MATA37 Week 10

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1 Improper integrals

In the past when calculating $\int_a^b f(x)\,dx$, we have assumed that:
(1) the interval $[a, b]$ is bounded ($a, b \neq \infty$)
(2) $f(x)$ has no vertical asymptotes (VAs) in $[a, b]$ (range($f(x)$) is bounded)

If the condition(s) (1) and/or (2) are broken, $\int_a^b f(x)\,dx$ is an **improper integral**.

Examples of **type I improper integrals** (integrals that break condition (1)):

$$\int_3^\infty \frac{\arctan(x)}{1+x^2}\,dx$$
$$\int_{-\infty}^1 \arctan(x)\,dx$$

Examples of **type II improper integrals** (integrals that break condition (2)):

$$\int_2^5 \frac{8}{\sqrt{x-2}}\,dx$$
$$\int_{\pi/2}^\pi \csc(x)\,dx$$
$$\int_{-1}^1 x^{-2}\,dx$$

1.1 Defining Type I and Type II improper integrals

Type I integrals have ∞ as one of the integration limits.

$$\int_a^\infty f(x)\,dx = \lim_{N\to\infty}\int_a^N f(x)\,dx$$

$$\int_{-\infty}^b f(x)\,dx = \lim_{M\to\infty}\int_M^b f(x)\,dx$$

$$\int_{-\infty}^\infty f(x)\,dx = \int_{-\infty}^c f(x)\,dx + \int_c^\infty f(x)\,dx\,, \text{ where } c \in \mathbb{R}$$

$$= \lim_{M\to\infty} \int_M^c f(x)\,dx + \lim_{N\to\infty} \int_c^N f(x)\,dx$$

The improper integral **converges** if the limit exists. It **diverges** if the limit doesn't exist.
In the third case, both limits must exist for the integral to converge.

In Type II integrals, the integrand has a discontinuity at the endpoints or within the interval of integration. Given $\int_a^b f(x)\,dx$:
If $f(b)$ is undefined:

$$\int_a^b f(x)\,dx = \lim_{A\to b^-} \int_a^A f(x)\,dx$$

If $f(a)$ is undefined:

$$\int_a^b f(x)\,dx = \lim_{A\to a^+} \int_A^b f(x)\,dx$$

If $f(c)$ is undefined for some $c \in (a,b)$:

$$\begin{aligned} \int_a^b f(x)\,dx &= \int_a^c f(x)\,dx + \int_c^b f(x)\,dx \\ &= \lim_{A\to c^-} \int_a^A f(x) + \lim_{A\to c^+} \int_A^b f(x) \end{aligned}$$

Example 1. Evaluate the following improper integrals.

$$\begin{aligned}
&(1) \quad \int_0^1 \frac{\ln(x)}{x}\,dx \\
&(2) \quad \int_1^\infty \frac{\ln(x)}{x}\,dx \\
&(3) \quad \int_1^\infty \frac{\ln(x)}{x^2}\,dx \\
&(4) \quad \int_{-\infty}^\infty xe^{-x}\,dx \\
&(6) \quad \int_{-\infty}^\infty x^2e^{-x^3}\,dx \\
&(7) \quad \int_0^4 \frac{1}{x^2+x-6}\,dx \\
&(8) \quad \int_2^\infty \frac{1}{\sqrt{x-2}}\,dx \\
&(9) \quad \int_{-2}^2 \frac{1}{x^2}\,dx \\
&(10) \quad \int_1^\infty \frac{\sqrt{1+\sqrt{x}}}{\sqrt{x}}\,dx
\end{aligned}$$

1.2 Comparison theorem

The comparison theorem gives a way of determining whether a specific kind of improper integral converges or diverges. Namely, given an integral of the form $\int_a^\infty f(x)\,dx$, we might be able to compare f to another function g where the behaviour of $\int_a^\infty g(x)\,dx$ is known.

Theorem 1. ***Comparison theorem (convergence)***
Let $a \in \mathbb{R}$. Suppose g, f are continuous on $[a, \infty)$.
IF (a) $0 \leq g(x) \leq f(x) \quad \forall x \in [a, \infty)$
and (b) $\int_a^\infty f(x)$ converges
THEN $\int_a^\infty g(x)$ also converges.

Theorem 2. ***Comparison theorem (divergence)***
Let $a \in \mathbb{R}$. Suppose g, f are continuous on $[a, \infty)$.
IF (a) $0 \leq f(x) \leq g(x) \quad \forall x \in [a, \infty)$
and (b) $\int_a^\infty f(x)$ diverges
THEN $\int_a^\infty g(x)$ also diverges.

The same theorems can be used on Type II improper integrals, just by replacing the interval of integration. **Make sure you verify both hypotheses before using comparison theorem.**

Example 2. Determine whether each of the following integrals converges or diverges.

$$
\begin{aligned}
&(1) \quad \int_0^\infty e^{-x^2}\,dx \\
&(2) \quad \int_1^\infty \frac{1+e^{-2x}}{x}\,dx \\
&(3) \quad \int_1^\infty \frac{dx}{x+e^{2x}} \\
&(4) \quad \int_1^\infty \frac{1}{\sqrt{x^3+1}}\,dx \\
&(5) \quad \int_0^1 \frac{1}{\csc^2 x\sqrt{x}}\,dx
\end{aligned}
$$

2 Sequences

We generally think of sequences as an ordered list of numbers, e.g. {1,2,3,4...} or {1, 1, 2, 3, 5, 8, 13, ...} or {3, 1, 4, 1, 5, 6, ...}. The precise definition of a sequence is as follows:

Definition 1. *A* ***sequence*** *of real numbers is a function whose domain is the natural numbers,* $\mathbb{N} = \{1, 2, 3, \ldots\}$.
Can be denoted by

- $a_n = f(n)$
- $\{a_n\}_{n=1}^\infty$
- $\{a_n\}$

a_n is the **general term** of the sequence—it's a function that takes a natural number n as an input and outputs a real number.
The definition of convergence of a sequence is pretty much the same as the definition of convergence of a function at infinity, as seen in MATA31.

Definition 2. *Let $l \in \mathbb{R}$. The sequence $\{a_n\}$* ***converges*** *to l iff*

$$\forall \epsilon > 0, \exists N > 0 \text{ s.t. } \forall n \in \mathbb{N}, \text{ if } n > N, \text{ then } |a_n - l| < \epsilon$$

Denoted by

- $a_n \to l$ *as* $n \to \infty$
- $a_n \to l$
- $\lim_{n\to\infty} a_n = l$

If $\{a_n\}$ does not converge, we say it ***diverges****. i.e. there does not exist any such l.*

Proving $\{a_n\}$ converges to l:

- Take arbitrary $\epsilon > 0$
- Want to find some $N \in \mathbb{R}$ where, if $n \in \mathbb{N}$ and $n > N$, then $|a_n - l| < \epsilon$. Usually N is in terms of ϵ.
- To find this N, write $|a_n - l|$ and try simplifying or finding upper bounds for this function. You usually get something like $|a_n - l| \leq \frac{1}{n^p}$ for some $p \in \mathbb{R}$.
- If you get to something like $\frac{1}{n^p}$, you can use the hypothesis $n > N$ to say $\frac{1}{n^p} < \frac{1}{N^p}$, then substitute in a choice of N to conclude $|a_n - l| \leq \frac{1}{n^p} < \epsilon$.
- To complete the proof, begin by taking arbitrary $\epsilon > 0$. Define N as you determined the previous step. Then suppose n is some natural number $> N$.
- Simplify the expression $|a_n - l|$ and show that under the above assumptions, $|a_n - l| < \epsilon$.

Example 1. Prove that $\left\{\dfrac{\sin(n)}{\sqrt[3]{n}}\right\}_{n=1}^{\infty}$ converges to 0.
Sol. Want to prove the following:

$$\text{For all } \epsilon > 0, \text{ there exists } N > 0 \text{ s.t. for all } n \in \mathbb{N}, \text{ if } n > N, \text{ then } \left|\frac{\sin(n)}{\sqrt[3]{n}}\right| < \epsilon$$

Let $\epsilon > 0$ be arbitrary.
Choose $N = \qquad\qquad > 0$.
Assume $n \in \mathbb{N}$, $n > N$.

$$\begin{aligned} \left|\frac{\sin(n)}{\sqrt[3]{n}}\right| &= \frac{|\sin(n)|}{|\sqrt[3]{n}|} && \text{abs. value properties} \\ &\leq \frac{1}{|\sqrt[3]{n}|} && \text{since } |\sin(n)| \leq 1 \text{ for all } n \end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{\sqrt[3]{n}} \quad \text{since } \sqrt[3]{n} \text{ is positive for } n > 1 \\
&< \frac{1}{\sqrt[3]{N}}
\end{aligned}
$$

Since we assumed $n > N$, it follows $\sqrt[3]{n} > \sqrt[3]{N}$ and hence $1/\sqrt[3]{n} < 1/\sqrt[3]{N}$.
Now, if we choose $N = \frac{1}{\epsilon^3}$, then $\frac{1}{\sqrt[3]{N}} = \epsilon$.Then we can complete the proof with the choice of N:
Let $\epsilon > 0$ be arbitrary.
Choose $N = \frac{1}{\epsilon^3} > 0$.
Assume $n \in \mathbb{N}$, $n > N$.

$$
\begin{aligned}
\left|\frac{\sin(n)}{\sqrt[3]{n}}\right| &= \frac{|\sin(n)|}{|\sqrt[3]{n}|} \quad \text{abs. value properties} \\
&\leq \frac{1}{|\sqrt[3]{n}|} \quad \text{since } |\sin(n)| \leq 1 \text{ for all } n \\
&= \frac{1}{\sqrt[3]{n}} \quad \text{since } \sqrt[3]{n} \text{ is positive for } n > 1 \\
&< \frac{1}{\sqrt[3]{N}} \quad \text{by } n > N \text{, it follows } \sqrt[3]{n} > \sqrt[3]{N} \text{ and hence } 1/\sqrt[3]{n} < 1/\sqrt[3]{N}. \\
&= \frac{1}{\sqrt[3]{1/\epsilon^3}} \quad \text{by choice of } N \\
&= \frac{1}{1/\epsilon} \\
&= \epsilon
\end{aligned}
$$

Hence, if $n > N$, then $\left|\frac{\sin(n)}{\sqrt[3]{n}}\right| < \epsilon$, as wanted. $\square$

2.1 Properties of convergent sequences

Let $\{a_n\}$ and $\{b_n\}$ be sequences. Let $a, b, k \in \mathbb{R}$.

Theorem 3. *Suppose $a_n \to a$ and $b_n \to b$. Then*

$$
\begin{aligned}
&(a) \quad \lim_{n\to\infty}(a_n + b_n) = \lim_{n\to\infty} a_n + \lim_{n\to\infty} b_n = a + b && (1) \\
&(b) \quad \lim_{n\to\infty} k(a_n) = ka && (2) \\
&(c) \quad \lim_{n\to\infty}(a_n)(b_n) = \big(\lim_{n\to\infty} a_n\big)\big(\lim_{n\to\infty} b_n\big) = ab && (3) \\
&(d) \quad \lim_{n\to\infty}\big(\frac{a_n}{b_n}\big) = \frac{a}{b} && (4) \\
&\qquad \textit{(Provided that } b \neq 0 \textit{ and } b_n \neq 0\ \forall n \in \mathbb{N}\textit{)} && (5)
\end{aligned}
$$

Theorem 4. ***Uniqueness of limits***
IF $\{a_n\}$ converges
THEN its limit is unique.

Definition 3. *A sequence is* ***bounded*** *if*

$$\exists c \in \mathbb{R}^+ \ s.t. \ |a_n| \leq c \quad \forall n \in \mathbb{N} \tag{6}$$

Note that $|a_n| \leq c$ *equivalently means* $-c \leq a_n \leq c$.

Theorem 5. ***Convergent sequences are bounded***
IF $\{a_n\}$ *converges*
THEN $\{a_n\}$ *is bounded.*

Example 2. Determine whether the following sequences converge or diverge and provide a complete $\epsilon - N$ proof.

$$\begin{aligned}
&(1) \quad \{\frac{n \arctan(n) - 1}{n^2 + 6\sqrt{n}}\} \\
&(2) \quad \{\frac{2n-1}{n+3}\} \\
&(3) \quad \{n^{-3/2}\} \\
&(4) \quad \{1 + 2(-1)^n\}
\end{aligned}$$

(1) Hint: use the fact $\arctan(x) < \pi/2$ for all x.

Example 3. Prove by $\epsilon - N$ definition that if $\{a_n\}$ converges to 0 and $\{b_n\}$ is bounded, then $\{a_n b_n\}$ converges.

3 Answers to selected examples

Example 1. Determine whether each of the following integrals converges or diverges.

$$
\begin{aligned}
&(1) \quad \int_0^\infty e^{-x^2}\,dx \\
&(2) \quad \int_1^\infty \frac{1+e^{-2x}}{x}\,dx \\
&(3) \quad \int_1^\infty \frac{dx}{x+e^{2x}} \\
&(4) \quad \int_{-1}^\infty \frac{1}{\sqrt{x^3+1}}\,dx \\
&(5) \quad \int_0^1 \frac{1}{\csc^2 x\sqrt{x}}\,dx
\end{aligned}
$$

(1) Use comparison test with e^{-x} to show convergence.

(2) Use comparison test with $1/x$ to show divergence.
(3) Use comparison test with $1/e^{2x}$ to show convergence. (Note: you could try using $1/x$, but this won't lead you closer to the answer.)

Sol. Comparison test with $1/e^{2x}$:
(a) For all $x \in [1,\infty)$, it holds that $1/e^{2x} \geq 0$ since $e^{2x} \geq 0$. ✓
(b) Since $x \in [1,\infty)$, it follows that

$$x + e^{2x} > e^{2x}$$

Hence,

$$\frac{1}{x+e^{2x}} < \frac{1}{e^{2x}} \quad ✓$$

Now, if we prove the integral of $1/e^{2x}$ over $[1,\infty)$ converges, we can conclude that the original integral converges by CT.

$$
\begin{aligned}
\int_1^\infty e^{-2x}\,dx &= \lim_{N\to\infty}\int_1^N e^{-2x}\,dx \\
&= \lim_{N\to\infty}\Big[-\frac{1}{2}e^{-2x}\Big]_1^N \\
&= \lim_{N\to\infty} -\frac{1}{2}(e^{-2N}-e^{-2}) \\
&= \frac{1}{2}e^{-2} \quad \text{since as } N\to\infty,\ e^{-2N}\to 0, \text{ and } \frac{1}{2}e^{-2} \text{ is constant}
\end{aligned}
$$

Therefore, $\int_1^\infty e^{-2x}\,dx$ converges, hence by CT, $\int_1^\infty \frac{dx}{x+e^{2x}}$ converges. □
Note: it's also true that $\frac{1}{x+e^{2x}} < \frac{1}{x}$. But $\int_1^\infty \frac{1}{x}\,dx$ diverges, so CT can't tell you anything about the original integral.

(4) Use comparison test with $1/\sqrt{x^3}$ to show convergence. Don't forget to split the integral first since the function also has a discontinuity at $x=-1$.

(5) Recall $\csc x = 1/\sin x$ and use the fact that $0 \leq \sin^2 x \leq 1$.