Exam 2

To avoid any confusion, unless specified otherwise, vector spaces are complex vector spaces, inner-products are complex inner-products, and matrices are complex matrices.

Part I: True/False

Each problem is points for a total of 50 points. (5 points each.)

(1) ______ A is unitary iff $A^H = A^{-1}$.

(2) ______ A is unitary iff A preserves inner-products, that is, $\langle \boldsymbol{x}, \boldsymbol{y} \rangle = \langle A\boldsymbol{x}, A\boldsymbol{y} \rangle$.

(3) ______ If A preserves the L^2-norm, that is, $\|\boldsymbol{x}\|_2 = \|A\boldsymbol{x}\|_2$, then A preserves the inner-product.

(4) ______ If A is diagonalizable and for all eigenvalues, λ of A, $|\lambda| = 1$, then A is unitary.

(5) ______ If λ is an eigenvalue of A, then $\bar{\lambda}$ is an eigenvalue of A^H.

(6) ______ If $\boldsymbol{v}$ is an eigenvector of A, then $\bar{\boldsymbol{v}}$ is an eigenvector of A^H.

(7) ______ $\langle A, B \rangle = \operatorname{tr}(B^H A)$ is an inner product on $\mathbb{C}^{n \times n}$.

(8) ______ For all Hermitian matrices A, there is a matrix B so that $B^H B = A$.

(9) ______ There are linear maps $L : \mathbb{R}^5 \to \mathbb{R}^4$ such that $\dim(\ker(L)) = 2 = \dim(\operatorname{rng}(L))$.

(10) ______ For $k \leq \min\{m, n\}$, the space of matrices of rank k is a subspace of $\mathbb{C}^{m \times n}$.

Part II: Computational (45 points)

(1) (30 points) Find (by hand) he singular value decomposition of

$$A = \begin{bmatrix} \sqrt{2} & 1 & -\sqrt{2}/2 \\ -\sqrt{2} & 1 & \sqrt{2}/2 \\ -\sqrt{2}/2 & -1 & \sqrt{2} \\ \sqrt{2}/2 & -1 & -\sqrt{2} \end{bmatrix}$$

You should be able to complete each step by hand.

(a) Find the eigenvalues of $A^T A$, $\lambda_1 \geq \lambda_2 \geq \lambda_3 \geq 0$.

(b) Find a complete orthonormal set of eigenvectors $\{\boldsymbol{v_1}, \boldsymbol{v_2}, \boldsymbol{v_3}\}$, where $\boldsymbol{v}_i$ is an eigenvector for λ_i.

(c) Set up the 4×3 matrix Σ with $\Sigma_{ii} = \sigma_i = \sqrt{\lambda_i}$ (the i^{th} singular value) and all other $\Sigma_{ij} = 0$.

(d) Find $\boldsymbol{u}_i$ the left singular vectors. Recall $\boldsymbol{u}_i = \frac{1}{\sigma_i} A\boldsymbol{v}_i$ for $i = 1, 2, 3$ and $\boldsymbol{u}_4$ is a basis for $\text{NS}(A^T)$.

(e) Let $U = \begin{bmatrix} \boldsymbol{u}_1 & \boldsymbol{u}_2 & \boldsymbol{u}_3 & \boldsymbol{u}_4 \end{bmatrix}$ and $V = \begin{bmatrix} \boldsymbol{v}_1 & \boldsymbol{v}_2 & \boldsymbol{v}_3 \end{bmatrix}$.

(f) Verify that $A = U\Sigma V^T$.

This all works out very nicely for this carefully chosen matrix A.

This page blank (for work)

(2) (15 points) Find the best rank 2 approximation to A from (1) with respect to $\|\cdot\|_F$.

Part III: Theory and Proofs (60 points; 15 points each)

(1) Use the Spectral Theorem to show that for a Hermitian matrix A, A is positive definite iff $A = B^H B$ for some matrix B.

In some sense B is the correct notion of the *square-root* of A.

(2) A map $B : V \times V \to \mathbb{R}$ is a **sesquilinear form** iff

- $B(\alpha_1 \boldsymbol{x}_1 + \alpha_2 \boldsymbol{x}_2, \boldsymbol{y}) = \alpha_1 B(\boldsymbol{x}_1, \boldsymbol{y}) + \alpha_2 B(\boldsymbol{x}_2, \boldsymbol{y})$
- $B(\boldsymbol{x}, \beta_1 \boldsymbol{y}_1 + \beta_2 \boldsymbol{y}_2) = \bar{\beta}_1 B(\boldsymbol{x}, \boldsymbol{y}_1) + \bar{\beta}_2 B(\boldsymbol{x}, \boldsymbol{y}_2)$

Show that for any basis of V and bi-linear form $B : V \times V \to \mathbb{C}$, there is a matrix representation of B. In particular let $\mathcal{C} = \{\boldsymbol{c}_1, \ldots, \boldsymbol{c}_n\}$ be a basis for V. Define the matrix $[B]_{\mathcal{C}}$ by

$$([B]_{\mathcal{C}})_{i,j} = B(\boldsymbol{c}_j, \boldsymbol{c}_i)$$

Show that

$$B(\boldsymbol{x}, \boldsymbol{y}) = [\boldsymbol{y}]_{\mathcal{C}}^H [B]_{\mathcal{C}} [\boldsymbol{x}]_{\mathcal{C}}$$

(3) Recall that a complex inner product is a sesquilinear form $B : V \times V \to \mathbb{C}$ that is

(conjugate) symmetric: For all $\boldsymbol{x}, \boldsymbol{y} \in V$, $B(\boldsymbol{x}, \boldsymbol{y}) = \overline{B(\boldsymbol{y}, \boldsymbol{x})}$.

positive definite For all $\boldsymbol{x} \neq \mathbf{0}$, $B(\boldsymbol{x}, \boldsymbol{x}) \in \mathbb{R}^{+}$.

Usually, we write $\langle \boldsymbol{x}, \boldsymbol{y} \rangle$ instead of $B(\boldsymbol{x}, \boldsymbol{y})$, when we're thinking of the sesquilinear form as an inner product.

Use (2) to show that for any complex inner-product space V with inner-product $\langle \boldsymbol{x}, \boldsymbol{y} \rangle_V$ and any basis $\mathcal{C} = \{\boldsymbol{c}_1, \ldots, \boldsymbol{c}_n\}$ for V, there is a positive definite Hermitian matrix A so that

$$\langle \boldsymbol{x}, \boldsymbol{y} \rangle_V = [y]_{\mathcal{C}}^{H} A [\boldsymbol{x}]_{\mathcal{C}}$$

A Hermitian matrix $A \in \mathbb{C}^{n \times n}$ is **positive definite** iff $\boldsymbol{x}^H A \boldsymbol{x} \in \mathbb{R}^{+}$ for $\boldsymbol{x} \neq \mathbf{0}$.

(4) Use (3) and the spectral theorem to show that for any complex inner-product $\langle \cdot, \cdot \rangle_V$ on a complex vector space V, there is an orthonormal basis $\mathcal{U} = \{\boldsymbol{u}_1, \ldots, \boldsymbol{u}_n\}$ and a diagonal real matrix D with all diagonal entries positive so that

$$\langle \boldsymbol{x}, \boldsymbol{y} \rangle_V = [\boldsymbol{y}]_{\mathcal{U}}^H D [\boldsymbol{x}]_{\mathcal{U}}$$

In other words $D = \mathrm{diag}(d_1, \ldots, d_n)$ with $d_i \in \mathbb{R}^+$ and $\boldsymbol{x} = \sum_{i=1}^n \alpha_i \boldsymbol{u}_i$ and $\boldsymbol{y} = \sum_{i=1}^n \beta_i \boldsymbol{u}_i$, then

$$\langle \boldsymbol{x}, \boldsymbol{y} \rangle_V = \sum_{i=1}^n \bar{\beta}_i \alpha_i d_i$$

A standard *weighted* inner-product.