C++ Project

PRICING VARIANCE SWAPS UNDER HESTON MODEL

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Part I

Description

Chapter 1

Variance Swaps

A Variance Swap contract is a financial instrument whose payoff depends on the realized variance of an asset price over a period of time $[0, T]$ with fixing dates $t_0 = 0$, t_1, ..., $t_N = T$.
We denote $(S_t)_{t \in [t_0, t_N]}$ the underlying spot price along the period $[t_0, t_N]$.

1.1 Log-return

For $i \in 1, \ldots, N$, the price log-return over $[t_{i-1}, t_i]$ is defined by:

$$R(t_{i-1}, t_i) = \ln(\frac{S_{t_i}}{S_{t_{i-1}}}) \tag{1.1}$$

1.2 Realized variance over $[t_{i-1}, t_i]$

From now on, we use the following convention:
The realized **total variance** of the asset price over the period $[t_{i-1}, t_i]$ is defined by the square of the log-return over this period:

$$\sigma_{i-1}^2 \times (t_i - t_{i-1}) = R(t_{i-1}, t_i)^2 \tag{1.2}$$

where σ_{i-1}^2 is the so-called **annualized** realized variance over $[t_{i-1}, t_i]$.

1.3 Annualized Realized Variance over $[0, T]$

Consequently, as the realized total variance is additive, we can define the annualized realized variance $\sigma_{[0,T]}^2$ such as:

$$\sigma_{[0,T]}^2 \times T = \sum_{i=1}^{N} R(t_{i-1}, t_i)^2 \tag{1.3}$$

Hence:

$$\sigma_{[0,T]}^2 = \frac{1}{T} \sum_{i=1}^{N} \ln^2(\frac{S_{t_i}}{S_{t_{i-1}}}) \tag{1.4}$$

1.4 Payoff

The payoff function at time $t_N = T$ of this financial contract is given by:

$$V(\{t_0, t_1, \ldots, t_N\}) = L \times (\sigma^2_{[0,T]} - \sigma^2_K) \tag{1.5}$$

where:

- L is the notional of the contract.
- σ^2_K is the the variance strike of the contract.

The value of this contract at inception $t_0 = 0$ is therefore given by:

$$V_0 = \mathbb{E}^{\mathbb{Q}}[D(0,T)\ V(\{t_0, t_1, \ldots, t_M\})] \tag{1.6}$$

where $D(0,T)$ is the discount factor process at inception.
In our case, we have a deterministic discount rate r so that

$$D(0,T) = e^{-rT}$$

1.5 Objective

Finding the variance strike σ^2_K which makes the value of the contract at inception worth zero.
It is easy to see that in that case, the choice for that strike is:

$$\sigma^2_K = \mathbb{E}^{\mathbb{Q}}[\sigma^2_{[0,T]}] \tag{1.7}$$

Chapter 2

Heston Model

We assume from now that the spot process $(S_t)_{t\in[t_0,t_N]}$ follows the following SDE (Stochastic Differential Equation) under the risk-neutral measure $\mathbb{Q}$:

$$\begin{cases} \frac{dS_t}{S_t} = \mu dt + \sqrt{v_t}dW_t^S \\ dv_t = \kappa(\theta - v_t)dt + \sigma_v\sqrt{v_t}dW_t^V \\ d < W_t^S, W_t^V >= \rho dt \end{cases} \tag{2.1}$$

$(v_t)_{t\in[t_0,t_N]}$ is called the variance process, $(W_t^S)_{t\in[t_0,t_N]}$ and $(W_t^V)_{t\in[t_0,t_N]}$ are standard Brownian motions, correlated by the constant $\rho \in [-1, 1]$.

Part II

Analytical Formula

Chapter 3

Introducing a new variable I_t

We are interested in the computation of the following term:

$$\mathbb{E}[ln^2(\frac{S_{t_i}}{S_{t_{i-1}}})] \tag{3.1}$$

We are facing a 2-dimensional problem where the payoff depends on both times t_i and t_{i-1}.
To remedy this problem, we introduce the following variable:

$$I_t = \int_{u=0}^{t} \delta(t_{i-1} - u) S_u du \tag{3.2}$$

We can notice that:

$$I_t = \begin{cases} S_{t_{i-1}} & \text{if } t \in [t_{i-1}, t_i] \\ 0 & \text{if } t \in [t_0, t_{i-1}] \end{cases} \tag{3.3}$$

Chapter 4

Feynman-Kac PDE

For any contingent claim $U_i = U_i(t, S_t, I_t, v_t)$ which has an European payoff function ϕ_i at t_i of the form $\phi_i = \phi_i(S_{t_i}, I_{t_i}, v_{t_i})$, its PDE is given by (using Ito's Lemma and martingale property under the risk-neutral measure):

$$\begin{cases} \frac{\partial U_i}{\partial t} + \mu S \frac{\partial U_i}{\partial S} + \kappa(\theta - v)\frac{\partial U_i}{\partial v} + \frac{1}{2} v S^2 \frac{\partial^2 U_i}{\partial S^2} + \frac{1}{2}\sigma_v^2 v \frac{\partial^2 U_i}{\partial v^2} + \rho\sigma_v v S \frac{\partial^2 U_i}{\partial S \partial v} + \delta(t_{i-1} - t) S \frac{\partial U_i}{\partial I} - rU_i = 0 \\ U_i(t_i, S, I, v) = \phi_i(S, I, v) \end{cases} \tag{4.1}$$

The Feynman-Kac theorem states then that the solution of this PDE verifies:

$$U_i(0, S_0, I_0, v_0) = e^{-rt_i}\ \mathbb{E}^{\mathbb{Q}}[\phi_i(S_{t_i}, I_{t_i}, v_{t_i})] \tag{4.2}$$

The PDE described above shows a Dirac delta function on the singular point t_{i-1}.
This PDE can therefore be decomposed into one PDE to be solved in the interval $]t_{i-1}, t_i]$ whose payoff at t_i is given by ϕ_i, and into another PDE to be solved in the interval $[0, t_{i-1}[$ whose payoff limit at t_{i-1} is given by the limit at t_{i-1} of the solution of the former PDE.

To summarize:

- In the interval $]t_{i-1}, t_i]$, the PDE verifies:

$$\begin{cases} \frac{\partial U_i}{\partial t} + \mu S \frac{\partial U_i}{\partial S} + \kappa(\theta - v)\frac{\partial U_i}{\partial v} + \frac{1}{2} v S^2 \frac{\partial^2 U_i}{\partial S^2} + \frac{1}{2}\sigma_v^2 v \frac{\partial^2 U_i}{\partial v^2} + \rho\sigma_v v S \frac{\partial^2 U_i}{\partial S \partial v} - rU_i = 0 \\ U_i(t_i, S, I, v) = \phi_i(S, I, v) \end{cases} \tag{4.3}$$

- In the interval $[0, t_{i-1}[$, the PDE verifies:

$$\begin{cases} \frac{\partial U_i}{\partial t} + \mu S \frac{\partial U_i}{\partial S} + \kappa(\theta - v)\frac{\partial U_i}{\partial v} + \frac{1}{2} v S^2 \frac{\partial^2 U_i}{\partial S^2} + \frac{1}{2}\sigma_v^2 v \frac{\partial^2 U_i}{\partial v^2} + \rho\sigma_v v S \frac{\partial^2 U_i}{\partial S \partial v} - rU_i = 0 \\ lim_{t \nearrow t_{i-1}} U_i(t, S, I, v) = lim_{t \searrow t_{i-1}} U_i(t, S, I, v) \end{cases} \tag{4.4}$$

4.1 Solving the PDE on $]t_{i-1}, t_i]$

Let's remind that in our case, the function payoff ϕ_i is given by:

$$\phi_i(S, v, I) = \ln^2(\frac{S}{I}) \tag{4.5}$$

Since the variable I does not appear anymore in the PDE expression, the latter can be written under the following form:

$$\begin{cases} \frac{\partial U}{\partial t} + \mu S \frac{\partial U}{\partial S} + \kappa(\theta - v)\frac{\partial U}{\partial v} + \frac{1}{2} v S^2 \frac{\partial^2 U}{\partial S^2} + \frac{1}{2}\sigma_v^2 v \frac{\partial^2 U}{\partial v^2} + \rho \sigma_v v S \frac{\partial^2 U}{\partial S \partial v} - rU = 0 \\ \\ U(T, S, v) = H(S) \end{cases} \tag{4.6}$$

where

$$\begin{cases} T = t_i \\ U = U_i \\ H(S) = \ln^2(S/I) \\ I \quad \text{is considered as a constant} \end{cases}$$

Using the following transformations:

$$\begin{cases} \tau = T - t \\ x = \ln(S) \end{cases}$$

The PDE is converted into the following form:

$$\begin{cases} \frac{\partial U}{\partial \tau} = (\mu - \frac{1}{2} v)\frac{\partial U}{\partial x} + \kappa(\theta - v)\frac{\partial U}{\partial v} + \frac{1}{2} v \frac{\partial^2 U}{\partial x^2} + \frac{1}{2}\sigma_v^2 v \frac{\partial^2 U}{\partial v^2} + \rho \sigma_v v \frac{\partial^2 U}{\partial x \partial v} - rU \\ \\ U(0, x, v) = H(e^x) \end{cases} \tag{4.7}$$

Applying the Fourier transform on this equation with respect to the variable x, we obtain the following problem:

$$\begin{cases} \frac{\partial \tilde{U}}{\partial \tau} = \frac{1}{2}\sigma_v^2 v \frac{\partial^2 \tilde{U}}{\partial v^2} + (\kappa\theta + (\rho\sigma_v j\omega - \kappa)v)\frac{\partial \tilde{U}}{\partial v} + (\mu j\omega - r - \frac{1}{2}(j\omega + \omega^2)v)\tilde{U} \\ \\ \tilde{U}(0, \omega, v) = \mathcal{F}[H(e^x)] \end{cases} \tag{4.8}$$

According to Heston's paper, the solution of the above PDE can be assumed to be an affine exponential function of v of the following form:

$$\tilde{U}(\tau, \omega, v) = e^{C(\tau,\omega) + D(\tau,\omega)\ v}\ \tilde{U}(0, \omega, v) \tag{4.9}$$

Substituting this form into the PDE, since the variable v can take any value in $\mathbb{R}_+$, it leads to 2 ordinary differential equations (ODE):

$$\begin{cases} \frac{\partial D}{\partial \tau} = \frac{1}{2}\sigma_v^2 D^2 + (\rho\sigma_v j\omega - \kappa)D - \frac{1}{2}(j\omega + \omega^2) \\ \\ \frac{\partial C}{\partial \tau} = \kappa\theta D + \mu j\omega - r \end{cases} \tag{4.10}$$

With initial conditions:

$$\begin{cases} C(0, \omega) = 0 \\ \\ D(0, \omega) = 0 \end{cases}$$

4.1.1 First ODE

The first ODE is a so-called Riccati equation.
To solve it, we first look for a constant solution D_0 for D:

$$\frac{1}{2}\sigma_v^2 D_0^2 + (\rho\sigma_v j\omega - \kappa)D_0 - \frac{1}{2}(j\omega + \omega^2) = 0 \tag{4.11}$$

We define a, Δ and b such as:

$$\begin{cases} a = \kappa - \rho\sigma_v j\omega \\ \\ \Delta = a^2 + \sigma_v^2\ (j\omega + \omega^2) \\ \\ b = \sqrt{\Delta} \end{cases}$$

In that case, solving this famous quadratic equation gives the following solution for D_0:

$$\boxed{D_0 = \frac{a - b}{\sigma_v^2}} \tag{4.12}$$

Then denoting $\tilde{D} = D - D_0$, we get the following equation:

$$\frac{\partial \tilde{D}}{\partial \tau} = \frac{1}{2}\sigma_v^2 \tilde{D}^2 - b\tilde{D} \tag{4.13}$$

Dividing by $\tilde{D}^2$, we get the following equation:

$$\frac{\partial(\frac{1}{\tilde{D}})}{\partial \tau} + \frac{1}{2}\sigma_v^2 - b\frac{1}{\tilde{D}} = 0 \tag{4.14}$$

This equation can be rewritten the following way:

$$\frac{\partial(\frac{1}{\tilde{D}} - \frac{1}{2b}\sigma_v^2)}{\partial \tau} - b(\frac{1}{\tilde{D}} - \frac{1}{2b}\sigma_v^2) = 0 \tag{4.15}$$

The solution is of the form:

$$\frac{1}{\tilde{D}} - \frac{1}{2b}\sigma_v^2 = Ke^{b\tau} \tag{4.16}$$

Taking $\tau = 0$, it gives:

$$K = -(\frac{1}{2b}\sigma_v^2 + \frac{1}{D_0}) \tag{4.17}$$

So that the general expression for D is therefore:

$$\begin{aligned} D(\tau, \omega) &= D_0 + \frac{1}{-(\frac{1}{2b}\sigma_v^2 + \frac{1}{D_0})e^{b\tau} + \frac{1}{2b}\sigma_v^2} \\ &= D_0 \left(1 - \frac{2b}{(\sigma_v^2 D_0 + 2b)e^{b\tau} - \sigma_v^2 D_0}\right) \\ &= D_0\ (a+b)\frac{e^{b\tau} - 1}{(a+b)e^{b\tau} - (a-b)} \end{aligned} \tag{4.18}$$

$$\boxed{D(\tau, \omega) = \frac{a - b}{\sigma_v^2}\frac{1 - e^{-b\tau}}{1 - ge^{-b\tau}}} \tag{4.19}$$

Where:

$$g = \frac{a - b}{a + b}$$

4.1.2 Second ODE

$$C(\tau,\omega) = (\mu j\omega - r)\tau + \kappa\theta\frac{a-b}{\sigma_v^2}\int_0^{\tau}\frac{1-e^{-bu}}{1-ge^{-bu}}du \tag{4.20}$$

Noticing the following:

$$\begin{aligned}\frac{1-e^{-bu}}{1-ge^{-bu}} &= 1 - \frac{g-1}{b}\ \frac{-gbe^{bu}}{1-ge^{bu}} \\ &= 1 - \frac{2}{a-b}\ \frac{gbe^{-bu}}{1-ge^{-bu}}\end{aligned} \tag{4.21}$$

We finally have:

$$\boxed{C(\tau,\omega) = (\mu j\omega - r)\tau + \frac{\kappa\theta}{\sigma_v^2}\left[(a-b)\tau - 2\ln(\frac{1-ge^{-b\tau}}{1-g})\right]} \tag{4.22}$$

4.1.3 Solution U

The solution U we are looking for in our problem verifies:

$$U(\tau,x,v) = \mathcal{F}^{-1}[e^{C(\tau,\omega)+D(\tau,\omega)\ v}\mathcal{F}[H(e^x)]] \tag{4.23}$$

With the function H being:

$$H(e^x) = (x - \ln(I))^2 = x^2 - 2\ln(I)x + \ln^2(I) \tag{4.24}$$

The generalized Fourier transform of x^n for any $n \in \mathbb{N}$ is $\mathcal{F}(x^n) = 2\pi j^n\delta^{(n)}$ where $\delta^{(n)}$ is the n-th order derivative of the generalized delta function verifying:

$$\int_{-\infty}^{+\infty}\delta^{(n)}(\omega)\Phi(\omega)d\omega = (-1)^n\Phi^{(n)}(0) \tag{4.25}$$

We then deduce:

$$\mathcal{F}[H(e^x)] = 2\pi[-\delta^{(2)}(\omega) - 2j\ln(I)\delta^{(1)}(\omega) + \ln^2(I)\delta(\omega)] \tag{4.26}$$

The solution U is finally expressed as:

$$\begin{cases} U(\tau,x,v) = -f^{(2)}(0) + 2j\ln(I)f^{(1)}(0) + \ln^2(I)f(0) \\ \\ f(\omega) = e^{C(\tau,\omega)+D(\tau,\omega)v+j\omega x}\end{cases} \tag{4.27}$$

We need to compute the derivatives of f at the point zero:

$$\begin{cases} f(0) = e^{C(\tau,0)+D(\tau,0)v} = e^{-r\tau} \\ \\ f^{(1)}(0) = [C^{(1)}(\tau,0) + D^{(1)}(\tau,0)v + jx]\ f(0) \\ \\ f^{(2)}(0) = [(C^{(2)}(\tau,0) + D^{(2)}(\tau,0)v) + (C^{(1)}(\tau,0) + D^{(1)}(\tau,0)v + jx)^2]\ f(0)\end{cases} \tag{4.28}$$

When t tends to t_{i-1}, we have x_t which tends to $\ln(I_{t_{i-1}})$. So let's rewrite the solution U at time $t = t_{i-1}$ with $x = \ln(I)$:

$$\begin{aligned}
U(\tau = t_i - t_{i-1}, x, v) &= e^{-r\tau}\ [-f^{(2)}(0) + 2j\ln(I)f^{(1)}(0) + \ln^2(I)f(0)] \\
&= e^{-r\tau}\ [-(C^{(2)}(\tau,0) + D^{(2)}(\tau,0)v) - (C^{(1)}(\tau,0) + D^{(1)}(\tau,0)v + jx)^2 \\
&+ 2jx(C^{(1)}(\tau,0) + D^{(1)}(\tau,0)v + jx) + x^2] \\
&= G(v) \\
&= e^{-r(t_i - t_{i-1})}g(v)
\end{aligned} \tag{4.29}$$

Where:

$$\boxed{g(v) = [(D^{(1)})^2v^2 + (2C^{(1)}D^{(1)} + D^{(2)})v + ((C^{(1)})^2 + C^{(2)})](t_i - t_{i-1}, 0)} \tag{4.30}$$

4.2 Solving the PDE on $[0, t_{i-1}[$

We consider the following PDE:

$$\begin{cases}
\frac{\partial U}{\partial t} + \mu S\frac{\partial U}{\partial S} + \kappa(\theta - v)\frac{\partial U}{\partial v} + \frac{1}{2}vS^2\frac{\partial^2 U}{\partial S^2} + \frac{1}{2}\sigma_v^2 v\frac{\partial^2 U}{\partial v^2} + \rho\sigma_v vS\frac{\partial^2 U}{\partial S\partial v} - rU = 0 \\
U(T, S, v) = G(v)
\end{cases} \tag{4.31}$$

where

$$\begin{cases}
T = t_{i-1} \\
U = U_i
\end{cases}$$

According to Feynman-Kac, the solution U of this PDE verifies (for $t \leq t_{i-1}$):

$$\begin{aligned}
U(t, S_t, v_t) &= \mathbb{E}^{\mathbb{Q}}[e^{-r(t_{i-1}-t)} \times G(v_{t_{i-1}})|\mathcal{F}_t] \\
&= \mathbb{E}^{\mathbb{Q}}[e^{-r(t_{i-1}-t)} \times e^{-r(t_i - t_{i-1})} \times g(v_{t_{i-1}})|\mathcal{F}_t] \\
&= e^{-r(t_i - t)} \int_0^{+\infty} g(v_{t_{i-1}})\ p(v_{t_{i-1}}|v_t)\ dv_{t_{i-1}}
\end{aligned} \tag{4.32}$$

Where

$$p(v_{t_{i-1}}|v_t) = ce^{-\frac{2W+V}{2}}(\frac{V}{2W})^{\frac{q}{2}}I_q(\sqrt{2W\ V})$$

with:

$$\begin{cases}
c = \frac{2\kappa}{\sigma_v^2(1 - e^{-\kappa(t_{i-1}-t)})} \\
V = 2cv_{t_{i-1}} \\
q = \frac{2\kappa\theta}{\sigma_v^2} - 1 \\
W = cv_te^{-\kappa(t_{i-1}-t)}
\end{cases} \tag{4.33}$$

I_q is the modified Bessel function of the first kind of order q.
We can notice that the transition probability density $p(v_{t_{i-1}}|v_t)$ verifies:

$$p(v_{t_{i-1}}|v_t) = 2cp_{\chi^2(2q+2,2W)}(2cv_{t_{i-1}}) \tag{4.34}$$

Applying the change of variable $v = 2cv_{t_{i-1}}$ in the integral, we get:

$$\begin{aligned} U(t, S_t, v_t) &= e^{-r(t_i-t)} \int_0^{+\infty} g(\frac{v}{2c})\ p_{\chi^2(2q+2,2W)}(v)dv \\ &= -e^{-r(t_i-t)}\ \left[(D^{(1)})^2 \frac{\mathbb{E}[V^2]}{4c^2} + (2C^{(1)}D^{(1)} + D^{(2)})\frac{\mathbb{E}[V]}{2c} + ((C^{(1)})^2 + C^{(2)})\right] \end{aligned} \tag{4.35}$$

where

$$V \sim \chi^2(2q+2, 2W)$$

Which leads to the following result:

$$\begin{aligned} U(t, S_t, v_t)e^{r(t_i-t)} &= -(D^{(1)})^2 \frac{(q+1+2W) + (q+1+W)^2}{c^2} \\ &- (2C^{(1)}D^{(1)} + D^{(2)})\ \frac{q+1+W}{c} \\ &- ((C^{(1)})^2 + C^{(2)}) \end{aligned} \tag{4.36}$$

At time $t = 0$, we finally get:

$$\begin{aligned} U(0, S_0, v_0)e^{rt_i} &= \mathbb{E}[\ln^2(\frac{S_{t_i}}{S_{t_{i-1}}})] \\ &= -(D^{(1)})^2 \frac{(\tilde{q}+2W_i) + (\tilde{q}+W_i)^2}{c_i^2} \\ &- (2C^{(1)}D^{(1)} + D^{(2)})\ \frac{\tilde{q}+W_i}{c_i} \\ &- ((C^{(1)})^2 + C^{(2)}) \end{aligned} \tag{4.37}$$

Where:

$$\begin{cases} c_i = \frac{2\kappa}{\sigma_v^2(1-e^{-\kappa t_{i-1}})} \\ \tilde{q} = \frac{2\kappa\theta}{\sigma_v^2} \\ W_i = c_i v_0 e^{-\kappa t_{i-1}} \end{cases} \tag{4.38}$$

4.3 Special case where $i = 1$

In the case $i = 1$, we notice that $S_{t_{i-1}} = S_0$ is known, so we can just apply the formula of the first PDE:

$$U_1(0, S_0, v_0) = G(v_0) = e^{-rt_1} \times g(v_0) \tag{4.39}$$