MATA37 Week 10

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1 Trigonometric substitution

Some integrals involving certain expressions become easier when you perform substitution with a trigonometric function, which are based on certain trigonometric identities. The following table summarizes certain expressions that can signal where to use a trig substitution. Note that a^2 represents a real number that appears in the integral, while x is the integration variable. The functions also need to be restricted to certain domains so that they are one-to-one. **Don't forget to include the restriction on θ.** Also, remember the fact that $\sqrt{a^2} = |a|$, NOT just $\sqrt{a^2} = a$. If you get something like $\sqrt{\cos^2(\theta)} = |\cos(\theta)|$, you can use the restriction on theta to simplify the answer (see sample answer). Also note that when $\theta \in [0, \frac{\pi}{2})$, $\sec\theta > 0$, and when $\theta \in (\frac{\pi}{2}, \pi]$, $\sec\theta < 0$.

Expression	Substitution	Restriction	Identity
$a^2 - x^2$	$x = a\sin\theta$	$\theta \in [-\frac{\pi}{2}, \frac{\pi}{2}]$	$1 - \sin^2\theta = \cos^2\theta$
$a^2 + x^2$	$x = a\tan\theta$	$\theta \in (-\frac{\pi}{2}, \frac{\pi}{2})$	$1 + \tan^2\theta = \sec^2\theta$
$x^2 - a^2$	$x = a\sec\theta$	$\theta \in [0, \frac{\pi}{2}) \cup (\frac{\pi}{2}, \pi]$	$\sec^2\theta - 1 = \tan^2\theta$

Example 1. Evaluate the following integrals using trigonometric substitution:

$$(1) \quad \int_0^3 \frac{1}{4x^2+9}\,dx$$

$$(2) \quad \int \sqrt{x-1}\sqrt{3-x}\,dx$$

$$(3) \quad \int \ln\big(x^2+1\big)\,dx$$

$$(4) \quad \int \frac{\sqrt{x^2-2}}{x}\,dx, \quad x > \sqrt{2}$$

$$(5) \quad \int \frac{1}{\sqrt{x^2-5}}\,dx, \quad x < -\sqrt{5}$$

$$(6) \quad \int \sqrt{x^2-8x+25}\,dx$$

$$(7) \quad \int \sqrt{x^2+6x+18}\,dx$$

$$(8) \quad \int e^{2x}(1-e^{4x})^{3/2}\,dx$$

Hints:
(2) Use the fact that $\sqrt{a}\sqrt{b} = \sqrt{ab}$. Then complete the square.

(3) Use integration by parts first.
(8) Use substitution first.

2 Improper integrals

In the past when calculating $\int_a^b f(x)\,dx$, we have assumed that:
(1) the interval $[a, b]$ is bounded ($a, b \neq \infty$)
(2) $f(x)$ has no vertical asymptotes (VAs) in $[a, b]$ (range($f(x)$) is bounded)

If the condition(s) (1) and/or (2) are broken, $\int_a^b f(x)\,dx$ is an **improper integral**.

Examples of **type I improper integrals** (integrals that break condition (1)):

$$\int_3^\infty \frac{\arctan(x)}{1+x^2}\,dx$$
$$\int_{-\infty}^1 \arctan(x)\,dx$$

Examples of **type II improper integrals** (integrals that break condition (2)):

$$\int_2^5 \frac{8}{\sqrt{x-2}}\,dx$$
$$\int_{\pi/2}^{\pi} \csc(x)\,dx$$
$$\int_{-1}^1 x^{-2}\,dx$$

2.1 Defining Type I and Type II improper integrals

Type I integrals have ∞ as one of the integration limits.

$$\int_a^\infty f(x)\,dx = \lim_{N\to\infty}\int_a^N f(x)\,dx$$

$$\int_{-\infty}^b f(x)\,dx = \lim_{M\to\infty}\int_M^b f(x)\,dx$$

$$\begin{aligned}\int_{-\infty}^\infty f(x)\,dx &= \int_{-\infty}^c f(x)\,dx + \int_c^\infty f(x)\,dx\,,\ \text{where } c\in\mathbb{R}\\ &= \lim_{M\to\infty}\int_M^c f(x)\,dx + \lim_{N\to\infty}\int_c^N f(x)\,dx\end{aligned}$$

The improper integral **converges** if the limit exists. It **diverges** if the limit doesn't exist.
In the third case, both limits must exist for the integral to converge.

In Type II integrals, the integrand has a discontinuity at the endpoints or within the interval of integration. Given $\int_a^b f(x)\,dx$:
If $f(b)$ is undefined:

$$\int_a^b f(x)\,dx = \lim_{A\to b^-} \int_a^A f(x)\,dx$$

If $f(a)$ is undefined:

$$\int_a^b f(x)\,dx = \lim_{A\to a^+} \int_A^b f(x)\,dx$$

If $f(c)$ is undefined for some $c \in (a,b)$:

$$\begin{aligned}\int_a^b f(x)\,dx &= \int_a^c f(x)\,dx + \int_c^b f(x)\,dx \\ &= \lim_{A\to c^-} \int_a^A f(x) + \lim_{A\to c^+} \int_A^b f(x)\end{aligned}$$

In the third case, we need **both** limits to converge for the original integral to converge. If one of the parts diverges, **and we do not get an indeterminate form**, then the original integral diverges.

Example 1. Evaluate the following improper integrals.

(1) $\displaystyle\int_0^1 \frac{\ln(x)}{x}\,dx$

(2) $\displaystyle\int_1^\infty \frac{\ln(x)}{x}\,dx$

(3) $\displaystyle\int_1^\infty \frac{\ln(x)}{x^2}\,dx$

(4) $\displaystyle\int_0^\infty xe^{-x}\,dx$

(5) $\displaystyle\int_{-\infty}^\infty x\,dx$

(6) $\displaystyle\int_{-\infty}^\infty x^2e^{-x^3}\,dx$

(7) $\displaystyle\int_0^4 \frac{1}{x^2+x-6}\,dx$

(8) $\displaystyle\int_2^5 \frac{1}{\sqrt{x-2}}$

3 Sample answers

Trig substitution: (4)

$$\int \frac{\sqrt{x^2-2}}{x}\,dx, \quad x > \sqrt{2}$$

Note the expression x^2-2, which is of the form x^2-a^2 with $a=\sqrt{2}$. Use $x=\sqrt{2}\sec\theta$.

$$\begin{aligned} x &= \sqrt{2}\sec\theta \qquad \text{since } x > \sqrt{2}.\ \theta\in[0,\pi/2). \\ dx &= \sqrt{2}\sec\theta\tan\theta\,d\theta \end{aligned}$$

Rewriting the expression $\sqrt{x^2-2}$:

$$\begin{aligned} \sqrt{x^2-2} &= \sqrt{(\sqrt{2}\sec\theta)^2-2} \\ &= \sqrt{2\sec^2\theta-2} \\ &= \sqrt{2(\sec^2\theta-1)} \\ &= \sqrt{2}\sqrt{sec^2\theta-1} \\ &= \sqrt{2}\sqrt{\tan^2\theta} \\ &= \sqrt{2}|\tan\theta| \\ &= \sqrt{2}\tan\theta \qquad (\text{Since } \theta\in[0,\pi/2),\ \tan\theta>0) \end{aligned}$$

Rewriting the integral,

$$\begin{aligned} \int \frac{\sqrt{x^2-2}}{x}\,dx &= \int \frac{\sqrt{2}\tan\theta}{\sqrt{2}\sec\theta}\sqrt{2}\sec\theta\tan\theta\,d\theta \\ &= \int \sqrt{2}\tan^{\theta}, d\theta \\ &= \sqrt{2}\int \sec^2\theta-1\,d\theta \\ &= \sqrt{2}(\tan\theta-\theta)+C \end{aligned}$$

You can draw a right triangle with angle θ and $\sec\theta = x/\sqrt{2}$ to solve for $\tan\theta$. I'm not good enough with LaTeX to include the diagram but it results in $\tan\theta=\sqrt{x^2-2}$. Also note that $x=\sqrt{2}\sec\theta$ implies $\theta=\sec^{-1}(\frac{x}{\sqrt{2}})$. (The -1 denotes the inverse secant function, NOT a negative exponent.)

$$\begin{aligned} &= \sqrt{2}(\tan\theta-\theta)+C \\ &= \sqrt{2}\big(\sqrt{x^2-2}-\sec^{-1}(\frac{x}{\sqrt{2}})\big)+C \end{aligned}$$

Improper integrals by definition: (1)

$$\int_0^1 \frac{\ln(x)}{x}\,dx$$

The integral is over the interval $[0,1]$, and the integrand has a discontinuity/is not defined at $x=0$, which is in the interval. This is a Type II improper integral.

$$\begin{aligned}
\int_0^1 \frac{\ln(x)}{x}\,dx &= \lim_{A\to 0^+} \int_A^1 \frac{\ln(x)}{x}\,dx \\
&= \lim_{A\to 0^+} \int_{\ln A}^0 u\,du \qquad \text{Use substitution } u=\ln(x) \implies du = \frac{1}{x}\,dx \text{ and change int. limits} \\
&= \lim_{A\to 0^+} \left[\frac{1}{2}u^2\right]_{\ln A}^0 \\
&= \lim_{A\to 0^+} -\frac{1}{2}(\ln A)^2
\end{aligned}$$

Note that as $A \to 0^+$, $\ln A \to -\infty$ (picture the graph of $\ln(x)$.) Hence, $\lim_{A\to 0^+} -\frac{1}{2}(\ln A)^2 = -\infty$, so the integral diverges.