MATA37 Week 4

Kevin Santos

June 1, 2022

1 Darboux definition of the definite integral

Consider the **Dirichlet function**:

$$f(x) = \begin{cases} 1 & \text{if } x \in \mathbb{Q} \\ 0 & \text{if } x \notin \mathbb{Q} \end{cases}$$

Is f integrable? We can find out with the Darboux definition of the definite integral.

Let $a, b \in \mathbb{R}$, $a < b$. Let f be some function that's **bounded** on $[a, b]$. That is,

$$\exists c \in \mathbb{R} \text{ s.t. } |f(x)| \leq c \quad \forall x \in [a, b]$$

Let $P = \{x_i\}_{i=0}^{n}$ be *any* partition of $[a, b]$. For each $i = 1, 2, ..., n$ define:

$$m_i = \inf\{f(x) \mid x \in [x_{i-1}, x_i]\}$$
$$M_i = \sup\{f(x) \mid x \in [x_{i-1}, x_i]\}$$

With these, we can define the upper sum and lower sum.

Definition 1. *The* ***upper sum*** *of f over P is*

$$U(f, P) = \sum_{i=1}^{n} M_i(x_i - x_{i-1})$$

The ***lower sum*** *of f over P is*

$$L(f, P) = \sum_{i=1}^{n} m_i(x_i - x_{i-1})$$

Calculating the upper sum $U(f, P)$ means taking the width of each subinterval defined by P, multiplying it by the supremum of $f(x)$ within that subinterval, and adding these up.

Example 1. (A2 Exercise)
Calculate the $L(f, P)$ for the following function and partition:

$$f(x) = 2x \qquad x \in [0, 1] \qquad P = \{0, \frac{1}{4}, \frac{1}{2}, 1\}$$

Sol. First, I'll find each m_i, then substitute that into my equation.
With the given partition, let $x_0 = 0$, $x_1 = \frac{1}{4}$, $x_2 = \frac{1}{2}$, $x_3 = 1$. Since the last term is x_3, we have $n = 3$ for

this partition.
For example, for m_1, we are looking at the range of values that $f(x)$ takes when x ranges between 0 and 1/4. When x ranges from 0 to 1/4, $f(x)$ ranges from 0 to 1/2. Hence, m_1 is the infimum of the interval $[0, 1/2]$.

$$\begin{aligned} m_1 &= \inf\{f(x) \mid x \in [x_0, x_1]\} \\ &= \inf\{f(x) \mid x \in [0, 1/4]\} \\ &= \inf\{f(x) \mid f(x) \in [0, 1/2]\} \\ &= 0 \end{aligned} \qquad \begin{aligned} m_2 &= \inf\{f(x) \mid x \in [x_1, x_2]\} \\ &= \inf\{f(x) \mid x \in [1/4, 1/2]\} \\ &= \inf\{f(x) \mid f(x) \in [1/2, 1]\} \\ &= 1/2 \end{aligned} \qquad \begin{aligned} m_3 &= \inf\{f(x) \mid x \in [x_2, x_3]\} \\ &= \inf\{f(x) \mid x \in [1/2, 1]\} \\ &= \inf\{f(x) \mid f(x) \in [1, 2]\} \\ &= 1 \end{aligned}$$

Now we can calculate $L(f, P)$:

$$\begin{aligned} L(f, P) &= \sum_{i=1}^{n} m_i(x_i - x_{i-1}) && \text{(By def. of } L(f, P)\text{)} \\ &= \sum_{i=1}^{3} m_i(x_i - x_{i-1}) \\ &= m_1(x_1 - x_0) + m_2(x_2 - x_1) + m_3(x_3 - x_2) && \text{(Expanding sigma notation)} \\ &= 0(1/4 - 0) + 1/2(1/2 - 1/4) + 1(1 - 1/2) \\ &= 0 + 1/8 + 1/2 \\ &= 5/8 \end{aligned}$$

Example 2. Calculating an upper sum
Calculate the upper sum of the Dirichlet function for any partition over $[0, 2]$.

Let $P = \{x_i\}_{i=0}^{n}$ be an arbitrary partition. Define M_i as above: For each $i = 1, 2, ..., n$, define $M_i = \sup\{f(x) \mid x \in [x_{i-1}, x_i]\}$.

We can use the fact that **the rational numbers are dense in the real numbers**. Loosely, this means that between any two rational numbers, there is an irrational number, and between any two irrational numbers, there must be a rational number. In particular, **any interval must contain at least one rational number and at least one irrational number**. Therefore, each of the intervals $[x_{i-1}, x_i]$ (as defined by the given partition) must contain an irrational and a rational. This means, for all possible values of i, $1 \leq i \leq n$:

$$\begin{aligned} &\exists c_1 \in [x_{i-1}, x_i] \quad \text{s.t.} \quad c_1 \in \mathbb{Q} \quad \implies \quad \exists c_1 \in [x_{i-1}, x_i] \quad \text{s.t.} \quad f(c_1) = 1 \\ &\exists c_2 \in [x_{i-1}, x_i] \quad \text{s.t.} \quad c_2 \in \mathbb{I} \quad \implies \quad \exists c_2 \in [x_{i-1}, x_i] \quad \text{s.t.} \quad f(c_2) = 0 \end{aligned}$$

Therefore, we can find the value of M_i for all possible values of i, $1 \leq i \leq n$:

$$\begin{aligned} M_i &= \sup\{f(x) \mid x \in [x_{i-1}, x_i]\} \\ &= \sup\{0, 1\} \\ M_i &= 1 \end{aligned}$$

Calculating the upper sum:

$$\begin{aligned}
U(f,P) &= \sum_{i=1}^{n} M_i(x_i - x_{i-1}) \\
&= \sum_{i=1}^{n} (1)(x_i - x_{i-1}) \qquad \text{(since } M_i = 1 \text{ for all } i,\ 1 \le i \le n\text{)} \\
&= \sum_{i=1}^{n} (x_i - x_{i-1}) \\
&= (x_1 - x_0) + (x_2 - x_1) + (x_3 - x_2) + ... + (x_{n-1} - x_{n-2}) + (x_n - x_{n-1}) \quad \text{(Expanding sum)} \\
&= (\cancel{x_1} - x_0) + (\cancel{x_2} - \cancel{x_1}) + (\cancel{x_3} - \cancel{x_2}) + ... + (\cancel{x_{n-1}} - \cancel{x_{n-2}}) + (x_n - \cancel{x_{n-1}}) \\
&= x_n - x_0 \qquad \text{(This is an example of a \textbf{telescoping sum} — the terms cancel out)} \\
&= 2 - 0 \\
&= 2
\end{aligned}$$

Example 3. We'll calculate the upper sum of the Dirichlet function for any partition over $[0, 2]$.

Let $P = \{x_i\}_{i=0}^{n}$ be an arbitrary partition. Define M_i as above: For each $i = 1, 2, ..., n$, define $M_i = \sup\{f(x) \mid x \in [x_{i-1}, x_i]\}$.

We can use the fact that **the rational numbers are dense in the real numbers**. Loosely, this means that between any two rational numbers, there is an irrational number, and between any two irrational numbers, there must be a rational number. In particular, **any interval must contain at least one rational number and at least one irrational number**. Therefore, each of the intervals $[x_{i-1}, x_i]$ (as defined by the given partition) must contain an irrational and a rational. This means, for all possible values of i, $1 \le i \le n$:

$$\begin{aligned}
\exists c_1 \in [x_{i-1}, x_i] \quad \text{s.t.} \quad c_1 \in \mathbb{Q} \quad &\implies \quad \exists c_1 \in [x_{i-1}, x_i] \quad \text{s.t.} \quad f(c_1) = 1 \\
\exists c_2 \in [x_{i-1}, x_i] \quad \text{s.t.} \quad c_2 \in \mathbb{I} \quad &\implies \quad \exists c_2 \in [x_{i-1}, x_i] \quad \text{s.t.} \quad f(c_2) = 0
\end{aligned}$$

This means that for any interval $[x_{i-1}, x_i]$, the values that f takes along the interval are 0 and 1. That is, the set $\{f(x) \mid x \in [x_{i-1}, x_i]\} = \{0, 1\}$.
Therefore, we can find the value of M_i for all possible values of i, $1 \le i \le n$:

$$\begin{aligned}
M_i &= \sup\{f(x) \mid x \in [x_{i-1}, x_i]\} \\
&= \sup\{0, 1\} \\
M_i &= 1
\end{aligned}$$

Calculating the upper sum:

$$
\begin{aligned}
U(f,P) &= \sum_{i=1}^{n} M_i(x_i - x_{i-1}) \\
&= \sum_{i=1}^{n} (1)(x_i - x_{i-1}) \qquad \text{(since } M_i = 1\text{)} \\
&= \sum_{i=1}^{n} (x_i - x_{i-1}) \\
&= (x_1 - x_0) + (x_2 - x_1) + (x_3 - x_2) + ... + (x_{n-1} - x_{n-2}) + (x_n - x_{n-1}) \\
&= (\cancel{x_1} - x_0) + (\cancel{x_2} - \cancel{x_1}) + (\cancel{x_3} - \cancel{x_2}) + ... + (\cancel{x_{n-1}} - \cancel{x_{n-2}}) + (x_n - \cancel{x_{n-1}}) \\
&= x_n - x_0 \qquad \text{(This is an example of a } \textbf{telescoping sum} \text{ — the terms cancel out)} \\
&= 2 - 0 \\
&= 2
\end{aligned}
$$

Since P was taken to be an arbitrary partition, this means for all partitions P, $U(f,P) = 2$.

1.1 Darboux definition of integrability

We can define whether a function is integrable using upper and lower sums instead of Riemann sums.

Let $a, b \in \mathbb{R}$, $a < b$. Let f be a bounded function on $[a,b]$. Let P be an arbitrary partition of $[a,b]$.

Definition 2. ***Darboux definition of integrability***
f is integrable on $[a,b]$ $\iff$
$\sup\{L(f,P) \mid \forall \text{ partitions } P \text{ of } [a,b]\} = \inf\{U(f,P) \mid \forall \text{ partitions } P \text{ of } [a,b]\}$

This means f is integrable if the supremum of all possible lower sums is equal to the infimum of all possible upper sums.
To show a function **isn't** integrable, show that the two values in the definition are **not** equal.

Definition 3. ***Integrability reforumulation***
f is integrable on $[a,b]$ $\iff$
$\forall \epsilon > 0, \exists$ *a Partition P of $[a,b]$ s.t.* $U(f,P) - L(f,P) < \epsilon$

In other words, f is integrable on $[a,b]$ if, for any given ϵ, we can choose a partition such that the difference between the upper and lower sums of f with respect to P is < 0.

Note: We mainly use this integrability reformulation to prove a function **isn't** integrable. To show that a function isn't integrable, find a specific value of ϵ where the difference between the upper and lower sums for any given partition is greater than ϵ.

Example 4. Consider g(x) = $\begin{cases} 1/2 & x \in \mathbb{Q} \\ 1/4 & x \notin \mathbb{Q} \end{cases}$

Prove that g is not integrable on $[1, 2]$ using (a) the Darboux definition of the definite integral and (b) the integrability reformulation.

Sol. (a) The Darboux definition states that g is integrable on [1,2] iff
$\sup\{L(g, P) \mid \forall$ partitions P of $[a, b]\} = \inf\{U(g, P) \mid \forall$ partitions P of $[a, b]\}$.
To show that g is not interval, you would need to show that these two values are not equal. To do this, take an arbitrary partition and find all possible lower sums and all possible upper sums for an arbitrary partition. Then take the supremum of the possible lower sums and the infimum of the possible upper sums.

Proof. Let $P = \{x_i\}_{i=0}^{n}$ be any arbitrary partition of [1,2]. Every interval $[x_{i-1}, x_i]$ for $i = 1, ..., n$, by the density of the real numbers, contains an irrational number and a rational number. Therefore the function attains the values 1/2 and 1/4 within every interval. i.e.
for all $i = 1, ..., n$, $\exists c_1 \in [x_{i-1}, x_i]$ s.t. $c_1 \in \mathbb{Q}$ and $g(c_1) = 1/2$ and
for all $i = 1, ..., n$, $\exists c_2 \in [x_{i-1}, x_i]$ s.t. $c_1 \notin \mathbb{Q}$ and $g(c_2) = 1/4$
Let i be arbitrary, $i = 1, .., n$. Calculating M_i and m_i:

$$\begin{aligned} M_i &= \sup\{f(x) \mid x \in [x_{i-1}, x_i]\} \\ &= \sup\{1/2, 1/4\} \qquad \text{Since function attains 1/2 and 1/4 within each interval by density of rational numbers} \\ &= 1/2 \end{aligned}$$

$$\begin{aligned} m_i &= \inf\{f(x) \mid x \in [x_{i-1}, x_i]\} \\ &= \inf\{1/2, 1/4\} \qquad \text{Since } \exists c_1, c_2 \in [x_{i-1}, x_i] \text{ s.t. } g(c_1) = 1/2 \text{ and } g(c_2) = 1/4\text{, by density of rational numbers} \\ &= 1/4 \end{aligned}$$

Therefore for all $i = 1, ..., n$, $M_i = 1/2$ and $m_i = 1/4$.

Calculating $U(g,P)$ and $L(g,P)$:

$$\begin{aligned}
U(g,P) &= \sum_{i=1}^{n} M_i(x_i - x_{i-1}) \\
&= \sum_{i=1}^{n} (1/2)(x_i - x_{i-1}) \qquad \text{(since } M_i = 1/2 \quad \forall i = 1, ..., n) \\
&= \frac{1}{2}\sum_{i=1}^{n} (x_i - x_{i-1}) \qquad \text{By sigma property} \\
&= \frac{1}{2}[(x_1 - x_0) + (x_2 - x_1) + (x_3 - x_2) + ... + (x_{n-1} - x_{n-2}) + (x_n - x_{n-1})] \\
&= \frac{1}{2}[(\cancel{x_1} - x_0) + (\cancel{x_2} - \cancel{x_1}) + (\cancel{x_3} - \cancel{x_2}) + ... + (\cancel{x_{n-1}} - \cancel{x_{n-2}}) + (x_n - \cancel{x_{n-1}})] \\
&= \frac{1}{2}(x_n - x_0) \\
&= \frac{1}{2}(2 - 1) \\
&= \frac{1}{2}
\end{aligned}$$

$$\begin{aligned}
L(g,P) &= \sum_{i=1}^{n} m_i(x_i - x_{i-1}) \\
&= \sum_{i=1}^{n} (1/4)(x_i - x_{i-1}) \qquad \text{(since } m_i = 1/4 \quad \forall i = 1, ..., n) \\
&= \frac{1}{4}\sum_{i=1}^{n} (x_i - x_{i-1}) \qquad \text{By sigma property} \\
&= \frac{1}{4}[(x_1 - x_0) + (x_2 - x_1) + (x_3 - x_2) + ... + (x_{n-1} - x_{n-2}) + (x_n - x_{n-1})] \\
&= \frac{1}{4}[(\cancel{x_1} - x_0) + (\cancel{x_2} - \cancel{x_1}) + (\cancel{x_3} - \cancel{x_2}) + ... + (\cancel{x_{n-1}} - \cancel{x_{n-2}}) + (x_n - \cancel{x_{n-1}})] \\
&= \frac{1}{4}(x_n - x_0) \\
&= \frac{1}{4}(2 - 1) \\
&= \frac{1}{4}
\end{aligned}$$

Since P was taken to be arbitrary, for all partitions P, $U(g,P) = \frac{1}{2}$ and $L(g,P) = \frac{1}{4}$.
Therefore

$$\begin{aligned}
\sup\{L(g,P) \mid \forall \text{ partitions } P \text{ of } [a,b]\} &= \\
&= \sup\{1/4\} \\
&= 1/4
\end{aligned}
\qquad
\begin{aligned}
\inf\{U(g,P) \mid \forall \text{ partitions } P \text{ of } [a,b]\} &= \\
&= \inf\{1/2\} \\
&= 1/2
\end{aligned}$$

Thus $\sup\{L(g,P) \mid \forall \text{ partitions } P \text{ of } [a,b]\} \neq \inf\{U(g,P) \mid \forall \text{ partitions } P \text{ of } [a,b]\}$ so g is not integrable.

Sol. (b) The integrability reformulation states that g is integrable if, for any $\epsilon > 0, \exists$ a Partition P of $[a, b]$ s.t. $U(f, P) - L(f, P) < \epsilon$. In order to prove that g is not integrable, you need to identify an ϵ and partition P for which the definition doesn't hold, i.e. $U(f, P) - L(f, P) > \epsilon$. It's easier to calculate the possible lower and upper sums first, then choose an epsilon that is greater than the difference between them.

Proof. Let $P = \{x_i\}_{i=0}^{n}$ be any arbitrary partition of [1,2]. Every interval $[x_{i-1}, x_i]$ for $i = 1, ..., n$, by the density of the real numbers, contains an irrational number and a rational number. Therefore the function attains the values $1/2$ and $1/4$ within every interval.

Calculating the upper and lower sums $U(g, P)$ and $L(g, P)$ as above (not going to rewrite it here but you'd need to follow the same steps + justification) gives $U(g, P) = 1/2$ and $L(g, P) = 1/4$. Thus, for any partition P, $U(g, P) - L(g, P) = 1/2 - 1/4 = 1/4$.

Let $\epsilon = 1/8$. Then for any partition P of [1,2], $U(g, P) - L(g, P) = 1/2 - 1/4 = 1/4 > \epsilon = 1/8$. Therefore, there exists an $\epsilon > 0$ and a partition P of [1,2] for which $U(g, P) - L(g, P) > \epsilon$, i.e. g is not integrable on [1,2].

Example 5. Evaluate the following sums:

$$(1) \quad \sum_{n=2}^{100} \ln\left(1 + \frac{1}{n}\right)$$

$$(2) \quad \sum_{n=5}^{200} \frac{2}{n^2 - 1}$$

Solutions on next page.

Evaluate the following sums:

$$(1) \quad \sum_{n=2}^{100} \ln\left(1+\frac{1}{n}\right)$$
$$(2) \quad \sum_{n=5}^{200} \frac{2}{n^2-1}$$

Sol. (1)

$$\begin{aligned}
\sum_{n=2}^{100} \ln\left(1+\frac{1}{n}\right) &= \sum_{n=2}^{100} \ln\left(\frac{n+1}{n}\right) \\
&= \sum_{n=2}^{100} \ln(n+1)-\ln(n) \qquad \text{Property of ln} \\
&= (\ln(2+1)-\ln(2)) + (\ln(3+1)-\ln(3)) + ... + (\ln(99+1)-\ln(99)) + (\ln(100+1)-\ln(100)) \\
&= (\ln(3)-\ln(2)) + (\ln(4)-\ln(3)) + (\ln(5)-\ln(4)) + ... + (\ln(100)-\ln(99)) + (\ln(101)-\ln(100)) \\
&= (\cancel{\ln(3)}-\ln(2)) + (\cancel{\ln(4)}-\cancel{\ln(3)}) + (\cancel{\ln(5)}-\cancel{\ln(4)}) + ... + (\cancel{\ln(100)}-\cancel{\ln(99)}) + (\ln(101)-\cancel{\ln(100)}) \\
&= \ln(101)-\ln(2)
\end{aligned}$$

Sol. (2)

$$\begin{aligned}
\sum_{n=5}^{200} \frac{2}{n^2-1} &= \sum_{n=5}^{200} \frac{2}{(n+1)(n-1)} \\
&= \sum_{n=5}^{200} \frac{1}{n+1} - \frac{1}{n-1} \\
&= \left(\frac{1}{5+1}-\frac{1}{5-1}\right) + \left(\frac{1}{6+1}-\frac{1}{6-1}\right) + ... + \left(\frac{1}{199+1}-\frac{1}{199-1}\right) + \left(\frac{1}{200+1}-\frac{1}{200-1}\right) \\
&= \left(\frac{1}{6}-\frac{1}{4}\right) + \left(\frac{1}{7}-\frac{1}{5}\right) + \left(\frac{1}{8}-\frac{1}{6}\right) + \left(\frac{1}{9}-\frac{1}{7}\right) + ... + \left(\frac{1}{199}-\frac{1}{197}\right) + \left(\frac{1}{200}-\frac{1}{198}\right) + \left(\frac{1}{201}-\frac{1}{199}\right) \\
&= \left(\cancel{\frac{1}{6}}-\frac{1}{4}\right) + \left(\cancel{\frac{1}{7}}-\frac{1}{5}\right) + \left(\cancel{\frac{1}{8}}-\cancel{\frac{1}{6}}\right) + \left(\cancel{\frac{1}{9}}-\cancel{\frac{1}{7}}\right) + ... + \left(\cancel{\frac{1}{199}}-\cancel{\frac{1}{197}}\right) + \left(\frac{1}{200}-\cancel{\frac{1}{198}}\right) + \left(\frac{1}{201}-\cancel{\frac{1}{199}}\right) \\
&= -\frac{1}{4}-\frac{1}{5}+\frac{1}{200}+\frac{1}{201}
\end{aligned}$$