MATA37 Week 8

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1 Substitution

Let $a, b \in \mathbb{R}$, $a < b$.

Theorem 1. ***Substitution rule***
IF f and g' are continuous functions,
THEN, letting $u = g(x)$, which implies $du = g(x)\,dx$:

$$(1) \quad \int f(g(x))g'(x)\,dx = \int f(u)du$$

$$(2) \quad \int_a^b f(g(x))g'(x)\,dx = \int_{g(a)}^{g(b)} f(u)du$$

2 Integration by parts

Let $a, b \in \mathbb{R}$, $a < b$.

Theorem 2. ***Integration by parts***
IF $f(x)$ and $g(x)$ are differentiable functions,
THEN (rewriting $u = f(x)$ and $v = g(x)$):

$$(1) \quad \int udv = uv - \int vdu$$

$$\textit{and} \quad (2) \quad \int_a^b udv = \left[uv\right]_a^b - \int_a^b vdu$$

When doing integration by parts, it helps to first choose/identify what you want u and dv to be, then compute du and v.
Usually you choose u as a polynomial, log, or inverse trig function, since calculating their derivatives du usually results in a simpler integral.
Make sure that your choice of dv makes it easy to get its antiderivative, v.

3 Partial fraction decomposition

Sometimes we need to break down rational functions into the sum of two fractions.
We do this to **proper** rational functions, i.e. rational functions where the degree of the numerator is less

than the degree of the denominator.
The number of resulting fractions is the total multipilicity of factors in the denominator—just add up the exponents of the factors.
The numerator of each fraction is the general polynomial of one degree less than the factor in the denominator.
Steps:
1. Make sure the fraction is in proper form, i.e. the degree of the numerator ¡ degree of the denominator. (If it's not, then you have to do some polynomial long division.)
2. Fully factor the denominator.
3. Write out the denominators of the fractions. The denominator of each fraction will be an individual factor of the denominator in the original fraction. If you have a factor, say $(x-1)$ with multiplicity $k>1$, i.e. the denominator has $(x-1)^k$, you need k fractions with denominators of increasing power: $(x-1)$, $(x-1)^2$, $(x-1)^3$, ... , $(x-1)^k$.
4. Fill in the numerators of each fraction. The numerator is the general polynomial of degree one less then the degree of the factor in the denominator. Use different letters for each coefficient.
Some examples of rational functions rewritten in P.F.D. form:

$$\begin{aligned}
\frac{1}{(x-1)(x+2)(x-3)} &= \frac{1}{x-1}+\frac{1}{x+2}+\frac{1}{x-3} \\
\frac{x+4}{x^3(x^2-3)} &= \frac{A}{x}+\frac{B}{x^2}+\frac{C}{x^3}+\frac{Dx+E}{x^2-3} \\
\frac{1}{(x-2)^2(x^2+4)^2} &= \frac{A}{x-2}+\frac{B}{(x-2)^2}+\frac{Cx+D}{x^2+4}+\frac{Ex+F}{(x^2+4)^2}
\end{aligned}$$

Example 1.
Write the following in PFD form:

$$\frac{x^3}{x(x^2-3)(x-1)}$$
$$\frac{1}{x^4+6x^2+9}$$
$$\frac{x+2}{x^3-1}$$
$$\frac{2x+3}{(x+1)(x^2-1)}$$
$$\frac{2}{x(x^4+5)}$$

Example 2. A bunch of integrals for practice. Any of the above methods can be applied. You might have to use substitution + algebra.

$$(a) \quad \int \frac{e^x}{e^x+1}\,dx$$
$$(b) \quad \int_0^{13} \frac{dx}{\sqrt[3]{(1+2x)^2}}$$
$$(c) \quad \int_1^2 x\sqrt{x-1}\,dx$$

$$(d)\quad \int x^2 \sin 3x \, dx$$

$$(e)\quad \int \sin(\ln x) \, dx$$

$$(f)\quad \int (\ln(x))^2 \, dx$$

$$(g)\quad \int_1^4 \sqrt{x} \ln(x) \, dx$$

$$(h)\quad \int e^{2x} \sin(3x) \, dx$$

$$(i)\quad \int_3^7 \frac{1}{(x+1)(x-2)} \, dx$$

$$(j)\quad \int \frac{dx}{x(x+1)(2x+3)}$$

$$(k)\quad \int \frac{x^2}{(x+1)^3} \, dx$$

$$(l)\quad \int \frac{\sqrt{x+4}}{x} \, dx$$

Example 3. Use integration by parts to show the following:

$$\int f(x) \, dx = xf(x) - \int xf'(x) \, dx$$

Answers to writing the following in PFD form:

$$\begin{aligned}
\frac{x^3}{x(x^2-3)(x-1)} &= \frac{A}{x} + \frac{Bx+C}{x^2-3} + \frac{D}{x-1} \\
\frac{1}{x^4+6x^2+9} &= \frac{1}{(x^2+3)^2} \\
&= \frac{Ax+B}{x^2+3} + \frac{Cx+D}{(x^2+3)^2} \\
\frac{x+2}{x^3-1} &= \frac{x+2}{(x-1)(x^2+x+1)} \\
&= \frac{A}{x-1} + \frac{Bx+C}{x^2+x+1} \\
\frac{2x+3}{(x+1)(x^2-1)} &= \frac{2x+3}{(x+1)(x+1)(x-1)} \\
&= \frac{2x+3}{(x+1)^2(x-1)} \\
&= \frac{A}{x+1} + \frac{B}{(x+1)^2} + \frac{C}{x-1} \\
\frac{2}{x(x^4+2x+5)} &= \frac{A}{x} + \frac{Bx^3+Cx^2+Dx+E}{x^4+2x+5}
\end{aligned}$$

Hints for the integrals:

(a) $\int \frac{e^x}{e^x+1}\,dx$ — use substitution

(b) $\int_0^{13} \frac{dx}{\sqrt[3]{(1+2x)^2}}$ — use substitution. Often when you have a square root term, you can let $u =$ the entire square root term and u will reappear after you differentiate it.

(c) $\int_1^2 x\sqrt{x-1}$ — use substitution, see (b).

(d) $\int x^2 \sin 3x\,dx$ — use integration by parts twice. ($u = x^2$, then $u = x$ on the resulting integral)

(e) $\int \sin(\ln x)\,dx$ — this is one of those examples where you need to do integration by parts twice, which will get you back to the original integral. Then you can rearrange the equation to solve for the original integral. (the same process for solving e.g. $\int e^x \cos(x)\,dx$) Make sure your choice of u and dv are consistent in both integration by parts.

(f) $\int (\ln(x))^2\,dx$ — do integration by parts twice. Let $u = (\ln(x))^2$, $dv = dx$ and go from there.

(g) $\int_1^4 \sqrt{x}\ln(x)\,dx$ — do substitution first, then integration by parts. You can also use integration by parts directly. Either approach is fine; try going through both.

(h) $\int e^{2x}\sin(3x)$ — do integration by parts twice, get back the original integral, solve for the original integral. (see (e))

(i) $\int_3^7 \frac{1}{(x+1)(x-2)}\,dx$ — use PFD

(j) $\int \frac{dx}{x(x+1)(2x+3)}$ — use PFD

(k) $\int \frac{\sqrt{x+4}}{x}\,dx$ — use PFD

(l) $\int \frac{\sqrt{x+4}}{x}$ — use substitution, then PFD.
(let $u = \sqrt{x+4}$. Then $u^2 = x + 4$, so $x = u^2 - 4$ and $dx = 2udu$, etc.)