Crocheting Hyperbolic Discs

This is a paper for two groups of people. One group are those who want to learn how to crochet hyperbolic discs but don't want to pay for Daina Taimina's book "Crocheting Adventures with Hyperbolic Planes". The second group are those who *have* purchased her book, have read the appendix where she gives the instructions, and want to see a derivation that's a little cleaner and more in-depth. (By "cleaner" I mean free of unnecessary parameters— we don't really need, for example, physical stitch height or the "radius" of the hyperbolic plane in the derivation or instructions.)

Instead of giving the details on how to actually crochet, I will direct the reader to this youtube video by CodeParade: `https://youtu.be/xtlDND7NVp8?si=PngMpbUamKCqs4py`. In it, he teaches just enough about how to crochet if all you ever want to crochet are hyperbolic discs. I must warn you, however, the procedure he lays out for constructing a hyperbolic disc is incorrect, and I will explain how below. (As a side note, his video game, Hyperbolica, in which the player explores a 3-dimensional hyperbolic world, nails hyperbolic geometry and is very fun, if a little nauseating.)

The Hyperbolic Plane

The hyperbolic plane is a 2-dimensional Riemmanian manifold with Ricci curvature scalar of -2 everywhere. (Yes, I know it's actually $R = -2/\mathcal{R}^2$, where $\mathcal{R}$ is the "radius" of the hyperbolic plane. I am setting $\mathcal{R} = 1$ because it makes no difference to the geometry— it only becomes pertinent if you care about specific physical measurements. And if you're upset that I'm using Ricci instead of Gauss, then you'll have to forgive me— my love of geometry arises from the way in which our universe can be modeled as a 4-dimensional pseudo-Reimannian manifold.) With standard polar coordinates, its line element is $ds^2 = dr^2 + \sinh^2 r\ d\theta^2$.

We will be talking about discs in this paper, so let me define what I mean by "disc": for a particular point P in a 2-dimensional manifold $\mathcal{M}$, a disc of radius ρ is the set of all points $Q \in \mathcal{M}$, for which there exists a curve that connects P and Q such that its length is less than or equal to ρ. (I believe this is the two dimensional case of what mathematicians would define as a "ball", but with us only working in two dimensions, I thought "disc" would be less confusing.) Since translation and rotation are isometries of the hyperbolic plane, for our purposes it really doesn't matter where P is— we can simply call it the origin of our polar coordinates, in which case the disc is simply all points with $r \leq \rho$. It turns out that a hyperbolic disc, *i.e.* a disc in the hyperbolic plane, can be physically constructed using crochet. In fact, any disc with any arbitrary geometry can be physically constructed with crochet.

A Very Simplified Overview of Crochet

(If you didn't watch the youtube video and you're a maniac who's reading this without any intention of doing any actual crocheting, this section is for you. If you understand the basics of crochet, you can skip this section.) A single stitch makes a loop of yarn (not a simple loop, but a loop nonetheless) that connects to other stitches. Think of chain mail, where each ring is like a single stitch. Projects are constructed stitch by stitch, row by row, bottom to top. As with chain mail, a single stitch is typically connected to four other stitches: the two on either side in the same row, the stitch in the row below, and it's lastly connected to the stitch in the row above.

Let's call a stitch that's only connected to three others, whose topside has not yet been connected to a fourth stitch "open", and a stitch that has four other stitches connected to it "closed". When making a new stitch, you don't have to connect it to the next open one in the row below— you can instead connect the new one back to the previously closed one so that that stitch below would then be connected to five stitches: one below, two on either side, and *two* above. The fifth stitch added is called an "increase" because it increases the number of stitches in that row compared to the one below. (Alternatively, you could "decrease", *i.e.* skip the next open stitch and connect to the second open one in that row below, and the resulting row would be shorter than the previous, but this will not be pertinent to us in this paper.)

Overview of the Disc

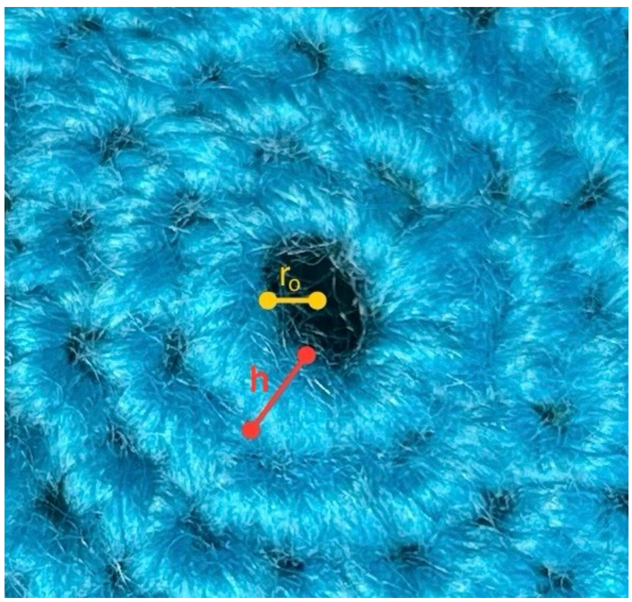
To make a disc, you start off by making a small circular ring of stitches (I will derive the optimal number of stitches for this starting ring below). This makes the first row. You continue to add stitches around the first row thereby making a second row, and just continue spiraling out. An important consideration that must be explicitly stated is the fact that the width of a stitch is pretty much only dependent upon the type of yarn being used (assuming consistent crocheting technique). Since only a single length of one yarn type is used for any particular disc, we can safely say with sufficient accuracy that stitch width is the same everywhere in any given disc. It's then easy to see that after the first ring of stitches, if we never add increase stitches, the resulting shape made from spiraling will just be a cylinder. Clearly, if we're wanting to make any kind of disc, we will have to include some increase stitches so that the circumference grows as we spiral out.

There are few sources that give instructions on how to crochet a hyperbolic disc, but the (free) ones that I have found either don't explicitly tell you how often to increase, or else claim that making a hyperbolic disc is as simple as spiraling out with a fixed ratio of increase stitches to normal stitches after making the initial ring (I'm looking at you, CodeParade). The (presumed) rationale is that for a hyperbolic plane, the circumference of a circle at $r = \rho + \Delta\rho$ is m times greater than that of a circle at $r = \rho$, where m is some constant greater than 1, and $\Delta\rho$ is some fixed change in r. We may think of $\Delta\rho$ as the radial height of a row of stitches (which is also constant, again assuming consistent crocheting technique), so this is saying that each subsequent row around the perimeter of the disc needs to have m times as many stitches as the one below it. This then implies a fixed ratio of increase to normal stitches. This paper has two purposes: One is to show that this is only exactly true in the limit where $r \to \infty$, and in fact is very wrong at relatively small r. The other purpose is to describe how exactly the ratio does need to change as you spiral out in order to adequately approximate the hyperbolic plane. Doing this will involve a bit of notation, so let's cover that first.

Definitions and Derivations

Like I said, having a fixed ratio of increase to normal stitches works in the limit where $r \to \infty$, or equivalently, where the number of rows approaches infinity, but in practice, once you've crocheted enough rows, using that fixed ratio will be fine. The final ratio of increase to normal stitches that the crocheter wants to end up using once they've spiraled out far enough we will call $X : Y$. You could choose to increase with every stitch, *i.e.* $X : Y = 1{:}1$, but a disc whose circumference is doubling each time you go around gets out of control pretty quickly; you'd end up with something resembling a little ball of ruffles. Most crocheters will probably want a final ratio somewhere between 1:10 and 1:3. This choice goes hand in hand with the radial height of a single row of stitches, we'll call h— choosing one determines the other, so technically you could choose the row height and it will determine your final stitch ratio, but the final ratio is a more straightforward choice for the crocheter to make. (It's important to note that the height to which I'm referring is not a physical height in this context, it is the coordinate height, *i.e.* the change in r from the top edge of a row to its bottom edge. The physical height of a row of stitches depends only on the yarn type. If we assign physical units to our coordinate r, then h would be *related* to the physical height, but it still wouldn't be numerically equivalent in general.) h roughly tells you how quickly your disc is going to ruffle up, *i.e.* the ruffle rate— the greater h, the greater the ruffle rate (this is not going to be a quantitatively explicit term, I just thought it sounded good). This should make intuitive sense if you think about the fact that Riemannian manifolds are locally Euclidean, and as h gets bigger, the more quickly you get further away from what might be considered "local". It's easier to see how $X : Y$ more directly informs the ruffle rate— obviously increasing $X : Y$ causes a rise in ruffle rate. Therefore, loosely speaking, the greater $X : Y$ is, the greater h is, but I will give the exact relation later. (A different way to think about ruffle rate is how "zoomed out" your view of the hyperbolic plane is. If we are very much zoomed in, then things will look more Euclidean, which equates to a low ruffle rate, and vice versa.)

Let's use C_n to refer to the circumference of the middle of row n (where r is halfway between the bottom and top edges of that row of stitches), and we will define $\mu(n)$ to be the ratio of the circumference of row n to that of the row below it, *i.e.* $\mu(n) \coloneqq C_n/C_{n-1}$. The implicit assumption made by those who instruct us to use a fixed $X : Y$ throughout the entire disc is that this μ is equivalent to their m and ought to be a constant rather than dependent upon n. But if we assume for now that it is in fact dependent, it should be obvious that the mathematical expression for my claim above would be $m \equiv \lim_{n\to\infty} \mu(n)$. Before I derive an equation for C_n, I need to briefly describe the first row of stitches, *i.e.* the starting ring. There will be an empty circle in the middle with radius r_o whose outer border is the bottom of the first row of stitches. Then there will be the first row of stitches themselves with a height of h. Let's say that $r_o = {}^1\!/\!{}_2\ h$. This is probably as good of an approximation as you can get given the variability in different people's crochet technique, but it will also make the math cleaner. Anyway, its exact value becomes increasingly unimportant the further out you crochet.



So now we can derive an equation for C_n. Because we are assuming $r_o = {}^1\!/\!{}_2\ h$, the middle of row n will have $r = n \cdot h$. The circumference of a circle in the hyperbolic plane with radius r, which we can read directly from the line element, is $2\pi \sinh r$. Therefore, $C_n = 2\pi \sinh(n \cdot h)$, which also gives us $\mu(n) = \sinh(n \cdot h)/\sinh(n \cdot h - h)$. We should immediately notice that μ is indeed dependent upon n. This at least shows that the assumption $\mu =$ const. is clearly false, but we have yet to show that $\lim_{n\to\infty} \mu(n) =$ const. Since $\sinh r \equiv (e^{2r} - 1)/2e^r$ we have

$$\mu(n) = \frac{e^{2n\cdot h} - 1}{2e^{n\cdot h}} \cdot \frac{2e^{n\cdot h - h}}{e^{2(n\cdot h - h)} - 1}$$

Then in the limit where $n \to \infty$ it's easy to see that this simplifies to

$$\lim_{n\to\infty} \mu(n) = \frac{e^{n\cdot h}}{2} \cdot \frac{2}{e^{n\cdot h - h}} = e^h$$

So there we have it. Once we've crocheted out far enough, each row should have about e^h times as many stitches as the row below. And if $X : Y$ is the final ratio of increase to normal stitches, then we can say the exact relationship between X, Y, and h and is $e^h = (X + Y)/Y$, or more usefully, $h = \ln(X + Y) - \ln Y$. If we define $x_n : y_n$ as the ratio of increase to normal stitches any arbitrary row n ought to have, then in fact we have $\mu(n) = (x_n + y_n)/y_n$, or again, more usefully, $x_n/y_n = \mu(n) - 1$. What this gives you is a number which, if expressed as a ratio of integers, is the exact ratio of increase to normal stitches that you ought to have for any given row n.

Instructions for Determining Increase Rate

So what you would do in practice is choose X and Y, use the expression $\ln(X+Y) - \ln Y$ to get h, then plug that and n into the equation above to get x_n/y_n. At this point, you will need to make a choice because the number you get will most likely not be something nice like 0.250. You'll have to find a nice fraction yourself that is *close* to x_n/y_n and use that as $x_n : y_n$. For the first several rows, you might also want to keep track of the actual number of stitches and increases you made, and whether that actual ratio was over or under the fraction you chose. But even if you're a little off, you're still going to end up with a disc that more closely resembles a hyperbolic disc than you would have by simply going with a fixed ratio throughout.

How far is far enough?

I said earlier that using the final ratio $X : Y$ is perfectly reasonable once you crochet out far enough, but how far is far enough? Easy: once you get to a row n where X/Y is the most reasonable fraction near x_n/y_n, then from that row on, just use $X : Y$. Let's look at an example where $X : Y = 1 : 6$ and $h = 0.154$. For row 11, $x_{11}/y_{11} = 0.182$, which is pretty close to $2/11$— that's a good ratio, I would just do five stitches, increase, six stitches, increase, and repeat. Then row 12 has $x_{12}/y_{12} = 0.177$. This is close to $3/17$, but for me, it's not worth the effort. At this point, I may as well drop down to $1/6$. This makes $C_{12} = e^h \cdot C_{11}$, when ideally it would be equal to $\mu(12) \cdot C_{11}$. If we divide the actual circumference by the ideal circumference and multiply by 100, we get 99.3%. That's good enough for me.

Derivation of Number of Stitches in Starting Ring

Let's figure out how many stitches the starting ring ought to be. Like I said earlier, most crocheters will probably want $X : Y$ to be somewhere between 1:10 and 1:3. This puts h between 0.095 and 0.288. For $h = 0.288$, the circumference, $2\pi \sinh(r_o + h/2) = 2\pi \sinh(h)$, ought to be 1.83. And for $h = 0.095$, $C = 0.598$. We will assume that stitch width, w, is equal to h, which is a very reasonable assumption. In order to find the number of stitches in the first row, we simply divide the circumference by w. So for $h = 0.288$, number of stitches is about 6.4, and for $h = 0.095$, we have 6.3 stitches, so 6, maybe 7 stitches will do it for any reasonable choice of $X : Y$.

How wrong is using a fixed ratio?

The last thing I'll do is show just how wrong using a fixed ratio can be for the early rows, so let's continue the example with $X : Y = 1 : 6$ and $h = 0.154$. For row 2, $x_2/y_2 = 1.02$. Rounding down to 1, this means we need to increase for each stitch in the second row. Compare that to the prescribed ratio of increasing once every six stitches. Keep in mind, this is still only the second row, where we're still looking pretty Euclidean. To be sure, let's figure out exactly what the ratio would be for a flat disc. First, $C_n = 2\pi n \cdot h$, and therefore $\mu(n) = n/(n-1)$. So $\mu(2) = 2$, which means the second row needs to be exactly twice as long as the first row, *i.e.* it needs to have twice as many stitches, *i.e.* the second row for a flat disc should also increase for each stitch. This means that using such a drastically low ratio here will actually give a circumference *less* than you should have even on a flat plane. This is actually what happens when you have a decent amount of *positive* curvature.

Crocheting Tips

- Use a hook that is skinnier than the yarn recommends. This minimizes the size of the holes in your disc which gives it a nicer, more continuous look.

- Make tight stitches. This makes the piece less flexible. The difference between a disc with tight stitches and one with loose stitches is much more easily felt than seen— it's kind of like the difference between a sheet of paper that's dry and one that's wet. And if we're only crocheting out of our love for geometry, we want inflexibility because that means stricter adherence to the intrinsic geometry of the manifold.

- Before you reach your final stitch ratio, use a stitch marker to identify the beginning of the row that you're currently on so that you can easily see when it's time to start the next row. You can buy stitch markers, or hairpins work just as well (lucky for me, my wife leaves them all over the house).

- CodeParade mentions cutting the starting tail in his video. Don't do that. Leave it. And when you are finished with your project, give yourself an ending tail as well. Then use a yarn needle to "weave in your ends", as they say. This will prevent your project from unravelling.