the fact we have already discussed that for any invertible matrix A, $(A^T)^{-1} = (A^{-1})^T$.

Problem 8 (20 pts)**.** Let A be an $n \times m$ matrix, show that

$$A = O \text{ (the zero matrix)} \iff A\boldsymbol{x} = \mathbf{0} \text{ for all } \boldsymbol{x}$$

Problem 9 (20 pts)**.** Let A be an $m \times n$ matrix, show that for any $\boldsymbol{x} \in \mathbb{R}^n$,

$$A\boldsymbol{x} = \mathbf{0} \iff A^T A\boldsymbol{x} = \mathbf{0}.$$

Hint: There are several ways to do this, but you might use that if $A^T A\boldsymbol{x} = \mathbf{0}$, then $\boldsymbol{x}^T A^T A\boldsymbol{x} = (A\boldsymbol{x})^T (A\boldsymbol{x}) = \mathbf{0}$. When can $\boldsymbol{y}^T \boldsymbol{y} = \mathbf{0}$?

Problem 10 (20 pts)**.** Let A be an 3×5 matrix given by rows as:

$$A = \begin{bmatrix} \boldsymbol{a}_1 & \boldsymbol{a}_2 \cdots & \boldsymbol{a}_5 \end{bmatrix}$$

Let

$$\operatorname{rref}(A) = \begin{bmatrix} 1 & -2 & 0 & 3 & 0 \\ 0 & 0 & 1 & -4 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

Explain how we know that $\boldsymbol{a}_2 = -2\boldsymbol{a}_1$ and $\boldsymbol{a}_4 = 3\boldsymbol{a}_1 - 4\boldsymbol{a}_3$ and hence that

$$A = \begin{bmatrix} \boldsymbol{a}_1 & \boldsymbol{a}_3 & \boldsymbol{a}_5 \end{bmatrix} \begin{bmatrix} 1 & -2 & 0 & 3 & 0 \\ 0 & 0 & 1 & -4 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

Hint: What is the solution to $A\boldsymbol{x} = \mathbf{0}$? This can be read off of $\operatorname{rref}(A)$.