MATA37 Week 10

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1 Trigonometric substitution

Some integrals involving certain expressions become easier when you perform substitution with a trigonometric function, which are based on certain trigonometric identities. The following table summarizes certain expressions that can signal where to use a trig substitution. Note that a^2 represents a real number that appears in the integral, while x is the integration variable. The functions also need to be restricted to certain domains so that they are one-to-one. **Don't forget to include the restriction on θ.** Also, remember the fact that $\sqrt{a^2} = |a|$, NOT just $\sqrt{a^2} = a$. If you get something like $\sqrt{\cos^2(\theta)} = |\cos(\theta)|$, you can use the restriction on theta to simplify the answer (see sample answer). Also note that when $\theta \in [0, \frac{\pi}{2})$, $\sec\theta > 0$, and when $\theta \in (\frac{\pi}{2}, \pi]$, $\sec\theta < 0$.

Expression	Substitution	Restriction	Identity
$a^2 - x^2$	$x = a\sin\theta$	$\theta \in [-\frac{\pi}{2}, \frac{\pi}{2}]$	$1 - \sin^2\theta = \cos^2\theta$
$a^2 + x^2$	$x = a\tan\theta$	$\theta \in (-\frac{\pi}{2}, \frac{\pi}{2})$	$1 + \tan^2\theta = \sec^2\theta$
$x^2 - a^2$	$x = a\sec\theta$	$\theta \in [0, \frac{\pi}{2}) \cup (\frac{\pi}{2}, \pi]$	$\sec^2\theta - 1 = \tan^2\theta$

1.1 Trigonometric integrals

After appplying trigonometric substitution, you will often end up with a trigonometric integral. A trigonometric integral is one that involves integrating some product of trigonometric functions. The trigonometric integrals we'll encounter will come in the following forms.

Let $m, n, l, k \in \mathbb{Z}^{\geq 0}$. (Note that $\mathbb{Z}^{\geq 0}$ means some of these integers can be 0.) Let $a, b \in \mathbb{R}$, $a < b$.

Definition 1. *A* ***trigonometric integral*** *is an integral that is one of the following forms:*

$$\int \sin^l(x)\cos^k(x)\,dx \quad or \quad \int_a^b \sin^l(x)\cos^k(x)\,dx$$

$$\int \sec^m(x)\tan^n(x)\,dx \quad or \quad \int_a^b \sec^m(x)\tan^n(x)\,dx$$

When evaluating trig integrals, it helps if you have the following identities memorized (A represents a real number) :

$$\boxed{\sin^2 A + \cos^2 A = 1} \tag{1}$$

$$\tan^2 A + 1 = \sec^2 A \tag{2}$$

$$\boxed{\sin^2 A = \frac{1-\cos(2A)}{2}} \tag{3}$$

$$\cos^2 A = \frac{1+\cos(2A)}{2} \tag{4}$$

$$\boxed{\sin(2A) = 2\sin A\cos A} \tag{5}$$

If you want to, you only really need to memorize the boxed identities.
If you take (1) and divide both sides by $\cos^2 A$, you get (2). (Remember that $\tan A = \frac{\sin A}{\cos A}$ and $\sec A = \frac{1}{\cos A}$.)
If you remember (3), then you can remember the "cos" version in (4) by changing the sign from "-" to "+".

You should also remember how the reciprocal trig functions csc, sec, and cot are defined:

$$\sec(x) = \frac{1}{\cos(x)}$$
$$\csc(x) = \frac{1}{\sin(x)}$$
$$\cot(x) = \frac{1}{\tan(x)}\left(= \frac{\cos(x)}{\sin(x)}\right)$$

As well, recall these derivatives of trigonometric functions:

$$\frac{\mathrm{d}}{\mathrm{d}x}\sin(x) = \cos(x)$$
$$\frac{\mathrm{d}}{\mathrm{d}x}\cos(x) = -\sin(x)$$
$$\frac{\mathrm{d}}{\mathrm{d}x}\tan(x) = \sec^2(x)$$
$$\frac{\mathrm{d}}{\mathrm{d}x}\sec(x) = \tan(x)\sec(x)$$

1.2 sin/cos trig integrals

The different approaches to solving a trig integral with sin and/or cos can be split into two cases:

Example 1. Even/odd exponents and odd/odd exponents
Evaluate $\int \cos^2 x \, \sin^5 x \, dx$.

Sol.

$$\begin{aligned}
\int \cos^2 x \, \sin^5 x \, dx &= \int \cos^2 x \sin^4 x \, \sin x \, dx \\
&= \int \cos^2 x \, (\sin^2 x)^2 \sin x \, dx \\
&= \int \cos^2 x \, (1-\cos^2 x)^2 \sin x \, dx \qquad \text{Let } u = \cos x. \implies du = -\sin x \, dx \\
&= \int u^2(1-u^2)^2 \, (-du) \\
&= -\int u^2(u^4 - 2u^2 + 1) \; du
\end{aligned}$$

$$\begin{aligned}
&= -\int u^6 - 2u^4 + u^2 \, du \\
&= -\left(\frac{u^2}{7} - \frac{2u^5}{5} + \frac{u^3}{3}\right) + C \\
&= -\frac{1}{7}\cos^7 x + \frac{2}{5}\cos^5 x - \frac{1}{3}\cos^3 x + C \qquad \square
\end{aligned}$$

The term with the odd exponent, $\sin^5 x$, was split up into $\sin^4 x$ and $\sin x$. Then $\sin^4 x$ was written in terms of $\sin^2 x$ so that we can apply the identity. It becomes easy to apply substitution after that.
A similar procedure can be applied **when the exponents are both odd**. You can split up one of the terms as shown above. Splitting up either term should both give the same answer. Try it out on, for example, $\int_0^\pi \cos^5 x \sin^3 x \, dx$.

Example 2. Even/even exponents
Evaluate $\int \cos^2(2x)\sin^4(2x)\,dx$.

Sol.

$$\begin{aligned}
\int \cos^2(2x)\sin^4(2x)\,dx &= \int \cos^2(2x)(\sin^2(2x))^2\,dx \\
&= \int \left(\frac{1-\cos(4x)}{2}\right)\left(\frac{1+\cos(4x)}{2}\right)^2 dx \\
&= \frac{1}{8}\int (1-\cos(4x))(1+\cos(4x))(1+\cos(4x))\,dx \\
&= \frac{1}{8}\int (1-\cos^2(4x))(1+\cos(4x))\,dx \\
&= \frac{1}{8}\int 1 - \cos^2(4x) + \cos(4x) - \cos^3(4x)\,dx \\
&= \frac{1}{8}\left(\int 1\,dx - \int \cos^2(4x)\,dx + \int \cos(4x)\,dx - \int \cos^3(4x)\,dx\right) \\
&= \ldots
\end{aligned}$$

1.3 sec/tan trig integrals

Recall that the derivative of $\tan x$ is $\sec^2 x$ and the derivative of $\sec x$ is $\sec x \tan x$.
You can do a u-substitution on a sec/tan trig integral if you can rearrange the terms to give something of the following forms:

$$\begin{aligned}
&\int \left(\text{just tangent stuff}\right)\sec^2 x\,dx \qquad u = \tan x \implies du = \sec^2 x\,dx \\
&\int \left(\text{just secant stuff}\right)\sec x \tan x\,dx \qquad u = \sec x \implies du = \sec x \tan x\,dx
\end{aligned}$$

Example 3. Evaluate the following integrals using trigonometric substitution:

$$\begin{aligned}
&(1) \quad \int_0^3 \frac{1}{4x^2+9}\,dx \\
&(2) \quad \int \sqrt{x-1}\sqrt{3-x}\,dx \\
&(3) \quad \int \ln\big(x^2+1\big)\,dx \\
&(4) \quad \int \frac{\sqrt{x^2-2}}{x}\,dx, \quad x > \sqrt{2} \\
&(5) \quad \int \frac{1}{\sqrt{x^2-5}}\,dx, \quad x < -\sqrt{5} \\
&(6) \quad \int \sqrt{x^2-8x+25}\,dx \\
&(7) \quad \int \sqrt{x^2+6x+18}\,dx \\
&(8) \quad \int e^{2x}(1-e^{4x})^{3/2}\,dx
\end{aligned}$$

Hints:
(2) Use the fact that $\sqrt{a}\sqrt{b} = \sqrt{ab}$. Then complete the square.
(3) Use integration by parts first.
(8) Use substitution first.

2 Improper integrals

In the past when calculating $\int_a^b f(x)\,dx$, we have assumed that:
(1) the interval $[a, b]$ is bounded ($a, b \neq \infty$)
(2) $f(x)$ has no vertical asymptotes (VAs) in $[a, b]$ (range($f(x)$) is bounded)

If the condition(s) (1) and/or (2) are broken, $\int_a^b f(x)\,dx$ is an **improper integral**.

Examples of **type I improper integrals** (integrals that break condition (1)):

$$\int_3^\infty \frac{\arctan(x)}{1+x^2}\,dx$$

$$\int_{-\infty}^1 \arctan(x)\,dx$$

Examples of **type II improper integrals** (integrals that break condition (2)):

$$\int_2^5 \frac{8}{\sqrt{x-2}}\,dx$$

$$\int_{\pi/2}^\pi \csc(x)\,dx$$

$$\int_{-1}^1 x^{-2}\,dx$$

2.1 Defining Type I and Type II improper integrals

Type I integrals have ∞ as one of the integration limits.

$$\int_a^\infty f(x)\,dx = \lim_{N\to\infty}\int_a^N f(x)\,dx$$

$$\int_{-\infty}^b f(x)\,dx = \lim_{M\to\infty}\int_M^b f(x)\,dx$$

$$\begin{aligned}\int_{-\infty}^\infty f(x)\,dx &= \int_{-\infty}^c f(x)\,dx + \int_c^\infty f(x)\,dx\,, \text{ where } c\in\mathbb{R}\\ &= \lim_{M\to\infty}\int_M^c f(x)\,dx + \lim_{N\to\infty}\int_c^N f(x)\,dx\end{aligned}$$

The improper integral **converges** if the limit exists. It **diverges** if the limit doesn't exist.
In the third case, both limits must exist for the integral to converge.

In Type II integrals, the integrand has a discontinuity at the endpoints or within the interval of integration. Given $\int_a^b f(x)\,dx$:
If $f(b)$ is undefined:

$$\int_a^b f(x)\,dx = \lim_{A\to b^-}\int_a^A f(x)\,dx$$

If $f(a)$ is undefined:

$$\int_a^b f(x)\,dx = \lim_{A\to a^+}\int_A^b f(x)\,dx$$

If $f(c)$ is undefined for some $c\in(a,b)$:

$$\begin{aligned}\int_a^b f(x)\,dx &= \int_a^c f(x)\,dx + \int_c^b f(x)\,dx\\ &= \lim_{A\to c^-}\int_a^A f(x) + \lim_{A\to c^+}\int_A^b f(x)\end{aligned}$$

In the third case, we need **both** limits to converge for the original integral to converge. If one of the parts diverges, **and we do not get an indeterminate form**, then the original integral diverges.

Example 1. Evaluate the following improper integrals.

(1) $\displaystyle\int_0^1 \frac{\ln(x)}{x}\,dx$

(2) $\displaystyle\int_1^\infty \frac{\ln(x)}{x}\,dx$

(3) $\displaystyle\int_1^\infty \frac{\ln(x)}{x^2}\,dx$

(4) $\displaystyle\int_0^\infty xe^{-x}\,dx$

(5) $\displaystyle\int_{-\infty}^{\infty} xe^{-x}\,dx$

(6) $\displaystyle\int_{-\infty}^{\infty} x^2e^{-x^3}\,dx$

(7) $\displaystyle\int_{0}^{4} \frac{1}{x^2+x-6}\,dx$

(8) $\displaystyle\int_{2}^{5} \frac{1}{\sqrt{x-2}}$

3 Sample answers

Trig substitution: (4)

$$\int \frac{\sqrt{x^2-2}}{x}\,dx, \quad x > \sqrt{2}$$

Note the expression x^2-2, which is of the form x^2-a^2 with $a=\sqrt{2}$. Use $x=\sqrt{2}\sec\theta$.

$$\begin{aligned} x &= \sqrt{2}\sec\theta \qquad \text{since } x > \sqrt{2}.\ \theta \in [0, \pi/2). \\ dx &= \sqrt{2}\sec\theta\tan\theta\,d\theta \end{aligned}$$

Rewriting the expression $\sqrt{x^2-2}$:

$$\begin{aligned} \sqrt{x^2-2} &= \sqrt{(\sqrt{2}\sec\theta)^2-2} \\ &= \sqrt{2\sec^2\theta-2} \\ &= \sqrt{2(\sec^2\theta-1)} \\ &= \sqrt{2}\sqrt{sec^2\theta-1} \\ &= \sqrt{2}\sqrt{\tan^2\theta} \\ &= \sqrt{2}|\tan\theta| \\ &= \sqrt{2}\tan\theta \qquad (\text{Since } \theta\in[0,\pi/2),\ \tan\theta>0) \end{aligned}$$

Rewriting the integral,

$$\begin{aligned} \int \frac{\sqrt{x^2-2}}{x}\,dx &= \int \frac{\sqrt{2}\tan\theta}{\sqrt{2}\sec\theta}\sqrt{2}\sec\theta\tan\theta\,d\theta \\ &= \int \sqrt{2}\tan^{\theta}, d\theta \\ &= \sqrt{2}\int \sec^2\theta - 1\,d\theta \\ &= \sqrt{2}(\tan\theta-\theta)+C \end{aligned}$$

You can draw a right triangle with angle θ and $\sec\theta = x/\sqrt{2}$ to solve for $\tan\theta$. I'm not good enough with LaTeX to include the diagram but it results in $\tan\theta = \sqrt{x^2-2}$. Also note that $x=\sqrt{2}\sec\theta$ implies $\theta = \sec^{-1}(\frac{x}{\sqrt{2}})$. (The -1 denotes the inverse secant function, NOT a negative exponent.)

$$\begin{aligned} &= \sqrt{2}(\tan\theta-\theta)+C \\ &= \sqrt{2}\big(\sqrt{x^2-2} - \sec^{-1}(\frac{x}{\sqrt{2}})\big)+C \end{aligned}$$

Improper integrals by definition: (1)

$$\int_0^1 \frac{\ln(x)}{x}\,dx$$

The integral is over the interval $[0,1]$, and the integrand has a discontinuity/is not defined at $x=0$, which is in the interval. This is a Type II improper integral.

$$
\begin{aligned}
\int_0^1 \frac{\ln(x)}{x}\,dx &= \lim_{A\to 0^+}\int_A^1 \frac{\ln(x)}{x}\,dx \\
&= \lim_{A\to 0^+}\int_{\ln A}^0 u\,du \qquad \text{Use substitution } u=\ln(x) \implies du=\frac{1}{x}\,dx \text{ and change int. limits} \\
&= \lim_{A\to 0^+}\left[\frac{1}{2}u^2\right]_{\ln A}^0 \\
&= \lim_{A\to 0^+} -\frac{1}{2}(\ln A)^2
\end{aligned}
$$

Note that as $A\to 0^+$, $\ln A\to -\infty$ (picture the graph of $\ln(x)$.) Hence, $\lim_{A\to 0^+} -\frac{1}{2}(\ln A)^2 = -\infty$, so the integral diverges.