CO 444/644 Algebraic Graph Theory

Keven Qiu
Instructor: Jane Gao

Chapter 1

Introduction

We will focus on simple, undirected graphs without loops unless explicitly stated. We use $X = (V, E)$ to denote graphs and G for groups. $V(X)$ and $E(X)$ are the sets of vertices and edges of graph X respectively and $\deg(v)$ to denote the degree of a vertex $v \in V(X)$.

Definition: Isomorphism

An isomorphism between graphs X, Y is a function $f : V(X) \to V(Y)$ such that $uv \in E(X)$ if and only if $f(u)f(v) \in E(Y)$.

1.1 Automorphisms

Definition: Automorphism

An automorphism of the graph X is an isomorphism $f : X \to X$.

$\mathrm{Aut}(X)$ is the set of all automorphisms of X.

$\mathrm{Sym}(V)$ is used to denote the symmetric group of permutations on V. In group theory, we may have used $V = [n]$. We may use this notation alongside $\mathrm{Sym}(n)$ when explicitly enumerating the vertices of a graph from 1 to n.

Proposition

$\mathrm{Aut}(X) \leq \mathrm{Sym}(V(X))$ with the group operation for $\sigma, \tau \in \mathrm{Aut}(X)$ defined $\sigma\tau := \tau \circ \sigma$.

For $g \in \mathrm{Sym}(V(X))$ and $v \in V(X)$, let v^g denote $g(v)$. Let S^g denote $\{g(v) : v \in S\}$ for set S.

Suppose $Y \subseteq X$ is a subgraph and $g \in \mathrm{Aut}(X)$. Y^g is the graph defined $V(Y^g) = V(Y)^g$ and $E(Y^g) = \{u^g v^g : uv \in E(Y)\}$.

E.g. The following is an example of graphs X and Y along with functions that are and are not automorphisms.

Let $X = (\{1,2,3,4\}, \{12,13,14,23,34\}), Y = (\{1,2,3\}, \{12,13,23\}), Y^g = (\{1,3,4\}, \{13,14,34\})$ where $g(1) = 3, g(2) = 4, g(3) = 1, g(4) = 2$. $f(1) = 1, f(2) = 4, f(3) = 3, f(4) = 2$ is an automorphism while Y^g where $f(1) = 3, f(2) = 3, f(3) = 4, f(4) = 1$ is not an automorphism.

Lemma

For $v \in V(X)$ and $g \in \mathrm{Aut}(X)$, $\deg(v) = \deg(v^g)$.

Proof. Let $Y(v)$ be the subgraph of X induced by v and the neighbors $N_X(v)$. Then

$$Y(v) \cong Y(v)^g = Y(v^g)$$

so $\deg(v) = \deg(v^g)$.

Lemma

Let $u, v \in V(X)$ and $g \in \mathrm{Aut}(X)$, then the length of the shortest paths are preserved, i.e. $d(u,v) = d(u^g, v^g)$.

Proof. Show that a shortest uv-path in X is mapped to a shortest $u^g v^g$-path by g.

1.2 Homomorphisms

Definition: Homomorphism

Let X and Y be graphs. We say $f : V(X) \to V(Y)$ is a homomorphism if $x \sim y$ implies $f(x) \sim f(y)$ in Y.

$\sim$ is for adjacency and $f : X \to Y$ instead of $f : V(X) \to V(Y)$ for simplicity.

Let $\chi(X)$ denote the chromatic number of X, the minimum number of colors needed to color the vertices of a graph such that no two adjacent vertices have the same color.

Let K_r denote the complete graph on r vertices where every pair of distinct vertices is connected by an edge. We say that K_r is a clique, where it generally is a subset of vertices of a graph such that every two distinct vertices in the clique are adjacent.

Lemma

$$\chi(X) = \min\{r \in \mathbb{N} : \exists \text{ homomorphism } X \to K_r\}$$

Proof. Let $k = \chi(X)$. We first show $k \geq \min\{r \in \mathbb{N} : \exists \text{ homomorphism } X \to K_r\}$. Let f be a k-coloring of X. Then f is a homomorphism from X to K_k.

Next, we show that $k \leq \min\{r \in \mathbb{N} : \exists \text{ homomorphism } X \to K_r\}$. Let $\overline{r} = \min\{r \in \mathbb{N} : \exists \text{ homomorphism } X \to K_r\}$. Let $h : X \to K_{\overline{r}}$ be a homomorphism. Then $h^{-1}(v)$ is an independent set. So, giving $h^{-1}(v)$ distinct colors yields an $\overline{r}$-coloring.

Definition: Retraction

A homomorphism $f : X \to Y$ such that

1. $Y \subseteq X$.
2. $f|_Y = id$, the identity map.

If a retraction from X to Y exists, we call Y a retract of X.

We use the notation $f|_Y$ to mean the function f when restricted to the domain of Y.

E.g. Suppose $K_r \cong Y \subseteq X$ and $\chi(X) = r$. We will prove that Y is a retract of X. The proof is as follows: let $f : V(X) \to [r]$ where $r = \chi(X)$ be an r-coloring of X. Then, Y receives distinct colors since $Y \cong K_r$. Without loss of generality, assume $V(Y) = [r]$. Then f is a homomorphism from X to K_r and $f|_Y = id$. Therefore, f is a retraction.

E.g. Recall that a cycle graph C_n is defined $V(C_n) = \{0, \ldots, n-1\}$ where $n \geq 3$ and $E(C_n) = \{ij : i - j \equiv \pm 1 \pmod n\}$. Let $g = (1, 2, \ldots, n-1, 0) \in \mathrm{Aut}(C_n)$. This can be viewed as a rotation. We can define the subgroup

$$R = \{g^m : 0 \leq m \leq n-1\} \leq \mathrm{Aut}(C_n)$$

to capture all possible rotations.

We can define reflections. Let h be defined $h(i) = -i \pmod n \in \mathrm{Aut}(C_n)$. We can see that R and Rh are disjoint cosets of $\mathrm{Aut}(C_n)$ and $Rh \leq \mathrm{Aut}(C_n)$. It follows that $|\mathrm{Aut}(C_n)| \geq 2n$.

Definition: Circulant Graph

Let $\mathbb{Z}_n = \{0, \ldots, n-1\}$ and $C \subseteq \mathbb{Z}_n \setminus \{0\}$ be closed under inverse, that is, $x \in C \implies -x \in C$. We define the circulant graph $X = X(\mathbb{Z}_n, C)$ where $V(X) = \mathbb{Z}_n, E(X) = \{ij : i - j \in C\}$.

One can show that the arguments from the previous example for C_n also hold for $X = X(\mathbb{Z}_n, C)$. That is, $|\mathrm{Aut}(X(\mathbb{Z}_n, C))| \geq 2n$. We can generalize this result for arbitrary groups using Cayley graphs.

Definition: Johnson Graph

Given $v \geq k \geq i$ as integers where $[v] = \{1, \ldots, v\}$, the Johnson graph $J = J(v, k, i)$ is defined $V(J) = \{S \subseteq [v] : |S| = k\}, E(J) = \{ST : |S \cap T| = i\}$.

$J(5, 2, 0)$ is the Peterson graph. $J(v, k, 0)$ is the Kneser graph.

Proposition

There exists a subgroup of $\mathrm{Aut}(J(v, k, i))$ that is isomorphic to $\mathrm{Sym}(v)$.

Proof. For $g \in \mathrm{Sym}(v)$, define $\tau_g : \binom{[v]}{k} \to \binom{[v]}{k}$ as $\tau(S) = S^g$. Note that $|S \cap T| = |S^g \cap T^g|$ for vertices $S, T \in J(v, k, i)$ since we are essentially just relabeling elements of S and T, so

$\tau_g \in \mathrm{Aut}(J(v,k,i))$. We can conclude that

$$\{\tau_g : g \in \mathrm{Sym}(v)\} \cong \mathrm{Sym}(v)$$

Chapter 2

Groups

Definition: Homomorphism

Given groups G and H, $f : G \to H$ is a homomorphism if for all $g, h \in G$,

$$f(gh) = f(g)f(h)$$

Definition: Kernel

The kernel of a homomorphism f is defined

$$\ker(f) = f^{-1}(1)$$

Definition: Group Action

Suppose G is a group and V is a set. A homomorphism $f : G \to \text{Sym}(V)$ is a permutation representation of G and we call it an action of G on V.

E.g. Let X be a graph and take $V = V(X)$. Let $G = \text{Aut}(X)$. Then $f : G \to \text{Sym}(V)$ defined $f(g) = g$ for $g \in G$ is an action.

E.g. Let G be a group. Let $f : G \to \text{Sym}(V)$ where V is arbitrary be defined $f(g) = id$ where id is the identity permutation. f is an action.

Definition: Faithful Action

The action f is faithful if $\ker(f) = \{1\}$.

We can see that the first action example above is faithful, but not the second.

Let group G act on V, via $f : G \to \text{Sym}(V)$. Let $g \in G$, we use the notation

$$x^g := g^{f(g)} \text{ and } S^G := S^{f(g)}$$

where S is an arbitrary set.

Definition: G-Invariant

Let group G act on V and $g \in G$. S is G-invariant if $S = S^g$ for all $g \in G$.

Definition: Orbit

Let group G act on V. The orbit of $x \in V$ is

$$x^G := \{x^g : g \in G\}$$

One may show that V is partitioned into disjoint orbits and each orbit is G-invariant and transitive (for every x, y in the same orbit, there exists $g \in G$ where $x^g = y$).

Definition: Stabilizer

Let $G \leq \mathrm{Sym}(V)$ and $x \in V$. The stabilizer of x is

$$G_x := \{g \in G : x^g = x\}$$

Lemma

Let $G \leq \mathrm{Sym}(V)$ and $x \in V$, then $G_x \leq G$.

Lemma

Let $G \leq \mathrm{Sym}(V)$ and let S be an orbit of G. Let $x, y \in S$, then

$$H := \{h \in G : x^h = y\}$$

is a right coset of G_x. Conversely, if H is a right coset of G_x, then for all $h, h' \in H$, $x^h = x^{h'}$.

Proof. ($\implies$) G is transitive on S, so there exists $g \in G$ where $x^g = y$. For any $h \in H$, $x^h = y$ by the definition of H. So, $x^h = x^g$. Then,

$$x^{hg^{-1}} = x \implies hg^{-1} \in G_x \implies h \in G_x g$$

($\impliedby$) Assume $H = G_x g$ for some $g \in G$. Let $h, h' \in H$ where $h = \sigma g$ and $h' = \sigma' g$ for some $\sigma, \sigma' \in G_x$. We have

$$x^h = x^{\sigma g} = x^g = x^{\sigma' g} = x^{h'}$$

Lemma (Orbit-Stabilizer)

Let G be a permutation group acting on V with $x \in V$. Then

$$|G_x| \left|x^G\right| = |G|$$

Proof. Let $\mathcal{H}$ be the set of right cosets of G_x and define $f : x^G \to \mathcal{H}$ as

$$f(y) = \{g \in G : x^g = y\}$$

The previous lemma shows that f is a bijection. Therefore, $|\mathcal{H}| = \left|x^G\right|$. Since the right cosets of G_x partition G, we have

$$|G| = |G_x|\,|\mathcal{H}| = |G_x|\left|x^G\right|$$

Definition: Conjugate

Let G be a permutation group and let $g, h \in G$. g is conjugate to h if there is some $\sigma \in G$ such that

$$g = \sigma h \sigma^{-1}$$

Proposition

If H is a subgroup of G and $g \in G$, then $gHg^{-1} \leq G$ and $gHg^{-1} \cong H$.

Lemma

If $y \in x^G$, then G_x and G_y are conjugate.

Proof. Suppose $y = x^g$ where $g \in G$. We will prove that $g^{-1}G_x g = G_y$.

($\subseteq$) Note that $y^{g^{-1}} = x$. For every $h \in G_x$, $y^{g^{-1}hg} = x^{hg} = g^g = y$.

($\supseteq$) For $h \in G_y$, $x^{ghg^{-1}} = y^{hg^{-1}} = y^{g^{-1}} = x$. Then $ghg^{-1} \in G_x$, rearranging gives $h \in g^{-1}G_x g$.

Definition: Fix

Let $G \leq \mathrm{Sym}(V)$ and $g \in G$. Then

$$\mathrm{fix}(g) = \{v \in V : v^g = v\}$$

Lemma (Burnside)

Let $G \leq \mathrm{Sym}(V)$. Then

$$\#\text{ of orbits of } G = \frac{1}{|G|}\sum_{g \in G}|\mathrm{fix}(g)|$$

Proof. Let $\Lambda = \{(g, x) : g \in G, x \in V, x \in \mathrm{fix}(g)\}$. We will apply a double-counting argument. Observe that

$$|\Lambda| = \sum_{g \in G}|\mathrm{fix}(g)| = \sum_{x \in V}|G_x|$$

Equating these gives

$$\begin{aligned}
\sum_{g \in G} |\mathrm{fix}(g)| &= \sum_{x \in V} |G_x| \\
&= \sum_{x \in V} \frac{|G|}{|x^G|} && \text{(Orbit-Stabilizer)} \\
&= |G| \sum_{x \in V} \frac{1}{|x^G|} \\
&= |G| \, (\# \text{ of orbits of } G)
\end{aligned}$$