Model Summary

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1 Definitions

k_t: number of new infections on day t.
λ_t: underlying daily rate of infection (and expected number of new infections) on day t.

2 Assumptions

Firstly, that k_t depends on λ_t via a negative binomial distribution with dispersion parameter r:

$$\mathrm{P}(k_t \mid r, \lambda_t) = \mathrm{NB}\left(k_t \,\middle|\, r, \frac{\lambda}{r+\lambda} = p_t\right).$$

Secondly, that the posterior on p_t is a beta distribution (which is the conjugate prior for the negative binomial with known dispersion):

$$\begin{aligned} \mathrm{P}(p_t \mid k_1, \dots, k_t) &= \mathrm{B}(p_t \mid \alpha_t, \beta_t) \\ \Leftrightarrow \mathrm{P}\left(\frac{\lambda_t}{r} \,\middle|\, k_1, \dots, k_t\right) &= \mathrm{BP}\left(\frac{\lambda_t}{r} \,\middle|\, \alpha_t, \beta_t\right). \end{aligned}$$

As indicated here, this is equivalent to the assumption of a beta prime prior on λ_t/r.

Thirdly, we assume that the predictive prior on p_t is related to the posterior on p_{t-1} as follows:

$$\begin{aligned} \mathrm{P}(p_t \mid k_1, \dots, k_{t-1}) &= \mathrm{B}\left(p_t \,\middle|\, \frac{\alpha_{t-1}}{c}, \frac{\beta_{t-1}}{c}\right) \\ \Leftrightarrow \mathrm{P}\left(\frac{\lambda_t}{r} \,\middle|\, k_1, \dots, k_{t-1}\right) &= \mathrm{BP}\left(\frac{\lambda_t}{r} \,\middle|\, \frac{\alpha_{t-1}}{c}, \frac{\beta_{t-1}}{c}\right). \end{aligned}$$

3 Results

The parameters of the posterior distributions satisfy

$$\begin{aligned}
\alpha_t &= \frac{\alpha_{t-1}}{c} + k_t = \frac{a_1}{c^{t-1}} + \sum_{i=0}^{t-1} \frac{k_{t-i}}{c^i}, \\
\beta_t &= \frac{\beta_{t-1}}{c} + r = \frac{b_1}{c^{t-1}} + \sum_{i=0}^{t-1} \frac{r}{c_i},
\end{aligned}$$

and the marginal likelihood of the model parameters r and c is

$$\begin{aligned}
\mathrm{P}(k_1, \dots, k_t \mid r, c) &= \prod_{t=1}^{t} \mathrm{P}(k_t \mid k_1, \dots, k_{t-1}, r, c) \\
&= \prod_{t=1}^{t} \frac{1}{k_t \, \mathrm{B}(k_t, r)} \frac{\mathrm{B}(a_t + k_t, b_t + r)}{\mathrm{B}(a_t, b_t)} \quad (\text{if } k_t > 0).
\end{aligned}$$