

MATA37 Week 11

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1 Improper integrals

In the past when calculating $\int_a^b f(x) dx$, we have assumed that:

- (1) the interval $[a, b]$ is bounded ($a, b \neq \infty$)
- (2) $f(x)$ has no vertical asymptotes (VAs) in $[a, b]$ ($\text{range}(f(x))$ is bounded)

If the condition(s) (1) and/or (2) are broken, $\int_a^b f(x) dx$ is an **improper integral**.

Examples of **type I improper integrals** (integrals that break condition (1)):

$$\int_3^\infty \frac{\arctan(x)}{1+x^2} dx$$
$$\int_{-\infty}^1 \arctan(x) dx$$

Examples of **type II improper integrals** (integrals that break condition (2)):

$$\int_2^5 \frac{8}{\sqrt{x-2}} dx$$
$$\int_{\pi/2}^\pi \csc(x) dx$$
$$\int_{-1}^1 x^{-2} dx$$

1.1 Defining Type I and Type II improper integrals

Type I integrals have ∞ as one of the integration limits.

$$\int_a^\infty f(x) dx = \lim_{N \rightarrow \infty} \int_a^N f(x) dx$$

$$\int_{-\infty}^b f(x) dx = \lim_{M \rightarrow \infty} \int_M^b f(x) dx$$

$$\int_{-\infty}^\infty f(x) dx = \int_{-\infty}^c f(x) dx + \int_c^\infty f(x) dx, \text{ where } c \in \mathbb{R}$$

$$= \lim_{M \rightarrow \infty} \int_M^c f(x) dx + \lim_{N \rightarrow \infty} \int_c^N f(x) dx$$

The improper integral **converges** if the limit exists. It **diverges** if the limit doesn't exist.

In the third case, both limits must exist for the integral to converge.

In Type II integrals, the integrand has a discontinuity at the endpoints or within the interval of integration.
Given $\int_a^b f(x) dx$:

If $f(b)$ is undefined:

$$\int_a^b f(x) dx = \lim_{A \rightarrow b^-} \int_a^A f(x) dx$$

If $f(a)$ is undefined:

$$\int_a^b f(x) dx = \lim_{A \rightarrow a^+} \int_A^b f(x) dx$$

If $f(c)$ is undefined for some $c \in (a, b)$:

$$\begin{aligned} \int_a^b f(x) dx &= \int_a^c f(x) dx + \int_c^b f(x) dx \\ &= \lim_{A \rightarrow c^-} \int_a^A f(x) + \lim_{A \rightarrow c^+} \int_A^b f(x) \end{aligned}$$

Example 1. Evaluate the following improper integrals.

$$(1) \int_0^1 \frac{\ln(x)}{x} dx$$

$$(2) \int_1^\infty \frac{\ln(x)}{x} dx$$

$$(3) \int_1^\infty \frac{\ln(x)}{x^2} dx$$

$$(4) \int_{-\infty}^\infty x e^{-x} dx$$

$$(6) \int_{-\infty}^\infty x^2 e^{-x^3} dx$$

$$(7) \int_0^4 \frac{1}{x^2 + x - 6} dx$$

$$(8) \int_2^\infty \frac{1}{\sqrt{x-2}} dx$$

$$(9) \int_{-2}^2 \frac{1}{x^2} dx$$

$$(10) \int_1^\infty \frac{\sqrt{1+\sqrt{x}}}{\sqrt{x}} dx$$

1.2 Comparison theorem

The comparison theorem gives a way of determining whether a specific kind of improper integral converges or diverges. Namely, given an integral of the form $\int_a^\infty f(x) dx$, we might be able to compare f to another function g where the behaviour of $\int_a^\infty g(x) dx$ is known.

Theorem 1. Comparison theorem (convergence)

Let $a \in \mathbb{R}$. Suppose g, f are continuous on $[a, \infty)$.

IF (a) $0 \leq g(x) \leq f(x) \quad \forall x \in [a, \infty)$

and (b) $\int_a^\infty f(x) dx$ converges

THEN $\int_a^\infty g(x) dx$ also converges.

Theorem 2. Comparison theorem (divergence)

Let $a \in \mathbb{R}$. Suppose g, f are continuous on $[a, \infty)$.

IF (a) $0 \leq f(x) \leq g(x) \quad \forall x \in [a, \infty)$

and (b) $\int_a^\infty f(x) dx$ diverges

THEN $\int_a^\infty g(x) dx$ also diverges.

The same theorems can be used on Type II improper integrals, just by replacing the interval of integration.

Make sure you verify both hypotheses before using comparison theorem.

Example 2. Determine whether each of the following integrals converges or diverges.

- (1) $\int_0^\infty e^{-x^2} dx$
- (2) $\int_1^\infty \frac{1 + e^{-2x}}{x} dx$
- (3) $\int_1^\infty \frac{dx}{x + e^{2x}}$
- (4) $\int_{-1}^\infty \frac{1}{\sqrt{x^3 + 1}} dx$
- (5) $\int_0^1 \frac{1}{\csc^2 x \sqrt{x}} dx$

2 Sequences

We generally think of sequences as an ordered list of numbers, e.g. $\{1, 2, 3, 4, \dots\}$ or $\{1, 1, 2, 3, 5, 8, 13, \dots\}$ or $\{3, 1, 4, 1, 5, 6, \dots\}$. The precise definition of a sequence is as follows:

Definition 1. A **sequence** of real numbers is a function whose domain is the natural numbers,

$\mathbb{N} = \{1, 2, 3, \dots\}$.

Can be denoted by

- $a_n = f(n)$
- $\{a_n\}_{n=1}^\infty$
- $\{a_n\}$

a_n is the **general term** of the sequence—it's a function that takes a natural number n as an input and outputs a real number.

The definition of convergence of a sequence is pretty much the same as the definition of convergence of a function at infinity, as seen in MATA31.

Definition 2. Let $l \in \mathbb{R}$. The sequence $\{a_n\}$ **converges** to l iff

$$\forall \epsilon > 0, \exists N > 0 \text{ s.t. } \forall n \in \mathbb{N}, \text{ if } n > N, \text{ then } |a_n - l| < \epsilon$$

Denoted by

- $a_n \rightarrow l$ as $n \rightarrow \infty$
- $a_n \rightarrow l$
- $\lim_{n \rightarrow \infty} a_n = l$

If $\{a_n\}$ does not converge, we say it **diverges**. i.e. there does not exist any such l .

Proving $\{a_n\}$ converges to l :

- Take arbitrary $\epsilon > 0$
- Want to find some $N \in \mathbb{R}$ where, if $n \in \mathbb{N}$ and $n > N$, then $|a_n - l| < \epsilon$. Usually N is in terms of ϵ .
- To find this N , write $|a_n - l|$ and try simplifying or finding upper bounds for this function. You usually get something like $|a_n - l| \leq \frac{1}{n^p}$ for some $p \in \mathbb{R}$.
- If you get to something like $\frac{1}{n^p}$, you can use the hypothesis $n > N$ to say $\frac{1}{n^p} < \frac{1}{N^p}$, then substitute in a choice of N to conclude $|a_n - l| \leq \frac{1}{N^p} < \epsilon$.
- To complete the proof, begin by taking arbitrary $\epsilon > 0$. Define N as you determined the previous step. Then suppose n is some natural number $> N$.
- Simplify the expression $|a_n - l|$ and show that under the above assumptions, $|a_n - l| < \epsilon$.

Example 1. Prove that $\left\{ \frac{\sin(n)}{\sqrt[3]{n}} \right\}_{n=1}^{\infty}$ converges to 0.

Sol. Want to prove the following:

$$\text{For all } \epsilon > 0, \text{ there exists } N > 0 \text{ s.t. for all } n \in \mathbb{N}, \text{ if } n > N, \text{ then } \left| \frac{\sin(n)}{\sqrt[3]{n}} \right| < \epsilon$$

Let $\epsilon > 0$ be arbitrary.

Choose $N =$ > 0 .

Assume $n \in \mathbb{N}, n > N$.

$$\begin{aligned} \left| \frac{\sin(n)}{\sqrt[3]{n}} \right| &= \frac{|\sin(n)|}{|\sqrt[3]{n}|} \quad \text{abs. value properties} \\ &\leq \frac{1}{\sqrt[3]{n}} \quad \text{since } |\sin(n)| \leq 1 \text{ for all } n \end{aligned}$$

$$\begin{aligned}
&= \frac{1}{\sqrt[3]{n}} \quad \text{since } \sqrt[3]{n} \text{ is positive for } n > 1 \\
&< \frac{1}{\sqrt[3]{N}}
\end{aligned}$$

Since we assumed $n > N$, it follows $\sqrt[3]{n} > \sqrt[3]{N}$ and hence $1/\sqrt[3]{n} < 1/\sqrt[3]{N}$.

Now, if we choose $N = \frac{1}{\epsilon^3}$, then $\frac{1}{\sqrt[3]{N}} = \epsilon$. Then we can complete the proof with the choice of N :

Let $\epsilon > 0$ be arbitrary.

Choose $N = \frac{1}{\epsilon^3} > 0$.

Assume $n \in \mathbb{N}$, $n > N$.

$$\begin{aligned}
\left| \frac{\sin(n)}{\sqrt[3]{n}} \right| &= \frac{|\sin(n)|}{|\sqrt[3]{n}|} \quad \text{abs. value properties} \\
&\leq \frac{1}{|\sqrt[3]{n}|} \quad \text{since } |\sin(n)| \leq 1 \text{ for all } n \\
&= \frac{1}{\sqrt[3]{n}} \quad \text{since } \sqrt[3]{n} \text{ is positive for } n > 1 \\
&< \frac{1}{\sqrt[3]{N}} \quad \text{by } n > N, \text{ it follows } \sqrt[3]{n} > \sqrt[3]{N} \text{ and hence } 1/\sqrt[3]{n} < 1/\sqrt[3]{N}. \\
&= \frac{1}{\sqrt[3]{1/\epsilon^3}} \quad \text{by choice of } N \\
&= \frac{1}{1/\epsilon} \\
&= \epsilon
\end{aligned}$$

Hence, if $n > N$, then $\left| \frac{\sin(n)}{\sqrt[3]{n}} \right| < \epsilon$, as wanted. \square

2.1 Properties of convergent sequences

Let $\{a_n\}$ and $\{b_n\}$ be sequences. Let $a, b, k \in \mathbb{R}$.

Theorem 3. Suppose $a_n \rightarrow a$ and $b_n \rightarrow b$. Then

$$(a) \quad \lim_{n \rightarrow \infty} (a_n + b_n) = \lim_{n \rightarrow \infty} a_n + \lim_{n \rightarrow \infty} b_n = a + b \quad (1)$$

$$(b) \quad \lim_{n \rightarrow \infty} k(a_n) = ka \quad (2)$$

$$(c) \quad \lim_{n \rightarrow \infty} (a_n)(b_n) = \left(\lim_{n \rightarrow \infty} a_n \right) \left(\lim_{n \rightarrow \infty} b_n \right) = ab \quad (3)$$

$$(d) \quad \lim_{n \rightarrow \infty} \left(\frac{a_n}{b_n} \right) = \frac{a}{b} \quad (4)$$

$$(Provided that b \neq 0 \text{ and } b_n \neq 0 \forall n \in \mathbb{N}) \quad (5)$$

Theorem 4. Uniqueness of limits

IF $\{a_n\}$ converges

THEN its limit is unique.

Definition 3. A sequence is **bounded** if

$$\exists c \in \mathbb{R}^+ \text{ s.t. } |a_n| \leq c \quad \forall n \in \mathbb{N} \quad (6)$$

Note that $|a_n| \leq c$ equivalently means $-c \leq a_n \leq c$.

Theorem 5. *Convergent sequences are bounded*

IF $\{a_n\}$ converges

THEN $\{a_n\}$ is bounded.

Example 2. Determine whether the following sequences converge or diverge and provide a complete $\epsilon - N$ proof.

$$(1) \quad \left\{ \frac{n \arctan(n) - 1}{n^2 + 6\sqrt{n}} \right\}$$

$$(2) \quad \left\{ \frac{2n - 1}{n + 3} \right\}$$

$$(3) \quad \{n^{-3/2}\}$$

$$(4) \quad \{1 + 2(-1)^n\}$$

(1) Hint: use the fact $\arctan(x) < \pi/2$ for all x .

Example 3. Prove by $\epsilon - N$ definition that if $\{a_n\}$ converges to 0 and $\{b_n\}$ is bounded, then $\{a_n b_n\}$ converges.

3 Answers to selected examples

Example 1. Determine whether each of the following integrals converges or diverges.

- (1) $\int_0^{\infty} e^{-x^2} dx$
- (2) $\int_1^{\infty} \frac{1 + e^{-2x}}{x} dx$
- (3) $\int_1^{\infty} \frac{dx}{x + e^{2x}}$
- (4) $\int_{-1}^{\infty} \frac{1}{\sqrt{x^3 + 1}} dx$
- (5) $\int_0^1 \frac{1}{\csc^2 x \sqrt{x}} dx$

(1) Use comparison test with e^{-x} to show convergence.

(2) Use comparison test with $1/x$ to show divergence.

(3) Use comparison test with $1/e^{2x}$ to show convergence. (Note: you could try using $1/x$, but this won't lead you closer to the answer.)

Sol. Comparison test with $1/e^{2x}$:

(a) For all $x \in [1, \infty)$, it holds that $1/e^{2x} \geq 0$ since $e^{2x} \geq 0$. ✓

(b) Since $x \in [1, \infty)$, it follows that

$$x + e^{2x} > e^{2x}$$

Hence,

$$\frac{1}{x + e^{2x}} < \frac{1}{e^{2x}} \quad \checkmark$$

Now, if we prove the integral of $1/e^{2x}$ over $[1, \infty)$ converges, we can conclude that the original integral converges by CT.

$$\begin{aligned} \int_1^{\infty} e^{-2x} dx &= \lim_{N \rightarrow \infty} \int_1^N e^{-2x} dx \\ &= \lim_{N \rightarrow \infty} \left[-\frac{1}{2} e^{-2x} \right]_1^N \\ &= \lim_{N \rightarrow \infty} -\frac{1}{2} (e^{-2N} - e^{-2}) \\ &= \frac{1}{2} e^{-2} \quad \text{since as } N \rightarrow \infty, e^{-2N} \rightarrow 0, \text{ and } \frac{1}{2} e^{-2} \text{ is constant} \end{aligned}$$

Therefore, $\int_1^{\infty} e^{-2x} dx$ converges, hence by CT, $\int_1^{\infty} \frac{dx}{x + e^{2x}}$ converges. \square

Note: it's also true that $\frac{1}{x + e^{2x}} < \frac{1}{x}$. But $\int_1^{\infty} \frac{1}{x} dx$ diverges, so CT can't tell you anything about the original integral.

(4) Use comparison test with $1/\sqrt{x^3}$ to show convergence. Don't forget to split the integral first since the function also has a discontinuity at $x = -1$.

(5) Recall $\csc x = 1/\sin x$ and use the fact that $0 \leq \sin^2 x \leq 1$.