MATA37 Week 10

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1 Series

Let $\{a_n\}$ be a sequence.
A sequence can be used to define an **infinite series**, in which you add up all possible terms of the sequence:

$$\sum_{n=1}^{\infty} a_n = a_1 + a_2 + a_3 + a_4 + ...$$

The general term of the series $\sum a_n$ is a_n.
The convergence of an infinite series can be related to the partial sums of its general term.

Definition 1. *For each $n \in \mathbb{N}$, the **nth partial sum** is the finite sum*

$$S_n = a_1 + a_2 + a_3 + ... + a_n$$

Definition 2. *A series $\sum a_n$ **converges** to some $s \in \mathbb{R}$ if the sequence of partial sums $\{S_n\} = \{S_1, S_2, S_3, ...\}$ converges to some real number s. i.e.*

$$\lim_{n\to\infty} S_n = s$$

If a series doesn't converge (i.e. if this limit doesn't exist) we say it diverges.

1.1 Geometric Series

Definition 3. *A **geometric series** is a series of the form*

$$a + ar + ar^2 + ar^3 + ... + ar^n + ... = \sum_{n=0}^{\infty} ar^n$$

*r is the **common ratio** of the series.*

To show that a series is a geometric series, show that it can be written in the form $\sum_{n=0}^{\infty} ar^n$ and identify what a and the common ratio r is.

1.2 Geometric Series Test

Let $\sum ar^n$ be a geometric series.

- If $|r| < 1$, $\sum ar^n$ converges and $\sum ar^n = \frac{a}{1-r}$

- If $|r| \geq 1$, $\sum ar^n$ diverges

1.3 Properties of convergent series

Let $\sum a_n$ and $\sum b_n$ be infinite series. Suppose both $\sum a_n = s$ and $\sum b_n = t$ (i.e. $\sum a_n$ and $\sum b_n$ both converge). Then the following properties hold:

- For any $c \in \mathbb{R}$, $\sum ca_n$ converges and $\sum ca_n = cs$
- $\sum(a_n \pm b_n)$ converges with sum $s \pm t$
- (Vanishing Condition) $\lim_{n\to\infty} a_n = 0$

1.3.1 Divergence Test

Note the Vanishing Condition above:

$$\sum a_n \text{ converges} \implies \lim_{n\to\infty} a_n = 0$$

The **contrapositive** of this statement gives a criterion for a series diverging:

$$\lim_{n\to\infty} a_n \neq 0 \implies \sum a_n \text{ diverges}$$

Note that the **converse** of the Vanishing Condition is **not necessarily true**.

$$\lim_{n\to\infty} a_n = 0 \not\Longrightarrow \sum a_n \text{ converges}$$

2 Summary of convergence tests

Check pages 648-649 of your text for another nice summary of convergence tests.

2.1 Geometric Series Test (GS Test)

Given a geometric series with first term a and common ratio r, $\sum_{n=0}^{\infty} ar^n$:

- If $|r| < 1$ then the series converges with sum $\frac{a}{1-r}$.
- If $|r| \geq 1$ then the series diverges.

2.2 Divergence Test (Div Test)

Given an (infinite) series $\sum a_n$,
If $\lim_{n\to\infty} a_n \neq 0$
then $\sum a_n$ diverges.

2.3 Integral Test (IT)

Given a series $\sum a_n$, define $f(n) = a_n$.
If $f(x)$ is continuous, positive, and decreasing on $[k, \infty)$
then

$$\sum_{n=k}^{\infty} a_n \text{ converges} \iff \int_k^{\infty} f(x)\,dx \text{ converges}$$

Note that when we use integral test, we define a function f on the real numbers based on the sequence a_n. We can take the derivative of f to determine if the function is decreasing, and then we can conclude that a_n is decreasing.
To prove that the corresponding integral converges, calculate the integral **by definition**—**don't** use anything like Thm. 5.21 (pg. 470)
Note: IT doesn't tell you the sum of a convergent series—it only tells you whether the series converges or diverges.

2.4 p-series Test

A p-series is a series of the form $\sum_{n=1}^{\infty} \frac{1}{n^p}$ where $p \in \mathbb{R}$, $p > 0$.

Given a p-series $\sum_{n=1}^{\infty} \frac{1}{n^p}$,

- If $p > 1$ then the series converges.
- If $p \in (0, 1]$ then the series diverges.

In order to use the p-series test, you need to show that you have a p-series, i.e. identify the value of p.
Note: p-series test also doesn't tell you the sum of a convergent series.

2.5 Comparison Theorem (CT)

Let $\sum a_n$, $\sum b_n$, and $\sum c_n$ be series.
If $0 \le a_n \le b_n$ for all $n \in \mathbb{N}$ and $\sum b_n$ converges
then $\sum a_n$ converges.

If $0 \le c_n \le a_n$ for all $n \in \mathbb{N}$ and $\sum c_n$ diverges
then $\sum a_n$ diverges.

2.6 Alternating Series Test (AST)

An alternating series is a series of the form $\sum(-1)^{n+1} b_n$ or $\sum(-1)^n b_n$, where $b_n > 0$ for all $n \in \mathbb{N}$

If $b_n \ge b_{n+1} > 0$ for all $n \in \mathbb{N}$ (i.e. b_n is decreasing and positive)
and $\lim_{n\to\infty} b_n = 0$
then $\sum(-1)^n b_n$ converges.

Unlike with the integral test, you can't take the derivative of b_n to determine whether it is decreasing. You should try using bounds to prove it.

2.7 Ratio Test

Definition: A series $\sum a_n$ is **absolutely convergent** if $\sum a_n$ converges and $\sum |a_n|$ converges.
A series $\sum a_n$ is **conditionally convergent** if $\sum a_n$ converges but $\sum |a_n|$ diverges.

Let $\sum a_n$ be a series, $a_n \neq 0$ for any $n \in \mathbb{N}$.
Define $\rho = \lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right|$.

If $\rho < 1$, then $\sum a_n$ is absolutely convergent (and convergent).
If $\rho > 1$, then $\sum a_n$ diverges.
If $\rho = 1$, the test is inconclusive.

Ratio test is usually used on series with factorials or exponentials.

3 Examples

Example 1. Does $\sum_{n=1}^{\infty} \frac{(-1)^n}{n}$ absolutely converge (AC), conditionally converge (CC), or div? Prove.

Example 2. Determine whether the following series converge or diverge. If the series is convergent find its sum.

$$(a) \quad \sum_{n=1}^{\infty} \frac{e^n+1}{ne^n+1}$$

$$(b) \quad \sum_{n=1}^{\infty} \frac{cos(n)+1}{e^n}$$

$$(c) \quad \sum_{n=1}^{\infty} \frac{2^{n+1}-n}{3^n}$$

$$(d) \quad \sum_{n=1}^{\infty} \frac{4^{n+1}}{3^n+n}$$

$$(e) \quad \sum_{n=2}^{\infty} \frac{1}{n(\ln(n))^2}$$

$$(f) \quad \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{1+k^2}$$

Example 3. Find all positive values of a for which the following series converge:

$$(a) \quad \sum_{n=0}^{\infty} (a-3)^n$$

1. $\sum_{n=1}^{\infty} \frac{(-1)^n}{n}$ AC, CC, div?

AST: alternating series with $b_n = \frac{1}{n}$

$\sum_{n=1}^{\infty} (-1)^n \frac{1}{n}$ WTS b_n is decreasing ($b_n \geq b_{n+1}$) + positive

① $\forall n \in \mathbb{N}$, $n < n+1$

$$\frac{1}{n} > \frac{1}{n+1}$$

$b_n > b_{n+1}$ by def. of b_n

$\therefore$ b_n decreasing

② $\forall n \in \mathbb{N}$, $\frac{1}{n} > 0$

③ $\lim_{n\to\infty} \frac{1}{n} = 0$ $\forall \varepsilon > 0 \; \exists N > 0$ s.t. if $n > N$ $\left|\frac{1}{n}\right| < \varepsilon$

Let $\varepsilon > 0$ be arbitrary

Let $N = \underline{1/\varepsilon}$ Suppose $n > N$

$\left|\frac{1}{n}\right| = \frac{1}{n}$ since $\frac{1}{n} > 0$

$\overline{N}$ since $n > N \Rightarrow \frac{1}{n} < \frac{1}{N}$

$= \frac{1}{1/\varepsilon} = \varepsilon$ $\therefore \left|\frac{1}{n}\right| < \varepsilon$

$\lim_{n\to\infty} \frac{1}{n} = 0$ ✓

$\therefore \sum \frac{(-1)^n}{n}$ conv. by AST

Consider $\sum_{n=1}^{\infty} \left|\frac{(-1)^n}{n}\right| = \sum_{n=1}^{\infty} \frac{1}{n}$

By p-series test

Since $p = 1$, $p \in (0, 1]$

$\therefore \sum_{n=1}^{\infty} \frac{1}{n}$ diverges

$\therefore \sum \frac{(-1)^n}{n}$ conv.

but $\sum \left|\frac{(-1)^n}{n}\right|$ div.

$\therefore \sum \frac{(-1)^n}{n}$ CC

$$\sum_{n=1}^{\infty} \frac{e^n+1}{ne^n+1} \qquad \frac{e^n+1}{ne^n+1} \approx \frac{\cancel{e^n}}{n\cancel{e^n}} = \frac{1}{n}$$

$\sum \frac{1}{n}$ div.

Comparison

- smaller
- div

① $\frac{e^n+1}{ne^n+1} > 0$ ✓

$e^n+1 > 0 \ \forall n$

$ne^n+1 > 0 \ \forall n$

② $$\frac{e^n+1}{ne^n+1} > \frac{e^n+1}{ne^n+n}$$ max. denom.

$\frac{3}{5}$ → $\frac{2}{5}$ min. num.

→ $\frac{3}{10}$ max. denom.

$$= \frac{e^n+1}{n(e^n+1)}$$

$$= \frac{1}{n} \qquad \sum_{n=1}^{\infty} \frac{1}{n} \text{ div.}$$

By CT

$$\sum_{n=1}^{\infty} \frac{e^n+1}{ne^n+1} \text{ div.}$$

by p-series test since $p=1$

$p \in (0,1]$

$$\sum_{n=1}^{\infty} \frac{\cos^2(n)+1}{e^n}$$

$$\frac{\cos^2(n)+1}{e^n} \approx \frac{1}{e^n}$$

Comparison $\sum \frac{1}{e^n}$ conv.

- bigger
- conv.

$$\frac{\cos^2(n)+1}{e^n} > 0$$

$\cos^2(n)+1 > 0$, $e^n > 0$

$$\frac{\cos^2(n)+1}{e^n} \leq \frac{1+1}{e^n} = \frac{2}{e^n}$$

$\cos^2(n) \leq 1$

$\sum_{n=1}^{\infty} \frac{2}{e^n} \leftarrow$ conv by IT

Let $f(x) = \frac{2}{e^x} = 2e^{-x}$

$f'(x) = -2e^{-x} < 0$ since $e^x > 0$

$\therefore f$ is decreasing

$f(x) = \frac{2}{e^x} > 0 \ \forall x \in [1, \infty)$

f is continuous since e^{-x} is exp func.

$$\int_1^\infty \frac{2}{e^x}\,dx = \lim_{A\to\infty}\int_1^A 2e^{-x}\,dx$$

$$= \lim_{A\to\infty}\left[-2e^{-x}\right]_1^A$$

$$= \lim_{A\to\infty} -\frac{2}{e^A} + \frac{2}{e}$$

$e^A \to \infty$

$\to 0$

$$= \frac{2}{e} \qquad \therefore \int_1^\infty \frac{2}{e^x}\,dx \text{ conv.} \quad ✓$$

$\therefore \sum \frac{2}{e^n}$ conv. by IT

$$\sum_{n=1}^\infty \frac{2}{e^n} \text{ conv.}$$

$$= \sum_{n=1}^\infty 2\left(\frac{1}{e}\right)^n \qquad r = \frac{1}{e} \quad |r| < 1$$

$$\therefore \sum_{n=1}^\infty \frac{\cos^2(n)+1}{e^n} \text{ conv. by CT}$$

$$\sum_{n=1}^{\infty} \frac{2^{n+1}-n}{3^n}$$

$$\frac{2^{n+1}-n}{3^n} \approx \frac{2^{n+1}}{3^n} = \left(\frac{2}{3}\right)^n 2$$

Comparison test

$$\sum_{n=2}^{\infty} \frac{1}{n(\ln n)^2}$$

$$\int_2^{\infty} \frac{1}{x(\ln x)^2}\,dx$$

$u = \ln x$

$du = \frac{1}{x}\,dx$

- positive
- decreasing

$f' < 0$ $[2, \infty)$

2:04

2:14

$$\sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{1+k^2} \qquad a_k = \frac{1}{1+k^2}$$

AST

- positive
- decreasing $\underline{a_n > a_{n+1}}$
- $\lim_{a_k \to \infty} a_k = 0 \qquad n < n+1$

$(a)\ \sum_{n=0}^{\infty}(a-3)^n$

$$(b) \quad \sum_{n=0}^{\infty} 3^n a^n$$

$$(c) \quad \sum_{n=0}^{\infty} n^{-a^2/2}$$

$$(d) \quad \sum_{n=1}^{\infty} a^{\ln(n)}$$

Example 4. Prove by definition of convergent series that if $\sum a_n$ and $\sum b_n$ converge and $\sum a_n = s$ and $\sum b_n = t$ then $\sum(a_n + b_n) = s + t$

Example 5. Consider the following sequence, defined recursively:

$$a_1 = 1$$
$$a_{n+1} = 3 - \frac{1}{a_n}$$

Prove that the function is bounded above and increasing, then conclude that it converges using BMCT. Then find the limit of $\{a_n\}$.

Example 6. Consider the following sequence:

$$a_n = \frac{n!}{n^n}$$

Show that $\{a_n\}$ converges using BMCT.

Example 7. A10 Exercise #7
Prove or disprove the following statements:

(a) If $\sum a_n$ converges and $c \in \mathbb{R}$ is any constant then $\sum ca_n$ converges.

(f) If $\sum a_n$ converges and $\sum b_n$ diverges then $\sum(a_n b_n)$ diverges.

Example 8. A10 Exercise #8
Suppose $\sum a_n$ is convergent and $\sum b_n$ is divergent. Prove that $\sum a_n + b_n$ is divergent. (Hint: Do a proof by contradiction.)

Example 9. True or false: If $\sum a_n$ and $\sum b_n$ are divergent series, then $\sum a_n + b_n$ is divergent.

Example 10. Use a geometric series to prove that $0.9999... = 1$

$$\sum_{n=0}^{\infty} (a-3)^n$$

GS with $r = a-3$

conv if $|r| < 1$

$$|a-3| < 1$$

$$-1 < a-3 < 1$$

$$2 < a < 4$$

series conv. if

$$2 < a < 4$$

$$\sum_{n=0}^{\infty} n^{-\frac{a^2}{2}}$$

$$= \sum_{n=0}^{\infty} \frac{1}{n^{a^2/2}}$$

This is a p-series with $p = \frac{a^2}{2}$

conv. if $p > 1 \iff \frac{a^2}{2} > 1$

$$a^2 > 2$$

$a < -\sqrt{2}$

$a > \sqrt{2}$

$$\sum_{n=0}^{\infty} 3^n a^n = \sum_{n=0}^{\infty} (3a)^n$$

GS with $r = 3a$

conv. if $|r| < 1$

$$|3a| < 1$$

$$-1 < 3a < 1$$

by def. of $|\,|$

$$-\frac{1}{3} < a < \frac{1}{3}$$

the series conv. if

$$0 < a < 1/3$$

$$\sum_{n=1}^{\infty} a^{\ln(n)}$$

$$= \sum_{n=1}^{\infty} n^{\ln(a)} = n^{-(-\ln(a))}$$

$$= \sum_{n=1}^{\infty} \frac{1}{n^{-\ln(a)}}$$

This is a p-series with $p = -\ln(a)$

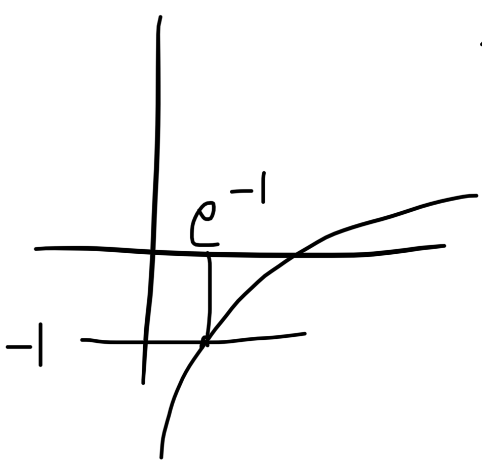
Conv. if $p > 1$

$$-\ln(a) > 1$$

$$\ln_e(a) < -1$$

$$a < e^{-1}$$

$$a < \frac{1}{e}$$



$$a^{\ln b} = b^{\ln a}$$

$\forall x \quad x = e^{\ln x}$

$$a^{\ln b} = e^{\ln(a^{\ln b})}$$

$\ln(x^y) = y \ln x$

$$= e^{\ln b \ln a}$$

$$= b^{\ln a}$$

$\log_b a = x$

$b^x = a$

4 Answers to selected examples

Example 1. Determine whether the following series converge or diverge. If the series is convergent find its sum.

$$(a) \quad \sum_{n=1}^{\infty} \frac{e^n+1}{ne^n+1}$$
$$(b) \quad \sum_{n=1}^{\infty} \frac{cos(n)+1}{e^n}$$
$$(c) \quad \sum_{n=1}^{\infty} \frac{2^{n+1}-n}{3^n}$$
$$(d) \quad \sum_{n=1}^{\infty} \frac{4^{n+1}}{3^n+n}$$
$$(e) \quad \sum_{n=2}^{\infty} \frac{1}{n(\ln(n))^2}$$
$$(f) \quad \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{1+k^2}$$

Hints:

(a) use comparison test. note that $\frac{e^n+1}{ne^n+1} \approx \frac{e^n}{ne^n} = \frac{1}{n}$, so you can guess the series will diverge.

(b) Use comparison test to compare with $\sum 1/e^n$ and integral test (to show $\sum 1/e^n$ converges).

(c) Use comparison test. note the series is $\approx \sum \frac{2^{n+1}}{3^n}$—does this conv or div?

(d) Use comparison test. note the series is $\approx \sum \frac{4^{n+1}}{3^n+n}$—does this conv or div?

(d) Use integral test.

(e) Use alternating series test.