

# CO 444/644 Algebraic Graph Theory

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# Chapter 1

## Introduction

We will focus on simple, undirected graphs without loops unless explicitly stated. We use  $X = (V, E)$  to denote graphs and  $G$  for groups.  $V(X)$  and  $E(X)$  are the sets of vertices and edges of graph  $X$  respectively and  $\deg(v)$  to denote the degree of a vertex  $v \in V(X)$ .

### Definition: Isomorphism

An isomorphism between graphs  $X, Y$  is a function  $f : V(X) \rightarrow V(Y)$  such that  $uv \in E(X)$  if and only if  $f(u)f(v) \in E(Y)$ .

## 1.1 Automorphisms

### Definition: Automorphism

An automorphism of the graph  $X$  is an isomorphism  $f : X \rightarrow X$ .

$\text{Aut}(X)$  is the set of all automorphisms of  $X$ .

$\text{Sym}(V)$  is used to denote the symmetric group of permutations on  $V$ . In group theory, we may have used  $V = [n]$ . We may use this notation alongside  $\text{Sym}(n)$  when explicitly enumerating the vertices of a graph from 1 to  $n$ .

### Proposition

$\text{Aut}(X) \leq \text{Sym}(V(X))$  with the group operation for  $\sigma, \tau \in \text{Aut}(X)$  defined  $\sigma\tau := \tau \circ \sigma$ .

For  $g \in \text{Sym}(V(X))$  and  $v \in V(X)$ , let  $v^g$  denote  $g(v)$ . Let  $S^g$  denote  $\{g(v) : v \in S\}$  for set  $S$ .

Suppose  $Y \subseteq X$  is a subgraph and  $g \in \text{Aut}(X)$ .  $Y^g$  is the graph defined  $V(Y^g) = V(Y)^g$  and  $E(Y^g) = \{u^g v^g : uv \in E(Y)\}$ .

**E.g.** The following is an example of graphs  $X$  and  $Y$  along with functions that are and are

not automorphisms.

Let  $X = (\{1, 2, 3, 4\}, \{12, 13, 14, 23, 34\})$ ,  $Y = (\{1, 2, 3\}, \{12, 13, 23\})$ ,  $Y^g = (\{1, 3, 4\}, \{13, 14, 34\})$  where  $g(1) = 3, g(2) = 4, g(3) = 1, g(4) = 2$ .  $f(1) = 1, f(2) = 4, f(3) = 3, f(4) = 2$  is an automorphism while  $Y^g$  where  $f(1) = 3, f(2) = 3, f(3) = 4, f(4) = 1$  is not an automorphism.

#### Lemma

For  $v \in V(X)$  and  $g \in \text{Aut}(X)$ ,  $\deg(v) = \deg(v^g)$ .

**Proof.** Let  $Y(v)$  be the subgraph of  $X$  induced by  $v$  and the neighbors  $N_X(v)$ . Then

$$Y(v) \cong Y(v)^g = Y(v^g)$$

so  $\deg(v) = \deg(v^g)$ .

#### Lemma

Let  $u, v \in V(X)$  and  $g \in \text{Aut}(X)$ , then the length of the shortest paths are preserved, i.e.  $d(u, v) = d(u^g, v^g)$ .

**Proof.** Show that a shortest  $uv$ -path in  $X$  is mapped to a shortest  $u^g v^g$ -path by  $g$ .

## 1.2 Homomorphisms

#### Definition: Homomorphism

Let  $X$  and  $Y$  be graphs. We say  $f : V(X) \rightarrow V(Y)$  is a homomorphism if  $x \sim y$  implies  $f(x) \sim f(y)$  in  $Y$ .

$\sim$  is for adjacency and  $f : X \rightarrow Y$  instead of  $f : V(X) \rightarrow V(Y)$  for simplicity.

Let  $\chi(X)$  denote the chromatic number of  $X$ , the minimum number of colors needed to color the vertices of a graph such that no two adjacent vertices have the same color.

Let  $K_r$  denote the complete graph on  $r$  vertices where every pair of distinct vertices is connected by an edge. We say that  $K_r$  is a clique, where it generally is a subset of vertices of a graph such that every two distinct vertices in the clique are adjacent.

#### Lemma

$$\chi(X) = \min\{r \in \mathbb{N} : \exists \text{ homomorphism } X \rightarrow K_r\}$$

**Proof.** Let  $k = \chi(X)$ . We first show  $k \geq \min\{r \in \mathbb{N} : \exists \text{ homomorphism } X \rightarrow K_r\}$ . Let  $f$  be a  $k$ -coloring of  $X$ . Then  $f$  is a homomorphism from  $X$  to  $K_k$ .

Next, we show that  $k \leq \min\{r \in \mathbb{N} : \exists \text{ homomorphism } X \rightarrow K_r\}$ . Let  $\bar{r} = \min\{r \in \mathbb{N} : \exists \text{ homomorphism } X \rightarrow K_r\}$ . Let  $h : X \rightarrow K_{\bar{r}}$  be a homomorphism. Then  $h^{-1}(v)$  is an independent set. So, giving  $h^{-1}(v)$  distinct colors yields an  $\bar{r}$ -coloring.

**Definition: Retraction**

A homomorphism  $f : X \rightarrow Y$  such that

1.  $Y \subseteq X$ .
2.  $f|_Y = id$ , the identity map.

If a retraction from  $X$  to  $Y$  exists, we call  $Y$  a retract of  $X$ .

We use the notation  $f|_Y$  to mean the function  $f$  when restricted to the domain of  $Y$ .

**E.g.** Suppose  $K_r \cong Y \subseteq X$  and  $\chi(X) = r$ . We will prove that  $Y$  is a retract of  $X$ . The proof is as follows: let  $f : V(X) \rightarrow [r]$  where  $r = \chi(X)$  be an  $r$ -coloring of  $X$ . Then,  $Y$  receives distinct colors since  $Y \cong K_r$ . Without loss of generality, assume  $V(Y) = [r]$ . Then  $f$  is a homomorphism from  $X$  to  $K_r$  and  $f|_Y = id$ . Therefore,  $f$  is a retraction.

**E.g.** Recall that a cycle graph  $C_n$  is defined  $V(C_n) = \{0, \dots, n-1\}$  where  $n \geq 3$  and  $E(C_n) = \{ij : i - j \equiv \pm 1 \pmod{n}\}$ . Let  $g = (1, 2, \dots, n-1, 0) \in \text{Aut}(C_n)$ . This can be viewed as a rotation. We can define the subgroup

$$R = \{g^m : 0 \leq m \leq n-1\} \leq \text{Aut}(C_n)$$

to capture all possible rotations.

We can define reflections. Let  $h$  be defined  $h(i) = -i \pmod{n} \in \text{Aut}(C_n)$ . We can see that  $R$  and  $Rh$  are disjoint cosets of  $\text{Aut}(C_n)$  and  $Rh \leq \text{Aut}(C_n)$ . It follows that  $|\text{Aut}(C_n)| \geq 2n$ .

**Definition: Circulant Graph**

Let  $\mathbb{Z}_n = \{0, \dots, n-1\}$  and  $C \subseteq \mathbb{Z}_n \setminus \{0\}$  be closed under inverse, that is,  $x \in C \implies -x \in C$ . We define the circulant graph  $X = X(\mathbb{Z}_n, C)$  where  $V(X) = \mathbb{Z}_n, E(X) = \{ij : i - j \in C\}$ .

One can show that the arguments from the previous example for  $C_n$  also hold for  $X = X(\mathbb{Z}_n, C)$ . That is,  $|\text{Aut}(X(\mathbb{Z}_n, C))| \geq 2n$ . We can generalize this result for arbitrary groups using Cayley graphs.

**Definition: Johnson Graph**

Given  $v \geq k \geq i$  as integers where  $[v] = \{1, \dots, v\}$ , the Johnson graph  $J = J(v, k, i)$  is defined  $V(J) = \{S \subseteq [v] : |S| = k\}, E(J) = \{ST : |S \cap T| = i\}$ .

$J(5, 2, 0)$  is the Peterson graph.  $J(v, k, 0)$  is the Kneser graph.

**Proposition**

There exists a subgroup of  $\text{Aut}(J(v, k, i))$  that is isomorphic to  $\text{Sym}(v)$ .

**Proof.** For  $g \in \text{Sym}(v)$ , define  $\tau_g : \binom{[v]}{k} \rightarrow \binom{[v]}{k}$  as  $\tau(S) = S^g$ . Note that  $|S \cap T| = |S^g \cap T^g|$  for vertices  $S, T \in J(v, k, i)$  since we are essentially just relabeling elements of  $S$  and  $T$ , so

$\tau_g \in \text{Aut}(J(v, k, i))$ . We can conclude that

$$\{\tau_g : g \in \text{Sym}(v)\} \cong \text{Sym}(v)$$

# Chapter 2

## Groups

### Definition: Homomorphism

Given groups  $G$  and  $H$ ,  $f : G \rightarrow H$  is a homomorphism if for all  $g, h \in G$ ,

$$f(gh) = f(g)f(h)$$

### Definition: Kernel

The kernel of a homomorphism  $f$  is defined

$$\ker(f) = f^{-1}(1)$$

### Definition: Group Action

Suppose  $G$  is a group and  $V$  is a set. A homomorphism  $f : G \rightarrow \text{Sym}(V)$  is a permutation representation of  $G$  and we call it an action of  $G$  on  $V$ .

**E.g.** Let  $X$  be a graph and take  $V = V(X)$ . Let  $G = \text{Aut}(X)$ . Then  $f : G \rightarrow \text{Sym}(V)$  defined  $f(g) = g$  for  $g \in G$  is an action.

**E.g.** Let  $G$  be a group. Let  $f : G \rightarrow \text{Sym}(V)$  where  $V$  is arbitrary be defined  $f(g) = id$  where  $id$  is the identity permutation.  $f$  is an action.

### Definition: Faithful Action

The action  $f$  is faithful if  $\ker(f) = \{1\}$ .

We can see that the first action example above is faithful, but not the second.

Let group  $G$  act on  $V$ , via  $f : G \rightarrow \text{Sym}(V)$ . Let  $g \in G$ , we use the notation

$$x^g := g^{f(g)} \text{ and } S^G := S^{f(g)}$$

where  $S$  is an arbitrary set.

**Definition:  $G$ -Invariant**

Let group  $G$  act on  $V$  and  $g \in G$ .  $S$  is  $G$ -invariant if  $S = S^g$  for all  $g \in G$ .

**Definition: Orbit**

Let group  $G$  act on  $V$ . The orbit of  $x \in V$  is

$$x^G := \{x^g : g \in G\}$$

One may show that  $V$  is partitioned into disjoint orbits and each orbit is  $G$ -invariant and transitive (for every  $x, y$  in the same orbit, there exists  $g \in G$  where  $x^g = y$ ).

**Definition: Stabilizer**

Let  $G \leq \text{Sym}(V)$  and  $x \in V$ . The stabilizer of  $x$  is

$$G_x := \{g \in G : x^g = x\}$$

**Lemma**

Let  $G \leq \text{Sym}(V)$  and  $x \in V$ , then  $G_x \leq G$ .

**Lemma**

Let  $G \leq \text{Sym}(V)$  and let  $S$  be an orbit of  $G$ . Let  $x, y \in S$ , then

$$H := \{h \in G : x^h = y\}$$

is a right coset of  $G_x$ . Conversely, if  $H$  is a right coset of  $G_x$ , then for all  $h, h' \in H$ ,  $x^h = x^{h'}$ .

**Proof.** ( $\implies$ )  $G$  is transitive on  $S$ , so there exists  $g \in G$  where  $x^g = y$ . For any  $h \in H$ ,  $x^h = y$  by the definition of  $H$ . So,  $x^h = x^g$ . Then,

$$x^{hg^{-1}} = x \implies hg^{-1} \in G_x \implies h \in G_x g$$

( $\impliedby$ ) Assume  $H = G_x g$  for some  $g \in G$ . Let  $h, h' \in H$  where  $h = \sigma g$  and  $h' = \sigma' g$  for some  $\sigma, \sigma' \in G_x$ . We have

$$x^h = x^{\sigma g} = x^g = x^{\sigma' g} = x^{h'}$$

**Lemma (Orbit-Stabilizer)**

Let  $G$  be a permutation group acting on  $V$  with  $x \in V$ . Then

$$|G_x| |x^G| = |G|$$

**Proof.** Let  $\mathcal{H}$  be the set of right cosets of  $G_x$  and define  $f : x^G \rightarrow \mathcal{H}$  as

$$f(y) = \{g \in G : x^g = y\}$$



The previous lemma shows that  $f$  is a bijection. Therefore,  $|\mathcal{H}| = |x^G|$ . Since the right cosets of  $G_x$  partition  $G$ , we have

$$|G| = |G_x| |\mathcal{H}| = |G_x| |x^G|$$

### Definition: Conjugate

Let  $G$  be a permutation group and let  $g, h \in G$ .  $g$  is conjugate to  $h$  if there is some  $\sigma \in G$  such that

$$g = \sigma h \sigma^{-1}$$

### Proposition

If  $H$  is a subgroup of  $G$  and  $g \in G$ , then  $gHg^{-1} \leq G$  and  $gHg^{-1} \cong H$ .

### Lemma

If  $y \in x^G$ , then  $G_x$  and  $G_y$  are conjugate.

**Proof.** Suppose  $y = x^g$  where  $g \in G$ . We will prove that  $g^{-1}G_xg = G_y$ .

( $\subseteq$ ) Note that  $y^{g^{-1}} = x$ . For every  $h \in G_x$ ,  $y^{g^{-1}hg} = x^{hg} = g^g = y$ .

( $\supseteq$ ) For  $h \in G_y$ ,  $x^{ghg^{-1}} = y^{hg^{-1}} = y^{g^{-1}} = x$ . Then  $ghg^{-1} \in G_x$ , rearranging gives  $h \in g^{-1}G_xg$ .

### Definition: Fix

Let  $G \leq \text{Sym}(V)$  and  $g \in G$ . Then

$$\text{fix}(g) = \{v \in V : v^g = v\}$$

### Lemma (Burnside)

Let  $G \leq \text{Sym}(V)$ . Then

$$\# \text{ of orbits of } G = \frac{1}{|G|} \sum_{g \in G} |\text{fix}(g)|$$

**Proof.** Let  $\Lambda = \{(g, x) : g \in G, x \in V, x \in \text{fix}(g)\}$ . We will apply a double-counting argument. Observe that

$$|\Lambda| = \sum_{g \in G} |\text{fix}(g)| = \sum_{x \in V} |G_x|$$

Equating these gives

$$\begin{aligned}
\sum_{g \in G} |\text{fix}(g)| &= \sum_{x \in V} |G_x| \\
&= \sum_{x \in V} \frac{|G|}{|x^G|} && \text{(Orbit-Stabilizer)} \\
&= |G| \sum_{x \in V} \frac{1}{|x^G|} \\
&= |G| (\# \text{ of orbits of } G)
\end{aligned}$$

### Definition: Asymmetric Graph

A graph  $X$  is asymmetric if  $\text{Aut}(X) = \{id\}$ .

### Theorem

Let  $\mathcal{G}_n = \{X \text{ on } [n]\}$  and  $X \in \mathcal{G}_n$  be chosen uniformly random, then

$$\lim_{n \rightarrow \infty} \Pr(X \text{ is asymmetric}) = 1$$

**Proof.** Let  $X \in \mathcal{G}_n$ ,  $\text{Iso}(X) = \{Y \in \mathcal{G}_n : X \cong Y\}$ .

Lemma:  $|\text{Iso}(X)| = \frac{n!}{|\text{Aut}(X)|}$ .

**Proof.** (Lemma) Let  $G = \text{Sym}([n])$ . For  $g \in G$ , let  $\tau_g : \mathcal{G}_n \rightarrow \mathcal{G}_n$  where  $X \mapsto X^g$ . Let  $H := \{\tau_g : g \in G\}$  acts on  $\mathcal{G}_n$  and  $H \cong G$ .

$$n! = |G| = |H| = |H_X| \cdot |X^H| = |\text{Aut}(X)| |\text{Iso}(X)|$$

Let  $\mathcal{H}$  be the set of isomorphism classes of graph on  $[n]$ . Let  $\mathcal{H} \in \mathcal{H}$ . If  $X \in \mathcal{C}$  is asymmetric, then  $|\mathcal{C}| = n!$ . If  $X$  is symmetric, then  $|\mathcal{C}| \leq \frac{n!}{2}$ .

Let  $\rho$  be the proportion of  $\mathcal{C} \in \mathcal{H}$  such that  $|\mathcal{C}| = n!$ . Now,

$$2^{\binom{n}{2}} = |\mathcal{G}_n| = \sum_{\mathcal{C} \in \mathcal{H}} |\mathcal{C}| \leq \rho |\mathcal{H}| n! + (1 - \rho) |\mathcal{H}| \frac{n!}{2}$$

Claim:  $|\mathcal{H}| = (1 + o(1)) \frac{2^{\binom{n}{2}}}{n!}$ , where  $o(1)$  denotes some  $x_n \in \mathbb{R}$  such that  $\lim_{n \rightarrow \infty} x_n = 0$ .

By claim,  $2^{\binom{n}{2}} \leq (1 + o(1)) \frac{2^{\binom{n}{2}}}{n!} \cdot n! \left( \rho + \frac{1-\rho}{2} \right) = (1 + o(1)) 2^{\binom{n}{2}} \cdot \frac{1+\rho}{2}$ .

Thus,  $\rho = 1 + o(1)$ . Then the proportion of asymmetric graphs in  $\mathcal{G}_n$  is  $\rho |\mathcal{H}| n! / 2^{\binom{n}{2}} = 1 + o(1)$ .

**Proof.** (Claim) Consider  $\mathcal{P} = \{\tau_g : g \in \text{Sym}([n])\}$  acting on  $\mathcal{G}_n$  where  $\tau_g(x) = x^g$ . The set of orbits is  $\mathcal{H}$ . Burnside's Lemma tells us  $|\mathcal{H}| = \frac{1}{|G|} \sum_{g \in G} |\text{fix}(\tau_g)|$ .

Observation: Every  $g$  induces a permutation  $G_g$  on  $E(K_{[n]})$ . Let  $C$  be an orbit under  $\sigma_g$ . Then, if  $X$  is fixed by  $\tau_g$ , then  $X$  either contains all edges in  $C$  or no edges in  $C$ .

Let  $\text{orb}_2(\sigma_g)$  be the number of orbits under  $\sigma_g$ . Thus,  $|\text{fix}(\tau_g)| = 2^{\text{orb}_2(\sigma_g)}$ . If  $g = \text{id}$ , then  $\text{orb}_2(\sigma_g) = \binom{n}{2}$ . If  $g = (i, j)$  for some  $i, j \in [n]$ ,  $\text{orb}_2(g) = (n-2) + \binom{n}{2} - 2(n-2)$ .

The contribution to Burnside's Lemma from a simple transposition is  $\binom{n}{2} 2^{\binom{n}{2} - (n-2)} = 2^{\binom{n}{2}}$ .  $\binom{n}{2} 2^{-(n-2)}$ . With some technical work we skip, we can show that  $\sum_{g \in G, g \neq \text{id}} |\text{fix}(\tau_g)| = o(1) \cdot |\text{fix}(\tau_{\text{id}})|$

$$\frac{1}{n!} |\text{fix}(\tau_{\text{id}})| \leq |\mathcal{H}| = \frac{1}{n!} (1 + o(1)) |\text{fix}(\tau_{\text{id}})| \implies |\mathcal{H}| = (1 + o(1)) \frac{2^{\binom{n}{2}}}{n!}$$

### Definition: Block of Imprimitivity

Let  $G$  be a transitive permutation group on  $V$  and  $S \subseteq V$ .  $S$  is a block of imprimitivity for  $G$  if  $S \neq \emptyset$  and  $\forall g \in G$ ,  $S^g = S$  or  $S^g \cap S = \emptyset$ .

$S = \{u\}$  for all  $u \in V$  and  $S = V$  are trivial blocks of imprimitivity.

### Definition: Primitive

$G$  is primitive if there does not exist non-trivial blocks of imprimitivity. Otherwise,  $G$  is imprimitive.

Remark: We assume transitivity since if  $G$  has an orbit  $S = x^G$  such that  $|S| \geq 2, S \neq V$ , then  $S$  is a block of imprimitivity.

**E.g.** If  $G = \text{Aut}(K_n)$ ,  $G$  is primitive.

**E.g.** Let  $G = \text{Aut}(C_4)$ ,  $G$  is not primitive.

**E.g.** Let  $G = \text{Aut}(C_{2n})$

### Lemma

Let  $G$  be a transitive permutation group on  $V$ . Let  $x \in V$ . Then,  $G$  is primitive if and only if  $G_x$  is a maximal subgroup of  $G$  (no  $K$  such that  $G_x < K < G$ ).

**Proof.** We prove  $G$  is imprimitive if and only if there exists  $K$  such that  $G_x < K < G$ .

( $\implies$ ) Let  $S$  be a block of imprimitivity with  $2 \leq |S| < |V|$ . With loss of generality, we may assume that  $x \in S$  since  $G$  is transitive. Let  $G_S = \{g \in G : S^g = S\}$  which is a subgroup of  $G$ . We prove that  $G_x < G_S$ .

Let  $g \in G_x$ . Then  $x \in S \cap S^g$ , so  $S^g = S$  (by definition of block of imprimitivity). Since  $|S| \geq 2$ ,  $\exists y \in S, y \neq x$ . Let  $h \in G$  such that  $x^h = y$ , this implies  $h \notin G_x$ . Then,  $y \in S \cap S^h \implies S = S^h \implies h \in G_S$ . These two points give us  $G_x < G_S$ .  $G_S < G$  since  $S = S^g$  for all  $g \in G_S$  but  $G$  is transitive.

( $\impliedby$ ) Suppose there exists  $K$  with  $G_x < K < G$ . Let  $S = x^K$ .  $2 \leq |S| < |V|$  (assignment).

Claim: For all  $g \in G$ , if  $S \cap S^g \neq \emptyset$ , then  $g \in K$  and  $S = S^g$ .

**Proof.** (Claim) Assume  $y \in S \cap S^g$ .  $y \in S \implies \exists h \in K : y = x^h$ .  $y \in S^g \implies \exists h' \in K : y = x^{h'g}$ . Combining, we get  $x = x^{h'gh^{-1}} \implies h'gh^{-1} \in G_x < K \implies g \in (h')^{-1}Kh \in K$ .

**E.g.** Consider  $K_3$  and the vertex 1.  $G_1 = \{id, (1)(23)\}$ ,  $G = \text{Aut}(K_3)$ . There is no bigger subgroup, so  $G_1$  is maximal.

**E.g.** Consider  $C_4$  and 1.  $G_1 = \{id, (1)(3)(24)\}$ ,  $K = \{id, (1)(3)(24), (13)(24), (13)(2)(4)\}$ . Here  $G_1 < K < \text{Aut}(C_4)$ . We constructed  $K = \{g \in \text{Aut}(C_4) : \{1, 3\}^g = \{1, 3\}\}$ .

# Chapter 3

## Transitive Graphs

### 3.1 Vertex-Transitive Graphs

#### Definition: Vertex-Transitive Graphs

$X$  is vertex-transitive if  $\text{Aut}(X)$  acts transitively on  $V(X)$ .

#### Definition: $k$ -Cube $Q_k$

$V(Q_k) = 2^{[k]}$ ,  $E(Q_k) = \{ij : H(i, j) = 1\}$  where  $H$  is the Hamming distance (positions where the binary string is different).

#### Lemma

$Q_k$  is vertex-transitive.

**Proof.** For all  $v \in 2^{[k]}$ , define  $\rho_v : 2^{[k]} \rightarrow 2^{[k]}$  such that  $x \mapsto x + v$ . Since  $H(x, y) = H(x + v, y + v)$ ,  $\rho_v \in \text{Aut}(Q_k)$ . So  $\{\rho_v : v \in 2^{[k]}\} \leq \text{Aut}(Q_k)$ , which acts transitively on  $V(Q_k)$ .

**Proof.** For all  $v \in \text{Sym}([k])$ , define  $\tau_v : 2^{[k]} \rightarrow 2^{[k]}$ ,  $S \mapsto S^v$ . Since  $H(x, y) = H(\tau_v(x), \tau_v(y))$ ,  $\{\tau_v : v \in \text{Sym}([k])\} \leq \text{Aut}(Q_k)$ .

Note  $\{\rho_v : v \in 2^{[k]}\} \cap \{\tau_v : v \in \text{Sym}([k])\} = \{id\}$ .  $\{\rho_v : v \in 2^{[k]}\}\{\tau_v : v \in \text{Sym}([k])\} \subseteq \text{Aut}(Q_k)$  and  $|\{\rho_v : v \in 2^{[k]}\}\{\tau_v : v \in \text{Sym}([k])\}| = \frac{|\{\rho_v : v \in 2^{[k]}\}| |\{\tau_v : v \in \text{Sym}([k])\}|}{|\{\rho_v : v \in 2^{[k]}\} \cap \{\tau_v : v \in \text{Sym}([k])\}|} = 2^k k!$ .

Remark: Cycles and Circulant graphs are vertex-transitive.

**Definition: Cayley Graph**

Given group  $G$  and  $C \subseteq G$  satisfying

- $1 \notin C$
- $x \in C \implies x^{-1} \in C$

Then  $X = X(G, C)$  such that  $V(X) = G$  and  $E(X) = \{gh : hg^{-1} \in C\} = \{gh : gh^{-1} \in C\}$ .

**Lemma**

Cayley graphs are vertex-transitive.

**Proof.** For any  $v \in G$ , define  $\rho_v : G \rightarrow G, x \mapsto xv$ .  $xy \in E(X(G, C)) \Leftrightarrow xy^{-1} \in C \Leftrightarrow (xv)(yv)^{-1} \in C \Leftrightarrow \{\rho_v(x), \rho_v(y)\} \in E(X(G, C))$ .

**Lemma**

Johnson graphs are vertex-transitive.

## 3.2 Edge-Transitive Graphs

A group acting on  $V$  naturally induces an action on

$$\binom{V}{2} \& (V)_2 = \{ij \in V^2 : i \neq j\}$$

by  $\{u, v\}^g := \{u^g, v^g\}$  and  $(u, v)^g = (u^g, v^g)$ .

**Definition: Edge-Transitive Graph**

$X$  is edge-transitive if  $\text{Aut}(X)$  acts transitively on  $E(X)$ .

**Definition: Arc-Transitive Graph**

$X$  is arc-transitive if  $\text{Aut}(X)$  acts transitively on  $\{ij : ij \in E(X)\}$

**Proposition**

Arc-transitive  $\implies$  vertex-transitive and edge-transitive.

**Proposition**

There exist graphs that are edge-transitive, but not vertex-transitive.

### Proposition

There exist graphs vertex-transitive, but not edge-transitive.

### Theorem

Edge-transitive graphs that are not vertex-transitive with no isolated vertices are bipartite.

**Proof.** Without loss of generality, we may assume that  $X$  has no isolated vertices.

2-orbits: Let  $xy \in E(X)$ . For  $w \in V(X)$ ,  $wz \in E(X)$  for some  $z \in V(X)$ . There exists  $\sigma \in \text{Aut}(X)$ ,  $\{x^\sigma, y^\sigma\} = \{w, z\}$ . This implies every vertex in  $X$  is either in  $x^G$  or  $y^G$ . However,  $X$  is not vertex-transitive,  $x^G \neq y^G$ , this gives the bipartition.

If  $wz \in E(X)$  and  $wz \in x^G$  (or  $wz \in y^G$ ), this implies no  $\sigma \in \text{Aut}(X)$  would map  $xy$  to  $wz$  since  $x^G \cap y^G = \emptyset$ .

### Theorem

If  $X$  is vertex, edge-transitive,  $k$ -regular,  $k$ -odd, then  $X$  is arc-transitive.

### Lemma

If  $X$  is a vertex, edge-transitive,  $k$ -regular, not arc-transitive, then  $k$  is even.

**Proof.** Define  $D(X)$  with  $V(D(X)) = V(X)$  and  $E(D(X)) = \{(x, y) : xy \in E(X)\}$ . Let  $xy \in E(X)$ ,  $\Omega_1 = (x, y)^G$ ,  $\Omega_2 = (y, x)^G$ ,  $G = \text{Aut}(X)$ .  $X$  is edge-transitive implies  $\Omega_1 \cup \Omega_2 = E(D(X))$ .  $X$  is not arc-transitive implies  $\Omega_1 \cap \Omega_2 = \emptyset$ .

Thus,  $\forall uv \in E(X)$ ,  $(u, v) \in \Omega_1 \implies (v, u) \in \Omega_2$ .  $\text{Aut}(X) = \text{Aut}(\Omega_1)$  which acts transitively on  $V(D(X)) = V(\Omega_1)$ , so  $d_{\Omega_1}^+ = d_{\Omega_1}^- = d_{\Omega_2}^+ = d_{\Omega_2}^-$  where  $+$  means in-degree and  $-$  means out-degree. Therefore,  $k = d_{\Omega_1}^+ + d_{\Omega_1}^- \equiv 0 \pmod{2}$ .

## 3.3 Edge-Connectivity

### Definition: Edge Atom

An edge atom of  $X$  is a minimum  $S \subseteq V(X)$  such that  $|\delta(S)| = \kappa_1(X)$ .

In this course  $\partial(S) = \delta(S)$ .

### Lemma

Any two distinct edge atoms are disjoint.

**Proof.** Let  $\kappa = \kappa_1(X)$ . Let  $A, B$  be distinct edge atoms. By minimality,  $|A|, |B| \leq \frac{|V(X)|}{2}$ . Suppose  $A \cap B \neq \emptyset$ :

Case 1:  $A \cup B = V(X)$ , then  $|A| = |B| = \frac{|V(X)|}{2}$  implies  $A \cap B = \emptyset$ , a contradiction.

Case 2:  $A \cup B \subsetneq V(X)$ , then  $|\partial(A \cup B)| \geq \kappa, |\partial(A \cap B)| \geq \kappa + 1$ .

$$\kappa + \kappa + 1 \leq |\partial(A \cup B)| + |\partial(A \cap B)| \leq |\partial(A)| + |\partial(B)| = 2\kappa$$

This is a contradiction.

### Lemma

Suppose  $S$  is a block of imprimitivity under  $\text{Aut}(X)$ , then  $X[S]$  is regular.

**Proof.** Let  $u, v \in S, u \neq v$ . Let  $Y = X[S]$ .  $X$  is vertex-transitive by assumption, this implies  $\exists g \in \text{Aut}(X), u^g = v \implies S = S^g$ . Hence,  $\{g|_S : g \in \text{Aut}(X)\} \subseteq \text{Aut}(Y)$ .  $\deg_Y(u) = \deg_Y(u^g) = \deg_Y(v)$  since automorphism preserves degree.

### Theorem

If  $X$  is connected,  $k$ -regular, and vertex-transitive, then  $\kappa_1(X) = k$ .

**Proof.** Obviously,  $\kappa_1(X) \leq k$ . For  $\kappa_1(X) \geq k$ , let  $S$  be an edge atom. Let  $g \in \text{Aut}(X)$  and  $B = S^g$ . Then by the first lemma, either  $S = B$  or  $S \cap B = \emptyset$ . So,  $S$  is a block of imprimitivity.

The second lemma implies  $X[S]$  is  $\ell$ -regular for some  $0 \leq \ell \leq k - 1$  because  $X$  is connected. Thus,  $|\partial(S)| = |S|(k - \ell)$  such that  $|S| \geq \ell + 1$ .  $|\partial(S)| \geq k$  (proof omitted).

This is  $|\partial(S)| = k$  when  $|S| = 1, \ell = 0$  or  $|S| = k, \ell = k - 1$ .

### Theorem

If  $X$  is connected and vertex-transitive, then

- (a)  $X$  has a matching missing  $\leq 1$  vertex.
- (b) Every edge in  $X$  is contained in a maximum matching.

**Proof.** (a) A vertex is critical if it is saturated by every maximum matching.

Case 1: There exists a critical vertex.

Every vertex is critical by vertex-transitivity, so  $X$  has a perfect matching.

Case 2: No critical vertex.

We prove  $\forall u, v$ , a maximum matching misses at most one of them by induction on  $\ell = d(u, v)$ .

Base case:  $\ell = 1$ , this is trivially true.

Assume  $\ell \geq 2$ . Inductive hypothesis applies to  $(x, y)$  where  $d(x, y) \leq \ell - 1$ . Take  $uv$ -path  $P$  with  $|P| = \ell \geq 2$ . There exists  $x \notin \{u, v\}$  on  $P$ .  $x$  is not critical means there exists a maximum matching  $M_x$  missing  $x$ . The inductive hypothesis applies  $(u, x)$  and  $(v, x)$  implies  $M_x$  saturates  $u$  and  $v$ .



Suppose on the contrary, there exists a maximum matching  $M$  that misses both  $u$  and  $v$ . There exists an alternating  $ux$ -path and  $vx$ -path in  $M \Delta M_x$  by claim (below).  $u = v$ , a contradiction.

Claim: Suppose  $(z, w)$  is a pair of vertices such that a maximum matching cannot miss both of them. Then  $M_z \Delta M_w$  must contain an alternating  $zw$ -path.

**Proof.** (Claim) Suppose on the contrary that  $z$  and  $w$  lies in distinct components of  $M_z \Delta M_w$ .  $M := M_w \Delta P$  is a maximum matching missing both  $z, w$ , a contradiction.

(b) By strong induction on number of vertices and number of edges.

Base case: Empty graph, this is trivial.

Inductive hypothesis: Suppose on the contrary that  $\exists e \in E(X)$  that  $e$  is not in any maximum matching of  $X$ . This implies  $X$  is not edge-transitive.

Let  $Y$  be the subgraph of  $X$  induced by  $e^{\text{Aut}(X)}$ .  $Y$  is vertex, edge-transitive, so  $Y \neq X$ . Inductive hypothesis applies to every component of  $Y$ .

Case 1:  $Y$  is connected.

By part (a) and that  $Y$  is vertex, edge-transitive,  $e$  is contained in a maximum matching of  $Y$  (which is a maximum matching of  $X$ ).

Case 2:  $Y$  contains multiple components  $C_i$ .

Claim:  $V(C_i)$  is a block of imprimitivity under  $\text{Aut}(X)$ .  $C_i \cong C_j$  for all  $i, j \in [m]$ .

Inductive hypothesis applies to each  $C_i$ . Case 2(a): each  $C_i$  has a perfect matching, this contradicts case 1. Case 2(b): each  $C_i$  has a matching missing 1 vertex.

Define  $Z$  where  $V(Z) = \{C_1, \dots, C_m\}$ ,  $E(Z) = \{C_i C_j : \exists e xy \in E(X), x \in C_i, y \in C_j\}$ . It is easy to show that  $Z$  is connected and vertex-transitive. Part (a) implies  $Z$  has a matching missing  $\leq 1$  vertex. We have found a maximum matching of  $X$  containing  $e$ . A contradiction.

## 3.4 Cayley Graphs

### Definition: Regular Group

A permutation group acting on  $V$  is regular if

- $G_x = \{1\}$  for all  $x \in V$  (semi-regular)
- $G$  is transitive.

### Proposition

If  $G$  acts on  $V$  is regular, then  $|G| = |V|$ .

**Proof.**  $|G| = |G_x| \cdot |x^G| = 1 \cdot |x^G| = |V|$ .

### Theorem

Let  $G$  be a group and  $C \subseteq G \setminus \{1\}$  inverse-closed. Then,  $\text{Aut}(X(G, C))$  contains a regular subgroup isomorphic to  $G$ .

**Proof.** (a) Let  $X = X(G, C)$ . Define  $\tau_g : V(X) \rightarrow V(X), \sigma \rightarrow \sigma g$  for all  $\sigma \in V(X) = G$ .

- $\{\tau_g : g \in G\} \leq \text{Aut}(X)$ .
- $\{\tau_g : g \in G\}$  acts transitively on  $G$ .
- $\{\tau_g : g \in G\} \cong G$ .
- $\{\tau_g : g \in G\}$  is semi-regular.

### Theorem

Suppose  $X$  is a graph. If  $G \leq \text{Aut}(X)$  acts regularly on  $V(X)$ , then  $X \cong X(G, C)$  for some inverse-closed  $C \subseteq G \setminus \{1\}$ .

**Proof.**  $G$  is regular, so  $|G| = |V(X)|$ . Fix  $u \in V(X)$ .  $\exists$  a unique  $g \in G$  such that  $u^g = v$  for all  $v \in V(X)$ . Call this  $g$  as  $g_v$ . Let  $C = \{g_v : v \sim u\}$ .

First  $1 \notin C$ ,  $u \approx u$ . Next, we prove  $X \cong X(G, C)$  by isomorphism  $f(x) = g_x, \forall x \in V(X)$ .  $xy \in E(X)$  if and only if  $\{x^{g_x^{-1}}, y^{g_x^{-1}}\} \in E(X)$  if and only if  $\{u, u^{g_y g_x^{-1}}\} \in E(X)$  if and only if  $g_y g_x^{-1} \in C$  since  $u^{g_x} = x, u^{g_y} = y, g_x, g_y \in G \leq \text{Aut}(X)$ .

By symmetric proof using  $g_y^{-1}$ , we obtain  $xy \in E(X)$  if and only if  $\{u, u^{g_x g_y^{-1}}\} \in E(X)$  if and only if  $g_x g_y^{-1} \in C$ , so  $C$  is inverse-closed.

### Theorem

- (a) If  $\theta : G \rightarrow G$  is an automorphism, then  $X(G, C) \cong X(G, \theta(C))$  and  $C \subseteq G \setminus \{1\}$  is inverse-closed.
- (b)  $\exists(G, C_1, C_2)$  such that  $X(G, C_1) \cong X(G, C_2)$ , but there is no automorphism  $\theta$  on  $G$  such that  $C_2 = \theta(C_1)$ .

**Proof.** (a) We prove that  $\theta : V(X) \rightarrow V(X), X = X(G, C)$  is an isomorphism.

$$\begin{aligned} hg^{-1} \in C &\Leftrightarrow \theta(hg^{-1}) \in \theta(C) \\ &\Leftrightarrow \theta(h)\theta(g)^{-1} \in \theta(C) \\ &\Leftrightarrow \theta(h)\theta(g^{-1}) \in \theta(C) \end{aligned}$$

### Definition: Generating Set

Let  $G$  be a group. We say a subset  $C \subseteq G$  be generating for  $G$  if every element in  $G$  can be expressed as a product of elements in  $C$ .

### Proposition

$X(G, C)$  is connected if and only if  $C$  is generating for  $G$ .

### Theorem

Every connected vertex-transitive graph is isomorphic to a retract of a Cayley graph.

**Proof.** Let  $x \in V(X)$ ,  $C = \{g \in \text{Aut}(X) : x^g \sim x\}$ , and  $G$  be the subgroup of  $\text{Aut}(X)$  that is generated by  $C$ .  $G$  acts transitively on  $V(X)$ . Let  $Y = X(G, C)$ . For every  $y \in V(X)$ , let  $C_y := \{g \in G : x^g = y\}$ .  $C_y$  is a right coset of  $G_x$ .  $C = \bigcup_{y \sim x} C_y$ ,  $C \cap G_x = \emptyset$  since  $x \not\sim x$ .

Moreover, for any  $a, b \in G$ ,  $x^a \sim x^b \Leftrightarrow x \sim x^{ba^{-1}} \Leftrightarrow ba^{-1} \in C$ .

Claim 1:  $C = G_x C G_x$ .

Let  $A_1, \dots, A_k$  be the set of right cosets of  $G_x$ . Let  $a_1 \in A_1, \dots, a_k \in A_k$ .

Claim 2: In  $Y = X(G, C)$ ,  $\forall 1 \leq i < j \leq k$ ,  $e(A_i, A_j) = 0$  or  $e(A_i, A_j) = |A_i| |A_j|$ . Moreover,  $\forall 1 \leq i \leq k$ ,  $e(A_i) = 0$ .

Claim 3:  $Y[a_1, \dots, a_k] \cong X$ .

Claim 4:  $Y[a_1, \dots, a_k]$  is a retract of  $Y$ .

**Proof.** (Claim 1)  $\subseteq$  is obvious. ( $\supseteq$ ) Let  $h, h' \in G_x$  and  $g \in C$ . Then  $x \sim x^g$ . Since  $x^h = x = x^{h'} \implies x = x^h \sim x^{gh} = x^{h'gh}$ . So we know that  $h'gh \in C \implies G_x C G_x \subseteq C$ .

**Proof.** (Claim 2) For any  $g' \in G$ ,  $g' \in A_j$  for some  $j$ .  $G' = ga_j$  for some  $g \in G_x$ . Suppose  $g, h \in G_x$ , then  $ga_i \sim ha_j \Leftrightarrow ga_i(ha_j)^{-1} \in C \Leftrightarrow ga_i a_j^{-1} h^{-1} \in C \Leftrightarrow a_i a_j^{-1} \in g^{-1} C h \in G_x C G_x = C$  by claim 1.

Statement 2: Is implied immediately by  $1 \notin C$  since  $a_i = a_j$  in this case and  $a_i a_i^{-1} = 1 \notin C$ .

**Proof.** (Claim 3) As shown in claim 2,  $\forall 1 \leq i < j \leq k$ ,  $a_i \sim a_j$  in  $Y[a_1, \dots, a_j]$  if and only if  $a_i a_j^{-1} \in C$ .

Let  $\rho : V(X) \rightarrow \{a_1, \dots, a_k\}$ ,  $y \mapsto a_j$  where  $a_j \in C_y$ . Verify that  $\rho$  is an isomorphism.

**Proof.** (Claim 4) Let  $\tau : V(Y) \rightarrow \{a_1, \dots, a_k\}$ ,  $g \mapsto a_j$  if  $g \in A_j$ . Claim 2 implies  $\tau$  is a homomorphism,  $\tau|_{\{a_1, \dots, a_k\}} = id$ .

# Chapter 4

## Generalized Polygons

### Definition: Incidence Structure

Given a set  $\mathcal{P}$  of points and a set  $\mathcal{L}$  of lines, and incidence relation  $I \subseteq \mathcal{P} \times \mathcal{L}$ . If  $(p, L) \in I$ , then the point  $p$  is in line  $L$ . The triple  $\mathcal{I} = (\mathcal{P}, \mathcal{L}, I)$  defines an incidence structure.

### Definition: Dual Incidence Structure

The triple  $\mathcal{I}^* = (\mathcal{L}, \mathcal{P}, I^*)$  where

$$I^* = \{(L, p) \in \mathcal{L} \times \mathcal{P} : (p, L) \in I\}$$

is called the dual of  $\mathcal{I}$ .

### Definition: Incidence Graph $X(\mathcal{I})$

Given  $\mathcal{I} = (\mathcal{P}, \mathcal{L}, I)$ ,  $X(\mathcal{I})$  is the incidence graph defined by the bipartite graph on  $\mathcal{P} \cup \mathcal{L}$  such that  $\{(p, L) \in E(X) : (p, L) \in I\}$ .

$$X(\mathcal{I}^*) \cong X(\mathcal{I}).$$

### Definition: Automorphism of $\mathcal{I}$

An automorphism of  $(\mathcal{P}, \mathcal{L}, I)$  is a permutation  $\sigma$  on  $\mathcal{P} \cup \mathcal{L}$  such that  $\mathcal{P}^\sigma = \mathcal{P}$ ,  $\mathcal{L}^\sigma = \mathcal{L}$  and  $(p^\sigma, L^\sigma) \in I \Leftrightarrow (p, L) \in I$ .

### Definition: Partial Linear Space

$\mathcal{I} = (\mathcal{P}, \mathcal{L}, I)$  is a partial linear space if for any  $x, y \in \mathcal{P}, x \neq y$ , there is at most one line  $L \in \mathcal{L}$  such that  $(x, L) \in I$  and  $(y, L) \in I$ .

We say  $x, y$  are joined by  $L$  and  $x, y$  are collinear.

**Lemma**

If  $\mathcal{I}$  is a partial linear space, then any two lines are incident with at most one point.

**Lemma**

If  $\mathcal{I}$  is a partial linear space, then  $X(\mathcal{I})$  has girth  $\geq 6$ .

**Proof.** If  $X$  contains a 4-cycle  $p, L, q, M$ , then  $p$  and  $q$  are incident to 2 lines, which is forbidden by partial linear space. Since the girth of  $X$  is even (bipartite) and it cannot be 4, then the girth is at least 6.

**Definition: Projective Planes**

A partial linear space satisfying

- (1) Any two lines meet at a unique point.
- (2) Any two points are joined by a unique line.
- (3) There exists three non-collinear points (a triangle).

**Theorem**

A partial linear space  $\mathcal{I}$  is a projective plane if and only if  $X(\mathcal{I})$  has diameter 3 and girth 6.

**Proof.** ( $\implies$ ) Let  $\mathcal{I} = (\mathcal{P}, \mathcal{L}, I)$  be a projective plane.