MATA37 Week 6

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1 Fundamental Theorem of Calculus (part 2)

The FTOC pt.2 essentially says that all continuous functions have antiderivatives. It gives a way of constructing said antiderivative for a given continuous function f.

Theorem 1. ***Fundamental Theorem of Calculus (part 2)***
IF f is continuous on $[a, b]$,
Define the function $F(x)$ where $x \in [a, b]$:

$$F(x) = \int_a^x f(t)dt$$

THEN (a) F is differentiable on (a, b) and continuous on $[a, b]$
and (b) $F'(x) = f(x) \quad \forall x \in [a, b]$
i.e. F is an antiderivative of f.
i.e. $\frac{\mathrm{d}}{\mathrm{d}x}F = \frac{\mathrm{d}}{\mathrm{d}x}\left(\int_a^x f(t)\,dt\right) = f(x)$

Note: before using the FTOC, Pt.2, you must check whether your function is continuous and show that it's continuous. (You don't need to do this when using FTOC Pt.1.)

Example 1. Find the derivative of the following function:

$$F(x) = \int_2^x e^{\sin\left(u^2\right)}\,du$$

Sol. The function $f(u) = e^{\sin\left(u^2\right)}$ is continuous since it's the composition of the continuous functions e^u, $\sin(u)$, and u^2. Therefore we can apply the FTOC pt. 2, taking the given $F(x)$ and $f(u) = e^{\sin\left(u^2\right)}$:

$$F'(x) = \frac{\mathrm{d}}{\mathrm{d}x}\left(\int_2^x e^{\sin\left(u^2\right)}\,du\right) = f(x) = e^{\sin\left(x^2\right)}$$

2 Integration technique: Substitution

Let $a, b \in \mathbb{R}$, $a < b$.

Theorem 2. ***Substitution rule***
IF f and g' are continuous functions,
THEN, letting $u = g(x)$, which implies $du = g'(x)\,dx$:

$$(1) \quad \int f(g(x))g'(x)\,dx = \int f(u)du$$

$$(2) \quad \int_a^b f(g(x))g'(x)\,dx = \int_{g(a)}^{g(b)} f(u)du$$

Note that in order to use this theorem, f should be continuous in the range of g.

Proof of substitution rule (definite integral):
Suppose f and g' are continuous on $[a, b]$.
WTS $\int_a^b f(g(x))g'(x)\,dx = \int_{g(a)}^{g(b)} f(u)\,du$
Let F be an antiderivative of f, i.e. $F' = f$. (Since f is continuous, it has an antiderivative by FTOC Pt. 2, so such an F must exist.)
Want to find an antiderivative of $f(g(x))g'(x)$ in order to use FTOC Pt. 1 on the LHS.
Claim: $F(g(x))$ is an antiderivative of $f(g(x))g'(x)$ on $[a, b]$.
Proof of claim: Let $x \in [a, b]$,

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}x}F(g(x)) &= F'(g(x))g'(x) \quad \text{by chain rule} \\ &= f(g(x))g'(x) \quad \text{since } F \text{ is an antiderivative of } f\end{aligned}$$

Therefore we can apply the FTOC Pt. 1 to the LHS.

$$\begin{aligned}LHS = \int_a^b f(g(x))g'(x)\,dx &= \Big[F(g(x))\Big]_a^b \quad \text{by FTOC Pt. 1 \& } F(g(x)) \text{ is an antideriv. of } f(g(x))g'(x) \text{ by proof of claim} \\ &= F(g(b)) - F(g(a))\end{aligned}$$

Applying FTOC Pt. 1 to the RHS:

$$\begin{aligned}RHS = \int_{g(a)}^{g(b)} f(u)\,du &= \Big[F(u)\Big]_{g(a)}^{g(b)} \quad \text{by FTOC Pt. 1 since } F \text{ is an antiderivative of } f \\ &= F(g(b)) - F(g(a))\end{aligned}$$

Therefore $LHS = RHS = F(g(b)) - F(g(a))$. $\square$

3 Example questions

Example 1. Using FTOC Pt.2
Find the derivatives of the following functions:

$$(1) \quad F(x) = \int_0^x \frac{1}{t+1}\,dt$$
$$(2) \quad F(x) = \int_2^{3x} u^2 + u\,du$$
$$(3) \quad F(x) = \int_\pi^{\sqrt{x}} \sin(t)\,dt$$
$$(4) \quad F(x) = \int_x^{x^2} \frac{2}{s^2+2}\,ds$$
$$(5) \quad F(x) = x^2 \int_0^{x^2} e^t\,dt$$

Example 2. Prove that the following function is increasing when $x > 0$:

$$H(x) = \int_0^{x^2} \frac{e^u}{u^2 - 2u + 2}\,du$$

Example 3. Let $A(x) = \int_0^x f(t)\,dt$, where f is positive, decreasing, and continuous for $x > 0$. Find $A''(x)$ to prove that $A(x)$ is concave down for $x > 0$.

Example 4. Find a function f such that for any real number x,

$$\int_0^x f(t)\,dt = \frac{\cos(x)}{1+x^2} - 1$$

Example 5. Let $x > 0$. Show that the following expression does not depend on x, i.e. that it is constant with respect to x.

$$\int_0^x \frac{1}{1+t^4}\,dt + \frac{1}{3}\int_0^{\frac{1}{x^3}} \frac{1}{1+t^{4/3}}\,dt$$

Example 6. Evaluate the following integrals using substitution:

$$(1) \quad \int \tan(x)\sec^2(x)\,dx$$
$$(2) \quad \int \sqrt{x}\,dx$$
$$(3) \quad \int \frac{x}{1+x^4}\,dx$$

(4) $\displaystyle\int \sec(x)\tan(x)\sqrt{1+\sec(x)}\,dx$

(5) $\displaystyle\int_1^e \frac{\ln x}{x}\,dx$

(6) $\displaystyle\int_0^1 x^2(2^{-x^3})\,dx$

(7) $\displaystyle\int_0^{\pi/2} \cos(x)\sin(\sin(x))\,dx$

Example 7. Suppose f is continuous on [0,1]. Prove that

$$\int_0^1 f(x)\,dx = \int_0^1 f(1-x)\,dx$$

4 Answers to examples

Example 1. Using FTOC Pt.2
Find the derivatives of the following functions:

$$(1)\quad F(x) = \int_0^x \frac{1}{t+1}\,dt$$
$$(2)\quad H(x) = \int_2^{3x} u^2 + u\,du$$
$$(3)\quad G(x) = \int_\pi^{\sqrt{x}} \sin(t)\,dt$$
$$(4)\quad \frac{\mathrm{d}}{\mathrm{d}x}\Big(\int_x^{x^2} \frac{2}{s^2+2}\,ds\Big)$$
$$(5)\quad F(x) = x^2 \int_0^{x^2} e^t\,dt$$

Sol. (1) Let $f(t) = \frac{1}{t+1}$. $f(t)$ is continuous on [0,b] for any $b > 0$. This is because it is a rational function, so it is continuous everwhere except for where its denominator=0, so its only discontinuity is at $t = -1$. Therefore we can apply the FTOC Pt.2, taking $F(x)$ as given and $f(t)$ as defined. By FTOC Pt.2,

$$F'(x) = f(x) = \frac{1}{x+1} \quad \square$$

Sol. (2) Let $f(u) = u^2 + u$. Since f is a polynomial, it is continuous everywhere. **In particular,** it is continuous on $[2, 3x]$ for any real number x. **But the given H has $3x$ as the upper limit of the integral, while FTOC Pt.2 requires the upper limit to be just x.** If you take $F(x)$ to be $\int_2^x u^2 + u\,du$ and $G(x) = 3x$, the given H is actually the composition of F and G, $H = F(G(x)) = F(3x)$. Since we know how to get the derivatives of F and $3x$, so we can apply the chain rule. By the FTOC Pt.2:

$$\begin{aligned} H'(x) = \frac{\mathrm{d}}{\mathrm{d}x}[F(G(x))] &= \frac{\mathrm{d}}{\mathrm{d}x}[F(3x)] \\ &= F'(3x)\cdot(3x)' \\ \text{By FTOC Pt.2, } F'(x) = f(x) = x^2 + x\text{, so } F'(3x) = (3x)^2 + (3x)\text{:} \quad &= [(3x)^2 + (3x)]\cdot 3 \\ &= 27x^2 + 9x \quad \square \end{aligned}$$

Sol. (3) Let $f(t) = \sin(t)$. $f(t)$ is a trigonometric function so it is continuous over its domain (which is $\mathbb{R}$). In particular it is continuous over $[\pi, \sqrt{x}]$. Hence we can apply FTOC Pt. 2 for any real number x. Again, the given integral doesn't match the statement of FTOC Pt.2, since the upper limit is $\sqrt{x}$ instead of just x. So the given function $G(x)$ is a composition of functions: If you let $F(x) = \int_\pi^{\sqrt{x}} \sin(t)\,dt$, then you can write G as $F(\sqrt{x})$ and we can apply the chain rule as above. By FTOC Pt.2:

$$\begin{aligned} G'(x) = \frac{\mathrm{d}}{\mathrm{d}x}F(\sqrt{x}) &= F'(\sqrt{x})\cdot(\sqrt{x})' \\ \text{By FTOC Pt.2, } F'(x) = f(x) = \sin(x)\text{, so } F'(\sqrt{x}) = \sin\big(\sqrt{x}\big)\text{:} \quad &= \sin\big(\sqrt{x}\big)\cdot\frac{1}{2\sqrt{x}} \quad \square \end{aligned}$$

Sol. (4) Let $f(s) = \frac{2}{s^2+2}$. $f(s)$ is a rational function, so it is continuous everywhere since its denominator s^2+2 is > 0 for all $s \in \mathbb{R}$. In particular, $f(s)$ is continuous on $[x, x^2]$.
You need to split this integral up before you can apply FTOC Pt.2.
Let $c \in \mathbb{R}$ be a constant s.t. $c \in [x, x^2]$. By the union interval property,

$$\int_x^{x^2} \frac{2}{s^2+2}\,ds = \int_x^c \frac{2}{s^2+2}\,ds + \int_c^{x^2} \frac{2}{s^2+2}\,ds$$

By another integral property,

$$\int_x^{x^2} \frac{2}{s^2+2}\,ds = -\int_c^x \frac{2}{s^2+2}\,ds + \int_c^{x^2} \frac{2}{s^2+2}\,ds$$

Let $F(x) = \int_c^x \frac{2}{s^2+2}$, the form of function that the FTOC Pt.2 can take, so we know the derivative of this function: $F'(x) = f(x) = \frac{2}{x^2+2}$. The derivative of the whole function can be rewritten:

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}x}\Big(\int_x^{x^2} \frac{2}{s^2+2}\,ds\Big) &= \frac{\mathrm{d}}{\mathrm{d}x}\Big(-\int_c^x \frac{2}{s^2+2}\,ds + \int_c^{x^2} \frac{2}{s^2+2}\,ds\Big) \\
&= \frac{\mathrm{d}}{\mathrm{d}x}\big(-F(x) + F(x^2)\big) \\
\text{Applying the chain rule:}\quad &= -F'(x) + F'(x^2)(x^2)' \\
&= -f(x) + f(x^2)2x \\
&= -\frac{2}{x^2+2} + \frac{2}{(x^2)^2+2}(2x) \\
&= -\frac{2}{x^2+2} + \frac{4x}{x^4+2} \quad \square
\end{aligned}$$

Sol. (5) Note e^t is continuous over all $t \in \mathbb{R}$, in particular it is cont. over $[0, x^2]$. Thus we can apply the FTOC Pt.2. FTOC Pt.2 requires that the upper limit of the integral be x, so let $G(x) = \int_0^x e^t\,dt$, and let $g(t) = e^t$. Therefore the original function can be rewritten:

$$F(x) = x^2 \int_0^{x^2} e^t\,dt = x^2 G(x^2)$$

We can apply the product rule to get the derivative.

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}x}F(x) &= \frac{\mathrm{d}}{\mathrm{d}x}\big(x^2 G(x^2)\big) \\
&= \frac{\mathrm{d}}{\mathrm{d}x}\big(x^2\big)G(x^2) + x^2\frac{\mathrm{d}}{\mathrm{d}x}\big(G(x^2)\big) \\
&= 2xG(x^2) + x^2(G'(x^2))(x^2)') \quad \text{By chain rule} \\
&= 2xG(x^2) + x^2(e^{x^2}2x) \quad \text{By FTOC Pt.2, } G'(x^2) = g(x^2) = e^{x^2} \\
&= 2x\int_0^{x^2} e^t\,dt + 2x^3 e^{x^2}
\end{aligned}$$

Example 2. Prove that the following function is increasing when $x > 0$:

$$H(x) = \int_0^{x^2} \frac{e^u}{u^2 - 2u + 2}\, du$$

A function is increasing if its derivative is > 0. Want to show $H'(x) > 0$ for $x > 0$.
Sol. First, let $f(u) = \frac{e^u}{u^2-2u+2}$. Show this function is continuous. $f(u)$ is the product of the functions e^u and $\frac{1}{u^2-2u+2}$.

- e^u is an exponential function so it is cont. everywhere
- $\frac{1}{u^2-2u+2}$ is a rational function, so it is cont. everywhere, since $u^2-2u+2 = (u-1)^2+1$ and $(u-1)^2 \geq 0$ for all u, so $(u-1)^2 + 1 > 0$ for all $u \in \mathbb{R}$.
- The product of two functions cont. on $\mathbb{R}$ is cont. on $\mathbb{R}$.
- Thus f is cont. on $\mathbb{R}$, so in particular it is cont. on $[0, x^2]$ where $x > 0$.

By the FTOC Pt.2, taking $F(x) = \int_0^x \frac{e^u}{u^2-2u+2}\, du$ and $f(u) = \frac{e^u}{u^2-2u+2}$, and applying the chain rule:

$$\begin{aligned} H'(x) &= F'(x^2)(x^2)' \\ &= f(x^2)(2x) \\ &= \frac{e^{x^2}}{(x^2)^2 - 2x^2 + 2}(2x) \\ &= \frac{2xe^{x^2}}{(x^2-1)^2 + 1} \end{aligned}$$

Since $(x^2-1)^2$ is always > 0, for all x, the denominator $(x^2-1)^2 > 0$ for all x, in particular, when $x > 0$.
As well, $e^{x^2} > 0$ for all x, and $2x > 0$ when $x > 0$, so the numerator $2xe^{x^2} > 0$ for all $x > 0$.
Since the numerator and denominator are both positive whenever $x > 0$, we can conclude that $H'(x) = \frac{2xe^{x^2}}{(x^2-1)^2+1} > 0$ for $x > 0$, i.e., $H(x)$ is increasing when $x > 0$.

Example 3. Let $A(x) = \int_0^x f(t)\, dt$, where f is continuous, positive, and decreasing for $x > 0$. Find $A''(x)$ to prove that $A(x)$ is concave down for $x > 0$.

Sol. Recall from A31 that $A(x)$ is concave down for $x > 0$ if its second derivative, $A''(x) < 0$ when $x > 0$.
Let f be a continuous function that's positive and decreasing when $x > 0$. Since f is continuous, we can apply the FTOC Pt.2 to $A(x) = \int_0^x f(t)\, dt$ and find its derivative.
By FTOC Pt.2:

$$A'(x) = f(x)$$

Furthermore, we can find its second derivative:

$$A''(x) = f'(x)$$

Since f is decreasing when $x > 0$, we know $f'(x) < 0$ when $x > 0$. It follows that $A''(x) < 0$ when $x > 0$, since $A''(x) = f'(x)$. Therefore $A''(x)$ is concave down when $x > 0$.

Example 4. Let $x > 0$. Show that the following expression does not depend on x, i.e. that it is constant with respect to x.

$$\int_0^x \frac{1}{1+t^4}\,dt + \frac{1}{3}\int_0^{\frac{1}{x^3}} \frac{1}{1+t^{4/3}}\,dt$$

Sol. Recall that if a function f is constant with respect to x, its derivative taken with respect to x is 0, i.e. $\frac{\mathrm{d}}{\mathrm{d}x}f = 0$. Thus we just need to take the derivative of the given expression and show that it evaluates to 0.
To take the derivative, we can apply FTOC Pt. 2.

$$\frac{\mathrm{d}}{\mathrm{d}x}\Big(\int_0^x \frac{1}{1+t^4}\,dt + \frac{1}{3}\int_0^{\frac{1}{x^3}} \frac{1}{1+t^{4/3}}\,dt\Big) = \frac{\mathrm{d}}{\mathrm{d}x}\Big(\int_0^x \frac{1}{1+t^4}\,dt\Big) + \frac{1}{3}\frac{\mathrm{d}}{\mathrm{d}x}\Big(\int_0^{\frac{1}{x^3}} \frac{1}{1+t^{4/3}}\,dt\Big)$$

We'll calculate each derivative separately.
(1) Since $f(t) = \frac{1}{1+t^4}$ is continuous everywhere (because $1 + t^4 > 0$ for all $t \in \mathbb{R}$, we can apply FTOC Pt. 2:

$$\frac{\mathrm{d}}{\mathrm{d}x}\Big(\int_0^x \frac{1}{1+t^4}\,dt\Big) = \frac{1}{1+x^4} \quad \text{By FTOC Pt. 2}$$

(2) Since $x > 0$, $1 + t^{4/3} > 0$ for all $t \in [0, 1/x^3]$. Thus, $\frac{1}{1+t^{4/3}}$ is continuous on $[0, 1/x^3]$. We can apply FTOC Pt. 2. Take $F(x)$ to be the function $\int_0^x \frac{1}{1+t^{4/3}}\,dt$. Then the integral in question, $\int_0^{\frac{1}{x^3}} \frac{1}{1+t^{4/3}}\,dt$, can be written as $F(1/x^3)$, equivalently $F(x^{-3})$.

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}x}\int_0^{x^{-3}} \frac{1}{1+t^{4/3}}\,dt &= \frac{\mathrm{d}}{\mathrm{d}x}F(x^{-3}) \\
&= F'(x^{-3}) \cdot \left(x^{-3}\right)' \quad \text{By chain rule} \\
&= \frac{1}{1+(x^{-3})^{4/3}} \cdot (-3x^{-4}) \quad \text{By FTOC Pt. 2, } F'(x) = \frac{1}{1+t^{4/3}}\text{. Therefore } F'(x^{-3}) = \frac{1}{1+(x^{-3})^{4/3}} \\
&= \frac{-3x^{-4}}{1+x^{-4}} \\
&= \frac{-3}{x^4(1+x^{-4})} \\
&= \frac{-3}{x^4+1}
\end{aligned}$$

Therefore, the original expression simplifies to:

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}x}\Big(\int_0^x \frac{1}{1+t^4}\,dt + \frac{1}{3}\int_0^{\frac{1}{x^3}} \frac{1}{1+t^{4/3}}\,dt\Big) &= \frac{\mathrm{d}}{\mathrm{d}x}\Big(\int_0^x \frac{1}{1+t^4}\,dt\Big) + \frac{1}{3}\frac{\mathrm{d}}{\mathrm{d}x}\Big(\int_0^{\frac{1}{x^3}} \frac{1}{1+t^{4/3}}\,dt\Big) \\
&= \frac{1}{1+x^4} + \frac{1}{3}\frac{-3}{x^4+1}
\end{aligned}$$

$$= \frac{1}{1+x^4} - \frac{1}{x^4+1}$$
$$= 0$$

Therefore the derivative of the expression with respect to x is 0, hence the expression is constant with respect to x. $\square$

Example 5. Evaluate the following integrals using substitution:

(1) $\displaystyle\int \tan(x)\sec^2(x)\,dx$

(2) $\displaystyle\int x^2\sqrt{x+2}\,dx$

(3) $\displaystyle\int \frac{x}{1+x^4}\,dx$

(4) $\displaystyle\int \sec(x)\tan(x)\sqrt{1+\sec(x)}\,dx$

(5) $\displaystyle\int_1^e \frac{\ln x}{x}\,dx$

(6) $\displaystyle\int_0^1 x^2(2^{-x^3})\,dx$

(7) $\displaystyle\int_0^{\pi/2} \cos(x)\sin(\sin(x))\,dx$

Sol. (1) Let $u = \tan(x)$. Then

$$u = \tan(x)$$
$$du = \sec^2(x)\,dx$$

$$\text{Thus, } \int \tan(x)\sec^2(x)\,dx = \int u\,du$$
$$= \frac{1}{2}u^2 + C$$
$$= \frac{1}{2}\tan^2(x) + C \quad \square$$

(2) Let $u = \sqrt{x+2}$. Then

$$u = \sqrt{x+2}$$
$$du = \frac{1}{2\sqrt{x+2}}\,dx$$
$$= \frac{1}{2u}\,dx$$
$$2u\,du = dx$$

Note also that $u = \sqrt{x}$ implies $u^4 = x^2$

$$\begin{aligned}\int x^2\sqrt{x+2}\,dx &= \int u^4(2u\,du)\\ &= \int 2u^5\,du\\ &= \frac{2}{6}u^6 + C\\ &= \frac{1}{3}(\sqrt{x+2})^6 + C\\ &= \frac{1}{3}(x+2)^3 + C \quad \square\end{aligned}$$

(3) Recall that an antiderivative of $\frac{1}{1+x^2}$ is $\arctan x$. The integrand is in a similar form, except its denominator is $1 + x^4$, but this can be written as $1 + (x^2)^2$. Let $u = x^2$. Then

$$\begin{aligned}u &= x^2\\ du &= 2x\,dx\\ \frac{1}{2}\,du &= x\,dx\end{aligned}$$

$$\begin{aligned}\int \frac{x}{1+x^4}\,dx &= \int \frac{1}{1+(x^2)^2}x\,dx\\ &= \int \frac{1}{1+u^2}\frac{1}{2}\,du\\ &= \frac{1}{2}\int \frac{1}{1+u^2}\,du\\ &= \frac{1}{2}\arctan(u) + C\\ &= \frac{1}{2}\arctan\left(x^2\right) + C\end{aligned}$$

(4) Let $u = \sec(x)$. Then

$$\begin{aligned}u &= \sec(x)\\ du &= \sec(x)\tan(x)\,dx\end{aligned}$$

$$\int \sec(x)\tan(x)\sqrt{1+\sec(x)} = \int \sqrt{1+u}\,du$$

Let $v = \sqrt{1+u}$. Then you have

$$\begin{aligned}v &= \sqrt{1+u}\\ dv &= \frac{1}{2\sqrt{1+u}}\,du\\ dv &= \frac{1}{2v}\,du\\ 2v\,dv &= du\end{aligned}$$

$$\begin{aligned}\int \sqrt{1+u}\,du &= \int v \cdot 2v\,dv \\ &= \int 2v^2\,dv \\ &= \frac{2}{3}v^3 + C \\ &= \frac{2}{3}(\sqrt{1+u})^3 + C \\ &= \frac{2}{3}(\sqrt{1+\sec(x)})^3 + C \quad \square\end{aligned}$$

(5) Let $u = \ln(x)$. When $x = 1$, $u = 0$, and when $x = e$, $u = 1$. So the new integration limits are 0 to 1.

$$\begin{aligned}u &= \ln x \\ du &= \frac{1}{x}\,dx\end{aligned}$$

$$\begin{aligned}\int_1^e \frac{\ln x}{x}\,dx &= \int_1^e \ln x \frac{1}{x}\,dx \\ &= \int_0^1 u\,du \\ &= \left[\frac{1}{2}u^2\right]_0^1 \\ &= \frac{1}{2}\end{aligned}$$

(6) Let $u = -x^3$. Then $du = -3x^2\,dx$ and when $x = 0$, $u = 0$, and when $x = 1$, $u = -1$.

$$\begin{aligned}\int_0^1 x^2(2^{-x^3})\,dx &= \int_0^{-1} -\frac{1}{3}2^u\,du \\ &= -\frac{1}{3}\int_0^{-1} 2^u\,du \\ &= \frac{1}{3}\int_{-1}^0 2^u\,du \\ &= \frac{1}{3}\left[\frac{2^u}{\ln(2)}\right]_{-1}^0 \\ &= \frac{1}{3}\left(\frac{2^0}{\ln(2)} - \frac{2^{-1}}{\ln(2)}\right) \\ &= \frac{1}{3}\left(\frac{1}{\ln(2)} - \frac{1}{2\ln(2)}\right) \\ &= \frac{1}{3}\left(\frac{1}{2\ln(2)}\right) \\ &= \frac{1}{6\ln(2)}\end{aligned}$$

(7) Let $u = \sin(x)$. Then when $x = 0$, $u = 0$, and when $x = \pi/2$, $u = 1$.

$$\begin{aligned} u &= \sin(x) \\ du &= \cos(x) \end{aligned}$$

$$\begin{aligned} \int_0^{\pi/2} \cos(x)\sin(\sin(x))\,dx &= \int_0^1 \sin(u)\,du \\ &= \big[-\cos(u)\big]_0^1 \\ &= -\cos(1) + \cos(0) \\ &= 1 - \cos(1) \quad \square \end{aligned}$$

Example 6. Suppose f is continuous on [0,1]. Prove that

$$\int_0^1 f(x)\,dx = \int_0^1 f(1-x)\,dx$$

Sol. First, since f is cont., we know an antiderivative exists. Let F be an antiderivative of f. We can apply substitution to evaluate the integral on the right. Let $u = 1 - x$. Then $du = -\,dx$. Also, when $x = 0$, $u = 1$, and when $x = 1$, $u = 0$.

$$\begin{aligned} \int_0^1 f(1-x)\,dx &= \int_1^0 f(u) - du \\ &= -\int_1^0 f(u)\,du \\ &= -\Big(-\int_0^1 f(u)\,du\Big) \\ &= \int_0^1 f(u)\,du \qquad\qquad = F(1) - F(0) \text{since } F \text{ is an antiderivative of } f, \text{ by FTOC Pt.1} \end{aligned}$$

Considering the RHS, using the antiderivative of f and the FTOC Pt.1, the integral evaluates to $F(1) - F(0)$. Therefore LHS=RHS=$F(1) - F(0)$. $\square$