MATA37 Week 5

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This week is about the Fundamental Theorem of Calculus, but you can refresh your memory with the following definitions. See Week 3 notes for more examples.

1 The indefinite integral

Definition 1. *An* ***antiderivative*** *of a function f is a function F where the derivative of F equals f, i.e. $F' = f$.*

For example, let $f(x) = \cos x$. Examples of antiderivatives of f are $F_1(x) = \sin x$ and $F_2(x) = \sin x + 5$, since $F_1'(x) = \cos x = F_2'(x)$.
Suppose f is continuous everywhere.

Definition 2. *The* ***indefinite integral*** *of f, denoted $\int f(x)\,dx$, is the collection of all antiderivatives of f.*

$$\int f(x)\,dx = F(x) + C$$

where F is an antiderivative of f, i.e. a function for which $F' = f$, and
C is an arbitrary constant called the ***integration constant***.

Note: at this point, from what we know so far, the indefinite integral is unrelated from the definite integral: the indefinite integral represents antiderivatives of a function, while the definite integral represents the signed area between a function and the x-axis. The relationship between these two seemingly unrelated concepts is revealed in the Fundamental Theorem of Calculus.

Also note: evaluating an indefinite integral gives a **function**, while evaluating a definite integral gives a **number**.

1.1 Properties of the indefinite integral

Suppose f and g are continuous functions.

$$(1) \quad \int [f(x) \pm g(x)]\,dx = \int f(x)\,dx \pm \int g(x)\,dx$$

$$(2) \quad \int kf(x)\,dx = k \int f(x)\,dx$$

Example 1. Proof of (1)
Suppose f and g are continuous functions. Let $F(x)$ and $G(x)$ be functions such that $F'(x) = f(x)$ and $G(x) = g'(x)$. Then, by definition of the indefinite integral,

$$\begin{aligned} \int f(x)\,dx \pm \int g(x)\,dx &= (F(x) + C_1) \pm (G(x) + C_2) \\ &= F(x) \pm G(x) + (C_1 \pm C_2) \\ \text{Let } C \text{ be the arbitrary constant } C = C_1 \pm C_2 : &= F(x) \pm G(x) + C \\ \text{By def. of indef. integral and by } [F(x) \pm G(x)]' = F'(x) \pm G'(x) = f(x) \pm g(x) : &= \int f(x) \pm g(x)\,dx \end{aligned}$$

Therefore $\int f(x)\,dx \pm \int g(x)\,dx = \int f(x) \pm g(x)\,dx$ as wanted.

2 Fundamental Theorem of Calculus (part 1)

The FTOC pt.1 relates the concept of antiderivatives to the concept of integration. It gives a way to calculate integrals without going through the Riemann sum definition.

Theorem 1. ***Fundamental Theorem of Calculus (Part 1)***
Let $a, b \in \mathbb{R}$, $a < b$.
IF (a) f is continuous on $[a, b]$
and (b) F is any antiderivative of f,
THEN
$\int_a^b f(x)\,dx = [F(x)]_a^b = F(b) - F(a)$

3 Fundamental Theorem of Calculus (part 2)

The FTOC pt.2 essentially says that all continuous functions have antiderivatives. It gives a way of constructing said antiderivative for a given continuous function f.

Theorem 2. ***Fundamental Theorem of Calculus (part 2)***
IF f is continuous on $[a, b]$,
Define the function $F(x)$ where $x \in [a, b]$:

$$F(x) = \int_a^x f(t)dt$$

THEN (a) F is differentiable on (a, b) and continuous on $[a, b]$
and (b) $F'(x) = f(x) \quad \forall x \in [a, b]$
i.e. F is an antiderivative of f.
i.e. $\frac{\mathrm{d}}{\mathrm{d}x}F = \frac{\mathrm{d}}{\mathrm{d}x}\left(\int_a^x f(t)\,dt\right) = f(x)$

Note: before using the FTOC, Pt.2, you must check whether your function is continuous and show that it's continuous. See the example(s) for the amount of detail you should have for this. (You don't need to do this when using FTOC Pt.1.)

Example 1. Find the derivative of the following function:

$$F(x) = \int_2^x e^{\sin\left(u^2\right)}\, du$$

Sol. The function $f(u) = e^{\sin\left(u^2\right)}$ is the composition of the functions e^u, $\sin(u)$, and u^2.

- e^u is cont. on its domain since it is an exponential function, so it is cont. on $\mathbb{R}$
- $\sin(u)$ is cont. on its domain since it is a trig function, so it is cont. on $\mathbb{R}$.
- u^2 is a polynomial so it is cont. everywhere.
- The composition of functions that are cont. on $\mathbb{R}$ is a function that is cont. on $\mathbb{R}$.
- Hence f is cont. on $\mathbb{R}$. **In particular,** it is cont. on $[2, x]$ for any $x \in \mathbb{R}$.

Therefore we can apply the FTOC pt. 2, with $F(x)$ and $f(u) = e^{\sin\left(u^2\right)}$. Compare the following with the statement of FTOC pt. 2, which says $F'(x) = f(x)$:

$$F'(x) = \frac{\mathrm{d}}{\mathrm{d}x}\left(\int_2^x e^{\sin\left(u^2\right)}\, du\right) = f(x) = e^{\sin\left(x^2\right)}$$

4 Example questions

Example 1. Using FTOC Pt.1

Evaluate the following integrals:

$$\begin{aligned}
&(1) \quad \int_0^{2\pi} \sin x\, dx \\
&(2) \quad \int_1^2 x\sqrt{x}\, dx \\
&(3) \quad \int_0^{1/\sqrt{3}} \frac{1}{4+12x^2}\, dx \\
&(4) \quad \int_{-2}^3 |x^2 - 1|\, dx
\end{aligned}$$

Example 2. Prove the following using the Fundamental Theorem of Calculus:

$$\begin{aligned}
&(1) \quad \int_a^b x\, dx = \frac{1}{2}(b^2 - a^2) \\
&(2) \quad \int_a^b kf(x)\, dx = k\int_a^b f(x)\, dx
\end{aligned}$$

Example 3. Using FTOC Pt.2
Find the derivatives of the following functions:

$$\begin{aligned}
&(1) \quad F(x) = \int_0^x \frac{1}{t+1}\,dt \\
&(2) \quad F(x) = \int_2^{3x} u^2 + u\,du \\
&(3) \quad F(x) = \int_\pi^{\sqrt{x}} \sin(t)\,dt \\
&(4) \quad F(x) = \int_x^{x^2} \frac{2}{s^2+2}\,ds
\end{aligned}$$

Example 4. Evaluate:

$$\int_0^1 \left(\frac{\mathrm{d}}{\mathrm{d}x}\Big(\int_0^1 e^{x^2}\,dx\Big)\right)dx$$

Example 5. Prove that the following function is increasing when $x > 0$:

$$H(x) = \int_0^{x^2} \frac{e^u}{u^2 - 2u + 2}\,du$$

Example 6. Let $A(x) = \int_0^x f(t)\,dt$, where f is positive, decreasing, and continuous for $x > 0$. Find $A''(x)$ to prove that $A(x)$ is concave down for $x > 0$.

Example 7. Find a function f such that for any real number x,

$$\int_0^x f(t)\,dt = \frac{\cos(x)}{1+x^2} - 1$$

5 Answers to examples

Example 1. Using FTOC Pt.1
Evaluate the following integrals:

$$\begin{aligned}
&(1) \quad \int_0^{2\pi} \sin x \, dx \\
&(2) \quad \int_1^2 x\sqrt{x} \, dx \\
&(3) \quad \int_0^{1/\sqrt{3}} \frac{1}{4+12x^2} \, dx \\
&(4) \quad \int_{-2}^{3} |x^2-1| \, dx
\end{aligned}$$

Sol. (1) An antiderivative of $\sin x$ is $-\cos x$, since $(-\cos x)' = -(-\sin x) = \sin x$. Therefore, by FTOC Pt. 1,

$$\begin{aligned}
\int_0^{2\pi} \sin x \, dx &= \Big[-\cos x\Big]_0^{2\pi} \\
&= -\cos(2\pi) - (-\cos(0)) \\
&= -1 + 1 \\
&= 0 \quad \square
\end{aligned}$$

Sol. (2) By the previous example (see Week 3), an antiderivative of $x\sqrt{x}$ is $\frac{2}{5}x^{5/2}$. Therefore, by FTOC Pt. 1,

$$\begin{aligned}
\int_1^2 x\sqrt{x} &= \left[\frac{2}{5}x^{5/2}\right]_1^2 \\
&= \frac{2}{5}(2)^{5/2} - \frac{2}{5}(1)^{5/2} \\
&= \frac{2}{5}(4\sqrt{2}) - \frac{2}{5} \\
&= \frac{2}{5}(4\sqrt{2}-1) \quad \square
\end{aligned}$$

Sol. (3) By the previous example (see Week 3), an antiderivative of $\frac{1}{4+12x^2}$ is $\frac{1}{4\sqrt{3}}\arctan\big(\sqrt{3}x\big)$. By FTOC Pt.1,

$$\begin{aligned}
\int_0^{1/\sqrt{3}} \frac{1}{4+12x^2} &= \left[\frac{1}{4\sqrt{3}}\arctan\Big(\sqrt{3}x\Big)\right]_0^{1/\sqrt{3}} \\
&= \frac{1}{4\sqrt{3}}\arctan\left(\sqrt{3}\frac{1}{\sqrt{3}}\right) - \frac{1}{4\sqrt{3}}\arctan\Big(0\sqrt{3}\Big) \\
&= \frac{1}{4\sqrt{3}}\arctan(1) - \frac{1}{4\sqrt{3}}\arctan(0) \\
&= \frac{1}{4\sqrt{3}}\Big(\frac{\pi}{4}\Big) - 0 \\
&= \frac{\pi}{16\sqrt{3}} \quad \square
\end{aligned}$$

Sol. (4) To deal with the absolute value, you have to split up the integral by figuring out where the function changes. Remember that $|a| = a$ when $a > 0$ and $|a| = -a$ when $a < 0$.
Looking at the function $f(x) = x^2 - 1$, $x^2 - 1 > 0$ when $x^2 > 1$, i.e. when $x < -1$ and $x > 1$. Therefore:

$$|x^2 - 1| = \begin{cases} x^2 - 1 & \text{when } x < -1 \\ -(x^2 - 1) & \text{when } -1 \leq x \leq 1 \\ x^2 - 1 & \text{when } x > 1 \end{cases}$$

Hence:

$$\begin{aligned}\int_{-2}^{3} |x^2 - 1|\, dx &= \int_{-2}^{-1} x^2 - 1\, dx + \int_{-1}^{1} -(x^2 - 1)\, dx + \int_{1}^{3} x^2 - 1\, dx \quad \text{By integral property and def. of absolute value} \\ &= \Big[\frac{1}{3}x^3 - x\Big]_{-2}^{-1} + \Big[-\frac{1}{3}x^3 + x\Big]_{-1}^{1} + \Big[\frac{1}{3}x^3 - x\Big]_{1}^{3} \\ &= (-1/3 + 1) - (-8/3 + 2) + (-1/3 + 1) - (1/3 - 1) + (9 - 3) - (1/3 - 1) \\ &= 28/3 \quad \square\end{aligned}$$

Example 2. Prove the following using the Fundamental Theorem of Calculus:

$$(1) \quad \int_a^b x\, dx = \frac{1}{2}(b^2 - a^2)$$

$$(2) \quad \int_a^b kf(x)\, dx = k\int_a^b f(x)\, dx$$

Proof. (1) Let $f(x) = x$. An antiderivative of x is $\frac{1}{2}x^2$. Therefore, by the FTOC Pt.1,

$$\int_a^b f(x)\, dx = \Big[\frac{1}{2}x^2\Big]_a^b = \frac{1}{2}b^2 - \frac{1}{2}a^2 = \frac{1}{2}(b^2 - a^2) \quad \square$$

Proof. (2) Suppose $f(x)$ is continuous. Let $F(x)$ be some antiderivative of $f(x)$, i.e. let $F(x)$ be such that $F'(x) = f(x)$. Note that an antiderivative of $kf(x)$ is $kF(x)$, since $[kF(x)]' = kf(x)$ by differentiation properties. Simplifying the LHS by the FTOC Pt.1:

$$\int_a^b kf(x)\, dx = \Big[kF(x)\Big]_a^b = kF(b) - kF(a)$$

SImplifying the RHS by the FTOC Pt.1:

$$k\int_a^b f(x)\, dx = k\Big[F(x)\Big]_a^b = k(F(b) - F(a)) = kF(b) - kF(a)$$

Therefore LHS=RHS=$kF(b) - kF(a)$. $\square$

Example 3. Using FTOC Pt.2

Find the indicated derivatives of the following functions:

$$
\begin{aligned}
&(1) \quad F(x) = \int_0^x \frac{1}{t+1}\,dt, \quad F'(x) \\
&(2) \quad H(x) = \int_2^{3x} u^2 + u\,du, \quad H'(x) \\
&(3) \quad G(x) = \int_\pi^{\sqrt{x}} \sin(t)\,dt, \quad G'(x) \\
&(4) \quad \frac{\mathrm{d}}{\mathrm{d}x}\Big(\int_x^{x^2} \frac{2}{s^2+2}\,ds\Big)
\end{aligned}
$$

Sol. (1) Let $f(t) = \frac{1}{t+1}$. $f(t)$ is continuous on [0,b] for any $b > 0$. This is because it is a rational function, so it is continuous everwhere except for where its denominator=0, so its only discontinuity is at $t = -1$. Therefore we can apply the FTOC Pt.2, taking $F(x)$ as given and $f(t)$ as defined. By FTOC Pt.2,

$$F'(x) = f(x) = \frac{1}{x+1} \quad \square$$

Sol. (2) Let $f(u) = u^2 + u$. Since f is a polynomial, it is continuous everywhere. **In particular,** it is continuous on $[2, 3x]$ for any real number x. **But the given H has $3x$ as the upper limit of the integral, while FTOC Pt.2 requires the upper limit to be just x.** If you take $F(x)$ to be $\int_2^x u^2 + u\,du$ and $G(x) = 3x$, the given H is actually the composition of F and G, $H = F(G(x)) = F(3x)$. Since we know how to get the derivatives of F and $3x$, so we can apply the chain rule. By the FTOC Pt.2:

$$
\begin{aligned}
H'(x) &= \frac{\mathrm{d}}{\mathrm{d}x}[F(G(x))] = \frac{\mathrm{d}}{\mathrm{d}x}[F(3x)] \\
&= F'(3x) \cdot (3x)' \\
\text{By FTOC Pt.2, } F'(x) = f(x) = x^2 + x, \text{ so } F'(3x) = (3x)^2 + (3x): \quad &= [(3x)^2 + (3x)] \cdot 3 \\
&= 27x^2 + 9x \quad \square
\end{aligned}
$$

Sol. (3) Let $f(t) = \sin(t)$. $f(t)$ is a trigonometric function so it is continuous over its domain (which is $\mathbb{R}$). In particular it is continuous over $[\pi, \sqrt{x}]$. Hence we can apply FTOC Pt. 2 for any real number x. Again, the given integral doesn't match the statement of FTOC Pt.2, since the upper limit is $\sqrt{x}$ instead of just x. So the given function $G(x)$ is a composition of functions: If you let $F(x) = \int_\pi^{\sqrt{x}} \sin(t)\,dt$, then you can write G as $F(\sqrt{x})$ and we can apply the chain rule as above. By FTOC Pt.2:

$$
\begin{aligned}
G'(x) &= \frac{\mathrm{d}}{\mathrm{d}x} F(\sqrt{x}) = F'(\sqrt{x}) \cdot (\sqrt{x})' \\
\text{By FTOC Pt.2, } F'(x) = f(x) = \sin(x), \text{ so } F'(\sqrt{x}) = \sin\big(\sqrt{x}\big): \quad &= \sin\big(\sqrt{x}\big) \cdot \frac{1}{2\sqrt{x}} \quad \square
\end{aligned}
$$

Sol. (4) Let $f(s) = \frac{2}{s^2+2}$. $f(s)$ is a rational function, so it is continuous everywhere since its denominator $s^2 + 2$ is > 0 for all $s \in \mathbb{R}$. In particular, $f(s)$ is continuous on $[x, x^2]$.
You need to split this integral up before you can apply FTOC Pt.2.

Let $c \in \mathbb{R}$ be a constant s.t. $c \in [x, x^2]$. By the union interval property,

$$\int_x^{x^2} \frac{2}{s^2+2}\,ds = \int_x^c \frac{2}{s^2+2}\,ds + \int_c^{x^2} \frac{2}{s^2+2}\,ds$$

By another integral property,

$$\int_x^{x^2} \frac{2}{s^2+2}\,ds = -\int_c^x \frac{2}{s^2+2}\,ds + \int_c^{x^2} \frac{2}{s^2+2}\,ds$$

Let $F(x) = \int_c^x \frac{2}{s^2+2}$, the form of function that the FTOC Pt.2 can take, so we know the derivative of this function: $F'(x) = f(x) = \frac{2}{x^2+2}$. The derivative of the whole function can be rewritten:

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}x}\Big(\int_x^{x^2} \frac{2}{s^2+2}\,ds\Big) &= \frac{\mathrm{d}}{\mathrm{d}x}\Big(-\int_c^x \frac{2}{s^2+2}\,ds + \int_c^{x^2} \frac{2}{s^2+2}\,ds\Big) \\
&= \frac{\mathrm{d}}{\mathrm{d}x}\big(-F(x) + F(x^2)\big) \\
\text{Applying the chain rule:} \quad &= -F'(x) + F'(x^2)(x^2)' \\
&= -f(x) + f(x^2)2x \\
&= -\frac{2}{x^2+2} + \frac{2}{(x^2)^2+2}(2x) \\
&= -\frac{2}{x^2+2} + \frac{4x}{x^4+2} \quad \square
\end{aligned}$$

Example 4. Evaluate:

$$\int_0^1 \left(\frac{\mathrm{d}}{\mathrm{d}x}\Big(\int_0^1 e^{x^2}\,dx\Big)\right)dx$$

Sol. The integral $\int_0^1 e^{x^2}$ must evaluate to some real number, say c. Then the derivative $\frac{\mathrm{d}}{\mathrm{d}x}(c)$ is 0, since the derivative of a constant (with respect to x) is 0.

$$\begin{aligned}
\int_0^1 \left(\frac{\mathrm{d}}{\mathrm{d}x}\Big(\int_0^1 e^{x^2}\,dx\Big)\right)dx &= \int_0^1 0\,dx \\
&= 0\int_0^1 1\,dx \\
&= 0(1-0) \\
&= 0 \quad \square
\end{aligned}$$

Example 5. Prove that the following function is increasing when $x > 0$:

$$H(x) = \int_0^{x^2} \frac{e^u}{u^2 - 2u + 2}\,du$$

A function is increasing if its derivative is > 0. Want to show $H'(x) > 0$ for $x > 0$.
Sol. First, let $f(u) = \frac{e^u}{u^2-2u+2}$. Show this function is continuous. $f(u)$ is the product of the functions e^u and $\frac{1}{u^2-2u+2}$.

- e^u is an exponential function so it is cont. everywhere
- $\frac{1}{u^2-2u+2}$ is a rational function, so it is cont. everywhere, since $u^2-2u+2=(u-1)^2+1$ and $(u-1)^2 \geq 0$ for all u, so $(u-1)^2+1>0$ for all $u \in \mathbb{R}$.
- The product of two functions cont. on $\mathbb{R}$ is cont. on $\mathbb{R}$.
- Thus f is cont. on $\mathbb{R}$, so in particular it is cont. on $[0,x^2]$ where $x>0$.

By the FTOC Pt.2, taking $F(x)=\int_0^x \frac{e^u}{u^2-2u+2}\,du$ and $f(u)=\frac{e^u}{u^2-2u+2}$, and applying the chain rule:

$$\begin{aligned}
H'(x) &= F'(x^2)(x^2)' \\
&= f(x^2)(2x) \\
&= \frac{e^{x^2}}{(x^2)^2-2x^2+2}(2x) \\
&= \frac{2xe^{x^2}}{(x^2-1)^2+1}
\end{aligned}$$

Since $(x^2-1)^2$ is always >0, for all x, the denominator $(x^2-1)^2>0$ for all x, in particular, when $x>0$. As well, $e^{x^2}>0$ for all x, and $2x>0$ when $x>0$, so the numerator $2xe^{x^2}>0$ for all $x>0$.
Since the numerator and denominator are both positive whenever $x>0$, we can conclude that $H'(x)=\frac{2xe^{x^2}}{(x^2-1)^2+1}>0$ for $x>0$, i.e., $H(x)$ is increasing when $x>0$.

Example 6. Let $A(x)=\int_0^x f(t)\,dt$, where f is continuous, positive, and decreasing on $[0,x]$ for $x>0$. Find $A''(x)$ to prove that $A(x)$ is concave down for $x>0$.

Sol. Recall from A31 that $A(x)$ is concave down for $x>0$ if its second derivative, $A''(x)<0$ when $x>0$. Let f be a continuous function that's positive and decreasing when $x>0$. Since f is continuous on $[0,x]$, we can apply the FTOC Pt.2 to $A(x)=\int_0^x f(t)\,dt$ and find its derivative.
By FTOC Pt.2:

$$A'(x)=f(x)$$

Furthermore, we can find its second derivative:

$$A''(x)=f'(x)$$

Since f is decreasing when $x>0$, we know $f'(x)<0$ when $x>0$. It follows that $A''(x)<0$ when $x>0$, since $A''(x)=f'(x)$. Therefore $A(x)$ is concave down when $x>0$.