Series Solutions of Laplace Problems

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Abstract

Laplace's equation is an important partial differential equation with numerous applications in applied mathematics, physics, and engineering. For most Laplace problems, the solutions may not always be found in closed form. The goal of this study is to investigate a numerical method for approximating the solution, based on series expansions. This method, although not new, appears not to be widely known in literature compared to other numerical methods for solving Laplace problems, such as conformal mapping. The simple alternative presented by the series method, in two dimensions on smooth domains, is to seek for coefficients in certain series expansions by over sampling points on the boundary of the given solution domain. The over-determined linear system of equations which is formed on the boundary is then solved using the method of least-squares. The mathematical ideas and foundations of the series method dates back to Runge's Theorem of 1885, and the method demonstrates exponential convergence if the Laplace problem is sufficiently smooth.

Declaration

I, the undersigned, hereby declare that the work contained in this research project is my original work, and that any work done by others or by myself previously has been acknowledged and referenced accordingly.

Ketty Muwowo, 5 August 2021

Contents

1. Introduction

Laplace problems and potential theory are important in a wide range of applications in applied mathematics, physics, and engineering. The areas of application include fluid mechanics, electrostatics, magnetism, and elasticity (Boyce, 2007). Exact solutions to Laplaces problems are only known in a few special cases. As a result, we use numerical methods such as finite difference, finite element, and boundary integral method, as well as semi-analytical techniques such as conformal mapping and series solutions to find approximate solutions (Read, 1993).

In this essay, we consider series solutions of the Laplace problem. In general, series solutions are of great importance as they help solve many different problems that cannot be solved explicitly by expressing the unknown solution in terms of finite combinations of simple familiar functions, such as polynomials and rational fuctions (Burden and Faires, 2010). The series that results is limited to a finite number of terms, thus resulting in an approximation of the function. The following are some of examples of series methods: Taylor series, which is a polynomial series based on a function's derivative at a single point; Maclaurin series, which is a polynomial series and a special case of the Taylor series centered at zero; Laurent series, which involves rational functions and is an extension of the Taylor series allowing negative exponent values; Fourier series, which is a trigonometric series that describes periodic functions as a series of sine and cosine functions (libretexts.org).

The concepts that underpin series expansion methods for the Laplace problem can be traced back to Runge's Theorem of 1885 (encyclopediaofmath.org). Despite this, it appears that the elementary approach of series expansions, in which the coefficients are found by solving an overdetermined linear system of equations on the boundary data, is not commonly known in literature (Trefethen, 2018). An early example of the use of the series method for the Laplace problem was in 1892 by Lord Rayleigh, who employed series to solve the problems in a plane where a periodic array of circular holes had been eliminated (Rayleigh, 1892). For seepage problems in porous media, series solutions for Laplace's equation with homogeneous boundary conditions specified on regular and irregular boundaries have since been obtained (Read, 1993). In a more recent paper, the potential within a wire cage with a 2D configuration of a ring of disks was investigated using the series expansion method to analyze the Faraday cage phenomena (Chapman et al., 2015). In another very recent paper, the series method was used in the computation of the logarithmic capacity cap(E), which is the basic measure of the size of a set E in the complex plane. In which they employed a new "log-lighting" method based on reciprocal-log approximations in the complex plane (Baddoo and Trefethen, 2021). The series method has not only been applied to Laplace equations but also to the Helmholtz equations among others (Gopal and Trefethen, 2019).

1.1 Aims and Objectives

In this essay, we aim to understand how the series expansion method can be applied to various Laplace problems and implement the proposed method in Python. We begin by describing the series method for the more well-known problem of polynomial interpolation. We then investigate the convergence of the method on different Laplace problems. We will consider Laplace problems in the form of the computation of a Green's function having a logarithmic singularity at the origin. For example, we will consider finding the Green's function on the exterior of a disk of radius r. We will then extend the idea to the solution on the exterior of multiple disks. Our approach follows that suggested by Trefethen (2018).

1.2 Outline

In section 2, we introduce some concepts on polynomial approximation, definitions and theorems which helps us understand why the method of least-squares is more appropriate in determining the coefficients in the series expansion method. In section 3, we analyse the method of solving Laplace problems by series expansion using the proposed numerical method and present the results and findings, before concluding and discussing possibilities for future work in section 4.

1.3 Background on the Laplace equations

The Laplace equation is a partial differential equation of second order, mostly used in mathematics and physics. Pierre-Simon Laplace (1746-1827), a French mathematician, was the first to study its features (Boyce, 2007). Laplace's equation states that, the sum of the second partial derivatives of some unknown function u equals zero. That is, in two dimensions,

$$\Delta u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0. \tag{1.3.1}$$

The solutions of Laplace's equations are harmonic functions, which are the focus of study in potential theory (Kellogg, 1953). Laplace's equations have numerous applications in mathematical physics. For example, in electrostatics, the electric potential function must satisfy the Laplace's equation in a dielectric medium having no electric charge. In gravitation, the potential function must satisfy the Laplace's equation for a particle in free space acted on by gravitational forces only. They are also applied in the study of steady-state temperatures in heat conduction where the heat equation is used (Boyce, 2007).

A generalisation of Laplace's equations is the Poisson equation. We have a Poisson equation if the right hand side of (1.3.1) is specified as a given function $f(x, y)$, that is $\Delta u = f$. The Laplace equation and the Poisson equation are examples of the simplest elliptic partial differential equations. They are also a special case of Helmholtz equation, $\Delta u = -k^2 u$, which is the eigenvalue problem for the Laplacian operator, where k^2 represents the eigenvalue and u is the eigenfunction (Boyce, 2007).

1.4 Preliminaries

1.4.1 Definition. Complex differentiable (Bruna and Cufí, 2013)

Let U be an open set in the plane, f a complex function defined on U, and $a \in U$. The function f is said to be *complex differentiable* at a point a if there exists the limit

$$\lim_{z \to a,\ z \in U} \frac{f(z) - f(a)}{z - a} = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h} = f'(a).$$

The complex number $f'(a)$ is called the *complex derivative* of f at the point a.

1.4.2 Definition. Holomorphic functions (Bruna and Cufí, 2013)

A function $f : \omega \to \mathbb{C}$ is *holomorphic* on the domain ω if and only if it is complex differentiable and f' is continuous on ω.

Another term applied to such a function f is that it is *complex analytic*. All holomorphic functions are complex analytic functions and vice versa.

1.4.3 Definition. Harmonic functions (Fisher, 1999)

A continuous complex-valued function u with continuous first and second partial derivatives on an open set D in the plane is said to be *harmonic* on D if it satisfies Laplace's equation:

$$\Delta u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \quad \text{on } D.$$

1.4.4 Definition. Green's Functions (Wolfram)

Given a linear differential operator $\mathcal{L} = \mathcal{L}(x)$ acting over a subset Ω, a Green's function $G = G(x, s)$ at a point $s \in \Omega$ is any solution of

$$\mathcal{L}G(x, s) = \delta(x - s),$$

where δ denotes the Dirac delta function.

Equivalently, let D be a domain of $\mathbb{C}_\infty$, that is, a connected open subset of the Riemann sphere. A Green's function on D is a function $G_D : D \times D \to (-\infty, \infty]$ such that for all $w \in D$,

(a) $z \to G_D(z, w)$ is harmonic on $D\backslash\{w\}$;

(b) $G_D(z, w) = \infty$, and as $z \to w$

$$G_D(z, w) = \begin{cases} \log|z| + \mathcal{O}(1)\ , & \text{if } w = \infty \\ -\log|z - w| + \mathcal{O}(1)\ , & \text{if } w \neq \infty; \end{cases}$$

(c) $G_D(z, w) \to 0$ as $z \to \zeta$ for all $\zeta \in \partial_\infty D$. Where $\partial_\infty D$ is the boundary of D taken with respect to the sphere (Rostand, 1997).

Green's functions are important because they assist in obtaining solutions for non-homogeneous boundary value problems. Thus, they are useful in solving the heat equation, wave equation, and Helmholtz's equation, among other problems. Therefore, they are commonly utilized in areas such as quantum field theory, electrodynamics, and aerodynamics where precise solutions of the appropriate differential operators are difficult or impossible to find (Duffy, 2015).

1.4.5 Theorem. *(Fisher, 1999)*

A function $f = u + iv$ is complex differentiable (holomorphic) if and only if its real and imaginary parts satisfy the Cauchy–Riemann equations:

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}. \tag{1.4.1}$$

Proof. Let $f(z) = u(x, y) + iv(x, y)$ be an analytic function in a region $\mathbb{R}$ and $z = x + iy$. Then

$$\frac{f(z + \delta z) - f(z)}{\delta z} = \frac{(u + \delta u) + i(v + \delta v) - (u + iv)}{\delta z} = \frac{\delta u + i\delta v}{\delta z} = \frac{\delta u}{\delta z} + i\frac{\delta v}{\delta z}.$$

It follows that

$$\frac{df}{dz} = f'(z) = \lim_{\delta z \to 0}\left(\frac{\delta u}{\delta z} + i\frac{\delta v}{\delta z}\right). \tag{1.4.2}$$

Along the real axis; $y = 0$ and $\delta y = 0$, hence $z = x$ and $\delta z = \delta x$. Then equation (1.4.2) becomes

$$f'(z) = \lim_{\delta x \to 0}\left(\frac{\delta u}{\delta x} + i\frac{\delta v}{\delta x}\right) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}. \tag{1.4.3}$$

Along the imaginary axis; $x = 0$ and $\delta x = 0$, hence $z = iy$ and $\delta z = i\delta y$. Then equation (1.4.2) becomes

$$f'(z) = \lim_{\delta y \to 0}\left(\frac{\delta u}{i\delta y} + i\frac{\delta v}{i\delta y}\right) = -i\frac{\partial u}{\partial y} + \frac{\partial v}{\partial y}. \tag{1.4.4}$$

If $f(z)$ is complex differentiable, then the two expressions for $f'(z)$ in (1.4.3) and (1.4.4) must be equal. Hence we have that:

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad \text{and} \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}.$$

□

1.4.6 Theorem. *(Fisher, 1999)*

The real and imaginary parts of an analytic function are harmonic.

Proof. Let $f = u + iv$ be a holomorphic function in some open set of the complex plane, then

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = \frac{\partial}{\partial x}\frac{\partial u}{\partial x} + \frac{\partial}{\partial y}\frac{\partial u}{\partial y}. \tag{1.4.5}$$

Therefore, substituting (1.4.1) in (1.4.5) yields

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = \frac{\partial}{\partial x}\frac{\partial v}{\partial y} - \frac{\partial}{\partial y}\frac{\partial v}{\partial x} = \frac{\partial^2 v}{\partial x \partial y} - \frac{\partial^2 v}{\partial y \partial x} = 0.$$

It can be shown in a similar calculation that v is also harmonic.

Thus, this shows that for any holomorphic function, the real and imaginary parts yields harmonic functions on $\mathbb{R}^2$. □

We now give some examples of holomorphic and harmonic functions, that will be useful to us later.

1.4.7 Proposition. The function $f(z) = \log(z)$ is analytic in $\mathbb{C}\setminus\{0\}$.

Proof. To prove that $f(z) = \log(z)$ is analytic, we proceed by showing that $f(z)$ satisfies the Cauchy–Riemann conditions. Letting $z = re^{i\theta}$, we have that

$$f(z) = \log(z) = \log(re^{i\theta}) = \log(r) + \log(e^{i\theta}) = \log|z| + i\theta = \log\left(\sqrt{x^2+y^2}\right) + i\tan^{-1}(y/x),$$

where $|z| = \sqrt{x^2+y^2}$ and $\theta = \tan^{-1}(y/x)$. Let

$$u = \log\left(\sqrt{x^2+y^2}\right), \quad v = \tan^{-1}(y/x),$$

then differentiating with respect to x using chain rule yields

$$\frac{\partial u}{\partial x} = \frac{1}{\sqrt{x^2+y^2}} \times \frac{1}{2\sqrt{x^2+y^2}} \times 2x = \frac{x}{x^2+y^2},$$

and

$$\frac{\partial v}{\partial x} = \frac{1}{1+\left(\frac{y}{x}\right)^2} \times -\frac{y}{x^2} = -\frac{y}{x^2+y^2}.$$

Differentiating u and v with respect to y yields

$$\frac{\partial u}{\partial y} = \frac{1}{\sqrt{x^2+y^2}} \times \frac{1}{2\sqrt{x^2+y^2}} \times 2y = \frac{y}{x^2+y^2},$$

and

$$\frac{\partial v}{\partial y} = \frac{1}{1+\left(\frac{y}{x}\right)^2} \times \frac{1}{x} = \frac{x}{x^2+y^2}.$$

Thus, we now have that

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad \text{and} \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}.$$

Therefore $f(z) = \log(z)$ satisfies the Cauchy–Riemann conditions. Hence $\log(z)$ is analytic in $\mathbb{C}\setminus\{0\}$. □

By Theorem 1.4.6, we can conclude that $\log|z|$ is harmonic on $\mathbb{C}\setminus\{0\}$, as it is the real part of an analytic function $f(z) = \log(z)$. It follows also that $\log|z-c|$ is harmonic in $\mathbb{C}\setminus\{c\}$.

1.4.8 Proposition. The function $f(z) = \frac{1}{z}$ is analytic in $\mathbb{C}\setminus\{0\}$.

Proof. Let $f(z) = \frac{1}{z}$, where $z = x+iy$. We have that

$$f(z) = \frac{1}{z} = \frac{1}{x+iy} = \frac{x-iy}{x^2+y^2}.$$

Thus the function $f(z)$ can be written as $f(z) = u+iv$, where

$$u = \frac{x}{x^2+y^2} \quad , \quad v = \frac{-y}{x^2+y^2},$$

hence

$$\frac{\partial u}{\partial x} = \frac{1(x^2+y^2)-x(2x)}{(x^2+y^2)^2} = \frac{y^2-x^2}{(x^2+y^2)^2}, \quad \frac{\partial v}{\partial y} = \frac{-1(x^2+y^2)-(-y)2y}{(x^2+y^2)^2} = \frac{y^2-x^2}{(x^2+y^2)^2}.$$

Thus we have that

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \tag{1.4.6}$$

and

$$\frac{\partial u}{\partial y} = \frac{-2xy}{(x^2+y^2)^2}, \quad \frac{\partial v}{\partial x} = \frac{2xy}{(x^2+y^2)^2}.$$

This implies that

$$\frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}. \tag{1.4.7}$$

As the Cauchy–Riemann equations are satisfied by equations (1.4.6) and (1.4.7), we can conclude that $\frac{1}{z}$ is analytic in $\mathbb{C}\setminus\{0\}$. □

Similarly to proposition (1.4.8), we may deduce that **Re** $\{\frac{1}{z-c}\}$ and **Im** $\{\frac{1}{z-c}\}$ are harmonic functions in $\mathbb{C}\setminus\{c\}$. We may also infer that $f(z) = \frac{1}{(z-c)^k}$ is holomorphic in $\mathbb{C}\setminus\{c\}$ $\forall\, k \geq 1$ and hence **Re** $\left\{\frac{1}{(z-c)^k}\right\}$ and **Im** $\left\{\frac{1}{(z-c)^k}\right\}$ are harmonic on the same domain.

2. Fundamentals

In this section, we introduce the idea of series expansions in the form of polynomial interpolation and polynomial approximation in a least-squares fitting. The concepts, definitions, and theorems that we describe will be needed in section 3 for solving the Laplace problem.

2.1 Polynomial Interpolation

Interpolation is a method of fitting data points by a suitable chosen basis to approximate a function. Suppose we are given some data of the form (x_i, y_i), $i = 0, 1, \cdots, n$, where $x_i, y_i \in \mathbb{C}$. The $x-$values are called interpolation nodes. The data can be fitted by an interpolating function $f(x)$. This can be achieved by satisfying the interpolation condition

$$f(x_i) = y_i,\ i = 0, 1, \cdots, n\,.$$

Motivation for using polynomials in approximation

The algebraic polynomials are well-known classes of functions that are most useful in mapping a set of real numbers to itself, i.e, the collection of functions that make up the form

$$P_n(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0,$$

where n is a nonnegative integer and $a_0, a_1, \cdots, a_n$ are constants.

In the approximation of functions, there are several motivating factors for considering the class of polynomials. The fact that they uniformly approximate continuous functions is one of the key reasons for their importance. This means that if we have any continuous functions defined on a closed and bounded interval, we can find a polynomial that is as "close" to it as possible (Burden and Faires, 2010). This result is expressed in the Weierstrass Approximation Theorem:

2.1.1 Theorem. *Weierstrass Approximation Theorem (Burden and Faires, 2010)*

Suppose that f is defined and continuous on $[a, b]$. For each $\epsilon > 0$, there exists a polynomial $P(x)$, with the property that

$$|f(x) - P(x)| < \epsilon, \text{ for all } x \text{ in } [a, b].$$

The other motivating factor for considering the polynomials is that, the derivative, indefinite integral, and roots can be determined easily. Thus, for these reasons, in the approximation of continuous functions, polynomials form the basis on many numerical methods.

2.1.2 Theorem. *Uniqueness (Tennenbaum and Pollard, 1985)*

Let $f(x_0), f(x_1), \cdots, f(x_n)$ be $n+1$ distinct values of a function $f(x)$. Then there is one and only one polynomial $P(x)$ of degree less than or equal to n which coincides with these $n+1$ values of $f(x)$.

2.1.3 Definition. (Tennenbaum and Pollard, 1985)

The unique polynomial $P(x)$ of degree less than or equal to n, whose values coincide with $n+1$ distinct points of a function $f(x)$, is called a *polynomial interpolating function* of $f(x)$.

2.1.4 Definition. (Boyd, 2000)

The *algebraic index of convergence* k is the largest number for which

$$\lim_{n\to\infty} |a_n| n^k < \infty, \; n \gg 1,$$

where the a_n are the coefficients of the series.

2.1.5 Definition. (Boyd, 2000)

If the algebraic index k is unbounded (i.e., a_n decreases faster than $1/n^k$ for any finite k), then the sequence converges *exponentially*.

Alternatively, the sequence convergences exponentially if for constants q and $r > 0$

$$a_n \sim \mathcal{O}(e^{-qn^r}), \; n \gg 1.$$

2.1.6 Definition. (Boyd, 2000)

A sequence a_n has supergeometric, geometric, subgeometric if

$$\lim_{n\to\infty} \log(|a_n|)/n = \begin{cases} \infty \;, & \text{supergeometric} \\ \text{Constant} \;, & \text{geometric} \\ 0 \;, & \text{subgeometric} \end{cases}$$

or, alternatively, $a_n \sim \begin{cases} \mathcal{O}(e^{-(n/j)\log(n)}) \;, & \text{supergeometric} \\ \mathcal{O}(e^{-qn}) \;, & \text{geometric} \\ \mathcal{O}(e^{-qn^r}), \forall\, r < 1 \;, & \text{subgeometric.} \end{cases}$

2.2 Interpolation Error

Suppose that P_n is the interpolant of the function f, that is

$$P_n(x_j) = f(x_j), \; j = 0, 1, \cdots, n,$$

and that the nodes are increasing and lie in an interval $[a, b]$, i.e,

$$a < x_0 < x_1 < x_2 < \cdots < x_n < b.$$

If P_n is to be a good approximation of f, we are concerned with how close $P_n(x)$ is to $f(x)$ for x in $[a, b]$ where we wish to use the interpolant. The infinity norm error of interpolation is given by

$$||f(x) - P_n(x)||_\infty = \max_{x\in[a,b]} |f(x) - P_n(x)|.$$

In general, we cannot predict this value exactly, but there are numerous techniques to obtain bounds or estimates.

2.2.1 Theorem. *(Burden and Faires, 2010)*

Let f be a function in $C^{n+1}[a, b]$, and let P be a polynomial of degree $\leq n$ that interpolates the function f at $n+1$ distinct points $x_0, x_1, \cdots, x_n \in [a, b]$. Then to each $x \in [a, b]$ there exists a point $\xi_x \in [a, b]$ such that

$$f(x) - P(x) = \frac{1}{(n+1)!} f^{(n+1)}(\xi_x) \prod_{i=0}^{n} (x - x_i) \;\; \textit{where } x_0 \leq \xi_x \leq x_n.$$

This theorem is useful theoretically, but since it requires the $n+1$ first derivative of f at some unknown point, ξ_x, it is difficult to use directly in practice.

If $x_0, x_1, \cdots, x_n$ are the Chebyshev points, then the following bounds can be obtained.

2.2.2 Theorem. *(Battles and Trefethen, 2004)*

Let f be a continuous function on $[-1,1], P_N$ its degree N polynomial interpolant in the Chebyshev points. Then

(i) if f has a kth derivative in $[-1,1]$ of bounded variation for some $k \geq 1$, $||f - P_N||_\infty = \mathcal{O}(N^{-k})$ as $N \to \infty$;

(ii) if f is analytic in a neighborhood of $[-1,1]$, $||f - P_N||_\infty = \mathcal{O}(C^N)$ as $N \to \infty$ for some $C < 1$; in particular we may take $C = 1/(M+m)$ if f is analytic in the closed ellipse with foci ± 1 and semimajor and semiminor axis length $M \geq 1$ and $m \geq 0$.

Geometric convergence is stated in case (ii) of Theorem 2.2.2.

2.3 Constructing the interpolants

One approach of determining the interpolating polynomial is to construct the Vandermonde matrix. We demonstrate this here by means of an example.

2.3.1 Example. We consider points $x_0, x_1, \cdots, x_n$, with interpolation condition given by

$$f(x_j) = P(x_j) \text{ for } j = 0, 1, \cdots, n. \tag{2.3.1}$$

Suppose we want to find an approximation for the function $f(x)$,

$$f(x) \approx \sum_{k=0}^{n} c_k x^k = P_n(x).$$

Using the interpolation condition (2.3.1), we have that;

$$\begin{aligned}
f(x_0) &= P_n(x_0) = c_0 x_0^0 + c_1 x_0^1 + c_2 x_0^2 + \cdots + c_n x_0^n \\
f(x_1) &= P_n(x_1) = c_0 x_1^0 + c_1 x_1^1 + c_2 x_1^2 + \cdots + c_n x_1^n \\
f(x_2) &= P_n(x_2) = c_0 x_2^0 + c_1 x_2^1 + c_2 x_2^2 + \cdots + c_n x_2^n \\
&\vdots \\
f(x_n) &= P_n(x_n) = c_0 x_n^0 + c_1 x_n^1 + c_2 x_n^2 + \cdots + c_n x_n^n
\end{aligned}$$

The interpolation condition leads to the linear system

$$V\mathbf{c} = \mathbf{f}, \tag{2.3.2}$$

with

$$V = \begin{bmatrix} 1 & x_0 & x_0^2 & \cdots & x_0^n \\ 1 & x_1 & x_1^2 & \cdots & x_1^n \\ 1 & x_2 & x_2^2 & \cdots & x_3^n \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ 1 & x_n & x_n^2 & \cdots & x_n^n \end{bmatrix}, \mathbf{c} = \begin{bmatrix} c_0 \\ c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix}, \text{and} \quad \mathbf{f} = \begin{bmatrix} f(x_0) \\ f(x_1) \\ f(x_2) \\ \vdots \\ f(x_n) \end{bmatrix}.$$

The matrix V is referred to as the Vandermonde matrix and **c** is the vector of coefficients. To find the polynomial $P_n(x)$ that approximates the function e^x, we solve the linear system (2.3.2) to find the coefficient vector **c**. This can be done numerically using techniques such as Gaussian elimination and LU factorisation. The code to illustrate this can be accessed in Series solutions of Laplace problems Python codes (Muwowo, August, 2021).

2.3.2 Example. Suppose we have the following data and wish to construct the interpolating polynomial

x_i	0	1	-1	2	-2
f_i	-5	-3	-15	39	-9

For the data set given, the polynomial interpolation problem results into the linear system given by

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 & 1 \\ 1 & 2 & 4 & 8 & 16 \\ 1 & -2 & 4 & -8 & 16 \end{bmatrix} \begin{bmatrix} c_0 \\ c_1 \\ c_2 \\ c_3 \\ c_4 \end{bmatrix} = \begin{bmatrix} -5 \\ -3 \\ -15 \\ 39 \\ -9 \end{bmatrix}$$

The solution of the above system is given by $(c_0, c_1, c_2, c_3, c_4) = (-5, 4, -7, 2, 3)$ which results in the interpolating polynomial $P(x) = -5 + 4x - 7x^2 + 2x^3 + 3x^4$. The code for the computation of the coefficients was done using Python software, and can be accessed in Series solutions of Laplace problems Python codes (Muwowo, August, 2021).

A downside of this approach is that the matrix V is often ill-conditioned, especially for poorly selected points (such as uniformly spaced). One way to fix this is by employing more points than coefficients, leading to an overdetermined system.

2.4 Over-determined systems

A linear system $A\mathbf{x} = \mathbf{b}$, with $A \in \mathbb{R}^{m\times n}, b \in \mathbb{R}^m$ is called overdetermined if $m > n$, where m is the number of equations and n is the number of unknowns. Such matrices are said to be "tall and skinny."

This system almost always has no exact solution in general. Thus, what we would like is the solution with the smallest amount of error. The error for an overdetermined linear system

$$A\mathbf{x} \simeq \mathbf{b}, \tag{2.4.1}$$

can be defined by considering the difference between the considered vector $A\mathbf{x}$ and the vector $\mathbf{b}$, that is

$$\mathbf{r} = \mathbf{b} - A\mathbf{x}.$$

This vector **r** is known as the 'residual' and its size can be measured using a norm. Thus, we can solve an overdetermined system (2.4.1) by minimising $||A\mathbf{x} - \mathbf{b}||$ for some norm.

The method of least squares can be used to solve an overdetermined system. More precisely, given an overdetermined linear system (2.4.1) solving this system using "least squares" means finding a vector $\mathbf{x}^*$ with the property

$$||A\mathbf{x}^* - \mathbf{b}||_2 \leq ||A\mathbf{x} - \mathbf{b}||_2, \text{ for all vectors } \mathbf{x}.$$

There is a simple method which can be used for solving overdetermined systems using least squares. Given the system (2.4.1), with A an $m \times n$, $m > n$, we proceed by multiplying both sides of (2.4.1) by the transpose A^T. This yields $(A^T A)\mathbf{x} = A^T\mathbf{b}$. Thus we now have a square system of linear equations, called the 'normal equations'. We now can solve this system using Gaussian elimination (Nair and Singh, 2018). Another approach, used by the Python function lsqsolve, is to form a QR factorisation of the matrix A.

In polynomial interpolation with uniform points, it is often necessary to 'oversample' the function, leading to an over-determined system, to avoid the Runge-phenomenon (Epperson, 1987).

2.5 Example

We now observe the error in the polynomial interpolation when we use a square Vandermonde matrix $(n \times n)$ for $m = n$ and when we use an overdetermined system $2n \times n$ with $m = 2n > n$ which is a least squares case. We perform the experiment for the functions $f(x) = e^x$ and $f(x) = \frac{1}{1+25x^2}$.

For each value of n, we estimate the error $||f(x) - P_n(x)||_\infty$ by finding the maximum difference between the function $f(x)$ and the interpolating polynomial $P_n(x)$ on the interval $[-1, 1]$ at a large number of points.

Table 2.1 and Figure 2.1 summarise the findings for the $n \times n$ case and the overdetermined $2n \times n$ case for the overdetermined system.

Interpolation		Least squares	
n $\times$ n	Error	2n$\times$ n	Error
2 $\times$ 2	5.5760 $\times 10^{-1}$	4 $\times$ 2	3.1168 $\times 10^{-1}$
3 $\times$ 3	7.8526 $\times 10^{-2}$	6 $\times$ 3	5.4158 $\times 10^{-2}$
4 $\times$ 4	9.9847 $\times 10^{-3}$	8 $\times$ 4	7.1316 $\times 10^{-3}$
5 $\times$ 5	1.1244 $\times 10^{-3}$	10 $\times$ 5	7.5545 $\times 10^{-4}$
6 $\times$ 6	1.1216 $\times 10^{-4}$	12 $\times$ 6	6.7067 $\times 10^{-5}$
7 $\times$ 7	9.9800 $\times 10^{-6}$	14 $\times$ 7	5.1380 $\times 10^{-6}$
8 $\times$ 8	7.9889 $\times 10^{-7}$	16 $\times$ 8	3.4702 $\times 10^{-7}$
9 $\times$ 9	5.8009 $\times 10^{-8}$	18 $\times$ 9	2.1009$\times 10^{-8}$
10 $\times$ 10	3.8499 $\times 10^{-9}$	20 $\times$ 10	1.1548 $\times 10^{-9}$
11 $\times$ 11	2.3508 $\times 10^{-10}$	22 $\times$ 11	5.8224 $\times 10^{-11}$

Table 2.1: Polynomial interpolation of e^x on $[-1, 1]$. We see that each time we increase n by 1, the error decreases by an order of magnitude. This is indicative of geometric convergence. Both approaches perform well here.

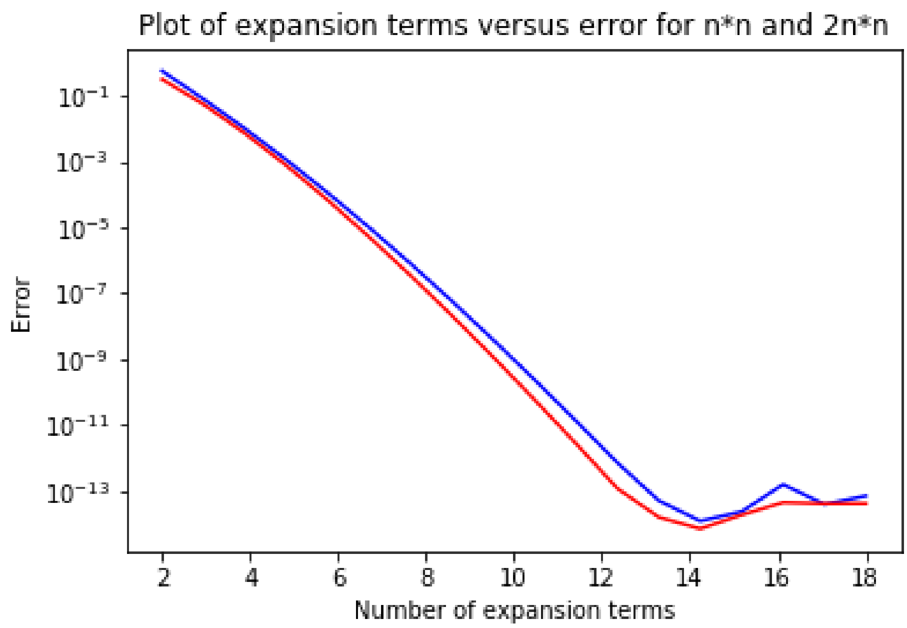

Figure 2.1: Plot of Error for case $1\,\&\,2$. The straight line on this linear-log plot indicate geometric convergence. It flatterns after $n = 15$ because of the ill conditioning of the Vandermonde matrix as the value of n increases.

In general, increasing the number of nodes in order to try and improve the interpolant is very dangerous as the interpolation polynomials do not converge to the function as $n \to \infty$. Particularly if the nodes are uniform as the accuracy is not always improved by going to higher degree. We illustrate this for a seemingly nice function,

$$f(x) = \frac{1}{1 + 25x^2}.$$

Let $P_n(x)$ be the interpolating polynomial in the interval $[-1, 1]$ with equally spaced nodes. The code for the interpolation computations can be found in Series solutions of Laplace problems Python codes (Muwowo, August, 2021).

Figure 2.2 and 2.3 shows the plots of the interpolating polynomial $P_n(x)$ using interpolation and least-squares respectively.

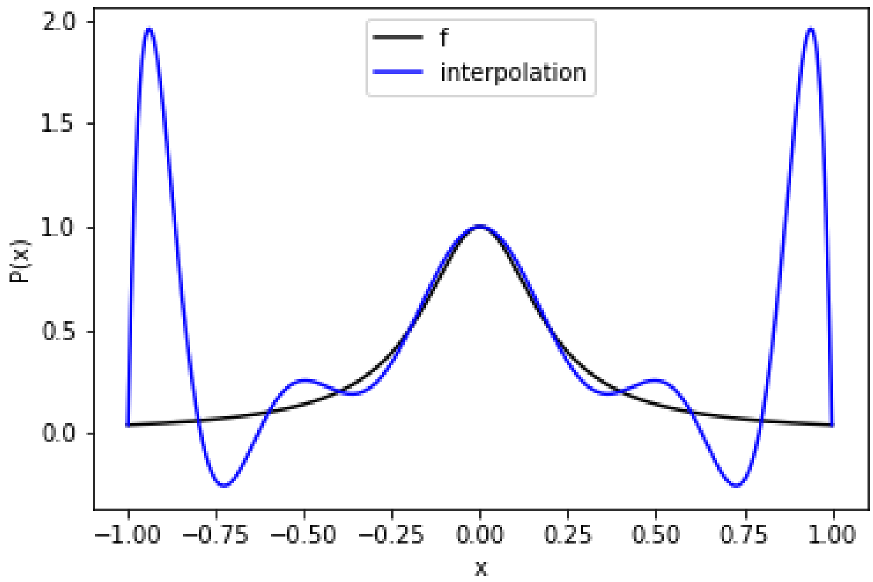

Figure 2.2: Plot of the interpolation of $f(x) = \frac{1}{1+25x^2}$ on 10 uniformly spaced nodes. The oscillations at the ends are the Runge-phenomenon.

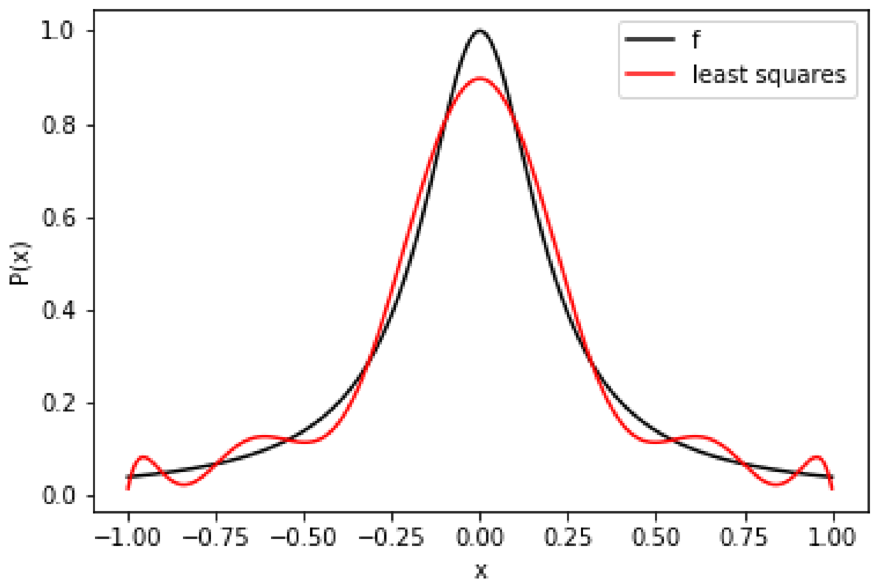

Figure 2.3: Plot of the interpolation of $f(x) = \frac{1}{1+25x^2}$ on 10 uniformly spaced nodes. The plot shows that the approximation for $P_n(x)$ using least-squares gives a better approximation as seen in the plot in red.

Figure 2.4 shows the plot of the error against the number of expansion terms using interpolation and least-squares.

The "Runge-phenomenon" or Runge example, is an issue of wide oscillations that occur along the edges of a set of uniform nodes when employing interpolation of high degree polynomials (Epperson, 1987). The Runge example is a valuable illustration of why polynomial interpolation with uniform interpolation points of high degree is not a good approximation technique in general, as the accuracy is not always improved by going to higher degree.

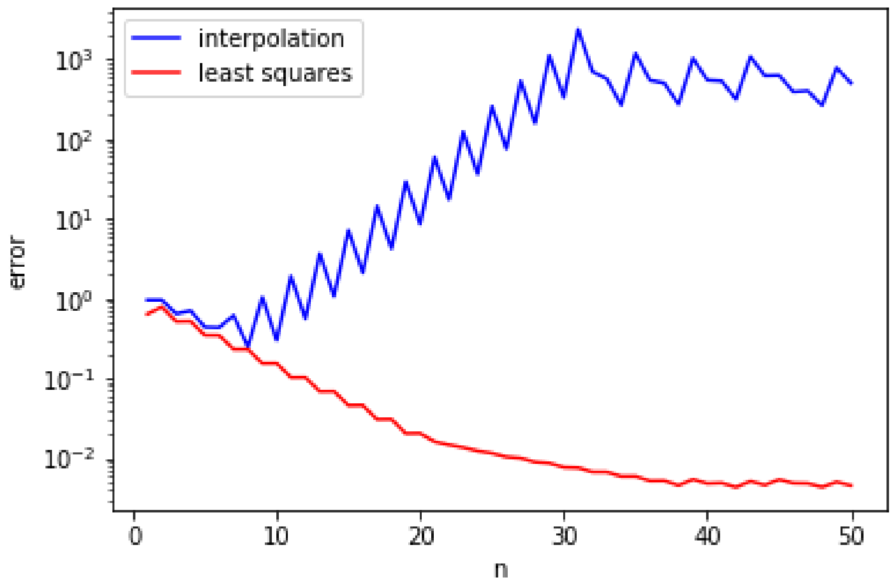

Figure 2.4: Plot of error against the number of expansion terms. From the log-linear plot in blue, we see that the error is growing exponentially larger for interpolation as the value of n increases. When we use least-squares the error becomes more stable and decays.

Observations

For the example on interpolation of e^x, we observe that the error decreases as we take more terms in the series $P_n(x) = \sum_{k=0}^{n} c_k x^k$. It is also observed that there is an exponential convergence in the error as the number of terms increases. This is evident from the straight line on the linear-log plot. Both plots $(n \times n)$ in blue and $(2n \times n)$ in red shows exponential convergence but for the overdetermined case, the convergence is marginally faster as can be seen from the plot in red from Figure 2.1. Both approaches perform well in this example.

For the Runge's function in Figure 2.2 it is observed that, the interpolating polynomial $P_n(x)$ in blue oscillates widely near the end points, and as n is increased it keeps on getting worse instead of getting better. The divergence of the interpolating polynomial is as a result of the Runge's phenomenon (Epperson, 1987). The maximum error

$$\max_{x\in[-1,1]} |f(x) - P_n(x)|,$$

grows exponentially as $n \to \infty$. This is clearly seen from the plot in blue in Figure 2.4. To solve this kind of problem, there some ways of getting around this. One of which is by using the Chebyshev points as described in Theorem 2.2.2. However, we see in Figure 2.3 from the plot in red, that by 'oversampling' and solving the least squares problem, we also fix the issue as we get a better approximation for the interpolating polynomial.

In interpolation, some major distribution and convergence challenges that arise are as a result of the square matrices that are used. When we sample more points and employ the method of least-square, the matrices become rectangular, and the difficulties vanish (Trefethen, 2018). This explains why, in the next section, we are going to consider using more points on the boundary of the given solution domain when solving the Laplace problems using the method of series expansion.

3. Methodology

As earlier mentioned, few Laplace problems have solutions which can be expressed explicitly or implicitly in terms of elementary functions. Finding a solution in such a case is not completely hopeless as the series method can be used.

We consider the following problem for our analysis of the method of solving Laplace problems by series expansion with least-squares matching of boundary data. This problem and method of solution is adopted from the paper by (Trefethen, 2018).

3.1 Green function for a disk

Trefethen asks the question in particular *"What is the Green function $u(z)$ in the exterior of a disk of radius r centered at position $z = c$ in the complex plane?"* (Trefethen, 2018). Precisely, we look for a function u satisfying

$$\Delta u = 0, \quad u(z) = 0 \quad \text{for} \quad |z - c| = r, \quad u(z) \sim \log |z| \quad \text{as} \quad z \to 0, \tag{3.1.1}$$

in the region of the complex plane exterior to $z = 0$ and circle $|z - c| = r < |c|$ (see Figure 3.1), with regular behaviour at $z = \infty$, that is, $u(z) \to u_\infty$ as $z \to \infty$ for some constant u_∞. To solve, (3.1.1), we approximate u by a series expansion of rational functions. The motivation for the use of rational functions $(z - c)^{-k}$ in approximation comes from Runge Theorem of 1885.

3.1.1 Theorem. *(Runge 1885) (encyclopediaofmath.org)*

Suppose K is a compact subset of $\mathbb{C}$ and f is analytic on K; further, let $\epsilon > 0$. Then there exists a rational function R with poles in K^C such that

$$|f(z) - R(z)| < \epsilon, (z \in K).$$

This is similar to Weierstrass Approximation Theorem 2.1.1, which states that, we can uniformly approximate every continuous function on a closed interval by polynomials.

Hence we seek a solution of the form

$$u(z) = \log |z| - \log |z - c| + C + \sum_{k=1}^{N} [a_k \textbf{Re}((z - c)^{-k}) + b_k \textbf{Im}((z - c)^{-k})]. \tag{3.1.2}$$

Alternatively, with $z - c = re^{i\theta}$ and for $|z - c| = r$, we have that

$$u(z) = \log |z| - \log r + C + \sum_{k=1}^{N} [a_k \textbf{Re}((re^{i\theta})^{-k}) + b_k \textbf{Im}((re^{i\theta})^{-k})],$$

by Euler's formular $e^{i\theta} = \cos\theta + i \sin\theta$, thus we have that

$$u(z) = \log |z| - \log r + C + \sum_{k=1}^{N} r^{-k} [a_k \textbf{Re}((\cos(k\theta) - i \sin(k\theta))) + b_k \textbf{Im}((\cos(k\theta) - i \sin(k\theta)))],$$

which can be written in real form as

$$u(z) = \log|z| - \log r + C + \sum_{k=1}^{N} r^{-k}[a_k \cos(k\theta) - b_k \sin(k\theta)]. \tag{3.1.3}$$

It is worth noting that the factor $1/2\pi$ which appear often in Green's function definition has been omitted for simplicity. We first check the validity of the series expansion used. To do this, we need to show that each of the terms in the series expansion of $u(z)$ in (3.1.2) are harmonic in the solution domain. This was earlier shown in proposition (1.4.7) and (1.4.8), therefore, all the terms in (3.1.2) are harmonic. It follows that, since Δ is a linear operator, if each of the terms are harmonic, then so is their sum.

The coefficients in the series expansion (3.1.2) are chosen to satisfy $u(z) = 0$ for $z = c + re^{i\theta}$ as closely as possible using the method of least squares. In order to obtain these unknown coefficients a_k and b_k, we solve an over-determined linear system of equations on the boundary of the domain, i.e, the circle $|z - c| = r$. This means, we oversample by taking more points on the boundary. We use $npts = 3N$ to avoid the 'Runge-phenomenon' effects, where $npts$ is the number of sample points and N is the number of expansion terms. If the number of points sampled on the boundary is sufficiently large, and if the boundary data are sufficiently smooth, one can hope that the approximation of the solution will be good and therefore generate a small error (Rostand, 1997).

The codes for the all the computations can be accessed in Muwowo (August, 2021). We use Matplotlib's contour plot in Python to construct a contour plot of the solution of the Laplace problem. The level curves computed are shown in Figure 3.1.

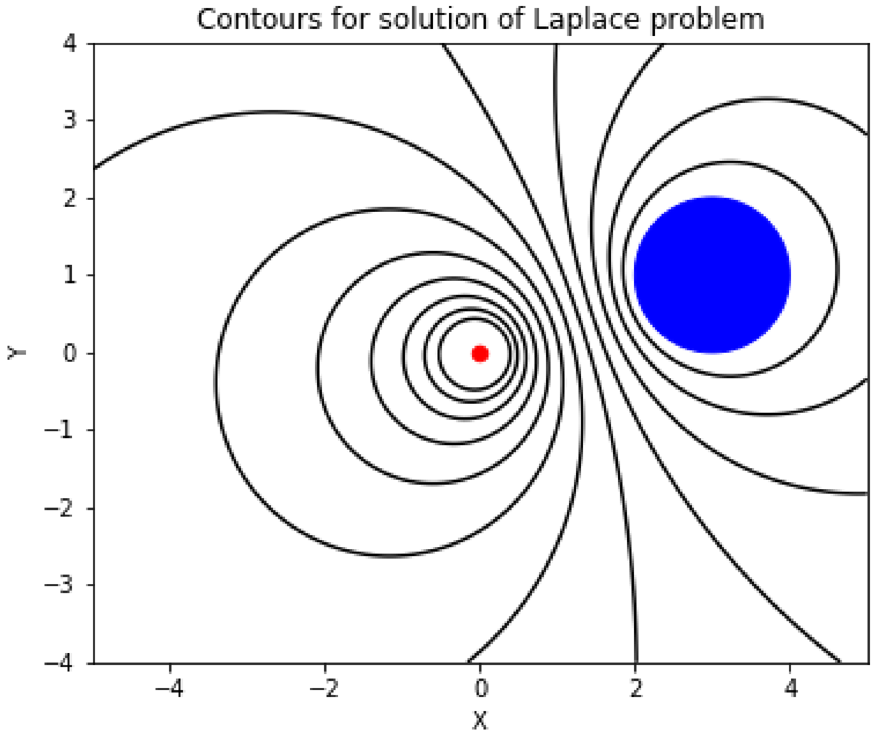

Figure 3.1: Green function $u(z)$ on the exterior of a disk. The level curves are equipotentials $u(z) =$ constant and are spaced regularly between 0 on the boundary of the disk and $-\infty$ at the singularity at $z = 0$.

The contour plot helps us to visualize a three-dimensional surface on a two-dimensional plane. In physics, the contour lines commonly indicate elevations that have the same electric potential. From the contour plot, we are able to observe how some of the z values changes as a function of two inputs x and y.

Figure 3.2 shows the surface plot for the Green function $u(z)$.

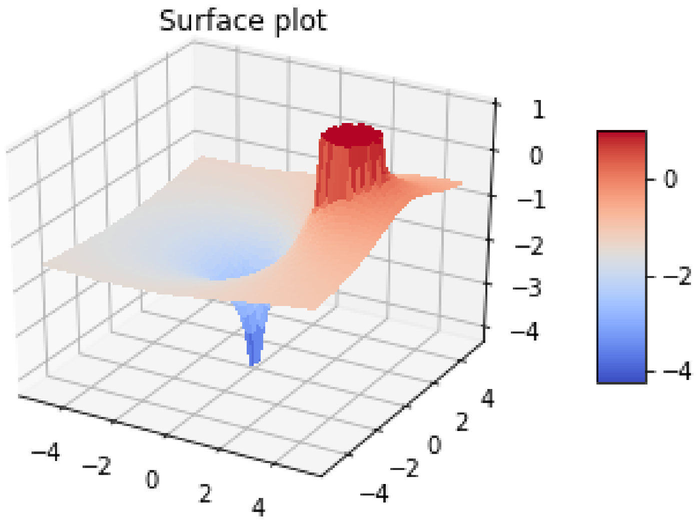

Figure 3.2: Surface plot for the Green function $u(z)$ on the exterior of a disk. The surface plot shows the three-dimensional view of the data. We can clearly see the pole at the singularity at $z = 0$.

3.2 Green function for several disks

We consider another example which is the same as the one before. The only exception that now arise is that instead of one disk defined by $|z-c| = r$, we now have J disks defined by $|z-c_j| = r_j$, $1 \leq j \leq J$. Specifically, we seek a function u in the region of the complex plane exterior to $z = 0$ and circles $|z - c_j| = r_j < |c_j|$. We approximate u by series approximation, which is a generalization of (3.1.2).

$$u(z) = \log|z| + C + \sum_{j=1}^{J}\left\{d_j \log|z - c_j| + \sum_{k=1}^{N} \left[a_{jk}\, \mathbf{Re}((z - c_j)^{-k}) + b_{jk}\, \mathbf{Im}((z - c_j)^{-k})\right]\right\}, \quad (3.2.1)$$

along with condition

$$\sum_{j=1}^{J} d_j = -1. \quad (3.2.2)$$

This final condition imposed ensures that we have nonsingular behaviour as $z = \infty$. In this series expansion, the coefficients d_j are now unknowns also. All the terms in the series expansion (3.2.1) are harmonic in the problem domain as earlier shown in Proposition (1.4.7) and (1.4.8).

The computations for all the coefficients of the series expansion (3.2.1) can be accessed in Series solutions of Laplace problems Python codes (Muwowo, August, 2021). We collect all these coefficients in a vector X. Using the method of least-squares, we obtain the unknowns which are $C, d_1, \cdots, d_J, a_{jk}, b_{jk}$. The overdetermined matrix A used has dimensions $1 + (J \times npts)$ by $1 + (J \times (2N + 1))$, where N is the number of expansion terms which was taken to be 10, $J = 4$ disks, and $npts$ is the number of sample points which was taken to be $3N$ as suggested by Trefethen (2018). Therefore, A is a (121×85) overdetermined matrix. We sample more points on the boundary so as to improve the accuracy and stability of the solution. The contour plot for the solution of the Green function exterior to several disks is shown in Figure 3.3.

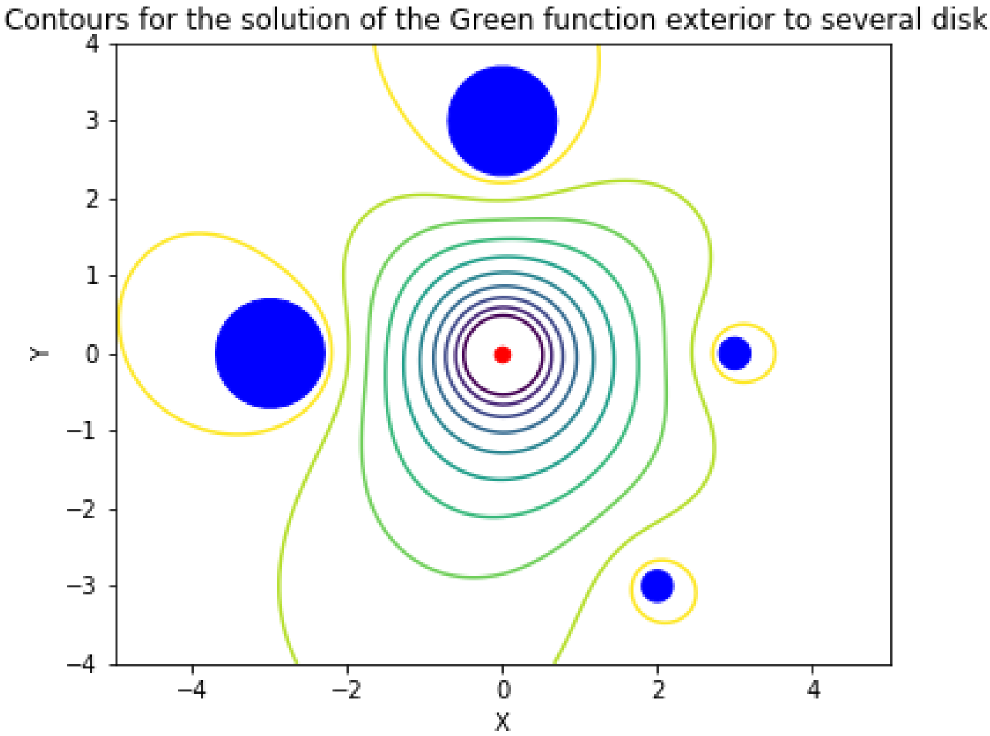

Figure 3.3: Green's function $u(z)$ on the exterior of several disks. The level curves are equipotentials $u(z)$ = constant and are spaced regularly between 0 on the boundary of the disk and $-\infty$ at the singularity at $z = 0$.

Figure 3.3 shows that the level curves are equipotentials and it shows the four disks two small and two big. The coefficients d_1, d_2, d_3 and d_4 for each disk are recorded below accurate to six digits:

$$d_1 = -0.342977, \ d_2 = -0.315902, \ d_3 = -0.182248, \ d_4 = -0.158873$$

The disks are numbered starting with the big one on the left in a clockwise order.

3.3 Convergence

We analyse the accuracy of this method by computing the single disk solution with larger values of N, where N is the number of expansion terms. We consider an arbitrary value $u(1)$, where $z = 1$ is a point outside the circle $|z - c| = r$, at which to measure the error. Upon computation with $N = 20$, we find $u(1) \approx -1.80545589563221$ which appears to be accurate to around fourteen digits, and we take to be the exact solution for means of computing the error. In Table 3.1, $npts$ represents the number of points sampled on the boundary. In the computation, $npts$ was taken to be $(3 \times N)$ as suggested by (Trefethen, 2018). Table 3.1 illustrates the accuracy at different values of N.

All the codes for the computations can be found in Series solutions of Laplace problems Python codes (Muwowo, August, 2021).

N	npts	u(1)	Error	Level of Accuracy
2	6	<u>-1.80</u>69109888037	1.452 $\times 10^{-3}$	3 digits
4	12	<u>-1.8054</u>690503953	1.009 $\times 10^{-5}$	5 digits
6	18	<u>-1.80545</u>90519972	9.568 $\times 10^{-8}$	6 digits
8	24	<u>-1.80545895</u>70998	7.777 $\times 10^{-10}$	9 digits
10	30	<u>-1.80545895632</u>15	5.254 $\times 10^{-13}$	12 digits
12	36	<u>-1.80545895632</u>18	2.334 $\times 10^{-13}$	12 digits

Table 3.1: Level of accuracy at different values of N. The table shows that as the error reduces with the increase in the value of N, there is an increase in the level of accuracy. Correct digits in the third column are underlined for clarity.

From Table 3.1, we also observe that the approximation of the solution improves as we take more points on the boundary. This confirms with literature that if we sample more points on the boundary and if the boundary data are sufficiently smooth, one expects that the approximation of the solution will be good and therefore generate a small error (Rostand, 1997).

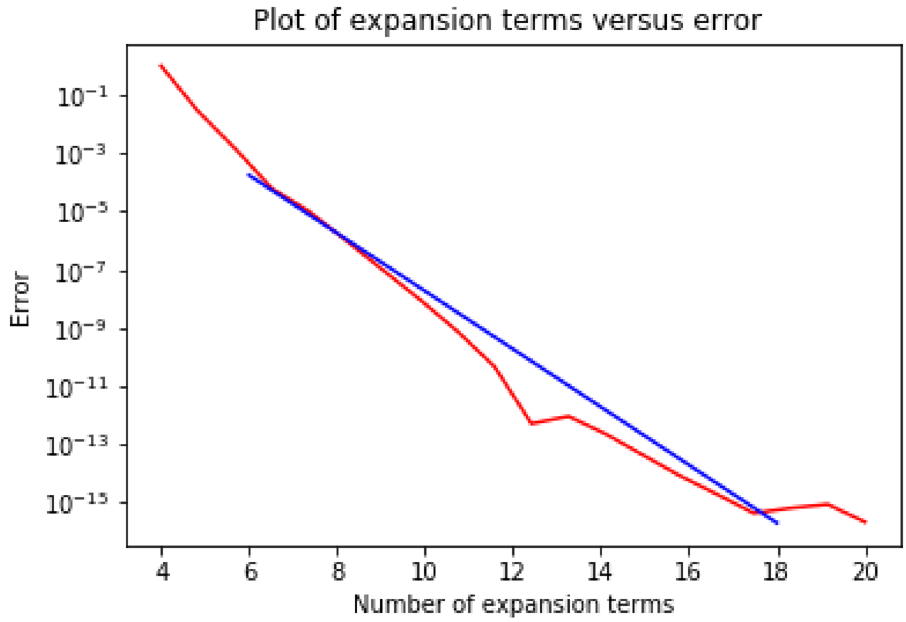

Figure 3.4: Plot of Error versus the number of expansion terms. The almost straight line on the linear-log plot indicates geometric convergence.

Figure 3.4 shows that geometric convergence occurs as a function of N. This can be observed from the almost straight line on the linear-log plot. To estimate the slope, a linear regression line of best fit from $n = 6$ to $n = 18$ data is shown in blue. The slope of the semilogy gives the convergence rate in base 10 with $err_N \approx \mathcal{O}(10^{-N})$. This confirms with the literature that the method demonstrates an exponential convergence if the Laplace problem is sufficiently smooth (Trefethen, 2018).

4. Conclusion and future work

4.1 Conclusion

This study focused on the numerical method of solving Laplace's equations by the method of approximating the solution based on series expansion, in which the expansion coefficients were determined using a least squares method on the boundary data.

We considered the Laplace problems in the form of the computation of a Green's function having a logarithmic singularity at the origin. We investigated the convergence of the method on the Green's function on the exterior of a disk and multiple disks domains. Using the series expansion method, we approximated the solution for the Laplace problem by sampling more points on the boundary, thereby forming an overdetermined linear system of equation on the boundary data. We then solved this overdetermined system using the method of least-squares in order to determine the coefficients of the series expansion. We implemented the proposed method using Python software. We also showed as to why the least-squares method was preferable in determining the coefficients, by carrying out the experiments on interpolation. In which we observed that interpolation formulation of square matrices poses some major distribution and convergence challenges. This is evident from the Runge's example in Section 2. The series method is useful in approximation of solutions as exact solutions to Laplace's problems are only known in a few special cases.

It was observed that the series solution method demonstrates an exponential convergence and this confirms with literature (Trefethen, 2018).

4.2 Future work

We implemented the series expansion method to approximate the solutions over a disk and multiple disks domains. The idea can also be extended to the solutions on the exterior of a slit plus disk, multiple slits, and more general smooth domains.

Further work can be done on the method of solving Laplace's equation on domains with corners using rational approximation. The numerical algorithms used quickly and accurately solves the 2D Laplace and Helmholtz equations despite the corner singularities (Gopal and Trefethen, 2019). Finally, an in-depth study can be undertaken to investigate the findings from a recent study which shows that application of the new "log-lighting" method opens up the possibility of solving Laplace problems in more general domains with an efficiency not achievable by previous methods. This method is based on reciprocal-log approximations in the complex plane (Baddoo and Trefethen, 2021).

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