CO 450/650 Combinatorial Optimization

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Part I

Introduction

Chapter 1

Introduction

Definition: Combinatorial Optimization

A subfield of mathematical optimization which involves searching for an optimal object in a finite collection of objects.

Typically, the collection has a concise representation, while the number of objects is large. Objects include graphs, networks, and matroids.

The main tool in combinatorial optimization is linear programming duality.

Chapter 2

Linear Programming

Definition: Linear Programming

The problem of finding a vector x that maximizes a given linear function $c^T x$, where x ranges over all vectors satisfying a given system $Ax \leq b$ of linear inequalities.

2.1 Farkas' Lemma

Lemma (Farkas' Lemma for Inequalities)

The system $Ax \leq b$ has a solution x if and only if there is no vector y satisfying $y \geq 0$, $y^T A = 0$, and $y^T b < 0$.

Proof. Suppose $Ax \leq b$ has a solution $\overline{x}$ and suppose there exists a vector $\overline{y} \geq 0$ satisfying $\overline{y}^T A = 0$ and $\overline{y}^T b < 0$. Then we obtain the contradiction

$$0 > \overline{y}^T b \geq \overline{y}^T (A\overline{x}) = (\overline{y}^T A)\overline{x} = 0$$

Now suppose that $Ax \leq b$ has no solution. If A has only one column, then the result is easy. Otherwise, apply Fourier-Motzkin elimination to obtain a system $A'x' \leq b'$ with one less variable. Since $A'x' \leq b'$ also has no solution, we can assume by induction that there exists a vector $y' \geq 0$ satisfying $y'^T A' = 0$ and $y'^T b' < 0$. Now since each inequality in $A'x' \leq b'$ is the sum of positive multiples of inequalities in $Ax \leq b$, we can use y' to construct a vector y satisfying the conditions in the theorem. $\square$

Lemma (Farkas' Lemma)

The system $Ax = b$ has a nonnegative solution if and only if there is no vector y satisfying $y^T A \geq 0$ and $y^T b < 0$.

Proof. Define

$$A' = \begin{bmatrix} A \\ -A \\ -I \end{bmatrix}, b' = \begin{bmatrix} b \\ -b \\ 0 \end{bmatrix}$$

Then $Ax = b$ has a nonnegative solution x if and only if $A'x' \leq b'$ has a solution x'. Applying Farkas' Lemma for Inequalities to $A'x' \leq b'$ gives the result. □

Corollary

Suppose the system $Ax \leq b$ has at least one solution. Then every solution x of $Ax \leq b$ satisfies $c^T x \leq \delta$ if and only if there exists a vector $y \geq 0$ such that $y^T A = c$ and $y^T b \leq \delta$.

2.2 Duality

Consider the LP:

$$\begin{aligned} \max \quad & c^T x \\ \text{s.t.} \quad & Ax \leq b \end{aligned}$$

and dual LP

$$\begin{aligned} \min \quad & y^T b \\ \text{s.t.} \quad & y^T A = c^T \\ & y \geq 0 \end{aligned}$$

Theorem (Weak Duality)

Let A be an $m \times n$ matrix, $b \in \mathbb{R}^m$, $c \in \mathbb{R}^n$. Suppose that $\overline{x}$ is a feasible solution to $Ax \leq b$ and $\overline{y}$ is a feasible solution to $y \geq 0, y^T A = c^T$. Then

$$c^T \overline{x} \leq \overline{y}^T b$$

Proof.

$$c^T \overline{x} = (\overline{y}^T A)\overline{x} = \overline{y}^T (A\overline{x}) \leq \overline{y}^T b$$

□

Theorem (Strong Duality)

Let A be an $m \times n$ matrix, $b \in \mathbb{R}^m$, $c \in \mathbb{R}^n$, then

$$\max\{c^T x : Ax \leq b\} = \min\{y^T b : y^T A = c^T, y \geq 0\}$$

provided that both sets are nonempty.

Corollary

Let A be an $m \times n$ matrix, $b \in \mathbb{R}^m$, $c \in \mathbb{R}^n$, then

$$\max\{c^T x : Ax \leq b, x \geq 0\} = \min\{y^T b : y^T A \geq c^T\}$$

provided that both sets are nonempty.

Definition: Complementary Slackness Conditions

For each $i \in \{1, \ldots, m\}$, either $y_i^* = 0$ or $a_i x^* = b_i$.

Theorem (Complementary Slackness Theorem)

Let x^* be a feasible solution of $\max\{c^T x : Ax \leq b\}$ and let y^* be a feasible solution of $\min\{y^T b : y^T A = c^T, y \geq 0\}$. Then x^* and y^* are optimal solutions for the maximum and minimum respectively if and only if the complementary slackness conditions hold.

Chapter 3

Graph Theory

Definition: Graph

A graph $G = (V, E)$ is a set of vertices/nodes V and a set of edges E.
We define $n = |V|$ and $m = |E|$.

Definition: Subgraph

$H = (W, F)$ of $G = (V, E)$ where $W \subseteq V$ and $F \subseteq E$.

Definition: Spanning Subgraph

H is spanning if $V(H) = V(G)$.

Definition: Path

A sequence $P = v_0, e_1, v_1, \ldots, e_k, v_k$ where $v_0, \ldots, v_k \in V(G)$, $e_1, \ldots, e_k \in E(G)$, and $e_i = v_{i-1}v_i$.

We call P a v_0v_1-path. P is called edge-simple if all e_i are distinct and simple if all v_i are distinct.

The length of P is the number of edges in P.

Definition: Circuit/Cycle

An edge-simple closed path.

Definition: Connected

A graph is connected if every pair of vertices is joined by a path.

Definition: Cut Vertex

A vertex v of a connected graph G where $G - v$ is not connected.

Definition: Forest

A graph with no circuits.

Definition: Tree

A connected forest.

Definition: Cut

Let $R \subseteq V$, then

$$\delta(R) = \{vw : vw \in E, v \in R, w \notin R\}$$

Definition: rs-Cut

A cut for which $r \in R, s \notin R$.

Part II

Polyhedral Combinatorics

Chapter 4

Integrality of Polyhedra

4.1 Convex Hull

Definition: Convex Combination

$x = \lambda_1 v_1 + \cdots + \lambda_k v_k$ for some vectors $v_1, \ldots, v_k$ and nonnegative scalars $\lambda_1, \ldots, \lambda_k$ such that $\lambda_1 + \cdots + \lambda_k = 1$.

Definition: Convex Hull

The convex hull of a finite set S, denoted $conv.hull(S)$, is the set of all vectors that can be written as a convex combination of S.

Proposition

Let $S \subseteq \mathbb{R}^n$ be a finite set and let $w \in \mathbb{R}^n$. Then

$$\max\{w^T x : x \in S\} = \max\{w^T x : x \in conv.hull(S)\}$$

4.2 Polytopes

Definition: Polyhedron

A set of the form $\{x : Ax \leq b\}$.

In combinatorial optimization, we typically have $x \geq 0$ as a constraint, so we have polyhedra of the form $\{x : Ax \leq b, x \geq 0\}$.

Definition: Polytope

A polyhedron $P \subseteq \mathbb{R}^n$ is a polytope if there exists $\ell, u \in \mathbb{R}^n$ such that $\ell \leq x \leq u$ for all $x \in P$.

Definition: Convex Set

Let P be a polyhedron, $x_1, x_2 \in P$, and $0 \leq \lambda \leq 1$. If $\lambda x_1 + (1-\lambda)x_2 \in P$, then P is a convex set.

Definition: Valid Inequality

An inequality $w^T x \leq t$ is valid for a polyhedron P if $P \subseteq \{x : w^T x \leq t\}$.

Definition: Hyperplane

The solution set of $w^T x = t$ where $w \neq 0$.

Definition: Supporting Hyperplane

With respect to a polyhedron P, a hyperplane is supporting if $w^T x \leq t$ is valid for P and $P \cap \{x : w^T x = t\} \neq \emptyset$.

Definition: Face

The intersection of a polyhedron with one of its supporting hyperplanes.

The null set and the polyhedron itself is a face.

Definition: Proper Face

Faces which are not the null set or the polyhedron itself.

Proposition

A nonempty set $F \subseteq P = \{x : Ax \leq b\}$ is a face of P if and only if for some subsystem $A^\circ x \leq b^\circ$ of $Ax \leq b$, we have $F = \{x \in P : A^\circ x = b^\circ\}$.

Proof. ($\Longrightarrow$) Suppose F is a face of P. Then there exists a valid inequality $w^T x \leq t$ such that $F = \{x \in P : w^T x = t\}$.

Consider the LP problem $\max\{w^T x : Ax \leq b\}$. The set of optimal solutions is precisely F. Now let y^* be an optimal solution to the dual problem $\min\{y^T b : y^T A = w, y \geq 0\}$ and let $A^\circ x \leq b^\circ$ be those inequalities $a_i^T x \leq b_i$ whose corresponding dual variable y_i^* is positive. By complementary slackness, we have $F = \{x : Ax \leq b, A^\circ x = b^\circ\}$ as required.

($\Longleftarrow$) Conversely, if $F = \{x \in P : A^\circ x = b^\circ\}$ for some subsystem $A^\circ x \leq b^\circ$ of $Ax \leq b$, then add the inequalities $A^\circ x \leq b^\circ$ to obtain an inequality $w^T x \leq t$. Every $x \in F$ satisfies $w^T x = t$ and every $x \in P \setminus F$ satisfies $w^T x < t$ as required. $\square$

Proposition

Let F be a minimal nonempty face of $P = \{x : Ax \leq b\}$. Then $F = \{x : A^{\circ}x = b^{\circ}\}$ for some subsystem $A^{\circ}x \leq b^{\circ}$ of $Ax \leq b$.
Moreover, the rank of the matrix A° is equal to the rank of A.

Definition: Vertex/Extreme Point

A vector $x \in P$ is called a vertex/extreme point if $\{x\}$ is a face of P.
Equivalently, $x \in P$ is a vertex/extreme point if x cannot be written as $\frac{1}{2}x_1 + \frac{1}{2}x_2$ for points $x_1, x_2 \in P, x_1 \neq x_2$.

Note: Not all polyhedra have vertices, but if $P \subseteq \mathbb{R}^n_+$, then P has vertices.

LP Fact

If a polyhedron P has vertices, then the set of optimal LP solutions contains at least one vertex of P.
Moreover, if all vertices of P are integral, then the LP always has an integral optimal solution.

Definition: Pointed Polyhedron

A polyhedron P is pointed if it has at least one vertex.

$\{(x_1, x_2) \in \mathbb{R}^2 : x_1 \geq 0\}$ is a polyhedron with no vertex.

Proposition

If a polyhedron P is pointed, then every minimal nonempty face of P is a vertex.

Proposition

Let $P = \{x : Ax \leq b\}$ and $v \in P$. Then v is a vertex of P if and only if v cannot be written as a convex combination of vectors in $P \setminus \{v\}$.

Theorem

A polytope is equal to the convex hull of its vertices.

Proof. Let P be a nonempty polytope. Since P is bounded, P must be pointed. Let $v_1, \ldots, v_k$ be the vertices of P. Clearly, $conv.hull(\{v_1, \ldots, v_k\}) \subseteq P$. So suppose there exists

$$u \in P \setminus conv.hull(\{v_1, \ldots, v_k\})$$

Then by proposition, there exists an inequality $w^T x \leq t$ that separates u from

$$conv.hull(\{v_1, \ldots, v_k\})$$

Let $t^* = \max\{w^T x : x \in P\}$ and consider the face $F = \{x \in P : w^T x = t^*\}$. Since $u \in P$, we have $t^* > t$. So F contains no vertex of P, a contradiction. $\square$

Theorem

A set P is a polytope if and only if there exists a finite set V such that P is the convex hull of V.

4.3 Integral Polytopes

Definition: Rational Polyhedron

A polyhedron that can be defined by rational linear systems.

Definition: Integral Polyhedron

A rational polyhedron where every nonempty face contains an integral vector.

Definition: Pointed Integral Polyhedron

A pointed rational polyhedron is integral if and only if all its vertices are integral.

Theorem

A rational polytope P is integral if and only if for all integral vectors w, the optimal value of $\max\{w^T x : x \in P\}$ is an integer.

Proof. To prove sufficiency, suppose that for all integral vectors w, the optimal value of $\max\{w^T x : x \in P\}$ is an integer. Let $v = (v_1, \ldots, v_n)^T$ be a vertex of P and let w be an integral vector such that v is the unique optimal solution to $\max\{w^T x : x \in P\}$. By multiplying w by a large positive integer if necessary, we may assume $w^T v > w^T u + u_1 - v_1$ for all vertices u of P other than v. This implies that if we let $\overline{w} = (w_1+1, w_2, \ldots, w_n)^T$, then v is an optimal solution to $\max\{\overline{w}^T x : x \in P\}$. So $\overline{w}^T v = w^T v + v_1$. But, by assumption, $w^T v$ and $\overline{w}^T v$ are integers. Thus, v_1 is an integer. We can repeat this for each component of v, so v must be integral. $\square$

4.4 Total Unimodularity

Proposition

Let A be an integral, nonsingular, $m \times n$ matrix. Then $A^{-1}b$ is integral for every integral vector $b \in \mathbb{R}^m$ if and only if $\det(A) = 1$ or -1.

Proof. ($\Longleftarrow$) Suppose $\det(A) = \pm 1$. By Cramer's Rule, we know that A^{-1} is integral, which implies $A^{-1}b$ is integral for every integral b.

($\Longrightarrow$) Conversely, suppose $A^{-1}b$ is integral for all integral vectors b. Then, in particular, $A^{-1}e_i$ is integral for all $i = 1, \ldots, m$. This means that A^{-1} is integral. So $\det(A)$ and

$\det(A^{-1})$ are both integers. But, $\det(A) \cdot \det(A^{-1}) = 1$, this implies $\det(A) = \pm 1$. $\square$

Definition: Unimodular

A matrix A of full row rank is unimodular if A is integral and each basis of A has determinant ± 1.

Theorem (Veinott & Dantzig 1968)

Let A be an integral $m \times n$ matrix of full row rank. Then the polyhedron defined by $Ax = b, x \geq 0$ is integral for every integral vector $b \in \mathbb{R}^m$ if and only if A is unimodular.

Proof. ($\Longleftarrow$) Suppose A is unimodular. Let $b \in \mathbb{R}^m$ be an integral vector and let $\overline{x}$ be a vertex of $\{x : Ax = b, x \geq 0\}$. The nonnegativity constraints implies the polyhedron has vertices. Then there are n linearly independent constraints satisfied by $\overline{x}$ with inequality. It follows that the columns of A corresponding to the nonzero components of $\overline{x}$ are linearly independent. Extending these columns to a basis B of A, we have the nonzero components of $\overline{x}$ are contained in the integral vector $B^{-1}b$. So $\overline{x}$ is integral.

($\Longrightarrow$) Conversely, suppose $\{x : Ax = b, x \geq 0\}$ is integral for all integral vectors b. Let B be a basis of A and let v be an integral vector in $\mathbb{R}^m$. By previous proposition, it suffices to show that $B^{-1}v$ is integral. Let y be an integral vector such that $y + B^{-1}v \geq 0$ and let $b = B(y + B^{-1}v)$. Note b is integral. Furthermore, by adding zero components to the vector $y + B^{-1}v$, we can obtain a vector $z \in \mathbb{R}^n$ such that $Az = b$. Then, z is a vertex of $\{x : Ax = b, x \geq 0\}$, since z is a polyhedron and satisfies n linearly independent constraints with equality: the m equations $Ax = b$ and the $n - m$ equations $x_i = 0$ for the columns i outside B. So z is integral, and thus, $B^{-1}v$ is integral. $\square$

Definition: Totally Unimodular (TU)

A matrix is totally unimodular if all of its square submatrices have determinant 0, 1, or -1.

It is easy to see that A is totally unimodular if and only if $\begin{bmatrix} A & I \end{bmatrix}$ is unimodular where $I \in \mathbb{R}^{m \times m}$.

Theorem (Hoffman-Kruskal)

Let A be an $m \times n$ integral matrix. Then the polyhedron defined by $Ax \leq b, x \geq 0$ is integral for every integral vector $b \in \mathbb{R}^m$ if and only if A is totally unimodular.

Proof. Applying the linear programming trick of adding slack variables, we have that for any integral b, the polyhedron $\{x : Ax \leq b, x \geq 0\}$ is integral if and only if the polyhedron $\{z : \begin{bmatrix} A & I \end{bmatrix} z = b, z \geq 0\}$ is integral. So the result follows from previous theorem. $\square$

Theorem

Let A be an $m \times n$ totally unimodular matrix and let $b \in \mathbb{R}^m$ be an integral vector. Then the polyhedron defined by $Ax \leq b$ is integral.

Proof. Let F be a minimal face of $\{x : Ax \leq b\}$. Then, by proposition, $F = \{x : A^\circ x = b^\circ\}$ for some subsystem $A^\circ x \leq b^\circ$ of $Ax \leq b$, with A° having full row rank. By reordering the columns, if necessary, we may write A° as $\begin{bmatrix} B & N \end{bmatrix}$ where B is a basis of A°. It follows

$$\overline{x} = \begin{bmatrix} B^{-1} b^\circ \\ 0 \end{bmatrix}$$

is an integral vector in F. □

Theorem

Let A be a $0, \pm 1$ valued matrix where each column has at most one $+1$ and at most -1. Then A is totally unimodular.

Proof. Let N be a $k \times k$ submatrix of A. If $k = 1$, then $\det(N)$ is either 0 or ± 1. So we may suppose that $k \geq 2$ and proceed by induction on k. If N has a column having at most one nonzero, then expanding the determinant along this column, we have that $\det(N)$ is either 0 or ± 1, by induction. On the other hand, if every column of N has both a $+1$ and a -1, then the sum of the rows of N is 0 and hence $\det(N) = 0$. □

Let $D = (V, E)$ be a digraph and let A be its incidence matrix. Then A is totally unimodular.

Definition: Network Matrix

Let $T = (V, E')$ be a spanning tree of D and define the matrix M having rows indexed by E' and columns indexed by E, where $e = (u, v) \in E$ and $e' \in E'$.

$$M_{e',e} = \begin{cases} +1 & \text{if } uv\text{-path in } T \text{ uses } e' \text{ in forward direction} \\ -1 & \text{if } uv\text{-path in } T \text{ uses } e' \text{ in backward direction} \\ 0 & \text{if } uv\text{-path in } T \text{ does not use } e' \end{cases}$$

Theorem (Tutte 1965)

Network matrices are totally unimodular.

Proposition

A is totally unimodular if and only if A^T is totally unimodular.

4.5 Edmonds' Matching Polytope

For a graph $G = (V, E)$, let $\mathcal{PM}(G) \subseteq \mathbb{R}^E$ denote the set of characteristic vectors of its perfect matchings.

Theorem (Perfect Matching Polytope Theorem)

For any graph $G = (V, E)$, the convex hull of $\mathcal{PM}(G)$ is identical to the set of solutions of the linear system

$$\begin{aligned} x(\delta(v)) &= 1,\ \forall v \in V \\ x(\delta(S)) &\geq 1,\ \forall S \subseteq V, |S| \geq 3 \text{ and odd} \\ x_e &\geq 0,\ \forall e \in E \end{aligned}$$

Theorem (Birkhoff)

Let G be a bipartite graph. Then the convex hull of the perfect matchings of G is defined by

$$\begin{aligned} x(\delta(v)) &= 1,\ \forall v \in V \\ x_e &\geq 0,\ \forall e \in E \end{aligned}$$

Theorem (Fractional Matching Polytope Theorem)

Let G be a graph and let $x \in FPM(G)$. Then x is a vertex of $FPM(G)$ if and only if $x_e \in \left\{0, \frac{1}{2}, 1\right\}$ for all $e \in E$ and the edges e for which $x_e = \frac{1}{2}$ form vertex-disjoint odd circuits.

Proof. (Perfect Matching Polytope Theorem – Schrijver)

Part III

Optimal Trees and Paths

Chapter 5

Minimum Spanning Trees

5.1 Problem

Definition: Spanning Tree

A subgraph $T \subseteq G$ where $V(T) = V(G)$, T is connected, and T is acyclic.

Lemma

An edge $e = uv$ of G is an edge of a circuit of G if and only if there is a path in $G \setminus e$ from u to v.

Minimum Spanning Tree Problem (MST)

Given a connected graph G and a real cost c_e for each $e \in E$, find a minimum cost spanning tree of G.

Lemma

A spanning connected subgraph of G is a spanning tree if and only if it has exactly $n - 1$ edges.

5.2 Algorithm

Theorem

A graph G is connected if and only if there is no set $A \subseteq V$ where $\emptyset \neq A \neq V$ with $\delta(A) = \emptyset$.

Algorithm 1 Kruskal's Algorithm for MST

sort E to $\{e_1, \ldots, e_m\}$ so that $c_{e_1} \leq \cdots \leq c_{e_m}$
$H = (V, F), F = \emptyset$
for $i = 1$ to m **do**
  if ends of e_i are in different components of H **then**
    $F \leftarrow F \cup \{e_i\}$
return H

5.3 Linear Programming Relaxation

Definition: $\kappa : E \to \mathbb{N}$

$\kappa(A)$ is the number of components in the subgraph (V, A) of G.

We can formulate the MST problem as an IP.

$$\begin{aligned} \min \quad & c^T x \\ \text{s.t.} \quad & x(A) \leq |V| - \kappa(A), \ \forall A \subset E \\ & x(E) = |V| - 1 \\ & x_e \in \{0, 1\}, \ \forall e \in E \end{aligned}$$

We can relax the integer program to get the following linear program.

Definition: MST LP

$$\begin{aligned} \min \quad & c^T x \\ \text{s.t.} \quad & x(A) \leq |V| - \kappa(A), \ \forall A \subset E \\ & x(E) = |V| - 1 \\ & x_e \geq 0, \ \forall e \in E \end{aligned}$$

We replace the minimization with a maximization in the primal to write the dual.

Definition: MST Dual LP

$$\begin{aligned} \min \quad & \sum_{A \subseteq E} (|V| - \kappa(A)) y_A \\ \text{s.t.} \quad & \sum (y_A : e \in A) \geq -c_e, \ \forall e \in E \\ & y_A \geq 0, \ \forall A \subset E \end{aligned}$$

Theorem (Edmonds 1971)

Let x^* be the characteristic vector of an MST with respect to costs c_e. Then x^* is an optimal solution of the linear program.

Proof. We show that x^* is optimal for the LP and x^* is the characteristic vector generated by Kruskal's algorithm. y_E is not required to be nonnegative.

Let $e_1, \ldots, d_m$ be the order in which Kruskal's algorithm considers the edges. Let $R_i = \{e_1, \ldots, e_i\}$ for $1 \leq i \leq m$. Let y^* be the be the dual solution. We denote $y^*_A = 0$ unless A is one of the R_i, $y^*_{R_i} = c_{e_{i+1}} - c_{e_i}$ for $1 \leq i \leq m-1$, and $y^*_{R_m} = -c_{e_m}$. It follows from the ordering of the edges, $y^*_A \geq 0$ for $A \neq E$. Now consider the first constraint, then where $e = e_i$, we have

$$\sum_{A:e\in A} y^*_A = \sum_{j=i}^{m} y^*_{R_j} = \sum_{j=i}^{m-1} (c_{e_{i+1}} - c_{e_i}) = -c_{e_i} = -c_e$$

All of the inequalities hold with equality. So the complementary slackness conditions ($x^*_e > 0 \implies \sum_{A:e\in A} y_A = c_e$) are satisfied.

We want to show now that the second constraint also satisfies complementary slackness conditions ($y^*_A > 0 \implies x(A) \leq |V| - \kappa(A)$). We know $A = R_i$ for some i. If the primal constraint does not hold with equality for R_i, then there is some edge of R_i whose addition to $E(T) \cap R_i$ would decrease the number of components of $(V, E(T) \cap R_i)$. But this edge would have ends in two different components of $(V, E(T) \cap R_i)$, and therefore would have been added to T by Kruskal's algorithm.

Therefore, x^* and y^* satisfy complementary slackness conditions. So, x^* is an optimal solution to the LP. $\square$

Chapter 6

Shortest Paths

Shortest Path Problem

Given a digraph G, a vertex $r \in V$, and a real cost vector $(c_e : e \in E)$, find for each $v \in V$, a dipath from r to v of least cost.

Let y_v for $v \in V$ be the least cost of a dipath to v, then y s

Definition: Feasible Potential

$y = (y_v : v \in V)$ is a feasible potential if it satisfies $y_v + c_{vw} \geq y_w$ for all $vw \in E$ and $y_r = 0$.

Proposition

Let y be a feasible potential and let P be a dipath from r to v. Then $c(P) \geq y_v$.

Proof. Suppose that P is $v_0, e_1, v_1, \ldots, e_k, v_k$ where $v_0 = r$ and $v_k = v$. Then

$$c(P) = \sum_{i=1}^{k} c_{e_i} \geq \sum_{i=1}^{k} (y_{v_i} - y_{v_{i-1}}) = y_{v_k} - y_{v_0} = y_v$$

□

6.1 Linear Programming

Theorem

Let G be a digraph, $r, s \in V$, and $c \in \mathbb{R}^E$. If there exists a least-cost dipath from r to v for every $v \in V$, then

$$\min\{c(P) : P \text{ an } rs\text{-dipath}\} = \max\{y_s : y \text{ a feasible potential}\}$$

Definition: Shortest Path LP

$$
\begin{aligned}
\min \quad & \sum(c_e x_e : e \in E) \\
\text{s.t.} \quad & \sum(x_{wv} : w \in V, wv \in E) - \sum(x_{vw} : w \in V, vw \in E) = b_v,\ \forall v \in V \\
& x_{vw} \geq 0,\ \forall vw \in E
\end{aligned}
$$

Definition: Shortest Path Dual LP

$$
\begin{aligned}
\max \quad & y_s - y_r \\
\text{s.t.} \quad & y_w - y_v \leq c_{vw},\ \forall vw \in E
\end{aligned}
$$

Part IV

Network Flows

Chapter 7

Maximum Flow

7.1 Problem

Definition: Net Flow/Excess

$$f_x(v) = x(\delta(\overline{v})) - x(\delta(v)) = \sum(x_{wv} : w \in V, wv \in E) - \sum(x_{vw} : w \in V, vw \in E)$$

Definition: rs-Flow

A vector x that satisfies $f_x(v) = 0$ for all $v \in V$.

Definition: Value of rs-Flow

$$f_x(s)$$

Maximum Flow Problem

Given a digraph $G = (V, E)$, with source r and sink s, find an rs-flow of maximum value.

Proposition

There exists a family $(P_1, \ldots, P_k)$ of rs-dipaths such that $|\{i : e \in P_i\}| \leq u_e$ for all $e \in E$ if and only if there exists an integral feasible rs-flow of value k.

Proof. ($\Longrightarrow$) We have seen family of dipaths determines a corresponding flow.

($\Longleftarrow$) Let x be a flow. We assume that x is acyclic, that is, there is no dicircuit C, each of whose arcs e has $x_e > 0$. If a dicircuit does exist, we can decrease x_e by 1 on all arcs of C. The new x remains feasible of value k.

If $k \geq 1$, we can find an arc vs with $x_{vs} \geq 1$. Then, if $v \neq r$, it follows that there is an arc wv with $x_{wv} \geq 1$ by the constraint $f_x(v) = 0$. If $w \neq r$, then the argument can be repeated producing distinct vertices, since x is acyclic, so we get a simple rs-dipath P_k on each arc e with $x_e \geq 1$. We can decrease x_e by 1 for each $e \in P_k$. The new x is an integral feasible flow of value $k-1$, and the process is repeated. □

7.2 Maximum Flows and Minimum Cuts

Definition: Maximum Flow LP

$$\begin{aligned} \max \quad & f_x(s) \\ \text{s.t.} \quad & f_x(v) = 0,\ \forall v \in V \setminus \{r, s\} \\ & 0 \leq x_e \leq u_e,\ \forall e \in E \end{aligned}$$

Definition: Path Flow

A vector $x \in \mathbb{R}^E$ such that for some rs-dipath P and some $\alpha \in \mathbb{R}$, $x_e = \alpha$ for each $e \in P$ and $x_e = 0$ for every other arc of G.

Definition: Circuit Flow

A vector $x \in \mathbb{R}^E$ such that for some rs-dicircuit C and some $\alpha \in \mathbb{R}$, $x_e = \alpha$ for each $e \in C$ and $x_e = 0$ for every other arc of G.

Proposition

Every rs-flow of nonnegative value is the sum of at most m flows, each of which is a path flow or a circuit flow.

Proposition

For any rs-cut $\delta(R)$ and any rs-flow x, we have

$$f_x(s) = x(\delta(R)) - x(\delta(\overline{R}))$$

Proof. We add the equations $f_x(v) = 0$ for all $v \in \overline{R} \setminus \{s\}$ as well as the identity $f_x(s) = f_x(s)$. The right hand side sums to $f_x(s)$.

For any arc vw with $v, w \in R$, x_{vw} occurs in none of the equations, so it does not occur in the sum. If $v, w \in \overline{R}$, then x_{vw} occurs in the equation for v with a coefficient of -1, and in the equation for w with a coefficient of $+1$, so it has a coefficient of 0 in the sum. If $v \in R, w \notin R$, then x_{vw} occurs in the equation for w with a coefficient of 1, and so has coefficient 1 in the sum. If $v \notin R, w \in R$, then x_{vw} occurs in the sum with a coefficient of -1. So, the left hand side sums to $x(\delta(R)) - x(\delta(\overline{R}))$, as required. □

Corollary

For any feasible rs-flow x and any rs-cut $\delta(R)$,

$$f_x(s) \leq u(\delta(R))$$

Proof. Using previous proposition, since $x(\delta(R)) \leq u(\delta(R))$ and $x(\delta(\overline{R})) \geq 0$. $\square$

Definition: Incrementing Path

A path is x-incrementing if every forward arc e has $x_e < u_e$ and every reverse arc e has $x_e > 0$.

Definition: Augmenting Path

An rs-path that is x-incrementing.

Theorem Maximum-Flow Minimum-Cut

If there is a maximum rs-flow, then

$$\max\{f_x(s) : x \text{ is a feasible } rs\text{-flow}\} = \min\{u(\delta(R)) : \delta(R) \text{ is an } rs\text{-cut}\}$$

Proof. By previous corollary, we need only show that there exists a feasible flow x and a cut $\delta(R)$ such that $f_x(s) = u(\delta(R))$. Let x be a flow of maximum value. Let $R = \{v \in V :$ there exists an x-incrementing rv-path$\}$. Clearly $r \in R$ and $s \notin R$, since there can be no x-augmenting path.

For every arc $vw \in \delta(R)$, we must have $x_{vw} = u_{vw}$, since otherwise adding vw to the x-incrementing rv-path would yield such a path to w, but $w \notin R$. Similar, for every arc $vw \in \delta(\overline{R})$, we have $x_{vw} = 0$. Then by proposition, $f_x(s) = x(\delta(R)) - x(\delta(\overline{R})) = u(\delta(R))$. $\square$

Theorem

A feasible flow x is maximum if and only if there is not x-augmenting path.

Proof. ($\implies$) If x is maximum, there is no x-augmenting path.

($\impliedby$) If there is no x-augmenting path, then the construction of the proof of Max-Flow Min-Cut yields a cut $\delta(R)$ with $f_x(s) = u(\delta(R))$, so x is maximum, by corollary. $\square$

Theorem

If u is integral and there exists a maximum flow, then there exists a maximum flow that is integral.

Proof. Choose an integral flow x of maximum value. If there is an x-augmenting path, then since x and u are integral, the new flow can be chosen integral, contradicting the choice of x. Hence there is no x-augmenting path, so x is a maximum flow, by previous theorem. $\square$

Corollary

If x is a feasible rs-flow and $\delta(R)$ is an rs-cut, then x is maximum and $\delta(R)$ is minimum if and only if $x_e = u_e$ for all $e \in \delta(R)$ and $x_e = 0$ for all $e \in \delta(\overline{R})$.

Proof. Combine Max-Flow Min-Cut theorem with the proof of corollary. □

7.3 Augmenting Path Algorithm

Algorithm 2 Ford-Fulkerson Algorithm

$x = 0$
while there is an x-augmenting path P **do**
  $\varepsilon_1 = \min(u_e - x_e : e \text{ forward in } P)$
  $\varepsilon_2 = \min(x_e : e \text{ reverse in } P)$
  $\varepsilon = \min(\varepsilon_1, \varepsilon_2)$ // x-width of P
  if $\varepsilon = \infty$ **then**
    no maximum flow
return x is maximum flow, set R of vertices reachable by an x-incrementing path from r is minimum cut

Definition: Auxiliary Digraph

$G(x)$, depending on G, u, x, where $V(G(x)) = V$ and $vw \in E(G(x))$ if and only if $vw \in E$ and $x_{vw} < u_{vw}$ or $wv \in E$ and $x_{wv} > 0$.

rs-dipaths in $G(x)$ corresponding to x-augmenting paths in G. Each iteration of Ford-Fulkerson can be performed in $O(m)$ time, using breadth-first search.

Theorem

If u is integral and the maximum flow value is $K < \infty$, then the maximum flow algorithm terminates after at most K augmentations.

7.3.1 Shortest Augmenting Paths

Theorem (Dinits 1970, Edmonds & Karp 1972)

If each augmentation of the augmenting path algorithm on a shortest augmenting path, then there are at most nm augmentations.

Corollary

The augmenting path algorithm with breadth-first search solves the maximum flow problem in time $O(nm^2)$.

Let $d_x(v,w)$ be the least length of a vw-dipath in $G(x)$. $d_x(v,w)=\infty$ if no vw-dipath exists.

Consider a typical augmentation from flow x to flow x' determined by the augmenting path P having vertex-sequence $v_0,\dots,v_k$.

Lemma

For each $v\in V$, $d_{x'}(r,v)\geq d_x(r,v)$ and $d_{x'}(v,s)\geq d_x(v,s)$.

Proof. Suppose that there exists a vertex v such that $d_{x'}(r,v)<d_x(r,v)$ and choose such v so that $d_{x'}(r,v)$ is as small as possible. Clearly, $d_{x'}(r,v)>0$. Let P' be a rv-dipath in $G(x')$ of length $d_{x'}(r,v)$ and let w be the second-last vertex of P'. Then

$$d_x(r,v)>d_{x'}(r,v)=d_{x'}(r,w)+1\geq d_x(r,w)+1$$

It follows that wv is an arc of $G(x')$, but not of $G(x)$, otherwise $d_x(r,v)\leq d_x(r,w)+1$, so $w=v_i$ and $v=v_{i-1}$ for some i. But, this implies that $i-1>i+1$, a contradiction. The second statement is similar. □

Definition: $\tilde{E}(x)$

$$\tilde{E}(x)=\{e\in E: e \text{ is an arc of a shortest } x\text{-augmenting path}\}$$

Lemma

If $d_{x'}(r,s)=d_x(r,s)$, then $\tilde{E}(x')\subsetneq\tilde{E}(x)$.

Proof. Let $k=d_x(r,s)$ and suppose that $e\in\tilde{E}(x')$. Then e induces an arc vw of $G(x')$ and $d_{x'}(r,v)=i-1$, $d_{x'}(ws)=k-i$ for some i. Therefore, $d_x(r,v)+d_x(w,s)\leq k-1$ by previous lemma. Now suppose that $e\notin\tilde{E}(x)$, then $x_e\neq x'_e$, so e is an arc of P, a contradiction. This proves $\tilde{E}(x')\subseteq\tilde{E}(x)$.

There is an arc e of P such that e is forward and $x'_e=u_e$ or e is reverse and $x'_e=0$. Therefore, any x'-augmenting path using e must use it in the opposite direction from P, so its length, for some i, will be at least $i+k-i+1+1=k+23$, so $e\notin\tilde{E}(x')$. □

Proof. (Dinits, Edmonds, Karp) It follows from previous lemma that there can be at most m augmentations per stage. Since there are at most $n-1$ stages, there are at most nm augmentations in all.

7.4 Applications

7.4.1 Flow Feasibility

Flow Feasibility Problem

Given a digraph G, $u \in \mathbb{R}^E_+$, and $b \in \mathbb{R}^V$, find, if possible, $x \in \mathbb{R}^E$ such that

$$f_x(v) = b_v, \ \forall v \in V$$

and

$$0 \leq x_e \leq u_e, \ \forall e \in E$$

Theorem (Gale 1957)

There exists a solution to the flow feasibility problem if and only if $b(V) = 0$ and for every $A \subseteq V$, $b(A) \leq u(\delta(\overline{A}))$.
If b and u are integral, then there is an integral solution.

Corollary

Given a digraph G and $b \in \mathbb{R}^V$, there exists $x \in \mathbb{R}^E$ with

$$f_x(v) = b_v, \ \forall v \in V$$

$$x_e \geq 0, \ \forall e \in E$$

if and only if $b(V) = 0$ and for every $A \subseteq V$ with $\delta(\overline{A}) = \emptyset$, we have $b(A) \leq 0$.

Definition: Circulation

A vector $x \in \mathbb{R}^E$ with $f_x(v) = 0$ for all $v \in V$.

Theorem (Hoffman's Circulation Theorem 1960)

Given a digraph G, $\ell \in (\mathbb{R} \cup \{-\infty\})^E$, and $u \in (\mathbb{R} \cup \{\infty\})^E$, with $\ell \leq u$, there is a circulation x with $\ell \leq x \leq u$ if and only if every $A \subseteq V$ satisfies $u(\delta(\overline{A})) \geq \ell(\delta(A))$.

Part V

Matchings

Chapter 8

Matchings

Definition: Matching

A set $M \subseteq E$ such that no vertex of G is incident with more than one edge in M.

Definition: M-Covered

A vertex v is covered by M if some edge of M is incident with v.

Definition: M-Exposed

A vertex v is exposed if v is not M-covered.

The number of vertices covered by M is $2\,|M|$ and number of M-exposed vertices is $|V| - 2\,|M|$.

Definition: Maximum Matching

A matching of maximum cardinality, denoted $\nu(G)$.

Definition: Deficiency

The minimum number of exposed vertices for any matching of G, denoted by $\operatorname{def}(G)$.

Note $\operatorname{def}(G) = |V| - 2\nu(G)$.

Definition: Perfect Matching

A matching that covers all vertices.

8.1 Bipartite Matching

Definition: Bipartite

$G = (V, E)$ is bipartite if $V = V_1 \cup V_2$, where V_1, V_2 disjoint and every edge has one end in V_1 and the other end in V_2.

Definition: Cover

A set $C \subseteq V$ such that every edge has at least one in C.

Lemma

If M is a matching and C is a cover, then $|M| \leq |C|$.

Proof. Every $e \in M$ has at least one end in C. No vertex in C meets more than one edge in M. □

Definition: Minimum Cover

A cover of minimum cardinality, denoted $\tau(G)$.

Theorem (König)

If G is bipartite, $\nu(G) = \tau(G)$.

Proof. We note that $\nu(G) \leq \nu^*(G)$ and $\tau(G) \geq \tau^*(G)$. By using LP duality and the matching LP (*Matching LP*), we show that $\nu(G) = \nu^*(G)$. We also have the matching LP in the form of $Mx^+ = (1, \ldots, 1)^T$. Since M is totally unimodular, then M^T is also totally unimodular. So the dual LP has all integral vertices, implying $\tau(G) = \tau^*(G)$. So,

$$\nu(G) = \nu^*(G) = \tau^*(G) = \tau(G)$$

□

8.2 Alternating Paths

Definition: M-Alternating

A path P is M-alternating if its edges are alternately in and not in M.

Definition: M-Augmenting

An M-alternating path P is M-augmenting if the ends of P are distinct and are both M-exposed.

Definition: Symmetric Difference

For sets S and T, let $S\Delta T$ denote the symmetric difference, which is defined as

$$S\Delta T = (S \cup T) \setminus (S \cap T)$$

Let a path P be an M-augmenting path. Then we can obtain a larger matching $M' = M\Delta E(P)$ with $|M'| = |M| + 1$.

Theorem (Petersen 1891, Berge 1957)

A matching M in a graph G is maximum if and only if there is no M-augmenting path.

Proof. ($\implies$) Suppose there exists an M-augmenting path P joining v and w. Then $N = M\Delta E(P)$ is a matching that covers all vertices covered by M, plus v and w. So, M is not maximum.

($\impliedby$) Conversely, suppose that M is not maximum and some other matching N satisfies $|N| > |M|$. Let $J = N\Delta M$. Each vertex of G is incident with at most two edges of J, so J is the edge set of some vertex disjoint paths and circuits of G. For each such path or circuit, the edges alternately belong to M or N. Therefore, all circuits are even and contain the same number of edges of M and N. Since $|N| > |M|$, there must be at least one path with more edges of N than M. This path is an M-augmenting path. $\square$

8.3 Matching LP

Definition: Matching LP

P is the set of solutions to

$$\begin{aligned} x(\delta(v)) &\leq 1, \ \forall v \in V \\ x_e &\geq 0, \ \forall e \in E \end{aligned}$$

Let $\overline{x}$ be a vertex of P. We show that $\overline{x}$ is integral, which implies that $M = \{e \in E : \overline{x}_e = 1\}$ is a matching and $\nu(G) = \nu^*(G)$.

Recall that for a polyhedron $P = \{x : Ax \leq b\} \subseteq \mathbb{R}^n$, $\overline{x} \in P$ is a vertex if and only if $\overline{x}$ is the unique solution to $A'x = b'$ for some subset of n inequalities $A'x \leq b'$ from $Ax \leq b$.

For our matching P, let $E^+ := \{e : \overline{x}_e > 0\}$ and $E^0 := \{x : \overline{x}_e = 0\}$. We write $\overline{x} = (\overline{x}^+, \overline{x}^0)$ split by (E^+, E^0).

Since $\overline{x}$ is a vertex, there exists $V^+ \subseteq V$ such that $\overline{x}$ is the unique solution to

$$\begin{aligned} \sum(x_e : e \in \delta(v) \cap E^+) &= 1, \ \forall v \in V^+ \\ x_e &= 0, \ \forall e \in E^0 \end{aligned}$$

Restricting to E^+, we can write the system of equations as

$$Mx^+ = (1, \ldots, 1)^T$$

By Cramer's Rule, the solution to the system is $(\overline{x}_1^+, \ldots, \overline{x}_k^+)$, where

$$\overline{x}_j^+ = \frac{\det(M^j)}{\det(M)}$$

with M^j obtained from M by replacing the jth column by $(1, \ldots, 1)^T$.

Claim: $\det(M) = 1$ or $\det(M) = -1$.

This gives that $\overline{x}_j^+$ is integer for all j, so $\overline{x}$ is integer. Thus, $\nu(G) = \nu^*(G)$.

Lemma

Let $G = (V, E)$ be a bipartite graph. Let A be the $|V| \times |E|$ matrix $\left[A_{ve}\right]$ with

$$A_{ve} = \begin{cases} 1 & \text{if } e \in \delta(v) \\ 0 & \text{if } e \notin \delta(v) \end{cases}$$

then A is totally unimodular.

Proof. By induction of the number of rows k of the submatrix B of A. If B is 1×1, then this is obvious.

Suppose it is true for $k = 1, \ldots, t-1$ and let B be a $t \times t$ submatrix of A.

1. If B has a column of all 0's, then $\det(B) = 0$.
2. If a column of B has exactly one 1, then we compute $\det(B)$ by expanding on that column and use induction.
3. Otherwise, every column of B has exactly two 1's.

We can partition the rows of B into W_1 and W_2, so that every column has exactly one 1 in W_1 and exactly one 1 in W_2 (W_1 are vertices in V_1, W_2 in V_2 from G being bipartite).

Now multiplying each row in W_1 by 1 and each row in W_2 by -1 and summing, we get the row vector of all 0's. So $\det(B) = 0$. □

8.4 Tutte-Berge Formula

Definition: Vertex Cover

A set A of vertices such that every edge has at least one end in A.

Let A be a subset of the vertices which $G - A$ has k components $H_1, \ldots, H_k$ having an odd number of vertices. Let M be a matching of G. For each i, either H_i has an M-exposed vertex or M contains an edge having just one end in $V(H_i)$. All such edges have their other ends in A and since M is a matching, all these ends must be distinct. Therefore, there can be at most $|A|$ edges and so the number of M-exposed vertices is at least $k - |A|$.

Definition: $\text{oc}(H)$

The number of odd components of a graph H.

Thus, for any $A \subseteq V$,

$$\nu(G) \leq \frac{1}{2}(|V| - \text{oc}(G - A) + |A|)$$

If A is a cover of G, then there are $|V| - |A|$ odd components of $G - A$ (each is a single vertex), so the right hand side reduces to $|A|$. This bound is at least as strong as that provided by covers.

Theorem (Tutte-Berge Formula)

For a graph $G = (V, E)$, we have

$$\max\{|M| : M \text{ a matching}\} = \min\left\{\frac{1}{2}(|V| - \text{oc}(G - A) + |A|) : A \subseteq V\right\}$$

Theorem (Tutte's Matching Theorem 1947)

A graph $G = (V, E)$ has a perfect matching if and only if for every $A \subseteq V$, $\text{oc}(G - A) \leq |A|$.

Definition: Shrink

Let C be an odd circuit in G. Define $G' = G \times C$ as the subgraph obtained from G by shrinking C; G' has vertex set $(V - V(C)) \cup \{C\}$ and edge set $E \setminus \gamma(V(C))$.

Proposition

Let C be an odd circuit of G, let $G' = G \times C$, and let M' be a matching of G'. Then here is a matching M of G such that $M \subseteq M' \cup E(C)$ and the number of M-exposed vertices of G is the same as the number of M'-exposed vertices of G'.

Proof. Choose a vertex $w \in V(C)$ as follows. If C is covered by $e \in M'$, then choose w to be the vertex in $V(C)$ that is an end of e, and otherwise, choose w arbitrarily. Deleting w from C results in a subgraph having a perfect matching M''. Take $M = M' \cup M''$. M has the required properties. $\square$

The previous proposition gives the inequality

$$\nu(G) \geq \nu(G \times C) + \frac{|V(C)| - 1}{2}$$

or equivalently,

$$\operatorname{def}(G) \leq \operatorname{def}(G \times C)$$

Definition: Tight Odd Circuit

An odd circuit C is tight if $\nu(G) = \nu(G \times C) + \frac{|V(C)|-1}{2}$.

Definition: Inessential

A vertex v of G is inessential if there is a maximum matching of G that does not cover v.

Definition: Essential

A vertex not inessential.

Let A be a set that satisfies the Tutte-Berge formula. Let $v \in A$ and consider $G' = G - v$. Then, $G' - (A \setminus \{v\})$ has the same odd components as $G - A$, so $\nu(G') < \nu(G)$, i.e. every $v \in A$ is essential.

Lemma

Let $G = (V, E)$ be a graph and let $vw \in E$. If v, w are both inessential, then there is a tight odd circuit C using vw. Moreover, C is an inessential vertex of $G \times C$.

Definition: Gallai-Edmonds Partition

Let B be the set of inessential vertices of $G = (V, E)$, C be the set of vertices not in B but adjacent to at least one element of B, and D be $V \setminus (B \cup C)$, then (B, C, D) is the Gallai-Edmonds Partition of G.

Proposition

Let (B, C, D) be the Gallai-Edmonds Paritition for G.
C is a minimizer in the Tutte-Berge formula.

Proposition

Let (B, C, D) be the Gallai-Edmonds Paritition for G.
For every maximum matching M and every vertex $v \in C$, there is an edge $vw \in M$ with $w \in B$.

Proposition

Let (B, C, D) be the Gallai-Edmonds Paritition for G.
Every maximum matching contains a perfect matching of $G[D]$.

8.5 Maximum Matching

Maximum Matching Problem

Given a graph G, find a maximum matching of G.

Definition: Maximum Matching ILP

$$\begin{aligned} \max \quad & \sum(x_e : e \in E) \\ \text{s.t.} \quad & x(\delta(v)) \leq 1, \ \forall v \in V \\ & x_e \geq 0, \ \forall e \in E \\ & x_e \text{ integer}, \ \forall e \in E \end{aligned}$$

Definition: Maximum Matching LP Relaxation

$$\begin{aligned} \max \quad & \sum(x_e : e \in E) \\ \text{s.t.} \quad & x(\delta(v)) \leq 1, \ \forall v \in V \\ & x_e \geq 0, \ \forall e \in E \end{aligned}$$

Definition: Minimum Cover Dual LP

$$\begin{aligned} \min \quad & \sum(y_v : v \in V) \\ \text{s.t.} \quad & y_u + y_v \geq 1, \ \forall e = (u, v) \in E \\ & y_v \geq 0, \ \forall v \in V \end{aligned}$$

Let M be a matching and C be a cover, then

$$x_e^M = \begin{cases} 1 & \text{if } e \in M \\ 0 & \text{if } e \notin M \end{cases}, y_v^C = \begin{cases} 1 & \text{if } v \in C \\ 0 & \text{if } v \notin C \end{cases}$$

So, $\nu(G) \leq \nu^*(G)$ and $\tau(G) \geq \tau^*(G)$, and by LP duality, we have

$$\nu(G) \leq \nu^*(G) = \tau^*(G) \leq \tau(G)$$

8.6 Perfect Matching

8.6.1 Alternating Trees

Suppose we have a matching M of G and a fixed M-exposed vertex r of G. We can iteratively build up sets A, B of vertices such that each vertex in A is the other end of an odd-length M-alternating path beginning at r, and each vertex in B is the other end of an even-length M-alternating path beginning at r.

Begin with $A = \emptyset, B = \{r\}$, and use the rule: if $vw \in E, v \in B, w \notin A \cup B, wz \in M$, then add w to A, z to B. The set $A \cup B$ and edges in the construction form a tree T rooted at r.

Definition: Alternating Tree

A tree T such that

- every vertex of T other than r is covered by an edge of $M \cap E(T)$;
- for every vertex v of T, the path in T from v to r is M-alternating.

We let the vertex sets at odd and even distances from the root as $A(T)$ and $B(T)$ respectively. Note that $|B(T)| = |A(T)| + 1$ since all other vertices other than r come in matched pairs, one in $A(T)$ and one in $B(T)$.

Using vw to Extend T

Input: A matching M' of a graph G', an M'-alternating tree T, and an edge vw of G' such that $v \in B(T)$, $w \notin V(T)$, and w is M'-covered.
Algorithm: Let wz be the edge in M' covering w (but z is not a vertex of T). Replace T by the tree having edge set $E(T) \cup \{vw, wz\}$.

Use vw to Augment M'

Input: A matching M' of a graph G', an M'-alternating tree T of G' with root r, and an edge vw of G' such that $v \in B(T)$, $w \notin V(T)$, and w is M'-exposed.
Algorithm: Let P be the path obtained by attaching vw to the path from r to v in T. Replace M' by $M' \Delta E(P)$.

Definition: Frustrated

An M-alternating tree T in a graph G is frustrated if every edge of G has one end in $B(T)$ and the other end in $A(T)$.

Proposition

Suppose that G has a matching M and an M-alternating tree T that is frustrated. Then G has no perfect matching.

Proof. Clearly, every element of $B(T)$ is a single-vertex odd component of $G \setminus A(T)$. Since $|A(T)| < |B(T)|$, then G has no perfect matching. $\square$

Algorithm 3 Bipartite Perfect Matching Algorithm

$M = \emptyset$
Choose an M-exposed vertex r
$T = (\{r\}, \emptyset)$
while there exists $vw \in E$ with $v \in B(T), w \notin V(T)$ **do**
  if w is M-exposed **then**
    Use vw to augment M
    if there is no M-exposed vertex in G **then**
      return Perfect matching M
    else
      $T = (\{r\}, \emptyset)$, where r is M-exposed
  else
    Use vw to extend T
return G has no perfect matching

8.6.2 Bipartite Perfect Matching Algorithm

Proposition

Suppose that G is bipartite, M is a matching of G, and T is an M-alternating tree such that no edge of G joins a vertex in $B(T)$ to a vertex not in $V(T)$. Then T is frustrated, and hence G has no perfect matching.

Proof. We show that every edge having an end in $B(T)$ has an end in $A(T)$. From the hypothesis, the only possible exception would be an edge joining two vertices in $B(T)$. But this edge, together with the paths joining them to the root of T, would form a closed path of odd length, which contradicts G being bipartite. Hence T is frustrated, and so by previous proposition, G has no perfect matching. $\square$

8.6.3 Blossom Algorithm for Perfect Matching

Definition: Derived Graph

A graph G' obtained from G by a sequence of odd-circuit shrinkings.

Definition: Original Vertex

A vertex in the derived graph G' that is in G.

Definition: Pseudonode

A vertex in the derived graph G' not in G.

Definition: $S(v)$

Given a vertex v of G', there corresponds a set $S(v)$ of vertices of G, where

$$S(v) = \begin{cases} v & \text{if } v \in V(G) \\ \bigcup_{w \in V(C)} S(w) & \text{if } v = C \text{ is a pseudonode} \end{cases}$$

Proposition

Let G' be a derived graph of G, M' be a matching of G', and T be an M'-alternating tree of G' such that no element of $A(T)$ is a pseudonode. If T is frustrated, then G has no perfect matching.

Proof. When deleting $A(T)$ from G, we get a component with vertex set $S(v)$ for each $v \in B(T)$. Therefore, $\mathrm{oc}(G \setminus A(T)) > |A(T)|$, so G has no perfect matching by Tutte's theorem. □

Definition: Blossom

Let $v, w \in B(T)$ and $vw \in E(G)$. The odd circuit in $T + vw$ is a blossom.

When we shrink a blossom, we get a pseudonode and the new graph is a derived graph.

Use vw to Shrink and Update M' and T

Input: A matching M' of a graph G', an M'-alternating tree T, and an edge vw of G' such that $vw \in B(T)$.
Algorithm: Let C the circuit formed by vw with the vw-path in T. Replace G' with $G' \times C$, M' by $M' \setminus E(C)$, and T by the tree in G' having edge set $E(T) \setminus E(C)$.

Proposition

After application of the shrinking subroutine, M' is a matching of G', T is an M'-alternating tree of G', and $C \in B(T)$.

Algorithm 4 Blossom Algorithm for Perfect Matching

Input: Graph G and matching M of G
$M' = M$
$G' = G$
Choose an M'-exposed vertex r of G'
$T = (\{r\}, \emptyset)$
while there exists $vw \in E'$ with $v \in B(T), w \notin A(T)$ **do**
    Case: w is M'-exposed
        Use vw to augment M'
        Extend M' to a matching M of G
        Replace M' by M, G' by G
        if there is no M'-exposed vertex in G' **then**
            return Perfect matching M'
        else
            $T = (\{r\}, \emptyset)$, where r is M'-exposed
    Case: $w \notin V(T)$, w is M'-covered
        Use vw to extend T
    Case: $w \in B(T)$
        Use vw to shrink and update M' and T
return G', M', T, G has no perfect matching

Theorem

The Blossom Algorithm terminates after $O(n)$ augmentations, $O(n^2)$ shrinking steps, and $O(n^2)$ tree-extension steps.
Moreover, it determines correctly whether G has a perfect matching.

8.7 Blossom Algorithm for Maximum Matching

We can extend the Blossom algorithm for perfect matchings to maximum matchings.

Theorem

The Blossom Algorithm can be implemented to run in time $O(nm \log n)$.

Algorithm 5 Blossom Algorithm for Maximum Matching

1: **Input**: Graph G and matching M of G
2: $M' = M$
3: $G' = G$
4: $\mathcal{T} = \emptyset$
5: Choose an M'-exposed vertex r of G'
6: $T = (\{r\}, \emptyset)$
7: **while** there exists $vw \in E'$ with $v \in B(T), w \notin A(T)$ **do**
8:     **Case**: w is M'-exposed
9:         Use vw to augment M'
10:         Extend M' to a matching M of G
11:         Replace M' by M, G' by G
12:         **if** there is no M'-exposed vertex in G' **then**
13:             **return** Perfect matching M'
14:         **else**
15:             $T = (\{r\}, \emptyset)$, where r is M'-exposed
16:     **Case**: $w \notin V(T)$, w is M'-covered
17:         Use vw to extend T
18:     **Case**: $w \in B(T)$
19:         Use vw to shrink and update M' and T
20: $\mathcal{T} = \mathcal{T} \cup \{T\}$
21: $G' = G \setminus V(T)$
22: $M' = M \setminus E(T)$
23: **if** there exists an M'-exposed vertex **then**
24:     Go to line 5
25: Restore the matching M
26: **return** M

Chapter 9

Weighted Matchings

9.1 Minimum-Weight Perfect Matching

Definition: Minimum-Weight Perfect Matching ILP

$$\begin{aligned} \min \quad & \sum (c_e x_e : e \in E) \\ \text{s.t.} \quad & x(\delta(v)) = 1, \ \forall v \in V \\ & x_e \in \{0, 1\}, \ \forall e \in E \end{aligned}$$

Definition: Minimum-Weight Perfect Matching LP Relaxation

$$\begin{aligned} \min \quad & \sum (c_e x_e : e \in E) \\ \text{s.t.} \quad & x(\delta(v)) = 1, \ \forall v \in V \\ & x_e \geq 0, \ \forall e \in E \end{aligned}$$

Definition: Minimum-Weight Perfect Matching Dual LP

$$\begin{aligned} \max \quad & \sum (y_v : v \in V) \\ \text{s.t.} \quad & y_u + y_v \leq c_e, \ \forall e = uv \in E \end{aligned}$$

Definition: Complementary Slackness Conditions for Minimum-Weight Perfect Matching

If $x_e > 0$, then $\bar{c}_e = c_e - y_u - y_v = 0$ for all $e \in E$.

9.2 Minimum-Weight Perfect Matching in Bipartite Graphs

Theorem (Birkhoff)

Let G be a bipartite graph and let $c \in \mathbb{R}^E$. Then G has a perfect matching if and only if the Minimum-Weight Perfect Matching LP Relaxation has a feasible solution. Moreover, if G has a perfect matching, then the minimum weight of a perfect matching is equal to the optimal value of the LP relaxation.

Definition: $E_=$

$E_= = \{e \in E : \bar{c}_e = 0\}$.

Algorithm 6 Bipartite Minimum-Weight Perfect Matching Algorithm

Let y be a feasible solution to the dual LP
M is a matching of $G_= = (V, E_=)$
$T = (\{r\}, \emptyset)$, where r is an M-exposed vertex of G
while True **do**
    while there exists $vw \in E_=$ with $v \in B(T), w \notin V(T)$ **do**
        if w is M-exposed **then**
            Use vw to augment M
            if there is no M-exposed vertex in G **then**
                return Perfect matching M
            else
                $T = (\{r\}, \emptyset)$, where r is M-exposed
        else
            Use vw to extend T
    if every $vw \in E$ with $v \in B(T)$ has $w \in A(T)$ **then**
        return G has no perfect matching
    else
        $\varepsilon = \min\{\bar{c}_{vw} : v \in B(T), w \notin V(T)\}$
        $y_v = y_v + \varepsilon$ for $v \in B(T)$
        $y_v = y_v - \varepsilon$ for $v \in A(T)$

9.3 Minimum-Weight Perfect Matching in General Graphs

Definition: Odd Cut

A set of the form $\delta(S)$ where S is an odd-cardinality set of vertices.

Definition: Blossom Inequality

If x is the characteristic vector of a perfect matching, then for every odd cut D of G,

$$x(D) \geq 1$$

Let $\mathcal{C}$ denote the set of all odd cuts of G that are *not* of the form $\delta(v)$ for some vertex v.

Definition: Minimum-Weight Perfect Matching LP – Stronger

$$\begin{aligned} \min \quad & \sum(c_e x_e : e \in E) \\ \text{s.t.} \quad & x(\delta(v)) = 1, \ \forall v \in V \\ & x(D) \geq 1, \ \forall D \in \mathcal{C} \\ & x_e \geq 0, \ \forall e \in E \end{aligned}$$

Definition: Minimum-Weight Perfect Matching Dual LP – Stronger

$$\begin{aligned} \max \quad & \sum(y_v : v \in V) + \sum(Y_D : D \in \mathcal{C}) \\ \text{s.t.} \quad & y_v + y_w + \sum(Y_D : e \in D \in \mathcal{C}) \leq c_e, \ \forall e = vw \in E \\ & Y_D \geq 0, \ \forall D \in \mathcal{C} \end{aligned}$$

Theorem

Let G be a graph and let $c \in \mathbb{R}^E$. Then G has a perfect matching if and only if the Minimum-Weight Perfect Matching LP has a feasible solution.
Moreover, if G has a perfect matching, then the minimum weight of a perfect matching is equla to the optimal value of the LP.

Chapter 10

T-Joins

Part VI

Matroids

Part VII

Traveling Salesman Problem