CO 444/644 Algebraic Graph Theory

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Chapter 1

Introduction

We will focus on simple, undirected graphs without loops unless explicitly stated. We use $X = (V, E)$ to denote graphs and G for groups. $V(X)$ and $E(X)$ are the sets of vertices and edges of graph X respectively and $\deg(v)$ to denote the degree of a vertex $v \in V(X)$.

Definition: Isomorphism

An isomorphism between graphs X, Y is a function $f : V(X) \to V(Y)$ such that $uv \in E(X)$ if and only if $f(u)f(v) \in E(Y)$.

1.1 Automorphisms

Definition: Automorphism

An automorphism of the graph X is an isomorphism $f : X \to X$.

$\mathrm{Aut}(X)$ is the set of all automorphisms of X.

$\mathrm{Sym}(V)$ is used to denote the symmetric group of permutations on V. In group theory, we may have used $V = [n]$. We may use this notation alongside $\mathrm{Sym}(n)$ when explicitly enumerating the vertices of a graph from 1 to n.

Proposition

$\mathrm{Aut}(X) \leq \mathrm{Sym}(V(X))$ with the group operation for $\sigma, \tau \in \mathrm{Aut}(X)$ defined $\sigma\tau := \tau \circ \sigma$.

For $g \in \mathrm{Sym}(V(X))$ and $v \in V(X)$, let v^g denote $g(v)$. Let S^g denote $\{g(v) : v \in S\}$ for set S.

Suppose $Y \subseteq X$ is a subgraph and $g \in \mathrm{Aut}(X)$. Y^g is the graph defined $V(Y^g) = V(Y)^g$ and $E(Y^g) = \{u^g v^g : uv \in E(Y)\}$.

E.g. The following is an example of graphs X and Y along with functions that are and are not automorphisms.

Let $X = (\{1,2,3,4\}, \{12,13,14,23,34\}), Y = (\{1,2,3\}, \{12,13,23\}), Y^g = (\{1,3,4\}, \{13,14,34\})$ where $g(1) = 3, g(2) = 4, g(3) = 1, g(4) = 2$. $f(1) = 1, f(2) = 4, f(3) = 3, f(4) = 2$ is an automorphism while Y^g where $f(1) = 3, f(2) = 3, f(3) = 4, f(4) = 1$ is not an automorphism.

Lemma

For $v \in V(X)$ and $g \in \mathrm{Aut}(X)$, $\deg(v) = \deg(v^g)$.

Proof. Let $Y(v)$ be the subgraph of X induced by v and the neighbors $N_X(v)$. Then

$$Y(v) \cong Y(v)^g = Y(v^g)$$

so $\deg(v) = \deg(v^g)$.

Lemma

Let $u, v \in V(X)$ and $g \in \mathrm{Aut}(X)$, then the length of the shortest paths are preserved, i.e. $d(u,v) = d(u^g, v^g)$.

Proof. Show that a shortest uv-path in X is mapped to a shortest $u^g v^g$-path by g.

1.2 Homomorphisms

Definition: Homomorphism

Let X and Y be graphs. We say $f : V(X) \to V(Y)$ is a homomorphism if $x \sim y$ implies $f(x) \sim f(y)$ in Y.

$\sim$ is for adjacency and $f : X \to Y$ instead of $f : V(X) \to V(Y)$ for simplicity.

Let $\chi(X)$ denote the chromatic number of X, the minimum number of colors needed to color the vertices of a graph such that no two adjacent vertices have the same color.

Let K_r denote the complete graph on r vertices where every pair of distinct vertices is connected by an edge. We say that K_r is a clique, where it generally is a subset of vertices of a graph such that every two distinct vertices in the clique are adjacent.

Lemma

$$\chi(X) = \min\{r \in \mathbb{N} : \exists \text{ homomorphism } X \to K_r\}$$

Proof. Let $k = \chi(X)$. We first show $k \geq \min\{r \in \mathbb{N} : \exists \text{ homomorphism } X \to K_r\}$. Let f be a k-coloring of X. Then f is a homomorphism from X to K_k.

Next, we show that $k \leq \min\{r \in \mathbb{N} : \exists \text{ homomorphism } X \to K_r\}$. Let $\overline{r} = \min\{r \in \mathbb{N} : \exists \text{ homomorphism } X \to K_r\}$. Let $h : X \to K_{\overline{r}}$ be a homomorphism. Then $h^{-1}(v)$ is an independent set. So, giving $h^{-1}(v)$ distinct colors yields an $\overline{r}$-coloring.

Definition: Retraction

A homomorphism $f : X \to Y$ such that

1. $Y \subseteq X$.
2. $f|_Y = id$, the identity map.

If a retraction from X to Y exists, we call Y a retract of X.

We use the notation $f|_Y$ to mean the function f when restricted to the domain of Y.

E.g. Suppose $K_r \cong Y \subseteq X$ and $\chi(X) = r$. We will prove that Y is a retract of X. The proof is as follows: let $f : V(X) \to [r]$ where $r = \chi(X)$ be an r-coloring of X. Then, Y receives distinct colors since $Y \cong K_r$. Without loss of generality, assume $V(Y) = [r]$. Then f is a homomorphism from X to K_r and $f|_Y = id$. Therefore, f is a retraction.

E.g. Recall that a cycle graph C_n is defined $V(C_n) = \{0, \dots, n-1\}$ where $n \geq 3$ and $E(C_n) = \{ij : i - j \equiv \pm 1 \pmod{n}\}$. Let $g = (1, 2, \dots, n-1, 0) \in \mathrm{Aut}(C_n)$. This can be viewed as a rotation. We can define the subgroup

$$R = \{g^m : 0 \leq m \leq n-1\} \leq \mathrm{Aut}(C_n)$$

to capture all possible rotations.

We can define reflections. Let h be defined $h(i) = -i \pmod{n} \in \mathrm{Aut}(C_n)$. We can see that R and Rh are disjoint cosets of $\mathrm{Aut}(C_n)$ and $Rh \leq \mathrm{Aut}(C_n)$. It follows that $|\mathrm{Aut}(C_n)| \geq 2n$.

Definition: Circulant Graph

Let $\mathbb{Z}_n = \{0, \dots, n-1\}$ and $C \subseteq \mathbb{Z}_n \setminus \{0\}$ be closed under inverse, that is, $x \in C \implies -x \in C$. We define the circulant graph $X = X(\mathbb{Z}_n, C)$ where $V(X) = \mathbb{Z}_n, E(X) = \{ij : i - j \in C\}$.

One can show that the arguments from the previous example for C_n also hold for $X = X(\mathbb{Z}_n, C)$. That is, $|\mathrm{Aut}(X(\mathbb{Z}_n, C))| \geq 2n$. We can generalize this result for arbitrary groups using Cayley graphs.

Definition: Johnson Graph

Given $v \geq k \geq i$ as integers where $[v] = \{1, \dots, v\}$, the Johnson graph $J = J(v, k, i)$ is defined $V(J) = \{S \subseteq [v] : |S| = k\}, E(J) = \{ST : |S \cap T| = i\}$.

$J(5, 2, 0)$ is the Peterson graph. $J(v, k, 0)$ is the Kneser graph.

Proposition

There exists a subgroup of $\mathrm{Aut}(J(v, k, i))$ that is isomorphic to $\mathrm{Sym}(v)$.

Proof. For $g \in \mathrm{Sym}(v)$, define $\tau_g : \binom{[v]}{k} \to \binom{[v]}{k}$ as $\tau(S) = S^g$. Note that $|S \cap T| = |S^g \cap T^g|$ for vertices $S, T \in J(v, k, i)$ since we are essentially just relabeling elements of S and T, so

$\tau_g \in \mathrm{Aut}(J(v,k,i))$. We can conclude that

$$\{\tau_g : g \in \mathrm{Sym}(v)\} \cong \mathrm{Sym}(v)$$

Chapter 2

Groups

2.1 Group Actions

Definition: Homomorphism

Given groups G and H, $f : G \to H$ is a homomorphism if for all $g, h \in G$,

$$f(gh) = f(g)f(h)$$

Definition: Kernel

The kernel of a homomorphism f is defined

$$\ker(f) = f^{-1}(1)$$

Definition: Group Action

Suppose G is a group and V is a set. A homomorphism $f : G \to \mathrm{Sym}(V)$ is a permutation representation of G and we call it an action of G on V.

E.g. Let X be a graph and take $V = V(X)$. Let $G = \mathrm{Aut}(X)$. Then $f : G \to \mathrm{Sym}(V)$ defined $f(g) = g$ for $g \in G$ is an action.

E.g. Let G be a group. Let $f : G \to \mathrm{Sym}(V)$ where V is arbitrary be defined $f(g) = id$ where id is the identity permutation. f is an action.

Definition: Faithful Action

The action f is faithful if $\ker(f) = \{1\}$.

We can see that the first action example above is faithful, but not the second.

Let group G act on V, via $f : G \to \mathrm{Sym}(V)$. Let $g \in G$, we use the notation

$$x^g := g^{f(g)} \text{ and } S^G := S^{f(g)}$$

where S is an arbitrary set.

Definition: G-Invariant

Let group G act on V and $g \in G$. S is G-invariant if $S = S^g$ for all $g \in G$.

Definition: Orbit

Let group G act on V. The orbit of $x \in V$ is

$$x^G := \{x^g : g \in G\}$$

One may show that V is partitioned into disjoint orbits and each orbit is G-invariant and transitive (for every x, y in the same orbit, there exists $g \in G$ where $x^g = y$).

Definition: Stabilizer

Let $G \leq \mathrm{Sym}(V)$ and $x \in V$. The stabilizer of x is

$$G_x := \{g \in G : x^g = x\}$$

Lemma

Let $G \leq \mathrm{Sym}(V)$ and $x \in V$, then $G_x \leq G$.

Lemma

Let $G \leq \mathrm{Sym}(V)$ and let S be an orbit of G. Let $x, y \in S$, then

$$H := \{h \in G : x^h = y\}$$

is a right coset of G_x. Conversely, if H is a right coset of G_x, then for all $h, h' \in H$, $x^h = x^{h'}$.

Proof. ($\implies$) G is transitive on S, so there exists $g \in G$ where $x^g = y$. For any $h \in H$, $x^h = y$ by the definition of H. So, $x^h = x^g$. Then,

$$x^{hg^{-1}} = x \implies hg^{-1} \in G_x \implies h \in G_x g$$

($\impliedby$) Assume $H = G_x g$ for some $g \in G$. Let $h, h' \in H$ where $h = \sigma g$ and $h' = \sigma' g$ for some $\sigma, \sigma' \in G_x$. We have

$$x^h = x^{\sigma g} = x^g = x^{\sigma' g} = x^{h'}$$

Lemma (Orbit-Stabilizer)

Let G be a permutation group acting on V with $x \in V$. Then

$$|G_x| \left|x^G\right| = |G|$$

Proof. Let $\mathcal{H}$ be the set of right cosets of G_x and define $f : x^G \to \mathcal{H}$ as

$$f(y) = \{g \in G : x^g = y\}$$

The previous lemma shows that f is a bijection. Therefore, $|\mathcal{H}| = \left|x^G\right|$. Since the right cosets of G_x partition G, we have

$$|G| = |G_x|\,|\mathcal{H}| = |G_x|\left|x^G\right|$$

Definition: Conjugate

Let G be a permutation group and let $g, h \in G$. g is conjugate to h if there is some $\sigma \in G$ such that

$$g = \sigma h \sigma^{-1}$$

Proposition

If H is a subgroup of G and $g \in G$, then $gHg^{-1} \leq G$ and $gHg^{-1} \cong H$.

Lemma

If $y \in x^G$, then G_x and G_y are conjugate.

Proof. Suppose $y = x^g$ where $g \in G$. We will prove that $g^{-1}G_x g = G_y$.

($\subseteq$) Note that $y^{g^{-1}} = x$. For every $h \in G_x$, $y^{g^{-1}hg} = x^{hg} = g^g = y$.

($\supseteq$) For $h \in G_y$, $x^{ghg^{-1}} = y^{hg^{-1}} = y^{g^{-1}} = x$. Then $ghg^{-1} \in G_x$, rearranging gives $h \in g^{-1}G_x g$.

Definition: Fix

Let $G \leq \mathrm{Sym}(V)$ and $g \in G$. Then

$$\mathrm{fix}(g) = \{v \in V : v^g = v\}$$

2.2 Burnside Lemma

Lemma (Burnside)

Let $G \leq \mathrm{Sym}(V)$. Then

$$\#\text{ of orbits of } G = \frac{1}{|G|}\sum_{g \in G} |\mathrm{fix}(g)|$$

Proof. Let $\Lambda = \{(g, x) : g \in G, x \in V, x \in \mathrm{fix}(g)\}$. We will apply a double-counting argument. Observe that

$$|\Lambda| = \sum_{g \in G} |\mathrm{fix}(g)| = \sum_{x \in V} |G_x|$$

Equating these gives

$$\begin{aligned}\sum_{g\in G}|\mathrm{fix}(g)| &= \sum_{x\in V}|G_x| \\ &= \sum_{x\in V}\frac{|G|}{|x^G|} && \text{(Orbit-Stabilizer)} \\ &= |G|\sum_{x\in V}\frac{1}{|x^G|} \\ &= |G|\,(\#\text{ of orbits of } G)\end{aligned}$$

Definition: Asymmetric Graph

A graph X is asymmetric if $\mathrm{Aut}(X) = \{id\}$.

Theorem

Let $\mathcal{G}_n = \{X \text{ on } [n]\}$ and $X \in \mathcal{G}_n$ be chosen uniformly random, then

$$\lim_{n\to\infty} \Pr(X \text{ is asymmetric}) = 1$$

Proof. Let $X \in \mathcal{G}_n$, $\mathrm{Iso}(X) = \{Y \in \mathcal{G}_n : X \cong Y\}$.

Lemma: $|\mathrm{Iso}(X)| = \frac{n!}{|\mathrm{Aut}(X)|}$.
Proof. (Lemma) Let $G = \mathrm{Sym}([n])$. For $g \in G$, let $\tau_g : \mathcal{G}_n \to \mathcal{G}_n$ where $X \mapsto X^g$. Let $H := \{\tau_g : g \in G\}$ acts on $\mathcal{G}_n$ and $H \cong G$.

$$n! = |G| = |H| = |H_X| \cdot \left|X^H\right| = |\mathrm{Aut}(X)|\,|\mathrm{Iso}(X)|$$

Let $\mathcal{H}$ be the set of isomorphism classes of graph on $[n]$. Let $\mathcal{H} \in \mathcal{H}$. If $X \in \mathcal{C}$ is asymmetric, then $|\mathcal{C}| = n!$. If X is symmetric, then $|\mathcal{C}| \leq \frac{n!}{2}$.

Let ρ be the proportion of $\mathcal{C} \in \mathcal{H}$ such that $|\mathcal{C}| = n!$. Now,

$$2^{\binom{n}{2}} = |\mathcal{G}_n| = \sum_{\mathcal{C}\in\mathcal{H}} |\mathcal{C}| \leq \rho\,|\mathcal{H}|\,n! + (1-\rho)\,|\mathcal{H}|\,\frac{n!}{2}$$

Claim: $|\mathcal{H}| = (1+o(1))\frac{2^{\binom{n}{2}}}{n!}$, where $o(1)$ denotes some $x_n \in \mathbb{R}$ such that $\lim_{n\to\infty} x_n = 0$.

By claim, $2^{\binom{n}{2}} \leq (1+o(1))\frac{2^{\binom{n}{2}}}{n!} \cdot n!\left(\rho + \frac{1-\rho}{2}\right) = (1+o(1))2^{\binom{n}{2}} \cdot \frac{1+\rho}{2}$.

Thus, $\rho = 1 + o(1)$. Then the proportion of asymmetric graphs in $\mathcal{G}_n$ is $\rho\,|\mathcal{H}|\,n!/2^{\binom{n}{2}} = 1 + o(1)$.

Proof. (Claim) Consider $\mathcal{P} = \{\tau_g : g \in \mathrm{Sym}([n])\}$ acting on $\mathcal{G}_n$ where $\tau_g(x) = x^g$. The set of orbits is $\mathcal{H}$. Burnside's Lemma tells us $|\mathcal{H}| = \frac{1}{|G|}\sum_{g\in G}|\mathrm{fix}(\tau_g)|$.

Observation: Every g induces a permutation G_g on $E(K_{[n]})$. Let C be an orbit under σ_g. Then, if X is fixed by τ_g, then X either contains all edges in C or no edges in C.

Let $\text{orb}_2(\sigma_g)$ be the number of orbits under σ_g. Thus, $|\text{fix}(\tau_g)| = 2^{\text{orb}_2(\sigma_g)}$. If $g = id$, then $\text{orb}_2(\sigma_g) = \binom{n}{2}$. If $g = (i,j)$ for some $i, j \in [n]$, $\text{orb}_2(g) = (n-2) + \binom{n}{2} - 2(n-2)$.

The contribution to Burnside's Lemma from a simple transposition is $\binom{n}{2} 2^{\binom{n}{2}-(n-2)} = 2^{\binom{n}{2}} \cdot \binom{n}{2} 2^{-(n-2)}$. With some technical work we skip, we can show that $\sum_{\substack{g \in G \\ g \neq id}} |\text{fix}(\tau_g)| = o(1) \cdot |\text{fix}(\tau_{id})|$

$$\frac{1}{n!} |\text{fix}(\tau_{id})| \leq |\mathcal{H}| = \frac{1}{n!}(1+o(1)) |\text{fix}(\tau_{id})| \implies |\mathcal{H}| = (1+o(1)) \frac{2^{\binom{n}{2}}}{n!}$$

2.3 Block of Imprimitivity

Definition: Block of Imprimitivity

Let G be a transitive permutation group on V and $S \subseteq V$. S is a block of imprimitivity for G if $S \neq \emptyset$ and $\forall g \in G$, $S^g = S$ or $S^g \cap S = \emptyset$.

$S = \{u\}$ for all $u \in V$ and $S = V$ are trivial blocks of imprimitivity.

Definition: Primitive

G is primitive if there does not exist non-trivial blocks of imprimitivity. Otherwise, G is imprimitive.

Remark: We assume transitivity since if G has an orbit $S = x^G$ such that $|S| \geq 2, S \neq V$, then S is a block of imprimitivity.

E.g. If $G = \text{Aut}(K_n)$, G is primitive.

E.g. Let $G = \text{Aut}(C_4)$, G is not primitive.

E.g. Let $G = \text{Aut}(C_{2n})$

Lemma

Let G be a transitive permutation group on V. Let $x \in V$. Then, G is primitive if and only if G_x is a maximal subgroup of G (no K such that $G_x < K < G$).

Proof. We prove G is imprimitive if and only if there exists K such that $G_x < K < G$.

($\implies$) Let S be a block of imprimitivity with $2 \leq |S| < |V|$. With loss of generality, we may assume that $x \in S$ since G is transitive. Let $G_S = \{g \in G : S^g = S\}$ which is a subgroup of G. We prove that $G_x < G_S$.

Let $g \in G_x$. Then $x \in S \cap S^g$, so $S^g = S$ (by definition of block of imprimitivity. Since $|S| \geq 2$, $\exists y \in S, y \neq x$. Let $h \in G$ such that $x^h = y$, this implies $h \notin G_x$. Then, $y \in S \cap S^h \implies S = S^h \implies h \in G_S$. These two points give us $G_x < G_S$. $G_S < G$ since $S = S^g$ for all $g \in G_S$ but G is transitive.

($\Longleftarrow$) Suppose there exists K with $G_x < K < G$. Let $S = x^K$. $2 \le |S| < |V|$ (assignment).

Claim: For all $g \in G$, if $S \cap S^g \neq \emptyset$, then $g \in K$ and $S = S^g$.
Proof. (Claim) Assume $y \in S \cap S^g$. $y \in S \implies \exists h \in K : y = x^h$. $y \in S^g \implies \exists h' \in K : y = x^{h'g}$. Combining, we get $x = x^{h'gh^{-1}} \implies h'gh^{-1} \in G_x < K \implies g \in (h')^{-1}Kh \in K$.

E.g. Consider K_3 and the vertex 1. $G_1 = \{id, (1)(23)\}, G = \text{Aut}(K_3)$. There is no bigger subgroup, so G_1 is maximal.

E.g. Consider C_4 and 1. $G_1 = \{id, (1)(3)(24)\}, K = \{id, (1)(3)(24), (13)(24), (13)(2)(4)\}$. Here $G_1 < K < \text{Aut}(C_4)$. We constructed $K = \{g \in \text{Aut}(C_4) : \{1, 3\}^g = \{1, 3\}\}$.

Chapter 3

Transitive Graphs

3.1 Vertex-Transitive Graphs

Definition: Vertex-Transitive Graphs

X is vertex-transitive if $\mathrm{Aut}(X)$ acts transitively on $V(X)$.

Definition: k-Cube Q_k

$V(Q_k) = 2^{[k]}$, $E(Q_k) = \{ij : H(i,j) = 1\}$ where H is the Hamming distance (positions where the binary string is different).

Lemma

Q_k is vertex-transitive.

Proof. For all $v \in 2^{[k]}$, define $\rho_v : 2^{[k]} \to 2^{[k]}$ such that $x \mapsto x + v$. Since $H(x,y) = H(x+v, y+v)$, $\rho_v \in \mathrm{Aut}(Q_k)$. So $\{\rho_v : v \in 2^{[k]}\} \leq \mathrm{Aut}(Q_k)$, which acts transitively on $V(Q_k)$.

Proof. For all $v \in \mathrm{Sym}([k])$, define $\tau_v : 2^{[k]} \to 2^{[k]}$, $S \mapsto S^v$. Since $H(x,y) = H(\tau_v(x), \tau_v(y))$, $\{\tau_v : v \in \mathrm{Sym}([k])\} \leq \mathrm{Aut}(Q_k)$.

Note $\{\rho_v : v \in 2^{[k]}\} \cap \{\tau_v : v \in \mathrm{Sym}([k])\} = \{id\}$. $\{\rho_v : v \in 2^{[k]}\}\{\tau_v : v \in \mathrm{Sym}([k])\} \subseteq \mathrm{Aut}(Q_k)$ and $\left|\{\rho_v : v \in 2^{[k]}\}\{\tau_v : v \in \mathrm{Sym}([k])\}\right| = \frac{\left|\{\rho_v : v \in 2^{[k]}\}\right|\left|\{\tau_v : v \in \mathrm{Sym}([k])\}\right|}{\left|\{\rho_v : v \in 2^{[k]}\} \cap \{\tau_v : v \in \mathrm{Sym}([k])\}\right|} = 2^k k!$.

Remark: Cycles and Circulant graphs are vertex-transitive.

Definition: Cayley Graph

Given group G and $C \subseteq G$ satisfying

- $1 \notin C$
- $x \in C \implies x^{-1} \in C$

Then $X = X(G, C)$ such that $V(X) = G$ and $E(X) = \{gh : hg^{-1} \in C\} = \{gh : gh^{-1} \in C\}$.

Lemma

Cayley graphs are vertex-transitive.

Proof. For any $v \in G$, define $\rho_v : G \to G, x \mapsto xv$. $xy \in E(X(G, C)) \Leftrightarrow xy^{-1} \in C \Leftrightarrow (xv)(yv)^{-1} \in C \Leftrightarrow \{\rho_v(x), \rho_v(y)\} \in E(X(G, C))$.

Lemma

Johnson graphs are vertex-transitive.

3.2 Edge-Transitive Graphs

A group acting on V naturally induces an action on

$$\binom{V}{2} \& (V)_2 = \{ij \in V^2 : i \neq j\}$$

by $\{u, v\}^g := \{u^g, v^g\}$ and $(u, v)^g = (u^g, v^g)$.

Definition: Edge-Transitive Graph

X is edge-transitive if $\mathrm{Aut}(X)$ acts transitively on $E(X)$.

Definition: Arc-Transitive Graph

X is arc-transitive if $\mathrm{Aut}(X)$ acts transitively on $\{ij : ij \in E(X)\}$

Proposition

Arc-transitive $\implies$ vertex-transitive and edge-transitive.

Proposition

There exist graphs that are edge-transitive, but not vertex-transitive.

Proposition

There exist graphs vertex-transitive, but not edge-transitive.

Theorem

Edge-transitive graphs that are not vertex-transitive with no isolated vertices are bipartite.

Proof. Without loss of generality, we may assume that X has no isolated vertices.

2-orbits: Let $xy \in E(X)$. For $w \in V(X)$, $wz \in E(X)$ for some $z \in V(X)$. There exists $\sigma \in \text{Aut}(X)$, $\{x^\sigma, y^\sigma\} = \{w, z\}$. This implies every vertex in X is either in x^G or y^G. However, X is not vertex-transitive, $x^G \neq y^G$, this gives the bipartition.

If $wz \in E(X)$ and $wz \in x^G$ (or $wz \in y^G$), this implies no $\sigma \in \text{Aut}(X)$ would map xy to wz since $x^G \cap y^G = \emptyset$.

Theorem

If X is vertex, edge-transitive, k-regular, k-odd, then X is arc-transitive.

Lemma

If X is a vertex, edge-transitive, k-regular, not arc-transitive, then k is even.

Proof. Define $D(X)$ with $V(D(X)) = V(X)$ and $E(D(X)) = \{(x, y) : xy \in E(X)\}$. Let $xy \in E(X), \Omega_1 = (x, y)^G, \Omega_2 = (y, x)^G, G = \text{Aut}(X)$. X is edge-transitive implies $\Omega_1 \cup \Omega_2 = E(D(X))$. X is not arc-transitive implies $\Omega_1 \cap \Omega_2 = \emptyset$.

Thus, $\forall uv \in E(X)$, $(u, v) \in \Omega_1 \implies (v, u) \in \Omega_2$. $\text{Aut}(X) = \text{Aut}(\Omega_1)$ which acts transitively on $V(D(X)) = V(\Omega_1)$, so $d^+_{\Omega_1} = d^-_{\Omega_1} = d^+_{\Omega_2} = d^+_{\Omega_2}$ where $+$ means in-degree and $-$ means out-degree. Therefore, $k = d^+_{\Omega_1} + d^-_{\Omega_1} \equiv 0 \pmod 2$.

3.3 Edge-Connectivity

Definition: Edge Atom

An edge atom of X is a minimum $S \subseteq V(X)$ such that $|\delta(S)| = \kappa_1(X)$.

In this course $\partial(S) = \delta(S)$.

Lemma

Any two distinct edge atoms are disjoint.

Proof. Let $\kappa = \kappa_1(X)$. Let A, B be distinct edge atoms. By minimality, $|A|, |B| \leq \frac{|V(X)|}{2}$. Suppose $A \cap B \neq \emptyset$:

Case 1: $A \cup B = V(X)$, then $|A| = |B| = \frac{|V(X)|}{2}$ implies $A \cap B = \emptyset$, a contradiction.

Case 2: $A \cup B \subsetneq V(X)$, then $|\partial(A \cup B)| \geq \kappa, |\partial(A \cap B)| \geq \kappa + 1$.

$$\kappa + \kappa + 1 \leq |\partial(A \cup B)| + |\partial(A \cap B)| \leq |\partial(A)| + |\partial(B)| = 2\kappa$$

This is a contradiction.

Lemma

Suppose S is a block of imprimitivity under $\mathrm{Aut}(X)$, then $X[S]$ is regular.

Proof. Let $u, v \in S, u \neq v$. Let $Y = X[S]$. X is vertex-transitive by assumption, this implies $\exists g \in \mathrm{Aut}(X)$, $u^g = v \implies S = S^g$. Hence, $\{g|_S : g \in \mathrm{Aut}(X)\} \subseteq \mathrm{Aut}(Y)$. $\deg_Y(u) = \deg_Y(u^g) = \deg_Y(v)$ since automorphism preserves degree.

Theorem

If X is connected, k-regular, and vertex-transitive, then $\kappa_1(X) = k$.

Proof. Obviously, $\kappa_1(X) \leq k$. For $\kappa_1(X) \geq k$, let S be an edge atom. Let $g \in \mathrm{Aut}(X)$ and $B = S^g$. Then by the first lemma, either $S = B$ or $S \cap B = \emptyset$. So, S is a block of imprimitivity.

The second lemma implies $X[S]$ is ℓ-regular for some $0 \leq \ell \leq k-1$ because X is connected. Thus, $|\partial(S)| = |S|\,(k - \ell)$ such that $|S| \geq \ell + 1$. $|\partial(S)| \geq k$ (proof omitted).

This is $|\partial(S)| = k$ when $|S| = 1, \ell = 0$ or $|S| = k, \ell = k - 1$.

Theorem

If X is connected and vertex-transitive, then

(a) X has a matching missing ≤ 1 vertex.

(b) Every edge in X is contained in a maximum matching.

Proof. **(a)** A vertex is critical if it is saturated by every maximum matching.

Case 1: There exists a critical vertex.
Every vertex is critical by vertex-transitivity, so X has a perfect matching.

Case 2: No critical vertex.
We prove $\forall u, v$, a maximum matching misses at most one of them by induction on $\ell = d(u, v)$.

Base case: $\ell = 1$, this is trivially true.

Assume $\ell \geq 2$. Inductive hypothesis applies to (x, y) where $d(x, y) \leq \ell - 1$. Take uv-path P with $|P| = \ell \geq 2$. There exists $x \notin \{u, v\}$ on P. x is not critical means there exists a maximum matching M_x missing x. The inductive hypothesis applies (u, x) and (v, x) implies M_x saturates u and v.

Suppose on the contrary, there exists a maximum matching M that misses both u and v. There exists an alternating ux-path and vx-path in $M \Delta M_x$ by claim (below). $u = v$, a contradiction.

Claim: Suppose (z, w) is a pair of vertices such that a maximum matching cannot miss both of them. Then $M_z \Delta M_w$ must contain an alternating zw-path.
Proof. (Claim) Suppose on the contrary that z and w lies in distinct components of $M_z \Delta M_w$. $M := M_w \Delta P$ is a maximum matching missing both z, w, a contradiction.

(b) By strong induction on number of vertices and number of edges.

Base case: Empty graph, this is trivial.

Inductive hypothesis: Suppose on the contrary that $\exists e \in E(X)$ that e is not in any maximum matching of X. This implies X is not edge-transitive.

Let Y be the subgraph of X induced by $e^{\mathrm{Aut}(X)}$. Y is vertex, edge-transitive, so $Y \neq X$. Inductive hypothesis applies to every component of Y.

Case 1: Y is connected.
By part (a) and that Y is vertex, edge-transitive, e is contained in a maximum matching of Y (which is a maximum matching of X).

Case 2: Y contains multiple components C_i.
Claim: $V(C_i)$ is a block of imprimitivity under $\mathrm{Aut}(X)$. $C_i \cong C_j$ for all $i, j \in [m]$.

Inductive hypothesis applies to each C_i. Case 2(a): each C_i has a perfect matching, this contradicts case 1. Case 2(b): each C_i has a matching missing 1 vertex.

Define Z where $V(Z) = \{C_1, \ldots, C_m\}, E(Z) = \{C_i C_j : \exists exy \in E(X), x \in C_i, y \in C_j\}$. It is easy to show that Z is connected and vertex-transitive. Part (a) implies Z has a matching missing ≤ 1 vertex. We have found a maximum matching of X containing e. A contradiction.

3.4 Cayley Graphs

Definition: Regular Group

A permutation group acting on V is regular if

- $G_x = \{1\}$ for all $x \in V$ (semi-regular)
- G is transitive.

Proposition

If G acts on V is regular, then $|G| = |V|$.

Proof. $|G| = |G_x| \left|x^G\right| = 1 \cdot \left|x^G\right| = |V|$.

Theorem

Let G be a group and $C \subseteq G \setminus \{1\}$ inverse-closed. Then, $\mathrm{Aut}(X(G,C))$ contains a regular subgroup isomorphic to G.

Proof. **(a)** Let $X = X(G,C)$. Define $\tau_g : V(X) \to V(X), \sigma \to \sigma g$ for all $\sigma \in V(X) = G$.

- $\{\tau_g : g \in G\} \leq \mathrm{Aut}(X)$.
- $\{\tau_g : g \in G\}$ acts transitively on G.
- $\{\tau_g : g \in G\} \cong G$.
- $\{\tau_g : g \in G\}$ is semi-regular.

Theorem

Suppose X is a graph. If $G \leq \mathrm{Aut}(X)$ acts regularly on $V(X)$, then $X \cong X(G,C)$ for some inverse-closed $C \subseteq G \setminus \{1\}$.

Proof. G is regular, so $|G| = |V(X)|$. Fix $u \in V(X)$. $\exists$ a unique $g \in G$ such that $u^g = v$ for all $v \in V(X)$. Call this g as g_v. Let $C = \{g_v : v \sim u\}$.

First $1 \notin C$, $u \nsim u$. Next, we prove $X \cong X(G,C)$ by isomorphism $f(x) = g_x, \forall x \in V(X)$. $xy \in E(X)$ if and only if $\{x^{g_x^{-1}}, y^{g_x^{-1}}\} \in E(X)$ if and only if $\{u, u^{g_y g_x^{-1}}\} \in E(X)$ if and only if $g_y g_x^{-1} \in C$ since $u^{g_x} = x, u^{g_y} = y, g_x, y_y \in G \leq \mathrm{Aut}(X)$.

By symmetric proof using g_y^{-1}, we obtain $xy \in E(X)$ if and only if $\{u, u^{g_x g_y^{-1}}\} \in E(X)$ if and only if $g_x g_y^{-1} \in C$, so C is inverse-closed.

Theorem

(a) If $\theta : G \to G$ is an automorphism, then $X(G,C) \cong X(G,\theta(C))$ and $C \subseteq G \setminus \{1\}$ is inverse-closed.

(b) $\exists (G, C_1, C_2)$ such that $X(G,C_1) \cong X(G,C_2)$, but there is no automorphism θ on G such that $C_2 = \theta(C_1)$.

Proof. **(a)** We prove that $\theta : V(X) \to V(X)$, $X = X(G,C)$ is an isomorphism.

$$\begin{aligned} hg^{-1} \in C &\Leftrightarrow \theta(hg^{-1}) \in \theta(C) \\ &\Leftrightarrow \theta(h)\theta(g)^{-1} \in \theta(C) \\ &\Leftrightarrow \theta(h)\theta(g^{-1}) \in \theta(C) \end{aligned}$$

Definition: Generating Set

Let G be a group. We say a subset $C \subseteq G$ be generating for G if every element in G can be expressed as a product of elements in C.

Proposition

$X(G,C)$ if connected if and only if C is generating for G.

Theorem

Every connected vertex-transitive graph is isomorphic to a retract of a Cayley graph.

Proof. Let $x \in V(X)$, $C = \{g \in \mathrm{Aut}(X) : x^g \sim x\}$, and G be the subgroup of $\mathrm{Aut}(X)$ that is generated by C. G acts transitively on $V(X)$. Let $Y = X(G,C)$. For every $y \in V(X)$, let $C_y := \{g \in G : x^g = y\}$. C_y is a right coset of G_x. $C = \bigcup_{y \sim x} C_y, C \cap G_x = \emptyset$ since $x \nsim x$.

Moreover, for any $a, b \in G$, $x^a \sim x^b \Leftrightarrow x \sim x^{ba^{-1}} \Leftrightarrow ba^{-1} \in C$.

Claim 1: $C = GxCG_x$.

Let $A_1, \ldots, A_k$ be the set of right cosets of G_x. Let $a_1 \in A_1, \ldots, a_k \in A_k$.
Claim 2: In $Y = X(G,C)$, $\forall 1 \leq i < j \leq k$, $e(A_i, A_j) = 0$ or $e(A_i, A_j) = |A_i|\,|A_j|$. Moreover, $\forall 1 \leq i \leq k$, $e(A_i) = 0$.

Claim 3: $Y[a_1, \ldots, a_k] \cong X$.

Claim 4: $Y[a_1, \ldots, a_k]$ is a retract of Y.

Proof. (Claim 1) $\subseteq$ is obvious. ($\supseteq$) Let $h, h' \in G_x$ and $g \in C$. Then $x \sim x^g$. Since $x^h = x = x^{h'} \implies x = x^h \sim x^{gh} = x^{h'gh}$. So we know that $h'gh \in C \implies G_xCG_x \subseteq C$.

Proof. (Claim 2) For any $g' \in G$, $g' \in A_j$ for some j. $G' = ga_j$ for some $g \in G_x$. Suppose $g, h \in G_x$, then $ga_i \sim ha_j \Leftrightarrow ga_i(ha_j)^{-1} \in C \Leftrightarrow ga_ia_j^{-1}h^{-1} \in C \Leftrightarrow a_ia_j^{-1} \in g^{-1}Ch \in G_xCG_x = C$ by claim 1.

Statement 2: Is implied immediately by $1 \notin C$ since $a_i = a_j$ in this case and $a_ia_i^{-1} = 1 \notin C$.

Proof. (Claim 3) As shown in claim 2, $\forall 1 \leq i < j \leq k$, $a_i \sim a_j$ in $Y[a_1, \ldots, a_j]$ if and only if $a_ia_j^{-1} \in C$.

Let $\rho : V(X) \to \{a_1, \ldots, a_k\}, y \mapsto a_j$ where $a_j \in C_y$. Verify that ρ is an isomorphism.

Proof. (Claim 4) Let $\tau : V(Y) \to \{a_1, \ldots, a_k\}, g \mapsto a_j$ if $g \in A_j$. Claim 2 implies τ is a homomorphism, $\tau|_{\{a_1, \ldots, a_k\}} = id$.

Chapter 4

Generalized Polygons

4.1 Incidence Graphs

Definition: Incidence Structure

Given a set $\mathcal{P}$ of points and a set $\mathcal{L}$ of lines, and incidence relation $I \subseteq \mathcal{P} \times \mathcal{L}$. If $(p, L) \in I$, then the point p is in line L. The triple $\mathcal{I} = (\mathcal{P}, \mathcal{L}, I)$ defines an incidence structure.

Definition: Dual Incidence Structure

The triple $\mathcal{I}^* = (\mathcal{L}, \mathcal{P}, I^*)$ where

$$I^* = \{(L, p) \in \mathcal{L} \times \mathcal{P} : (p, L) \in I\}$$

is called the dual of $\mathcal{I}$.

Definition: Incidence Graph $X(\mathcal{I})$

Given $\mathcal{I} = (\mathcal{P}, \mathcal{L}, I)$, $X(\mathcal{I})$ is the incidence graph defined by the bipartite graph on $\mathcal{P} \cup \mathcal{L}$ such that $\{(p, L) \in E(X) : (p, L) \in I\}$.

$X(\mathcal{I}^*) \cong X(\mathcal{I})$.

Definition: Automorphism of $\mathcal{I}$

An automorphism of $(\mathcal{P}, \mathcal{L}, I)$ is a permutation σ on $\mathcal{P} \cup \mathcal{L}$ such that $\mathcal{P}^\sigma = \mathcal{P}, \mathcal{L}^\sigma = \mathcal{L}$ and $(p^\sigma, L^\sigma) \in I \Leftrightarrow (p, L) \in I$.

Definition: Partial Linear Space

$\mathcal{I} = (\mathcal{P}, \mathcal{L}, I)$ is a partial linear space if for any $x, y \in \mathcal{P}, x \neq y$, there is at most one line $L \in \mathcal{L}$ such that $(x, L) \in I$ and $(y, L \in I)$.

We say x, y are joined by L and x, y are collinear.

Lemma

If $\mathcal{I}$ is a partial linear space, then any two lines are incident with at most one point.

Lemma

If $\mathcal{I}$ is a partial linear space, then $X(\mathcal{I})$ has girth ≥ 6.

Proof. If X contains a 4-cycle p, L, q, M, then p and q are incident to 2 lines, which is forbidden by partial linear space. Since the girth of X is even (bipartite) and it cannot be 4, then the girth is at least 6.

4.2 Projective Planes

Definition: Projective Planes

A partial linear space satisfying

(1) Any two lines meet at a unique point.

(2) Any two points are joined by a unique line.

(3) There exists three non-collinear points (a triangle).

Theorem

A partial linear space $\mathcal{I}$ is a projective plane if and only if $X(\mathcal{I})$ has diameter 3 and girth 6.

Proof. ($\implies$) Let $\mathcal{I} = (\mathcal{P}, \mathcal{L}, I)$ be a projective plane.

Definition:

Let $\mathbb{F}_q$ be a finite field of order q. Let $V = \mathbb{F}_q^3$.

$$PG(2, q) = (\mathcal{P}, \mathcal{L}, I)$$

where $\mathcal{P} = \{\langle u \rangle : u \in V \setminus \{0\}\}$, $\mathcal{L} = \{\langle u, v \rangle : u, v \in V \text{ linearly independent}\}$, $I = \{(p, L) \in \mathcal{P} \times \mathcal{L} : p \subseteq L\}$.

We can also write $\mathcal{L} = \{\langle u \rangle^{\perp} : u \in V \setminus \{0\}\}$. V contains $q^3 - 1$ non-zero vectors. This implies $|P| = \frac{q^3-1}{q-1} = 1 + q + q^2$ and $|\mathcal{L}| = 1 + q + q^2$

Every line contains $q^2 - 1$ non-zero vectors, and each line is incident with $\frac{q^2-1}{q-1} = 1 + q$ points. Similarly, every point is incident with $1 + q$ lines.

The Fano plane is $PG(2,2)$.

Lemma

$PG(2,q)$ is a projective plane.

Proof. Let $L_1 = \langle u, v \rangle \in \mathcal{L}$ and $L_2 = \langle u', v' \rangle \in \mathcal{L}$ such that $L_1 \neq L_2$. $\dim(L_1 + L_2) = \dim(L_1) + \dim(L_2) - \dim(L_1 + L_2) \geq 2 + 2 + 3 = 1$, but $\dim(L_1 \cap L_2) \leq 1$ because $L_1 \neq L_2$, so $\dim(L_1 \cap L_2) = 1$.

Let $P_1 = \langle u \rangle \in P$ where $v \notin \langle u \rangle$. Suppose L is a line incident with both u and v. $\langle u, v \rangle \subseteq L$. Since $\dim(L) = 2$, $L = \langle u, v \rangle$.

Let u, v, w be linearly independent. Obviously $P_1 = \langle u \rangle, P_2 = \langle v \rangle, P_3 = \langle w \rangle$ form a triangle.

Definition: $GL(3,q)$

$$GL(3,q) = \{3 \times 3 \text{ invertible matrices over } \mathbb{F}_q\}$$

$GL(3,q)$ is a group and acts on P and $\mathcal{L}$.

Lemma

$GL(3,q) \leq \mathrm{Aut}(PG(2,q))$.

Proof. Take $A \in GL(3,q)$ and $p \sim L$ in $PG(2,q)$. Show that $p^A \sim L^A$.

Theorem

$X(PG(2,q))$ is arc-transitive.

Proof. For any (p_1, L_1) such that $p_1 \sim L_1$, (p_2, L_2) such that $p_2 \sim L_2$, write $p_1 = \langle u_1 \rangle, L_1 = \langle u_1, v_1 \rangle$ and $p_2 = \langle u_2 \rangle, L_2 = \langle u_2, v_2 \rangle$. There exists $A \in GL(3,q)$ where $Au_1 = u_2$ and $Av_1 = v_2$. This implies $(p_1, L_1)^A = (p_2, L_2)$. Define $\pi : P \times \mathcal{L} \to P \times \mathcal{L}$ where $\langle u \rangle \mapsto \langle u \rangle^\perp$ for all $u \in V \setminus \{0\}$ and $\langle v \rangle^\perp \mapsto \langle v \rangle$ for all $v \in V \setminus \{0\}$. Then prove $\pi : \mathrm{Aut}(X(PG(2,q)))$ and $P^\pi = \mathcal{L}$ and $\mathcal{L}^\pi = P$.

Chapter 5

Homomorphisms

We write $X \to Y$ to mean there exists a homomorphism from X to Y. Transitive means $X \to Y, Y \to Z$ implies $X \to Z$. Reflexive means $X \to X$.

Are homomorphisms symmetric, i.e. for all $X \neq Y$, $X \to Y \implies Y \to X$? No, take $X = K_2$ and $Y = K_3$.

Are homomorphisms anti-symmetric, i.e. for all $X \neq Y$, $X \to Y \implies Y \not\to X$? No, take $X =$ square graph and $Y = K_2$.

5.1 Cores

Definition: Core

A graph X is a core if every homomorphism from X to its subgraph is an automorphism.

Definition: Core of a Graph

A graph Y is a core of graph X if Y is a core and $X \to Y, Y \subseteq X$.

Lemma

If Y is a core of X, then Y is a retract of X.

Proof. Let $f : X \to Y$ be a homomorphism. Then $g := f|_Y$ is an automorphism. So $g^{-1} \circ f$ is a retraction.

E.g. K_n is a core. C_n is a core if n is odd.

Definition: Odd Girth

The odd girth of X is the length of a shortest odd cycle.

A bipartite graph's odd girth is ∞.

Lemma

Suppose $X \to Y$, then

(a) $\chi(X) \leq \chi(Y)$.

(b) Odd girth of $X \geq$ odd girth of Y.

Corollary

(a) $C_{2n+1} \not\to K_2$ and C_{2n+1} is a core.

(b) Petersen graph $\not\to C_4$.

(c) A graph is critical if its χ-number is strictly greater than the χ-number of its proper subgraphs.

Critical graphs are cores.

Lemma

Let X be connected. If every path of length 2 of X lies in a shortest odd cycle, then X is a core.

From this lemma, we see the Petersen graph is a core.

Proof. Suppose on the contrary X is not a core. This means there exists $Y \subseteq X, Y \neq X$, $f : X \to Y$ retraction. So $\exists u \sim v$, $v \in V(Y), u \notin V(Y)$. Let $w = f(u) \implies u \nsim w$ and $w \sim v$. w, v, w is a 2-path, so there exists a shortest cycle C using the path u, v, w. $f(C)$ is a walk of length $|C|$, but has repeated vertices. There exists a shorter odd cycle than C, a contradiction.

Lemma

Suppose Y_1, Y_2 are cores. Then, Y_1, Y_2 are homomorphically equivalent if and only if $Y_1 \cong Y_2$.

Proof. Let $f_1 : Y_1 \to Y_2, f_2 : Y_2 \to Y_1$ homomorphisms. Then, $f_1 \circ f_2$ and $f_2 \circ f_1$ are homomorphisms $Y_1 \to Y_1, Y_2 \to Y_2$. Y_1, Y_2 are cores implies $f_1 \circ f_2$ and $f_2 \circ f_1$ are surjective. Both have to be bijective homomorphisms, implying isomorphisms.

Definition: Homomorphically Equivalent

Two graphs X, Y are homomorphically equivalent if $X \to Y$ and $Y \to X$.

Theorem

Every graph has a unique core $X^\bullet$, up to isomorphism.

Proof. The existence is trivial. For uniqueness, let Y_1, Y_2 be two cores. $Y_1 \to X \to Y_2$ and

$Y_2 \to X \to Y_1$. So Y_1 and Y_2 are homomorphically equivalent. The lemma implies $Y_1 \cong Y_2$.

Theorem

Two graphs are homomorphically equivalent if and only if their cores are isomorphic.

Proof. ($\implies$) Suppose $X \to Y, Y \to X$. Then, $X^\bullet \to X \to Y \to Y^\bullet$ and $Y^\bullet \to Y \to X \to X^\bullet$. So $X^\bullet \cong Y^\bullet$.

Theorem

$\to$ defines a partial order on the family of cores.

Proof. $\to$ is reflective and transitive. Lemma implies $\to$ is anti-symmetric.

Definition: Lattice

For all $x \neq y$, $x \wedge y$ and $x \vee y$ exist where $\wedge$ is greatest lower bound and $\vee$ is the least upper bound.

5.2 Product Graphs

Definition: Product

Let Y, Z be graphs. $Y \times Z$ is defined by $V(Y \times Z) = V(Y) \times V(Z)$ and $(y, z) \sim (y', z')$ if $y \sim y'$ and $z \sim z'$.

Lemma

(a) Suppose Y and Z are connected, then $Y \times Z$ disconnected if and only if Y, Z are both bipartite.

(b) $(Y_1 + Y_2) \times Z \cong Y_1 \times Z + Y_2 \times Z$.

(c) $Y \times Z \cong Z \times Y$.

(d) $P_x : V(X \times Y) \to V(X), (x, y) \mapsto x$ and $P_y : V(X \times Y) \to V(Y), (x, y) \mapsto y$ are homomorphisms from $X \times Y$ to X and to Y.

Theorem

Let X, Y, Z be graphs. If $f : Z \to X$ and $g : Z \to Y$ are homomorphisms, then there exists a unique homomorphism $\phi : Z \to X \times Y$ such that $f = P_x \circ \phi$ and $g = P_y \circ \phi$.

Proof. Let $\phi(z) = (f(z), g(z))$ for all $z \in Z$. If $u \sim v$ in Z, then $f(u) \sim f(v), g(u) \sim g(v)$. Then $\phi(u) \sim \phi(v)$ implies ϕ is a homomorphism.

Since $f = P_x \circ \phi, g = P_y \circ \phi$, (f, g) determines ϕ.

We will denote ϕ by $\phi_{f,g}$ since it is uniquely determined by f and g.

Proposition

(a) $X \times Y \to X, X \times Y \to Y$.

(b) If $Z \to X, Z \to Y$, then $Z \to X \times Y$.

(c) $|\mathrm{Hom}(Z, X \times Y)| = |\mathrm{Hom}(Z, X)| \cdot |\mathrm{Hom}(Z, Y)|$.

Proof. (a) comes from Lemma (d).

(b) by previous theorem.

(c) $\varphi : \mathrm{Hom}(Z, X) \times \mathrm{Hom}(Z, Y) \to \mathrm{Hom}(Z, X \times Y)$. We take $(f, g) \to \phi_{f,g}$ unique is a bijection by previous theorem.

Theorem

$\to$ defines a lattice on the family of cores.

Proof. Least upper bound: $X \to X+Y \to (X+Y)^{\bullet}, Y \to X+Y \to (X+Y)^{\bullet}$, so $(X+Y)^{\bullet}$ is an upper bound.

To prove it is the least, suppose Z is a core such that $X \to Z, Y \to Z$. Then $X + Y \to Z$ implies $(X + Y)^{\bullet} \to Z \implies X \vee Y = (X + Y)^{\bullet}$.

Greatest lower bound: $X \times Y \to X$ and $X \times Y \to Y$ by proposition (a). This implies $(X \times Y)^{\bullet}$ is a lower bound for X and Y.

To prove it is the greatest, suppose Z is a core such that $Z \to X, Z \to Y$. By proposition (b), $Z \to (X \times Y) \to (X \times Y)^{\bullet} \implies X \wedge Y = (X \times Y)^{\bullet}$.

Chapter 6

Matrix Theory

6.1 Eigenvalues

Definition: Adjacency Matrix

Let X be an undirected, simple graph. Denote $A(X)$ as the adjacency matrix of X defined as

$$A(X) = (a_{ij})_{i,j \in V(X)}$$

where $a_{ij} = 1$ if $i \sim j$.

Definition: Eigenvalues of a Graph

The eigenvalues of X are the eigenvalues of $A(X)$.

Definition: Characteristic Polynomial

$$\phi(X, x) = \phi(A(X), x) = \det(xI - A(X))$$

The roots of $\phi(A(X), x)$ are the eigenvalues.

Definition: Spectrum of a Graph

The list of eigenvalues (counting algebraic multiplicities) of $A(X)$.

If $A(X)$ is real and symmetric, then there are n real eigenvalues $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$.

If $X \cong Y$, then X and Y have the same eigenvalues/spectrum. The converse is false, since there are two graphs with same spectrum/characteristic polynomial, but the graphs are not isomorphic.

Definition: Cospectral

Graphs that have the same spectrum, but may not be isomorphic.

Lemma

Let $A = A(X)$. Then

(a) $(A^r)_{uv} =$ the number of uv-walks of length r.

(b) $tr(A^r) =$ the number of closed r-walks.

(c) $tr(A) = 0, tr(A^2) = 2\left|E(X)\right|, tr(A^3) = 6 \cdot \#\triangle$s.

Definition: Incidence Matrix

Let X be an undirected, simple graph. Denote $B(X)$ as the incidence matrix of X defined as

$$B(X) = (b_{ij})_{i \in V(X), j \in E(X)}$$

where $b_{ij} = 1$ if $i \in j$.

Definition: Degree Matrix

A diagonal matrix $D(X)$ where $(D(X))_{i,i} = \deg(i)$ for all $i \in V(X)$.

Lemma

Let $B = B(X), A = A(X), D = D(X)$, then

(a) $BB^T = D(X) + A(X)$

(b) $B^T B = 2I + A(LG(X))$ where $LG(X)$ is the line graph of X by replacing each edge with a vertex and two edges are adjacent if there is a vertex incident to both.

Theorem

$$\text{rank}(B(X)) = n - \#\text{ bipartite components}$$

Proof. It suffices to show that $nul(B^T) =$ number of bipartite components. Suppose $B^T x = 0$ if and only if $x_u + x_v = 0$ for all $uv \in E(X)$. Thus, $x_u = (-1)^r x_v$ if u, v are joined by a path of length r. This implies $x_u = 0$ if u is in a nonbipartite component.

x takes inverse values on vertices from opposite class in a bipartite component. So $\ker(B^T) = \left\langle \begin{pmatrix} 1^{C_A} \\ -1^{C_B} \\ 0^{\overline{C}} \end{pmatrix} : \text{bipartite component } C = C_A \cup C_B \right\rangle$.

Lemma

If $C \in \mathbb{R}^{n\times m}$ and $D \in \mathbb{R}^{m\times n}$, then

(a) CD and DC have the same set of nonzero eigenvalues.

(b) $\det(I - CD) = \det(I - DC)$.

Proof. Let $X = \begin{pmatrix} I & C \\ D & I \end{pmatrix}$, $Y = \begin{pmatrix} I & 0 \\ -D & I \end{pmatrix}$. Then $XY = \begin{pmatrix} I-CD & C \\ 0 & I \end{pmatrix}$ and $YX = \begin{pmatrix} I & C \\ 0 & I-DC \end{pmatrix}$. Then $\det(I-CD) = \det(XY) = \det(YX) = \det(I-DC)$.

The spectrum of CD is the set of roots of $\det(xI - CD) = x^n \det(I - x^{-1}CD) = x^n \det(I - x^{-1}DC) = x^{n-m}\det(xI - DC)$.

Proposition

Let X be a k-regular graph and $L = LG(X)$, then $\phi(L,\lambda) = (\lambda+2)^{\frac{kn}{2}-n}\phi(X, \lambda-k+2)$.

Proof. Recall $BB^T = A(X) + D(X)$ and $B^TB = 2I + A(LG(X))$. let $C = \lambda^{-1}B^T$ and $D = B$.

$$\begin{aligned}
\det(I - CD) &= \det(I - \lambda^{-1}B^TB \\
&= \det(I - \lambda^{-1}BB^T) && (\det(I-DC) = \det(I-CD)) \\
\det(\lambda I - B^TB) &= \lambda^{\frac{kn}{2}-n}\det(\lambda I - BB^T) \\
\det((\lambda-2)I - A(L)) &= \lambda^{\frac{kn}{2}-n}\det((\lambda-k)I - A(X)) \\
\\
\phi(L, \lambda-2) &= \lambda^{\frac{kn}{2}-n}\phi(X, \lambda-k)
\end{aligned}$$

Definition: Laplacian Matrix

$$L(X) = D(X) - A(X)$$

Definition: Normalized Laplacian Matrix

$$N(X) = I - D^{-1/2}AD^{-1/2}$$

Definition: Walk Matrix

$$W(X) = A(X)D^{-1}(X)$$

$(W(X))_{ij} = \frac{A_{ij}}{\deg(j)} = \frac{1\{i\sim j\}}{\deg(j)}$.

6.2 Real Symmetric Matrices

Proposition

Let $A \in \mathbb{R}^{n \times n}$ be a real, symmetric matrix.

(a) If u and v are eigenvector with distinct eigenvalues, then $u^T v = 0$.

(b) All eigenvalues are real.

(c) Let U be a subspace of $\mathbb{R}^n$, then if U is A-invariant, then $U^\perp$ is A-invariant. (A-invariant is $Au \in U, \forall u \in U$).

(d) U is a nonzero A-invariant subspace of $\mathbb{R}^n$.

(e) $\mathbb{R}^n$ has an orthonormal basis consisting of eigenvectors of A.

(f) $A = PDP^T$ with P orthogonal.

(g) $A = \sum_{i=1}^{n} \lambda_i v_i v_i^T$ with $v_1, \ldots, v_n$ are orthogonal.

6.3 Eigenvectors of $A(X)$

Finding eigenvalues of $A = A(X)$ by finding $f : V(X) \to \mathbb{R}$ such that $Af = \lambda f$. By definition of A,

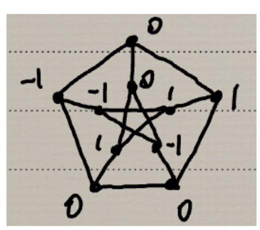
$$(Af)(u) = \sum_v A_{uv} f(v) = \sum_{v \sim u} f(v)$$

If we can find a function f such that

$$\sum_{v \sim u} f(v) = \lambda f(u), \forall u \in V(X)$$

then λ is an eigenvalue of $A(X)$.

E.g. Petersen graph. We have in the figure $\sum_{v \sim u} f(v) = f(u)$ for all $u \in V(X)$, so $\lambda = 1$ is an eigenvalue.



C_n. Let τ be an nth root of 1.

$$\sum_{v \sim u} f(v) = (\tau^{-1} + \tau)\tau^u, \forall u \in \{0, \ldots, n-1\}$$

So $\tau^{-1} + \tau$ is a real eigenvalue. There are n distinct eigenvalues.

k-regular graphs. Let $f(u) = 1$ for all u. Then $\sum_{v \sim u} f(u) = k$, so k is an eigenvalue.

Proposition

1 is an eigenvector if and only if X is regular.

Lemma

Let X be k-regular with n vertices and eigenvalues $k, \theta_2, \dots, \theta_n$. Then X and $\overline{X}$ have the same eigenvectors and the eigenvalues of $\overline{X}$ are $n-k-1, -\theta_2 - 1, \dots, -\theta_n - 1$.

Proof. $A(\overline{X}) = J - I - A(X)$ where J is the square all 1 matrix. 1 is an eigenvector of $A(X)$ corresponding to eigenvalue k.

$$A(\overline{X}) \cdot 1 = (J - I - A(X))1 = (n-1)1 - k \cdot 1 = (n-1-k)1$$

So 1 is the eigenvector of $A(\overline{X})$ corresponding to eigenvalue $n-1-k$.

Let $\{1, v_2, \dots, v_n\}$ be orthogonal eigenvectors of A. Fora ll $2 \leq j \leq n$,

$$\begin{cases} A(X) \cdot v_j = \theta_j v_j \\ 1^T v_j = 0 \end{cases}$$

So $A(\overline{X}) \cdot v_j = (J - I - A(X))v_j = -v_j - \theta_j v_j = (-1 - \theta_j)v_j$ for all $2 \leq j \leq n$.

6.4 Positive Semidefinite Matrices

Definition: Positive Semidefinite

A real symmetric matrix A is positive semidefinite if $u^T A u \geq 0$ for all $u \in \mathbb{R}^n$.

Definition: Positive Definite

A is positive definite if $u^T A u = 0$, i.e. $u = 0$.

Proposition

Let A be real and symmetric. The following are equivalent

(a) A is positive semidefinite.

(b) All eigenvalues of A are nonnegative.

(c) $A = B^T B$ for some B.

Lemma

If LG is a line graph, then $\lambda_{\min}(LG) \geq -2$.

Proof. Suppose LG is the line graph of X. Let $B = B(X)$. We know $B^T B = A(LG) + 2I$. $B^T B$ is PSD so $A(L) + 2I$ has minimum eigenvalue ≥ 0. Therefore, $\lambda_{\min}(LG) \geq -2$.

Lemma

Let X be a graph and Y be a vertex-induced subgraph of X, then

$$\lambda_{\min}(X) \leq \lambda_{\min}(Y) \leq \lambda_{\max}(Y) \leq \lambda_{\max}(X)$$

Proof. Let $A = A(X), \tilde{A} = A(Y)$. Let $\lambda = \lambda_{\max}(X)$. $\lambda I - A$ is PSD.

For any $f \in \mathbb{R}^{V(X)}$, where $f(u) = 0$ for all $u \in V(X) \setminus V(Y)$, let $\tilde{f} = f|_{V(Y)}$.

This implies $0 \leq f^T(\lambda I - A)f = \tilde{f}^T(\lambda I - \tilde{A})\tilde{f}$, so $\lambda I - \tilde{A}$ is PSD and $\lambda_{\max}(\tilde{A}) \leq \lambda$. Similarly, working on PSD matrix $A(X) - \lambda_{\min}(X)I$, $\lambda_{\min}(\tilde{A}) \geq \lambda_{\min}(A)$.

Proposition

The Laplacian matrix $L = L(X)$ is positive semidefinite.

Proof. Let $n = |V(X)|$. For any $x \in \mathbb{R}^n$,

$$\begin{aligned}
x^T L x &= \sum_{u,v} x_u L_{uv} x_v \\
&= \sum_{u} x_u^2 \deg(u) - \sum_{u} x_u \sum_{v \sim u} x_v \\
&= \sum_{uv \in E} (x_u^2 + x_v^2) - \sum_{uv \in E} 2 x_u x_v \\
&= \sum_{uv \in E} (x_u - x_v)^2 \geq 0
\end{aligned}$$

Remark: $x^T L x$ measures the smoothness of x on X.

$L(X)$ is PSD implies all eigenvalues of $L(X)$ are nonnegative. $(L(X)1)_u = \deg(u) - \deg(u) = 0$ for all $u \in V(X)$. 1 is an eigenvector of $L(X)$ with eigenvalue 0 and the minimum eigenvalue of $L(X)$ is 0.

Proposition

Let $L = L(X)$. Let $0 = \mu_1 \leq \mu_2 \leq \cdots \leq \mu_n$ be the eigenvalues of L. Then $\mu_2 > 0$ if and only if X is connected.

Proof. ($\implies$) That is, X is disconnected implies $\mu_2 = 0$. X is the union of 2 disjoint graphs X_1 and X_2. Then

$$L = \begin{pmatrix} L(X_1) & 0 \\ 0 & L(X_2) \end{pmatrix}$$

Then both $\begin{pmatrix} \mathbf{0} \\ \mathbf{1} \end{pmatrix}$ and $\begin{pmatrix} \mathbf{1} \\ \mathbf{0} \end{pmatrix}$ are eigenvectors with eigenvalue 0, so $\mu_2 = 0$.

($\Longleftarrow$) That is X is connected implies $\mu_2 > 0$. Suppose $f \in \mathbb{R}^{V(X)}$ is an eigenvector with eigenvalue 0.

$$\begin{aligned} & Lf = 0 \\ \Longrightarrow\ & f^T L f = 0 \\ \Longrightarrow\ & \sum_{uv \in E(X)} (f(u) - f(v))^2 = 0 \\ \Longrightarrow\ & f(u) = f(v), \forall uv \in E(X) \end{aligned}$$

X is connected implies f is constant on $V(X)$. The eigenspace corresponding to eigenvalue 0 has dimension 1. So $\mu_2 > 0$.

Proposition

Suppose X is k-regular. Let $A = A(X), L = L(X) = kI - A(X)$. Let $k = \lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$ be eigenvalues of A and $0 = \mu_1 \leq \mu_2 \leq \cdots \leq \mu_n$ be eigenvalues of L. Then $\mu_i = k - \lambda_i$ for all $1 \leq i \leq n$.

Theorem (Courant-Fisher)

Let A be an $n \times n$ real symmetric matrix and $\lambda_1 \geq \cdots \geq \lambda_n$ be eigenvalues of A. Then for all $1 \leq u \leq n$,

$$\lambda_u = \max_{\substack{S \subseteq \mathbb{R}^n \\ \dim(S) = u}} \min_{\substack{x \in S \\ x \neq 0}} \frac{x^T A x}{x^T x} = \min_{\substack{T \subseteq \mathbb{R}^n \\ \dim(T) = n-k+1}} \max_{\substack{x \in T \\ x \neq 0}} \frac{x^T A x}{x^T x}$$

where $\frac{x^T A x}{x^T x}$ is the Rayleigh Quotient.

Proof. (Courant-Fisher) We first prove

$$\max_{\substack{S \subseteq \mathbb{R}^n \\ \dim(S) = u}} \min_{\substack{x \in S \\ x \neq 0}} \frac{x^T A x}{x^T x} \geq \lambda_u$$

Let $f_1, \ldots, f_n$ be an orthonormal basis from set of eigenvectors of A, corresponding to $\lambda_1 \geq \cdots \geq \lambda_n$.

Let $S = \langle f_1, \ldots, f_u \rangle$. For all $x \in S, x \neq 0$,

$$x = \sum_{i=1}^{n} c_i f_i$$

for some $c_i \in \mathbb{R}$.

$$\frac{x^T A x}{x^T x} = \frac{\left(\sum_{i=1}^n c_i f_i\right)^T \left(\sum_{i=1}^n c_i \lambda_i f_i\right)}{x^T x} = \frac{\sum_{i=1}^n c_i^2 \lambda_i}{\sum_{i=1}^n c_i^2} \geq \lambda_k$$

Next, we show for all subspaces $S \subseteq \mathbb{R}^n$, $\dim(S) = k$.

$$\min_{\substack{x \in S \\ x \neq 0}} \frac{x^T A x}{x^T x} \leq \lambda_u$$

Let $T = \langle f_k, f_{k+1}, \ldots, f_n \rangle$, so $\dim(T) = n - k + 1$. $\dim(S \cap T) \geq 1$ implies

$$\begin{aligned}
\min_{\substack{x \in S \\ x \neq 0}} \frac{x^T A x}{x^T x} &\leq \min_{\substack{x \in S \cap T \\ x \neq 0}} \frac{x^T A x}{x^T x} \\
&= \max_{\substack{x \in T \\ x \neq 0}} \frac{x^T A x}{x^T x} \\
&\leq \lambda_u
\end{aligned}$$

Theorem

If A is an $n \times n$ symmetric, real, and $x \neq 0$ maximizes (or minimizes) $\frac{x^T A x}{x^T x}$, then x is an eigenvector of λ_1 (or λ_n).

Suppose x is an eigenvector with eigenvalue λ, then

$$\frac{x^T A x}{x^T x} = \frac{x^T \lambda x}{x^T x} = \lambda$$

Lemma

Let $\overline{d}$ and Δ are average degree and maximum degree of a graph X, then $\overline{d} \leq \lambda_1 \leq \Delta$.

Proof. (Lower-bound)

$$\lambda_1 = \max_{\substack{x \in \mathbb{R}^n \\ x \neq 0}} \frac{x^T A x}{x^T x} \geq \frac{1^T A 1}{n} = \frac{\sum_u \sum_{v: v \sim u} 1}{n} = \frac{2\,|E(X)|}{n} = \overline{d}$$

(Upper-bound) Let f be the eigenvector of λ_1. Let $u = \arg\max f$. Without loss of generality, $f(u) > 0$ (otherwise, consider $-f$). Then

$$\lambda_1 = \frac{(Af)(u)}{f(u)} = \frac{\sum_{v \sim u} f(v)}{f(u)} \leq \frac{\sum_{v \sim u} f(u)}{f(u)} = \deg(u) \leq \Delta$$

Lemma

If X is connected and $\lambda_1 = \Delta$, then X is Δ-regular.

Proof. The previous lemma holds with equality if and only if $f(v) = f(u)$ for all $v \sim u$ and $\deg(u) = \Delta$. Inductively, one can show that f is constant. This implies X is Δ-regular.

Theorem (Wilf 1967)

$$\chi(X) \leq \lfloor \lambda_1 \rfloor + 1$$

Proof. We prove it by induction on $n := |V(X)|$.

Suppose $n = 1$, then $\lambda_1 = 0$ and $\chi(X) = 1$.

IH: Assume $n \geq 1$ and the assertion holds for all graphs with n vertices. Let X be a graph $n+1$ vertices and let $\lambda_1 = \lambda_1(X)$. The first lemma says that $\overline{d} \leq \lambda_1$, so there exists a vertex u such that $\deg(u) \leq \lfloor \lambda_1 \rfloor$.

Let $S = V(X) \setminus \{u\}$ and $\lambda = \lambda_1(X[S])$. Recall that the largest eigenvalue of an induced subgraph is at most the largest eigenvalue of X. So, $\lambda \leq \lambda_1$.

By the inductive hypothesis, $\chi(X[S]) \leq \lfloor \lambda \rfloor + 1 \leq \lfloor \lambda_1 \rfloor + 1$. Since $\deg(u) \leq \lfloor \lambda_1 \rfloor$, we can extend an $(\lfloor \lambda_1 \rfloor + 1)$-coloring of $X[S]$ to an $(\lfloor \lambda_1 \rfloor + 1)$-coloring for X.

Definition: Principal Submatrix

Let A be a real symmetric matrix, then a submatrix B of A is called principal if B is obtained by deleting the same set of rows and columns of A.

Theorem (Cauchy)

Let A be an $n \times n$ real symmetric matrix and B be an $(n-1) \times (n-1)$ principal submatrix. Then $\alpha_1 \geq \beta_1 \geq \alpha_2 \geq \beta_2 \geq \cdots \geq \alpha_{n-1} \geq \beta_{n-1} \geq \alpha_n$ where α_i are eigenvalues of A and β_i are eigenvalues of B.

Proof. Without loss of generality, we may assume B is obtained by deleting the first row and column. Let $1 \leq k \leq n-1$.

By Courant-Fisher,

$$\alpha_k = \max_{\substack{S \subseteq \mathbb{R}^n \\ \dim(S)=k}} \min_{\substack{x \in S \\ x \neq 0}} \frac{x^T A x}{x^T x}$$

and

$$\beta_k = \max_{\substack{S \subseteq \mathbb{R}^{n-1} \\ \dim(S)=k}} \min_{\substack{x \in S \\ x \neq 0}} \frac{x^T B x}{x^T x} = \max_{\substack{S \subseteq \mathbb{R}^{n-1} \\ \dim(S)=k}} \min_{x \in S} \frac{\begin{pmatrix} 0 \\ x \end{pmatrix}^T A \begin{pmatrix} 0 \\ x \end{pmatrix}}{\begin{pmatrix} 0 \\ x \end{pmatrix}^T \begin{pmatrix} 0 \\ x \end{pmatrix}} = \max_{\substack{S \subseteq \mathbb{R}^n \\ \dim(S)=k \\ S \perp e_1}} \min_{x \in S} \frac{x^T A x}{x^T x} \leq \alpha_n$$

The same proof applies to $-A$ and $-B$ to get $\beta_k \geq \alpha_{k+1}$ for every $1 \leq k \leq n-1$.

Theorem (Perron-Frobenius Special Case)

Let X be a connected graph and $A = A(X)$. Let $\lambda_1 \geq \cdots \geq \lambda_n$ be the eigenvalues.

(a) λ_1 has a strictly positive eigenvector.

(b) $\lambda_2 < \lambda_1$.

(c) $\lambda_1 \geq -\lambda_n$ with equality if and only if X is bipartite.

Proof. (a) First show λ_1 has a nonnegative eigenvector. Assume f is an eigenvector of λ_1 and $\|f\|_2 = 1$. Let $f^+ |f|$, i.e. $f^+(u) = |f(u)|$ for every u. We show f^+ is an eigenvector of λ_1.

First $(f^+)^T f^+ = f^T f = \|f\|_2 = 1$. Moreover,

$$\lambda_1 = f^T A f = \sum_{uv \in E(X)} 2f(u)f(v) \leq \sum_{uv \in E(X)} 2\,|f(u)|\,|f(v)| = (f^+)^T A f^+$$

Courant-Fisher implies that $\lambda_1 = \max_{\substack{x \in \mathbb{R}^n \\ \|x\|_2 = 1}} x^T A x$ so $f^+ \in \arg\max_{x \in \mathbb{R}^n, \|x\|_2 = 1} x^T A x$. By the theorem, f^+ is an eigenvector of λ_1, so $Af^+ = \lambda_1 f^+$. Next we show that $f^+ > 0$.

Suppose not. There exists $vw \in E(X)$ such that $f^+(v) = 0$ and $f^+(w) > 0$.

$$\sum_{w : w \sim v} \underbrace{f^+(w)}_{>0} = \underbrace{\lambda_1 f^+(v)}_{=0}$$

A contradiction, so $f^+ > 0$.

(b) Let g be an eigenvector of λ_2. $\|g\|_2 = 1$. $g \perp f^+$. Then

$$\lambda_2 = g^T A g \leq |g|^T A\,|g| \leq \lambda_1 := \max_{h : \|h\|_2} h^T A h$$

Suppose $\lambda_2 = \lambda_1$.

1. By proof of (a), $|g|$ is strictly positive.
2. $\forall uv \in E(X) : g(u)g(v) = |g(u)g(v)|$.

$f^+ > 0$ and $g \perp f^+$. X is connected and by (1), there exists $uv \in E(X) : g(u) < 0 < g(v)$, contradicting (2).

(c) Let g be an eigenvector of λ_n. $\|g\|_2 = 1$.

$$|\lambda_n| = \left|g^T A g\right| = \left|\sum_{uv \in E(X)} 2g(u)g(v)\right| \leq \sum_{uv \in E(X)} 2\,|g(u)|\,|g(v)| = |g|^T A\,|g| \leq \lambda_1$$

It remains to show $=$ if and only if X is bipartite.

$(\Longleftarrow)$ We show that $\lambda_i = -\lambda_{n+1-i}$ for all i. Let S and T be the bipartition. Suppose X has eigenvalue λ with eigenvector f. Let

$$\overline{f} = \begin{cases} f(u) & \text{if } u \in S \\ -f(u) & \text{if } u \in T \end{cases}$$

This implies $A\overline{f} = (-\lambda)\overline{f}$.

$(\Longrightarrow)$ Suppose $\lambda_n = -\lambda_1$

$$\lambda_1 = |\lambda_n| = -\lambda_n = -g^T A g = -\sum_{uv \in E(X)} 2g(u)g(v) \leq \sum_{uv \in E(X)} 2\,|g(u)|\,|g(v)| \leq \lambda_1$$

1. Proof of part (a) implies $|g| > 0$.
2. $\forall uv \in E(X) : g(u)g(v) = -\,|g(u)|\,|g(v)|$.

Let $S = \{u : g(u) > 0\}$ and $T = \{v : g(v) < 0\}$. By 1, (S, T) is a bipartition. By 2, every $uv \in E(X)$ joins a vertex in S and a vertex in T.

Chapter 7

Strongly Regular Graphs

Definition: Strongly Regular Graph

X is a strongly regular with parameters (n, k, a, c) if

(a) $|V(X)| = n$.

(b) k-regular.

(c) every pair of adjacent vertices have a common neighbors.

(d) every pair of non-adjacent vertices have c common neighbors.

E.g. C_5 is $(5, 2, 0, 1)$-strongly regular.

E.g. If X is (n, k, a, c)-strongly regular, then $\overline{X}$ is $(n, n-1-k, \overline{a}, \overline{c})$-strongly regular with $\overline{a} = n - 2 - 2k + c$ and $\overline{c} = n - 2k + a$.

Definition: Primitive

A connected strongly-regular graph, otherwise it is imprimitive.

Lemma

Let X be (n, k, a, c)-strongly regular, excluding the complete graph. The following are equivalent:

(a) X is disconnected (i.e. imprimitive).

(b) $c = 0$.

(c) $a = k - 1$.

(d) $X \cong mK_{k+1}, m \geq 2$.

E.g. The line graph of K_n. $\left(\binom{n}{2}, 2(n-2), n-2, 4\right)$-strongly regular.

E.g. Line graph of $K_{n,n}$.

E.g. Paley graph(q).

E.g. Lattice graphs.

E.g. Latin square graphs.

Parameter relations: $c(n-k-1) = k(k-a-1)$. Pick a vertex u and create two sets: the set U of k neighbors and the set V of non-adjacent to u. $|U| = k$ and $|V| = n-k-1$.

For every $x \in V$, x has c neighbors in U, so $e(U, V) = c(n-k-1)$.

For every $y \in U$, y has $k-a-1$ neighbors in V, so $e(U, V) = k(k-a-1)$.

7.1 Eigenvalues

Lemma

Suppose X is a connected (n, k, a, c)-strongly regular graph and X is not complete. Let $A = A(X)$, then A has exactly 3 distinct eigenvalues k, θ, τ where

$$\theta = \frac{a-c+\sqrt{\Delta}}{2}$$
$$\tau = \frac{a-c-\sqrt{\Delta}}{2}$$
$$\Delta = (a-c)^2 + 4(k-c)$$

are roots of $\lambda^2 - (a-c)\lambda - (k-c) = 0$.
Moreover, the algebraic multiplicities of θ and τ are

$$m_\theta = \frac{1}{2}\left((n-1) - \frac{2k+(n-1)(a-c)}{\sqrt{\Delta}}\right), m_\tau = \frac{1}{2}\left((n-1) + \frac{2k+(n-1)(a-c)}{\sqrt{\Delta}}\right)$$

Proof. Consider A^2 where $(A^2)_{uv} :=$ the number of uv-walks of length 2.

$$(A^2)_{uv} = \begin{cases} k & \text{if } u = v \\ a & \text{if } u \sim v \\ c & \text{if } u \neq v, u \nsim v \end{cases}$$

So $A^2 = kI + aA + c(J - I - A)$ where $J = 11^T$ is the all 1-matrix. Suppose λ is an eigenvalue of A with eigenvector x. $\lambda \neq k \implies x \perp 1$.

$$A^2 x = (k-c)x + (a-c)Ax + cJx = 0 \implies \lambda^2 - (a-c)\lambda - (k-c) = 0$$

Also, $\lambda \in \{\theta, \tau\}$.

It remains to prove that $m_\theta, m_\tau > 0$ and $\theta, \tau \neq k$ (Perron-Frobenius) and $\theta \neq \tau$.

($\theta \neq \tau$) $\theta = \tau$ iff $\Delta = 0 \Leftrightarrow (a-c)^2 + 4(k-c) = 0$ iff $a = k = c$ which is impossible (a can be at most $k-1$).

(Verify m_θ, m_τ) n eigenvalues implies $m_\theta + m_\tau = n - 1$ by connectedness.

$tr(A) = 0 \implies k + \theta m_\theta + \tau m_\tau = 0$. Solving these gives m_θ, m_τ.

($m_\theta, m_\tau > 0$) Exercise, it uses $X \cong K_n$.

Lemma

A connected, regular graph with exactly 3 distinct eigenvalues is strongly regular.

Proof. X is k-regular and eigenvalues k, θ, τ. $A = A(X)$ and construct $M = (k-\theta)^{-1}(k-\tau)^{-1}(A-\theta I)(A-\tau I)$. First prove M has eigenvalue 1 with multiplicity 1 and eigenvalue 0 with multiplicity $n-1$. M is symmetric, so $M = \frac{1}{n}11^T = \frac{1}{n}J$.

$$(A-\theta I)(A-\tau I) = (k-\theta)(k-\tau)\frac{1}{n}J$$
$$A^2 - (\theta+\tau)A + \theta\tau I = (k-\theta)(k-\tau)\frac{1}{n}J$$

Thus, the number of length 2 walks joining u, v depends only on whether $u = v$ or $u \sim v$ or $u \neq v, u \nsim v$.

Theorem

Suppose X is a connected, regular, and not a complete graph, then X is strongly regular if and only if X has 3 distinct eigenvalues.

7.2 Paley Graphs

Definition: Paley Graph $P(q)$

Let q be a prime power and $q \equiv 1 \pmod 4$. The Paley graph $P(q)$ is defined by $V(P(q)) = GF(q)$ and $E(P(q)) = \{uv : u - v \text{ is a nonzero square}\}$.

E.g. Let $q = 5$. The nonzero squares are $\{\pm 1\}$, so $P(5) = C_5$.

Lemma

$P(q)$ is a simple, undirected graph.

Proof. There exists x such that $GF(q) \setminus \{0\} = \{x, x^2, \ldots, x^{q-1} = 1\}$. $0 = x^{q-1} - 1 = \left(x^{(q-1)/2} + 1\right)\left(x^{(q-1)/2} - 1\right) \implies x^{(q-1)/2} = -1$, so $x^{(q-1)/4}$ is a square root of -1 since $q \equiv 1 \pmod 4$. For all $u, v \in GF(q)$, $u - v$ is a nonzero square if and only if $v - u$ is a nonzero square.

$P(q)$ is $\frac{q-1}{2}$-regular.

Lemma

$P(q)$ is arc-transitive.

Proof. $\phi_{a,b} : V(P(q)) \to V(P(q)), x \mapsto ax + b.$

Claim: If a is a nonzero square, then $\phi_{a,b} \in \mathrm{Aut}(P(q))$.

Let (x, y) and (x', y') be fixed. Take $a = \frac{x'-y'}{x-y}$ and $b = x' - ax$, $(x, y)^{\phi_{a,b}} = (x', y')$.

Lemma

$P(q)$ is self-complementary, i.e. $P(q) \cong \overline{P(q)}$.

Proof. Consider $\sigma : P(q) \to \overline{P(q)}$ where $\sigma(x) = ax$. Fix a that is not a square. σ is an isomorphism, so $P(q) \cong \overline{P(q)}$.

Theorem

$P(q)$ is strongly regular.