# CO 450/650 Combinatorial Optimization

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# Part I

# Introduction

# Chapter 1

# Introduction

**Definition: Combinatorial Optimization**

A subfield of mathematical optimization which involves searching for an optimal object in a finite collection of objects.

Typically, the collection has a concise representation, while the number of objects is large. Objects include graphs, networks, and matroids.

The main tool in combinatorial optimization is linear programming duality.

# Chapter 2

# Linear Programming

**Definition: Linear Programming**

The problem of finding a vector $x$ that maximizes a given linear function $c^T x$, where $x$ ranges over all vectors satisfying a given system $Ax \leq b$ of linear inequalities.

## 2.1 Farkas' Lemma

**Lemma (Farkas' Lemma for Inequalities)**

The system $Ax \leq b$ has a solution $x$ if and only if there is no vector $y$ satisfying $y \geq 0$, $y^T A = 0$, and $y^T b < 0$.

***Proof.*** Suppose $Ax \leq b$ has a solution $\overline{x}$ and suppose there exists a vector $\overline{y} \geq 0$ satisfying $\overline{y}^T A = 0$ and $\overline{y}^T b < 0$. Then we obtain the contradiction

$$0 > \overline{y}^T b \geq \overline{y}^T (A\overline{x}) = (\overline{y}^T A)\overline{x} = 0$$

Now suppose that $Ax \leq b$ has no solution. If $A$ has only one column, then the result is easy. Otherwise, apply Fourier-Motzkin elimination to obtain a system $A'x' \leq b'$ with one less variable. Since $A'x' \leq b'$ also has no solution, we can assume by induction that there exists a vector $y' \geq 0$ satisfying $y'^T A' = 0$ and $y'^T b' < 0$. Now since each inequality in $A'x' \leq b'$ is the sum of positive multiples of inequalities in $Ax \leq b$, we can use $y'$ to construct a vector $y$ satisfying the conditions in the theorem. $\square$

**Lemma (Farkas' Lemma)**

The system $Ax = b$ has a nonnegative solution if and only if there is no vector $y$ satisfying $y^T A \geq 0$ and $y^T b < 0$.

***Proof.*** Define

$$A' = \begin{bmatrix} A \\ -A \\ -I \end{bmatrix}, b' = \begin{bmatrix} b \\ -b \\ 0 \end{bmatrix}$$

Then $Ax = b$ has a nonnegative solution $x$ if and only if $A'x' \leq b'$ has a solution $x'$. Applying Farkas' Lemma for Inequalities to $A'x' \leq b'$ gives the result. $\square$

**Corollary**

Suppose the system $Ax \leq b$ has at least one solution. Then every solution $x$ of $Ax \leq b$ satisfies $c^T x \leq \delta$ if and only if there exists a vector $y \geq 0$ such that $y^T A = c$ and $y^T b \leq \delta$.

## 2.2 Duality

Consider the LP:

$$\begin{aligned} \max \quad & c^T x \\ \text{s.t.} \quad & Ax \leq b \end{aligned}$$

and dual LP

$$\begin{aligned} \min \quad & y^T b \\ \text{s.t.} \quad & y^T A = c^T \\ & y \geq 0 \end{aligned}$$

**Theorem (Weak Duality)**

Let $A$ be an $m \times n$ matrix, $b \in \mathbb{R}^m$, $c \in \mathbb{R}^n$. Suppose that $\overline{x}$ is a feasible solution to $Ax \leq b$ and $\overline{y}$ is a feasible solution to $y \geq 0, y^T A = c^T$. Then

$$c^T \overline{x} \leq \overline{y}^T b$$

***Proof.***

$$c^T \overline{x} = (\overline{y}^T A)\overline{x} = \overline{y}^T (A\overline{x}) \leq \overline{y}^T b$$

$\square$

**Theorem (Strong Duality)**

Let $A$ be an $m \times n$ matrix, $b \in \mathbb{R}^m$, $c \in \mathbb{R}^n$, then

$$\max\{c^T x : Ax \leq b\} = \min\{y^T b : y^T A = c^T, y \geq 0\}$$

provided that both sets are nonempty.

**Corollary**

Let $A$ be an $m \times n$ matrix, $b \in \mathbb{R}^m$, $c \in \mathbb{R}^n$, then

$$\max\{c^T x : Ax \leq b, x \geq 0\} = \min\{y^T b : y^T A \geq c^T\}$$

provided that both sets are nonempty.

**Definition: Complementary Slackness Conditions**

For each $i \in \{1, \ldots, m\}$, either $y_i^* = 0$ or $a_i x^* = b_i$.

**Theorem (Complementary Slackness Theorem)**

Let $x^*$ be a feasible solution of $\max\{c^T x : Ax \leq b\}$ and let $y^*$ be a feasible solution of $\min\{y^T b : y^T A = c^T, y \geq 0\}$. Then $x^*$ and $y^*$ are optimal solutions for the maximum and minimum respectively if and only if the complementary slackness conditions hold.

# Chapter 3

# Graph Theory Terminology

**Definition: Graph**

A graph $G = (V, E)$ is a set of vertices/nodes $V$ and a set of edges $E$.
We define $n = |V|$ and $m = |E|$.

**Definition: Subgraph**

$H = (W, F)$ of $G = (V, E)$ where $W \subseteq V$ and $F \subseteq E$.

**Definition: Spanning Subgraph**

$H$ is spanning if $V(H) = V(G)$.

**Definition: Path**

A sequence $P = v_0, e_1, v_1, \ldots, e_k, v_k$ where $v_0, \ldots, v_k \in V(G)$, $e_1, \ldots, e_k \in E(G)$, and $e_i = v_{i-1}v_i$.

We call $P$ a $v_0v_1$-path. $P$ is called edge-simple if all $e_i$ are distinct and simple if all $v_i$ are distinct.

The length of $P$ is the number of edges in $P$.

**Definition: Circuit/Cycle**

A simple closed path.

**Definition: Connected**

A graph is connected if every pair of vertices is joined by a path.

**Definition: Cut Vertex**

A vertex $v$ of a connected graph $G$ where $G - v$ is not connected.

**Definition: Forest**

A graph with no circuits.

**Definition: Tree**

A connected forest.

**Definition: Cut**

Let $R \subseteq V$, then

$$\delta(R) = \{vw : vw \in E, v \in R, w \notin R\}$$

**Definition: $rs$-Cut**

A cut for which $r \in R, s \notin R$.

# Chapter 4

# Complexity Theory

**Definition: Decision Problem**

A problem with a yes-no answer.

**Definition: $\mathcal{P}$**

Decision problems that can be solved in polynomial time.

**Definition: $\mathcal{NP}$**

Decision problems in which we can certify the answer is yes in polynomial time.

**Definition: co-$\mathcal{NP}$**

Decision problems in which we can certify the answer is no in polynomial time.

A good characterization means the problem is in $\mathcal{NP} \cap co - \mathcal{NP}$.

**Definition: $\mathcal{NP}$-Hard**

A problem $X$ is $\mathcal{NP}$-hard if every other problem $Y$ in $\mathcal{NP}$ can be reduced to $X$.

S. Cook (1971) proved that the satisfiability problem (SAT) is $\mathcal{NP}$-hard. R. Karp (1972) used Cook's result to show 21 well-known combinatorial optimization problems are also $\mathcal{NP}$-hard.

To show that the traveling salesman problem (TSP) is $\mathcal{NP}$-hard, we show that any example of SAT can be formulated as a TSP, of size polynomial in the size of SAT. Then, since Cook shows SAT is $\mathcal{NP}$-hard, TSP is also $\mathcal{NP}$-hard.

# Part II

# Polyhedral Combinatorics

# Chapter 5

# Integrality of Polyhedra

## 5.1 Convex Hull

**Definition: Convex Combination**

$x = \lambda_1 v_1 + \cdots + \lambda_k v_k$ for some vectors $v_1, \ldots, v_k$ and nonnegative scalars $\lambda_1, \ldots, \lambda_k$ such that $\lambda_1 + \cdots + \lambda_k = 1$.

**Definition: Convex Hull**

The convex hull of a finite set $S$, denoted $conv.hull(S)$, is the set of all vectors that can be written as a convex combination of $S$.
It is also defined as the smallest convex set containing $S$.

**Proposition**

Let $S \subseteq \mathbb{R}^n$ be a finite set and let $w \in \mathbb{R}^n$. Then

$$\max / \min\{w^T x : x \in S\} = \max / \min\{w^T x : x \in conv.hull(S)\}$$

**Theorem (Minkowski)**

If $S$ is finite, then $conv.hull(S)$ is a polyhedron.

$$\begin{aligned} \max(w^T x : x \in S) &= \max(w^T x : x \in conv.hull(S)) \\ &= \max(w^T x : Ax \leq b) \\ &= \min(y^T b : y^T A = w^T, y \geq 0) \end{aligned}$$

So we can use LP duality to attack combinatorial problems. If we understand $Ax \leq b$, then the problem is in co-$\mathcal{NP}$. Thus, if we have an algorithm to produce the inequalities in $Ax \leq b$ (separation), then the problem is in $\mathcal{P}$ (Ellipsoid method).

## 5.2 Polytopes

**Definition: Polyhedron**

A set of the form $\{x : Ax \leq b\}$.

In combinatorial optimization, we typically have $x \geq 0$ as a constraint, so we have polyhedra of the form $\{x : Ax \leq b, x \geq 0\}$.

**Definition: Polytope**

A polyhedron $P \subseteq \mathbb{R}^n$ is a polytope if there exists $\ell, u \in \mathbb{R}^n$ such that $\ell \leq x \leq u$ for all $x \in P$.

**Definition: Convex Set**

Let $P$ be a polyhedron, $x_1, x_2 \in P$, and $0 \leq \lambda \leq 1$. If $\lambda x_1 + (1 - \lambda)x_2 \in P$, then $P$ is a convex set.

**Definition: Valid Inequality**

An inequality $w^T x \leq t$ is valid for a polyhedron $P$ if $P \subseteq \{x : w^T x \leq t\}$.

**Definition: Hyperplane**

The solution set of $w^T x = t$ where $w \neq 0$.

**Definition: Supporting Hyperplane**

With respect to a polyhedron $P$, a hyperplane is supporting if $w^T x \leq t$ is valid for $P$ and $P \cap \{x : w^T x = t\} \neq \emptyset$.

**Definition: Face**

The intersection of a polyhedron with one of its supporting hyperplanes.

The null set and the polyhedron itself is a face.

**Definition: Proper Face**

Faces which are not the null set or the polyhedron itself.

**Proposition**

A nonempty set $F \subseteq P = \{x : Ax \leq b\}$ is a face of $P$ if and only if for some subsystem $A^\circ x \leq b^\circ$ of $Ax \leq b$, we have $F = \{x \in P : A^\circ x = b^\circ\}$.

***Proof.*** ( $\Longrightarrow$ ) Suppose $F$ is a face of $P$. Then there exists a valid inequality $w^T x \leq t$ such that $F = \{x \in P : w^T x = t\}$.

Consider the LP problem $\max\{w^T x : Ax \leq b\}$. The set of optimal solutions is precisely $F$. Now let $y^*$ be an optimal solution to the dual problem $\min\{y^T b : y^T A = w, y \geq 0\}$ and let $A^\circ x \leq b^\circ$ be those inequalities $a_i^T x \leq b_i$ whose corresponding dual variable $y_i^*$ is positive. By complementary slackness, we have $F = \{x : Ax \leq b, A^\circ x = b^\circ\}$ as required.

( $\Longleftarrow$ ) Conversely, if $F = \{x \in P : A^\circ x = b^\circ\}$ for some subsystem $A^\circ x \leq b^\circ$ of $Ax \leq b$, then add the inequalities $A^\circ x \leq b^\circ$ to obtain an inequality $w^T x \leq t$. Every $x \in F$ satisfies $w^T x = t$ and every $x \in P \setminus F$ satisfies $w^T x < t$ as required. $\square$

**Proposition**

Let $F$ be a minimal nonempty face of $P = \{x : Ax \leq b\}$. Then $F = \{x : A^\circ x = b^\circ\}$ for some subsystem $A^\circ x \leq b^\circ$ of $Ax \leq b$.
Moreover, the rank of the matrix $A^\circ$ is equal to the rank of $A$.

**Definition: Vertex/Extreme Point**

A vector $x \in P$ is called a vertex/extreme point if $\{x\}$ is a face of $P$.
Equivalently, $x \in P$ is a vertex/extreme point if $x$ cannot be written as $\frac{1}{2}x_1 + \frac{1}{2}x_2$ for points $x_1, x_2 \in P, x_1 \neq x_2$.

Note: Not all polyhedra have vertices, but if $P \subseteq \mathbb{R}^n_+$, then $P$ has vertices.

**LP Fact**

If a polyhedron $P$ has vertices, then the set of optimal LP solutions contains at least one vertex of $P$.
Moreover, if all vertices of $P$ are integral, then the LP always has an integral optimal solution.

**Definition: Pointed Polyhedron**

A polyhedron $P$ is pointed if it has at least one vertex.

$\{(x_1, x_2) \in \mathbb{R}^2 : x_1 \geq 0\}$ is a polyhedron with no vertex.

**Proposition**

If a polyhedron $P$ is pointed, then every minimal nonempty face of $P$ is a vertex.

**Proposition**

Let $P = \{x : Ax \leq b\}$ and $v \in P$. Then $v$ is a vertex of $P$ if and only if $v$ cannot be written as a convex combination of vectors in $P \setminus \{v\}$.

**Theorem**

A polytope is equal to the convex hull of its vertices.

***Proof.*** Let $P$ be a nonempty polytope. Since $P$ is bounded, $P$ must be pointed. Let

$v_1, \ldots, v_k$ be the vertices of $P$. Clearly, $conv.hull(\{v_1, \ldots, v_k\}) \subseteq P$. So suppose there exists

$$u \in P \setminus conv.hull(\{v_1, \ldots, v_k\})$$

Then by proposition, there exists an inequality $w^T x \leq t$ that separates $u$ from

$$conv.hull(\{v_1, \ldots, v_k\})$$

Let $t^* = \max\{w^T x : x \in P\}$ and consider the face $F = \{x \in P : w^T x = t^*\}$. Since $u \in P$, we have $t^* > t$. So $F$ contains no vertex of $P$, a contradiction. $\square$

**Theorem**

A set $P$ is a polytope if and only if there exists a finite set $V$ such that $P$ is the convex hull of $V$.

## 5.3 Integral Polytopes

**Definition: Rational Polyhedron**

A polyhedron that can be defined by rational linear systems.

**Definition: Integral Polyhedron**

A rational polyhedron where every nonempty face contains an integral vector.

**Definition: Pointed Integral Polyhedron**

A pointed rational polyhedron is integral if and only if all its vertices are integral.

**Theorem**

A rational polytope $P$ is integral if and only if for all integral vectors $w$, the optimal value of $\max\{w^T x : x \in P\}$ is an integer.

***Proof.*** To prove sufficiency, suppose that for all integral vectors $w$, the optimal value of $\max\{w^T x : x \in P\}$ is an integer. Let $v = (v_1, \ldots, v_n)^T$ be a vertex of $P$ and let $w$ be an integral vector such that $v$ is the unique optimal solution to $\max\{w^T x : x \in P\}$. By multiplying $w$ by a large positive integer if necessary, we may assume $w^T v > w^T u + u_1 - v_1$ for all vertices $u$ of $P$ other than $v$. This implies that if we let $\overline{w} = (w_1 + 1, w_2, \ldots, w_n)^T$, then $v$ is an optimal solution to $\max\{\overline{w}^T x : x \in P\}$. So $\overline{w}^T v = w^T v + v_1$. But, by assumption, $w^T v$ and $\overline{w}^T v$ are integers. Thus, $v_1$ is an integer. We can repeat this for each component of $v$, so $v$ must be integral. $\square$

## 5.4 Total Unimodularity

**Proposition**

Let $A$ be an integral, nonsingular, $m \times n$ matrix. Then $A^{-1}b$ is integral for every integral vector $b \in \mathbb{R}^m$ if and only if $\det(A) = 1$ or $-1$.

***Proof.*** ( $\Longleftarrow$ ) Suppose $\det(A) = \pm 1$. By Cramer's Rule, we know that $A^{-1}$ is integral, which implies $A^{-1}b$ is integral for every integral $b$.

( $\Longrightarrow$ ) Conversely, suppose $A^{-1}b$ is integral for all integral vectors $b$. Then, in particular, $A^{-1}e_i$ is integral for all $i = 1, \dots, m$. This means that $A^{-1}$ is integral. So $\det(A)$ and $\det(A^{-1})$ are both integers. But, $\det(A) \cdot \det(A^{-1}) = 1$, this implies $\det(A) = \pm 1$. □

**Definition: Unimodular**

A matrix $A$ of full row rank is unimodular if $A$ is integral and each basis of $A$ has determinant $\pm 1$.

**Theorem (Veinott & Dantzig 1968)**

Let $A$ be an integral $m \times n$ matrix of full row rank. Then the polyhedron defined by $Ax = b, x \geq 0$ is integral for every integral vector $b \in \mathbb{R}^m$ if and only if $A$ is unimodular.

***Proof.*** ( $\Longleftarrow$ ) Suppose $A$ is unimodular. Let $b \in \mathbb{R}^m$ be an integral vector and let $\overline{x}$ be a vertex of $\{x : Ax = b, x \geq 0\}$. The nonnegativity constraints implies the polyhedron has vertices. Then there are $n$ linearly independent constraints satisfied by $\overline{x}$ with inequality. It follows that the columns of $A$ corresponding to the nonzero components of $\overline{x}$ are linearly independent. Extending these columns to a basis $B$ of $A$, we have the nonzero components of $\overline{x}$ are contained in the integral vector $B^{-1}b$. So $\overline{x}$ is integral.

( $\Longrightarrow$ ) Conversely, suppose $\{x : Ax = b, x \geq 0\}$ is integral for all integral vectors $b$. Let $B$ be a basis of $A$ and let $v$ be an integral vector in $\mathbb{R}^m$. By previous proposition, it suffices to show that $B^{-1}v$ is integral. Let $y$ be an integral vector such that $y + B^{-1}v \geq 0$ and let $b = B(y + B^{-1}v)$. Note $b$ is integral. Furthermore, by adding zero components to the vector $y + B^{-1}v$, we can obtain a vector $z \in \mathbb{R}^n$ such that $Az = b$. Then, $z$ is a vertex of $\{x : Ax = b, x \geq 0\}$, since $z$ is a polyhedron and satisfies $n$ linearly independent constraints with equality: the $m$ equations $Ax = b$ and the $n - m$ equations $x_i = 0$ for the columns $i$ outside $B$. So $z$ is integral, and thus, $B^{-1}v$ is integral. □

**Definition: Totally Unimodular (TU)**

A matrix is totally unimodular if all of its square submatrices have determinant 0, 1, or $-1$.

It is easy to see that $A$ is totally unimodular if and only if $\begin{bmatrix} A & I \end{bmatrix}$ is unimodular where $I \in \mathbb{R}^{m \times m}$.

**Theorem (Hoffman-Kruskal)**

Let $A$ be an $m \times n$ integral matrix. Then the polyhedron defined by $Ax \leq b, x \geq 0$ is integral for every integral vector $b \in \mathbb{R}^m$ if and only if $A$ is totally unimodular.

***Proof.*** Applying the linear programming trick of adding slack variables, we have that for any integral $b$, the polyhedron $\{x : Ax \leq b, x \geq 0\}$ is integral if and only if the polyhedron $\{z : \begin{bmatrix} A & I \end{bmatrix} z = b, z \geq 0\}$ is integral. So the result follows from previous theorem. $\square$

**Theorem**

Let $A$ be an $m \times n$ totally unimodular matrix and let $b \in \mathbb{R}^m$ be an integral vector. Then the polyhedron defined by $Ax \leq b$ is integral.

***Proof.*** Let $F$ be a minimal face of $\{x : Ax \leq b\}$. Then, by proposition, $F = \{x : A^\circ x = b^\circ\}$ for some subsystem $A^\circ x \leq b^\circ$ of $Ax \leq b$, with $A^\circ$ having full row rank. By reordering the columns, if necessary, we may write $A^\circ$ as $\begin{bmatrix} B & N \end{bmatrix}$ where $B$ is a basis of $A^\circ$. It follows

$$\overline{x} = \begin{bmatrix} B^{-1}b^\circ \\ 0 \end{bmatrix}$$

is an integral vector in $F$. $\square$

**Theorem**

Let $A$ be a $0, \pm 1$ valued matrix where each column has at most one $+1$ and at most $-1$. Then $A$ is totally unimodular.

***Proof.*** Let $N$ be a $k \times k$ submatrix of $A$. If $k = 1$, then $\det(N)$ is either $0$ or $\pm 1$. So we may suppose that $k \geq 2$ and proceed by induction on $k$. If $N$ has a column having at most one nonzero, then expanding the determinant along this column, we have that $\det(N)$ is either $0$ or $\pm 1$, by induction. On the other hand, if every column of $N$ has both a $+1$ and a $-1$, then the sum of the rows of $N$ is $0$ and hence $\det(N) = 0$. $\square$

Let $D = (V, E)$ be a digraph and let $A$ be its incidence matrix. Then $A$ is totally unimodular.

**Definition: Network Matrix**

Let $T = (V, E')$ be a spanning tree of $D$ and define the matrix $M$ having rows indexed by $E'$ and columns indexed by $E$, where $e = (u, v) \in E$ and $e' \in E'$.

$$M_{e',e} = \begin{cases} +1 & \text{if } uv\text{-path in } T \text{ uses } e' \text{ in forward direction} \\ -1 & \text{if } uv\text{-path in } T \text{ uses } e' \text{ in backward direction} \\ 0 & \text{if } uv\text{-path in } T \text{ does not use } e' \end{cases}$$

**Theorem (Tutte 1965)**

Network matrices are totally unimodular.

**Proposition**

$A$ is totally unimodular if and only if $A^T$ is totally unimodular.

## 5.5 Edmonds' Matching Polytope

For a graph $G = (V, E)$, let $\mathcal{PM}(G) \subseteq \mathbb{R}^E$ denote the set of characteristic vectors of its perfect matchings.

**Theorem (Perfect Matching Polytope Theorem)**

For any graph $G = (V, E)$, the convex hull of $\mathcal{PM}(G)$ is identical to the set of solutions of the linear system

$$\begin{aligned} x(\delta(v)) &= 1, \ \forall v \in V \\ x(\delta(S)) &\geq 1, \ \forall S \subseteq V, |S| \geq 3 \text{ and odd} \\ x_e &\geq 0, \ \forall e \in E \end{aligned}$$

**Theorem (Birkhoff)**

Let $G$ be a bipartite graph. Then the convex hull of the perfect matchings of $G$ is defined by

$$\begin{aligned} x(\delta(v)) &= 1, \ \forall v \in V \\ x_e &\geq 0, \ \forall e \in E \end{aligned}$$

**Theorem (Fractional Matching Polytope Theorem)**

Let $G$ be a graph and let $x \in FPM(G)$. Then $x$ is a vertex of $FPM(G)$ if and only if $x_e \in \left\{0, \frac{1}{2}, 1\right\}$ for all $e \in E$ and the edges $e$ for which $x_e = \frac{1}{2}$ form vertex-disjoint odd circuits.

***Proof.*** (Perfect Matching Polytope Theorem – Schrijver)

# Part III

# Optimal Trees and Paths

# Chapter 6

# Minimum Spanning Trees

## 6.1 Problem

**Definition: Spanning Tree**

A subgraph $T \subseteq G$ where $V(T) = V(G)$, $T$ is connected, and $T$ is acyclic.

**Lemma**

An edge $e = uv$ of $G$ is an edge of a circuit of $G$ if and only if there is a path in $G \setminus e$ from $u$ to $v$.

**Minimum Spanning Tree Problem (MST)**

Given a connected graph $G$ and a real cost $c_e$ for each $e \in E$, find a minimum cost spanning tree of $G$.

**Lemma**

A spanning connected subgraph of $G$ is a spanning tree if and only if it has exactly $n - 1$ edges.

## 6.2 Algorithm

**Theorem**

A graph $G$ is connected if and only if there is no set $A \subseteq V$ where $\emptyset \neq A \neq V$ with $\delta(A) = \emptyset$.

**Algorithm 1** Kruskal's Algorithm for MST

sort $E$ to $\{e_1, \ldots, e_m\}$ so that $c_{e_1} \leq \cdots \leq c_{e_m}$
$H = (V, F), F = \emptyset$
**for** $i = 1$ to $m$ **do**
    **if** ends of $e_i$ are in different components of $H$ **then**
        $F \leftarrow F \cup \{e_i\}$
**return** $H$

## 6.3 Linear Programming Relaxation

**Definition:** $\kappa : E \to \mathbb{N}$

$\kappa(A)$ is the number of components in the subgraph $(V, A)$ of $G$.

We can formulate the MST problem as an IP.

$$\begin{aligned} \min \quad & c^T x \\ \text{s.t.} \quad & x(A) \leq |V| - \kappa(A), \ \forall A \subset E \\ & x(E) = |V| - 1 \\ & x_e \in \{0, 1\}, \ \forall e \in E \end{aligned}$$

We can relax the integer program to get the following linear program.

**Definition: MST LP**

$$\begin{aligned} \min \quad & c^T x \\ \text{s.t.} \quad & x(A) \leq |V| - \kappa(A), \ \forall A \subset E \\ & x(E) = |V| - 1 \\ & x_e \geq 0, \ \forall e \in E \end{aligned}$$

We replace the minimization with a maximization in the primal to write the dual.

**Definition: MST Dual LP**

$$\begin{aligned} \min \quad & \sum_{A \subseteq E} (|V| - \kappa(A)) y_A \\ \text{s.t.} \quad & \sum (y_A : e \in A) \geq -c_e, \ \forall e \in E \\ & y_A \geq 0, \ \forall A \subset E \end{aligned}$$

**Theorem (Edmonds 1971)**

Let $x^*$ be the characteristic vector of an MST with respect to costs $c_e$. Then $x^*$ is an optimal solution of the linear program.

***Proof.*** We show that $x^*$ is optimal for the LP and $x^*$ is the characteristic vector generated by Kruskal's algorithm. $y_E$ is not required to be nonnegative.

Let $e_1, \ldots, d_m$ be the order in which Kruskal's algorithm considers the edges. Let $R_i = \{e_1, \ldots, e_i\}$ for $1 \leq i \leq m$. Let $y^*$ be the be the dual solution. We denote $y^*_A = 0$ unless $A$ is one of the $R_i$, $y^*_{R_i} = c_{e_{i+1}} - c_{e_i}$ for $1 \leq i \leq m-1$, and $y^*_{R_m} = -c_{e_m}$. It follows from the ordering of the edges, $y^*_A \geq 0$ for $A \neq E$. Now consider the first constraint, then where $e = e_i$, we have

$$\sum_{A:e \in A} y^*_A = \sum_{j=i}^{m} y^*_{R_j} = \sum_{j=i}^{m-1} (c_{e_{i+1}} - c_{e_i}) = -c_{e_i} = -c_e$$

All of the inequalities hold with equality. So the complementary slackness conditions ($x^*_e > 0 \implies \sum_{A:e \in A} y_A = c_e$) are satisfied.

We want to show now that the second constraint also satisfies complementary slackness conditions ($y^*_A > 0 \implies x(A) \leq |V| - \kappa(A)$). We know $A = R_i$ for some $i$. If the primal constraint does not hold with equality for $R_i$, then there is some edge of $R_i$ whose addition to $E(T) \cap R_i$ would decrease the number of components of $(V, E(T) \cap R_i)$. But this edge would have ends in two different components of $(V, E(T) \cap R_i)$, and therefore would have been added to $T$ by Kruskal's algorithm.

Therefore, $x^*$ and $y^*$ satisfy complementary slackness conditions. So, $x^*$ is an optimal solution to the LP. $\square$

# Chapter 7

# Shortest Paths

**Shortest Path Problem**

Given a digraph $G$, a vertex $r \in V$, and a real cost vector $(c_e : e \in E)$, find for each $v \in V$, a dipath from $r$ to $v$ of least cost.

Let $y_v$ for $v \in V$ be the least cost of a dipath to $v$, then $y$ s

**Definition: Feasible Potential**

$y = (y_v : v \in V)$ is a feasible potential if it satisfies $y_v + c_{vw} \geq y_w$ for all $vw \in E$ and $y_r = 0$.

**Proposition**

Let $y$ be a feasible potential and let $P$ be a dipath from $r$ to $v$. Then $c(P) \geq y_v$.

***Proof.*** Suppose that $P$ is $v_0, e_1, v_1, \ldots, e_k, v_k$ where $v_0 = r$ and $v_k = v$. Then

$$c(P) = \sum_{i=1}^{k} c_{e_i} \geq \sum_{i=1}^{k} (y_{v_i} - y_{v_{i-1}}) = y_{v_k} - y_{v_0} = y_v$$

□

## 7.1 Linear Programming

**Theorem**

Let $G$ be a digraph, $r, s \in V$, and $c \in \mathbb{R}^E$. If there exists a least-cost dipath from $r$ to $v$ for every $v \in V$, then

$$\min\{c(P) : P \text{ an } rs\text{-dipath}\} = \max\{y_s : y \text{ a feasible potential}\}$$

**Definition: Shortest Path LP**

$$\begin{aligned} \min \quad & \sum(c_e x_e : e \in E) \\ \text{s.t.} \quad & \sum(x_{wv} : w \in V, wv \in E) - \sum(x_{vw} : w \in V, vw \in E) = b_v, \ \forall v \in V \\ & x_{vw} \geq 0, \ \forall vw \in E \end{aligned}$$

**Definition: Shortest Path Dual LP**

$$\begin{aligned} \max \quad & y_s - y_r \\ \text{s.t.} \quad & y_w - y_v \leq c_{vw}, \ \forall vw \in E \end{aligned}$$

# Part IV

# Network Flows

# Chapter 8

# Maximum Flow

## 8.1 Problem

**Definition: Net Flow/Excess**

$$f_x(v) = x(\delta(\overline{v})) - x(\delta(v)) = \sum(x_{wv} : w \in V, wv \in E) - \sum(x_{vw} : w \in V, vw \in E)$$

**Definition: $rs$-Flow**

A vector $x$ that satisfies $f_x(v) = 0$ for all $v \in V$.

**Definition: Value of $rs$-Flow**

$$f_x(s)$$

**Maximum Flow Problem**

Given a digraph $G = (V, E)$, with source $r$ and sink $s$, find an $rs$-flow of maximum value.

**Proposition**

There exists a family $(P_1, \ldots, P_k)$ of $rs$-dipaths such that $|\{i : e \in P_i\}| \leq u_e$ for all $e \in E$ if and only if there exists an integral feasible $rs$-flow of value $k$.

***Proof.*** ( $\Longrightarrow$ ) We have seen family of dipaths determines a corresponding flow.

( $\Longleftarrow$ ) Let $x$ be a flow. We assume that $x$ is acyclic, that is, there is no dicircuit $C$, each of whose arcs $e$ has $x_e > 0$. If a dicircuit does exist, we can decrease $x_e$ by 1 on all arcs of $C$. The new $x$ remains feasible of value $k$.

If $k \geq 1$, we can find an arc $vs$ with $x_{vs} \geq 1$. Then, if $v \neq r$, it follows that there is an arc $wv$ with $x_{wv} \geq 1$ by the constraint $f_x(v) = 0$. If $w \neq r$, then the argument can be repeated producing distinct vertices, since $x$ is acyclic, so we get a simple $rs$-dipath $P_k$ on each arc $e$ with $x_e \geq 1$. We can decrease $x_e$ by 1 for each $e \in P_k$. The new $x$ is an integral feasible flow of value $k-1$, and the process is repeated. □

## 8.2 Maximum Flows and Minimum Cuts

**Definition: Maximum Flow LP**

$$\begin{aligned} \max \quad & f_x(s) \\ \text{s.t.} \quad & f_x(v) = 0, \ \forall v \in V \setminus \{r, s\} \\ & 0 \leq x_e \leq u_e, \ \forall e \in E \end{aligned}$$

**Definition: Path Flow**

A vector $x \in \mathbb{R}^E$ such that for some $rs$-dipath $P$ and some $\alpha \in \mathbb{R}$, $x_e = \alpha$ for each $e \in P$ and $x_e = 0$ for every other arc of $G$.

**Definition: Circuit Flow**

A vector $x \in \mathbb{R}^E$ such that for some $rs$-dicircuit $C$ and some $\alpha \in \mathbb{R}$, $x_e = \alpha$ for each $e \in C$ and $x_e = 0$ for every other arc of $G$.

**Proposition**

Every $rs$-flow of nonnegative value is the sum of at most $m$ flows, each of which is a path flow or a circuit flow.

**Proposition**

For any $rs$-cut $\delta(R)$ and any $rs$-flow $x$, we have

$$f_x(s) = x(\delta(R)) - x(\delta(\overline{R}))$$

***Proof.*** We add the equations $f_x(v) = 0$ for all $v \in \overline{R} \setminus \{s\}$ as well as the identity $f_x(s) = f_x(s)$. The right hand side sums to $f_x(s)$.

For any arc $vw$ with $v, w \in R$, $x_{vw}$ occurs in none of the equations, so it does not occur in the sum. If $v, w \in \overline{R}$, then $x_{vw}$ occurs in the equation for $v$ with a coefficient of $-1$, and in the equation for $w$ with a coefficient of $+1$, so it has a coefficient of 0 in the sum. If $v \in R, w \notin R$, then $x_{vw}$ occurs in the equation for $w$ with a coefficient of 1, and so has coefficient 1 in the sum. If $v \notin R, w \in R$, then $x_{vw}$ occurs in the sum with a coefficient of $-1$. So, the left hand side sums to $x(\delta(R)) - x(\delta(\overline{R}))$, as required. □

**Corollary**

For any feasible $rs$-flow $x$ and any $rs$-cut $\delta(R)$,

$$f_x(s) \leq u(\delta(R))$$

***Proof.*** Using previous proposition, since $x(\delta(R)) \leq u(\delta(R))$ and $x(\delta(\overline{R})) \geq 0$. $\square$

**Definition: Incrementing Path**

A path is $x$-incrementing if every forward arc $e$ has $x_e < u_e$ and every reverse arc $e$ has $x_e > 0$.

**Definition: Augmenting Path**

An $rs$-path that is $x$-incrementing.

**Theorem (Maximum-Flow Minimum-Cut)**

If there is a maximum $rs$-flow, then

$$\max\{f_x(s) : x \text{ is a feasible } rs\text{-flow}\} = \min\{u(\delta(R)) : \delta(R) \text{ is an } rs\text{-cut}\}$$

***Proof.*** By previous corollary, we need only show that there exists a feasible flow $x$ and a cut $\delta(R)$ such that $f_x(s) = u(\delta(R))$. Let $x$ be a flow of maximum value. Let $R = \{v \in V :$ there exists an $x$-incrementing $rv$-path$\}$. Clearly $r \in R$ and $s \notin R$, since there can be no $x$-augmenting path.

For every arc $vw \in \delta(R)$, we must have $x_{vw} = u_{vw}$, since otherwise adding $vw$ to the $x$-incrementing $rv$-path would yield such a path to $w$, but $w \notin R$. Similar, for every arc $vw \in \delta(\overline{R})$, we have $x_{vw} = 0$. Then by proposition, $f_x(s) = x(\delta(R)) - x(\delta(\overline{R})) = u(\delta(R))$. $\square$

**Theorem**

A feasible flow $x$ is maximum if and only if there is not $x$-augmenting path.

***Proof.*** ( $\implies$ ) If $x$ is maximum, there is no $x$-augmenting path.

( $\impliedby$ ) If there is no $x$-augmenting path, then the construction of the proof of Max-Flow Min-Cut yields a cut $\delta(R)$ with $f_x(s) = u(\delta(R))$, so $x$ is maximum, by corollary. $\square$

**Theorem**

If $u$ is integral and there exists a maximum flow, then there exists a maximum flow that is integral.

***Proof.*** Choose an integral flow $x$ of maximum value. If there is an $x$-augmenting path, then since $x$ and $u$ are integral, the new flow can be chosen integral, contradicting the choice of $x$. Hence there is no $x$-augmenting path, so $x$ is a maximum flow, by previous theorem. $\square$

**Corollary**

If $x$ is a feasible $rs$-flow and $\delta(R)$ is an $rs$-cut, then $x$ is maximum and $\delta(R)$ is minimum if and only if $x_e = u_e$ for all $e \in \delta(R)$ and $x_e = 0$ for all $e \in \delta(\overline{R})$.

***Proof.*** Combine Max-Flow Min-Cut theorem with the proof of corollary. □

## 8.3 Augmenting Path Algorithm

**Algorithm 2** Ford-Fulkerson Algorithm

$x = 0$
**while** there is an $x$-augmenting path $P$ **do**
  $\varepsilon_1 = \min(u_e - x_e : e$ forward in $P)$
  $\varepsilon_2 = \min(x_e : e$ reverse in $P)$
  $\varepsilon = \min(\varepsilon_1, \varepsilon_2)$ // $x$-width of $P$
  **if** $\varepsilon = \infty$ **then**
    no maximum flow
**return** $x$ is maximum flow, set $R$ of vertices reachable by an $x$-incrementing path from $r$ is minimum cut

**Definition: Auxiliary Digraph**

$G(x)$, depending on $G, u, x$, where $V(G(x)) = V$ and $vw \in E(G(x))$ if and only if $vw \in E$ and $x_{vw} < u_{vw}$ or $wv \in E$ and $x_{wv} > 0$.

$rs$-dipaths in $G(x)$ corresponding to $x$-augmenting paths in $G$. Each iteration of Ford-Fulkerson can be performed in $O(m)$ time, using breadth-first search.

**Theorem**

If $u$ is integral and the maximum flow value is $K < \infty$, then the maximum flow algorithm terminates after at most $K$ augmentations.

### 8.3.1 Shortest Augmenting Paths

**Theorem (Dinits 1970, Edmonds & Karp 1972)**

If each augmentation of the augmenting path algorithm on a shortest augmenting path, then there are at most $nm$ augmentations.

**Corollary**

The augmenting path algorithm with breadth-first search solves the maximum flow problem in time $O(nm^2)$.

Let $d_x(v,w)$ be the least length of a $vw$-dipath in $G(x)$. $d_x(v,w)=\infty$ if no $vw$-dipath exists.

Consider a typical augmentation from flow $x$ to flow $x'$ determined by the augmenting path $P$ having vertex-sequence $v_0,\ldots,v_k$.

**Lemma**

For each $v \in V$, $d_{x'}(r,v) \geq d_x(r,v)$ and $d_{x'}(v,s) \geq d_x(v,s)$.

***Proof.*** Suppose that there exists a vertex $v$ such that $d_{x'}(r,v) < d_x(r,v)$ and choose such $v$ so that $d_{x'}(r,v)$ is as small as possible. Clearly, $d_{x'}(r,v) > 0$. Let $P'$ be a $rv$-dipath in $G(x')$ of length $d_{x'}(r,v)$ and let $w$ be the second-last vertex of $P'$. Then

$$d_x(r,v) > d_{x'}(r,v) = d_{x'}(r,w) + 1 \geq d_x(r,w) + 1$$

It follows that $wv$ is an arc of $G(x')$, but not of $G(x)$, otherwise $d_x(r,v) \leq d_x(r,w)+1$, so $w = v_i$ and $v = v_{i-1}$ for some $i$. But, this implies that $i-1 > i+1$, a contradiction. The second statement is similar. $\square$

**Definition: $\tilde{E}(x)$**

$$\tilde{E}(x) = \{e \in E : e \text{ is an arc of a shortest } x\text{-augmenting path}\}$$

**Lemma**

If $d_{x'}(r,s) = d_x(r,s)$, then $\tilde{E}(x') \subsetneq \tilde{E}(x)$.

***Proof.*** Let $k = d_x(r,s)$ and suppose that $e \in \tilde{E}(x')$. Then $e$ induces an arc $vw$ of $G(x')$ and $d_{x'}(r,v) = i-1$, $d_{x'}(ws) = k-i$ for some $i$. Therefore, $d_x(r,v)+d_x(w,s) \leq k-1$ by previous lemma. Now suppose that $e \notin \tilde{E}(x)$, then $x_e \neq x'_e$, so $e$ is an arc of $P$, a contradiction. This proves $\tilde{E}(x') \subseteq \tilde{E}(x)$.

There is an arc $e$ of $P$ such that $e$ is forward and $x'_e = u_e$ or $e$ is reverse and $x'_e = 0$. Therefore, any $x'$-augmenting path using $e$ must use it in the opposite direction from $P$, so its length, for some $i$, will be at least $i+k-i+1+1 = k+23$, so $e \notin \tilde{E}(x')$. $\square$

***Proof.*** (Dinits, Edmonds, Karp) It follows from previous lemma that there can be at most $m$ augmentations per stage. Since there are at most $n-1$ stages, there are at most $nm$ augmentations in all.

## 8.4 Applications

### 8.4.1 Flow Feasibility

**Flow Feasibility Problem**

Given a digraph $G$, $u \in \mathbb{R}_+^E$, and $b \in \mathbb{R}^V$, find, if possible, $x \in \mathbb{R}^E$ such that

$$f_x(v) = b_v, \ \forall v \in V$$

and

$$0 \leq x_e \leq u_e, \ \forall e \in E$$

**Theorem (Gale 1957)**

There exists a solution to the flow feasibility problem if and only if $b(V) = 0$ and for every $A \subseteq V$, $b(A) \leq u(\delta(\overline{A}))$.
If $b$ and $u$ are integral, then there is an integral solution.

**Corollary**

Given a digraph $G$ and $b \in \mathbb{R}^V$, there exists $x \in \mathbb{R}^E$ with

$$f_x(v) = b_v, \ \forall v \in V$$

$$x_e \geq 0, \ \forall e \in E$$

if and only if $b(V) = 0$ and for every $A \subseteq V$ with $\delta(\overline{A}) = \emptyset$, we have $b(A) \leq 0$.

**Definition: Circulation**

A vector $x \in \mathbb{R}^E$ with $f_x(v) = 0$ for all $v \in V$.

**Theorem (Hoffman's Circulation Theorem 1960)**

Given a digraph $G$, $\ell \in (\mathbb{R} \cup \{-\infty\})^E$, and $u \in (\mathbb{R} \cup \{\infty\})^E$, with $\ell \leq u$, there is a circulation $x$ with $\ell \leq x \leq u$ if and only if every $A \subseteq V$ satisfies $u(\delta(\overline{A})) \geq \ell(\delta(A))$.

# Part V

# Matchings

# Chapter 9

# Matchings

**Definition: Matching**

A set $M \subseteq E$ such that no vertex of $G$ is incident with more than one edge in $M$.

**Definition: $M$-Covered**

A vertex $v$ is covered by $M$ if some edge of $M$ is incident with $v$.

**Definition: $M$-Exposed**

A vertex $v$ is exposed if $v$ is not $M$-covered.

The number of vertices covered by $M$ is $2|M|$ and number of $M$-exposed vertices is $|V| - 2|M|$.

**Definition: Maximum Matching**

A matching of maximum cardinality, denoted $\nu(G)$.

**Definition: Deficiency**

The minimum number of exposed vertices for any matching of $G$, denoted by $\operatorname{def}(G)$.

Note $\operatorname{def}(G) = |V| - 2\nu(G)$.

**Definition: Perfect Matching**

A matching that covers all vertices.

## 9.1 Bipartite Matching

**Definition: Bipartite**

$G = (V, E)$ is bipartite if $V = V_1 \cup V_2$, where $V_1, V_2$ disjoint and every edge has one end in $V_1$ and the other end in $V_2$.

**Definition: Cover**

A set $C \subseteq V$ such that every edge has at least one in $C$.

**Lemma**

If $M$ is a matching and $C$ is a cover, then $|M| \leq |C|$.

***Proof.*** Every $e \in M$ has at least one end in $C$. No vertex in $C$ meets more than one edge in $M$. □

**Definition: Minimum Cover**

A cover of minimum cardinality, denoted $\tau(G)$.

**Theorem (König)**

If $G$ is bipartite, $\nu(G) = \tau(G)$.

***Proof.*** We note that $\nu(G) \leq \nu^*(G)$ and $\tau(G) \geq \tau^*(G)$. By using LP duality and the matching LP (*Matching LP*), we show that $\nu(G) = \nu^*(G)$. We also have the matching LP in the form of $Mx^+ = (1, \ldots, 1)^T$. Since $M$ is totally unimodular, then $M^T$ is also totally unimodular. So the dual LP has all integral vertices, implying $\tau(G) = \tau^*(G)$. So,

$$\nu(G) = \nu^*(G) = \tau^*(G) = \tau(G)$$

□

## 9.2 Alternating Paths

**Definition: $M$-Alternating**

A path $P$ is $M$-alternating if its edges are alternately in and not in $M$.

**Definition: $M$-Augmenting**

An $M$-alternating path $P$ is $M$-augmenting if the ends of $P$ are distinct and are both $M$-exposed.

**Definition: Symmetric Difference**

For sets $S$ and $T$, let $S\Delta T$ denote the symmetric difference, which is defined as

$$S\Delta T = (S \cup T) \setminus (S \cap T)$$

Let a path $P$ be an $M$-augmenting path. Then we can obtain a larger matching $M' = M\Delta E(P)$ with $|M'| = |M| + 1$.

**Theorem (Petersen 1891, Berge 1957)**

A matching $M$ in a graph $G$ is maximum if and only if there is no $M$-augmenting path.

***Proof.*** ( $\implies$ ) Suppose there exists an $M$-augmenting path $P$ joining $v$ and $w$. Then $N = M\Delta E(P)$ is a matching that covers all vertices covered by $M$, plus $v$ and $w$. So, $M$ is not maximum.

( $\impliedby$ ) Conversely, suppose that $M$ is not maximum and some other matching $N$ satisfies $|N| > |M|$. Let $J = N\Delta M$. Each vertex of $G$ is incident with at most two edges of $J$, so $J$ is the edge set of some vertex disjoint paths and circuits of $G$. For each such path or circuit, the edges alternately belong to $M$ or $N$. Therefore, all circuits are even and contain the same number of edges of $M$ and $N$. Since $|N| > |M|$, there must be at least one path with more edges of $N$ than $M$. This path is an $M$-augmenting path. $\square$

## 9.3 Matching LP

**Definition: Matching LP**

$P$ is the set of solutions to

$$\begin{aligned} x(\delta(v)) &\leq 1, \ \forall v \in V \\ x_e &\geq 0, \ \forall e \in E \end{aligned}$$

Let $\overline{x}$ be a vertex of $P$. We show that $\overline{x}$ is integral, which implies that $M = \{e \in E : \overline{x}_e = 1\}$ is a matching and $\nu(G) = \nu^*(G)$.

Recall that for a polyhedron $P = \{x : Ax \leq b\} \subseteq \mathbb{R}^n$, $\overline{x} \in P$ is a vertex if and only if $\overline{x}$ is the unique solution to $A'x = b'$ for some subset of $n$ inequalities $A'x \leq b'$ from $Ax \leq b$.

For our matching $P$, let $E^+ := \{e : \overline{x}_e > 0\}$ and $E^0 := \{x : \overline{x}_e = 0\}$. We write $\overline{x} = (\overline{x}^+, \overline{x}^0)$ split by $(E^+, E^0)$.

Since $\overline{x}$ is a vertex, there exists $V^+ \subseteq V$ such that $\overline{x}$ is the unique solution to

$$\begin{aligned} \sum(x_e : e \in \delta(v) \cap E^+) &= 1, \ \forall v \in V^+ \\ x_e &= 0, \ \forall e \in E^0 \end{aligned}$$

Restricting to $E^+$, we can write the system of equations as

$$Mx^+ = (1, \dots, 1)^T$$

By Cramer's Rule, the solution to the system is $(\overline{x}_1^+, \dots, \overline{x}_k^+)$, where

$$\overline{x}_j^+ = \frac{\det(M^j)}{\det(M)}$$

with $M^j$ obtained from $M$ by replacing the $j$th column by $(1, \dots, 1)^T$.

Claim: $\det(M) = 1$ or $\det(M) = -1$.

This gives that $\overline{x}_j^+$ is integer for all $j$, so $\overline{x}$ is integer. Thus, $\nu(G) = \nu^*(G)$.

**Lemma**

Let $G = (V, E)$ be a bipartite graph. Let $A$ be the $|V| \times |E|$ matrix $\left[A_{ve}\right]$ with

$$A_{ve} = \begin{cases} 1 & \text{if } e \in \delta(v) \\ 0 & \text{if } e \notin \delta(v) \end{cases}$$

then $A$ is totally unimodular.

***Proof.*** By induction of the number of rows $k$ of the submatrix $B$ of $A$. If $B$ is $1 \times 1$, then this is obvious.

Suppose it is true for $k = 1, \dots, t-1$ and let $B$ be a $t \times t$ submatrix of $A$.

1. If $B$ has a column of all 0's, then $\det(B) = 0$.
2. If a column of $B$ has exactly one 1, then we compute $\det(B)$ by expanding on that column and use induction.
3. Otherwise, every column of $B$ has exactly two 1's.

We can partition the rows of $B$ into $W_1$ and $W_2$, so that every column has exactly one 1 in $W_1$ and exactly one 1 in $W_2$ ($W_1$ are vertices in $V_1$, $W_2$ in $V_2$ from $G$ being bipartite).

Now multiplying each row in $W_1$ by 1 and each row in $W_2$ by $-1$ and summing, we get the row vector of all 0's. So $\det(B) = 0$. □

## 9.4 Tutte-Berge Formula

**Definition: Vertex Cover**

A set $A$ of vertices such that every edge has at least one end in $A$.

Let $A$ be a subset of the vertices which $G - A$ has $k$ components $H_1, \dots, H_k$ having an odd number of vertices. Let $M$ be a matching of $G$. For each $i$, either $H_i$ has an $M$-exposed vertex or $M$ contains an edge having just one end in $V(H_i)$. All such edges have their other ends in $A$ and since $M$ is a matching, all these ends must be distinct. Therefore, there can be at most $|A|$ edges and so the number of $M$-exposed vertices is at least $k - |A|$.

**Definition:** $\mathrm{oc}(H)$

The number of odd components of a graph $H$.

Thus, for any $A \subseteq V$,

$$\nu(G) \leq \frac{1}{2}(|V| - \mathrm{oc}(G - A) + |A|)$$

If $A$ is a cover of $G$, then there are $|V| - |A|$ odd components of $G - A$ (each is a single vertex), so the right hand side reduces to $|A|$. This bound is at least as strong as that provided by covers.

**Theorem (Tutte-Berge Formula)**

For a graph $G = (V, E)$, we have

$$\max\{|M| : M \text{ a matching}\} = \min\left\{\frac{1}{2}(|V| - \mathrm{oc}(G - A) + |A|) : A \subseteq V\right\}$$

**Theorem (Tutte's Matching Theorem 1947)**

A graph $G = (V, E)$ has a perfect matching if and only if for every $A \subseteq V$, $\mathrm{oc}(G - A) \leq |A|$.

**Definition: Shrink**

Let $C$ be an odd circuit in $G$. Define $G' = G \times C$ as the subgraph obtained from $G$ by shrinking $C$; $G'$ has vertex set $(V - V(C)) \cup \{C\}$ and edge set $E \setminus \gamma(V(C))$.

**Proposition**

Let $C$ be an odd circuit of $G$, let $G' = G \times C$, and let $M'$ be a matching of $G'$. Then here is a matching $M$ of $G$ such that $M \subseteq M' \cup E(C)$ and the number of $M$-exposed vertices of $G$ is the same as the number of $M'$-exposed vertices of $G'$.

***Proof.*** Choose a vertex $w \in V(C)$ as follows. If $C$ is covered by $e \in M'$, then choose $w$ to be the vertex in $V(C)$ that is an end of $e$, and otherwise, choose $w$ arbitrarily. Deleting $w$ from $C$ results in a subgraph having a perfect matching $M''$. Take $M = M' \cup M''$. $M$ has the required properties. □

The previous proposition gives the inequality

$$\nu(G) \geq \nu(G \times C) + \frac{|V(C)| - 1}{2}$$

or equivalently,

$$\operatorname{def}(G) \leq \operatorname{def}(G \times C)$$

**Definition: Tight Odd Circuit**

An odd circuit $C$ is tight if $\nu(G) = \nu(G \times C) + \frac{|V(C)|-1}{2}$.

**Definition: Inessential**

A vertex $v$ of $G$ is inessential if there is a maximum matching of $G$ that does not cover $v$.

**Definition: Essential**

A vertex not inessential.

Let $A$ be a set that satisfies the Tutte-Berge formula. Let $v \in A$ and consider $G' = G - v$. Then, $G' - (A \setminus \{v\})$ has the same odd components as $G - A$, so $\nu(G') < \nu(G)$, i.e. every $v \in A$ is essential.

**Lemma**

Let $G = (V, E)$ be a graph and let $vw \in E$. If $v, w$ are both inessential, then there is a tight odd circuit $C$ using $vw$. Moreover, $C$ is an inessential vertex of $G \times C$.

**Definition: Gallai-Edmonds Partition**

Let $B$ be the set of inessential vertices of $G = (V, E)$, $C$ be the set of vertices not in $B$ but adjacent to at least one element of $B$, and $D$ be $V \setminus (B \cup C)$, then $(B, C, D)$ is the Gallai-Edmonds Partition of $G$.

**Proposition**

Let $(B, C, D)$ be the Gallai-Edmonds Paritition for $G$.
$C$ is a minimizer in the Tutte-Berge formula.

**Proposition**

Let $(B, C, D)$ be the Gallai-Edmonds Paritition for $G$.
For every maximum matching $M$ and every vertex $v \in C$, there is an edge $vw \in M$ with $w \in B$.

**Proposition**

Let $(B, C, D)$ be the Gallai-Edmonds Paritition for $G$.
Every maximum matching contains a perfect matching of $G[D]$.

## 9.5 Maximum Matching

**Maximum Matching Problem**

Given a graph $G$, find a maximum matching of $G$.

**Definition: Maximum Matching ILP**

$$\begin{aligned} \max \quad & \sum (x_e : e \in E) \\ \text{s.t.} \quad & x(\delta(v)) \leq 1, \ \forall v \in V \\ & x_e \geq 0, \ \forall e \in E \\ & x_e \text{ integer}, \ \forall e \in E \end{aligned}$$

**Definition: Maximum Matching LP Relaxation**

$$\begin{aligned} \max \quad & \sum (x_e : e \in E) \\ \text{s.t.} \quad & x(\delta(v)) \leq 1, \ \forall v \in V \\ & x_e \geq 0, \ \forall e \in E \end{aligned}$$

**Definition: Minimum Cover Dual LP**

$$\begin{aligned} \min \quad & \sum (y_v : v \in V) \\ \text{s.t.} \quad & y_u + y_v \geq 1, \ \forall e = (u, v) \in E \\ & y_v \geq 0, \ \forall v \in V \end{aligned}$$

Let $M$ be a matching and $C$ be a cover, then

$$x_e^M = \begin{cases} 1 & \text{if } e \in M \\ 0 & \text{if } e \notin M \end{cases}, y_v^C = \begin{cases} 1 & \text{if } v \in C \\ 0 & \text{if } v \notin C \end{cases}$$

So, $\nu(G) \leq \nu^*(G)$ and $\tau(G) \geq \tau^*(G)$, and by LP duality, we have

$$\nu(G) \leq \nu^*(G) = \tau^*(G) \leq \tau(G)$$

## 9.6 Perfect Matching

### 9.6.1 Alternating Trees

Suppose we have a matching $M$ of $G$ and a fixed $M$-exposed vertex $r$ of $G$. We can iteratively build up sets $A, B$ of vertices such that each vertex in $A$ is the other end of an odd-length $M$-alternating path beginning at $r$, and each vertex in $B$ is the other end of an even-length $M$-alternating path beginning at $r$.

Begin with $A = \emptyset, B = \{r\}$, and use the rule: if $vw \in E, v \in B, w \notin A \cup B, wz \in M$, then add $w$ to $A$, $z$ to $B$. The set $A \cup B$ and edges in the construction form a tree $T$ rooted at $r$.

**Definition: Alternating Tree**

A tree $T$ such that

- every vertex of $T$ other than $r$ is covered by an edge of $M \cap E(T)$;
- for every vertex $v$ of $T$, the path in $T$ from $v$ to $r$ is $M$-alternating.

We let the vertex sets at odd and even distances from the root as $A(T)$ and $B(T)$ respectively. Note that $|B(T)| = |A(T)| + 1$ since all other vertices other than $r$ come in matched pairs, one in $A(T)$ and one in $B(T)$.

**Using $vw$ to Extend $T$**

**Input**: A matching $M'$ of a graph $G'$, an $M'$-alternating tree $T$, and an edge $vw$ of $G'$ such that $v \in B(T)$, $w \notin V(T)$, and $w$ is $M'$-covered.
**Algorithm**: Let $wz$ be the edge in $M'$ covering $w$ (but $z$ is not a vertex of $T$). Replace $T$ by the tree having edge set $E(T) \cup \{vw, wz\}$.

**Use $vw$ to Augment $M'$**

**Input**: A matching $M'$ of a graph $G'$, an $M'$-alternating tree $T$ of $G'$ with root $r$, and an edge $vw$ of $G'$ such that $v \in B(T)$, $w \notin V(T)$, and $w$ is $M'$-exposed.
**Algorithm**: Let $P$ be the path obtained by attaching $vw$ to the path from $r$ to $v$ in $T$. Replace $M'$ by $M' \Delta E(P)$.

**Definition: Frustrated**

An $M$-alternating tree $T$ in a graph $G$ is frustrated if every edge of $G$ has one end in $B(T)$ and the other end in $A(T)$.

**Proposition**

Suppose that $G$ has a matching $M$ and an $M$-alternating tree $T$ that is frustrated. Then $G$ has no perfect matching.

***Proof.*** Clearly, every element of $B(T)$ is a single-vertex odd component of $G \setminus A(T)$. Since $|A(T)| < |B(T)|$, then $G$ has no perfect matching. $\square$

### 9.6.2 Bipartite Perfect Matching Algorithm

The following algorithm is to obtain a perfect matching in a bipartite graph based on alternating trees or outputs no perfect matching.

**Algorithm 3** Bipartite Perfect Matching Algorithm

```
M = ∅
Choose an M-exposed vertex r
T = ({r}, ∅)
while there exists vw ∈ E with v ∈ B(T), w ∉ V(T) do
    if w is M-exposed then
        Use vw to augment M
        if there is no M-exposed vertex in G then
            return Perfect matching M
        else
            T = ({r}, ∅), where r is M-exposed
    else
        Use vw to extend T
return G has no perfect matching
```

**Proposition**

Suppose that $G$ is bipartite, $M$ is a matching of $G$, and $T$ is an $M$-alternating tree such that no edge of $G$ joins a vertex in $B(T)$ to a vertex not in $V(T)$. Then $T$ is frustrated, and hence $G$ has no perfect matching.

***Proof.*** We show that every edge having an end in $B(T)$ has an end in $A(T)$. From the hypothesis, the only possible exception would be an edge joining two vertices in $B(T)$. But this edge, together with the paths joining them to the root of $T$, would form a closed path of odd length, which contradicts $G$ being bipartite. Hence $T$ is frustrated, and so by previous proposition, $G$ has no perfect matching. □

### 9.6.3 Blossom Algorithm for Perfect Matching

**Definition: Derived Graph**

A graph $G'$ obtained from $G$ by a sequence of odd-circuit shrinkings.

**Definition: Original Vertex**

A vertex in the derived graph $G'$ that is in $G$.

**Definition: Pseudonode**

A vertex in the derived graph $G'$ not in $G$.

**Definition:** $S(v)$

Given a vertex $v$ of $G'$, there corresponds a set $S(v)$ of vertices of $G$, where

$$S(v) = \begin{cases} v & \text{if } v \in V(G) \\ \bigcup_{w \in V(C)} S(w) & \text{if } v = C \text{ is a pseudonode} \end{cases}$$

**Proposition**

Let $G'$ be a derived graph of $G$, $M'$ be a matching of $G'$, and $T$ be an $M'$-alternating tree of $G'$ such that no element of $A(T)$ is a pseudonode. If $T$ is frustrated, then $G$ has no perfect matching.

***Proof.*** When deleting $A(T)$ from $G$, we get a component with vertex set $S(v)$ for each $v \in B(T)$. Therefore, $\text{oc}(G \setminus A(T)) > |A(T)|$, so $G$ has no perfect matching by Tutte's theorem. □

**Definition: Blossom**

Let $v, w \in B(T)$ and $vw \in E(G)$. The odd circuit in $T + vw$ is a blossom.

When we shrink a blossom, we get a pseudonode and the new graph is a derived graph.

**Use** $vw$ **to Shrink and Update** $M'$ **and** $T$

**Input**: A matching $M'$ of a graph $G'$, an $M'$-alternating tree $T$, and an edge $vw$ of $G'$ such that $vw \in B(T)$.
**Algorithm**: Let $C$ the circuit formed by $vw$ with the $vw$-path in $T$. Replace $G'$ with $G' \times C$, $M'$ by $M' \setminus E(C)$, and $T$ by the tree in $G'$ having edge set $E(T) \setminus E(C)$.

**Proposition**

After application of the shrinking subroutine, $M'$ is a matching of $G'$, $T$ is an $M'$-alternating tree of $G'$, and $C \in B(T)$.

**Algorithm 4** Blossom Algorithm for Perfect Matching

**Input**: Graph $G$ and matching $M$ of $G$
$M' = M$
$G' = G$
Choose an $M'$-exposed vertex $r$ of $G'$
$T = (\{r\}, \emptyset)$
**while** there exists $vw \in E'$ with $v \in B(T), w \notin A(T)$ **do**
  **Case**: $w$ is $M'$-exposed
    Use $vw$ to augment $M'$
    Extend $M'$ to a matching $M$ of $G$
    Replace $M'$ by $M$, $G'$ by $G$
    **if** there is no $M'$-exposed vertex in $G'$ **then**
      **return** Perfect matching $M'$
    **else**
      $T = (\{r\}, \emptyset)$, where $r$ is $M'$-exposed
  **Case**: $w \notin V(T)$, $w$ is $M'$-covered
    Use $vw$ to extend $T$
  **Case**: $w \in B(T)$
    Use $vw$ to shrink and update $M'$ and $T$
**return** $G', M', T$, $G$ has no perfect matching

**Theorem**

The Blossom Algorithm terminates after $O(n)$ augmentations, $O(n^2)$ shrinking steps, and $O(n^2)$ tree-extension steps.
Moreover, it determines correctly whether $G$ has a perfect matching.

## 9.7 Blossom Algorithm for Maximum Matching

We can extend the Blossom algorithm for perfect matchings to maximum matchings.

**Algorithm 5** Blossom Algorithm for Maximum Matching

1: **Input**: Graph $G$ and matching $M$ of $G$
2: $M' = M, G' = G, \mathcal{T} = \emptyset$
3: Choose an $M'$-exposed vertex $r$ of $G'$
4: $T = (\{r\}, \emptyset)$
5: **while** there exists $vw \in E'$ with $v \in B(T), w \notin A(T)$ **do**
6:     **Case**: $w$ is $M'$-exposed
7:         Use $vw$ to augment $M'$
8:         Extend $M'$ to a matching $M$ of $G$
9:         Replace $M'$ by $M$, $G'$ by $G$
10:         **if** there is no $M'$-exposed vertex in $G'$ **then**
11:             **return** Perfect matching $M'$
12:         **else**
13:             $T = (\{r\}, \emptyset)$, where $r$ is $M'$-exposed
14:     **Case**: $w \notin V(T)$, $w$ is $M'$-covered
15:         Use $vw$ to extend $T$
16:     **Case**: $w \in B(T)$
17:         Use $vw$ to shrink and update $M'$ and $T$
18: $\mathcal{T} = \mathcal{T} \cup \{T\}, G' = G \setminus V(T), M' = M \setminus E(T)$
19: **if** there exists an $M'$-exposed vertex **then**
20:     Go to line 5
21: Restore the matching $M$
22: **return** $M$

**Theorem**

The Blossom Algorithm can be implemented to run in time $O(nm \log n)$.

# Chapter 10

# Weighted Matchings

## 10.1 Minimum-Weight Perfect Matching

**Definition: Minimum-Weight Perfect Matching ILP**

$$\begin{aligned} \min \quad & \sum(c_e x_e : e \in E) \\ \text{s.t.} \quad & x(\delta(v)) = 1, \ \forall v \in V \\ & x_e \in \{0,1\}, \ \forall e \in E \end{aligned}$$

**Definition: Minimum-Weight Perfect Matching LP Relaxation**

$$\begin{aligned} \min \quad & \sum(c_e x_e : e \in E) \\ \text{s.t.} \quad & x(\delta(v)) = 1, \ \forall v \in V \\ & x_e \geq 0, \ \forall e \in E \end{aligned}$$

**Definition: Minimum-Weight Perfect Matching Dual LP**

$$\begin{aligned} \max \quad & \sum(y_v : v \in V) \\ \text{s.t.} \quad & y_u + y_v \leq c_e, \ \forall e = uv \in E \end{aligned}$$

**Definition: Complementary Slackness Conditions for Minimum-Weight Perfect Matching**

If $x_e > 0$, then $\bar{c}_e = c_e - y_u - y_v = 0$ for all $e \in E$.

## 10.2 Minimum-Weight Perfect Matching in Bipartite Graphs

**Theorem (Birkhoff)**

Let $G$ be a bipartite graph and let $c \in \mathbb{R}^E$. Then $G$ has a perfect matching if and only if the Minimum-Weight Perfect Matching LP Relaxation has a feasible solution. Moreover, if $G$ has a perfect matching, then the minimum weight of a perfect matching is equal to the optimal value of the LP relaxation.

**Definition: $E_=$**

$E_= = \{e \in E : \bar{c}_e = 0\}$.

**Algorithm 6** Bipartite Minimum-Weight Perfect Matching Algorithm

Let $y$ be a feasible solution to the dual LP
$M$ is a matching of $G_= = (V, E_=)$
$T = (\{r\}, \emptyset)$, where $r$ is an $M$-exposed vertex of $G$
**while** true **do**
    **while** there exists $vw \in E_=$ with $v \in B(T), w \notin V(T)$ **do**
        **if** $w$ is $M$-exposed **then**
            Use $vw$ to augment $M$
            **if** there is no $M$-exposed vertex in $G$ **then**
                **return** Perfect matching $M$
            **else**
                $T = (\{r\}, \emptyset)$, where $r$ is $M$-exposed
        **else**
            Use $vw$ to extend $T$
    **if** every $vw \in E$ with $v \in B(T)$ has $w \in A(T)$ **then**
        **return** $G$ has no perfect matching
    **else**
        $\varepsilon = \min\{\bar{c}_{vw} : v \in B(T), w \notin V(T)\}$
        $y_v = y_v + \varepsilon$ for $v \in B(T)$
        $y_v = y_v - \varepsilon$ for $v \in A(T)$

## 10.3 Minimum-Weight Perfect Matching in General Graphs

**Definition: Odd Cut**

A set of the form $\delta(S)$ where $S$ is an odd-cardinality set of vertices.

**Definition: Blossom Inequality**

If $x$ is the characteristic vector of a perfect matching, then for every odd cut $D$ of $G$,

$$x(D) \geq 1$$

Let $\mathcal{C}$ denote the set of all odd cuts of $G$ that are *not* of the form $\delta(v)$ for some vertex $v$.

**Definition: Minimum-Weight Perfect Matching LP – Stronger**

$$\begin{aligned} \min \quad & \sum(c_e x_e : e \in E) \\ \text{s.t.} \quad & x(\delta(v)) = 1, \ \forall v \in V \\ & x(D) \geq 1, \ \forall D \in \mathcal{C} \\ & x_e \geq 0, \ \forall e \in E \end{aligned}$$

**Definition: Minimum-Weight Perfect Matching Dual LP – Stronger**

$$\begin{aligned} \max \quad & \sum(y_v : v \in V) + \sum(Y_D : D \in \mathcal{C}) \\ \text{s.t.} \quad & y_v + y_w + \sum(Y_D : e \in D \in \mathcal{C}) \leq c_e, \ \forall e = vw \in E \\ & Y_D \geq 0, \ \forall D \in \mathcal{C} \end{aligned}$$

**Theorem**

Let $G$ be a graph and let $c \in \mathbb{R}^E$. Then $G$ has a perfect matching if and only if the Minimum-Weight Perfect Matching LP has a feasible solution.
Moreover, if $G$ has a perfect matching, then the minimum weight of a perfect matching is equal to the optimal value of the LP.

**Change $y$**

**Input**: A derived pair $(G', c')$, a feasible solution $y$ of stronger dual LP for this pair, a matching $M'$ of $G'$ consisting of equality edges, and an $M'$-alternating tree $T$ consisting of equality edges in $G'$.
**Algorithm**:

1. $\varepsilon_1 = \min(\bar{c}_e : e$ joins in $G'$ a vertex in $B(T)$ to a vertex not in $V(T))$
2. $\varepsilon_2 = \min(\bar{c}_e/2 : e$ joins in $G'$ two vertices in $B(T))$
3. $\varepsilon_3 = \min(y_v : v \in A(T), v$ is a pseudonode of $G')$
4. $\varepsilon = \min(\varepsilon_1, \varepsilon_2, \varepsilon_3)$
5. Replace
$$y_v = \begin{cases} y_v + \varepsilon & \text{if } v \in B(T) \\ y_v - \varepsilon & \text{if } v \in A(T) \\ y_v & \text{otherwise} \end{cases}$$

**Expand Odd Pseudonode $v$ and Update $M', T, c'$**

**Input**: A matching $M'$ consisting of equality edges of a derived graph $G'$, an $M'$-alternating tree $T$ consisting of equality edges, and an odd pseudonode $v$ of $G'$ such that $y_v = 0$.
**Algorithm**: Let $f, g$ be the edges of $T$ incident with $v$, let $C$ be the circuit that was shrunk to form $v$, let $u, w$ be the ends of $f, g$ in $V(C)$, and let $P$ be the even-length path in $C$ joining $u$ to $w$.
Replace $G'$ by the graph obtained by expanding $C$. Replace $M'$ by the matching obtained by extending $M'$ to a matching of $G'$. Replace $T$ by the tree having edge set $E(T) \cup E(P)$. For each edge $st$ with $s \in V(C)$ and $t \notin V(C)$, replace $c'_{st}$ by $c'_{st} + y_s$.

**Proposition**

After the application of the expand subroutine, $M'$ is a matching contained in $E_=$, and $T$ is an $M'$-alternating tree whose edges are all contained in $E_=$.

**Theorem**

The Blossom Algorithm terminates after $O(n)$ augmentation steps and $O(n^2)$ tree-extension, shrinking, expanding, and dual change steps.
Moreover, it returns a minimum-weight perfect matching or determines correctly that $G$ has no perfect matching.

**Algorithm 7** Blossom Algorithm for Minimum-Weight Perfect Matching

Let $y$ be a feasible solution to the dual LP, $M'$ a matching of $G_=$, $G' = G$<br>
$T = (\{r\}, \emptyset)$, where $r$ is an $M'$-exposed vertex of $G'$<br>
**while** true **do**<br>
    **Case**: There exists $e \in E_=$ whose ends in $G'$ are $v \in B(T)$ and an $M'$-exposed vertex $w \notin V(T)$<br>
        Use $vw$ to augment $M'$<br>
        **if** there is no $M'$-exposed vertex in $G'$ **then**<br>
            Extend $M'$ to a perfect matching $M$ of $G$ and return $M$<br>
        **else**<br>
            $T = (\{r\}, \emptyset)$, where $r$ is $M'$-exposed.<br>
    **Case**: There exists $e \in E_=$ whose ends in $G'$ are $v \in B(T)$ and an $M'$-covered vertex $w \notin V(T)$<br>
        Use $vw$ to extend $T$<br>
    **Case**: There exists $e \in E_=$ whose ends in $G'$ are $v, w \in B(T)$<br>
        Use $vw$ to shrink and update $M', T, c'$<br>
    **Case**: There is a pseudonode $v \in A(T)$ with $y_v = 0$<br>
        Expand $v$ and update $M', T, c'$<br>
    **Case**: None of the above<br>
        **if** every $e \in E$ incident in $G'$ with $v \in B(T)$ has its other end in $A(T)$ and $A(T)$ contains no pseudonode **then**<br>
            Stop, $G$ has no perfect matching<br>
        **else**<br>
            Change $y$

# Chapter 11

# T-Joins

**Definition: Euler Tour**

A closed edge-simple path $P$ such that $E(P) = E(G)$.

# Part VI

# Matroids

# Part VII

# Traveling Salesman Problem