MATA37 Week 12

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1 Sequences

We generally think of sequences as an ordered list of numbers, e.g. {1,2,3,4...} or {1, 1, 2, 3, 5, 8, 13, ...} or {3, 1, 4, 1, 5, 6, ...}. The precise definition of a sequence is as follows:

Definition 1. *A* ***sequence*** *of real numbers is a function whose* <u>*domain*</u> *is the natural numbers,*
$\mathbb{N} = \{1, 2, 3, ...\}$.
Can be denoted by

- $a_n = f(n)$
- $\{a_n\}_{n=1}^{\infty}$
- $\{a_n\}$

a_n is the **general term** of the sequence—it's a <u>function</u> that takes a natural number n as an input and outputs a real number.
The definition of convergence of a sequence is pretty much the same as the definition of convergence of a function at infinity, as seen in MATA31.

Definition 2. *Let $l \in \mathbb{R}$. The sequence $\{a_n\}$* ***converges*** *to l iff*

$$\forall \epsilon > 0, \exists N > 0 \ \textit{s.t.} \ \forall n \in \mathbb{N}, \ \textit{if } n > N, \ \textit{then } |a_n - l| < \epsilon$$

Denoted by

- $a_n \to l$ *as* $n \to \infty$
- $a_n \to l$
- $\lim_{n\to\infty} a_n = l$

If $\{a_n\}$ does not converge, we say it ***diverges****. i.e. there does not exist any such l.*

Proving $\{a_n\}$ converges to l:

- Take arbitrary $\epsilon > 0$
- Want to find some $N \in \mathbb{R}$ where, if $n \in \mathbb{N}$ and $n > N$, then $|a_n - l| < \epsilon$. Usually N is in terms of ϵ.

- To find this N, write $|a_n - l|$ and try simplifying or finding upper bounds for this function. You usually get something like $|a_n - l| \leq \frac{1}{n^p}$ for some $p \in \mathbb{R}$.
- If you get to something like $\frac{1}{n^p}$, you can use the hypothesis $n > N$ to say $\frac{1}{n^p} < \frac{1}{N^p}$, then substitute in a choice of N to conclude $|a_n - l| \leq \frac{1}{n^p} < \epsilon$.
- To complete the proof, begin by taking arbitrary $\epsilon > 0$. Define N as you determined the previous step. Then suppose n is some natural number $> N$.
- Simplify the expression $|a_n - l|$ and show that under the above assumptions, $|a_n - l| < \epsilon$.

Example 1. Prove that $\left\{\dfrac{\sin(n)}{\sqrt[3]{n}}\right\}_{n=1}^{\infty}$ converges to 0.

Sol. Want to prove the following:

$$\text{For all } \epsilon > 0, \text{ there exists } N > 0 \text{ s.t. for all } n \in \mathbb{N}, \text{ if } n > N, \text{ then } \left|\frac{\sin(n)}{\sqrt[3]{n}}\right| < \epsilon$$

Let $\epsilon > 0$ be arbitrary.
Choose $N = \qquad\qquad > 0$.
Assume $n \in \mathbb{N}$, $n > N$.

$$\begin{aligned}
\left|\frac{\sin(n)}{\sqrt[3]{n}}\right| &= \frac{|\sin(n)|}{|\sqrt[3]{n}|} && \text{abs. value properties} \\
&\leq \frac{1}{|\sqrt[3]{n}|} && \text{since } |\sin(n)| \leq 1 \text{ for all } n \\
&= \frac{1}{\sqrt[3]{n}} && \text{since } \sqrt[3]{n} \text{ is positive for } n > 1 \\
&< \frac{1}{\sqrt[3]{N}}
\end{aligned}$$

Since we assumed $n > N$, it follows $\sqrt[3]{n} > \sqrt[3]{N}$ and hence $1/\sqrt[3]{n} < 1/\sqrt[3]{N}$.
Now, if we choose $N = \dfrac{1}{\epsilon^3}$, then $\dfrac{1}{\sqrt[3]{N}} = \epsilon$.Then we can complete the proof with the choice of N:
Let $\epsilon > 0$ be arbitrary.
Choose $N = \dfrac{1}{\epsilon^3} > 0$.
Assume $n \in \mathbb{N}$, $n > N$.

$$\begin{aligned}
\left|\frac{\sin(n)}{\sqrt[3]{n}}\right| &= \frac{|\sin(n)|}{|\sqrt[3]{n}|} && \text{abs. value properties} \\
&\leq \frac{1}{|\sqrt[3]{n}|} && \text{since } |\sin(n)| \leq 1 \text{ for all } n \\
&= \frac{1}{\sqrt[3]{n}} && \text{since } \sqrt[3]{n} \text{ is positive for } n > 1 \\
&< \frac{1}{\sqrt[3]{N}} && \text{by } n > N, \text{ it follows } \sqrt[3]{n} > \sqrt[3]{N} \text{ and hence } 1/\sqrt[3]{n} < 1/\sqrt[3]{N}. \\
&= \frac{1}{\sqrt[3]{1/\epsilon^3}} && \text{by choice of } N \\
&= \frac{1}{1/\epsilon}
\end{aligned}$$

$$= \epsilon$$

Hence, if $n > N$, then $\left|\frac{\sin(n)}{\sqrt[3]{n}}\right| < \epsilon$, as wanted. $\square$

1.1 Properties of convergent sequences

Let $\{a_n\}$ and $\{b_n\}$ be sequences. Let $a, b, k \in \mathbb{R}$.

Theorem 1. *Suppose $a_n \to a$ and $b_n \to b$. Then*

$$(a) \quad \lim_{n\to\infty}(a_n + b_n) = \lim_{n\to\infty} a_n + \lim_{n\to\infty} b_n = a + b \tag{1}$$

$$(b) \quad \lim_{n\to\infty} k(a_n) = ka \tag{2}$$

$$(c) \quad \lim_{n\to\infty}(a_n)(b_n) = \big(\lim_{n\to\infty} a_n\big)\big(\lim_{n\to\infty} b_n\big) = ab \tag{3}$$

$$(d) \quad \lim_{n\to\infty}\big(\frac{a_n}{b_n}\big) = \frac{a}{b} \tag{4}$$

$$\textit{(Provided that } b \neq 0 \textit{ and } b_n \neq 0\ \forall n \in \mathbb{N}\textit{)} \tag{5}$$

Theorem 2. ***Uniqueness of limits***
IF $\{a_n\}$ converges
THEN its limit is unique.

Definition 3. *A sequence is **bounded** if*

$$\exists c \in \mathbb{R}^+ \ s.t.\ |a_n| \leq c \quad \forall n \in \mathbb{N} \tag{6}$$

Note that $|a_n| \leq c$ equivalently means $-c \leq a_n \leq c$.

Theorem 3. ***Convergent sequences are bounded***
IF $\{a_n\}$ converges
THEN $\{a_n\}$ is bounded.

2 Bounded Monotone Convergence Theorem

Definition 4. *A sequence $\{a_n\}$ is **monotone** if it is either decreasing or increasing $\forall n \in \mathbb{N}$.*
$\{a_n\}$ is decreasing when $\forall n \in \mathbb{N}, a_n \geq a_{n+1}$
$\{a_n\}$ is increasing when $\forall n \in \mathbb{N}, a_n \leq a_{n+1}$

Theorem 4. ***Bounded Monotone Convergence Theorem***
IF $\{a_n\}$ is <u>bounded</u> and <u>monotone</u>
THEN $\{a_n\}$ converges.
In particular, there are two cases:

(1) If $\{a_n\}$ is <u>increasing</u> and bounded <u>above</u>, then $\{a_n\}$ converges.

(2) If $\{a_n\}$ is <u>decreasing</u> and bounded <u>below</u>, then $\{a_n\}$ converges.

Definition 5. *A* ***Cauchy sequence*** *is a sequence* $\{a_n\}$ *where*

$$\forall \epsilon > 0, \exists N \in \mathbb{N} \text{ s.t. if } n, m > N, \text{ then } |a_n - a_m| < \epsilon$$

Theorem 5. ***Cauchy sequences and convergence***
Every Cauchy sequence converges, and every convergent sequence is a Cauchy sequence.
i.e., A sequence is Cauchy $\iff$ *it is convergent.*

Example 1. Determine whether the following sequences converge or diverge and provide a complete $\epsilon - N$ proof.

$$(1) \quad \{\frac{n \arctan(n) - 1}{n^2 + 6\sqrt{n}}\}$$
$$(2) \quad \{\frac{2n-1}{n+3}\}$$
$$(3) \quad \{n^{-3/2}\}$$
$$(4) \quad \{2 - 2(-1)^n\}$$
$$(5) \quad \{n^{3/2}\}$$
$$(6) \quad \{\frac{n+1}{3n-2}\}$$

(1) Hint: use the fact $\arctan(x) < \pi/2$ for all x.

(5) Hint: Prove that the sequence diverges to infinity. By definition, this means:

$$\forall M > 0,\, \exists N > 0 \text{ s.t. if } n \in N,\, n > N, \text{ then } a_n > M.$$

(6) You will need to use a "helper assumption"; see the notes in the module on Quercus.

Example 2. Prove by $\epsilon - N$ definition that if $\{a_n\}$ converges to 0 and $\{b_n\}$ is bounded, then $\{a_n b_n\}$ converges.

Example 3. Prove that if $\{a_n\}$ converges, $\{a_n\}$ is a Cauchy sequence.

Example 4. Suppose $\{a_n\}$ converges to a and $\{b_n\}$ diverges. Prove that $\{a_n + b_n\}$ diverges. (Hint: use proof by contradiction.)

Example 5. True or false?

a) Every convergent sequence is bounded.

b) Every bounded sequence is convergent.

c) Every Cauchy sequence is bounded.

d) If $\{a_n\}$ converges and $\{b_n\}$ converges, then $\{a_n b_n\}$ converges.

e) If $\{a_n\}$ diverges and $\{b_n\}$ diverges, then $\{a_n + b_n\}$ diverges.

f) If $\{a_n\}$ diverges and $\{b_n\}$ diverges, then $\{a_n b_n\}$ diverges.

Example 6. Consider the following sequence, defined recursively:

$$a_1 = 1$$
$$a_{n+1} = 3 - \frac{1}{a_n}$$

Prove that the function is bounded above and increasing, then conclude that it converges using BMCT. Then find the limit of $\{a_n\}$.

3 Answers to selected examples

Example 1. Prove $\{2(-1)^n\}$ diverges.

Sol. Let $a_n = 2 - 2(-1)^n$. Note that the sequence oscillates back and forth between the values 0 and 4; $a_n = 4$ when n is odd and $a_n = 0$ when n is even. That means the sequence diverges; it never approaches a single value.
Proof. Suppose for contradiction that a_n converges to some limit $l \in \mathbb{R}$.
Then,

$$\forall \epsilon > 0, \exists N > 0 \text{ s.t. } \forall n \in \mathbb{N}, \text{ if } n > N, \text{ then } |a_n - l| < \epsilon$$

Therefore, the definition should work for $\epsilon = 1$. i.e.,

$$\exists N_0 > 0 \text{ s.t. } \forall n \in \mathbb{N}, \text{ if } n > N_0, \text{ then } |2 - 2(-1)^n - l| < 1$$

take $n > N_0$, $n \in \mathbb{N}$ where n is even. It follows from the definition that

$$\begin{aligned} & |2 - 2(-1)^n - l| < 1 \\ \iff & |0 - l| < 1 \\ \iff & -1 < -l < 1 \end{aligned}$$

Now take $n < N_0$, $n \in \mathbb{N}$ where n is odd. The definition still holds:

$$\begin{aligned} & |2 - 2(-1)^n - l| < 1 \\ \iff & |4 - l| < 1 \\ \iff & -1 < 4 - l < 1 \\ \iff & 3 < l < 5 \end{aligned}$$

Hence, we have that l satisfies $-1 < l < 1$ AND $3 < l < 5$, which is not possible. Contradiction. Hence a_n cannot converge, i.e. a_n diverges.

Example 2. Suppose $\{a_n\}$ converges to a and $\{b_n\}$ diverges. Prove that $\{a_n + b_n\}$ diverges. (Hint: use proof by contradiction.)

Sol. For contradiction, suppose $a_n \to a$, b_n diverges, and $a_n + b_n$ *converges.*
Since a_n converges to a, by sequence properties, it follows that $(-1)a_n = -a_n$ is also convergent. Since the sum of convergent sequence is convergent, and by assumption $a_n + b_n$ converges, it follows that $(a_n + b_n) + (-a_n) = b_n$ is a convergent sequence, but this contradicts the assumption that b_n diverges. Contradiction, therefore $a_n + b_n$ must be a divergent sequence.

Example 3. True or false?

a) Every convergent sequence is bounded.
TRUE.

b) Every bounded sequence is convergent.
FALSE. Try to come up with a bounded sequence that diverges.

c) Every Cauchy sequence is bounded.
TRUE. Cauchy implies convergent and convergent implies bounded.

d) If $\{a_n\}$ converges and $\{b_n\}$ converges, then $\{a_n b_n\}$ converges.
TRUE. This is a property of sequences.

e) If $\{a_n\}$ diverges and $\{b_n\}$ diverges, then $\{a_n + b_n\}$ diverges.
FALSE. Try to come up with a counterexample, i.e. two sequences a_n and b_n that each diverge but $a_n + b_n$ *converges.*

f) If $\{a_n\}$ diverges and $\{b_n\}$ diverges, then $\{a_n b_n\}$ diverges.
FALSE. Try to come up with a counterexample, i.e. two sequences a_n and b_n the diverge but $a_n b_n$ *converges.*