Counting

Formula Sheet

1. The Rules of Sum and Product:

The Rule of Sum − If a sequence of tasks T1,T2,….,Tm can be done in w1,w2,..…wm ways respectively (the condition is that no tasks can be performed simultaneously), then the number of ways to do one of these tasks is w1+w2+⋯+wm. If we consider two tasks A and B which are disjoint (i.e. $A \cap B = \emptyset$), then mathematically $|A \cup B| = |A| + |B|$

The Rule of Product − If a sequence of tasks T1,T2,….,Tm can be done in w1,w2,..…wm ways respectively and every task arrives after the occurrence of the previous task, then there are w1×w2×⋯×wm ways to perform the tasks. Mathematically, if a task B arrives after a task A, then $|A \times B| = |A| \times |B|$

2. Pigeonhole Principle:

For any positive integer k, if k + 1 objects (pigeons) are placed in k boxes (pigeonholes), then at least one box contains two or more objects.

3. Extended Pigeonhole Principle:

It states that if n pigeons are assigned to m pigeonholes (The number of pigeons is very large than the number of pigeonholes), then one of the pigeonholes must contain at least [(n-1)/m]+1 pigeon.

4. Permutation and Combination:

A **permutation** is an arrangement of some elements in which order matters. In other words, a Permutation is an ordered Combination of elements.

The number of permutations of 'n' different things taken 'r' at a time is denoted by n_{P_r}

$$n_{P_r} = \frac{n!}{(n-r)!}$$

where $n! = 1.2.3.\ldots(n-1).n$

A **combination** is selection of some given elements in which order does not matter.

The number of all combinations of n things, taken r at a time is −

$$^nC_r = \frac{n!}{r!(n-r)!}$$

5. Inclusion and Exclusion Principle:

The **Inclusion-exclusion principle** computes the cardinal number of the union of multiple non-disjoint sets. For two sets A and B, the principle states −

$$|A \cup B| = |A| + |B| - |A \cap B|$$

For three sets A, B and C, the principle states −

$$|A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |A \cap C| - |B \cap C| + |A \cap B \cap C|$$

The generalized formula -

$$|\bigcup_{i=1}^{n} A_i| = \sum_{1 \le i < j < k \le n} |A_i \cap A_j| + \sum_{1 \le i < j < k \le n} |A_i \cap A_j \cap A_k| - \cdots + (-1)^{\pi-1} |A_1 \cap \cdots \cap A_2|$$

6. Linear Recurrence:

A recurrence relation is a functional relation between the independent variable x, dependent variable f(x) and the differences of various order of f (x). A recurrence relation is also called a difference equation, and we will use these two terms interchangeably.

Linear Recurrence Relations with Constant Coefficients:

A Recurrence Relations is called linear if its degree is one.

The general form of linear recurrence relation with constant coefficient is

$$C_0\ y_{n+r} + C_1\ y_{n+r-1} + C_2\ y_{n+r-2} + \cdots + C_r\ y_n = R\ (n)$$

Where $C_0, C_1, C_2 \ldots\ldots C_n$ are constant and R (n) is same function of independent variable n.

A solution of a recurrence relation in any function which satisfies the given equation.

Linear Homogeneous Recurrence Relations with Constant Coefficients:

A linear homogeneous difference equation with constant coefficients is given by

$$C_0 y_n+C_1 y_{n-1}+C_2 y_{n-2}+\cdots\ldots\ldots+C_r y_{n-r}=0 \text{ .... equation (i)}$$

Where $\mathbf{C_0, C_1, C_2 \ldots\ldots C_n}$ are constants.

The solution of the equation (i) is of the form $A\alpha_1^K$, where α_1 is the characteristics root and A is constant.

Substitute the values of A α^K for y_n in equation (1), we have

$$C_0 A\alpha^K+C_1 A\alpha^{K-1}+C_2 A\alpha^{K-2}+\cdots\ldots+C_r A\alpha^{K-r}=0 \text{.......equation (ii)}$$

After simplifying equation (ii), we have

$$C_0 \alpha^r+C_1 \alpha^{r-1}+C_2 \alpha^{r-2}+\cdots C_r=0 \text{.........equation (iii)}$$

The equation (iii) is called the characteristics equation of the difference equation.

If α_1 is one of the roots of the characteristics equation, then $A\alpha_1^K$ is a homogeneous solution to the difference equation.

To find the solution of the linear homogeneous difference equations, we have the four cases that are discussed as follows:

Case1: If the characteristic equation has n distinct real rootsα_1, α_2, α_3,.......α_n.

Thus, $\alpha_1^K, \alpha_2^K, \ldots\ldots \alpha_n^K$ are all solutions of equation (i).

Also, we have $A_1 \alpha_1^K, A_2 \alpha_1^K, \ldots\ldots\ldots A_n \alpha_n^K$ are all solutions of equation (i). The sums of solutions are also solutions.

Hence, the homogeneous solutions of the difference equation are

$$y_k = A_1 \alpha_1^K, A_2 \alpha_2^K, \ldots\ldots\ldots A_n \alpha_n^K.$$

Case2: If the characteristics equation has repeated real roots.

If $\propto_1=\propto_2$, then $(A_1+A_2\ K)\ \propto_1^K$ is also a solution.

If $\propto_1=\propto_2=\propto_3$ then $(A_1+A_2\ K+A_3\ K^2)\ \propto_1^K$ is also a solution.

Similarly, if root $\propto_1$ is repeated n times, then.

$$(A_1+A_2\ K+A_3\ K^2+......+A_n\ K_{n-1})\ \propto_1^K$$

The solution to the homogeneous equation.

Case3: If the characteristics equation has one imaginary root.

If $\alpha+i\beta$ is the root of the characteristics equation, then $\alpha-i\beta$ is also the root, where α and β are real.

Thus, $(\alpha+i\beta)^K$ and $(\alpha-i\beta)^K$ are solutions of the equations. This implies

$$(\alpha+i\beta)^K\ A_1+\alpha-i\beta)^K\ A_2$$

Is also a solution to the characteristics equation, where A_1 and A_2 are constants which are to be determined.

Case4: If the characteristics equation has repeated imaginary roots.

When the characteristics equation has repeated imaginary roots,

$$(C_1+C_2\ k)\ (\alpha+i\beta)^K+(C_3+C_4\ K)(\alpha-i\beta)^K$$

Is the solution to the homogeneous equation.

Non-Homogeneous Recurrence Relation and Particular Solutions:

A recurrence relation is called non-homogeneous if it is in the form

$$F_n = AF_{n-1} + BF_{n-2} + f(n) \quad \text{where} \quad f(n) \neq 0$$

Its associated homogeneous recurrence relation is $F_n = AF_{n-1} + BF_{n-2}$

The solution (a_n) of a non-homogeneous recurrence relation has two parts.

First part is the solution (a_h) of the associated homogeneous recurrence relation and the second part is the particular solution (a_t) .

$$a_n = a_h + a_t$$

Solution to the first part is done using the procedures discussed in the previous section.

To find the particular solution, we find an appropriate trial solution.

Let $f(n) = cx^n$; let $x^2 = Ax + B$ be the characteristic equation of the associated homogeneous recurrence relation and let x_1 and x_2 be its roots.

- If $x \neq x_1$ and $x \neq x_2$, then $a_t = Ax^n$
- If $x = x_1$, $x \neq x_2$, then $a_t = Anx^n$
- If $x = x_1 = x_2$, then $a_t = An^2x^n$