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Multivariate Fay–Herriot models for small area estimation

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ABSTRACT

Multivariate Fay–Herriot models for estimating small area indicators are introduced. Among the available procedures for fitting linear mixed models, the residual maximum likelihood (REML) is employed. The empirical best predictor (EBLUP) of the vector of area means is derived. An approximation to the matrix of mean squared crossed prediction errors (MSE) is given and four MSE estimators are proposed. The first MSE estimator is a plug-in version of the MSE approximation. The remaining MSE estimators combine parametric bootstrap with the analytic terms of the MSE approximation. Several simulation experiments are performed in order to assess the behavior of the multivariate EBLUP and for comparing the MSE estimators. The developed methodology and software are applied to data from the 2005 and 2006 Spanish living condition surveys. The target of the application is the estimation of poverty proportions and gaps at province level.

1. Introduction

Surveys are designed for obtaining reliable estimates in the whole population or in some subpopulations called planned domains. However, it is quite common in practice to use survey data for estimating indicators of non-planned domains (small areas) with small samples sizes. Small area estimation deals with inference problems for this kind of domains. In these cases, direct estimators might have large sampling errors. Direct estimators can be improved by assuming regression models that link all the sample data by introducing a relation between the variable of interest and a set of explanatory variables.

Linear mixed models use random area effects for the extra between-area variation of the data that is not explained by the auxiliary variables. Often auxiliary individual information is not available, but data aggregated to the small areas can be found in administrative registers. Then the model can be stated at the small area level. An area-level linear mixed model with random area effects was first proposed by Fay and Herriot (1979) to estimate average per-capita income in small places of the US. Since then, the Fay–Herriot model has been one of the most widely used models in small area estimation.

In recent years, many researchers have investigated applications of the Fay–Herriot model to small area estimation problems. Without being exhaustive, we cite some papers dealing with the Fay–Herriot model. Prasad and Rao (1990); Datta and Lahiri (2000); Das et al. (2004), González-Manteiga et al. (2010), Jiang et al. (2011), Datta et al. (2011a) and Kubokawa (2011) gave tools for measuring the uncertainty of model-based small area estimators. Datta et al. (2011b), Bell et al. (2013) and Pfeffermann et al. (2014) studied the problem of benchmarking. Ybarra and Lohr (2008) proposed a new small area estimator that accounts for sampling variability in the auxiliary information. Herrador et al. (2011) treated situations where small areas are divided into two groups and domain random effects have different variances across the groups. Slud and Maiti (2011) were interested on small area estimation based on left censored survey data.

Statisticians are often required to estimate correlated descriptive measures, like poverty or unemployment indicators. Multivariate models take into account for the correlation of several variables and typically fit to this kind of situations.

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Some papers can be found in the literature of small area estimation where multivariate linear mixed models are employed. Fay (1987) and Datta et al. (1991) compared the precision of small area estimators obtained from univariate models for each response variable with the ones obtained by a multivariate model. Datta et al. (1996) used also a multivariate Fay–Herriot model for obtaining hierarchical Bayes estimates of median income of four-person families for the US states. González-Manteiga et al. (2008b) studied a class of multivariate Fay–Herriot model with a common random effect for all the components of the target vector. They further introduced bootstrap approximations to prediction errors. This paper introduces a class of multivariate Fay–Herriot models with one random effect per component of the target vector and allowing for different covariance patterns between the components of the vector of random effects. This is a new and flexible class of multivariate models that does not contain the models of González-Manteiga et al. (2008b) as particular cases.

Historical data give relevant information that can be used to obtain better small area estimators. Several authors have proposed extensions of the Fay–Herriot model that borrow strength from time. Choudry and Rao (1989) introduced a model including several time instants and considering an autocorrelated structure for sampling errors. Rao and Yu (1994) proposed a model that borrows information across areas and over time. Ghosh et al. (1996) proposed a time correlated area level model to estimate the median income of four-person families for American states. Datta et al. (1999), You and Rao (2000), Datta et al. (2002), Esteban et al. (2011, 2012), Marhuenda et al. (2013) and Morales et al. (2015) gave some extensions of the Rao–Yu model with applications to the estimation of labor or poverty indicators. Singh et al. (2005) and Pfeffermann and Burck (1990) considered models with time-varying random slopes obeying an autoregressive process. This paper applies multivariate Fay–Herriot models to time correlated data. In this setup, the introduced multivariate models contain the models proposed by Esteban et al. (2011) as particular cases.

The paper is organized as follows. Section 2 introduces the multivariate Fay–Herriot model and gives a residual maximum likelihood (REML) fitting algorithm. Unlike the model introduced by González-Manteiga et al. (2008b) with a common random effect for all the components of the target variable, the new models have multivariate vectors of random effects with the same dimension as the target variable and allowing for different correlation structures. Section 3 approximates the matrix of mean squared crossed prediction errors (MSE) of the multivariate empirical best predictor (EBLUP) and gives some estimators. The first MSE estimator is a plug-in derivation of the MSE approximation. The remaining MSE estimators combine parametric bootstrap with analytic terms appearing in the MSE approximation. Section 4 presents three simulation experiments. The first simulation studies the behavior of the multivariate EBLUPs under different correlation structures. The second simulation compares the performances of the MSE estimators proposed in Section 3. The third simulation studies the robustness of the EBLUPs against departures from normality. Section 5 applies the developed methodology to data from the Spanish Living Conditions surveys of 2005 and 2006. Two applications are presented. The target of the first application is the estimation of 2006 poverty proportions and gaps. The second application jointly estimates 2005 and 2006 poverty proportions. Section 6 gives some concluding remarks. The Appendix contains detailed proofs of main results.

2. Multivariate Fay–Herriot models

Let U be a finite population partitioned into D domains $U_1, \dots, U_D$. Let $\mu_d = (\mu_{d1}, \dots \mu_{dR})'$ be a vector of characteristics of interest in the domain d and let $y_d = (y_{d1}, \dots y_{dR})'$ be a vector of direct estimators of μ_d. The multivariate Fay–Herriot model is defined in two stages. The sampling model is

$$y_d = \mu_d + e_d, \quad d = 1, \dots, D, \tag{1}$$

where the vectors $e_d \sim N(0, V_{ed})$ are independent and the $R \times R$ covariance matrices V_{ed} are known. Moreover, it is assumed that the μ_{dr}'s are linearly related to p_r explanatory variables associated to the rth characteristic in the domain d. Let $x_{dr} = (x_{dr1}, \dots, x_{drp_r})$ be a row vector containing the p_r explanatory variables for μ_{dr} and let $x_d = \operatorname{diag}(x_{d1}, \dots, x_{dR})_{R\times p}$ with $p = \sum_{r=1}^{R} p_r$. Let β_r be a column vector of size p_r containing the regression parameters for μ_{dr} and let $\beta = \left(\beta_1', \dots, \beta_r'\right)'_{p\times 1}$. González-Manteiga et al. (2008b) considered the linking model

$$\mu_d = x_d\beta + 1_R v_d, \quad v_d \overset{ind}{\sim} N(0, \sigma_v^2),\ d = 1, \dots, D, \tag{2}$$

where 1_n is the $n \times 1$ vector with all elements equal to 1. This paper introduces multivariate Fay–Herriot models by assuming (1) and substituting the condition (2) by the more realistic linking model

$$\mu_d = x_d\beta + u_d, \quad u_d \overset{ind}{\sim} N(0, V_{ud}),\ d = 1, \dots, D, \tag{3}$$

where the vectors u_d's are independent of the vectors e_d's. The $R \times R$ covariance matrices V_{ud} depend on m unknown parameters, $\theta_1, \dots, \theta_m$, with $1 \le m \le \frac{R(R-1)}{2} + R$. Let I_n be the $n \times n$ identity matrix, $\delta_{\ell d}$ be the Kronecker delta, $y = (y_1, \dots, y_d)'$ be the vector of response variables and define

$$u = \underset{1\le d\le D}{\operatorname{col}}(u_d), \qquad e = \underset{1\le d\le D}{\operatorname{col}}(e_d), \qquad u_d = \underset{1\le r\le R}{\operatorname{col}}(u_{dr}), \qquad e_d = \underset{1\le r\le R}{\operatorname{col}}(e_{dr}),$$

$$X = \underset{1\le d\le D}{\operatorname{col}}(x_d), \qquad Z_d = \underset{1\le \ell\le D}{\operatorname{col}}(\delta_{\ell d} I_R), \qquad Z = \underset{1\le d\le D}{\operatorname{col}'}(Z_d) = I_{DR}, \qquad V_u = \underset{1\le d\le D}{\operatorname{diag}}(V_{ud}),$$

where col and col$'$ are matrix operators stacking by columns and rows respectively.

In matrix form, the multivariate Fay–Herriot model (1) + (3) is

$$y = X\beta + Zu + e = X\beta + Z_1u_1 + \cdots + Z_Du_D + e, \tag{4}$$

where $e, u_1, \ldots, u_D$ are independent with distributions

$$e \sim N(0, V_e), \qquad u \sim N(0, V_u) \quad \text{and} \quad u_d \sim N(0, V_{ud}), \quad d = 1, \ldots, D.$$

Along this paper, we consider four particularizations of model (4). Model 0 is the product of independent marginal models that assumes (1), (3) and takes

$$V_{e_d} = \underset{1\le r\le R}{\text{diag}}(\sigma^2_{edr}), \qquad V_{u_d} = \underset{1\le r\le R}{\text{diag}}(\sigma^2_{ur}), \quad d = 1, \ldots, D, \tag{5}$$

where the sampling error variances σ^2_{edr}'s are known, $m = R$ and $\theta_r = \sigma^2_{ur}$, $r = 1, \ldots, R$. This is to say, the components of e_d and u_d are independent under Model 0. Model 1 is the multivariate Fay–Herriot model that assumes (1) and (3), with a known but not necessarily diagonal matrix V_e, and takes

$$V_{u_d} = \underset{1\le r\le R}{\text{diag}}(\sigma^2_{ur}), \quad d = 1, \ldots, D, \tag{6}$$

$m = R$ and $\theta_r = \sigma^2_{ur}$, $r = 1, \ldots, R$. Note that Model 0 is Model 1 with V_e diagonal. Model 2 is the AR(1) Fay–Herriot model that assumes (1), (3) and takes

$$V_{ud} = \sigma^2_u\Omega_d(\rho), \quad \Omega_d(\rho) = \frac{1}{1-\rho^2}\begin{pmatrix} 1 & \rho & \cdots & \rho^{R-1} \\ \rho & 1 & \cdots & \rho^{R-2} \\ \vdots & \vdots & & \vdots \\ \rho^{R-1} & \rho^{R-2} & \cdots & 1 \end{pmatrix}, \; d = 1, \ldots, D, \tag{7}$$

$m = 2, \theta_1 = \sigma^2_u, \theta_2 = \rho$. Model 3 is the HAR(1) Fay–Herriot model that is an heteroscedastic version of Model 2. The matrix V_{e_d} is not forced to be diagonal in Models 1–3. Under Model 3, the components of $u_d = (u_{d1}, \ldots, u_{dR})'$ fulfill

$$u_{dr} = \rho u_{dr-1} + a_{dr}, \qquad u_{d0} \sim N\left(0, \sigma^2_0\right), \qquad a_{dr} \sim N\left(0, \sigma^2_r\right), \quad r = 1, \ldots, R, \tag{8}$$

where $\sigma^2_0 = 1$, u_{d0}, a_{dr}, $r = 1, \ldots, R$, are independent, $m = R + 1$ and $\theta_1 = \sigma^2_1, \ldots, \theta_R = \sigma^2_R, \theta_{R+1} = \rho$. The elements σ_{udij}, $i, j = 1, \ldots, R$, of matrix V_{ud} are

$$\sigma_{udii} = \sum_{k=0}^{i} \rho^{2k}\sigma^2_{i-k}, \qquad \sigma_{udij} = \sum_{k=0}^{|j-i|} \rho^{2k+|j-i|}\sigma^2_{|j-i|-k}, \quad i \ne j.$$

If $(y_{d1}, \ldots, y_{dR})'$ is a vector of direct estimators at time periods $r = 1, \ldots, R$ and V_e is diagonal, then Model 2 can be written in the form of Esteban et al. (2011), i.e.

$$y_{dr} = x_{dr}\beta + u_{dr} + e_{dr}, \quad d = 1, \ldots, D, \; r = 1, \ldots, R,$$

where, for each domain d, the random effects $\{u_{dr}\}_{r=1}^R$ are assumed to follow an AR(1) stochastic process and the random errors $\{e_{dr}\}_{r=1}^R$ are i.i.d. $N(0, \sigma^2_e)$. However, the introduced class of multivariate Fay–Herriot models does not contain the Rao and Yu (1994) model or the González-Manteiga et al. (2008b) model as special cases.

Under model (4), it holds that

$$E(y) = X\beta \quad \text{and} \quad V = \text{var}(y) = Z'V_uZ + V_e = V_u + V_e = \underset{1\le d\le D}{\text{diag}}(V_d),$$

where $V_d = V_{ud} + V_{ed}$, $d = 1, \ldots, D$. Further, the best linear unbiased estimator (BLUE) of β, and the best linear unbiased predictors (BLUP) of u and μ are

$$\hat\beta_B = (X'V^{-1}X)^{-1}X'V^{-1}y, \qquad \hat u_B = V_uZ'V^{-1}(y - X\hat\beta_B), \qquad \hat\mu_B = X\hat\beta_B + Z\hat u_B. \tag{9}$$

The residual maximum likelihood (REML) method maximizes the joint probability density function of a vector of $DR - p$ independent contrasts $\omega = W'y$, where W is a $DR \times (DR - p)$ matrix with linearly independent columns and such that $W'W = I_{DR-p}$ and $W'X = 0$. It holds that ω is independent of the BLUE $\hat\beta_B$ given in (9). The joint probability density function of ω is the REML likelihood. The REML log-likelihood of model (4) is

$$l_{reml}(\theta) = -\frac{DR-p}{2}\log 2\pi + \frac{1}{2}\log|X'X| - \frac{1}{2}\log|V| - \frac{1}{2}\log|X'V^{-1}X| - \frac{1}{2}y'Py,$$

where $\theta = (\theta_1, \ldots, \theta_m)$, $P = V^{-1} - V^{-1}X(X'V^{-1}X)^{-1}X'V^{-1}$, $PVP = P$ and $PX = 0$. By taking partial derivatives of l_{reml} with respect to θ_ℓ, $\ell = 1, \ldots, m$, we obtain the score vector

$$S(\theta) = (S_1, \ldots, S_m)', \qquad S_\ell = S_\ell(\theta) = \frac{\partial l_{reml}}{\partial\theta_\ell} = -\frac{1}{2}\text{tr}(PV_\ell) + \frac{1}{2}y'PV_\ell Py,$$

where

$$V_{d\ell}=\frac{\partial V_d}{\partial \theta_\ell},\qquad V_\ell=\frac{\partial V}{\partial \theta_\ell}=\underset{1\le d\le D}{\text{diag}}\,(V_{d\ell}),\qquad P_\ell=\frac{\partial P}{\partial \theta_\ell}=-P\frac{\partial V}{\partial \theta_\ell}P=-PV_\ell P.$$

By taking again partial derivatives, changing the sign and taking expectations, we get the Fisher information matrix

$$F(\theta)=\left(F_{a,b}\right)_{a,b=1,\dots,m},\qquad F_{ab}=F_{ab}(\theta)=\frac{1}{2}\text{tr}(PV_aPV_b),\quad a,b=1,\dots,m.$$

The REML updating equation of the Fisher-scoring algorithm is

$$\theta^{k+1}=\theta^k+F^{-1}(\theta^k)S(\theta^k). \tag{10}$$

For Model 1, we propose the starting values $\hat\theta_{r,0}=\hat\sigma^2_{ur,0},\ r=1,\dots,R$, where $\hat\sigma^2_{ur,0}$ is the Prasad and Rao (1990) moment-based estimator of σ^2_{ur} in the rth marginal Fay–Herriot model. For Model 2, we propose $\hat\theta_{1,0}=\frac{1}{R}\sum_{r=1}^{R}\hat\sigma^2_{ur,0}$ and $\hat\theta_{2,0}=\hat\rho_0=0$. For Model 3, we propose $\hat\theta_{r,0}=\hat\sigma^2_{ur,0},\ r=1,\dots,R$, and $\hat\theta_{R+1,0}=\hat\rho_0=0$.

The output of algorithm (10), $\hat\theta$, is the REML estimator of θ. By plugging $\hat\theta$ in V_u, we get $\hat V_u=V_u(\hat\theta)$ and $\hat V=\hat V_u+V_e$. By substituting $\hat V_u$ in (9), we obtain the EBLUP of $\mu=X\beta+Zu$, i.e.

$$\hat\beta_E=(X'\hat V^{-1}X)^{-1}X'\hat V^{-1}y,\qquad \hat u_E=\hat V_uZ'\hat V^{-1}(y-X\hat\beta_E),\qquad \hat\mu_E=X\hat\beta_E+Z\hat u_E. \tag{11}$$

The asymptotic distributions of the REML estimators $\hat\theta$ and $\hat\beta$,

$$\hat\theta\sim N_m(\theta,F^{-1}(\theta)),\qquad \hat\beta\sim N_p(\beta,(X'V^{-1}X)^{-1}),$$

can be used to construct $(1-\alpha)$-level confidence intervals for θ_ℓ and β_j, i.e.

$$\hat\theta_\ell\pm z_{\alpha/2}\,\nu_{\ell\ell}^{1/2},\quad \ell=1,\dots,m,\qquad \hat\beta_j\pm z_{\alpha/2}\,q_{jj}^{1/2},\quad j=1,\dots,p,$$

where $F^{-1}(\hat\theta)=(\nu_{ab})_{a,b=1,\dots,m}$, $(X'V^{-1}(\hat\theta)X)^{-1}=(q_{ij})_{i,j=1,\dots,p}$ and z_α is the α-quantile of the $N(0,1)$ distribution. For $\hat\beta_j=\beta_0$, the p-value for testing the hypothesis $H_0:\ \beta_j=0$ is

$$p=2P_{H_0}(\hat\beta_j>|\beta_0|)=2P(N(0,1)>|\beta_0|/\sqrt{q_{jj}}).$$

3. The matrix of mean squared crossed errors

Prasad and Rao (1990) gave an approximation to the MSE of the EBLUP of μ_{dr} under the univariate Fay–Herriot model when their proposed moment-based estimator of the variance σ^2_{ur} is employed. Datta and Lahiri (2000) extended the results of Prasad and Rao (1990) to the case of the general longitudinal model. They further considered ML and REML estimators of the variance components. For the general linear model, Das et al. (2004) derived the MSE of the EBLUP under REML and ML. Their proof contains the general longitudinal model considered by Datta and Lahiri (2000) as a special case. However, none of the three papers study the approximation of the matrix of mean squared crossed errors of the EBLUP vector $\hat\mu_E$. They deal with the approximation of the MSEs of the components of $\hat\mu_E$. Although, the multivariate Fay–Herriot model (4) can be written in the form of the general linear mixed model considered by Das et al. (2004), the approximation of the matrix of mean squared crossed errors is not covered by this paper.

This section uses the hypotheses H1–H6 stated in the Appendix and introduces the notation $f(D)=O(D)_{m\times m}$ and $f(D)=o(D)_{m\times m}$ for matrix-valued functions such that $f(D)/D$ is element-wise uniformly bounded and converge to a zero $m\times m$ matrix, as $D\to\infty$, respectively. Similarly, $f(D)=O_p(D)$ and $f(D)=o_p(D)$ are used for convergence in probability when f and g are matrix-valued stochastic functions. This section approximates the matrix of mean squared crossed errors of the EBLUP, i.e.

$$MSE\left(\hat\mu_E\right)=E\left(\left(\hat\mu_E-\mu\right)\left(\hat\mu_E-\mu\right)'\right).$$

By adding and subtracting $\hat\mu_B$, we have $\hat\mu_E-\mu=\hat\mu_B-\mu+\hat\mu_E-\hat\mu_B$. Therefore,

$$\begin{aligned}\left(\hat\mu_E-\mu\right)\left(\hat\mu_E-\mu\right)'&=\left(\hat\mu_B-\mu\right)\left(\hat\mu_B-\mu\right)'+\left(\hat\mu_B-\mu\right)\left(\hat\mu_E-\hat\mu_B\right)'\\&\quad+\left(\hat\mu_E-\hat\mu_B\right)\left(\hat\mu_B-\mu\right)'+(\hat\mu_E-\hat\mu_B)(\hat\mu_E-\hat\mu_B)'.\end{aligned} \tag{12}$$

Under the assumption of normality on u and e and for unbiased and translation invariant estimators of θ, Kackar and Harville (1981) proved that the expectations of the last two terms in (12) are null. Therefore, by taking expectations, we get

$$MSE\left(\hat\mu_E\right)=MSE\left(\hat\mu_B\right)+E\left[\left(\hat\mu_E-\hat\mu_B\right)\left(\hat\mu_E-\hat\mu_B\right)'\right]. \tag{13}$$

By the general prediction theorem, we have

$$MSE\left(\hat{\mu}_B\right) = G_1(\theta) + G_2(\theta),$$

where $T = V_u - V_u Z' V^{-1} Z V_u$, $Q = (x'V^{-1}x)^{-1}$ and

$$G_1(\theta) = ZTZ', \qquad G_2(\theta) = \left(X - ZTZ'V_e^{-1}X\right) Q \left(X' - X'V_e^{-1}ZTZ'\right). \tag{14}$$

For calculating the second summand in (13), we write $\hat{\mu}_B = \hat{\mu}(\theta)$ and $\hat{\mu}_E = \hat{\mu}(\hat{\theta})$. A Taylor series expansion $\hat{\mu}(\hat{\theta})$ around θ yields to

$$(\hat{\mu}_E - \hat{\mu}_B)(\hat{\mu}_E - \hat{\mu}_B)' \approx S(\hat{\theta} - \theta)(\hat{\theta} - \theta)'S' + o_p(D^{-1}),$$

where $S = \left(\frac{\partial \hat{\mu}_{dr}}{\partial \hat{\theta}_j} : d = 1, \ldots, D, r = 1, \ldots, R;\ j = 1, \ldots, m\right)$ is a $DR \times m$ matrix, i.e.

$$S = \frac{\partial \hat{\mu}}{\partial \hat{\theta}} = \underset{1\le\ell\le m}{\mathrm{col}'}\,(S^{(\ell)}), \quad S^{(\ell)} = \frac{\partial \hat{\mu}}{\partial \theta_\ell} = \underset{1\le d\le D}{\mathrm{col}}\,(\underset{1\le r\le R}{\mathrm{col}}\,(s_{dr}^{(\ell)})),\ s_{dr}^{(\ell)} = \frac{\partial \hat{\mu}_{dr}}{\partial \theta_\ell}.$$

In the new notation, we have

$$\begin{aligned} S(\hat{\theta} - \theta)(\hat{\theta} - \theta)'S' &= \underset{1\le\ell\le m}{\mathrm{col}'}\,(S^{(\ell)})\ \underset{1\le\ell\le m}{\mathrm{col}}\,(\hat{\theta}_\ell - \theta_\ell)\ \underset{1\le\ell\le m}{\mathrm{col}'}\,(\hat{\theta}_\ell - \theta_\ell)\ \underset{1\le\ell\le m}{\mathrm{col}}\,(S^{(\ell)\prime}) \\ &= \sum_{i=1}^m S^{(i)}(\hat{\theta}_i - \theta_i) \sum_{l=j1}^m (\hat{\theta}_j - \theta_j) S^{(j)\prime} = \sum_{i=1}^m \sum_{j=1}^m (\hat{\theta}_i - \theta_i)(\hat{\theta}_j - \theta_j) S^{(i)} S^{(j)\prime}. \end{aligned}$$

By taking expectations, we get

$$E[S(\hat{\theta} - \theta)(\hat{\theta} - \theta)'S'] = \sum_{i=1}^m \sum_{j=1}^m E\left[(\hat{\theta}_i - \theta_i)(\hat{\theta}_j - \theta_j) S^{(i)} S^{(j)\prime}\right] + o(D^{-1}).$$

Theorem 1. *Let us assume that* H1–H4 *and* H5–H6 *holds, then*

$$E[(\hat{\theta}_i - \theta_i)(\hat{\theta}_j - \theta_j) S^{(i)} S^{(j)\prime}] = \mathrm{cov}(\hat{\theta}_i, \hat{\theta}_j) L^{(i)} V L^{(j)\prime} + o(D^{-1})_{DR\times DR}.$$

Proof. By Lemma 1 of Appendix, the components of the vectors $S^{(i)}$ and $S^{(j)}$ are linear functions of $V = Zu + e$, i.e.

$$s_{dr}^{(i)} = (F_{dr}^{(i)} + L_{dr})'V, \quad r = 1, \ldots, R,\ d = 1, \ldots, D,$$

where $S^{(i)} = \left(s_{11}^{(i)}, \ldots, s_{DR}^{(i)}\right)$, $F^{(i)} = \left(F_{11}^{(i)\prime}, \ldots, F_{DR}^{(i)\prime}\right)$ and $L^{(i)} = \left(l_{11}^{(i)\prime}, \ldots, l_{DR}^{(i)\prime}\right)$ are defined in (17) of Appendix and similarly for $S^{(j)}$. By H6, we have $\hat{\theta}_i = k + y'C_i y$, with $E[\hat{\theta}_i] = \theta_i$. As $\hat{\theta}_i$ is translation invariant and $v = Zu + e = y - x\beta$, we have $\hat{\theta}_i(y) = \hat{\theta}_i(y - x\beta) = \hat{\theta}_i(v)$ and $\hat{\theta}_i(v) = k + v'Cv$, with $\theta_i = k + E[v'Cv]$. By subtracting, we get $\hat{\theta}_i - \theta_i = v'A_i v - E[v'A_i v]$. By defining $q_i = v'A_i v$, we have $\hat{\theta}_i - \theta_i = q_i - E[q_i]$ and similarly for $\hat{\theta}_j$.

As $v \sim N(0, V)$, we apply Lemma 3 with $\lambda_1 = F_{dr_1}^{(i)} + l_{dr_1}^{(i)}$, $\lambda_2 = F_{dr_2}^{(j)} + l_{dr_2}^{(j)}$, $s_1 = s_{dr_1}^{(i)}$, $s_2 = s_{dr_2}^{(j)}$ and $q_1 = V'A_i V$, $q_2 = V'A_j V$. We obtain

$$E[s_{dr_1}^{(i)} s_{dr_2}^{(j)} (\hat{\theta}_i - \theta_i)(\hat{\theta}_j - \theta_j)] = \mathrm{cov}(s_{dr_1}^{(i)}, s_{dr_2}^{(j)}) \mathrm{cov}(\hat{\theta}_i, \hat{\theta}_j) + 8(F_{dr_1}^{(i)} + l_{dr_1}^{(i)})' V A_i V A_j V (F_{dr_2}^{(j)} + l_{dr_2}^{(j)}).$$

In matrix notation, we have

$$E[(\hat{\theta}_i - \theta_i)(\hat{\theta}_j - \theta_j) s^{(i)} s^{(j)\prime}] = \mathrm{cov}(s^{(i)}, s^{(j)}) \mathrm{cov}(\hat{\theta}_i, \hat{\theta}_j) + 8(F^{(i)} + L^{(i)}) V A_i V A_j V (F^{(j)} + L^{(j)})'.$$

From Lemma 4, we get

$$\mathrm{cov}(s^{(i)}, s^{(j)}) = L^{(i)} V L^{(j)\prime} + O(D^{-1})_{DR\times DR}.$$

By applying Lemmas 2 and A.3 of Prasad and Rao (1990), we finally obtain

$$(F^{(i)} + L^{(i)}) V A_i V A_j V (F^{(j)} + L^{(j)})' = O(D^{-2})_{DR\times DR}$$

and

$$E[(\hat{\theta}_i - \theta_i)(\hat{\theta}_j - \theta_j) s^{(i)} s^{(j)\prime}] = \mathrm{cov}(\hat{\theta}_i, \hat{\theta}_j) L^{(i)} V L^{(j)\prime} + o(D^{-1})_{DR\times DR}.$$

Corollary 1. *If* H1–H6 *holds, then*

$$E\left[\left(\hat{\mu}_E-\hat{\mu}_B\right)\left(\hat{\mu}_E-\hat{\mu}_B\right)'\right]=G_3\left(\theta\right)+o(D^{-1})_{DR\times DR}$$

and

$$MSE(\hat{\mu}_E)=G_1(\theta)+G_2(\theta)+G_3(\theta)+o(D^{-1})_{DR\times DR},$$

where

$$G_3\left(\theta\right)=\sum_{i=1}^{m}\sum_{j=1}^{m}\operatorname{cov}(\hat{\theta}_i,\hat{\theta}_j)L^{(i)}VL^{(j)\prime}.$$

Similarly as Prasad and Rao (1990), Datta and Lahiri (2000) and Das et al. (2004), we estimate $MSE(\hat{\mu}_E)$ with

$$mse(\hat{\mu}_E)=G_1(\hat{\theta})+G_2(\hat{\theta})+2G_3(\hat{\theta}). \tag{15}$$

We further consider the three bootstrap-based MSE estimators described in Section 4. They are the same bootstrap alternatives of González-Manteiga et al. (2008b).

4. Simulations

This section presents three simulation experiments. Simulation 1 is designed to analyze the behavior of the EBLUPs under Models 0–3. We recall that Models 0–3 are the particularizations of the multivariate Fay–Herriot model (1) + (3) to the conditions (5)–(8) respectively. Simulation 2 studies the performance of four MSE estimators under Model 1. Simulation 3 studies the robustness of the EBLUPs from Model 1 against departures from normality.

Let us write model (4) in the form

$$y_d=x_d\beta+u_d+e_d,\quad d=1,\ldots,D. \tag{16}$$

Take $R=2, p_1=p_2=1, p=2, \beta_1=\beta_2=1, \mu_{x1}=\mu_{x2}=10, \sigma_{x11}=1, \sigma_{x22}=2$ and $\rho_x=1/2$. For $r=1,2, d=1,\ldots,D$, generate $x_d=\operatorname{diag}(x_1,x_2)$, where

$$x_{d1}=\mu_{x1}+\sigma_{x11}^{1/2}U_{d1},\qquad x_{d2}=\mu_{x2}+\sigma_{x22}^{1/2}\left(\rho_xU_{d1}+\sqrt{1-\rho_x^2}U_{d2}\right),$$
$$U_{dr}=\frac{d-D}{D}+\frac{r}{R+1},\quad d=1,\ldots,D.$$

For $d=1,\ldots,D$, simulate $u_d\sim N_2(0,V_{ud})$ and $e_d\sim N_2(0,V_{ed})$, where $V_{ed}=\left(\sigma_{dij}\right)_{i,j=1,2}$, with $\sigma_{d11}=1, \sigma_{d22}=2$, $\sigma_{d12}=\rho_e\sqrt{\sigma_{d11}\sigma_{d22}}$, and V_{ud} is taken from Model k, $k=0,1,2,3$. For Models 1 and 3, we take $\sigma_{u_1}^2=2, \sigma_{u_2}^2=4$.

4.1. Simulation 1

The steps of Simulation 1 are

1. Repeat $I=10^4$ times ($i=1,\ldots,I$)
   1.1. Generate $\{(e_{dr}^{(i)},u_{dr}^{(i)},y_{dr}^{(i)},x_{dr}):d=1,\ldots,D,r=1,2\}$ from Model k, $k=0,1,2,3$.
   1.2. Calculate the EBLUPa, $\hat{\mu}_{Ed}^{(i,a)}$, derived from Model a, where $a=0,3$ if $k=0$ and $a=0,k$ otherwise.
2. Output: $MSE_r^{(a)}=\frac{1}{D}\sum_{d=1}^{D}MSE_{dr}$, $MSE_{dr}=\frac{1}{I}\sum_{i=1}^{I}(\hat{\mu}_{Edr}^{(i,a)}-\mu_{dr}^{(i)})^2$, $r=1,2$, $a=0,1$.

Table 4.1 presents the simulation results for the components $r=1,2$ and for the numbers of domains $D=50,100,200,400$. The three first columns of Table 4.1 indicates the correlation parameters of the generating model (ρ_e,ρ), the model generating the data (k) and the model deriving the EBLUP (a). If Model k, $k=1,2,3$, generates the data, then Table 4.1 presents the MSE for EBLUPs derived under the true Model k (EBLUPk) and under Model 0 (EBLUP0). If Model 0 generates the data, this table gives the corresponding performance measures for EBLUP0 and EBLUP3.

The results of simulation 1 give some intuition for answering the question if a multivariate Fay–Herriot model has advantage over the marginal univariate Fay–Herriot model for each component variable. If true generating data model is multivariate (models 1, 2, 3), Table 4.1 shows that the corresponding EBLUPs have lower MSE that the corresponding ones based on univariate models (model 0).

4.2. Simulation 2

The target of Simulation 2 is to investigate the behavior of the MSE estimator (15) and the three bootstrap alternatives considered by González-Manteiga et al. (2008b). For the sake of brevity, we restrict the simulations to Model 1 with $\rho_e=1/2$.

Table 4.1
$MSE_1^{(a)}$ (left) and $MSE_2^{(a)}$ (right) for models $k = 0, 1, 2, 3$.

ρ_e, ρ	k	a	$r=1$				$r=2$			
			50	100	200	400	50	100	200	400
-,-	0	0	0.686	0.679	0.673	0.669	1.379	1.358	1.343	1.337
		3	0.695	0.682	0.673	0.670	1.388	1.362	1.345	1.340
$\frac{1}{2}$,-	1	0	0.684	0.677	0.671	0.669	1.374	1.354	1.341	1.338
		1	0.647	0.640	0.633	0.631	1.299	1.275	1.266	1.263
$\frac{1}{2}, 0$	2	0	0.690	0.677	0.672	0.669	1.062	1.031	1.016	1.008
		2	0.634	0.621	0.615	0.612	1.013	0.985	0.972	0.965
$0, \frac{1}{2}$	2	0	0.750	0.734	0.732	0.729	1.201	1.167	1.157	1.150
		2	0.717	0.704	0.701	0.698	1.096	1.069	1.058	1.051
$\frac{1}{2}, \frac{1}{2}$	2	0	0.746	0.737	0.731	0.728	1.196	1.172	1.156	1.149
		2	0.739	0.731	0.725	0.722	1.179	1.159	1.145	1.139
$\frac{1}{2}, 0$	3	0	0.687	0.677	0.672	0.670	1.374	1.351	1.345	1.337
		3	0.658	0.644	0.637	0.632	1.319	1.286	1.274	1.263
$0, \frac{1}{2}$	3	0	0.712	0.702	0.697	0.695	1.431	1.408	1.401	1.395
		3	0.700	0.686	0.679	0.676	1.408	1.378	1.367	1.358
$\frac{1}{2}, \frac{1}{2}$	3	0	0.710	0.702	0.697	0.695	1.428	1.404	1.400	1.395
		3	0.714	0.702	0.695	0.692	1.434	1.403	1.395	1.389

The steps of Simulation 2 are

1. Repeat $I = 500$ times ($i = 1, \dots, 500$)
   1.1. Generate $\{(e_{dr}^{(i)}, u_{dr}^{(i)}, y_{dr}^{(i)}, x_{dr}) : d = 1, \dots, D, r = 1, 2\}$ from Model 1.
   1.2. Calculate $\mu_d^{(i)} = X_d\beta + I_2 u_d^{(i)}, \hat{\sigma}_{ur}^{2(i)}, \hat{\beta}_{Er}^{(i)}, d = 1, \dots, D, r = 1, 2.$
   1.3. For $d = 1, \dots, D$, calculate the MSE estimator (15), i.e.
   $$mse_d^{0(i)} = G_{1d}^{(i)}(\hat{\sigma}_{u1}^{2(i)}, \hat{\sigma}_{u2}^{2(i)}) + G_{2d}^{(i)}(\hat{\sigma}_{u1}^{2(i)}, \hat{\sigma}_{u2}^{2(i)}) + 2G_{3d}^{(i)}(\hat{\sigma}_{u1}^{2(i)}, \hat{\sigma}_{u2}^{2(i)}).$$
   1.4. Repeat $B = 200$ times ($b = 1, \dots, B$)
      1.4.1. Generate a bootstrap sample, $\{(e_{dr}^{*(ib)}, u_{dr}^{*(ib)}, y_{dr}^{*(ib)}, x_{dr}) : d = 1, \dots, D, r = 1, 2\}$, from Model 1, but taking $\hat{\sigma}_{ur}^{2(i)}$ and $\hat{\beta}_{Er}^{(i)}$ instead of σ_{ur}^2 and β_r, $r = 1, 2$.
      1.4.2. Calculate $\mu_d^{*(ib)} = X_d\hat{\beta}_E^{(i)} + u_d^{*(ib)}, \hat{\sigma}_{ur}^{2*(ib)}, \hat{\beta}_{Br}^{*(ib)}, \hat{\beta}_{Er}^{*(ib)}, d = 1, \dots, D, r = 1, 2.$
      1.4.3. For $d = 1, \dots, D$, calculate
      $$\hat{\mu}_{Bd}^{*(ib)} = X_d\hat{\beta}_B^{*(ib)} + I_2\hat{u}_{Bd}^{*(ib)}, \qquad \hat{\mu}_{Ed}^{*(ib)} = X_d\hat{\beta}_E^{*(ib)} + I_2\hat{u}_{Ed}^{*(ib)}.$$
      $$\delta_{Ed}^{*(ib)} = (\hat{\mu}_{Ed}^{*(ib)} - \mu_{Ed}^{*(ib)}), \qquad \delta_{Bd}^{*(ib)} = (\hat{\mu}_{Bd}^{*(ib)} - \mu_{Bd}^{*(ib)}),$$
      $$\delta_{EBd}^{*(ib)} = (\hat{\mu}_{Ed}^{*(ib)} - \hat{\mu}_{Bd}^{*(ib)}).$$
   1.5 For $d = 1, \dots, D$, calculate
   $$mse_d^{1(i)} = \frac{1}{B}\sum_{b=1}^{B} \delta_{Ed}^{*(ib)}\delta_{Ed}^{*(ib)t}$$
   $$mse_d^{2(i)} = G_{1d}^{(i)}(\hat{\sigma}_u^{2(i)}) + G_{2d}(i)(\hat{\sigma}_u^{2(i)}) + \frac{1}{B}\sum_{b=1}^{B} \delta_{EBd}^{*(ib)}\delta_{EBd}^{*(ib)t}$$
   $$mse_d^{3(i)} = 2[G_{1d}^{(i)}(\hat{\sigma}_u^{2(i)}) + G_{2d}^{(i)}(\hat{\sigma}_u^{2(i)})] - \frac{1}{B}\sum_{b=1}^{B}[G_1(\hat{\sigma}_u^{2*(ib)}) + G_2(\hat{\sigma}_u^{2*(ib)})] + \frac{1}{B}\sum_{b=1}^{B} \delta_{EBd}^{*(ib)}\delta_{EBd}^{*(ib)t}.$$
2. Calculate $mse_d^\ell = \frac{1}{I}\sum_{i=1}^{I} mse_d^{\ell(i)}, \ \ell = 0, 1, 2, 3.$
3. Take the $MSE_{dr}^{(1)}$'s from Simulation 1. For $\ell = 0, 1, 2, 3, \ r = 1, 2$, calculate
$$B_r^\ell = \frac{1}{ID}\sum_{d=1}^{D}\sum_{i=1}^{I}(mse_{dr}^{\ell(i)} - MSE_{dr}), \qquad E_r^\ell = \frac{1}{ID}\sum_{d=1}^{D}\sum_{i=1}^{I}(mse_{dr}^{\ell(i)} - MSE_{dr})^2.$$
$$RB_r^\ell = \frac{B_r^\ell}{\frac{1}{ID}\sum_{d=1}^{D}\sum_{i=1}^{I} MSE_{dr}}, \qquad RE_r^\ell = \frac{[E_r^\ell]^{1/2}}{\frac{1}{ID}\sum_{d=1}^{D}\sum_{i=1}^{I} MSE_{dr}}$$
where $mse_{dr}^{\ell(i)}$ is the rth element of $\mathrm{diag}(mse_d^{\ell(i)}), \ \ell = 0, 1, 2, 3.$

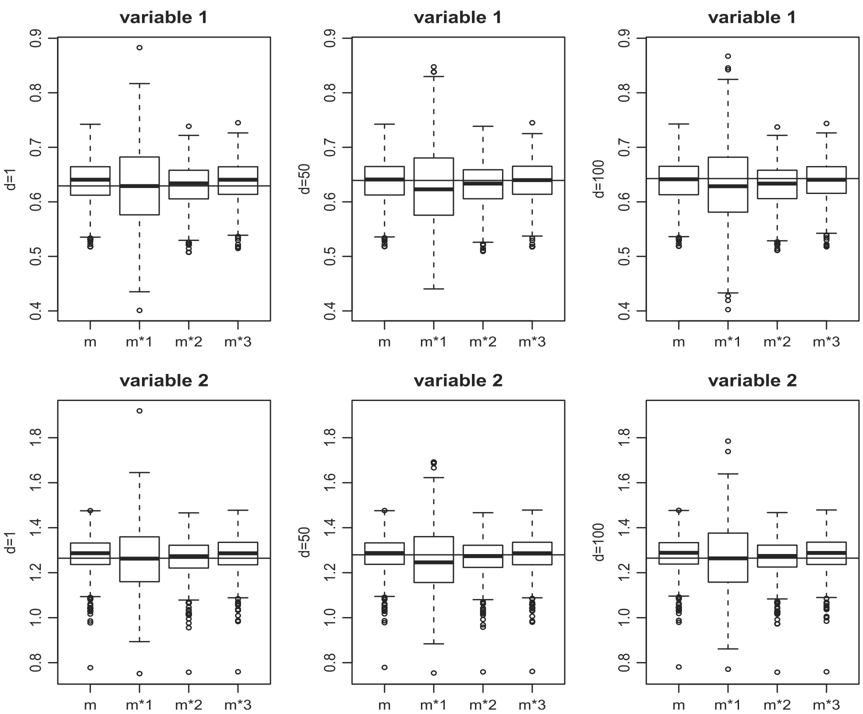

Fig. 4.1. Boxplots of $mse_d^{\ell(i)}$, $\ell = 0, 1, 2, 3$ for $D = 100$, $d = 1, 50, 100$, $r = 1, 2$.

Table 4.2
$10^3 E_r^\ell$ (top) and RE_r^ℓ in % (bottom), $\ell = 0, 1, 2, 3$, $r = 1, 2$.

D	E_1^0	E_1^1	E_1^2	E_1^3	E_2^0	E_2^1	E_2^2	E_2^3
50	3.45	7.83	3.67	3.46	13.28	31.19	14.77	13.37
100	1.72	5.81	1.85	1.72	7.55	23.92	8.05	7.68
200	0.83	4.84	0.85	0.83	3.90	20.02	3.94	3.93
400	0.51	4.47	0.51	0.51	2.02	18.01	2.06	2.01
50	45.419	69.185	47.847	46.558	44.341	69.083	48.006	45.790
100	32.080	59.598	33.997	32.841	33.438	60.503	35.436	34.700
200	22.307	54.397	22.982	22.828	24.024	55.351	24.803	24.836
400	17.420	52.284	17.912	17.919	17.298	52.495	17.918	17.761

Table 4.3
$10^3 B_r^\ell$ (top) and RB_r^ℓ in % (bottom), $\ell = 1, 2, 3$, $r = 1, 2$.

D	B_1^0	B_1^1	B_1^2	B_1^3	B_2^0	B_2^1	B_2^2	B_2^3
50	8.12	−4.95	−5.20	8.53	5.03	−21.31	−21.84	5.88
100	−2.33	−8.25	−8.95	−2.05	−1.30	−15.28	−14.63	−0.68
200	1.48	−1.93	−1.82	1.57	4.69	−1.66	−1.86	4.82
400	−0.13	−1.75	−1.77	−0.04	−1.47	−4.65	−4.78	−1.08
50	6.274	−3.869	−4.102	6.753	1.935	−8.337	−8.629	2.328
100	−1.804	−6.449	−7.068	−1.626	−0.501	−5.976	−5.780	−0.271
200	1.142	−1.505	−1.434	1.244	1.804	−0.651	−0.733	1.908
400	−0.104	−1.370	−1.398	−0.033	−0.567	−1.819	−1.887	−0.429

Fig. 4.1 presents the boxplots of $mse_d^{\ell(i)}$, $\ell = 0, 1, 2, 3$, $i = 1, \ldots, I$, for the case of $D = 100$ domains. As the values of x_{d1} and x_{d2} increase as d increases, Fig. 4.1 is divided into columns and rows. The columns are for the domains $d = 1$ (left), $d = 50$ (center) and $d = 100$ (right) and the rows are for the target variables $r = 1$ (top) and $r = 2$ (bottom). We observe that $msed^0$ (m), $msed^2$ (m^*2) and $msed^3$ (m^*3) behave better than $msed^1$ (m^*1). We also note that $msed^1$ tends to have a negative bias.

Tables 4.2 and 4.3 present the average empirical MSEs and biases, E_r^ℓ and B_r^ℓ (multiplied by 10^3), and their relative counterpart in %, RE_r^ℓ and RB_r^ℓ, of the four MSE estimators. Table 4.2 shows that the estimator $msed^1$ has the greatest MSEs and the estimators $msed$ and $msed^3$ have the lowest MSEs. Table 4.2 also shows that the MSEs of all the estimators decrease

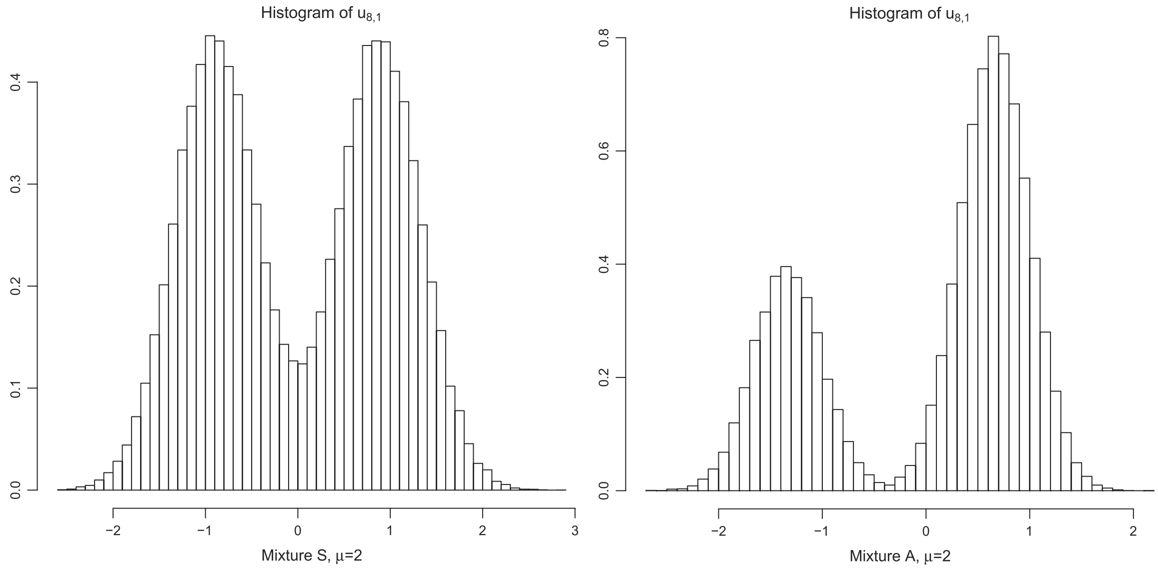

Fig. 4.2. Probability histograms of 10^5 realizations of $u_{8,1}$ under S_2 (left) and A_2 (right).

as the number of domains D increases. We observe in Table 4.3 that the average biases of estimators $msed^1$ and $msed^2$ are negative and therefore they tend to underestimate the MSEs of the EBLUPs.

The simulation results are coherent with similar studies in linear mixed models, where MSE estimators of type mse_d^ℓ, $\ell = 0, 1, 2, 3$, are employed. González-Manteiga et al. (2008b) carried out simulations under the nested error regression model and they obtained a small negative bias for the parametric bootstrap estimator $msed^1$. González-Manteiga et al. (2008a) established the consistency of the MSE estimators under a multivariate Fay–Herriot model. They showed that $msed^\ell$, $\ell = 0, 1, 2$, are $O(D^{-1})_{R\times R}$ and that $msed^3$ is $o(D^{-1})_{R\times R}$. Their simulation scenarios showed that the bias of $msed^0$ and $msed^3$ were in most cases lower than the corresponding ones of $msed^1$ and $msed^2$. Marhuenda et al. (2013) analyzed the behavior of $msed^1$ under a spatio-temporal Fay–Herriot model. They also found some small negative bias in their simulations.

4.3. Simulation 3

The target of the third simulation experiment is to investigate the robustness of the EBLUPs from Model 1 against departures from the normal distribution assumed for the random effects $\{u_{dr}\}$. As the direct estimators are weighted sums, the central limit theorem suggests that the sampling errors $\{e_{dr}\}$ should not be far from normality. This is why Simulation 3 does not study departures from normality of the sampling errors. For the sake of brevity, we restrict the simulations to Model 1 with $\rho_e = 1/2$. Simulation 3 repeats the steps of Simulation 1, but generating the u_{dr}'s from non normal distributions. We first generate i.i.d. random variables $Z_{1r}, \ldots, Z_{Dr}$ from a mixture of normal distributions. We take $X \sim \text{Bin}(1, p)$, i.e. $X = 1$ with probability p and $X = 0$ with probability $1 - p$. We further simulate $Z_{dr} \sim N(\mu_0, \sigma_0^2)$ if $X = 0$ and $Z_{dr} \sim N(\mu_1, \sigma_1^2)$ if $X = 1$. We implement two families of mixtures. The first mixture S_μ is defined by $\mu_0 = -\mu$, $\mu_1 = \mu$, $\sigma_0^2 = \sigma_1^2 = 1$ and $p = 1/2$. It holds that $E(Z_d) = 0$ and $\text{var}(Z_d) = 1 + \mu^2$. The second mixture A_μ is defined by $\mu_0 = -2\mu$, $\mu_1 = \mu$, $\sigma_0^2 = \sigma_1^2 = 1$ and $p = 2/3$. It holds that $E(Z_{dr}) = 0$ and $\text{var}(Z_{dr}) = 1 + 2\mu^2$. Finally, u_{dr} is obtained from Z_{dr} by the transformation

$$u_{dr} = \frac{\sigma_{u_r}}{\sqrt{\text{var}(Z_{dr})}} Z_{dr}, \quad d = 1, \ldots, D,$$

so that $E(u_{dr}) = 0$ and $\text{var}(u_{dr}) = \sigma_{u_r}^2$. To illustrate the type of deviation from normality, Fig. 4.2 plots the probability histograms of 10^5 realizations of $u_{d,1}$, $d = 8$, under the bimodal symmetric and asymmetric distributions S_2 and A_2 respectively.

Table 4.4 presents the average empirical MSEs of the EBLUPs, derived from Model 1 for the components $r = 1, 2$, under the alternative generating distribution, $F \in \mathcal{F} = \{A_\mu, S_\mu : \mu = 0.25, 0.5, 1, 1.5, 2\}$, indicated in the first row. The second and fourth rows contain the empirical MSEs of the EBLUPs for the components $r = 1, 2$. The third and fifth rows contain the relative loss of efficiency with respect to the normal distribution $N = N(0, 1)$ in %, i.e.

$$L_r^{(1)} = 100 \frac{MSE_r^{(1)}(F) - MSE_r^{(1)}(N)}{MSE_r^{(1)}(N)}, \quad r = 1, 2, \; F \in \mathcal{F}.$$

Table 4.4 shows that the average MSEs increases as the generating distribution gets far from normality. For small deviations from normality, $\mu = 0.25, 0.5$, the relative loss of efficiency is below 10%. For large deviations the relative loss of efficiency is

Table 4.4
$MSE_r^{(1)}$ and $L_r^{(1)}$, $r = 1, 2$, for $D = 100$.

Distribution	MSE_1	L_1	MSE_2	L_2
$N(0.1)$	0.640	0.00	1.275	0.00
$S_{0.25}$	0.646	0.80	1.301	2.01
$S_{0.5}$	0.668	4.26	1.347	5.61
S_1	0.732	14.33	1.464	14.81
$S_{1.5}$	0.782	22.14	1.571	23.21
S_2	0.833	30.12	1.659	30.05
$A_{0.25}$	0.653	2.04	1.310	2.75
$A_{0.5}$	0.690	7.69	1.381	8.31
A_1	0.778	21.42	1.555	21.94
$A_{1.5}$	0.836	30.60	1.665	30.53
A_2	0.867	35.41	1.731	35.70

over 10%. Therefore the EBLUPs, based on the multivariate Fay–Herriot model (4), are not robust against sensible deviations from normality.

5. Application to real data

Esteban et al. (2012) and Marhuenda et al. (2013) gave estimates of province poverty proportions and gaps by using data from the 2006 Spanish Living Condition Survey (SLCS). They calculated EBLUPs based on univariate temporal area-level linear mixed models. This section uses the same data as in the above cited papers for estimating poverty proportions and gaps, but calculates EBLUPs based on multivariate Fay–Herriot models. The target domains are the 52 Spanish provinces crossed by sex ($D = 104$). The target indicators are the poverty proportion ($\alpha = 0$) and gap ($\alpha = 1$),

$$\bar{Y}_{\alpha d} = \frac{1}{N_d}\sum_{j=1}^{N_d} y_{\alpha dj}, \quad y_{\alpha dj} = \left(\frac{z - E_{dj}}{z}\right)^{\alpha} I(E_{dj} < z),$$

where z is the poverty line and E_{dj} is the equivalised net income of individual j within domain $d, j = 1, \dots, N_d,\ d = 1, \dots, D$.

We denote the SLCS sample and the domain samples by s and s_d and the corresponding sample sizes by n and n_d respectively, so that $s = \cup_{d=1}^{D} s_d$ and $n = \sum_{d=1}^{D} n_d$. The direct estimator of a total $Y_{dr} = \sum_{j=1}^{N_d} y_{drj}$ is

$$\hat{Y}_{dr}^{dir} = \sum_{j\in s_d} w_{dj}\, y_{drj},$$

where s_d is the domain sample and the w_{dj}'s are the official calibrated sampling weights which take into account for non response. The estimated domain size is

$$\hat{N}_d^{dir} = \sum_{j\in s_d} w_{dj}.$$

A direct estimator of the domain mean $\bar{Y}_{dr}$ is $\bar{y}_{dr} = \hat{Y}_{dr}^{dir}/\hat{N}_d^{dir}$. The direct estimates of the domain means are used as responses in the area-level model. The design-based covariances of these estimators can be approximated by

$$\widehat{\text{cov}}_{\pi}(\hat{Y}_{dr_1}^{dir}, \hat{Y}_{dr_2}^{dir}) = \sum_{j\in s_d} w_{dj}(w_{dj} - 1)(y_{dr_1 j} - \bar{y}_{dr_1})(y_{dr_2 j} - \bar{y}_{dr_2}),$$

$$\sigma_{\pi,d,r_1,r_2} = \widehat{\text{cov}}_{\pi}(\bar{y}_{dr_1}, \bar{y}_{dr_2}) = \widehat{\text{cov}}_{\pi}(\hat{Y}_{dr_1}^{dir}, \hat{Y}_{dr_2}^{dir})/\hat{N}_d^2.$$

The last formulas are obtained from Särndal et al. (1992, pp. 43, 185 and 391), with the simplifications $w_{dj} = 1/\pi_{dj}$, $\pi_{dj,dj} = \pi_{dj}$ and $\pi_{di,dj} = \pi_{di}\pi_{dj},\ i \neq j$ in the second order inclusion probabilities. We take the σ_{π,d,r_1,r_2}'s as the known elements of the matrix V_{ed} in the multivariate Fay–Herriot models.

The available auxiliary variables are the domain proportions of people in the categories of the following classification variables: Age (*age*1: ≤ 15, *age*2: 16–24, *age*3: 25–49, *age*4: 50–64, *age*5: ≥ 65), Education (*edu*0: less than primary, *edu*1: primary, *edu*2: secondary, *edu*3: university), Citizenship (*cit*1: Spanish, *cit*2: not Spanish), Labor situation (*lab*0: ≤ 15, *lab*1: employed, *lab*2: unemployed, *lab*3: inactive). As the proportions of people in the categories of a classification variable sum up to one, we take the reference categories out of the data file of auxiliary variables. The reference categories are *age*5, *edu*3, *cit*2 and *lab*3.

This section presents two applications. The first application jointly estimates 2006 poverty proportions and gaps for provinces crossed by sex. The second application jointly estimates 2005 and 2006 poverty proportions for provinces crossed by sex.

Table 5.1
Regression parameters and p-values for Model 3, $\alpha = 0$, 2006.

Variables	Constant	*age1*	*age2*	*edu1*	*cit1*	*lab2*
β_1	−0.70357	0.95490	1.45541	0.74745	0.30873	1.50050
p-value	0.00000	0.00066	0.00165	0.00000	0.00137	0.00006

Table 5.2
Regression parameters and p-values for Model 3, $\alpha = 1$, 2006.

Variables	Constant	*edu0*	*edu1*	*edu2*	*cit1*	*lab1*
β_2	−0.37458	0.97049	0.34255	0.16551	0.152031	−0.06384
p-value	0.00001	0.00000	0.00001	0.11197	0.00104	0.02502

5.1. Application 1

For jointly estimating 2006 poverty proportions and gaps, we fit Model 3 to a subset of auxiliary variables. Tables 5.1 and 5.2 present the estimated regression parameters β_{rj}, $r = 1, 2, j = 1, \dots, p_r, p_1 = p_2 = 6$ and the p-values for testing $H_0 : \beta_{rj} = 0$. By observing the signs of the regression parameters we conclude that provinces having larger proportions of population in categories *age1*, *age2*, *edu1*, *cit1* and *lab2* have greater poverty proportion. On the other side, provinces having larger proportions of population in categories *edu0*, *edu1*, *edu2*, and *cit1* and smaller proportions of population in the category *lab1* have greater poverty gaps.

The estimates of the variance component parameters are $\hat{\sigma}^2_{u1} = 0.00138$, $\hat{\sigma}^2_{u2} = 0.00037$ and $\widehat{\rho} = 0.01859$. We test $H_0 : \sigma^2_{u1} = \sigma^2_{u2}$. The test statistics is

$$T_{12} = \frac{\widehat{\sigma}^2_{u1} - \widehat{\sigma}^2_{u2}}{\sqrt{\nu_{11} + \nu_{22} - 2\nu_{12}}} = 3.34588,$$

where ν_{ij}, $i, j = 1, 2, 3$ are the elements of the inverse of the REML Fisher information matrix of Model 3 evaluated at $\hat{\theta} = (\hat{\sigma}^2_1, \hat{\sigma}^2_2, \hat{\rho})$. As T_{12} has standard normal asymptotic distribution under H_0, the p-value is 0.00082. We conclude that random effects variances are different and we prefer Model 3 instead of Model 2. We also test $H_0 : \rho = 0$. The test statistics is

$$T_\rho = \frac{\widehat{\rho}}{\sqrt{\nu_{33}}} = 1.96464.$$

As T_ρ has standard normal asymptotic distribution under H_0, the p-value is 0.049456. Therefore, we conclude that both components (poverty proportion and gap) are positively correlated and we prefer Model 3 instead of Model 1.

Fig. 5.1 plots the EBLUPs, under Model 3, of poverty proportions and gaps in Spanish provinces during 2006. We observe that the poverty is more severe in the south-west of Spain and slightly higher for women than for men.

Fig. 5.2 plots the root-MSEs of direct and EBLUP (under Model 3) estimators of poverty proportions (left) and gaps (right) for men in Spanish provinces during 2006. The EBLUP estimators have lower MSEs than the direct ones. For the sake of brevity we do not present the corresponding plots for women, where similar results are obtained.

5.2. Application 2

For jointly estimating 2005 and 2006 poverty proportions, we fit Model 3 to a subset of auxiliary variables. Tables 5.3 and 5.4 present the estimated regression parameters β_{rj}, $r = 1, 2, j = 1, \dots, p_r, p_1 = p_2 = 6$ and the p-values for testing $H_0 : \beta_{rj} = 0$. By observing the signs of the regression parameters we conclude that provinces having larger proportions of population in categories *age1*, *age2*, *edu1*, *cit1* and *lab2* have greater poverty proportion in 2005 and 2006.
The estimates of the variance component parameters are $\hat{\sigma}^2_{u1} = 0.00256$, $\hat{\sigma}^2_{u2} = 0.00193$ and $\widehat{\rho} = 0.02105$. We test $H_0 : \sigma^2_{u1} = \sigma^2_{u2}$. The test statistics is

$$T_{12} = \frac{\widehat{\sigma}^2_{u1} - \widehat{\sigma}^2_{u2}}{\sqrt{\nu_{11} + \nu_{22} - 2\nu_{12}}} = 1.0756,$$

where ν_{ij}, $i, j = 1, 2, 3$ are the elements of the inverse of the REML Fisher information matrix of Model 3 evaluated at $\hat{\theta} = (\hat{\sigma}^2_1, \hat{\sigma}^2_2, \hat{\rho})$. As T_{12} has standard normal asymptotic distribution under H_0, the p-value is 0.28208. We cannot conclude that random effects variances are different and we prefer Model 2 instead of Model 3. Therefore, we fit Model 2 to the subset of auxiliary variables appearing in Tables 5.5 and 5.6.
We test $H_0 : \rho = 0$ under model 2. The test statistics is

$$T_\rho = \frac{\widehat{\rho}}{\sqrt{\nu_{22}}} = 16.72633,$$

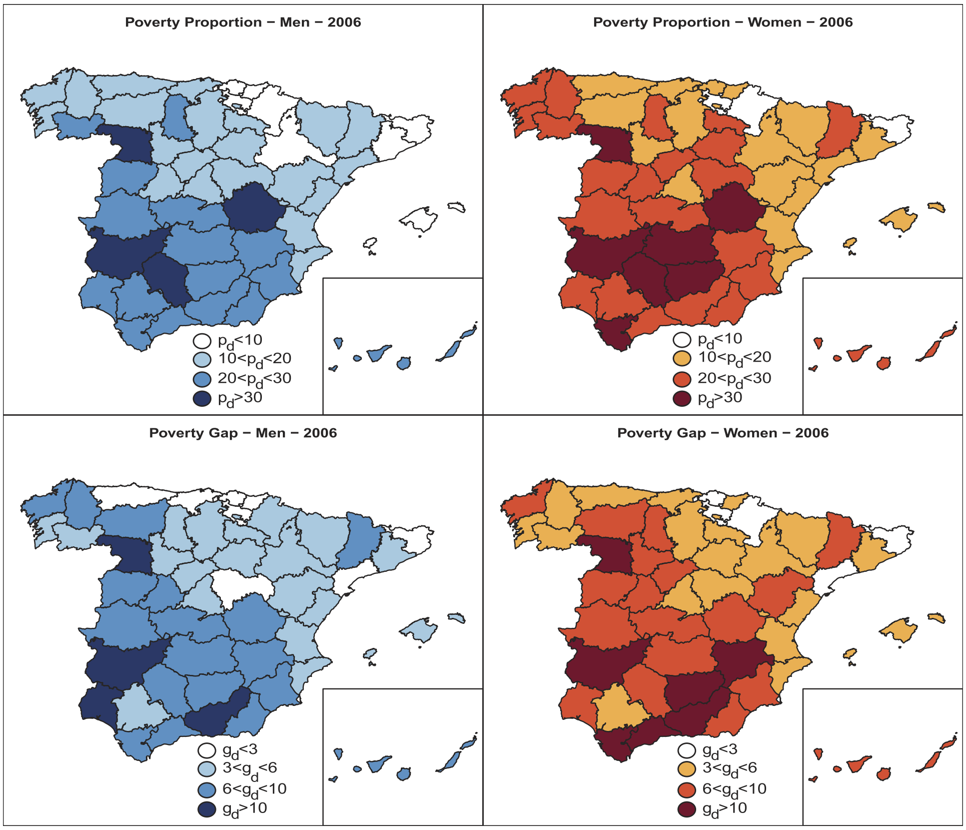

Fig. 5.1. Poverty proportions (top) and gaps (bottom) for men (left) and women (right) in Spanish provinces during 2006.

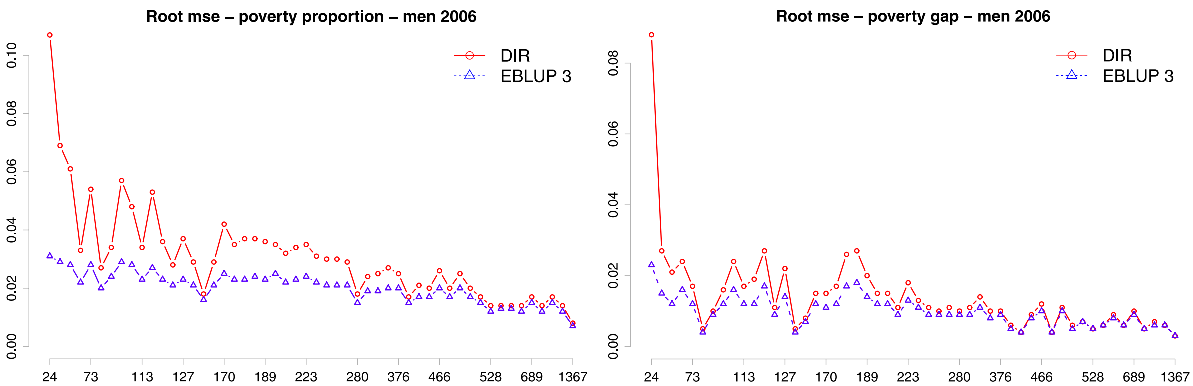

Fig. 5.2. Root-MSEs of direct and EBLUP (under Model 3) estimators of poverty proportions (left) and gaps (right) in Spanish provinces during 2006.

Table 5.3
Regression parameters and p-values for Model 3, $\alpha = 0$, 2005.

Variables	Constant	*age1*	*age2*	*edu1*	*cit1*	*lab2*
β	−0.65428	0.69780	2.38240	0.71074	0.25924	0.71268
p-value	0.00010	0.06540	0.00049	0.00000	0.08960	0.15129

Table 5.4
Regression parameters and p-values for Model 3, $\alpha = 0$, 2006.

Variables	Constant	*age1*	*age2*	*edu1*	*cit1*	*lab2*
β	−0.75278	0.88497	1.89752	0.79734	0.31471	2.04460
p-value	0.00000	0.00609	0.00047	0.00000	0.00414	0.00000

Table 5.5
Regression parameters and p-values for Model 2, $\alpha = 0$, 2005.

Variables	Constant	*age1*	*age2*	*edu1*	*cit1*	*lab2*
β	−0.53822	0.67365	1.74785	0.60288	0.23672	0.99025
p-value	0.00040	0.03876	0.00209	0.00000	0.08998	0.02351

Table 5.6
Regression parameters and p-values for Model 2, $\alpha = 0$, 2006.

Variables	Constant	*age1*	*age2*	*edu1*	*cit1*	*lab2*
β	−0.74083	0.90128	1.69006	0.68294	0.37468	1.78575
p-value	0.00000	0.00595	0.00127	0.00000	0.00163	0.00007

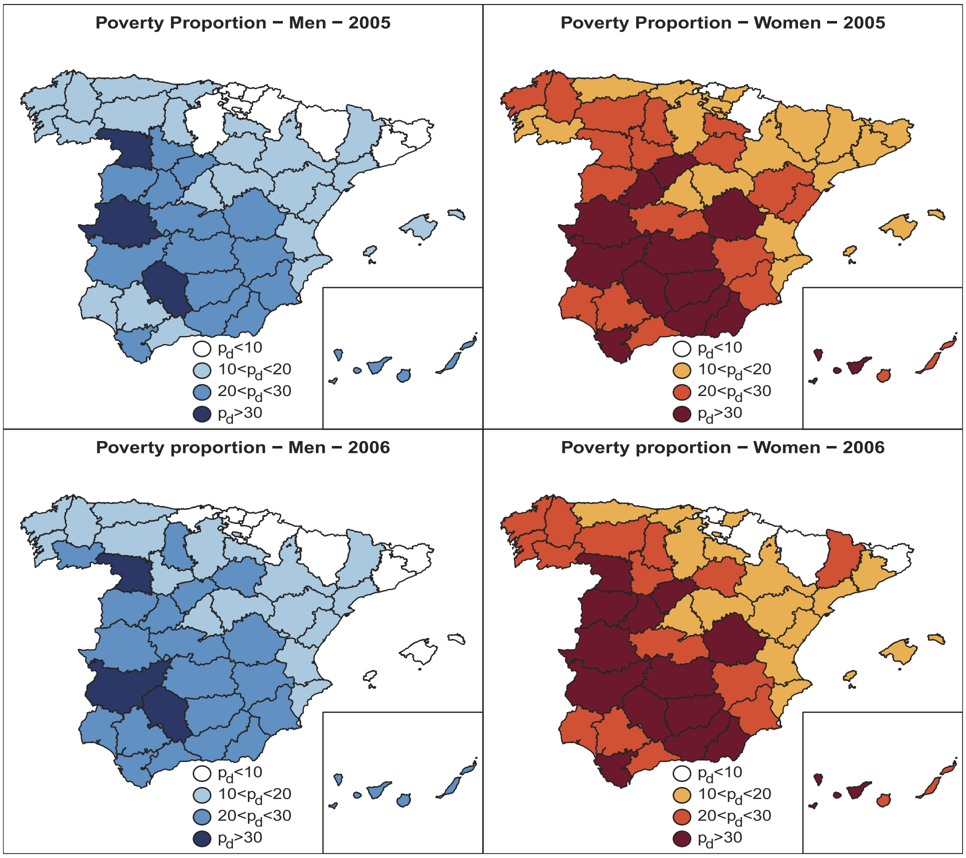

Fig. 5.3. Poverty proportions in 2005 (top) and 2006 (bottom) for men (left) and women (right) in Spanish provinces during 2006.

where ν_{ij}, $i, j = 1, 2$ are the elements of the inverse of the REML Fisher information matrix of Model 2 evaluated at $\hat{\theta} = (\hat{\sigma}^2, \hat{\rho})$. As T_ρ has standard normal asymptotic distribution under H_0, the p-value is 0.00. Therefore, we conclude that both components (2005 and 2006 poverty proportions) are positively correlated and we prefer Model 2 instead of Model 1.

Fig. 5.3 plots the EBLUPs, under Model 2, of poverty proportions in Spanish provinces during 2005 and 2006. The figure shows that poverty is more severe in the south-west of Spain and slightly higher for women than for men.

Fig. 5.4 plots the root-MSE of direct and EBLUP (under Model 2) estimators of poverty proportions for 2005 (left) and 2005 (right) in Spanish provinces. The EBLUP estimators have lower MSEs than the direct ones. For the sake of brevity we do not present the corresponding plots for women, where similar results are obtained.

5.3. Comparisons

This section compares the 2006 province poverty estimates of Esteban et al. (2012) with the corresponding ones obtained in Applications 1 and 2 (appearing in Sections 5.1 and 5.2 respectively) and with the direct estimates. For this sake, we consider separately the subpopulations of men and women. We sort the provinces by sample size and we look at the men

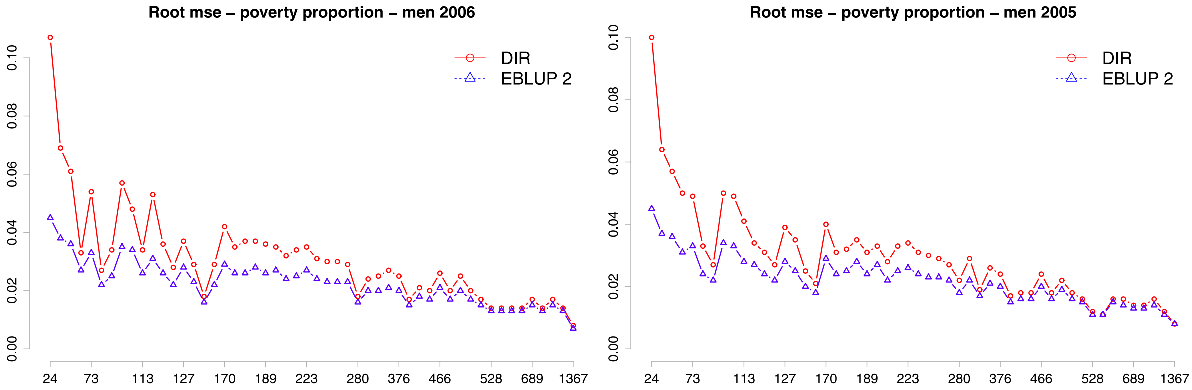

Fig. 5.4. Root-MSEs of direct and EBLUP (under Model 2) estimators of poverty proportions for 2006 (left) and 2005 (right) in Spanish provinces.

Table 5.7
Province poverty proportions and root-MSEs for men in 2006.

n	Dir	A0	A1	A2	E3	rDir	rA0	rA1	rA2	rE3
24	0.25	0.17	0.19	0.20	0.24	0.107	0.042	0.031	0.045	0.026
92	0.08	0.11	0.08	0.11	0.10	0.027	0.023	0.020	0.022	0.021
124	0.19	0.24	0.18	0.21	0.22	0.036	0.029	0.023	0.026	0.024
145	0.05	0.06	0.05	0.06	0.07	0.018	0.017	0.016	0.016	0.021
173	0.24	0.25	0.23	0.25	0.25	0.035	0.028	0.023	0.026	0.026
221	0.31	0.30	0.31	0.32	0.32	0.034	0.027	0.023	0.025	0.030
280	0.06	0.07	0.06	0.07	0.08	0.018	0.017	0.015	0.016	0.021
428	0.09	0.10	0.10	0.10	0.09	0.017	0.016	0.015	0.015	0.022
477	0.37	0.35	0.36	0.33	0.31	0.025	0.022	0.020	0.020	0.030
556	0.10	0.10	0.10	0.10	0.10	0.014	0.014	0.013	0.013	0.022
911	0.11	0.11	0.11	0.11	0.09	0.014	0.013	0.012	0.013	0.022
1367	0.08	0.08	0.08	0.08	0.09	0.008	0.007	0.007	0.007	0.022

Table 5.8
Province poverty proportions and root-MSEs for women in 2006.

n	Dir	A0	A1	A2	E3	rDir	rA0	rA1	rA2	rE3
18	0.60	0.23	0.53	0.27	0.30	0.126	0.043	0.032	0.048	0.034
86	0.16	0.20	0.15	0.19	0.16	0.041	0.031	0.025	0.028	0.023
124	0.25	0.27	0.25	0.24	0.25	0.040	0.030	0.025	0.027	0.025
138	0.07	0.08	0.07	0.08	0.08	0.023	0.021	0.019	0.019	0.021
193	0.29	0.29	0.28	0.29	0.28	0.037	0.029	0.024	0.027	0.026
233	0.31	0.32	0.31	0.32	0.32	0.033	0.027	0.023	0.025	0.029
292	0.10	0.10	0.10	0.10	0.10	0.020	0.019	0.017	0.018	0.022
448	0.13	0.14	0.13	0.14	0.12	0.020	0.018	0.017	0.017	0.022
517	0.39	0.37	0.39	0.36	0.33	0.025	0.022	0.019	0.020	0.031
577	0.14	0.14	0.13	0.14	0.12	0.017	0.016	0.015	0.015	0.022
1008	0.13	0.13	0.13	0.12	0.11	0.013	0.013	0.012	0.012	0.022
1494	0.11	0.11	0.11	0.11	0.11	0.008	0.008	0.008	0.008	0.022

results for provinces in the positions $5 \times k + 1$, $k = 1, \ldots, 10$. We also include the province of Barcelona with largest sample size.

Esteban et al. (2012) studies several univariate extensions of the Fay–Herriot model to temporal data. These authors recommended using their model 3 with random effects taking into account for AR(1) time correlation within each domain. They use past data from 2004 and 2005 for giving estimates of 2006. Tables 5.7 and 5.8 label their EBLUPs and their root-MSE estimates by E3 and rE3 respectively. These tables also include the poverty proportion EBLUPs obtained in Applications 1 and 2, which are labeled by A1 and A2. They also include the EBLUPs based on the univariate Fay–Herriot model for the poverty proportion, labeled by A0. The corresponding root-MSE estimates are labeled by rA0, rA1 and rA2 respectively. The direct estimates and their root-MSE estimates are labeled by Dir and rDir respectively. Finally n denotes the SLCS2006 province sample size.

We observe that the A1-estimates are in general more close to the Dir-estimates than the A0, A2 and E3 estimates. This is because the direct estimators of the poverty proportions and gaps are highly correlated and the fitted HAR(1) includes the sampling correlations of these estimates in the 2×2 covariance matrices of the vectors $e_d = (e_{d1}, e_{d2})'$ of sampling errors. The models deriving the EBLUPs A2 and E3 assume that sampling errors from different time periods are independent. They do not incorporate the corresponding design-based sampling correlations. The basic Fay–Herriot model incorporates less information than the three others. In summary, the model having a better fit to data is the one given in the Application 1.

We also observe that the E3 values are more alike to the A2 than to the A1. This is a natural fact, as E3 and A2 values are derived under temporal models. Tables 5.7 and 5.8 also present the estimated root-MSEs. We observe that the three EBLUPs have lower root-MSE than the direct estimator. We also observe that rA1 is in general lower than rA0, rA2 and rE3. In this sense, we are tempted to recommend A1 as best. Nevertheless, we do not conclude that A1 is preferable to A0, A2 or E3 because the root-MSEs are derived under different models. This is to say, they are not comparable. Unless we select one of the models as the true model by using model selection tools, we cannot nominate any of the EBLUPs as best. However, we have not been able to do that. By comparing the residuals from models for A0, A1, A2 and E3, we have concluded that the three models have a similar good fit to the data. So we equally recommend each of the considered EBLUPs.

6. Concluding remarks

Datta et al. (2002) considered current population survey (CPS) estimates of median income of four-person families for states of the US for nine years (1981–1989) to produce estimates for the year 1989. They chose 1989 since the corresponding estimates were available from the census which allowed them to compare their hierarchical Bayes (HB) estimates with a few multivariate HB estimates. This comparison showed that the more complex multivariate HB estimators did not perform better than their univariate HB estimator. Datta et al. (2002) conclusions were thus restricted to their case of study.

We have conducted an empirical research based on Monte Carlo simulations. If the true generating model is multivariate (models 1, 2, 3), Table 4.1 shows that their EBLUPs have lower MSE than the corresponding ones based on univariate models (model 0). To increase our intuition about the gain of precision obtained by using multivariate models, new Tables 5.7 and 5.8 contains the EBLUPs based on the univariate Fay–Herriot model (labeled with A0) and their estimated root-MSEs (labeled with rA0). These tables show that the rA0-values are greater than the root-MSEs (rA1 and rA2) of EBLUPs based on multivariate models. Here we simply gain some amount of intuition about the gain of precision obtained by using multivariate models. We recall that the root-MSEs are derived under different models and therefore they are not comparable.

Datta et al. (2002) also recommended the use of univariate methods because they are more simple to implement. This is true. However, once the methods are implemented and available (for example, in R or SAS code), we do not find great usability differences. Our opinion is that univariate models (simpler models in general) are good enough if we have a good set of auxiliary variables. If this is not the case, then more complex models, that takes into account additional data relationships, might provide estimators with a sensible gain of precision.

This paper introduces multivariate Fay–Herriot models for estimating small area parameters. Multivariate models incorporate the correlation of several target variables and borrow strength from auxiliary variables. The introduced models give some modeling flexibility as it is shown in Section 5, where they are applied to the estimation of poverty proportions and gaps in 2006 and to the estimation of poverty proportions in 2005 and 2006. The paper gives the EBLUPs under the multivariate models and four MSE estimators. The presented simulations give some indications about the behavior of the EBLUPs and the MSE estimators.

Acknowledgments

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Appendix

Let us consider the following hypotheses:

H1 $0 < p < \infty, 0 < r < \infty$.

H2 $V_{ed},\ d = 1, \ldots, D$, are positive definite matrices with uniformly bounded elements.

H3 $\left|x_{drj}\right| \le x < \infty, X'X = O\left(D\right)_{pR\times pR}$,

H4 $X'V_e^{-1}X = O\left(D\right)_{pR\times pR}, \sum_{d=1}^{D} 1_R' V_{ed} 1_R = O\left(D\right)$.

H5 $(X'V^{-1}X)^{-1} = O(D^{-1})_{pR\times pR}$,

H6 $\hat{\theta}_i = k + y'C_i y$ is an unbiased, consistent and translation invariant estimator of $\theta_i,\ i = 1, \ldots, m$, where $k = O(1)$ and

$$C_i = \text{diag}\left\{O(D^{-1})_{R\times R}, \ldots, O(D^{-1})_{R\times R}\right\} + O(D^{-2})_{DR\times DR}.$$

Lemma 1. *Let $v = Zu + e$ be the vector containing the random part of model (4). Under* H1–H2 *, it holds that*

$$s^{(i)} = \frac{\partial \hat{\mu}}{\partial \theta_i} = (F^{(i)} + L^{(i)})V,$$

where

$$F^{(i)} = -(I-R)XQX'\frac{\partial V^{-1}}{\partial \theta_i}A - \frac{\partial R}{\partial \theta_i}XQX'V^{-1}, \qquad L^{(i)} = \frac{\partial R}{\partial \theta_i}, \tag{17}$$
$$A = I - XQX'V^{-1}, \qquad R = V_uV^{-1}, \qquad Q = (X'V^{-1}X)^{-1}.$$

Proof. The BLUP of μ is

$$\hat{\mu}_B = X\hat{\beta}_B + R(y - X\hat{\beta}_B) = X\hat{\beta}_B - RX\hat{\beta}_B + Ry = XQX'V^{-1}y - RXQX'V^{-1}y + Ry.$$

By substituting y by $X\beta + v$, we have

$$\begin{aligned}\hat{\mu}_B &= XQX'V^{-1}X\beta - RXQX'V^{-1}X\beta + RX\beta + XQX'V^{-1}v \\ &\quad - RXQX'V^{-1}v + Rv = X\beta + XQX'V^{-1}v + RAv.\end{aligned}$$

By taking partial derivatives with respect to θ_i, we get

$$\begin{aligned} s^{(i)} &= \frac{\partial \hat{\mu}_B}{\partial \theta_i} = -XQX'\frac{\partial V^{-1}}{\partial \theta_i}XQX'v + XQX'\frac{\partial V^{-1}}{\partial \theta_i}v + \frac{\partial R}{\partial \theta_i}Av + R\frac{\partial A}{\partial \theta_i}v \\ &= XQX'\frac{\partial V^{-1}}{\partial \theta_i}(I - XQX'V^{-1})v + \frac{\partial R}{\partial \theta_i}AV + R\frac{\partial A}{\partial \theta_i}v \\ &= XQX'\frac{\partial V^{-1}}{\partial \theta_i}Av + \frac{\partial R}{\partial \theta_i}Av + R\frac{\partial A}{\partial \theta_i}v. \end{aligned} \tag{18}$$

The partial derivative of A with respect to θ_i is

$$\frac{\partial A}{\partial \theta_i} = XQX'\frac{\partial V^{-1}}{\partial \theta_i}XQX'V^{-1} - XQX'\frac{\partial V^{-1}}{\partial \theta_i} = -XQX'\frac{\partial V^{-1}}{\partial \theta_i}A.$$

Therefore,

$$\begin{aligned} s^{(i)} &= XQX'\frac{\partial V^{-1}}{\partial \theta_i}Av - RXQX'\frac{\partial V^{-1}}{\partial \theta_i}Av + \frac{\partial R}{\partial \theta_i}Av \\ &= \left[(I-R)XQX'\frac{\partial V^{-1}}{\partial \theta_i}A + \frac{\partial R}{\partial \theta_i}A\right]v \\ &= \left[(I-R)XQX'\frac{\partial V^{-1}}{\partial \theta_i}A - \frac{\partial R}{\partial \theta_i}XQX'V^{-1} + \frac{\partial R}{\partial \theta_i}\right]v = [F^{(i)} + L^{(i)}]v \end{aligned}$$

where

$$F^{(i)} = (I-R)XQX'\frac{\partial V^{-1}}{\partial \theta_i}A - \frac{\partial R}{\partial \theta_i}XQX'V^{-1}, \qquad L^{(i)} = \frac{\partial R}{\partial \theta_i}.$$

As the partial derivatives of V^{-1} and $R = V_uV^{-1}$ are

$$\begin{aligned} \frac{\partial V^{-1}}{\partial \theta_i} &= -V^{-1}\frac{\partial V}{\partial \theta_i}V^{-1} = -V^{-1}W_iV^{-1}, \\ \frac{\partial R}{\partial \theta_i} &= \frac{\partial V_u}{\partial \theta_i}V^{-1} + V_u\frac{\partial V}{\partial \theta_i} = W_iV^{-1} - V_uV^{-1}W_iV^{-1} = (I-R)W_iV^{-1}, \end{aligned}$$

we finally get

$$\begin{aligned} F^{(i)} &= -(I-R)XQX'V^{-1}W_iV^{-1}A - (I-R)W_iV^{-1}XQX'V^{-1}, \\ L^{(i)} &= (I-R)W_iV^{-1}. \end{aligned}$$

Lemma 2. *Under* H1–H4 *, it holds that*

(i) $L^{(i)} = \mathrm{diag}_{1\le d\le D}(L_d^{(i)})$, *with* $L_d^{(i)} = O(1)_{R\times R}$, $d = 1, \ldots, D$.
(ii) $F^{(i)} = O(D^{-1})_{DR\times DR}$.

Proof. As $L_d^{(i)} = W_{di}V_d^{-1} - V_{ud}V_d^{-1}W_{di}V_d^{-1}$, (i) follows from H1–H4. On the other hand,

$$F^{(i)} = (I-R)XQX'\frac{\partial V^{-1}}{\partial\theta_i}A - L^{(i)}XQX'V^{-1}.$$

From H3–H4, we have $Q = (X'V^{-1}X)^{-1} = O(D^{-1})_{p\times p}$ and $XQX' = O(D^{-1})_{DR\times DR}$. From $L^{(i)} = O(1)_{DR\times DR}$ and $V^{-1} = O(1)_{DR\times DR}$, the second summand of $F^{(i)}$ is

$$L^{(i)}XQX'V^{-1} = O(D^{-1})_{DR\times DR}.$$

Concerning the first summand of $F^{(i)}$, we have

$$I-R = O(1)_{DR\times DR},\qquad \frac{\partial V^{-1}}{\partial\theta_i} = O(1)_{DR\times DR},\qquad (I-R)XQX'\frac{\partial V^{-1}}{\partial\theta_i} = O(1)_{DR\times DR}.$$

If we post-multiply by $A = I - XQX'V^{-1}$, we get

$$(I-R)XQX'\frac{\partial V^{-1}}{\partial\theta_i}A = (I-R)XQX'\frac{\partial V^{-1}}{\partial\theta_i} - (I-R)XQX'\frac{\partial V^{-1}}{\partial\theta_i}XQX'V^{-1},$$

where

$$\begin{aligned} XQX'\frac{\partial V^{-1}}{\partial\theta_i}XQX' &= XQ\left(X'\frac{\partial V^{-1}}{\partial\theta_i}X\right)QX' \\ &= O(D^{-1})_{DR\times p}[O(D)]_{p\times p}O(D^{-1})_{p\times DR} = O(D^{-1})_{DR\times DR}. \end{aligned}$$

Therefore, the first summand of $F^{(i)}$ is

$$(I-R)XQX'\frac{\partial V^{-1}}{\partial\theta_i}XQX'V^{-1} = O(D^{-1})_{DR\times DR}.$$

Lemma A.1 (Prasad and Rao, 1990)**.** *Let A_1 and A_2 be nonstochastic matrices of order n and $y \sim N_n(\mathbf{0}, V)$, where V is positive definite. Then*

(a) $E[y(y'A_sy)y'] = tr(A_sV)V + 2VA_sV,\quad s = 1, 2,$

(b) $E[(y'A_1y)(y'A_2y)] = 2tr(A_1VA_2V) + tr(A_1V)tr(A_2V),$

(c) $$\begin{aligned} E[y(y'A_1y)(y'A_2y)y'] &= tr(A_1V)tr(A_2V)V + 2tr(A_1V)VA_2V \\ &\quad + 2tr(A_2V)VA_1V + 2tr(A_1VA_2V)V + 4VA_1VA_2V + 4VA_2VA_1V. \end{aligned}$$

Lemma A.2 (Prasad and Rao, 1990)**.** *Let $y \sim N_n(0, V)$, $z_j = \lambda_j'y$ and $q_j = y'A_jy$, $j = 1, \dots, p$, where λ_j and A_j are nonstochastic of order $n\times 1$ and $n\times n$ respectively. Let $\boldsymbol{z} = (z_1, \dots, z_p)'$, $q = (q_1, \dots, q_p)'$ have covariance matrices V_z and V_q respectively. Then*

$$E[(\boldsymbol{z}'(q - E[q]))^2] = tr(V_zV_q) + 4\sum_{j=1}^{p}\sum_{i=1}^{p}\{\lambda_j'VA_jVA_iV\lambda_i + \lambda_j'VA_iVA_jV\lambda_i\},$$

$$\begin{aligned} E[z_iz_j(q_i - E[q_i])(q_j - E[q_j])] &= \lambda_i'E[y(y'A_iy)(y'A_jy)y']\lambda_j - E[q_i]\lambda_i'E[y(y'A_jy)y']\lambda_j \\ &\quad - E[q_i]\lambda_i'E[y(y'A_iy)y']\lambda_j + E[q_i]E[q_j]\lambda_i'V\lambda_j. \end{aligned}$$

Lemma A.3 (Prasad and Rao, 1990)**.** *Let us assume* (a) $V = \text{diag}_{1\le d\le D}(V_d)$, (b) $C = \text{diag}_{1\le d\le D}\big(O(D^{-1})_{R\times R}\big) + O(D^{-2})_{DR\times DR}$, (c) $r = \text{col}_{1\le d\le D}\text{col}_{1\le j\le R}\big(O(D^{-1})\big)$, (d) $s_i = \text{col}_{1\le d\le D}\text{col}_{1\le j\le R}\big(\delta_{id}O(1)\big)$, *where V_d is an $R\times R$ matrix with bounded elements. Then the following results hold:* (e) $VCVCV = O(D^{-2})_{DR\times DR}$, (f) $s_i'\sum s_i = O(1)$, (g) $(r+s_i)'VCVCV(r+s_i) = O(D^{-2})$.

Lemma 3. *Let $v \sim N(0, V)$, $s_1 = \lambda_1'v$, $s_2 = \lambda_2'v$, $q_1 = v'A_1v$ and $q_2 = v'A_2v$, where λ_i and A_i, $i = 1, 2$, are nonstochastic vectors and matrices respectively. Then*

$$E[s_1s_2(q_1 - E[q_1])(q_2 - E[q_2])] = \text{cov}(q_1, q_2)\text{cov}(s_1, s_2) + 8\lambda_1'VA_1VA_2\lambda_2.$$

Proof. By applying Lemma A.2 of Prasad and Rao (1990), we have

$$\begin{aligned} E &= E[s_1s_2(q_1 - E[q_1])(q_2 - E[q_2])] = \lambda_1'E[v(v'A_1vv'A_2v)v']\lambda_2 \\ &\quad - E[q_1]\lambda_1'E[v(v'A_2v)v']\lambda_2 - E[q_2]\lambda_1'E[v(v'A_1v)v']\lambda_2 + E[q_1]E[q_2]\lambda_1'V\lambda_2. \end{aligned}$$

By applying Lemma A.1(c) of Prasad and Rao (1990), we have

$$E[v(v'A_1vv'A_2v)v'] = \mathrm{tr}(A_1V)\mathrm{tr}(A_2V)V + 2\mathrm{tr}(A_1V)VA_2V + 2\mathrm{tr}(A_2V)VA_1V + 2\mathrm{tr}(A_1VA_2V)V + 4VA_1VA_2V + 4VA_2VA_1V.$$

By applying Lemma A.1(a) of Prasad and Rao (1990), we have

$$E[v(v'A_iv)v'] = \mathrm{tr}(A_iV)V + 2VA_iV, \quad i = 1, 2.$$

Further, $E[q_i] = \mathrm{tr}(A_iV)$, $\mathrm{cov}(q_1, q_2) = 2\mathrm{tr}(A_1VA_2V)$ and $\mathrm{cov}(s_1, s_2) = \lambda_1'V\lambda_2$. By substitution, we get

$$\begin{aligned} E &= E[q_1]E[q_2]\lambda_1'V\lambda_2 + 2E[q_1]\lambda_1'VA_2V\lambda_2 + 2E[q_2]\lambda_1'VA_1V\lambda_2 + 8\lambda_1'VA_1VA_2V\lambda_2 \\ &\quad + 2\mathrm{tr}(A_1VA_2V)\lambda_1'V\lambda_2 - E[q_1]E[q_2]\lambda_1'V\lambda_2 - 2E[q_1]\lambda_1'VA_2V\lambda_2 - E[q_1]E[q_2]\lambda_1'V\lambda_2 \\ &\quad - 2E[q_2]\lambda_1'VA_1V\lambda_2 + E[q_1]E[q_2]\lambda_1'V\lambda_2 = 2\mathrm{tr}(A_1VA_2V)\lambda_1'V\lambda_2 + 8\lambda_1'VA_1VA_2V\lambda_2 \\ &= \mathrm{cov}(q_1, q_2)\mathrm{cov}(s_1, s_2) + 8\lambda_1'VA_1VA_2\lambda_2. \end{aligned}$$

Lemma 4. *Under* H1–H4 , *it holds*

$$\mathrm{cov}(s^{(i)}, s^{(j)}) = L^{(i)}VL^{(j)\prime} + O(D^{-1})_{DR\times DR}.$$

Proof. From Lemma 1, we have that $s^{(i)} = (L^{(i)} + F^{(i)})v$, where $v \sim N(0, V)$ and

$$F^{(i)} = O(D^{-1})_{DR\times DR}, \qquad L^{(i)} = \underset{1\le d\le D}{\mathrm{diag}}\,(L_d^{(i)}), \qquad L_d^{(i)} = O(1)_{R\times R}, \quad d = 1, \dots, D, \tag{19}$$

and similarly for $s^{(j)}$. On the other hand, we have

$$\begin{aligned} \mathrm{cov}(s^{(i)}, s^{(j)}) &= (L^{(i)} + F^{(i)})V(L^{(j)} + F^{(j)})' \\ &= L^{(i)}VL^{(j)\prime} + L^{(i)}VF^{(j)\prime} + F^{(i)}VL^{(j)\prime} + F^{(i)}VF^{(j)\prime}. \end{aligned}$$

From (19), we get $F^{(i)}VF^{(j)\prime} = O(D^{-1})_{DR\times DR}$ and

$$L^{(i)}VF^{(j)\prime} = O(D^{-1})_{DR\times DR}, \qquad F^{(i)}VL^{(j)\prime} = O(D^{-1})_{DR\times DR}.$$

Therefore, $\mathrm{cov}(s^{(i)}, s^{(j)}) = L^{(i)}VL^{(j)\prime} + O(D^{-1})_{DR\times DR}$.

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