MATA37 Week 12

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Definition 1. *A sequence is **bounded** if*

$$\exists c \in \mathbb{R}^+ \text{ s.t. } |a_n| \leq c \quad \forall n \in \mathbb{N} \tag{1}$$

Note that $|a_n| \leq c$ equivalently means $-c \leq a_n \leq c$.

Theorem 1. ***Convergent sequences are bounded***
IF $\{a_n\}$ converges
THEN $\{a_n\}$ is bounded.

1 Bounded Monotone Convergence Theorem

Definition 2. *A sequence $\{a_n\}$ is **monotone** if it is either decreasing or increasing $\forall n \in \mathbb{N}$.*
$\{a_n\}$ is decreasing when $\forall n \in \mathbb{N}, a_n \geq a_{n+1}$
$\{a_n\}$ is increasing when $\forall n \in \mathbb{N}, a_n \leq a_{n+1}$

Theorem 2. ***Bounded Monotone Convergence Theorem***
IF $\{a_n\}$ is bounded and monotone
THEN $\{a_n\}$ converges.
In particular, there are two cases:

(1) If $\{a_n\}$ is increasing and bounded above, then $\{a_n\}$ converges.

(2) If $\{a_n\}$ is decreasing and bounded below, then $\{a_n\}$ converges.

Example 1. Consider the following sequence, defined recursively:

$$a_1 = 1$$
$$a_{n+1} = 3 - \frac{1}{a_n}$$

Prove that the function is bounded above and increasing, then conclude that it converges using BMCT. As a challenge, try finding the limit of $\{a_n\}$.

2 Series

Let $\{a_n\}$ be a sequence.
A sequence can be used to define an **infinite series**, in which you add up all possible terms of the sequence:

$$\sum_{n=1}^{\infty} a_n = a_1 + a_2 + a_3 + a_4 + ...$$

The general term of the series $\sum a_n$ is a_n.
The convergence of an infinite series is defined using the partial sums of its general term.

Definition 3. *For each $n \in \mathbb{N}$, the **nth partial sum** is the finite sum*

$$S_n = a_1 + a_2 + a_3 + ... + a_n$$

Definition 4. *A series $\sum a_n$ **converges** to some $s \in \mathbb{R}$ if the sequence of partial sums $\{S_n\} = \{S_1, S_2, S_3, ...\}$ converges to some real number s. i.e.*

$$\lim_{n\to\infty} S_n = s$$

If a series doesn't converge (i.e. if this limit doesn't exist) we say it diverges.

2.1 Geometric Series

Definition 5. *A **geometric series** is a series of the form*

$$a + ar + ar^2 + ar^3 + ... + ar^n + ... = \sum_{n=0}^{\infty} ar^n$$

*r is the **common ratio** of the series.*

To show that a series is a geometric series, show that it can be written in the form $\sum_{n=0}^{\infty} ar^n$ and identify what a (the first term in the series) and the common ratio r is.

2.2 Geometric Series Test

Let $\sum ar^n$ be a geometric series.

- If $|r| < 1$, $\sum ar^n$ converges and $\sum ar^n = \frac{a}{1-r}$
- If $|r| \geq 1$, $\sum ar^n$ diverges

2.3 Properties of convergent series

Let $\sum a_n$ and $\sum b_n$ be infinite series. Suppose both $\sum a_n = s$ and $\sum b_n = t$ (i.e. $\sum a_n$ and $\sum b_n$ both converge). Then the following properties hold:

- For any $c \in \mathbb{R}$, $\sum ca_n$ converges and $\sum ca_n = cs$

- $\sum(a_n \pm b_n)$ converges with sum $s \pm t$

- (Vanishing Condition) $\lim_{n\to\infty} a_n = 0$

2.3.1 Divergence Test

Note the Vanishing Condition above:

$$\sum a_n \text{ converges} \implies \lim_{n\to\infty} a_n = 0$$

The **contrapositive** of this statement gives a criterion for a series diverging:

$$\lim_{n\to\infty} a_n \neq 0 \implies \sum a_n \text{ diverges}$$

That is, to show that sum $\sum a_n$ diverges, you can show that $\lim_{n\to\infty} a_n \neq 0$.

Note that the **converse** of the Vanishing Condition is **not necessarily true**.

$$\lim_{n\to\infty} a_n = 0 \not\Longrightarrow \sum a_n \text{ converges}$$

For example, $\sum \frac{1}{n}$ *diverges* even though $\lim_{n\to\infty} \frac{1}{n} = 0$.

Example 1. Determine whether the following series converge or diverge. If the series is convergent find its sum.

$$(a) \quad \sum_{n=0}^{\infty} \left(\frac{12}{(-5)^n}\right)$$

$$(b) \quad 20 - 4 + 0.8 - 0.16 + ...$$

$$(c) \quad \sum_{n=1}^{\infty} a_n \text{ where } S_n = a_1 + a_2 + ... + a_n = \frac{n^2+1}{4n^2+1}$$

$$(d) \quad \sum_{n=1}^{\infty} \frac{n}{n+1}$$

$$(e) \quad \sum_{n=1}^{\infty} \ln\left(\frac{n}{n+1}\right)$$

$$(f) \quad \sum_{n=1}^{\infty} \pi \frac{5^{n-1}(-1)^n}{6^{n+1}}$$

Example 2. Prove by definition of convergent series that if $\sum a_n$ and $\sum b_n$ converge and $\sum a_n = s$ and $\sum b_n = t$ then $\sum(a_n + b_n) = s + t$

Example 3.
True or false? If $\sum a_n$ converges and $\sum b_n$ diverges then $\sum(a_n b_n)$ diverges.

Example 4.
Suppose $\sum a_n$ is convergent and $\sum b_n$ is divergent. Prove that $\sum a_n + b_n$ is divergent. (Hint: Do a proof by contradiction.)

Example 5. Use a geometric series to prove that $0.33333... = \frac{1}{3}$.

3 Answers to selected examples

Example 1. Consider the following sequence, defined recursively:

$$a_1 = 1$$
$$a_{n+1} = 3 - \frac{1}{a_n}$$

Prove that the function is bounded above and increasing, then conclude that it converges using BMCT. As a challenge, try to find the limit of $\{a_n\}$.

Sol. Use induction to prove both hypotheses. Bounded above:
We can guess that each term in the sequence will be < 3 since each subsequent term is formed by subtracting from 3.
Base case:
$n = 1$: $a_1 = 1 < 3$ ✓
Inductive hypothesis:
Suppose $a_k < 3$ for some $k \in \mathbb{N}$.
WTS $a_{k+1} < 3$, which is equivalent to $3 - \frac{1}{a_k} < 3$.

$$\begin{aligned} & a_k < 3 \quad \text{by I.H.} \\ \iff & \frac{1}{a_k} > \frac{1}{3} \quad \text{(take reciprocal of both sides)} \\ \iff & -\frac{1}{a_k} < -\frac{1}{3} \quad \text{multiply both sides by -1} \\ \iff & 3 - \frac{1}{a_k} < 3 - \frac{1}{3} < 3 \end{aligned}$$

Hence $3 - \frac{1}{a_k} = a_{k+1} < 3$. ✓
By induction, a_n is bounded above by 3.

Increasing:
Base case:
$n = 1$: $a_1 = 1 < a_2 = 2$ ✓
Inductive hypothesis:
Suppose $a_k < a_{k+1}$ for some $k \in \mathbb{N}$.
WTS $a_{k+1} < a_{k+2}$

$$\begin{aligned} & a_k < a_{k+1} \quad \text{by I.H.} \\ \iff & \frac{1}{a_k} > \frac{1}{a_{k+1}} \quad \text{(take reciprocal of both sides)} \\ \iff & -\frac{1}{a_k} < -\frac{1}{a_{k+1}} \quad \text{(multiply both sides by -1)} \\ \iff & 3 - \frac{1}{a_k} < 3 - \frac{1}{a_{k+1}} \quad \text{(add 3 to both sides)} \end{aligned}$$

$$\iff a_{k+1} < a_{k+2} \quad \text{by def. of } a_n \quad \checkmark$$

By induction, a_n is increasing.
Therefore, since a_n is bounded above and increasing, a_n converges by BMCT.

Example 2. Determine whether the following series converge or diverge. If the series is convergent find its sum.

$$(a) \quad \sum_{n=0}^{\infty} \Big(\frac{12}{(-5)^n}\Big)$$

$$(b) \quad 20 - 4 + 0.8 - 0.16 + ...$$

$$(c) \quad \sum_{n=1}^{\infty} a_n \text{ where } S_n = a_1 + a_2 + ... + a_n = \frac{n^2+1}{4n^2+1}$$

$$(d) \quad \sum_{n=1}^{\infty} \frac{n}{n+1}$$

$$(e) \quad \sum_{n=1}^{\infty} \ln\Big(\frac{n}{n+1}\Big)$$

$$(f) \quad \sum_{n=1}^{\infty} \pi \frac{5^{n-1}(-1)^n}{6^{n+1}}$$

Sol. (a)

$$\begin{aligned}\sum_{n=0}^{\infty} \Big(\frac{12}{(-5)^n}\Big) &= \sum_{n=0}^{\infty} \Big(12 \frac{1}{(-5)^n}\Big) \\ &= \sum_{n=0}^{\infty} \Big(12 \big(\frac{(1)^n}{(-5)^n}\big)\Big) \\ &= \sum_{n=0}^{\infty} \Big(12 \big(-\frac{1}{5}\big)^n\Big)\end{aligned}$$

This is a geometric series with $r = -\frac{1}{5}$ and first term $a = 12\big(-\frac{1}{5}\big)^0 = 12$. Since $|r| < 1$, by GS test, the sum converges and has value $\frac{12}{1-(-1/5)} = 10$.

(b)

$$20 - 4 + 0.8 - 0.16 + ...$$

This is a geometric series with first term $a = 20$ and common ratio $r = -1/5$. Since $|r| < 1$, by GS test, the sum converges and has value $\frac{20}{1-(-1/5)} = \frac{50}{3}$.

(c)

$$\sum_{n=1}^{\infty} a_n \text{ where } S_n = a_1 + a_2 + ... + a_n = \frac{n^2+1}{4n^2+1}$$

Evaluating $\lim_{n\to\infty} S_n = \lim_{n\to\infty} \frac{n^2+1}{4n^2+1} = \frac{1}{4}$. (NOTE: on an assignment, you should try proving this using the $\epsilon - N$ definition of a sequence, just to be thorough.) Hence, by definition of series convergence, $sum_{n=1}^{\infty} a_n$

converges with sum $\frac{1}{4}$. (d)

$$\sum_{n=1}^{\infty} \frac{n}{n+1}$$

Let $a_n = \frac{n}{n+1}$. Since $\lim_{n\to\infty} a_n = 1 \neq 0$, the series diverges by divergence test. (Note: again, on an assignment, you should try proving $\lim_{n\to\infty} a_n = 1$ using the def. of a sequence, just to be thorough.)
One way to see intuitively why this series diverges is to write $\frac{n}{n+1} = 1 - \frac{1}{n+1}$. $\sum 1$ and $\sum \frac{1}{n+1}$ both diverge, so you can guess that the original series also diverges.

(e)

$$\sum_{n=1}^{\infty} \ln\left(\frac{n}{n+1}\right)$$

One of the first tests you should try is the divergence test, which will immediately tell you whether the series diverges. However, in this case, $\lim_{n\to\infty} \ln\left(\frac{n}{n+1}\right) = 0$, so we can't use the divergence test and we don't know yet whether the series converges or diverges.
Writing $\ln\left(\frac{n}{n+1}\right) = \ln(n) - \ln(n+1)$, we can see that the series is actually made up of telescoping sums, so we can try evaluating the series by definition using the partial sums.

$$\begin{aligned}
S_n &= a_1 + a_2 + a_3 + ... + a_n \\
&= (\ln(1) - \ln(2)) + (\ln(2) - \ln(3)) + (\ln(3) - \ln(4)) + ... + (\ln(n) - \ln(n+1)) \\
&= \ln(1) - \ln(n+1) \\
&= -\ln(n+1)
\end{aligned}$$

$\lim_{n\to\infty} S_n = \lim_{n\to\infty} -\ln(n+1)$ does not exist since it goes to $-\infty$, hence by definition of series, the original series diverges.

(f)

$$\begin{aligned}
\sum_{n=1}^{\infty} \pi \frac{5^{n-1}(-1)^n}{6^{n+1}} &= \sum_{n=1}^{\infty} \pi \frac{5^n 5^{-1}(-1)^n}{6^n 6} \\
&= \sum_{n=1}^{\infty} \pi \frac{5^n(-1)^n}{6^n \cdot 6 \cdot 5} \\
&= \sum_{n=1}^{\infty} \pi \Big(-\frac{5}{6}\Big)^n \Big(\frac{1}{30}\Big)
\end{aligned}$$

This is a geometric series with common ratio $r = -\frac{5}{6}$. The first term in the series is the $n = 1$ term, so $a = \pi(-5/6)^1(1/30) = -\pi/36$. Since $|r| < 1$, the series converges with sum $\frac{\pi/36}{1-(-5/6)} = \frac{\pi}{66}$

Example 3. Use a geometric series to prove that $0.33333... = \dfrac{1}{3}$.

0.333... can be written as a series:

$$0.3333... = 0.3 + 0.03 + 0.003 + 0.0003 + 0.00003 + ...$$

Equivalently,

$$0.3333... = \frac{3}{10} + \frac{3}{100} + \frac{3}{1000} + \frac{3}{10000} + ...$$

This is a geometric series with ratio $r = \frac{1}{10}$ and first term $a = \frac{3}{10}$. Since $|r| < 1$, the series converges and the sum is $\frac{3/10}{1-1/10} = \frac{3/10}{9/10} = \frac{1}{3}$.