Counting

Formula Sheet

1. The Rules of Sum and Product:

The Rule of Sum − If a sequence of tasks $T1, T2, \ldots, Tm$ can be done in $w1, w2, \ldots wm$ ways respectively (the condition is that no tasks can be performed simultaneously), then the number of ways to do one of these tasks is $w1+w2+\cdots+wm$. If we consider two tasks A and B which are disjoint (i.e. $A \cap B = \emptyset$), then mathematically $|A \cup B| = |A| + |B|$

The Rule of Product − If a sequence of tasks $T1, T2, \ldots, Tm$ can be done in $w1, w2, \ldots wm$ ways respectively and every task arrives after the occurrence of the previous task, then there are $w1 \times w2 \times \cdots \times wm$ ways to perform the tasks. Mathematically, if a task B arrives after a task A, then $|A \times B| = |A| \times |B|$

2. Pigeonhole Principle:

For any positive integer k, if k + 1 objects (pigeons) are placed in k boxes (pigeonholes), then at least one box contains two or more objects.

3. Extended Pigeonhole Principle:

It states that if n pigeons are assigned to m pigeonholes (The number of pigeons is very large than the number of pigeonholes), then one of the pigeonholes must contain at least [(n-1)/m]+1 pigeon.

4. Permutation and Combination:

A **permutation** is an arrangement of some elements in which order matters. In other words, a Permutation is an ordered Combination of elements.

The number of permutations of 'n' different things taken 'r' at a time is denoted by n_{P_r}

$$n_{P_r} = \frac{n!}{(n-r)!}$$

where $n! = 1.2.3.\ldots.(n-1).n$

A **combination** is selection of some given elements in which order does not matter.

The number of all combinations of n things, taken r at a time is −

$$^{n}C_{r} = \frac{n!}{r!(n-r)!}$$

5. Inclusion and Exclusion Principle:

The **Inclusion-exclusion principle** computes the cardinal number of the union of multiple non-disjoint sets. For two sets A and B, the principle states −

$$|A \cup B| = |A| + |B| - |A \cap B|$$

For three sets A, B and C, the principle states −

$$|A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |A \cap C| - |B \cap C| + |A \cap B \cap C|$$

6. Linear Recurrence:

A recurrence relation is a functional relation between the independent variable x, dependent variable f(x) and the differences of various order of f (x). A recurrence relation is also called a difference equation, and we will use these two terms interchangeably.

Linear Recurrence Relations with Constant Coefficients:

A Recurrence Relations is called linear if its degree is one.

The general form of linear recurrence relation with constant coefficient is

$$C_0\, y_{n+r} + C_1\, y_{n+r-1} + C_2\, y_{n+r-2} + \cdots + C_r\, y_n = R\ (n)$$

Where $C_0, C_1, C_2 \ldots\ldots C_n$ are constant and R (n) is same function of independent variable n.

A solution of a recurrence relation in any function which satisfies the given equation.

Linear Homogeneous Recurrence Relations with Constant Coefficients:

A linear homogeneous difference equation with constant coefficients is given by

$C_0\ y_n+C_1\ y_{n-1}+C_2\ y_{n-2}+\cdots\ldots\ldots+C_r\ y_{n-r}=0$ .... equation (i)

Where $\mathbf{C_0, C_1, C_2 \ldots\ldots C_n}$ are constants.

The solution of the equation (i) is of the form $A\alpha_1^K$, where α_1 is the characteristics root and A is constant.

Substitute the values of $A\ \alpha^K$ for y_n in equation (1), we have

$C_0\ A\alpha^K+C_1\ A\alpha^{K-1}+C_2\ A\alpha^{K-2}+\cdots\ldots.+C_r\ A\alpha^{K-r}=0$.......equation (ii)

After simplifying equation (ii), we have

$C_0\ \alpha^r+C_1\ \alpha^{r-1}+C_2\ \alpha^{r-2}+\cdots C_r=0$..........equation (iii)

The equation (iii) is called the characteristics equation of the difference equation.

If α_1 is one of the roots of the characteristics equation, then $A\alpha_1^K$ is a homogeneous solution to the difference equation.

To find the solution of the linear homogeneous difference equations, we have the four cases that are discussed as follows:

Case1: If the characteristic equation has n distinct real roots α_1, α_2, α_3,.......α_n.

Thus, $\alpha_1^K, \alpha_2^K, \ldots\ldots. \alpha_n^K$ are all solutions of equation (i).

Also, we have $A_1\ \alpha_1^K, A_2\ \alpha_1^K, \ldots\ldots\ldots A_n\ \alpha_n^K$ are all solutions of equation (i). The sums of solutions are also solutions.

Hence, the homogeneous solutions of the difference equation are

$$y_k= A_1\ \alpha_1^K, A_2\ \alpha_2^K, \ldots\ldots\ldots A_n\ \alpha_n^K.$$

Case2: If the characteristics equation has repeated real roots.

If $\propto_1=\propto_2$, then $(A_1+A_2\ K)\ \propto_1^K$ is also a solution.

If $\propto_1=\propto_2=\propto_3$ then $(A_1+A_2\ K+A_3\ K^2)\ \propto_1^K$ is also a solution.

Similarly, if root $\propto_1$ is repeated n times, then.

$$(A_1+A_2\ K+A_3\ K^2+......+A_n\ K_{n-1})\ \propto_1^K$$

The solution to the homogeneous equation.

Case3: If the characteristics equation has one imaginary root.

If $\alpha+i\beta$ is the root of the characteristics equation, then $\alpha-i\beta$ is also the root, where α and β are real.

Thus, $(\alpha+i\beta)^K$ and $(\alpha-i\beta)^K$ are solutions of the equations. This implies

$$(\alpha+i\beta)^K\ A_1+\alpha-i\beta)^K\ A_2$$

Is also a solution to the characteristics equation, where A_1 and A_2 are constants which are to be determined.

Case4: If the characteristics equation has repeated imaginary roots.

When the characteristics equation has repeated imaginary roots,

$$(C_1+C_2\ k)\ (\alpha+i\beta)^K+(C_3+C_4\ K)(\alpha-i\beta)^K$$

Is the solution to the homogeneous equation.

Non-Homogeneous Recurrence Relation and Particular Solutions:

A recurrence relation is called non-homogeneous if it is in the form

$$F_n = AF_{n-1} + BF_{n-2} + f(n) \quad \text{where} \quad f(n) \neq 0$$

Its associated homogeneous recurrence relation is $F_n = AF_{n-1} + BF_{n-2}$

The solution (a_n) of a non-homogeneous recurrence relation has two parts.

First part is the solution (a_h) of the associated homogeneous recurrence relation and the second part is the particular solution (a_t) .

$$a_n = a_h + a_t$$

Solution to the first part is done using the procedures discussed in the previous section.

To find the particular solution, we find an appropriate trial solution.

Let $f(n) = cx^n$; let $x^2 = Ax + B$ be the characteristic equation of the associated homogeneous recurrence relation and let x_1 and x_2 be its roots.

- If $x \neq x_1$ and $x \neq x_2$, then $a_t = Ax^n$
- If $x = x_1$, $x \neq x_2$, then $a_t = Anx^n$
- If $x = x_1 = x_2$, then $a_t = An^2x^n$

Reference Links:

1. The Rules of Sum and Product:

https://youtu.be/8JiWWvEoaoc

https://youtu.be/9HDdnbacDO4

2. Pigeonhole Principle:

https://youtu.be/B2A2pGrDG8I

https://youtu.be/4Dz4vNUxnZM

3. Extended Pigeonhole Principle:

https://youtu.be/cJH4T6iV3lE?t=103

https://youtu.be/3UeHl3UtmGI?t=287

4. Permutation and Combination:

https://youtu.be/0NAASclUm4k?t=3

https://youtu.be/J1m9sB5XZQc

5. Inclusion and Exclusion Principle:

https://youtu.be/51-b2mgZVNY

https://youtu.be/GS7dIWA6Hpo?t=10

6. Linear Recurrence:

https://youtu.be/DXjmwccnC-c

https://youtu.be/eAaP4XaB8hM

https://youtu.be/UCnl5DoUcWk

https://youtu.be/EfF_XSEX1Sk