Exam 1

This exam covers Topics 1 - 3, Topic 4 will not be covered here.

Part I: True/False (5 points each; 25 points)

For each of the following mark as true or false.

a) _____ $\mathrm{tr}(AB) = \mathrm{tr}(BA)$ for an $n \times n$ matrices A and B, where $\mathrm{tr}(C) \stackrel{\text{df}}{=} \sum_{i=1}^{n} C_{ii}$, the sum of the diagonal of C.

b) _____ $\mathrm{tr}(ABC) = \mathrm{tr}(BAC)$ for an $n \times n$ matrices A, B, and C.

c) _____ If W is a subspace of a vector space V, then there is a subspace U so that $V = W \oplus U$.

d) _____ If W is a subspace of a vector space V and $\mathcal{B}$ is a basis for V, then B can be restricted to a basis for W.

e) _____ If $B = EA$ where E is invertible, then $\mathrm{NS}(A) = \mathrm{NS}(B)$.

Part II: Definitions and Theorems (5 points each; 25 points)

a) Define what it means for a set of vectors $\mathcal{B} = \{\boldsymbol{v}_1, \ldots, \boldsymbol{v}_n\}$ from a real vector space V to span V.

b) Define what it means for a set of vectors $\{\boldsymbol{v}_1, \ldots, \boldsymbol{v}_n\}$ from a real vector space V to be linearly independent.

c) Define what it means for a set of vectors $\mathcal{B} = \{\boldsymbol{v}_1, \ldots, \boldsymbol{v}_n\}$ to be a basis for a vector space V.

d) State the Rank-Nullity Theorem.

e) What conditions must be checked to verify that $W \subseteq V$ is a subspace of a vector space. V

Part III: Computational (15 points each; 45 point)

a) Use row ops to find an echelon form of

$$A = \begin{bmatrix} 1 & 2 & 2 & -2 & 2 \\ 2 & 4 & 1 & -2 & 5 \\ 1 & 2 & -1 & 0 & 3 \end{bmatrix}$$

Make sure to write out your steps and indicate the row ops at each step.

b) Use the echelon matrix found above to find a basis for $\mathrm{RS}(A)$, $\mathrm{NS}(A)$, and $\mathrm{CS}(A)$. Give a brief reason for your choice.

Without a justification, you might just have a lucky guess and I will not accept this. Your justification can be short and use facts from the text or from the notes that I have provided.

c) Show that the upper-triangular $n \times n$ matrices form a subspace of all $n \times n$ matrices and find a basis for this subspace.

Part IV: Proofs (15 points each; 60 points)

Provide complete arguments/proofs for the following.

a) **Prove:** Let A and B be square matrices with $AB = I$. Show that A is invertible.

   You may refer to Theorem 1.5.2 or Theorem 2.2.2, but be clear and complete in your argument.

b) **Prove:** Let A be an $m \times n$ matrix, $\mathbb{R}^n = \mathrm{NS}(A) \oplus \mathrm{RS}(A)$.

c) **Prove:** If A and B are $m \times n$ matrices such that $A\boldsymbol{x} = B\boldsymbol{x}$ for all $\boldsymbol{x} \in \mathbb{R}^n$, then $A = B$.

d) **Prove:** If A is an $n \times n$ matrix and $A^k = \mathbf{0}$ for any k, then $A^n = \mathbf{0}$.