# MATA37 Week 6

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## 1 Fundamental Theorem of Calculus (part 2)

The FTOC pt.2 essentially says that all continuous functions have antiderivatives. It gives a way of constructing said antiderivative for a given continuous function $f$.

**Theorem 1.** ***Fundamental Theorem of Calculus (part 2)***
*IF $f$ is continuous on $[a, b]$,*
*Define the function $F(x)$ where $x \in [a, b]$:*

$$F(x) = \int_a^x f(t)dt$$

*THEN (a) $F$ is differentiable on $(a, b)$ and continuous on $[a, b]$*
*and (b) $F'(x) = f(x) \quad \forall x \in [a, b]$*
*i.e. $F$ is an antiderivative of $f$.*
*i.e. $\frac{\mathrm{d}}{\mathrm{d}x} F = \frac{\mathrm{d}}{\mathrm{d}x}\left( \int_a^x f(t)\, dt \right) = f(x)$*

Note: before using the FTOC, Pt.2, you must check whether your function is continuous and show that it's continuous. (You don't need to do this when using FTOC Pt.1.)

**Example 1.** Find the derivative of the following function:

$$F(x) = \int_2^x e^{\sin\left(u^2\right)}\, du$$

Sol. The function $f(u) = e^{\sin\left(u^2\right)}$ is continuous since it's the composition of the continuous functions $e^u$, $\sin(u)$, and $u^2$. Therefore we can apply the FTOC pt. 2, taking the given $F(x)$ and $f(u) = e^{\sin\left(u^2\right)}$:

$$F'(x) = \frac{\mathrm{d}}{\mathrm{d}x}\left( \int_2^x e^{\sin\left(u^2\right)}\, du \right) = f(x) = e^{\sin\left(x^2\right)}$$

## 2 Integration technique: Substitution

Let $a, b \in \mathbb{R}$, $a < b$.

**Theorem 2.** ***Substitution rule***
*IF $f$ and $g'$ are continuous functions,*
*THEN, letting $u = g(x)$, which implies $du = g'(x)\,dx$:*

$$(1) \quad \int f(g(x))g'(x)\,dx = \int f(u)du$$

$$(2) \quad \int_a^b f(g(x))g'(x)\,dx = \int_{g(a)}^{g(b)} f(u)du$$

*Note that in order to use this theorem, $f$ should be continuous in the range of $g$.*

Proof of substitution rule (definite integral):
Suppose $f$ and $g'$ are continuous on $[a, b]$.
WTS $\int_a^b f(g(x))g'(x)\,dx = \int_{g(a)}^{g(b)} f(u)\,du$
Let $F$ be an antiderivative of $f$, i.e. $F' = f$. (Since $f$ is continuous, it has an antiderivative by FTOC Pt. 2, so such an $F$ must exist.)
Want to find an antiderivative of $f(g(x))g'(x)$ in order to use FTOC Pt. 1 on the LHS.
Claim: $F(g(x))$ is an antiderivative of $f(g(x))g'(x)$ on $[a, b]$.
Proof of claim: Let $x \in [a, b]$,

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}x}F(g(x)) &= F'(g(x))g'(x) \quad \text{by chain rule} \\ &= f(g(x))g'(x) \quad \text{since } F \text{ is an antiderivative of } f \end{aligned}$$

Therefore we can apply the FTOC Pt. 1 to the LHS.

$$\begin{aligned} LHS = \int_a^b f(g(x))g'(x)\,dx &= \Big[F(g(x))\Big]_a^b \quad \text{by FTOC Pt. 1 \& } F(g(x)) \text{ is an antideriv. of } f(g(x))g'(x) \text{ by proof of claim} \\ &= F(g(b)) - F(g(a)) \end{aligned}$$

Applying FTOC Pt. 1 to the RHS:

$$\begin{aligned} RHS = \int_{g(a)}^{g(b)} f(u)\,du &= \Big[F(u)\Big]_{g(a)}^{g(b)} \quad \text{by FTOC Pt. 1 since } F \text{ is an antiderivative of } f \\ &= F(g(b)) - F(g(a)) \end{aligned}$$

Therefore $LHS = RHS = F(g(b)) - F(g(a))$. $\square$

## 3 Example questions

**Example 1. Using FTOC Pt.2**
Find the derivatives of the following functions:

$$(1) \quad F(x) = \int_0^x \frac{1}{t+1}\,dt$$
$$(2) \quad F(x) = \int_2^{3x} u^2 + u\,du$$
$$(3) \quad F(x) = \int_\pi^{\sqrt{x}} \sin(t)\,dt$$
$$(4) \quad F(x) = \int_x^{x^2} \frac{2}{s^2+2}\,ds$$
$$(5) \quad F(x) = x^2 \int_0^{x^2} e^t\,dt$$

**Example 2.** Prove that the following function is increasing when $x > 0$:

$$H(x) = \int_0^{x^2} \frac{e^u}{u^2 - 2u + 2}\,du$$

**Example 3.** Let $A(x) = \int_0^x f(t)\,dt$, where $f$ is positive, decreasing, and continuous for $x > 0$. Find $A''(x)$ to prove that $A(x)$ is concave down for $x > 0$.

**Example 4.** Find a function $f$ such that for any real number $x$,

$$\int_0^x f(t)\,dt = \frac{\cos(x)}{1+x^2} - 1$$

**Example 5.** Let $x > 0$. Show that the following expression does not depend on $x$, i.e. that it is constant with respect to $x$.

$$\int_0^x \frac{1}{1+t^4}\,dt + \frac{1}{3}\int_0^{\frac{1}{x^3}} \frac{1}{1+t^{4/3}}\,dt$$

**Example 6.** Evaluate the following integrals using substitution:

$$(1) \quad \int \tan(x)\sec^2(x)\,dx$$
$$(2) \quad \int \sqrt{x}\,dx$$
$$(3) \quad \int \frac{x}{1+x^4}\,dx$$

(4) $\displaystyle\int \sec(x)\tan(x)\sqrt{1+\sec(x)}\,dx$

(5) $\displaystyle\int_1^e \frac{\ln x}{x}\,dx$

(6) $\displaystyle\int_0^1 x^2(2^{-x^3})\,dx$

(7) $\displaystyle\int_0^{\pi/2} \cos(x)\sin(\sin(x))\,dx$

**Example 7.** Suppose $f$ is continuous on [0,1]. Prove that

$$\int_0^1 f(x)\,dx = \int_0^1 f(1-x)\,dx$$

# 4 Answers to examples

**Example 1. Using FTOC Pt.2**

Find the derivatives of the following functions:

$$
\begin{aligned}
&(1) \quad F(x) = \int_0^x \frac{1}{t+1}\,dt \\
&(2) \quad H(x) = \int_2^{3x} u^2 + u\,du \\
&(3) \quad G(x) = \int_\pi^{\sqrt{x}} \sin(t)\,dt \\
&(4) \quad \frac{\mathrm{d}}{\mathrm{d}x}\Big(\int_x^{x^2} \frac{2}{s^2+2}\,ds\Big) \\
&(5) \quad F(x) = x^2 \int_0^{x^2} e^t\,dt
\end{aligned}
$$

Sol. (1) Let $f(t) = \frac{1}{t+1}$. $f(t)$ is continuous on [0,b] for any $b > 0$. This is because it is a rational function, so it is continuous everwhere except for where its denominator=0, so its only discontinuity is at $t = -1$. Therefore we can apply the FTOC Pt.2, taking $F(x)$ as given and $f(t)$ as defined. By FTOC Pt.2,

$$F'(x) = f(x) = \frac{1}{x+1} \quad \square$$

Sol. (2) Let $f(u) = u^2 + u$. Since $f$ is a polynomial, it is continuous everywhere. **In particular,** it is continuous on $[2, 3x]$ for any real number $x$. **But the given $H$ has $3x$ as the upper limit of the integral, while FTOC Pt.2 requires the upper limit to be just $x$.** If you take $F(x)$ to be $\int_2^x u^2 + u\,du$ and $G(x) = 3x$, the given $H$ is actually the composition of $F$ and $G$, $H = F(G(x)) = F(3x)$. Since we know how to get the derivatives of $F$ and $3x$, so we can apply the chain rule. By the FTOC Pt.2:

$$
\begin{aligned}
H'(x) = \frac{\mathrm{d}}{\mathrm{d}x}[F(G(x))] &= \frac{\mathrm{d}}{\mathrm{d}x}[F(3x)] \\
&= F'(3x)\cdot(3x)' \\
\text{By FTOC Pt.2, } F'(x) = f(x) = x^2 + x, \text{ so } F'(3x) = (3x)^2 + (3x): \quad &= [(3x)^2 + (3x)]\cdot 3 \\
&= 27x^2 + 9x \quad \square
\end{aligned}
$$

Sol. (3) Let $f(t) = \sin(t)$. $f(t)$ is a trigonometric function so it is continuous over its domain (which is $\mathbb{R}$). In particular it is continuous over $[\pi, \sqrt{x}]$. Hence we can apply FTOC Pt. 2 for any real number $x$. Again, the given integral doesn't match the statement of FTOC Pt.2, since the upper limit is $\sqrt{x}$ instead of just $x$. So the given function $G(x)$ is a composition of functions: If you let $F(x) = \int_\pi^{\sqrt{x}} \sin(t)\,dt$, then you can write $G$ as $F(\sqrt{x})$ and we can apply the chain rule as above. By FTOC Pt.2:

$$
\begin{aligned}
G'(x) = \frac{\mathrm{d}}{\mathrm{d}x}F(\sqrt{x}) &= F'(\sqrt{x})\cdot(\sqrt{x})' \\
\text{By FTOC Pt.2, } F'(x) = f(x) = \sin(x), \text{ so } F'(\sqrt{x}) = \sin\big(\sqrt{x}\big): \quad &= \sin\big(\sqrt{x}\big)\cdot\frac{1}{2\sqrt{x}} \quad \square
\end{aligned}
$$

Sol. (4) Let $f(s) = \frac{2}{s^2+2}$. $f(s)$ is a rational function, so it is continuous everywhere since its denominator $s^2 + 2$ is $> 0$ for all $s \in \mathbb{R}$. In particular, $f(s)$ is continuous on $[x, x^2]$.
You need to split this integral up before you can apply FTOC Pt.2.
Let $c \in \mathbb{R}$ be a constant s.t. $c \in [x, x^2]$. By the union interval property,

$$\int_x^{x^2} \frac{2}{s^2+2}\,ds = \int_x^c \frac{2}{s^2+2}\,ds + \int_c^{x^2} \frac{2}{s^2+2}\,ds$$

By another integral property,

$$\int_x^{x^2} \frac{2}{s^2+2}\,ds = -\int_c^x \frac{2}{s^2+2}\,ds + \int_c^{x^2} \frac{2}{s^2+2}\,ds$$

Let $F(x) = \int_c^x \frac{2}{s^2+2}$, the form of function that the FTOC Pt.2 can take, so we know the derivative of this function: $F'(x) = f(x) = \frac{2}{x^2+2}$. The derivative of the whole function can be rewritten:

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}x}\Big(\int_x^{x^2} \frac{2}{s^2+2}\,ds\Big) &= \frac{\mathrm{d}}{\mathrm{d}x}\Big(-\int_c^x \frac{2}{s^2+2}\,ds + \int_c^{x^2} \frac{2}{s^2+2}\,ds\Big) \\
&= \frac{\mathrm{d}}{\mathrm{d}x}\big(-F(x) + F(x^2)\big) \\
\text{Applying the chain rule:} \quad &= -F'(x) + F'(x^2)(x^2)' \\
&= -f(x) + f(x^2)2x \\
&= -\frac{2}{x^2+2} + \frac{2}{(x^2)^2+2}(2x) \\
&= -\frac{2}{x^2+2} + \frac{4x}{x^4+2} \quad \square
\end{aligned}$$

Sol. (5) Note $e^t$ is continuous over all $t \in \mathbb{R}$, in particular it is cont. over $[0, x^2]$. Thus we can apply the FTOC Pt.2. FTOC Pt.2 requires that the upper limit of the integral be $x$, so let $G(x) = \int_0^x e^t\,dt$, and let $g(t) = e^t$. Therefore the original function can be rewritten:

$$F(x) = x^2 \int_0^{x^2} e^t\,dt = x^2 G(x^2)$$

We can apply the product rule to get the derivative.

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}x}F(x) &= \frac{\mathrm{d}}{\mathrm{d}x}\big(x^2 G(x^2)\big) \\
&= \frac{\mathrm{d}}{\mathrm{d}x}\big(x^2\big)G(x^2) + x^2\frac{\mathrm{d}}{\mathrm{d}x}\big(G(x^2)\big) \\
&= 2xG(x^2) + x^2(G'(x^2))(x^2)') \quad \text{By chain rule} \\
&= 2xG(x^2) + x^2(e^{x^2}2x) \quad \text{By FTOC Pt.2, } G'(x^2) = g(x^2) = e^{x^2} \\
&= 2x\int_0^{x^2} e^t\,dt + 2x^3 e^{x^2}
\end{aligned}$$

**Example 2.** Prove that the following function is increasing when $x > 0$:

$$H(x) = \int_0^{x^2} \frac{e^u}{u^2 - 2u + 2}\, du$$

A function is increasing if its derivative is $> 0$. Want to show $H'(x) > 0$ for $x > 0$.
Sol. First, let $f(u) = \frac{e^u}{u^2-2u+2}$. Show this function is continuous. $f(u)$ is the product of the functions $e^u$ and $\frac{1}{u^2-2u+2}$.

- $e^u$ is an exponential function so it is cont. everywhere
- $\frac{1}{u^2-2u+2}$ is a rational function, so it is cont. everywhere, since $u^2-2u+2 = (u-1)^2+1$ and $(u-1)^2 \geq 0$ for all $u$, so $(u-1)^2+1 > 0$ for all $u \in \mathbb{R}$.
- The product of two functions cont. on $\mathbb{R}$ is cont. on $\mathbb{R}$.
- Thus $f$ is cont. on $\mathbb{R}$, so in particular it is cont. on $[0, x^2]$ where $x > 0$.

By the FTOC Pt.2, taking $F(x) = \int_0^x \frac{e^u}{u^2-2u+2}\, du$ and $f(u) = \frac{e^u}{u^2-2u+2}$, and applying the chain rule:

$$\begin{aligned} H'(x) &= F'(x^2)(x^2)' \\ &= f(x^2)(2x) \\ &= \frac{e^{x^2}}{(x^2)^2 - 2x^2 + 2}(2x) \\ &= \frac{2xe^{x^2}}{(x^2-1)^2+1} \end{aligned}$$

Since $(x^2-1)^2$ is always $> 0$, for all $x$, the denominator $(x^2-1)^2 > 0$ for all $x$, in particular, when $x > 0$.
As well, $e^{x^2} > 0$ for all $x$, and $2x > 0$ when $x > 0$, so the numerator $2xe^{x^2} > 0$ for all $x > 0$.
Since the numerator and denominator are both positive whenever $x > 0$, we can conclude that $H'(x) = \frac{2xe^{x^2}}{(x^2-1)^2+1} > 0$ for $x > 0$, i.e., $H(x)$ is increasing when $x > 0$.

**Example 3.** Let $A(x) = \int_0^x f(t)\, dt$, where $f$ is continuous, positive, and decreasing for $x > 0$. Find $A''(x)$ to prove that $A(x)$ is concave down for $x > 0$.

Sol. Recall from A31 that $A(x)$ is concave down for $x > 0$ if its second derivative, $A''(x) < 0$ when $x > 0$. Let $f$ be a continuous function that's positive and decreasing when $x > 0$. Since $f$ is continuous, we can apply the FTOC Pt.2 to $A(x) = \int_0^x f(t)\, dt$ and find its derivative.
By FTOC Pt.2:

$$A'(x) = f(x)$$

Furthermore, we can find its second derivative:

$$A''(x) = f'(x)$$

Since $f$ is decreasing when $x > 0$, we know $f'(x) < 0$ when $x > 0$. It follows that $A''(x) < 0$ when $x > 0$, since $A''(x) = f'(x)$. Therefore $A''(x)$ is concave down when $x > 0$.

**Example 4.** Let $x > 0$. Show that the following expression does not depend on $x$, i.e. that it is constant with respect to $x$.

$$\int_0^x \frac{1}{1+t^4}\,dt + \frac{1}{3}\int_0^{\frac{1}{x^3}} \frac{1}{1+t^{4/3}}\,dt$$

Sol. Recall that if a function $f$ is constant with respect to $x$, its derivative taken with respect to $x$ is 0, i.e. $\frac{\mathrm{d}}{\mathrm{d}x} f = 0$. Thus we just need to take the derivative of the given expression and show that it evaluates to 0.

To take the derivative, we can apply FTOC Pt. 2.

$$\frac{\mathrm{d}}{\mathrm{d}x}\Big(\int_0^x \frac{1}{1+t^4}\,dt + \frac{1}{3}\int_0^{\frac{1}{x^3}} \frac{1}{1+t^{4/3}}\,dt\Big) = \frac{\mathrm{d}}{\mathrm{d}x}\Big(\int_0^x \frac{1}{1+t^4}\,dt\Big) + \frac{1}{3}\frac{\mathrm{d}}{\mathrm{d}x}\Big(\int_0^{\frac{1}{x^3}} \frac{1}{1+t^{4/3}}\,dt\Big)$$

We'll calculate each derivative separately.

(1) Since $f(t) = \frac{1}{1+t^4}$ is continuous everywhere (because $1 + t^4 > 0$ for all $t \in \mathbb{R}$, we can apply FTOC Pt. 2:

$$\frac{\mathrm{d}}{\mathrm{d}x}\Big(\int_0^x \frac{1}{1+t^4}\,dt\Big) = \frac{1}{1+x^4} \quad \text{By FTOC Pt. 2}$$

(2) Since $x > 0$, $1 + t^{4/3} > 0$ for all $t \in [0, 1/x^3]$. Thus, $\frac{1}{1+t^{4/3}}$ is continuous on $[0, 1/x^3]$. We can apply FTOC Pt. 2. Take $F(x)$ to be the function $\int_0^x \frac{1}{1+t^{4/3}}\,dt$. Then the integral in question, $\int_0^{\frac{1}{x^3}} \frac{1}{1+t^{4/3}}\,dt$, can be written as $F(1/x^3)$, equivalently $F(x^{-3})$.

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}x}\int_0^{x^{-3}} \frac{1}{1+t^{4/3}}\,dt &= \frac{\mathrm{d}}{\mathrm{d}x}F(x^{-3}) \\
&= F'(x^{-3}) \cdot \Big(x^{-3}\Big)' \quad \text{By chain rule} \\
&= \frac{1}{1+(x^{-3})^{4/3}} \cdot (-3x^{-4}) \quad \text{By FTOC Pt. 2, } F'(x) = \frac{1}{1+t^{4/3}}\text{. Therefore } F'(x^{-3}) = \frac{1}{1+(x^{-3})^{4/3}} \\
&= \frac{-3x^{-4}}{1+x^{-4}} \\
&= \frac{-3}{x^4(1+x^{-4})} \\
&= \frac{-3}{x^4+1}
\end{aligned}$$

Therefore, the original expression simplifies to:

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}x}\Big(\int_0^x \frac{1}{1+t^4}\,dt + \frac{1}{3}\int_0^{\frac{1}{x^3}} \frac{1}{1+t^{4/3}}\,dt\Big) &= \frac{\mathrm{d}}{\mathrm{d}x}\Big(\int_0^x \frac{1}{1+t^4}\,dt\Big) + \frac{1}{3}\frac{\mathrm{d}}{\mathrm{d}x}\Big(\int_0^{\frac{1}{x^3}} \frac{1}{1+t^{4/3}}\,dt\Big) \\
&= \frac{1}{1+x^4} + \frac{1}{3}\frac{-3}{x^4+1}
\end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{1+x^4} - \frac{1}{x^4+1} \\
&= 0
\end{aligned}
$$

Therefore the derivative of the expression with respect to $x$ is 0, hence the expression is constant with respect to $x$. $\square$

**Example 5.** Evaluate the following integrals using substitution:

$$
\begin{aligned}
&(1) \quad \int \tan(x)\sec^2(x)\,dx \\
&(2) \quad \int x^2\sqrt{x+2}\,dx \\
&(3) \quad \int \frac{x}{1+x^4}\,dx \\
&(4) \quad \int \sec(x)\tan(x)\sqrt{1+\sec(x)}\,dx \\
&(5) \quad \int_1^e \frac{\ln x}{x}\,dx \\
&(6) \quad \int_0^1 x^2(2^{-x^3})\,dx \\
&(7) \quad \int_0^{\pi/2} \cos(x)\sin(\sin(x))\,dx
\end{aligned}
$$

Sol. (1) Let $u = \tan(x)$. Then

$$
\begin{aligned}
u &= \tan(x) \\
du &= \sec^2(x)\,dx
\end{aligned}
$$

$$
\begin{aligned}
\text{Thus, } \int \tan(x)\sec^2(x)\,dx &= \int u\,du \\
&= \frac{1}{2}u^2 + C \\
&= \frac{1}{2}\tan^2(x) + C \quad \square
\end{aligned}
$$

(2) Let $u = \sqrt{x+2}$. Then

$$
\begin{aligned}
u &= \sqrt{x+2} \\
du &= \frac{1}{2\sqrt{x+2}}\,dx \\
&= \frac{1}{2u}\,dx \\
2u\,du &= dx
\end{aligned}
$$

Note also that $u = \sqrt{x}$ implies $u^4 = x^2$

$$\begin{aligned}\int x^2\sqrt{x+2}\,dx &= \int u^4(2u\,du)\\ &= \int 2u^5\,du\\ &= \frac{2}{6}u^6 + C\\ &= \frac{1}{3}(\sqrt{x+2})^6 + C\\ &= \frac{1}{3}(x+2)^3 + C \quad \square\end{aligned}$$

(3) Recall that an antiderivative of $\frac{1}{1+x^2}$ is $\arctan x$. The integrand is in a similar form, except its denominator is $1 + x^4$, but this can be written as $1 + (x^2)^2$. Let $u = x^2$. Then

$$\begin{aligned}u &= x^2\\ du &= 2x\,dx\\ \frac{1}{2}\,du &= x\,dx\end{aligned}$$

$$\begin{aligned}\int \frac{x}{1+x^4}\,dx &= \int \frac{1}{1+(x^2)^2}x\,dx\\ &= \int \frac{1}{1+u^2}\frac{1}{2}\,du\\ &= \frac{1}{2}\int \frac{1}{1+u^2}\,du\\ &= \frac{1}{2}\arctan(u) + C\\ &= \frac{1}{2}\arctan\big(x^2\big) + C\end{aligned}$$

(4) Let $u = \sec(x)$. Then

$$\begin{aligned}u &= \sec(x)\\ du &= \sec(x)\tan(x)\,dx\end{aligned}$$

$$\int \sec(x)\tan(x)\sqrt{1+\sec(x)} = \int \sqrt{1+u}\,du$$

Let $v = \sqrt{1+u}$. Then you have

$$\begin{aligned}v &= \sqrt{1+u}\\ dv &= \frac{1}{2\sqrt{1+u}}\,du\\ dv &= \frac{1}{2v}\,du\\ 2v\,dv &= du\end{aligned}$$

$$\begin{aligned}\int \sqrt{1+u}\,du &= \int v\cdot 2v\,dv\\ &= \int 2v^2\,dv\\ &= \frac{2}{3}v^3 + C\\ &= \frac{2}{3}(\sqrt{1+u})^3 + C\\ &= \frac{2}{3}(\sqrt{1+\sec(x)})^3 + C \quad \square\end{aligned}$$

(5) Let $u = \ln(x)$. When $x = 1$, $u = 0$, and when $x = e$, $u = 1$. So the new integration limits are 0 to 1.

$$\begin{aligned}u &= \ln x\\ du &= \frac{1}{x}\,dx\end{aligned}$$

$$\begin{aligned}\int_1^e \frac{\ln x}{x}\,dx &= \int_1^e \ln x \frac{1}{x}\,dx\\ &= \int_0^1 u\,du\\ &= \left[\frac{1}{2}u^2\right]_0^1\\ &= \frac{1}{2}\end{aligned}$$

(6) Let $u = -x^3$. Then $du = -3x^2\,dx$ and when $x = 0$, $u = 0$, and when $x = 1$, $u = -1$.

$$\begin{aligned}\int_0^1 x^2(2^{-x^3})\,dx &= \int_0^{-1} -\frac{1}{3}2^u\,du\\ &= -\frac{1}{3}\int_0^{-1} 2^u\,du\\ &= \frac{1}{3}\int_{-1}^0 2^u\,du\\ &= \frac{1}{3}\left[\frac{2^u}{\ln(2)}\right]_{-1}^0\\ &= \frac{1}{3}\left(\frac{2^0}{\ln(2)} - \frac{2^{-1}}{\ln(2)}\right)\\ &= \frac{1}{3}\left(\frac{1}{\ln(2)} - \frac{1}{2\ln(2)}\right)\\ &= \frac{1}{3}\left(\frac{1}{2\ln(2)}\right)\\ &= \frac{1}{6\ln(2)}\end{aligned}$$

(7) Let $u = \sin(x)$. Then when $x = 0$, $u = 0$, and when $x = \pi/2$, $u = 1$.

$$\begin{aligned} u &= \sin(x) \\ du &= \cos(x) \end{aligned}$$

$$\begin{aligned} \int_0^{\pi/2} \cos(x)\sin(\sin(x))\,dx &= \int_0^1 \sin(u)\,du \\ &= \big[-\cos(u)\big]_0^1 \\ &= -\cos(1) + \cos(0) \\ &= 1 - \cos(1) \quad \square \end{aligned}$$

**Example 6.** Suppose $f$ is continuous on $\mathbb{R}$. Prove that

$$\int_0^1 f(x)\,dx = \int_0^1 f(1-x)\,dx$$

Sol. First, since $f$ is cont., we know it is integrable. We can apply substitution to evaluate the integral on the right. Let $u = 1 - x$. Then $du = -\,dx$. Also, when $x = 0$, $u = 1$, and when $x = 1$, $u = 0$.

$$\begin{aligned} \int_0^1 f(1-x)\,dx &= \int_1^0 f(u) - du \\ &= -\int_1^0 f(u)\,du \\ &= -\Big(-\int_0^1 f(u)\,du\Big) \\ &= \int_0^1 f(u)\,du \end{aligned}$$

We also have that $\int_0^1 f(u)\,du = \int_0^1 f(x)\,dx$. This is an integral property. The variable of integration doesn't matter as long as the integrand function is the same. Hence, the right side $= \int_0^1 f(x)\,dx =$ the left side. $\square$