Quiz 8

Name: Sample

Student Number:

Problem 1 [10 marks]: Determine if the following improper integral converges or diverges. Make sure to fully justify your solution.

Let $f(x) = \frac{x\ln x + \sin^2 ex}{3x^3 \ln x + 5x}$

$$\int_5^\infty \frac{x\ln(x) + \sin^2(ex)}{3x^3\ln(x) + 5x}\,dx$$

CONV. or DIV.?

Guess: $\frac{x\ln x + \sin^2 ex}{3x^3\ln x + 5x} \approx \frac{x\ln x}{3x^3 \ln x}$ ← consider highest degree / "biggest" terms

$= \frac{1}{3x^2}$ and $\int_5^\infty \frac{1}{3x^2}dx$ CONV. $\therefore$ guess CONV.

① Check nonnegative

for $x \in [5, \infty)$, $\ln x > 0$
$\sin^2 ex \geq 0$
$x\ln x > 0$
$x^3 \ln x > 0$

$\therefore \frac{x\ln x + \sin^2(ex)}{3x^3\ln x + 5x} \geq 0$

② Find function $g(x)$ that is bigger and $\int_5^\infty g(x)$ CONV.
↳ should be easy to integrate

$\frac{x\ln x + \sin^2 ex}{3x^3\ln x + 5x} \leq \frac{x\ln x + 1}{3x^3\ln x + 5x}$ since $\sin^2 ex \leq 1$ $\forall x$

$\leq \frac{x\ln x + 1}{3x^3\ln x}$ min. denom.

$\leq \frac{x\ln x + x\ln x}{3x^3\ln x}$ max. denom. since $x\ln x \geq 1$ $\forall x \in [5, \infty)$

$= \frac{2x\ln x}{3x^3\ln x}$

$= \frac{2}{3x^2} = g(x)$

$\int_5^\infty \frac{2}{3x^2}dx = \lim_{A\to\infty}\int_5^A \frac{2}{3x^2}dx$

$= \lim_{A\to\infty}\left[-\frac{2}{3x}\right]_5^A$

$= \lim_{A\to\infty} -\frac{2}{3A} + \frac{2}{15}$, $\frac{2}{3A} \to 0$

$= \frac{2}{15}$

$\therefore \int_5^\infty \frac{2}{3x^2}dx$ converges

By ① and ②
$0 \leq f(x) \leq g(x)$
$\int_5^\infty g(x)dx$ converges

$\therefore \int_5^\infty f(x)dx$ converges by CT.

* You can use a different $g(x)$, but you need to be able to evaluate its integral AND $\int_5^\infty g(x)$ must converge
↳ if $\int_5^\infty g(x)$ diverges, it doesn't tell you anything about $\int_5^\infty f(x)$.

Problem 2 [10 marks]: Prove by defintion the following sequence converges.

$$a_n = \frac{\cos^2(n) + e^{2n} + 3n^2 + \sin^2(n)}{\pi e^{2n}}$$

WTS $\exists l \in \mathbb{R}\ \forall \varepsilon > 0\ \exists N > 0$ s.t. if $n \in \mathbb{N}, n > N$ then $|a_n - l| < \varepsilon$.

Guess l: $a_n = \frac{\cos^2 n + e^{2n} + 3n^2 + \sin^2 n}{\pi e^{2n}} \approx \frac{e^{2n}}{\pi e^{2n}} \Rightarrow \frac{1}{\pi}$

consider highest degree terms ("biggest" terms)

Let $l = \frac{1}{\pi}$

Let $\varepsilon > 0$ be arbitrary

Let $N = \frac{4}{\pi \varepsilon} > 0$ ← choice of N must be > 0 for ~~all~~ any $\varepsilon > 0$. e.g. if $N = \ln(\frac{1}{\varepsilon})$, N can be < 0.

Suppose $n > N$

$$|a_n - l| = \left| \frac{\cos^2 n + e^{2n} + 3n^2 + \sin^2 n}{\pi e^{2n}} - \frac{1}{\pi} \right| = \left| \frac{\cos^2 n + e^{2n} + 3n^2 + \sin^2 n}{\pi e^{2n}} - \frac{e^{2n}}{\pi e^{2n}} \right|$$

$= \left| \frac{1 + 3n^2}{\pi e^{2n}} \right|$ since $\cos^2 n + \sin^2 n = 1$

$= \frac{1 + 3n^2}{\pi e^{2n}}$ by abs. value def. since $1 + 3n^2 \geq 0, \pi e^{2n} \geq 0$

$\leq \frac{n^2 + 3n^2}{\pi e^{2n}}$ max numerator since $n^2 \geq 1\ \forall n \in \mathbb{N}$

$= \frac{4n^2}{\pi e^{2n}}$

$\leq \frac{4n^2}{\pi n^3}$ min. denom. since $n^3 \leq e^{2n}\ \forall n \in \mathbb{N}$

$= \frac{4}{\pi n}$ since $n > N$, $\frac{1}{n} < \frac{1}{N} \Rightarrow \frac{4}{\pi n} < \frac{4}{\pi N}$

$< \frac{4}{\pi N}$

$= \frac{4}{\pi \left(\frac{4}{\pi \varepsilon}\right)}$

$= \varepsilon$ i.e. if $n > N$, $|a_n - l| < \varepsilon$ as wanted.

We need to keep finding upper bounds until we get ~~a~~ something like $\frac{1}{n^p}$. That's why I used bounds to get a single power of n in the num. and denom.