MATA37 Week 2

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1 Sigma notation

Let m, n, k be integers ≥ 0.

Let a_k be a real-valued function in terms of k. (That is, $a_k = f(k)$ where f is a function). We can represent the sum of numbers $a_m + a_{m+1} + ... + a_n$ using sigma notation.

$$\sum_{k=m}^{n} a_k = a_m + a_{m+1} + ... + a_n$$

k is the index variable.

m is the initial value.

n is the end value.

Example 1. Representing a sum in sigma notation

Rewrite the following sums using sigma notation in two different ways.

(1) $\frac{4}{3} + \frac{5}{4} + \frac{6}{5} + \frac{7}{6} + \frac{8}{7} + \frac{9}{8} + \frac{10}{9} + \frac{11}{10} + \frac{12}{11}$

(2) $2 + 5 + 8 + 11 + 14$

(3) $5 + 5 + 5 + 5 + 5$

(4) $1 + 0 + -1 + 0 + 1 + 0 + -1 + 0 + 1 + 0 + -1 + 0$

1.1 Properties of sigma notation

Let $m, n, k, l \in \mathbb{Z}^{\geq 0}$ such that $l \leq k \leq n$ and $l \leq m$. Let c be a real number.

Let a_k and b_k be real valued functions of k.

Theorem 1. ***Properties of sigma notation***

$$(I) \quad \sum_{k=m}^{n}(a_k \pm b_k) = \sum_{k=m}^{n} a_k \pm \sum_{k=m}^{n} b_k$$
$$(II) \quad \sum_{k=m}^{n} c \cdot a_k = c\sum_{k=m}^{n} a_k$$
$$(III) \quad \sum_{k=m}^{n} a_k = \sum_{k=m}^{l-1} a_k + \sum_{k=l}^{n} a_k$$

(I) says that we can split up terms under a summation symbol when they're being added together.

(II) says that we can take any constants that are being multiplied by the general term "outside" the summation symbol. In general, we can do this with any multiplicative term that doesn't depend on the index variable.

(III) says that we can split up any summation from $k = m$ to $k = n$ into two separate summations by using some integer l that's in between m and n.

Example 2. Proof of Thm 1, Property (I)
Prove $\sum_{k=m}^{n}(a_k + b_k) = \sum_{k=m}^{n} a_k + \sum_{k=m}^{n} b_k$.

Example 3. Simplifying sum addition (pg. 326, #41 in text)
Rewrite the following as a single sum (you'll have to add on a few terms):

$$2\sum_{k=0}^{100} a_k + \sum_{k=3}^{101} a_k$$

The following series come up often and you should memorize their values. Let $c \in \mathbb{R}$ be a constant and let n be a positive integer.

Theorem 2. ***Useful sum formulas***

$$\begin{aligned}
&(a) \quad \sum_{i=1}^{n} 1 = n \\
&(b) \quad \sum_{i=1}^{n} c = cn \qquad \left(\sum_{i=1}^{n} c = \sum_{i=1}^{n} c \cdot 1 = c\sum_{i=1}^{n} 1 = cn\right) \\
&(c) \quad \sum_{i=1}^{n} i = \frac{n(n+1)}{2} \\
&(d) \quad \sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6} \\
&(e) \quad \sum_{i=1}^{n} i^3 = \left[\frac{n(n+1)}{2}\right]^2
\end{aligned}$$

Example 4.
Using the above properties, evaluate $\sum_{i=1}^{n} \frac{2}{n^3}(i+1)^2$.

2 Riemann Sums

Let a and b be real numbers such that $a < b$.

Definition 1. *A* ***partition****, P, of the interval $[a, b]$ is a collection of $n+1$ points $\{x_0, x_1, x_2, ...x_n\}$ such that $x_0 = a$, $x_n = b$, and the following holds:*

$$x_0 = a < x_1 < x_2 < ... < x_n = b$$

A partition, P, is an ordered set of $n+1$ points taken from $[a, b]$ whose endpoints x_0 and x_n are a and b. For example, a partition of the interval $[0, 2]$ might be $P = \{x_0 = 0, x_1 = \frac{1}{4}, x_2 = \frac{1}{2}, x_3 = 1, x_4 = \frac{3}{2}, x_5 = 2\}$.

We're mostly concerned with a more specific type of partition called the Riemann partition. Consider the same interval $[a, b]$ as above.

Definition 2. *A* ***Riemann partition*** *of $[a, b]$ is a partition $P = \{x_0, x_1, x_2, ..., x_n\}$ of $[a, b]$ such that $x_i = a + i\Delta x$ for all $i = 0, 1, ..., n$ where $\Delta x = \dfrac{b-a}{n}$.*

A Riemann partition is a partition that splits up the interval $[a, b]$ into n pieces. The width of each of these pieces is $\Delta x = \frac{b-a}{n}$. We can think of these n pieces as the intervals $[x_0, x_1], [x_1, x_2], ..., [x_{n-1}, x_n]$.

More succinctly, the partition splits the interval into the subintervals $[x_{i-1}, x_i]$, where $i = 0, 1, 2, ..n$.

We can use a Riemann partition to approximate the area between a function and the x-axis.

Let $[a, b]$ be an interval as above and let f be a given function. Also, let $P = \{x_0, x_1, x_2, ...x_n\}$ be a Riemann partition on $[a, b]$. Define $\Delta x = \dfrac{b-a}{n}$ and $x_i = a + i\Delta x$.

Definition 3. *A **Riemann sum** for f on $[a,b]$ is the following summation:*

$$\sum_{i=1}^{n} f(x_i^*)\Delta x$$

*where x_i^* is some **sample point** in the interval $[x_{i-1}, x_i]$.*

x_i^* is taken as some arbitrary sample point within the interval $[x_{i-1}, x_i]$. The Riemann sum represents the following calculation: At each sample point, we find the value of the function at the point, $f(x_i^*)$ then multiply that by the width of the interval,$\Delta x = \dfrac{b-a}{n}$; this represents the area of a rectangle. Adding these areas up gives an approximation of the area under the curve. We can be consistent in our choice for x_i^*. For example, we can take x_i^* to be the left endpoint of each interval $[x_{i-1}, x_i]$, in which case we take $x_i^* = x_{i-1}$. Taking the right endpoints gives $x_i^* = x_i$. We can also take the x_i^* to be the midpoint of each interval. (Finding the formula for this is a homework question on assignment 1.) These particular cases can be represented as follows:

Definition 4. *Taking the **right Riemann sum** of f:*

$$\begin{aligned} R_n &= \sum_{i=1}^{n} f(x_i)\Delta x \\ &= f(x_1)\Delta x + f(x_2)\Delta x + ... + f(x_n)\Delta x \end{aligned}$$

*Taking the **left Riemann sum** of f:*

$$\begin{aligned} L_n &= \sum_{i=1}^{n} f(x_{i-1})\Delta x \\ &= f(x_0)\Delta x + f(x_1)\Delta x + ... + f(x_{n-1})\Delta x \end{aligned}$$

Where $\Delta x = \dfrac{b-a}{n}$ and $x_i = a + i\Delta x$

Example 1. Calculating Riemann sums
For the function $f(x) = (x+1)^2$, find L_4 and R_4 over the interval $[1,3]$. i.e. find the left and right Riemann sums for f, with $n = 4$, over $[1,3]$.

Example 2. Recognizing an approximation as a Riemann sum
For the following sums, determine the type of approximation (left Riemann sum, right Riemann sum, etc.)

and identify $f(x)$, $[a, b]$, Δx, and x_i (and n where applicable).

$$(1) \qquad \sum_{i=1}^{3} \left(1 + \frac{i-1}{3}\right)\frac{1}{3}$$

$$(2) \qquad \frac{1}{10}\left[\left(\frac{1}{10}\right)^9 + \left(\frac{2}{10}\right)^9 + \left(\frac{3}{10}\right)^9 + \ldots + 1\right]$$

$$(3) \qquad \sum_{i=1}^{5} \left(\sin\left(2 + \frac{4i}{5}\right)\right)\frac{1}{5}$$

$$(4) \qquad \sum_{i=1}^{n} \frac{1}{n^2}(i+n)$$

Example 3. Proving statements related to Riemann sums

Let $a, b \in \mathbb{R}$, $a < b$. Let $n \in \mathbb{Z}^{\geq 0}$. Let f be a function defined on $[a, b]$. Prove the following:

(1) If f is a constant function on $[a, b]$, then the left Riemann sum L_n and the right Riemann sum R_n give the exact same value.

(2) If f is decreasing on $[a, b]$, then $R_n \leq L_n$.

3 Answers to examples

Example 1. Representing a sum in sigma notation
Rewrite the following sums using sigma notation in two different ways.
(1) $\frac{4}{3}+\frac{5}{4}+\frac{6}{5}+\frac{7}{6}+\frac{8}{7}+\frac{9}{8}+\frac{10}{9}+\frac{11}{10}+\frac{12}{11}$
(2) $2+5+8+11+14$
(3) $5+5+5+5+5$
(4) $1+0+-1+0+1+0+-1+0+1+0+-1+0$

Sol.

$$(1) \qquad \frac{4}{3}+\frac{5}{4}+\frac{6}{5}+\frac{7}{6}+\frac{8}{7}+\frac{9}{8}+\frac{10}{9}+\frac{11}{10}+\frac{12}{11}=\sum_{n=3}^{11}\frac{n+1}{n}=\sum_{n=4}^{12}\frac{n}{n-1}$$

$$(2) \qquad 2+5+8+11+14=\sum_{n=0}^{4}3n+2=\sum_{n=1}^{5}3n-1$$

$$(3) \qquad 5+5+5+5+5=\sum_{n=1}^{5}5=\sum_{n=0}^{n=4}5$$

$$(4) \qquad 1+0+-1+0+1+0+-1+0+1+0+-1+0=\sum_{k=0}^{12}\cos\left(\frac{k\pi}{2}\right)=\sum_{k=0}^{12}\sin\left(\frac{(k+1)\pi}{2}\right)$$

Example 2. Proof of Thm 1, Property (I)
Prove $\sum_{k=m}^{n}(a_k+b_k)=\sum_{k=m}^{n}a_k+\sum_{k=m}^{n}b_k$.

Sol.

$$\begin{aligned}\sum_{k=m}^{n}(a_k+b_k) &= (a_m+b_m)+(a_{m+1}+b_{m+1})+...+(a_n+b_n) \qquad \text{(Def. of sigma notation)}\\ &= (a_m+a_{m+1}+...+a_n)+(b_m+b_{m+1}+...+b_n) \qquad \text{(Commutativity of real numbers)}\\ &= \sum_{k=m}^{n}a_k+\sum_{k=m}^{n}b_k \qquad \text{(Def. of sigma notation)} \qquad \square\end{aligned}$$

Example 3. Simplifying sum addition (pg. 326, #41 in text)
Rewrite the following as a single sum (you'll have to add on a few terms):

$$2\sum_{k=0}^{100}a_k+\sum_{k=3}^{101}a_k$$

Sol.

$$
\begin{aligned}
2\sum_{k=0}^{100} a_k + \sum_{k=3}^{101} a_k &= 2\sum_{k=0}^{100} a_k + \sum_{k=0}^{101} a_k - (a_0 + a_1 + a_2) && \text{(By def. of sigma notation — changing initial value)} \\
&= \sum_{k=0}^{100} 2a_k + \sum_{k=0}^{101} a_k - (a_0 + a_1 + a_2) && \text{(By property of sigma notation)} \\
&= \sum_{k=0}^{101} 2a_k - 2a_{101} + \sum_{k=0}^{101} a_k - (a_0 + a_1 + a_2) && \text{By def. of sigma notation — changing end value)} \\
&= \sum_{k=0}^{101} (2a_k + a_k) - 2a_{101} - (a_0 + a_1 + a_2) && \text{(By property of sigma notation)} \\
&= \sum_{k=0}^{101} (3a_k) - 2a_{101} - a_0 - a_1 - a_2 && \square
\end{aligned}
$$

Example 4. Calculating Riemann sums

For the function $f(x) = (x+1)^2$, find L_4 and R_4 over the interval $[1,3]$. i.e. find the left and right Riemann sums for f, with $n = 4$, over $[1,3]$.

Sol. From the values given in the question, we have $a = 1$, $b = 3$, $n = 4$. We can use these to find Δx and x_i.

$$
\begin{aligned}
\Delta x &= \frac{b-a}{n} \\
&= \frac{3-1}{4} \\
&= \frac{1}{2}
\end{aligned}
\qquad\qquad
\begin{aligned}
x_i &= a + i\Delta x \\
&= 1 + i\frac{1}{2} \\
&= 1 + \frac{i}{2}
\end{aligned}
$$

Finding the right Riemann sum,

$$
\begin{aligned}
R_4 &= \sum_{i=1}^{4} f(x_i)\Delta x \qquad \text{(By def. of right Riemann sum)} \\
&= \sum_{i=1}^{4} f\Big(1+\frac{i}{2}\Big)\Big(\frac{1}{2}\Big) \qquad \text{(By the calculated values above)} \\
&= \sum_{i=1}^{4} \Big(\Big(1+\frac{i}{2}\Big)+1\Big)^2\Big(\frac{1}{2}\Big) \qquad \text{(Since the given } f \text{ is } f(x)=(x+1)^2\text{)} \\
&= \sum_{i=1}^{4} \Big(2+\frac{i}{2}\Big)^2\Big(\frac{1}{2}\Big) \\
&= \frac{1}{2}\sum_{i=1}^{4} \Big(2+\frac{i}{2}\Big)^2 \qquad \text{(By sigma property (II))} \\
&= \frac{1}{2}\sum_{i=1}^{4} 4+2i+\frac{i^2}{4} \\
&= \frac{1}{2}\left(\sum_{i=1}^{4} 4+\sum_{i=1}^{4} 2i+\sum_{i=1}^{4}\frac{i^2}{4}\right) \qquad \text{(By sigma property (I))} \\
&= \frac{1}{2}\left(4\sum_{i=1}^{4} 1+2\sum_{i=1}^{4} i+\frac{1}{4}\sum_{i=1}^{4} i^2\right) \qquad \text{(By sigma property (II)} \\
&= \frac{1}{2}\left(4(4)+2\frac{(4)(5)}{2}+\frac{1}{4}\Big(\frac{(4)(5)(9)}{6}\Big)\right) \qquad \text{(By useful summation formulas)} \\
&= \frac{1}{2}(16+20+\frac{45}{6}) \\
&= \frac{261}{12} \qquad \square
\end{aligned}
$$

Finding the left Riemann sum, we use $x_{i-1} = a + (i-1)\Delta x = 1 + \frac{i-1}{2}$ and apply the above process, with $f(x_{i-1})$ instead of $f(x_i)$.

Example 5. Recognizing an approximation as a Riemann sum

For the following sums, determine the type of approximation (left Riemann sum, right Riemann sum, etc.) and identify $f(x)$, $[a,b]$, Δx, x_i (and n where applicable).

$$(1) \qquad \sum_{i=1}^{3}\Big(1+\frac{i-1}{3}\Big)\frac{1}{3}$$

$$(2) \qquad \frac{1}{10}\left[\Big(\frac{1}{10}\Big)^9+\Big(\frac{2}{10}\Big)^9+\Big(\frac{3}{10}\Big)^9+...+1\right]$$

$$(3) \qquad \sum_{i=1}^{5}\Big(\sin\Big(2+\frac{4i}{5}\Big)\Big)\frac{1}{5}$$

$$(4) \qquad \sum_{i=1}^{n}\frac{1}{n^2}(i+n)$$

Sol. (1)

Looking at (1), the expression $i-1$ should stand out to you. That means we're probably dealing with a left Riemann sum. Compare (1) to the def. of L_n:

$$(1) \quad \sum_{i=1}^{3}\Big(1+\frac{i-1}{3}\Big)\frac{1}{3}$$

$$L_n = \sum_{i=1}^{n} f(x_{i-1})\Delta x$$

Looking at the end value, we conclude $n=3$. That means $\Delta x = \frac{b-a}{n} = \frac{b-a}{3}$. So Δx is some fraction with denominator 3. Comparing (1) to the def. of L_n again, it looks like we have $f(x_{i-1}) = (1+\frac{i-1}{3})$ and $\Delta x = \frac{1}{3}$. That means, by how we define $x_i = a + i\Delta x$, we have $x_{i-1} = a + (i-1)\frac{1}{3}$. (Substituting $i-1$ and $\Delta x = \frac{1}{3}$) And since we see $f(x_{i-1}) = (1+\frac{i-1}{3})$ and $x_{i-1} = a+(i-1)\frac{1}{3}$, we can let $a=1$.
Using $a=1$ and $\Delta x = \frac{b-a}{3}$, it follows $b=2$.
Now, we know

$$f(x_{i-1}) = f(1+\frac{i-1}{3}) \qquad \text{(By definition of } x_{i-1}\text{)}$$

$$f(x_{i-1}) = 1+\frac{i-1}{3} \qquad \text{(By inspection, comparing the def. of } L_n \text{ to the expression (1))}$$

That means $f(1+\frac{i-1}{3}) = 1+\frac{i-1}{3}$. What function is f if it takes in a value and outputs the same value? f must be $f(x)=x$.
In sum, we have

- $f(x)=x$
- $n=3$
- $a=1$, $b=2$
- $\Delta x = \frac{1}{3}$
- $x_i = 1+\frac{i}{3}$

(1) is the left Riemann sum, with $n=3$, of $f(x)=x$ over $[1,2]$. $\square$

Sol. (2)
First, we need to rewrite the sum part (the second part, in the square brackets) in sigma notation.

$$\frac{1}{10}\left[\Big(\frac{1}{10}\Big)^9 + \Big(\frac{2}{10}\Big)^9 + \Big(\frac{3}{10}\Big)^9 + ... + 1\right] = \frac{1}{10}\sum_{i=1}^{10}\Big(\frac{i}{10}\Big)^9$$

We can bring the $\frac{1}{10}$ part inside the sigma notation by property, so

$$\frac{1}{10}\sum_{i=1}^{10}\Big(\frac{i}{10}\Big)^9 = \sum_{i=1}^{10}\Big(\frac{i}{10}\Big)^9\frac{1}{10}$$

Compare this with the definition for R_n.

$$(2) \quad \sum_{i=1}^{10} \left(\frac{i}{10}\right)^9 \frac{1}{10}$$
$$R_n = \sum_{i=1}^{n} f(x_i)\Delta x$$

Looking at the end values, we have $n = 10$.
Let $f(x_i) = (\frac{i}{10})^9$ and $\Delta x = \frac{1}{10}$. Since $\Delta x = \frac{b-a}{n} = \frac{b-a}{10} = \frac{1}{10}$, we know $b - a = 1$.
Since $x_i = a + i\Delta x$, we want to have something looking like $x_i = a + i\frac{1}{10}$ in our expression $f(x_i) = (\frac{i}{10})^9$.
Notice $f(x_i) = (\frac{i}{10})^9 = (0 + \frac{i}{10})^9$.
So if we let $a = 0$: $\qquad x_i = a + i\Delta x = 0 + i\frac{1}{10} = \frac{i}{10} \qquad$ (If $a = 0$ then we also know $b = 1$, since $b - a = 1$)
So we have $f(x_i) = f(\frac{i}{10}) = (\frac{i}{10})^9$. So f takes in a number and outputs that number to the power of 9. That means $f(x) = x^9$.
In sum, we have

- $f(x) = x^9$
- $n = 10$
- $a = 0$, $b = 1$
- $\Delta x = \frac{1}{10}$
- $x_i = \frac{i}{10}$

(2) is the right Riemann sum, with $n = 10$, of $f(x) = x^9$ over $[0, 1]$. $\square$

Sol. (3) As before, let's compare (3) to the def. of R_n

$$(3) \quad \sum_{i=1}^{5} \left(\sin\left(2 + \frac{4i}{5}\right)\right)\frac{4}{5}$$
$$R_n = \sum_{i=1}^{n} f(x_i)\Delta x$$

We can see $n = 5$.
Let $f(x_i) = \sin(2 + \frac{4i}{5})$ and $\Delta x = \frac{4}{5}$. From $\Delta x = \frac{b-a}{5} = \frac{4}{5}$, we know $b - a = 4$.
We also know that x_i is of the form $a + i\Delta x = a + i\frac{4}{5}$.
We see that the expression $2 + \frac{4i}{5}$ pops up in $f(x_i) = \sin(2 + \frac{4i}{5})$, which looks similar to what x_i would look like, with $a = 2$.
Let's take $a = 2$. (This means $b = 6$)
Then $x_i = a + i\Delta x = 2 + \frac{4i}{5}$. And if $f(x_i) = \sin(2 + \frac{4i}{5})$, and we know $x_i = 2 + \frac{4i}{5}$, then $f(2 + \frac{4i}{5}) = \sin(2 + \frac{4i}{5})$.
So f must just be the sine function, $f(x) = \sin(x)$.
In sum, we have

- $f(x) = \sin(x)$
- $n = 5$

- $a = 2$, $b = 6$
- $\Delta x = \frac{4}{5}$
- $x_i = 2 + \frac{4i}{5}$

(3) is the right Riemann sum, with $n = 4$, of $f(x) = \sin(x)$ over $[2, 6]$. $\square$

Hint for (4): rewrite/expand and factor out $1/n$.

Example 6. Proving statements related to Riemann sums
Let $a, b \in \mathbb{R}$, $a < b$. Let $n \in \mathbb{Z}^{>0}$. Let f be a function defined on $[a, b]$. Prove the following:

(1) If f is a constant function on $[a, b]$, then the left Riemann sum L_n and the right Riemann sum R_n give the exact same value.

(2) If f is decreasing on $[a, b]$, then $R_n \leq L_n$.

Sol. Suppose f is a constant function on $[a, b]$. This means for all x in $[a, b]$, $f(x) = c$ for some $c \in \mathbb{R}$. Let n be some positive integer. Let $P = \{x_0 + x_1 + ... + x_n\}$ be a partition of $[a, b]$. Since f is a constant function, for all $i = 1, 2, ..., n$, we know $f(x_i) = f(x_{i-1})$.
We want to show $L_n = R_n$. Start at the left and work our way to the right.

$$\begin{aligned}
L_n &= \sum_{i=1}^{n} f(x_{i-1})\Delta x \qquad \text{(By def. of } L_n\text{)} \\
&= \sum_{i=1}^{n} f(x_i)\Delta x \qquad \text{(Since } f \text{ is a constant function, } f(x_i) = f(x_{i-1}) \text{ for all } i = 1, 2, ..., n\text{)} \\
&= R_n \qquad \text{(By def. of } R_n\text{)} \qquad \square
\end{aligned}$$

Sol. (2) First, note $\Delta x = (b - a)/n$ and Δx is a positive value since $a < b$ and $n \in \mathbb{Z}^{>0}$.
f is decreasing on $[a, b]$ means for any c_1, c_2 in $[a, b]$ where $c_1 \leq c_2$, we know $f(c_2) \leq f(c_1)$.
$x_{i-1} \leq x_i$ for all i (this is because by def. of x_i, $x_{i-1} + \Delta x = x_i$ and Δx is positive.)
Hence, $f(x_i) \leq f(x_{i-1})$ for all i, $1 \leq i \leq n$. Multiplying both sides of the inequality by Δx (note again $\Delta x > 0$) gives

$$f(x_i)\Delta x \leq f(x_{i-1})\Delta x \qquad \text{for all } 1 \leq i \leq n$$

Since the inequality is true for all $1 \leq i \leq n$, it follows that

$$\sum_{i=1}^{n} f(x_i)\Delta x \leq \sum_{i=1}^{n} f(x_{i-1})\Delta x$$

i.e. by def.,

$$R_n \leq L_n \qquad \square$$