# MATA37 Week 3

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## 1 The definite integral

As $n$ becomes larger, that is, as we take smaller and smaller subintervals of $[a, b]$, the Riemann sum becomes more accurate, giving a better approximation for the exact area under the curve.

The definite integral allows us to find this exact area.

**Definition 1.** *Let $a, b \in \mathbb{R}$ where $a < b$. Let $P = \{x_0, x_1, ..., x_n\}$ be a Riemann partition. Suppose $f$ is a continuous function on $[a, b]$. The* ***definite integral*** *of $f$ from $x = a$ to $x = b$ is*

$$\int_a^b f(x)\,dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*)\Delta x$$

*(Provided that the limit exists. If the limit exists, we say that $f$ is* ***integrable*** *over $[a, b]$)*

$a$ is the **lower integration limit**; $b$ is the **upper integration limit**.

The function $f(x)$ is called the **integrand**.

Geometrically, the definite integral represents the **signed area** between the graph of the function and the x-axis. When this area is below the x-axis, the definite integral gives a negative value.

**Example 1. Calculating a definite integral using Riemann sum definition**
Evaluate $\int_0^2 (x^2 + 3x + 5)\,dx$ using the Riemann sum definition.

**Example 2. Calculating a definite integral by interpreting it geometrically**
Give the exact value of the following integrals:

(a) $\int_{-1}^{1} \sqrt{1 - x^2}\,dx$

(b) $\int_2^5 f(x)\,dx$ where $f(x) = \begin{cases} 2 & \text{when } x < 4 \\ 8 - x & \text{when } x \geq 4 \end{cases}$

(c) $\int_2^5 f(x)\,dx$ where $f(x) = \begin{cases} 2 & \text{when } x < 4 \\ x - 8 & \text{when } x \geq 4 \end{cases}$

**Example 3. Converting a Riemann sum into a definite integral**
Rewrite the following sums as a definite integral.

$$(a) \qquad \lim_{n\to\infty}\sum_{i=1}^{n}\left[3\left(4+\frac{2i}{n}\right)^5-6\right]\frac{2}{n}$$

$$(b) \qquad \lim_{n\to\infty}\sum_{i=1}^{n}\frac{i^4}{n^5}$$

For (a), as an extra exercise, try writing as a definite integral in an alternate way by using the specified values: (i) $\Delta x = 6/n$, (ii) $a = 2$

## 1.1 Properties of the definite integral

Let $a, b \in \mathbb{R}$ where $a < b$. Suppose $f$ and $g$ are integrable on $[a, b]$.

*(i)* If $f(x) \geq 0$ on $[a, b]$, then $\displaystyle\int_a^b f(x)\,dx \geq 0$

If $f(x) \leq 0$ on $[a, b]$, then $\displaystyle\int_a^b f(x)\,dx \leq 0$

*(ii)* $\displaystyle\int_a^b (f(x) \pm g(x))\,dx = \int_a^b f(x)\,dx \pm \int_a^b g(x)\,dx$

*(iii)* Let $c \in \mathbb{R}$. Then $\displaystyle\int_a^b cf(x)\,dx = c\int_a^b f(x)\,dx$

*(iv)* $\displaystyle\int_a^a f(x)\,dx = 0$

*(v)* $\displaystyle\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx$

*(vi)* **Union interval property** : Let $d \in [a, b], d \in \mathbb{R}$. Then $\displaystyle\int_a^b f(x)\,dx = \int_a^d f(x)\,dx + \int_d^b f(x)\,dx$

*(vii)* If $f(x) \leq g(x)$ on $[a, b]$, then $\displaystyle\int_a^b f(x)\,dx \leq \int_a^b g(x)\,dx$

*(viii)* **Integral inequality**: IF $f$ is **bounded** above and below, that is, $\exists m, M \in \mathbb{R}$ s.t. $m \leq f(x) \leq M \quad \forall x \in [a, b]$

THEN $\displaystyle m(b-a) \leq \int_a^b f(x)\,dx \leq M(b-a)$

(i) states that if a function is non-negative (or non-positive) over $[a, b]$, then so is its integral over $[a, b]$.

(ii), (iii), and (vi) are analogous to certain properties of sigma notation. (In fact, (ii) and (iii) are a direct consequence of those sigma properties.)

(iv): When the integration limits are the same, the integral always evaluates to 0, no matter what function $f$ is given. It's kind of like finding the area of a single line connecting the function to the x-axis at a single point—lines have no area.

(v): You can switch the integration limits of an integral, but that changes its sign.

(vii) is easy to see if you sketch out a graph where one function $f$ is always less than another function $g$; the area under $f$ would be less than the area under $g$.

(viii): there's an A2 exercise question based on this property that I recommend trying out. It helps to sketch an example of the situation. Hint: the values on the left and right side of the inequality can be interpreted as the areas of a rectangle.

**Example 4.** (A2 Exercise)
If $f(x) \geq 0$ on $[a, b]$, then $\int_a^b f(x)\,dx \geq 0$.

**Example 5.** (A2 Exercise)
Evaluate $\int_7^7 e^{x^2}\,dx + \int_0^{\sqrt{2}} \frac{1}{3\sqrt{2}}\,dx$

**Example 6.** Let $a, b \in \mathbb{R}$, $a < b$. Let $c \in \mathbb{R}$. Prove the following using the Riemann sum definition:

$$\int_a^b c\,dx = c(b-a)$$

**Example 7.** Prove the following inequality:

$$\int_{\pi/2}^{3\pi/4} \sin^4 x\,dx \leq \frac{\pi}{4}$$

## 2 The indefinite integral

**Definition 2.** *An* ***antiderivative*** *of a function $f$ is a function $F$ where the derivative of $F$ equals $f$, i.e. $F' = f$.*

For example, let $f(x) = \cos x$. Examples of antiderivatives of $f$ are $F_1(x) = \sin x$ and $F_2(x) = \sin x + 5$, since $F_1'(x) = \cos x = F_2'(x)$.
Suppose $f$ is continuous everywhere.

**Definition 3.** *The* ***indefinite integral*** *of $f$, denoted $\int f(x)\,dx$, is the collection of all antiderivatives of $f$.*

$$\int f(x)\,dx = F(x) + C$$

*where $F$ is an antiderivative of $f$, i.e. a function for which $F' = f$, and*
*$C$ is an arbitrary constant called the* ***integration constant****.*

Note: at this point, from what we know so far, the indefinite integral is unrelated from the definite integral: the indefinite integral represents antiderivatives of a function, while the definite integral represents the signed area between a function and the x-axis. The relationship between these two seemingly unrelated concepts is revealed in the Fundamental Theorem of Calculus.

Also note: evaluating an indefinite integral gives a **function** (more precisely, not just one function but a family of functions), while evaluating a definite integral gives a **number**.

## 2.1 Properties of the indefinite integral

Suppose $f$ and $g$ are continuous functions.

$$\begin{aligned} &(1) \quad \int [f(x) \pm g(x)]\,dx = \int f(x)\,dx \pm \int g(x)\,dx \\ &(2) \quad \int kf(x)\,dx = k\int f(x)\,dx \end{aligned}$$

**Example 1. Proof of (1)**
Suppose $f$ and $g$ are continuous functions. Let $F(x)$ and $G(x)$ be functions such that $F'(x) = f(x)$ and $G(x) = g'(x)$. Then, by definition of the indefinite integral,

$$\begin{aligned} \int f(x)\,dx \pm \int g(x)\,dx &= (F(x) + C_1) \pm (G(x) + C_2) \\ &= F(x) \pm G(x) + (C_1 \pm C_2) \\ \text{Let } C \text{ be the arbitrary constant } C = C_1 \pm C_2\text{:} \quad &= F(x) \pm G(x) + C \end{aligned}$$

By def. of indef. integral and by $[F(x) \pm G(x)]' = F'(x) \pm G'(x) = f(x) \pm g(x)$, we have that $\int f(x) \pm g(x)\,dx = F(x) \pm G(x) + C$.
Therefore $\int f(x)\,dx \pm \int g(x)\,dx = \int f(x) \pm g(x)\,dx \quad (= F(x) \pm G(x) + C)$ as wanted.

**Example 2. Practicing guess and check to evaluate integrals**
Evaluate the following integrals by guess and check, i.e. by inspection, i.e. by using the known derivatives of functions. When showing your work, show the steps that the derivative of your answer gives the integrand.

$$\begin{aligned} &(a) \quad \int x\sqrt{x}\,dx \\ &(b) \quad \int (x^2 - \sqrt{x})^2\,dx \\ &(c) \quad \int \sqrt{x}\Big(x^2 - \frac{1}{x}\Big) + \sec x \tan x\,dx \\ &(d) \quad \int \frac{1}{4 + 12x^2}\,dx \\ &(e) \quad \int (4e^x + 8)(e^x + 2x)^3\,dx \\ &(f) \quad \int 3x\cos(x^2)\,dx \\ &(g) \quad \int \frac{x^2}{(1 - x^3)^2}\,dx \\ &(h) \quad \int \frac{1 + x^2}{x^4}\,dx \end{aligned}$$

Always check to see if your solution is correct!

Answers to the above example:

(a)

$$\begin{aligned}\int x\sqrt{x} &= \int x(x^{1/2})\,dx \\ &= \int x^{3/2}\,dx \\ &= \frac{2}{5}x^{5/2}+C\end{aligned}$$

Note the derivative $x^{5/2}$ would be $\frac{5}{2}x^{3/2}$. But we're trying to find the antiderivative of $x^{3/2}$. So to get rid of the 5/2 factor that results from differentiating $x^{5/2}$, we multiply $x^{5/2}$ by $\frac{2}{5}$. Then $(\frac{2}{5}x^{5/2})' = \frac{2}{5}(\frac{5}{2}x^{3/2}) = x^{3/2}$ as wanted.

(b)

$$\begin{aligned}\int (x^2-\sqrt{x})^2\,dx &= \int (x^2)^2 - 2x^2\sqrt{x} - (\sqrt{x})^2\,dx \\ &= \int x^4 - 2x^{5/2} - x\,dx \\ &= \int x^4\,dx - \int 2x^{5/2}\,dx - \int x\,dx && \text{By integral property} \\ &= \frac{1}{5}x^5 - \frac{2}{7}x^{7/2} - \frac{1}{2}x^2 + C && \text{You can group together the integration constant from each integral}\end{aligned}$$

There's an error in the above solution. Where is it?

(c)

$$\begin{aligned}\int \sqrt{x}\Big(x^2-\frac{1}{x}\Big) + \sec x\tan x\,dx &= \int x^{5/2} - (x^{1/2}x^{-1}) + \sec x\tan x\,dx \\ &= \int x^{5/2} - x^{-1/2} + \sec x\tan x\,dx \\ &= \int x^{5/2}\,dx - \int x^{-1/2}\,dx + \int \sec x\tan x\,dx && \text{By integral properties} \\ &= \frac{2}{7}x^{7/2} - 2x^{1/2} + \sec x + C\end{aligned}$$

(d)

$$\begin{aligned}\int \frac{1}{4+12x^2}\,dx &= \int \frac{1}{4}\frac{1}{1+3x^2}\,dx \\ &= \frac{1}{4}\int \frac{1}{1+3x^2}\,dx && \text{By integral properties} \\ &= \frac{1}{4}\int \frac{1}{1+(\sqrt{3}x)^2}\,dx \\ &= \frac{1}{4\sqrt{3}}\arctan(\sqrt{3}x)+C\end{aligned}$$

(e)

$$\begin{aligned}
\int (4e^x + 8)(e^x + 2x)^3 \, dx &= \int 4(e^x + 2)(e^x + 2x)^3 \, dx \\
&= \int (e^x + 2)[4(e^x + 2x)^3] \\
&= (e^x + x)^5 + C
\end{aligned}$$

Recognize $e^x + 2$ as the derivative of $e^x + 2x$ and apply the chain rule.
(f)

$$\int 3x \cos(x^2) \, dx$$

Note that by the chain rule, $[\sin(x^2)]' = 2x \cos(x^2)$. To remove the 2 coefficient and change it to be 3, need to multiply by by $3/2$.

$$\int 3x \cos(x^2) \, dx = \frac{3}{2} \sin(x^2) + C$$

(g)

$$\int \frac{6x^2}{(1 - x^3)^2} \, dx$$

Note that by the chain rule, the derivative of $\frac{1}{1-x^3}$ is $\frac{-3x^2}{(1-x^3)^2}$. In order to get the function $\frac{6x^2}{(1-x^3)^2}$, you'd need to multiply by a factor -2.

$$\int \frac{6x^2}{(1 - x^3)^2} \, dx = \frac{-2}{1 - x^3} + C$$

(h)

$$\int \frac{1 + x^2}{x^4} \, dx$$

Split the fraction into two fractions to evaulate each separately:

$$\begin{aligned}
\int \frac{1 + x^2}{x^4} \, dx &= \int \frac{1}{x^4} + \frac{x^2}{x^4} \, dx \\
&= \int \frac{1}{x^4} \, dx + \int \frac{x^2}{x^4} \, dx \quad \text{By integral properties} \\
&= \int \frac{1}{x^4} \, dx + \int \frac{1}{x^2} \, dx \\
&= \frac{1}{5} x^5 + \frac{1}{3} x^3 + C
\end{aligned}$$

# 3 Answers to examples

**Example 1. Calculating a definite integral using Riemann sum definition**
Evaluate $\int_0^2 (x^2+3x+5)\,dx$ using the Riemann sum definition.

Sol. Looking at the given integral, we have $a=0$, $b=2$, and $f(x)=x^2+3x+5$. We use these to find $\Delta x$ and $x_i$ for any Riemann partition $\{x_i\}_{i=0}^n$ of the interval $[0,2]$.

$$\Delta x = \frac{b-a}{n} = \frac{3-1}{n} = \frac{2}{n}$$
$$x_i = a + i\Delta x = 0 + i\frac{2}{n} = \frac{2i}{n}$$

By the definition, we need to choose a certain $x_i^*$ in each interval $[x_{i-1}, x_i]$. Choose $x_i^*$ to be the right endpoint, $x_i^* = x_i$. (Note that we could also choose the left endpoint, $x_i^* = x_{i-1} = \frac{2(i-1)}{n}$, and get the same answer.) So by definition,

$$\begin{aligned}
\int_0^2 (x^2+3x+5)\,dx &= \lim_{n\to\infty}\sum_{i=1}^n f(x_i^*)\Delta x \\
&= \lim_{n\to\infty}\sum_{i=1}^n f(x_i)\Delta x \qquad \text{(choosing right Riemann sum)} \\
&= \lim_{n\to\infty}\sum_{i=1}^n f\Big(\frac{2i}{n}\Big)\frac{2}{n} \qquad \text{(sub in } x_i \text{ and } \Delta x \text{ as calculated)} \\
&= \lim_{n\to\infty}\frac{2}{n}\sum_{i=1}^n f\Big(\frac{2i}{n}\Big) \qquad \text{(sigma property (II))} \\
&= \lim_{n\to\infty}\frac{2}{n}\sum_{i=1}^n \Big[\Big(\frac{2i}{n}\Big)^2 + 3\Big(\frac{2i}{n}\Big) + 5\Big] \qquad \text{(sub in } f(x)=x^2+3x+5 \text{ as given)} \\
&= \lim_{n\to\infty}\frac{2}{n}\sum_{i=1}^n \Big(\frac{4i^2}{n^2} + \frac{6i}{n} + 5\Big) \\
&= \lim_{n\to\infty}\frac{2}{n}\left(\sum_{i=1}^n \frac{4i^2}{n^2} + \sum_{i=1}^n \frac{6i}{n} + \sum_{i=1}^n 5\right) \qquad \text{(sigma property (I))} \\
&= \lim_{n\to\infty}\frac{2}{n}\left(\frac{4}{n^2}\sum_{i=1}^n i^2 + \frac{6}{n}\sum_{i=1}^n i + \sum_{i=1}^n 5\right) \qquad \text{(sigma property (II))} \\
&= \lim_{n\to\infty}\frac{2}{n}\left(\frac{4}{n^2}\Big(\frac{n(n+1)(2n+1)}{6}\Big) + \frac{6}{n}\Big(\frac{n(n+1)}{2}\Big) + 5n\right) \qquad \text{(Theorem 2 - useful sum formulas)} \\
&= \lim_{n\to\infty}\frac{2}{n}\left(\frac{4}{n}\Big(\frac{(n+1)(2n+1)}{6}\Big) + 6\Big(\frac{(n+1)}{2}\Big) + 5n\right) \\
&= \lim_{n\to\infty}\frac{2}{n}\left(\frac{4}{n}\Big(\frac{2n^2+3n+1}{6}\Big) + (3n+3) + 5n\right) \\
&= \lim_{n\to\infty}\frac{2}{n}\left(\frac{4}{n}\Big(\frac{n^2}{3} + \frac{n}{2} + \frac{1}{6}\Big) + 8n + 3\right)
\end{aligned}$$

$$
\begin{aligned}
&= \lim_{n\to\infty} \frac{2}{n}\left(\frac{4}{3}n + 2 + \frac{2}{3n} + 8n + 3\right)\\
&= \lim_{n\to\infty} \frac{2}{n}\left(\frac{28}{3}n + 5 + \frac{2}{3n}\right)\\
&= \lim_{n\to\infty} \left(\frac{56}{3} + \frac{10}{n} + \frac{4}{3n^2}\right)\\
&= \frac{56}{3} \qquad \left(\text{Recall } \lim_{n\to\infty}(1/n) = \lim_{n\to\infty}(1/n^2) = 0\right)
\end{aligned}
$$

**Example 2. Calculating a definite integral by interpreting it geometrically**
Give the exact value of the integrals:
(a) $\int_{-1}^{1} \sqrt{1-x^2}\,dx$

Sol. Sketch out the graph of the function—it represents the upper half circle of radius 1 centered at (0,0). The integral represented the signed area between the graph and the x-axis. Since the graph is entirely above the x-axis, the integral is positive, and we know the area of a semicircle of radius 1 is $\pi/2$. Therefore $\int_{-1}^{1} \sqrt{1-x^2}\,dx = \pi/2$.

(b) $\int_{2}^{5} f(x)\,dx$ where $f(x) = \begin{cases} 2 & \text{when } x < 4 \\ 8-x & \text{when } x \geq 4 \end{cases}$
Sketch the graph of $f(x)$ and look at the area between the graph and the $x$-axis. It consists of a rectangle $A_1$ with length 2 and height 2 and a triangle $A_2$ with base 1 and height 1 over a rectangle $A_3$ with length 1 and height 3. The area of $A_1$ is 4, the area of $A_2$ is 1/2 and the area of $A_3$ is 3. Adding these areas gives 15/2. Hence $\int_{2}^{5} f(x)\,dx = 15/2$.

(c) is the same as (b), except $A_2$ and $A_3$ are below the $x$-axis so their signed area is negative. Hence $\int_{2}^{5} f(x)\,dx = 4 - 1/2 - 3 = 1/2$.

**Example 3. Converting a Riemann sum into a definite integral**
Rewrite the following sums as a definite integral. (Part (2) is from A2, exercise #6)

$$(1) \qquad \lim_{n\to\infty} \sum_{i=1}^{n} \left[3\left(4+\frac{2i}{n}\right)^5 - 6\right]\frac{2}{n}$$

$$(2) \qquad \lim_{n\to\infty} \sum_{i=1}^{n} \frac{i^4}{n^5}$$

Sol. (1)
First we need to identify the Riemann sum. We can compare the given sum to the formula for the right

Riemann sum:

$$(1) \quad \sum_{i=1}^{n} \Big[3\big(4+\frac{2i}{n}\big)^5-6\Big]\frac{2}{n}$$
$$R_n = \sum_{i=1}^{n} f(x_i)\Delta x$$

We can match up what we think $f(x_i)$ and $\Delta x$ should be. $\Delta x$ usually looks like a fraction with denominator $n$. Let $\Delta x = \frac{2}{n}$ and $f(x_i) = [3(4+\frac{2i}{n})^5-6]$.
Recall that the expression for $x_i$ is of the form $a + i\Delta x$. Since $\Delta x = \frac{2}{n}$, $x_i$ will look something like $a + i\frac{2}{n}$.
The expression $(4+\frac{2i}{n})$ matches up with what $x_i = a + i\frac{2}{n}$ would look like! Let's let $a = 4$.
Since $\Delta x = \frac{b-a}{n} = \frac{2}{n}$ and we have $a = 4$, then $b$ should be 6.
Lastly, we need to figure out what $f$ is. Our expression for $f(x_i)$ looks like $f(x_i) = 3(4+\frac{2i}{n})^5-6$. Since we have that $f(x_i) = 4+\frac{2i}{n}$, we can look at this expression and replace every expression $4+\frac{2i}{n}$ with $x_i$. That gives $f(x_i) = 3(x_i)^5-6$. So the function is $f(x) = 3x^5-6$.
With the values of the Riemann sum $a = 4$, $b = 6$, and $f(x) = 3x^5-6$, when we take the limit of the Riemann sum at $n \to \infty$ and turn it into a definite integral, these values will stay the same.

$$\lim_{n\to\infty} \sum_{i=1}^{n} \Big[3\big(4+\frac{2i}{n}\big)^5-6\Big]\frac{2}{n} = \int_a^b f(x)dx = \int_4^6 3x^5-6\,dx \quad \square$$

**Note**: this answer is NOT UNIQUE. For example, what if we specified we need $a = 2$, not $a = 4$?
First, we still have $\Delta x = \frac{b-a}{n} = \frac{2}{n}$, so if $a = 2$ then $b$ must be 4. This would also change our expression for $x_i$. Preserving our value for $\Delta x = \frac{2}{n}$, we would get $x_i = 2 + i\frac{2}{n}$. So we'd need to figure out how to get this expression for $x_i$ to show up in our expression for $f(x_i)$. In other words, we need to get what $f(x_i) = 3(4+\frac{2i}{n})^5-6$ would be, in terms of $x_i = 2 + i\frac{2}{n}$. We can rewrite

$$f(x_i) = 3\Big(4+\frac{2i}{n}\Big)^5-6 = 3\Big[2+\Big(2+\frac{2i}{n}\Big)\Big]^5-6 = 3(2+x_i)^5-6$$

So the function in this case would be $f(x) = 3(2+x)^5-6$.
**If we're given that** $a = 2$, the integral becomes

$$\lim_{n\to\infty} \sum_{i=1}^{n} \Big[3\big(4+\frac{2i}{n}\big)^5-6\Big]\frac{2}{n} = \int_2^4 3(2+x)^5-6\,dx$$

This answer is also correct, but if we're allowed to come up with our $a$ and $b$ values ourselves, then we should choose easy ones to work with.

Sol. (2)

We need to do some factoring to the sum to make it look like a Riemann sum. Riemann sums always have the form $\sum f(x_i^*)\Delta x$, and $\Delta x$ is always a fraction with denominator $n$. We can factor out such a fraction in the sum.

$$\sum_{i=1}^{n} \frac{i^4}{n^5} = \sum_{i=1}^{n} \Big(\frac{i^4}{n^4}\Big)\frac{1}{n} = \sum_{i=1}^{n} \Big(\frac{i}{n}\Big)^4\frac{1}{n}$$

Now we can compare this to the definition of the right Riemann sum.

$$(2) \quad \sum_{i=1}^{n} \left(\frac{i}{n}\right)^4 \frac{1}{n}$$
$$R_n = \sum_{i=1}^{n} f(x_i)\Delta x$$

Let $\Delta x = \frac{1}{n}$ and let $f(x_i) = (\frac{i}{n})^4$.
Since we have what $\Delta x$ is, we can find what $x_i$ would look like—it'll be of the form $x_i = a + i\frac{1}{n} = a + \frac{i}{n}$. We see $\frac{i}{n}$ show up in the sum, but there's nothing added to it. So we can take $a$ to be 0! In that case $x_i = a + \frac{i}{n} = 0 + \frac{i}{n} = \frac{i}{n}$. Note that since we have $\Delta x = \frac{b-a}{n} = \frac{1}{n}$, so if we take $a = 0$ then $b = 1$.
Now that we have an expression for $x_i$, which is $x_i = \frac{i}{n}$ we can take a look at what we have for $f(x_i)$ and replace each $\frac{i}{n}$ with $x_i$ to figure out what our function $f$ is.

$$f(x_i) = \left(\frac{i}{n}\right)^4 = (x_i)^4$$

So $f(x) = x^4$.
Now when we take the limit of the Riemann sum at $\infty$ and convert it into a definite integral, we'll use the above values of $a = 0$, $b = 1$, and $f(x) = x^4$.

$$\lim_{n\to\infty} \sum_{i=1}^{n} \frac{i^4}{n^5} = \int_0^1 x^4\,dx \quad \square$$

**Example 4.** (A2 Exercise #13(a))
Prove property (i) above: If $f(x) \geq 0$ on $[a, b]$, then $\int_a^b f(x)\,dx \geq 0$.

Sol. Suppose $f(x)$ is a function where $f(x) \geq 0$ for $x \in [a, b]$ (and that its integral exists on $[a, b]$). Want to show $\int_a^b f(x)\,dx \geq 0$.

$$\int_a^b f(x)\,dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*)\Delta x$$

We know that $f(x) \geq 0$ for all $x \in [a, b]$. In particular this means, for all $i = 1, ..., n$, we have $f(x_i^*) \geq 0$. Therefore, we have that

$$\sum_{i=1}^{n} f(x_i^*)\Delta x \geq \sum_{i=1}^{n} (0)\Delta x$$

Simplifying the RHS,

$$\sum_{i=1}^{n} f(x_i^*)\Delta x \geq \sum_{i=1}^{n} 0$$
$$\implies \sum_{i=1}^{n} f(x_i^*)\Delta x \geq 0$$

And finally, take the limit as $n \to \infty$ of both sides.

$$\begin{aligned}
&\lim_{n\to\infty}\sum_{i=1}^{n} f(x_i^*)\Delta x \geq \lim_{n\to\infty} 0 \\
\implies &\lim_{n\to\infty}\sum_{i=1}^{n} f(x_i^*)\Delta x \geq 0 \\
&\implies \int_a^b f(x)\,dx \geq 0 \qquad \text{(By def. of definite integral)} \quad \square
\end{aligned}$$

**Example 5.** (A2 Exercise #14)
Evaluate $\int_7^7 e^{x^2}\,dx + \int_0^{\sqrt{2}} \frac{1}{3\sqrt{2}}\,dx$

Sol.

$$\begin{aligned}
\int_7^7 e^{x^2}\,dx + \int_0^{\sqrt{2}} \frac{1}{3\sqrt{2}}\,dx &= 0 + \int_0^{\sqrt{2}} \frac{1}{3\sqrt{2}}\,dx \qquad \left(\text{By integral property (iv) above, } \int_7^7 e^{x^2}\,dx = 0\right) \\
&= \int_0^{\sqrt{2}} \frac{1}{3\sqrt{2}}\,dx \\
&= \int_0^{\sqrt{2}} 1\cdot\frac{1}{3\sqrt{2}}\,dx \\
&= \frac{1}{3\sqrt{2}}\int_0^{\sqrt{2}} 1\,dx
\end{aligned}$$

Calculating the integral $\int_0^{\sqrt{2}} 1\,dx$ by the Riemann sum definition. We'll use the right Riemann sum. The integration limits are $a = 0$ and $b = \sqrt{2}$, so $\Delta x = \frac{\sqrt{2}}{n}$ and $x_i = a + i\Delta x = \frac{i\sqrt{2}}{n}$

$$\begin{aligned}
\int_0^{\sqrt{2}} 1\,dx &= \lim_{n\to\infty}\sum_{i=1}^{n} f(x_i)\Delta x \qquad \text{(Def. of definite integral)} \\
&= \lim_{n\to\infty}\sum_{i=1}^{n} f\left(\frac{i\sqrt{2}}{n}\right)\frac{\sqrt{2}}{n} \\
&= \lim_{n\to\infty}\sum_{i=1}^{n} 1\cdot\frac{\sqrt{2}}{n} \qquad \left(\text{Since } f \text{ is the constant function } f(x) = 1,\ f\left(\frac{i\sqrt{2}}{n}\right) = 1 \text{ for all } i = 1, ..., n\right) \\
&= \lim_{n\to\infty}\frac{\sqrt{2}}{n}\sum_{i=1}^{n} 1 \qquad \text{(By sigma property)} \\
&= \lim_{n\to\infty}\frac{\sqrt{2}}{n}(n) \qquad \left(\text{By useful summation formula, } \sum_{i=1}^{n} 1 = n\right) \\
&= \lim_{n\to\infty}\sqrt{2} \\
&= \sqrt{2} \qquad \text{(The limit of a constant is that constant.)}
\end{aligned}$$

Continuing the original calculation,

$$\int_7^7 e^{x^2}\,dx + \int_0^{\sqrt{2}} \frac{1}{3\sqrt{2}}\,dx = \frac{1}{3\sqrt{2}}\int_0^{\sqrt{2}} 1\,dx$$

$$= \frac{1}{3\sqrt{2}}(\sqrt{2})$$
$$= \frac{1}{3} \quad \square$$

**Example 6.** Let $a, b \in \mathbb{R}$, $a < b$. Let $c \in \mathbb{R}$. Prove the following using the Riemann sum definition:

$$\int_a^b c\,dx = c(b-a)$$

Sol. We will evaluate $\int_a^b c\,dx$ using the Riemann definition with the constant function $f(x) = c$.
Let $\{x_i\}_{i=0}^n$ be a Riemann partition of $[a, b]$. We have that

$$\Delta x = \frac{b-a}{n}$$
$$x_i = a + i\Delta x = a + i\frac{b-a}{n}$$

We will use a right Riemann sum, so we choose $x_i^*$ to be $x_i$. Hence by def.,

$$\begin{aligned}
\int_a^b c\,dx &= \lim_{n\to\infty} \sum_{i=1}^n f(x_i^*)\Delta x \\
&= \lim_{n\to\infty} \sum_{i=1}^n f(x_i)\Delta x \qquad \text{(choosing right Riemann sum)} \\
&= \lim_{n\to\infty} \sum_{i=1}^n f(x_i)\frac{b-a}{n} \\
&= \lim_{n\to\infty} \sum_{i=1}^n c\frac{b-a}{n} \qquad \text{(Since } f \text{ is a constant function, } f(x_i) = c \text{ for all } i = 0, ..., n) \\
&= \lim_{n\to\infty} c\frac{b-a}{n} \sum_{i=1}^n 1 \qquad \text{(By a } \Sigma \text{ property, since } c\frac{b-a}{n} \text{ is independent of the index variable } i.) \\
&= \lim_{n\to\infty} c\frac{b-a}{n}(n) \qquad \text{(By } \Sigma \text{ formula)} \\
&= \lim_{n\to\infty} c(b-a) \\
&= c(b-a) \qquad \text{(Limit of a constant)}
\end{aligned}$$

**Example 7.** Prove the following inequality:

$$\int_{\pi/2}^{3\pi/4} \sin^4 x\,dx \leq \frac{\pi}{4}$$

Sol. We know $0 \leq \sin x \leq 1$ for all $x \in \mathbb{R}$. Hence, $0 \leq \sin^4 x \leq 1$ for all $x \in \mathbb{R}$. Since $\sin^4 x \leq 1$, in particular for $x \in [\pi/2, 3\pi/4]$, by an integral property, we have

$$\int_{\pi/2}^{3\pi/4} \sin^4 x \, dx \leq \int_{\pi/2}^{3\pi/4} 1 \, dx$$

And as proven above,

$$\int_{\pi/2}^{3\pi/4} 1 \, dx = 1(\frac{3\pi}{4} - \frac{\pi}{2}) = \frac{\pi}{4}$$

Hence,

$$\int_{\pi/2}^{3\pi/4} \sin^4 x \, dx \leq \int_{\pi/2}^{3\pi/4} 1 \, dx = \frac{\pi}{4}$$

NOTE: I can only use the fact $\int_a^b c \, dx = c(b-a)$ since I proved it above. On an assignment or test, if you want to use a fact about integrals that isn't among the given properties, it's best that you prove it first.