Basic ray theory
Lagrange-Hamilton theory

KetilHokstad, January 2016

Introduction

- Seismic ray theory:
  - Solution to the elastic wave equation by the «method of characteristics»
  - High-frequency approximation

- Wave theory vs. ray theory
  - Wave theory: Heisenberg's uncertainty relation applies; x or p can be known exactly (but not both)
  - Ray theroy: Heisenberg is ignored; both x and p can be known exactly.

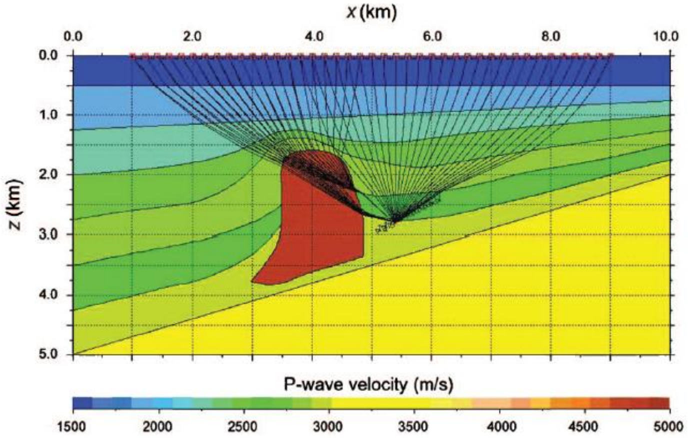


Figure 6. Synthetic velocity model with an embedded salt body and a simulated deviated well. Rays traced from level number nine to all surface sampling locations are indicated.

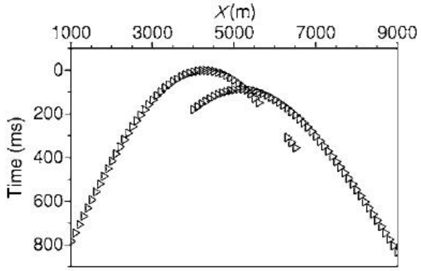


Figure 7. Simulated direct ray traveltime data. The data correspond to a drill-bit source at level number nine.

Acoustic wave equation for displacement

Newton and Hooke in acoustic medium:

$$\begin{aligned}\rho\partial_t^2 u_i &= \partial_i\sigma + f_i \\ \sigma &= M\partial_j u_j\end{aligned}$$

Wave equation for displacement:

$$\rho\partial_t^2 u_i = \partial_i(M\partial_j u_j) + f_i$$

Assume $f_i = 0$, $\partial_j\rho \simeq 0$ and $\partial_j M \simeq 0$ (smooth medium):

$$\begin{aligned}\frac{1}{v^2}\partial_t^2 u_i - \partial_i\partial_j u_j = 0 \\ v = \sqrt{M/\rho}\end{aligned}$$

Eikonal equation (1)

Ray ansatz:

$$u_i = A(\mathbf{x}) g_j e^{-i\omega(t-T)}$$

Derivatives:

$$\begin{aligned}
\partial_t^2 u_i &= -\omega^2 u_i \\
\partial_j u_j &= [\partial_j (A g_j) + A g_j (i\omega \partial_j T)] e^{-i\omega(t-T)}
\end{aligned}$$

$$\begin{aligned}
\partial_i \partial_j u_j &= [\partial_i \partial_j (A g_j) + \partial_j (A g_j)(i\omega \partial_i T) \\
&+ \partial_i (A g_j)(i\omega \partial_j T) + (A g_j)(i\omega \partial_i \partial_j T) \\
&- A g_j \omega^2 \partial_j T \partial_i T] e^{-i\omega(t-T)}
\end{aligned}$$

$$\partial_i \partial_j u_j = [f_0 + i\omega f_1 - \omega^2 A g_j \partial_j T \partial_i T] e^{-i\omega(t-T)}$$

Eikonal equation (2)

Substitute derivatives in wave equation:

$$(Ag_i\frac{\omega^2}{v^2} + \left[f_0 + i\omega f_1 - \omega^2 A g_j \partial_j T \partial_i T\right])e^{-i\omega(t-T)} = 0$$

Divide by ω^2, and let $\omega \to \infty$:

$$\lim_{\omega\to\infty}\left(Ag_i\frac{1}{v^2} - \left[\frac{f_0}{\omega^2} + \frac{if_1}{\omega} + Ag_j\partial_j T\partial_i T\right]\right) = 0$$

Finally, use normalization $g_i g_i = 1$ and $\nabla T \parallel$ **g** (P-waves):

$$\left(g_i\frac{1}{v^2} - [g_j\partial_j T\partial_i T]\right)g_i \quad = \quad 0$$

Eikonal equation: $\frac{1}{v^2} - |\nabla T|^2 \quad = \quad 0$

Lagrange-Hamilton theory

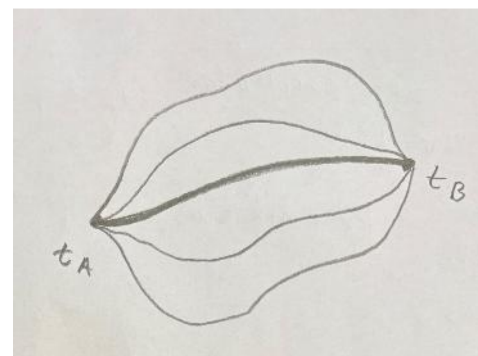
- Alternative formulation of mechanics (higher level of abstraction)
- Focused on energy rather than forces

Lagrangian:

$$\begin{aligned} L &= L(x_i, \dot{x}_i, t) = K - U \quad (i \in \{1, 2, 3, \cdots, 3n\}) \\ K &= \text{kinetic energy} \\ U &= \text{potential energy} \end{aligned}$$

Principle of least action:

$$\begin{aligned} S &= \int_{t_A}^{t_B} L(x_i, \dot{x}_i, t) dt \\ \Delta S &= 0 \quad \text{for the physical path} \end{aligned}$$



Reference: Goldstein, H., 1951, Classical Mechanics

Euler-Lagrange equations

Variation of the action:

$$\Delta S = \int_{t_A}^{t_B} \left[\frac{\partial L}{\partial x_i} \Delta x_i + \frac{\partial L}{\partial \dot{x}_i} \Delta \dot{x}_i \right] dt = 0$$

Integration by parts:

$$\Delta S = \int_{t_A}^{t_B} \left[\frac{\partial L}{\partial x_i} - \frac{d}{dt} \frac{\partial L}{\partial \dot{x}_i} \right] \Delta x_i dt + \cancel{\frac{\partial L}{\partial x_i} \Delta x_i \Big|_{t_A}^{t_B}} = 0$$

Euler-Lagrange equations:

$$\frac{\partial L}{\partial x_i} - \frac{d}{dt} \frac{\partial L}{\partial \dot{x}_i} = 0$$

Example:
Harmonic oscillator

$$K = \frac{1}{2} m \dot{x}^2$$
$$U = \frac{1}{2} k x^2$$

$$\frac{\partial L}{\partial x} = -kx$$
$$\frac{d}{dt} \frac{\partial L}{\partial \dot{x}} = m \ddot{x}$$

$$m \ddot{x} = -kx$$

Hamiltonian

Define momentum

$$p_i = \frac{\partial L}{\partial \dot{x}_i}$$

Then, using Euler-Lagrange:

$$\dot{p}_i = \frac{d}{dt}\frac{\partial L}{\partial \dot{x}_i} = \frac{\partial L}{\partial x_i}$$

Define Hamiltonian (Legendre transform):

$$H \doteq p_i \dot{x}_i - L$$

- 6D phase space: $\{x_i, p_i\}$

Example:
Harmonic oscillator

$$\begin{aligned} K &= \frac{1}{2} m \dot{x}^2 \\ U &= \frac{1}{2} k x^2 \end{aligned}$$

$$p = \frac{\partial L}{\partial \dot{x}} = m\dot{x}$$

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{x}} = \frac{\partial L}{\partial x}$$

$$\dot{p} = m\ddot{x} = -kx$$

$$\begin{aligned} H &= p\dot{x} - L \\ &= m\dot{x}^2 - (K - U) \\ &= K + U \end{aligned}$$

Differential, from definition of H:

$$\begin{aligned}
\Delta H &= \dot{x}_i \Delta p_i + p_i \Delta \dot{x}_i - \Delta L \\
&= \dot{x}_i \Delta p_i + p_i \Delta \dot{x}_i - \left[\frac{\partial L}{\partial x_i} \Delta x_i + \frac{\partial L}{\partial \dot{x}_i} \Delta \dot{x}_i + \frac{\partial L}{\partial t} \Delta t \right] \\
&= \dot{x}_i \Delta p_i + \cancel{\left[p_i - \frac{\partial L}{\partial \dot{x}_i} \right]} \Delta \dot{x}_i - \frac{\partial L}{\partial x_i} \Delta x_i - \frac{\partial L}{\partial t} \Delta t \\
&= \dot{x}_i \Delta p_i - \dot{p}_i \Delta x_i - \frac{\partial L}{\partial t} \Delta t
\end{aligned}$$

Also:

$$\Delta H = \frac{\partial H}{\partial p_i} \Delta p_i + \frac{\partial H}{\partial x_i} \Delta x_i + \frac{\partial H}{\partial t} \Delta t$$

Hamiltons equations:

$$\boxed{\begin{aligned}
\dot{x}_i &= \frac{\partial H}{\partial p_i} \\
\dot{p}_i &= -\frac{\partial H}{\partial x_i}
\end{aligned}}$$

$$\frac{\partial H}{\partial t} = \frac{dH}{dt} = -\frac{\partial L}{\partial t}$$

Time derivative of Hamiltonian

$$
\begin{aligned}
\frac{dH}{dt} &= \dot{x}_i \dot{p}_i + \ddot{x}_i p_i - \frac{dL}{dt} \\
&= \dot{x}_i \dot{p}_i + \ddot{x}_i p_i - \left[\frac{\partial L}{\partial x_i} \dot{x}_i + \frac{\partial L}{\partial \dot{x}_i} \ddot{x}_i + \frac{\partial L}{\partial t} \right] \\
&= \dot{x}_i \left(\cancel{\dot{p}_i - \frac{d}{dt} \frac{\partial L}{\partial \dot{x}_i}} \right) + \ddot{x}_i \left(\cancel{p_i - \frac{\partial L}{\partial \dot{x}_i}} \right) - \frac{\partial L}{\partial t} \\
&= -\frac{\partial L}{\partial t}
\end{aligned}
$$

Emmy Noether (1882-1935)

Why do we have conservation laws?

Invariance of the Lagrangian (action)

Continues transformations

$$\begin{aligned} x_i &\rightarrow x'_i = x_i + \eta_i \\ t &\rightarrow t' = t + \tau \end{aligned}$$

Invariance of the Lagrangian:

$$L(x_i, t) = L(x'_i, t')$$

$$\Downarrow$$

$$L(x_i, t) - L(x'_i, t') = \frac{\partial L}{\partial x}_i \eta_i + \frac{\partial L}{\partial t} \tau = 0$$

Noether's theorem

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{x}_i}\eta_i - \frac{dH}{dt}\tau = 0$$

$$\frac{d}{dt}[p_i\eta_i - H\tau] = 0$$

For every continuous symmetry transformation of the Lagrangian (action), there is a conservation law

Time translation symmetry => Conservation of energy

$$\begin{aligned} \eta_i &= 0 \\ \tau &> 0 \\ \tau\frac{dH}{dt} &= 0 \end{aligned}$$

Spatial translation symmetry => Conservation of momentum

$$\begin{aligned} \eta_1 &> 0 \\ \eta_2 &= \eta_3 = 0 \\ \tau &= 0 \\ \eta_1\frac{p_i}{dt} &= 0 \end{aligned}$$

Noether's theorem

Continuum and fields:
- Lagrangian density

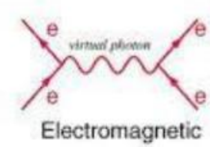
$$S = \int \mathcal{L}\, d^3x\, dt$$
$$L = \int \mathcal{L}\, d^3x$$

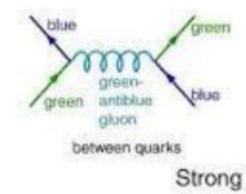
- For every continuous symmetry transformation of the Lagrangian (action), there is a conservation law

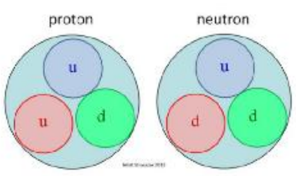
- The converse is possibly not true (?)

Examples:
- Time translation => conservation of energy
- Spatial translation => conservation of momentum (slowness)
- Rotation => conservation of angular momentum

- Spin
- Abelian gauge symmetry => conservation of charge
- Non-Abelian gauge symmetry => conservation of color (QCD)







Method of characteristics; Hamiltonian

Find $x_i = x_i(u)$ and $p_i(u)$ such that

$$\mathcal{H}(p_i, x_i) \quad = \quad 0, \ (i = 1, 2, 3)$$

(p_i, x_i) are coordinates in 6D **phase space**.

u is a monotonically increasing parameter, usually one of

$$\begin{aligned} u &= T, \ \text{(traveltime)} \\ u &= s, \ \text{(arc length)} \end{aligned}$$

Here we will use $u = T$; convenient in anisotropic media.
Hill (1990) uses $u = s$.

Different forms of the Hamiltonian

The Hamiltonian $\mathcal{H}$ can be defined in many ways:

$$\begin{aligned}\mathcal{H}_0 &= v^2 p_i p_i - 1 = 0 \\ \mathcal{H}_1 &= |\mathbf{p}| - \frac{1}{v} = 0 \\ \mathcal{H}_2 &= p_i p_i - \frac{1}{v^2} = 0\end{aligned}$$

All are OK; motivated by the eikonal equation.

Hamilton canonical equations (1)

Complete differential:

$$d\mathcal{H} = \frac{\partial \mathcal{H}}{\partial p_i} dp_i + \frac{\partial \mathcal{H}}{\partial x_i} dx_i = 0$$

$$\frac{\partial \mathcal{H}}{\partial p_i} dp_i = -\frac{\partial \mathcal{H}}{\partial x_i} dx_i$$

which is fullfilled if

$$\begin{aligned} dp_i &= \frac{\partial \mathcal{H}}{\partial x_i} du \\ dx_i &= -\frac{\partial \mathcal{H}}{\partial p_i} du \end{aligned}$$

where u is any monotonically increasing parameter.

Hamilton canonical equations (2)

Hamilton equations

$$\frac{dp_i}{du} = \frac{\partial \mathcal{H}}{\partial x_i}$$
$$\frac{dx_i}{du} = -\frac{\partial \mathcal{H}}{\partial p_i}$$

Let $u = T$ (traveltime along the ray):

$$\frac{dp_i}{dT} = \frac{\partial \mathcal{H}}{\partial x_i}$$
$$\frac{dx_i}{dT} = -\frac{\partial \mathcal{H}}{\partial p_i}$$

- The Hamilton equations are of fundamental importance in theoretical physics:
  - Classical mechanics
  - Classical field theory
  - Quantum field theory
  - Ray theory

- Some good books:
  - Landau and Lifshitz (1939): The Classical Theory of Fields
  - Goldstein (1951): Classical Mechanics
  - Morse and Feshbach (1953): Methods of Theoretical Physics.
  - Cerveny (2001): Seismic Ray Theory

Equation for T

Sometimes (if $u = s$) we need an equation for T

$$\begin{aligned}\frac{dT}{du} &= \frac{\partial T}{\partial x_i}\frac{dx_i}{du} \\ &= p_i \frac{dx_i}{du} \\ &= -p_i \frac{\partial \mathcal{H}}{\partial p_i}\end{aligned}$$

Choice of u vs Hamiltonian:

$$u = s \quad \Rightarrow \quad \frac{dT}{du} = \frac{dT}{ds} = \frac{1}{v} \Rightarrow \mathcal{H} = \mathcal{H}_1 \text{ (Hill)}$$

$$u = T \quad \Rightarrow \quad \frac{dT}{du} = \frac{dT}{dT} = 1 \Rightarrow \mathcal{H} = \mathcal{H}_0$$

Anisotropic elastic wave equation

Newton and Hooke:

$$\begin{aligned}\rho \partial_t^2 u_i &= \partial_j \sigma_{ij} \\ \sigma_{ij} &= c_{ijkl} \partial_k u_l\end{aligned}$$

Wave equation:

$$\rho \partial_t^2 u_i = \partial_j c_{ijkl} \partial_k u_l$$

Smooth medium ($\partial_j c_{ijkl} \simeq 0$):

$$\partial_t^2 u_i = \frac{c_{ijkl}}{\rho} \partial_j \partial_k u_l$$

Anisotropic elastic ray theory

Ray ansatz:

$$u_i(\mathbf{x}, t) \quad = \quad A g_j e^{-i\omega(t-T)}$$

Substitute in wave equation and let $\omega \to \infty$:

$$\frac{c_{ijkl}}{\rho} p_j p_l g_k - g_i \quad = \quad 0$$

Christoffel equation:

$$[\Gamma_{ik} - \delta_{ik}] g_k = 0$$

Christoffel tensor:

$$\Gamma_{ik} = \frac{c_{ijkl}}{\rho} p_j p_l$$

Properties of the Christoffel tensor:

$$\begin{aligned} \Gamma_{ik} &= \Gamma_{ki} \quad \text{(symmetric)} \\ \Gamma_{ik}(a\mathbf{p}) &= a^2 \Gamma_{ik}(\mathbf{p}) \end{aligned}$$

Homogeneous function of 2nd degree

Eigenvalues and eigenvectors

Eigenvalues:

$$\begin{aligned} \det[\mathbf{\Gamma} - G^{(m)}\mathbf{I}] &= 0 \\ G^{(m)}(\mathbf{p}) &= 1 \end{aligned}$$

Eigenvectors:

$$[\mathbf{\Gamma} - G^{(m)}\mathbf{I}]\mathbf{g}^{(m)} = 0 \qquad [\Gamma_{ik} - G^{(m)}\delta_{ik}]g_k^{(m)} = 0$$

Normalization:

$$\mathbf{g}^{(m)T}\mathbf{g}^{(m)} = 1 \qquad g_i^{(m)}g_i^{(m)} = 1$$

3 eigenvalues, 3 eigenvectors $\Rightarrow$ 3 wave modes:

$$m = \begin{cases} 1 & : \text{qS1 waves} \\ 2 & : \text{qS2 waves} \\ 3 & : \text{qP waves} \end{cases}$$

More about the eigenvalues

Multiply by $g_i^{(m)}$ and sun over i, j:

$$[\Gamma_{ik} - G^{(m)}\delta_{ik}]g_k^{(m)}g_i^{(m)} \quad = \quad 0$$

Explicit equation for eigenvalues:

$$G^{(m)} \quad = \quad \Gamma_{ik}g_k^{(m)}g_i^{(m)}$$

Using the homogeneous function property:

$$G^{(m)}(\mathbf{p}) \quad = \quad 1 \quad \text{(to fulfill Christoffel equation)}$$
$$G^{(m)}(\mathbf{n}) \quad = \quad v^2 \quad \text{(by definition of phase normal)}$$

Slowness and phase direction:

$$\mathbf{p} \quad = \quad \nabla T = \frac{\mathbf{n}}{v}$$
$$v \quad = \quad \text{phase velocity, } \mathbf{n} = \text{phase normal, } |\mathbf{n}| = 1$$

Hamiltonian and and Hamilton equations

Hamiltonian:

$$\mathcal{H} = \frac{1}{2}(G^{(m)} - 1) = 0$$

$$\mathcal{H} = \mathcal{H}_0$$

Hamilton equations:

$$\frac{\partial x_i}{\partial T} = \frac{\partial \mathcal{H}}{\partial p_i} = \frac{1}{2}\frac{\partial G^{(m)}}{\partial p_i}$$

$$\frac{\partial p_i}{\partial T} = -\frac{\partial \mathcal{H}}{\partial x_i} = -\frac{1}{2}\frac{\partial G^{(m)}}{\partial x_i}$$

Kinetic ray tracing system, 6 first-order PDEs, to be solved simultaneously

Group velocity (ray velocity):

$$V = \left|\frac{d\mathbf{x}}{dT}\right|$$

$$V = \sqrt{V_i V_i} = \sqrt{\frac{dx_i}{dT}\frac{dx_i}{dT}}$$

The group velocity vector is tangent to the ray

Kinematic initial value ray tracing (1)

1. Choose wavemode: $m = 1, 2$ or 3.

2. Choose source position $\mathbf{x}_0$ and initial phase direction (θ_0, ϕ_0),

$$\mathbf{n}_0 = (\sin\theta_0 \cos\phi_0, \sin\theta_0 \sin\phi_0, \cos\phi_0)$$

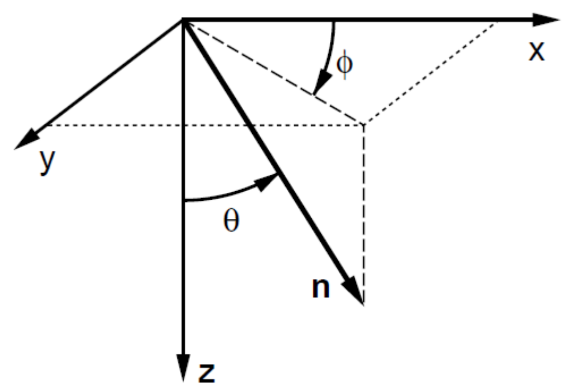
3. Get $c_{ijkl}(\mathbf{x}_0)$ and $\rho(\mathbf{x}_0)$ from elastic model and compute

$$v(\mathbf{x}_0, \theta_0, \phi_0) = \sqrt{(G^{(m)}(\mathbf{n}_0))}$$

(VTI: v is independent of ϕ)

4. Compute the initial slowness

$$\mathbf{p}_0 = \frac{\mathbf{n}_0}{v(\mathbf{x}_0, \theta_0, \phi_0)}$$



Kinematic initial value ray tracing (2)

5. Solve the Christoffel equation to get polarization $\mathbf{g}_0^{(m)}$

6. Set initial conditions for the ray:

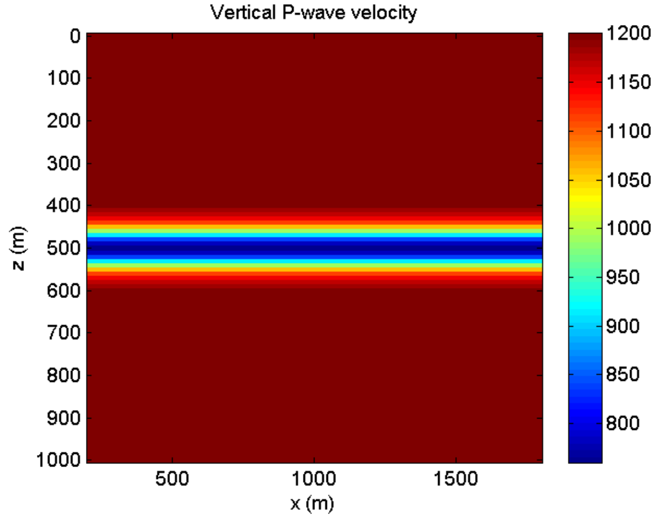
$$\begin{aligned} \mathbf{x}(T_0) &= \mathbf{x}_0 \\ \mathbf{p}(T_0) &= \mathbf{p}_0 \\ \mathbf{g}^{(m)}(T_0) &= \mathbf{g}_0^{(m)} \end{aligned}$$

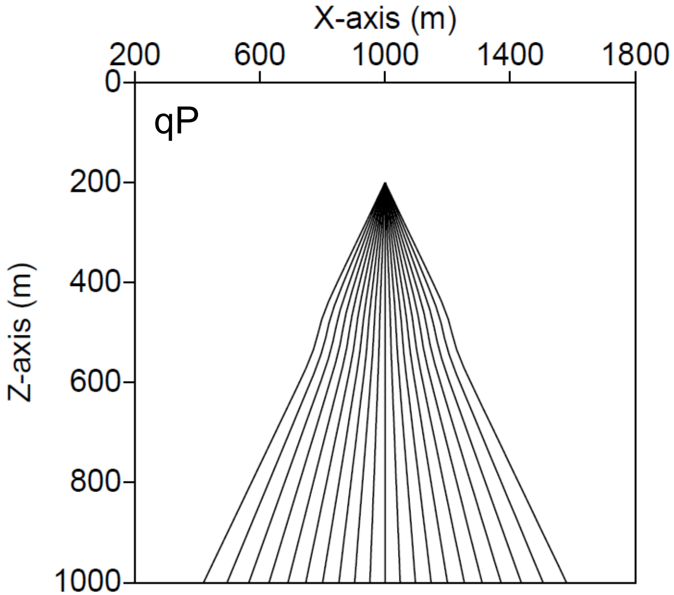
7. Solve ray tracing system by Runge-Kutta integration

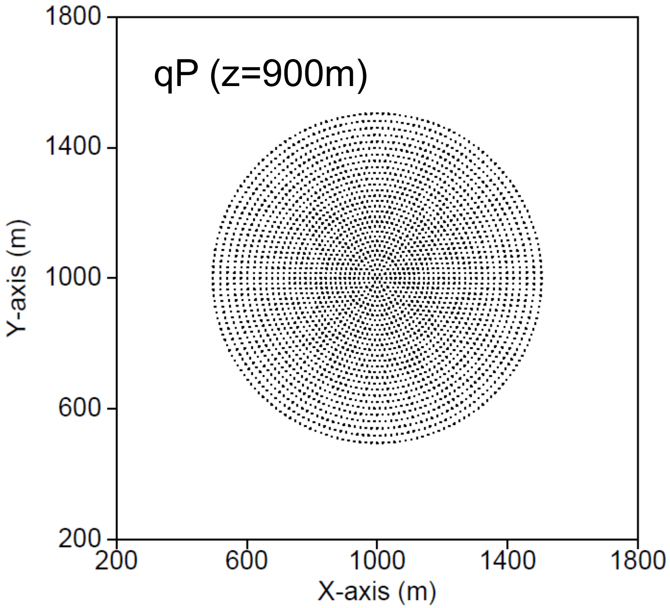
$$\begin{aligned} \dot{\mathbf{y}} &= \mathbf{f}(\mathbf{y}) \\ \mathbf{y} &= [x_i, p_i]^T \quad (i = 1, 2, 3) \end{aligned}$$

Numerical example

$$\frac{v_{P0}}{v_{S0}} = 2$$
$$\delta = 0.10$$
$$\epsilon = 0.20$$
$$\gamma = 0.05$$







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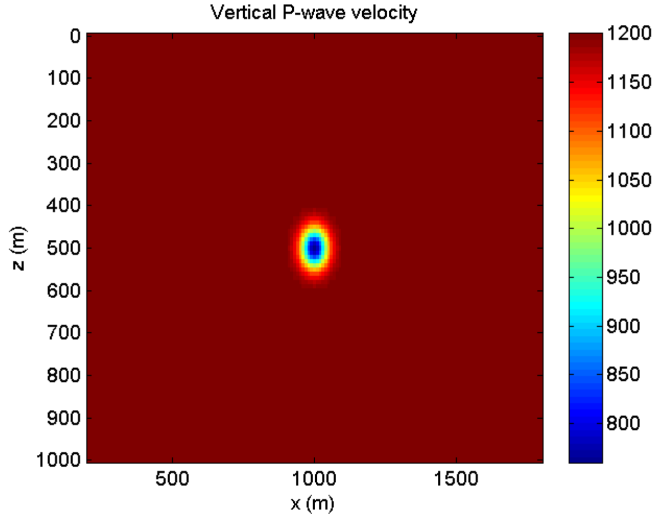

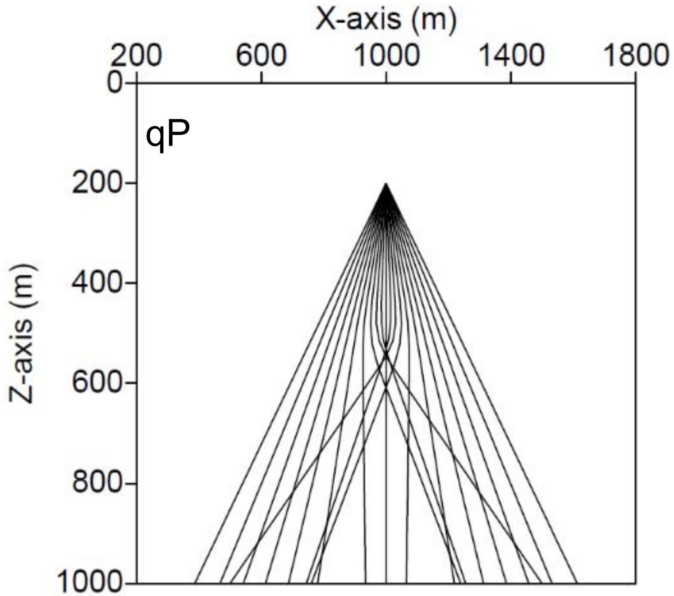

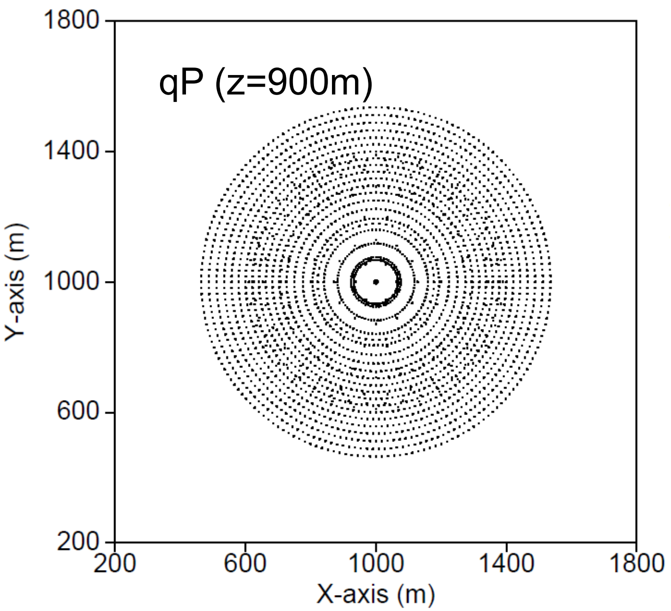

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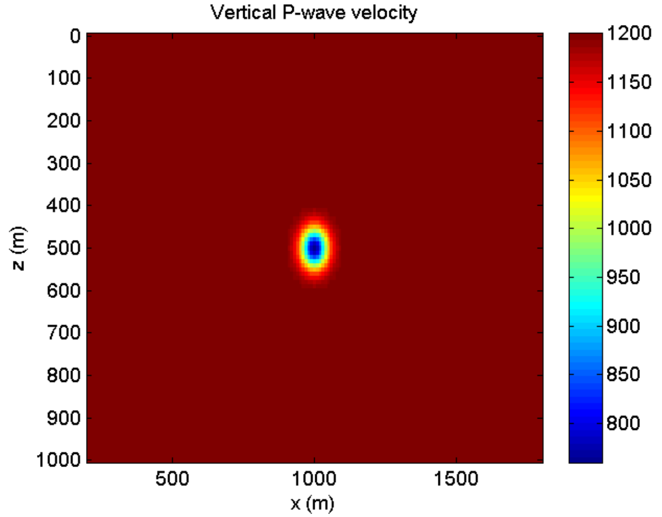

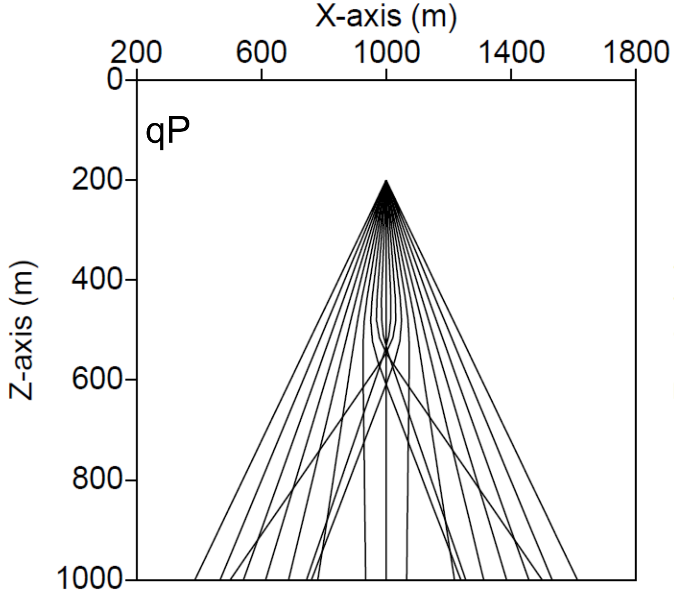

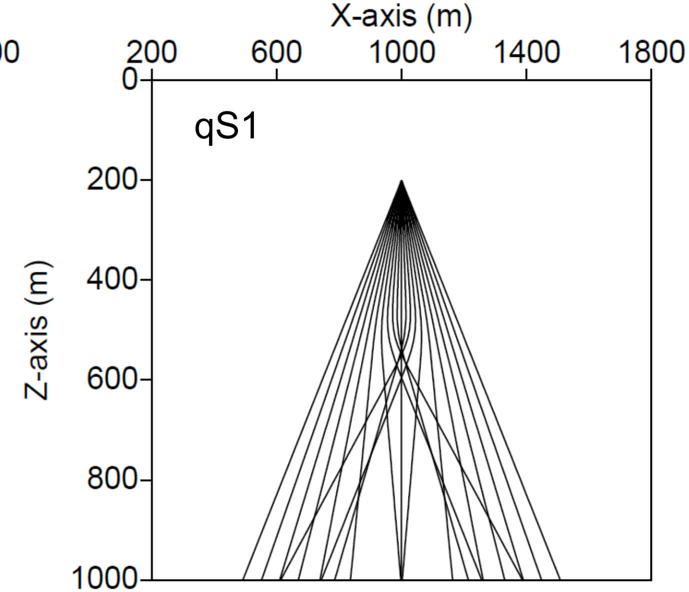

Numerical example

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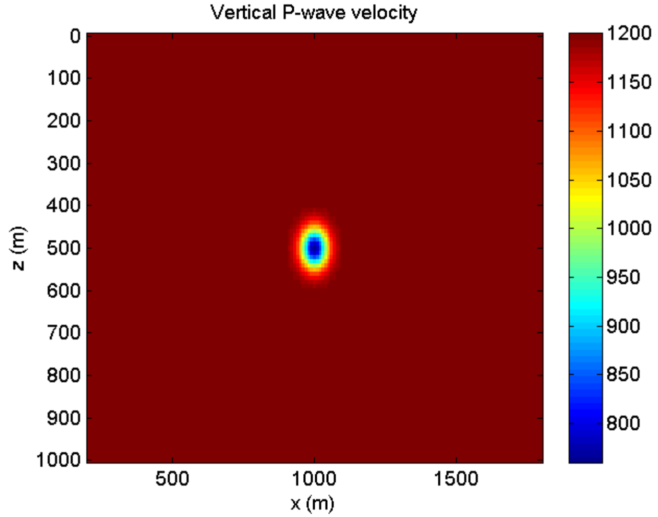
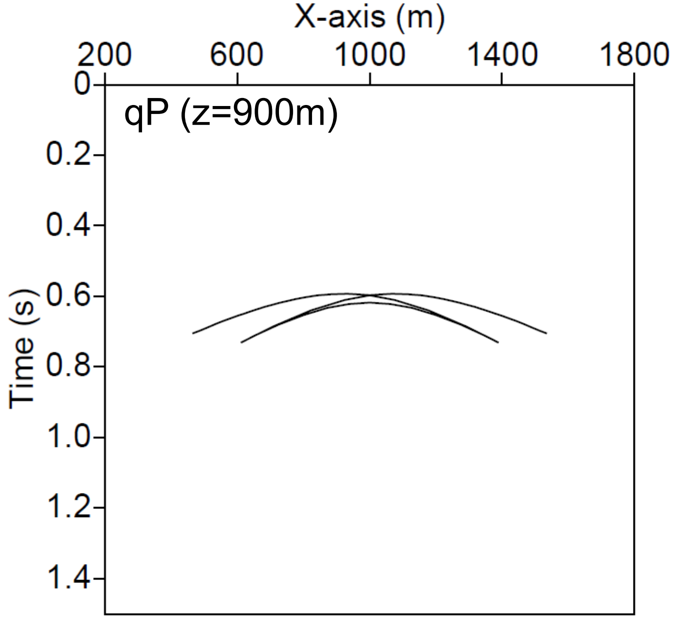
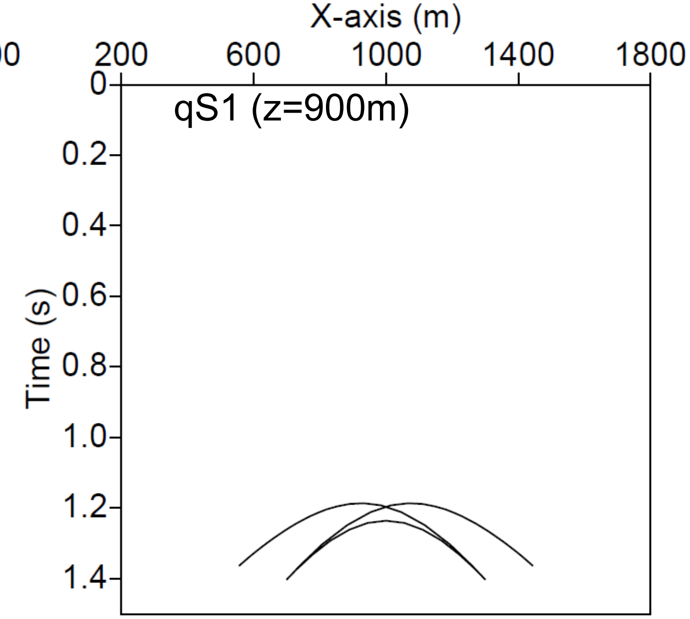






Dynamic ray tracing, geometrical spreading

Ray ansatz:

$$u_i(\mathbf{x}, t) \quad = \quad A g_j e^{-i\omega(t-T)}$$

Ray amplitude:

$$A(\mathbf{x}, \mathbf{x}_0) \quad = \quad \frac{\mathcal{R}^c(\mathbf{x}, \mathbf{x}_0) e^{i\frac{\pi}{2} sgn(\omega)\sigma}}{4\pi Z(\mathbf{x}) Z(\mathbf{x}_0) \mathcal{L}(\mathbf{x}, \mathbf{x}_0)}$$

$\sigma =$ KMAH index

$$Z(\mathbf{x}) \quad = \quad \rho(\mathbf{x}) v(\mathbf{x})$$

$$\mathcal{R}^c(\mathbf{x}, \mathbf{x}_0) \quad = \quad \prod_k R_k \left[\frac{(\rho V \cos\psi)_{k+}}{(\rho V \cos\psi)_{k-}} \right]^{1/2}$$

$$\mathcal{L}(\mathbf{x}, \mathbf{x}_0) \quad = \quad [\det \mathbf{Q}_2]^{1/2} = \left[\frac{\det \mathbf{Q}}{v} \right]^{1/2}$$

$v \quad = \quad$ phase velocity

$V \quad = \quad$ group velocity

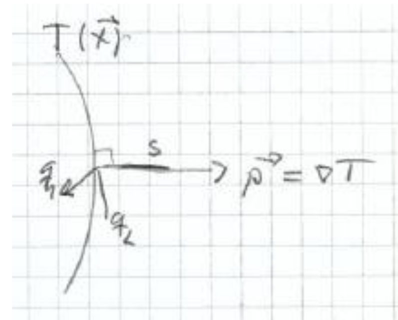
$\psi \quad = \quad$ group angle

What is $\mathbf{Q}_2$ and $\mathbf{Q}$?

Local phase and group (ray) coordinates

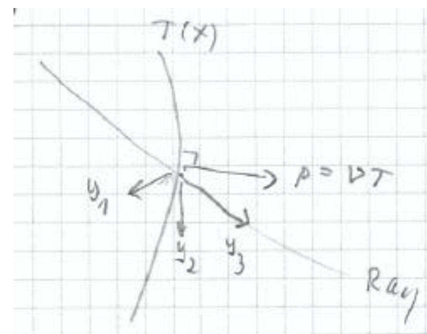
Local ray-centered phase coordinates:

$$\begin{aligned} \mathbf{q} &= (q_i, q_2, s) \\ \hat{s} &\parallel \mathbf{p} \\ \mathbf{p} &= (0,0,p) = (0,0,1/v) \end{aligned}$$



Local ray-centered group coordinates:

$$\begin{aligned} \mathbf{y} &= (y_1, y_2, y_3) \\ y_1 = y_2 &= 0 \text{ on the ray} \\ \mathbf{V} &= (0,0,V) \end{aligned}$$



Isotropic media:

$$(q_1, q_2, s) = (y_1, y_2, y_3)$$

Relative geometrical spreading

In ray-centered coordinates:

$$\mathcal{L}(\mathbf{x}, \mathbf{x}_0) = [\det \mathbf{Q}_2]^{1/2}, \quad (2 \times 2\,\text{matrix})$$

$$(Q_2)_{IJ} = \frac{\partial^2 T}{\partial q_I \partial q_J}, \quad (I, J = 1, 2)$$

In Cartesian coordinates:

$$\mathcal{L}(\mathbf{x}, \mathbf{x}_0) = = \left[\frac{\det \mathbf{Q}}{v}\right]^{1/2} \quad (3 \times 3\,\text{matrix})$$

$$Q_{iJ} = \frac{\partial x_i}{\partial \gamma_j}$$

$\gamma = (\theta, \phi) =$ phase direction

$$Q_{i3} = \frac{\partial x_i}{\partial T} = V_i$$

Q_{IJ}, $(J = 1, 2)$ are computed by dynamic ray tracing

Dynamic ray tracing

3×2 matrices $\mathbf{Q}$ and $\mathbf{P}$:

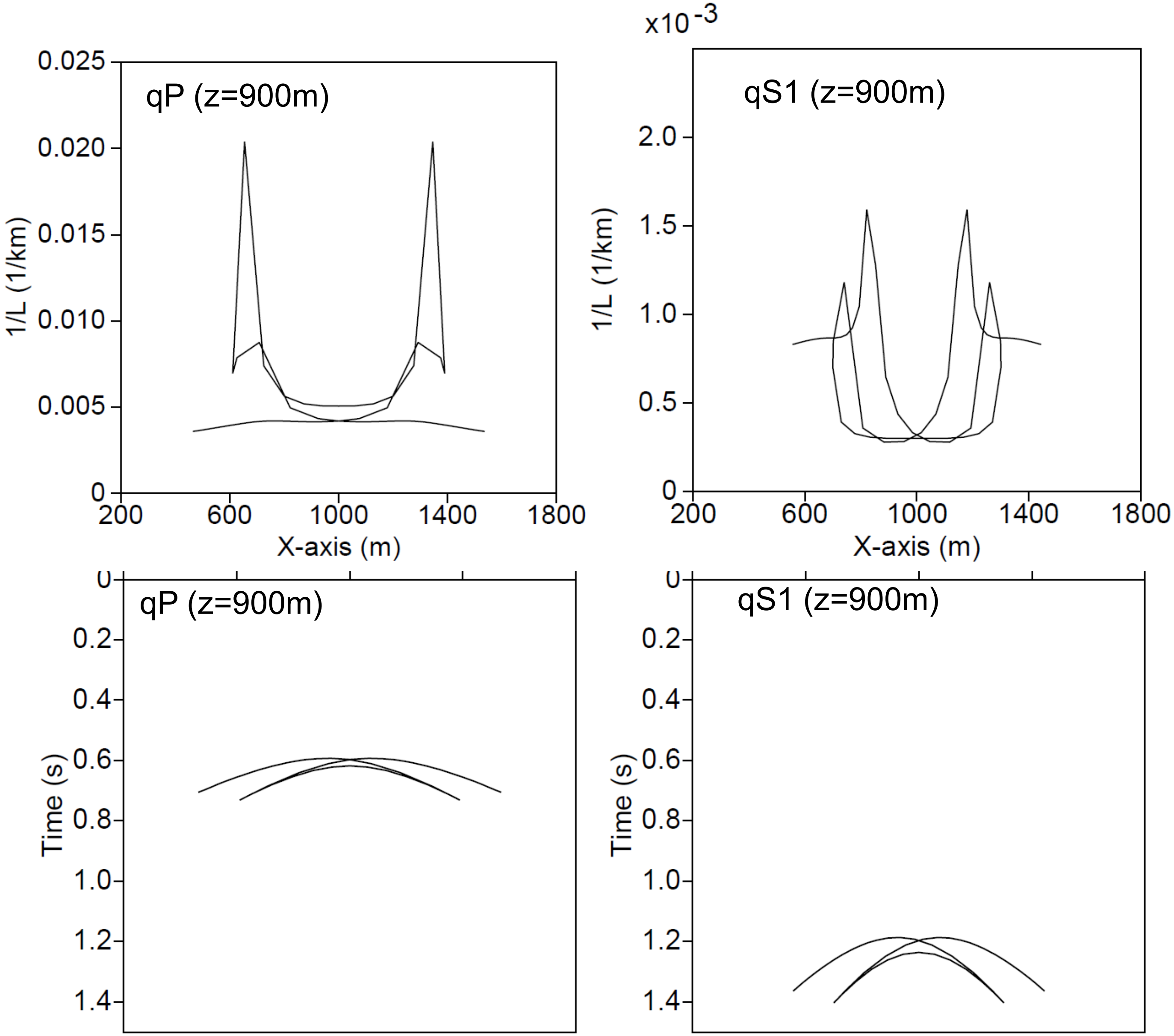
$$
\begin{aligned}
Q_{iK} &= \frac{\partial x_i}{\gamma_K} \\
P_{iK} &= \frac{\partial p_i}{\gamma_K}
\end{aligned}
$$

$\gamma = (\theta, \phi) =$ phase direction

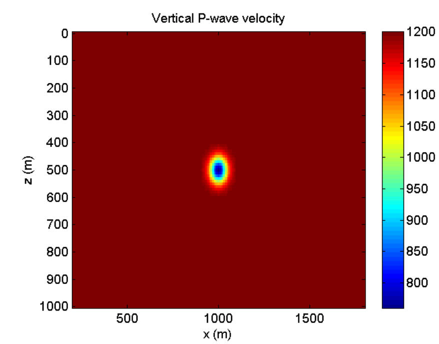
Dynamic ray tracing system:

$$
\begin{aligned}
\frac{\partial Q_{iK}}{\partial T} &= \frac{\partial^2 \mathcal{H}}{\partial p_i \partial x_j} Q_{jK} + \frac{\partial^2 \mathcal{H}}{\partial p_i \partial p_j} P_{jK} \\
\frac{\partial P_{iK}}{\partial T} &= -\frac{\partial^2 \mathcal{H}}{\partial x_i \partial x_j} Q_{jK} - \frac{\partial^2 \mathcal{H}}{\partial x_i \partial p_j} P_{jK}
\end{aligned}
$$

Dynamic ray tracing system; 12 first-order PDEs, to be solved simultaneously



Geometrical spreading



Traveltime

Seismograms from ray racing

Sum over events and FFT from frequency to time:

$$
\begin{aligned}
u(\mathbf{x}, t) &= \int d\omega \sum_n A_n(\mathbf{x}, \mathbf{x}_s) e^{-i\omega[t - T(\mathbf{x}, \mathbf{x}_s)]} \\
A_n(\mathbf{x}, \mathbf{x}_0) &= \frac{\mathcal{R}^c(\mathbf{x}, \mathbf{x}_0) e^{i\frac{\pi}{2} sgn(\omega)\sigma}}{4\pi Z(\mathbf{x}) Z(\mathbf{x}_0) \mathcal{L}(\mathbf{x}, \mathbf{x}_0)}
\end{aligned}
$$

- Trace events one by one
- Event specification by ray codes

Common-shot Kirchhoff migration revisited

Substitute U and D in imaging condition:

$$I(x;x_s) \quad = \quad \int d\omega \int\int dx_r dy_r p_z(x_r) \frac{\mathcal{L}(x,x_s)}{\mathcal{L}(x,x_r)} e^{-i\omega[T(x,x_r)+T(x,x_s)]} i\omega P(x_r,\omega)$$

The integral over ω is the inverse Fourier transform:

$$I(x;x_s) \quad = \quad -2 \int\int dx_r dy_r W(x,x_r,x_s) \partial_t P(x_r,\tau)$$

The integral over ω is the inverse Fourier transform:

$$\begin{aligned} \tau \quad &= \quad T(x,x_r)+T(x,x_s) = \text{total traveltime} \\ W \quad &= \quad \frac{\cos\theta}{v} \frac{\mathcal{L}(x,x_s)}{\mathcal{L}(x,x_r)} e^{-i\delta\tau} = \text{weight function} \\ \delta\tau \quad &= \quad \text{phase shift due to caustics} \end{aligned}$$

Wavefront construction

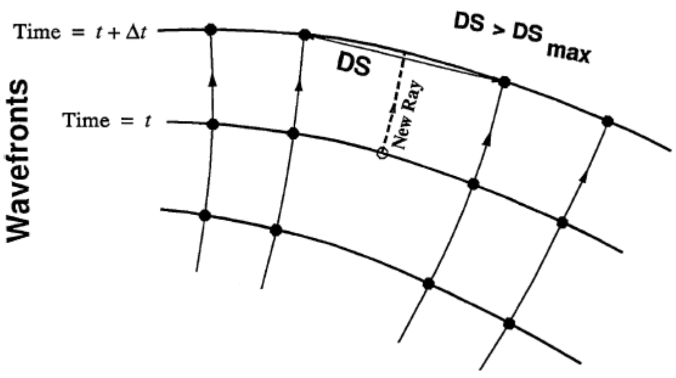
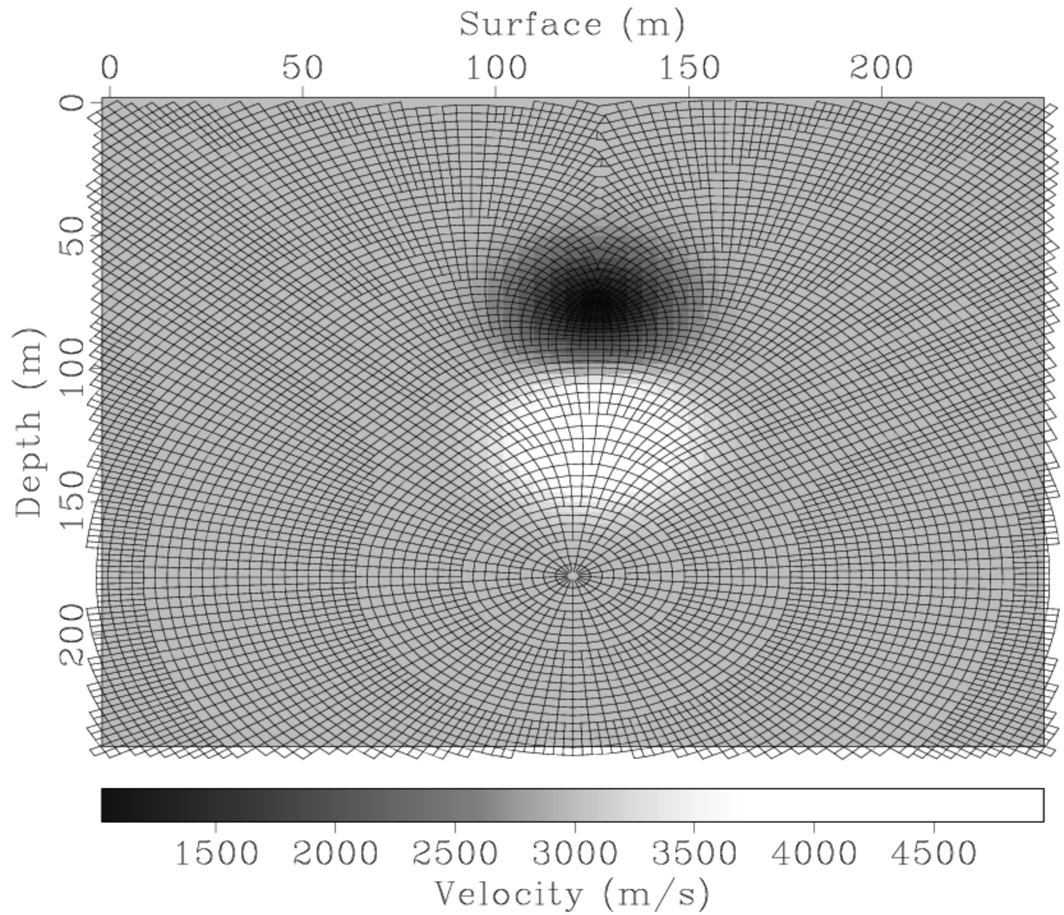


FIG. 6. Propagation of WFs and interpolation of a new ray in 2-D WF construction. When the difference between two rays becomes too large at the WF at time $t + \Delta t$, a new ray is interpolated on the WF at time t.

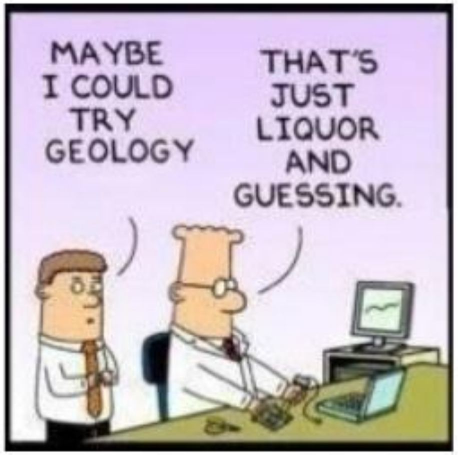
Vinje and Iversen, Geophysics (1999)



Applications of ray theory and ray tracing

- Applications of ray tracing:
  - Prestack depth migration (Kirchhoff family)
  - Seismic interpretation
  - Survey planning
  - Generation of synthetic seismic data
  - Theoretical investigations
- Non-seismic applications:
  - Electromagnetics
  - Thermal modeling

- Different types of ray tracing:
  - Initial value ray tracing
  - Two-point ray tracing
  - Wave-front construction



Thermal modeling; heat rays

Fourier's law (steady-state temperature):

$$\mathbf{q} = -k\nabla T$$

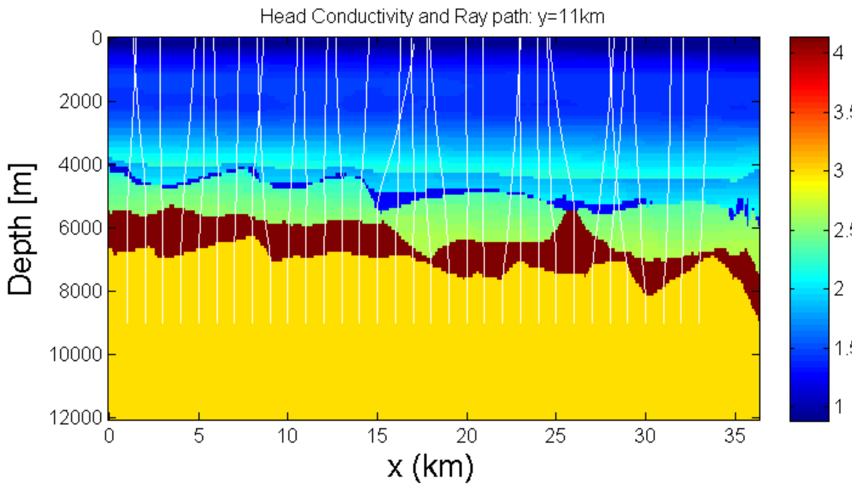
$$\begin{aligned} T &= \text{temperature} \\ k &= \text{thermal conductivity} \\ \mathbf{q} &= \text{heat flow} \end{aligned}$$

Eikonal equation for temperature:

$$\left(\frac{q}{k}\right)^2 = |\nabla T|^2$$

Can be solved with a seismic ray racer if

$$\frac{k}{q} \rightarrow v$$



Method of characteristics

«Cold -flow» ray-tracing - isotherms