

MATA37 Week 10

Kevin Santos

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1 Trigonometric substitution

Some integrals involving certain expressions become easier when you perform substitution with a trigonometric function, which are based on certain trigonometric identities. The following table summarizes certain expressions that can signal where to use a trig substitution. Note that a^2 represents a real number that appears in the integral, while x is the integration variable. The functions also need to be restricted to certain domains so that they are one-to-one. **Don't forget to include the restriction on θ .** Also, remember the fact that $\sqrt{a^2} = |a|$, NOT just $\sqrt{a^2} = a$. If you get something like $\sqrt{\cos^2(\theta)} = |\cos(\theta)|$, you can use the restriction on theta to simplify the answer (see sample answer). Also note that when $\theta \in [0, \frac{\pi}{2})$, $\sec \theta > 0$, and when $\theta \in (\frac{\pi}{2}, \pi]$, $\sec \theta < 0$.

Expression	Substitution	Restriction	Identity
$a^2 - x^2$	$x = a \sin \theta$	$\theta \in [-\frac{\pi}{2}, \frac{\pi}{2}]$	$1 - \sin^2 \theta = \cos^2 \theta$
$a^2 + x^2$	$x = a \tan \theta$	$\theta \in (-\frac{\pi}{2}, \frac{\pi}{2})$	$1 + \tan^2 \theta = \sec^2 \theta$
$x^2 - a^2$	$x = a \sec \theta$	$\theta \in [0, \frac{\pi}{2}) \cup (\frac{\pi}{2}, \pi]$	$\sec^2 \theta - 1 = \tan^2 \theta$

1.1 Trigonometric integrals

After applying trigonometric substitution, you will often end up with a trigonometric integral. A trigonometric integral is one that involves integrating some product of trigonometric functions. The trigonometric integrals we'll encounter will come in the following forms.

Let $m, n, l, k \in \mathbb{Z}^{\geq 0}$. (Note that $\mathbb{Z}^{\geq 0}$ means some of these integers can be 0.) Let $a, b \in \mathbb{R}$, $a < b$.

Definition 1. A **trigonometric integral** is an integral that is one of the following forms:

$$\begin{aligned} \int \sin^l(x) \cos^k(x) dx \quad \text{or} \quad \int_a^b \sin^l(x) \cos^k(x) dx \\ \int \sec^m(x) \tan^n(x) dx \quad \text{or} \quad \int_a^b \sec^m(x) \tan^n(x) dx \end{aligned}$$

When evaluating trig integrals, it helps if you have the following identities memorized (A represents a real number) :

$$\boxed{\sin^2 A + \cos^2 A = 1} \tag{1}$$

$$\tan^2 A + 1 = \sec^2 A \tag{2}$$

$$\boxed{\sin^2 A = \frac{1 - \cos(2A)}{2}} \quad (3)$$

$$\cos^2 A = \frac{1 + \cos(2A)}{2} \quad (4)$$

$$\boxed{\sin(2A) = 2 \sin A \cos A} \quad (5)$$

If you want to, you only really need to memorize the boxed identities.

If you take (1) and divide both sides by $\cos^2 A$, you get (2). (Remember that $\tan A = \frac{\sin A}{\cos A}$ and $\sec A = \frac{1}{\cos A}$.)

If you remember (3), then you can remember the "cos" version in (4) by changing the sign from "-" to "+".

You should also remember how the reciprocal trig functions csc, sec, and cot are defined:

$$\begin{aligned} \sec(x) &= \frac{1}{\cos(x)} \\ \csc(x) &= \frac{1}{\sin(x)} \\ \cot(x) &= \frac{1}{\tan(x)} \left(= \frac{\cos(x)}{\sin(x)} \right) \end{aligned}$$

As well, recall these derivatives of trigonometric functions:

$$\begin{aligned} \frac{d}{dx} \sin(x) &= \cos(x) \\ \frac{d}{dx} \cos(x) &= -\sin(x) \\ \frac{d}{dx} \tan(x) &= \sec^2(x) \\ \frac{d}{dx} \sec(x) &= \tan(x) \sec(x) \end{aligned}$$

1.2 sin/cos trig integrals

The different approaches to solving a trig integral with sin and/or cos can be split into two cases:

Example 1. Even/odd exponents and odd/odd exponents

Evaluate $\int \cos^2 x \sin^5 x \, dx$.

Sol.

$$\begin{aligned} \int \cos^2 x \sin^5 x \, dx &= \int \cos^2 x \sin^4 x \sin x \, dx \\ &= \int \cos^2 x (\sin^2 x)^2 \sin x \, dx \\ &= \int \cos^2 x (1 - \cos^2 x)^2 \sin x \, dx \quad \text{Let } u = \cos x. \implies du = -\sin x \, dx \\ &= \int u^2 (1 - u^2)^2 (-du) \\ &= - \int u^2 (u^4 - 2u^2 + 1) \, du \end{aligned}$$

$$\begin{aligned}
&= - \int u^6 - 2u^4 + u^2 du \\
&= - \left(\frac{u^7}{7} - \frac{2u^5}{5} + \frac{u^3}{3} \right) + C \\
&= -\frac{1}{7} \cos^7 x + \frac{2}{5} \cos^5 x - \frac{1}{3} \cos^3 x + C \quad \square
\end{aligned}$$

The term with the odd exponent, $\sin^5 x$, was split up into $\sin^4 x$ and $\sin x$. Then $\sin^4 x$ was written in terms of $\sin^2 x$ so that we can apply the identity. It becomes easy to apply substitution after that.

A similar procedure can be applied **when the exponents are both odd**. You can split up one of the terms as shown above. Splitting up either term should both give the same answer. Try it out on, for example, $\int_0^\pi \cos^5 x \sin^3 x dx$.

Example 2. Even/even exponents

Evaluate $\int \cos^2(2x) \sin^4(2x) dx$.

Sol.

$$\begin{aligned}
\int \cos^2(2x) \sin^4(2x) dx &= \int \cos^2(2x) (\sin^2(2x))^2 dx \\
&= \int \left(\frac{1 - \cos(4x)}{2} \right) \left(\frac{1 + \cos(4x)}{2} \right)^2 dx \\
&= \frac{1}{8} \int (1 - \cos(4x))(1 + \cos(4x))(1 + \cos(4x)) dx \\
&= \frac{1}{8} \int (1 - \cos^2(4x))(1 + \cos(4x)) dx \\
&= \frac{1}{8} \int 1 - \cos^2(4x) + \cos(4x) - \cos^3(4x) dx \\
&= \frac{1}{8} \left(\int 1 dx - \int \cos^2(4x) dx + \int \cos(4x) dx - \int \cos^3(4x) dx \right) \\
&= \dots
\end{aligned}$$

1.3 sec/tan trig integrals

Recall that the derivative of $\tan x$ is $\sec^2 x$ and the derivative of $\sec x$ is $\sec x \tan x$.

You can do a u-substitution on a sec/tan trig integral if you can rearrange the terms to give something of the following forms:

$$\begin{aligned}
\int \left(\text{just tangent stuff} \right) \sec^2 x dx & \quad u = \tan x \implies du = \sec^2 x dx \\
\int \left(\text{just secant stuff} \right) \sec x \tan x dx & \quad u = \sec x \implies du = \sec x \tan x dx
\end{aligned}$$

Example 3. Evaluate the following integrals using trigonometric substitution:

- (1) $\int_0^3 \frac{1}{4x^2 + 9} dx$
- (2) $\int \sqrt{x-1}\sqrt{3-x} dx$
- (3) $\int \ln(x^2 + 1) dx$
- (4) $\int \frac{\sqrt{x^2 - 2}}{x} dx, \quad x > \sqrt{2}$
- (5) $\int \frac{1}{\sqrt{x^2 - 5}} dx, \quad x < -\sqrt{5}$
- (6) $\int \sqrt{x^2 - 8x + 25} dx$
- (7) $\int \sqrt{x^2 + 6x + 18} dx$
- (8) $\int e^{2x}(1 - e^{4x})^{3/2} dx$

Hints:

- (2) Use the fact that $\sqrt{a}\sqrt{b} = \sqrt{ab}$. Then complete the square.
- (3) Use integration by parts first.
- (8) Use substitution first.

2 Improper integrals

In the past when calculating $\int_a^b f(x) dx$, we have assumed that:

- (1) the interval $[a, b]$ is bounded ($a, b \neq \infty$)
- (2) $f(x)$ has no vertical asymptotes (VAs) in $[a, b]$ ($\text{range}(f(x))$ is bounded)

If the condition(s) (1) and/or (2) are broken, $\int_a^b f(x) dx$ is an **improper integral**.

Examples of **type I improper integrals** (integrals that break condition (1)):

$$\int_3^\infty \frac{\arctan(x)}{1+x^2} dx$$

$$\int_{-\infty}^1 \arctan(x) dx$$

Examples of **type II improper integrals** (integrals that break condition (2)):

$$\int_2^5 \frac{8}{\sqrt{x-2}} dx$$

$$\int_{\pi/2}^\pi \csc(x) dx$$

$$\int_{-1}^1 x^{-2} dx$$

2.1 Defining Type I and Type II improper integrals

Type I integrals have ∞ as one of the integration limits. They are integrals over **unbounded intervals**.

$$\int_a^\infty f(x) dx = \lim_{N \rightarrow \infty} \int_a^N f(x) dx$$

$$\int_{-\infty}^b f(x) dx = \lim_{M \rightarrow \infty} \int_M^b f(x) dx$$

$$\begin{aligned} \int_{-\infty}^\infty f(x) dx &= \int_{-\infty}^c f(x) dx + \int_c^\infty f(x) dx, \text{ where } c \in \mathbb{R} \\ &= \lim_{M \rightarrow \infty} \int_M^c f(x) dx + \lim_{N \rightarrow \infty} \int_c^N f(x) dx \end{aligned}$$

The improper integral **converges** if the limit exists. It **diverges** if the limit doesn't exist.

In the third case, both limits must exist for the integral to converge.

In Type II integrals, the integrand has a discontinuity at the endpoints or within the interval of integration. These are integrals **over discontinuities**.

Given $\int_a^b f(x) dx$:

If $f(b)$ is undefined:

$$\int_a^b f(x) dx = \lim_{A \rightarrow b^-} \int_a^A f(x) dx$$

If $f(a)$ is undefined:

$$\int_a^b f(x) dx = \lim_{A \rightarrow a^+} \int_A^b f(x) dx$$

If $f(c)$ is undefined for some $c \in (a, b)$:

$$\begin{aligned} \int_a^b f(x) dx &= \int_a^c f(x) dx + \int_c^b f(x) dx \\ &= \lim_{A \rightarrow c^-} \int_a^A f(x) + \lim_{A \rightarrow c^+} \int_A^b f(x) \end{aligned}$$

In the third case, we need **both** limits to converge for the original integral to converge. If one of the parts diverges, **and we do not get an indeterminate form**, then the original integral diverges.

Example 1. Evaluate the following improper integrals.

- (1) $\int_0^1 \frac{\ln(x)}{x} dx$
- (2) $\int_1^\infty \frac{\ln(x)}{x} dx$
- (3) $\int_1^\infty \frac{\ln(x)}{x^2} dx$

$$(4) \int_0^{\infty} x e^{-x} dx$$

$$(5) \int_{-\infty}^{\infty} x e^{-x} dx$$

$$(6) \int_{-\infty}^{\infty} x^2 e^{-x^3} dx$$

$$(7) \int_0^4 \frac{1}{x^2 + x - 6} dx$$

$$(8) \int_2^5 \frac{1}{\sqrt{x-2}} dx$$

3 Sample answers

Trig substitution: (4)

$$\int \frac{\sqrt{x^2 - 2}}{x} dx, \quad x > \sqrt{2}$$

Note the expression $x^2 - 2$, which is of the form $x^2 - a^2$ with $a = \sqrt{2}$. Use $x = \sqrt{2} \sec \theta$.

$$x = \sqrt{2} \sec \theta \quad \text{since } x > \sqrt{2}. \quad \theta \in [0, \pi/2).$$

$$dx = \sqrt{2} \sec \theta \tan \theta d\theta$$

Rewriting the expression $\sqrt{x^2 - 2}$:

$$\begin{aligned} \sqrt{x^2 - 2} &= \sqrt{(\sqrt{2} \sec \theta)^2 - 2} \\ &= \sqrt{2 \sec^2 \theta - 2} \\ &= \sqrt{2(\sec^2 \theta - 1)} \\ &= \sqrt{2} \sqrt{\sec^2 \theta - 1} \\ &= \sqrt{2} \sqrt{\tan^2 \theta} \\ &= \sqrt{2} |\tan \theta| \\ &= \sqrt{2} \tan \theta \quad (\text{Since } \theta \in [0, \pi/2), \tan \theta > 0) \end{aligned}$$

Rewriting the integral,

$$\begin{aligned} \int \frac{\sqrt{x^2 - 2}}{x} dx &= \int \frac{\sqrt{2} \tan \theta}{\sqrt{2} \sec \theta} \sqrt{2} \sec \theta \tan \theta d\theta \\ &= \int \sqrt{2} \tan^2 \theta d\theta \\ &= \sqrt{2} \int \sec^2 \theta - 1 d\theta \\ &= \sqrt{2} (\tan \theta - \theta) + C \end{aligned}$$

You can draw a right triangle with angle θ and $\sec \theta = x/\sqrt{2}$ to solve for $\tan \theta$. I'm not good enough with LaTeX to include the diagram but it results in $\tan \theta = \sqrt{x^2 - 2}$. Also note that $x = \sqrt{2} \sec \theta$ implies $\theta = \sec^{-1}(\frac{x}{\sqrt{2}})$. (The -1 denotes the inverse secant function, NOT a negative exponent.)

$$\begin{aligned} &= \sqrt{2} (\tan \theta - \theta) + C \\ &= \sqrt{2} (\sqrt{x^2 - 2} - \sec^{-1}(\frac{x}{\sqrt{2}})) + C \end{aligned}$$

Improper integrals by definition: (1)

$$\int_0^1 \frac{\ln(x)}{x} dx$$

The integral is over the interval $[0, 1]$, and the integrand has a discontinuity/is not defined at $x = 0$, which

is in the interval. This is a Type II improper integral.

$$\begin{aligned}\int_0^1 \frac{\ln(x)}{x} dx &= \lim_{A \rightarrow 0^+} \int_A^1 \frac{\ln(x)}{x} dx \\ &= \lim_{A \rightarrow 0^+} \int_{\ln A}^0 u du \quad \text{Use substitution } u = \ln(x) \implies du = \frac{1}{x} dx \text{ and change int. limits} \\ &= \lim_{A \rightarrow 0^+} \left[\frac{1}{2} u^2 \right]_{\ln A}^0 \\ &= \lim_{A \rightarrow 0^+} -\frac{1}{2} (\ln A)^2\end{aligned}$$

Note that as $A \rightarrow 0^+$, $\ln A \rightarrow -\infty$ (picture the graph of $\ln(x)$.) Hence, $\lim_{A \rightarrow 0^+} -\frac{1}{2} (\ln A)^2 = -\infty$, so the integral diverges.