Math 571 - Exam 1 (20 points)

Richard Ketchersid

Question 1 (20 points)**.** For the exam, you need only indicate True or False. No justification is required. If you want to earn back some points, you can supply full justifications for **all** of the problems. You may earn back 50% of lost points.

(a) False Let $X = (0, 1] \subseteq \mathbb{R}$. In the induced metric, X is closed and bounded, so X is compact.

The intervals $\left(\frac{1}{n}, 1\right]$ gives an open cover with no subcover.

(b) True A discrete space is compact iff it is finite.

An open cover is just the cover by $\{x\}$ for each $x \in X$. If compact, there is a finite subcover, and hence X is finite. conversely, if X is finite, then any open cover is finite as the entire collection of open sets is finite.

(c) True $\mathrm{Cl}(A \cup B) = \mathrm{Cl}(A) \cup \mathrm{Cl}(B)$.

Trivially, $\mathrm{Cl}(A) \cup \mathrm{Cl}(B) \subseteq \mathrm{Cl}(A \cup B)$. Let $x \in \mathrm{Cl}(A \cup B)$. Suppose $x \notin \mathrm{Cl}(A)$, then there is open O with $x \in O$ and $O \cap A = \emptyset$. But then every open nbhd of x contained in O must intersect B and thus $x \in \mathrm{Cl}(B)$.

(d) False $\mathrm{Cl}(A \cap B) = \mathrm{Cl}(A) \cap \mathrm{Cl}(B)$.

Take A and B dense with $A \cap B = \emptyset$. For example, A could be all binary rationals in $(0, 1)$, i.e., $\alpha = \sum_{i=1}^{n} \frac{b_i}{2^{i+1}}$ where $b_i \in 2$ and some $b_i \neq 0$ and B could be all ternary rationals, i.e., $\alpha = \sum_{i=1}^{n} \frac{a_i}{3^{i+1}}$ where $a_i \in 3$ and some $a_i \neq 0$. Then $\mathrm{Cl}(A) \cap \mathrm{Cl}(B) = X \cap X = X$ while $\mathrm{Cl}(A \cap B) = \mathrm{Cl}(\emptyset) = \emptyset$.

(e) False For X a metric space, to show that a set $F \subseteq X$ is closed, it is necessary and sufficient to show that every sequence from F has a subsequence that converges to a point in F.

The requirement is that every convergent sequence converges to a point in x, not that every sequence converges. In particular, $(0, 1)$ satisfies the mentioned criterion but is not closed.

(f) False For X a metric space, to show that a set $K \subseteq X$ is compact, it is necessary and sufficient to show that every sequence from K has a subsequence that converges.

Here again, the required condition is that every sequence from K has a convergent subsequence converging to a point in K. The same counter-example as above suffices.

(g) False If A is connected, then ∂A is connected.

Consider the strip $A = [0,1] \times \mathbb{R}$ in $\mathbb{R}^2$. Then $\partial A =$ consists of the two lines $x = 0$ and $x = 1$.

It might be tempting to argue as follows. Suppose $C \cup D = \partial A$, $C \cap D = \emptyset$, $C \cap \partial A \neq \emptyset \neq D \partial A$, and C and D are open in ∂A. Then let $E = \text{Int}(A)$. Then $\text{Cl}(A) = \partial A \cup \text{Int}(A) = C \cup D \cup E$. The issue here is that C and D are not relatively open to $\text{Cl}(A)$, we know $C = C' \cap \partial A$, and $D = D' \cap \partial A$ where C' and D' are open. So we know $\text{Cl}(A) = C' \cup D' \cup E$, but now $C' \cap E \neq \emptyset \neq D' \cap E$.

(h) <u>False</u> Let (Y, d_Y) be a metric space and $f : X \to Y$. Define $d_f : X \times X \to [0, \infty)$ by $d_f(x, x') = d_Y(f(x), f(x'))$. d_f will always give a metric on X for all X, Y, and f.

(symmetry) $d_X(x,x') = d_X(x',x)$ and (triangle inequality) $d_X(x,x') \leq d_X(x,x'') + d_X(x'',x')$ are both clear. The only issue is the identity of indiscernibles. It is clear that

$$d_X(x,x') = 0 \iff d_Y(f(x), f(x')) = 0 \iff f(x) = f(x').$$

But we need $f(x) = f(x') \iff x = x'$, that is, we need f to be 1-1.

(i) <u>False</u> On $\mathbb{R}^* = \mathbb{R} - \{0\}$, $d^*(x,y) = \left|\frac{1}{x} - \frac{1}{y}\right| = \frac{|x-y|}{|xy|}$ is a metric on $\mathbb{R}^*$. In this metric, $\left(\frac{1}{n} \mid n = 1, 2, \ldots\right)$ has a limit.

For $m > 1$, $d^*\left(1, \frac{1}{m}\right) = m - 1$. This is not bounded so the sequence can't have a limit. Suppose $\frac{1}{m} \to x$, then $d^*(1,x) = d$ and thus $d^*(1,m) \leq d + d^*(m,x)$ so $d^*(m,x) \geq d^*(1,m) - d = m - d$.

Perhaps more interesting is that $(n \mid n = 1, 2, \ldots)$ is a Cauchy sequence with no limit.

(j) <u>True</u> Let $d(x,y) = |x - y|$ be the standard metric on $\mathbb{R}$ and let d^* be as in part (i). A little work gives that for $\delta |x_0| < 1$, letting $\delta' = |x_0|\left(1 - \frac{1}{\delta|x_0|+1}\right)$ and $\delta'' = |x_0|\left(\frac{1}{1-\delta|x_0|} - 1\right)$ we have that

$$|x - x_0| < \delta' \implies \left|\frac{1}{x} - \frac{1}{x_0}\right| < \delta$$

and

$$\left|\frac{1}{x} - \frac{1}{x_0}\right| < \delta \implies |x - x_0| < \delta''.$$

So $(\mathbb{R}^*, d^*)$ and $(\mathbb{R}^*, d)$ have the same open sets, and hence the two metrics induce the same topological space.

The given information indicates that $N_{\delta'}(x_0) \subseteq N^*_\delta(x_0)$ and $N^*_\delta(x_0) \subseteq N_{\delta''}(x_0)$. So in every d-nbhd there is a d^*-nbhd and vice versa.