Combinatorial Game Theory

Keven Qiu
Instructor: Ryan Hayward

1 Introduction

Definition 1.1: Combinatorial Game Theory (CGT)

The study of finite 2-player, alternating turns, deterministic rules/moves, perfect information games.

Definition 1.2: Normal Play Games

Whoever cannot move loses.

We can represent combinatorial games with a tree of all continuations. This is a directed acyclic graph where each node contains the game state and the player to move.

Games are played with **L**eft and **R**ight. In the tree, solid lines represent L moving and dotted represents R moving.

Definition 1.3: ptm

Player-to-move.

Definition 1.4: optm

Opponent-to-move.

Definition 1.5: WLD Game

A combinatorial game that can result in a win, loss, or draw.

Theorem 1.6 (Zermelo)

In a Win-Loss-Draw game, one of the following holds:

- ptm (first player) has a winning strategy.
- optm (second player) has a winning strategy.
- ptm and optm each have a draw strategy (a non-losing strategy).

Proof. Consider a Tree of all Continuations (ToaC). We prove this by induction. Assume the theorem holds for all continuations from the root $g_1, \dots, g_t$. We label each node of the each subtree with $+$, $-$ or 0. There are three cases:

Case 1 At least 1 of the g_i is labeled $-$, say g_j. $g_j = -$ means the player-to-move at g_j loses. Thus, if you are at g and you move to g_j, then opponent is the player-to-move at g_j and loses. You force a win, so ptm wins.

Cbse 2 Assume not case 1 and that some $g_j = 0$. All g_i's are 0 or +. You move to the $g_j = 0$ and therefore guarantee a ptm-draw. If opponent moves first, then they can either move to a + and we win, or they move to 0 and forces a draw. So optm-draw under optimal play.

Ccse 3 Every $g_i = +$. Every move ptm makes goes to +, so next player can always win. Thus, optm-win.

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1.1 Outcome Classes

Definition 1.7: $\mathcal{Y}$

Left wins, no matter if they are 1st or 2nd player.

Definition 1.8: $\mathcal{R}$

Right wins, no matter if they are 1st or 2nd player.

Definition 1.9: $\mathcal{P}$

The 2nd player wins (0 game).

Definition 1.10: $\mathcal{N}$

The 1st player wins.

2 Clobber

Clobber is a normal play combinatorial game. Each player either controls X or O and each turn they can capture an adjacent opponent symbol. The player who cannot move loses.

Linear clobber is when the game is played on a path graph. We can represent this in options notation:

- $x = \{|\} = 0$. The game tree is a single root node.
- $xo = \{0|0\} = *$.
- $xox = \{0, 0|0, 0\} = \{0, 0\} = *$.
- $xxo = \{0|*\} = \uparrow$.
- $oox = \{*|0\} = \downarrow$.
- $xoxo = \{0, *, xxo|0, *, oox\}$.

Definition 2.1: Equivalent Games

Two games G, H are equivalent if they have the same canonical form.

Definition 2.2: Sum of Games

For games A, B,

$$A + B = \{A^L + B, A + B^L | A^R + B, A + B^R\}$$

for all Left options A^L, B^L and Right options A^R, B^R.

Theorem 2.3 (Alternate Equivalent Games)

Two combinatorial games G, H are equivalent iff for all games X, $\text{oc}(G + X) = \text{oc}(H + X)$ (outcome class).

The negative clobber game swaps each x to o and o to x.

Definition 2.4: Negative Game $-G$

If $G = \{G^L | G^R\}$, then $-G = \{-G^R | -G^L\}$.

E.g. $G = \{xxo, xo, xxxo | oox, ox\}$, then

$$-G = \{xxo, xo | oox, ox, ooox\}$$

Theorem 2.5 (Alternate Equivalent Games 2)

Two combinatorial games G, H are equivalent if $G + -H \in \mathcal{P}$.

Proposition 2.6

$\downarrow + * = \downarrow * \equiv \{0 | *, 0\}$.

Proof. We have $\downarrow = \{* | 0\}$ and $* = \{0 | 0\}$. By sums of games definition,

$$\downarrow + * = \{* + *, \downarrow + 0 | 0 + *, \downarrow + 0\} = \{0, \downarrow | *, \downarrow\}$$

Define $G = \{0 | *, 0\}$ and $H = \downarrow + * = \{0, \downarrow | *, \downarrow\}$. To prove $G \equiv H$, we need to show $G + -H \in \mathcal{P}$.

First, $-H = \{*, \uparrow | 0, \uparrow\}$. We can compute $G + -H$, but there are two cases:

1. L plays first on $G + -H$, show R can win. If G plays first, then the three options of the tree are $-H, G + *, G + \uparrow$.

   If $-H$, R wins using 0.

   If $G + *$, then R can win by $* + * = 0$.

   If $G + \uparrow$, then R plays to $G + *$. L plays to either $*$ or G, which then R can both play to 0.

2. R plays first on $G + -H$, show L can win.

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3 Hex

A combinatorial game played on a grid of hexagons. The goal is for one player to connect their colored sides using their same symbol/rock.

Hex has no draws.

Theorem 3.1

For $n \times n$ hex, the first player always wins.

Proof. We prove by contradiction. Let Left make the first play of the game, and assume that Right has a winning strategy. With Left's first move she puts a stone on any cell, a placement called "extra". With this move Left becomes Second, and she henceforth follows the winning strategy that is available to Right. At some point, if this strategy calls for Left to place a stone where the extra sits; then she will simply make another arbitrary placement. Thus Left can win in contradiction to the hypotheses. ■

4 Go

Go is played on a graph, typically a grid. It is a combinatorial game where the winning condition is more stones and territory occupied. A legal move must be played on an empty cell and the block (a connected component) must have at least 1 liberty/free space.

5 Nim

The game of Nim involves piles of items and on each turn, a player picks at least 1 item from a pile. The last person to pick items is the winner.

Theorem 5.1 (Bouton)

There is a winning move iff the XOR sum is nonzero.

Consider the piles 6, 3, 1. In binary this is $110, 011, 001$. The XOR sum is 100, thus there is a winning move for ptm.

There is a legal move from each pile whose leftmost bit matches the leftmost 1 bit of XOR sum. For the example, we add 100 to 110 to get 010, so we pick 2 from the first pile of 6 which leaves 4.

Definition 5.2: $*k$

The game of Nimber (Nim with one pile) with k stones.

Definition 5.3: Impartial Game

A game where both Left and Right have the same move options.

Theorem 5.4

Every impartial game is equivalent to a Nimber (one pile Nim).

6 Domineering

Definition 6.1: Cram

A combinatorial, normal play game played on a grid of dots. On each move, you can cover 2 horizontally or vertically adjacent dots. Player who cannot make a move loses.

Definition 6.2: Domineering

A game of Cram, but Left can only play vertically and Right can only play horizontally.

The game $1 = \{0|\}$ cannot be realized as a clobber game since Left only has one move.

Similarly, $-1 = \{|0\}$.

Proposition 6.3

$$G + 1 \in \mathcal{L}, -1 + G \in \mathcal{R}$$

Proof. First $G + 1 \in \mathcal{L}$. There are two cases:

1. Assume Left is player 1. Play on G. If result $G \to 0$, so total game is $1 + 0 = 1$. It is Right's turn and Right has no moves by definition of 1 game.

   If result is $G \to G'$, so total game is $G' + 1$. It is now Right's turn and Right must play on G' since Right has no moves in 1. So on Right's turn, so $G' + 1 \to 0$ or $G' + 1 \to G'' + 1$. For the 0 case, Left can play on 1 and wins. For the second case, continue by induction on depth of tree.

2. Assume Right is player 1. Same as above, when Right plays on $G + 1$.

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Proposition 6.4

A smallest game g such that $g \equiv 3 \times 3$ domineering game is $g = \pm 1 = \{1| - 1\}$.

Proof. There are two cases:

1. $g \equiv \pm 1$: We need to show $g + - \pm 1 \in \mathcal{P}$. There are two cases:

   - Left plays first on $g + - \pm 1$, show Right wins. $- \pm 1 = \pm 1$ since you invert and negate. So we show $g + \pm 1 \in \mathcal{P}$.
   - Right plays first on $g + - \pm 1$, show Left wins.

   We would do this for more complicated proofs, but we can notice $g = \pm 1$.

2. There is no smaller game (tree depth): The game tree g has depth 2. We show that $g \not\equiv$ any game with depth 0 or 1. g is in $\mathcal{N}$, but 0 is in $\mathcal{P}$, so it is not equivalent to a depth 0 game. For depth 1, ...

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7 Canonical Form

Definition 7.1: Canonical Form

The minimally equivalent game.

Definition 7.2: $A < B$

$A < B$ if $B + -A \in \mathcal{L}$.

Definition 7.3: $A \leqq B$

$A \leqq B$ if $A \equiv B$ or $A < B$.

Theorem 7.4 (Canonical Form I [Conway, Berlekamp, Guy])

If $G = \{L_1, L_2, \dots | R_1, R_2, \dots\}$ and $L_1 \leqq L_2$, then

$$G \equiv \{L_2, \dots | R_1, R_2, \dots\}$$

If $G = \{L_1, L_2, \dots | R_1, R_2, \dots\}$ and $R_1 \geqq R_2$, then

$$G \equiv \{L_1, L_2, \dots | R_2, \dots\}$$

This shows to prune inferior options.

Proposition 7.5

$$0 < 1$$

Proposition 7.6

$$-1 < 1$$

Definition 7.7: $A > B$

$A > B$ if $B + -A \in \mathcal{R}$.

For example, $G = \{0, 1 | 1, -1\}$, we can prune 0 from the left options since $0 < 1$, so $G \equiv \{1 | 1, -1\}$. And also, $1 > -1$, so we can prune 1 from the right options to get $G \equiv \{1, -1\}$.

Definition 7.8: Fuzzy

A fuzzy game is a game in outcome class $\mathcal{N}$.

Definition 7.9: Reversible

For a game g, a left (right) option L_1 (R_1) is reversible if it has right (left) option LR_1 (RL_1) such that LR_1 (RL_1) is at least as good for Right (Left) as g

Let $g = \{0, *|*\}$. Say L_1 here is $*$ and LR_1 is 0. Does Right prefer LR_1 to g, i.e. $LR_1 = 0 \leqq g$?

Either $g \equiv 0$ leads to position $\mathcal{P}$ or $g > 0$ is in $\mathcal{R}$.

Yes, we can replace $*$ for left options to the game $\{0|*\}$.

The above is g is Right-preferable to h if $g \leqq h$. g is Left-preferable to h if $g \geqq h$.

Theorem 7.10 (Canonical Form II – Reversible)

If LR_1 is reversible, then we can replace L_1 with set of left options of LR_1.

Proposition 7.11

$$1 \leqq *$$

Proof. Consider $1 - *$, we show that Left has a 2-player win strategy. So we only need to consider Right first moves. Right moves to 1, and Left wins. ∎

Definition 7.12: $\frac{1}{2}$

$$\frac{1}{2} = \{0|1\}$$

The game $\frac{1}{2}$ is the same as the domineering game that looks like an L block since Left can play to either 0 or -1, and Right can play in the one spot, so it plays to 1.

Claim: $g + g + -1 \in \mathcal{P}$. This g is called $\frac{1}{2}$ from this.

7.1 Counting Minimal Games

The question here is to ask how many games g are there that $CF(g)$ has depth $\leq k$.

For depth ≤ 0, there is exactly one game with depth 0, the game 0.

For depth $= 1$, there are 3 games $1, -1, *$. So there are at most 4 games with depth ≤ 1.

For depth ≤ 2, there are at most $2^4 \times 2^4 = 256$ games. This is because there are 4 games of depth ≤ 1. There are 2^4 sets of left options.

Since the move options are dominant, there are only really 6 possible Left and 6 possible Right move options. Careful analysis shows there are only 22 unique games.

Table 1: Canonical form of games with depth ≤ 2

L/R	$\emptyset$	0	-1	1	$*$	$0,*$
$\emptyset$	0	-1	-2	0	0	-1
0	1	$*$	$\{0\|-1\}$	$\frac{1}{2}$	$\uparrow$	$\downarrow *$
1	2	$\{1\|0\}$	± 1			
-1	0	$-\frac{1}{2}$	$-1*$	0		
$*$	0	$\downarrow$	$\{*\|-1\}$	0	0	
$0,*$	1	$\uparrow *$	$\{0,*\|-1\}$	$\frac{1}{2}$	$\uparrow$	$*2$

8 Numbers

Definition 8.1: Number

A normal-play, combinatorial game with no loops but possibly infinite.

Definition 8.2: Dyadic Rationals

A normal-play, combinatorial game with no loops (finite).

Definition 8.3: Reals

A normal-play, combinatorial game with no loops (possibly infinite).

Definition 8.4: Surreals

A normal-play, combinatorial game with no loops (infinite).

Definition 8.5: Number (Recursive)

A game g where $\forall g^L$ is a number and $\forall g^R$ is a number, and no $g^L \geqq g^R$.

Here $\forall g^L$ means for each individual left option.

E.g. $0 = \{|\}$ is a number. $\{0|\}$, $\{|0\}$, $\{0|1\}$ are numbers, but $\{1|0\}$ is not a number.

$\{1, 2|4, 5\}$ is a number. By CF 1, this is equivalent to $\{2|4\}$.

Theorem 8.6

Numbers are not fuzzy.

Proof. Let $g = \{g^L|g^R\}$ is a number. Assume $g||0$. Then there is some $g^L \geqq 0$ (Left prefers larger numbers) and some $g^R \leqq 0$ (Right prefers smaller numbers). But $g^L \geqq 0 \geqq g^R$, which contradicts the definition of a number. ■

Theorem 8.7

Let $g = \{g^L | g^R\}$ be a number and $\forall h \in g^L, \forall k \in g^R$, then

$$h < g < k$$

Let a game of Hackenbush with one blue line on the ground and two red lines vertically on top of the blue line. This game is

$$\left\{0|\frac{1}{2}, 1\right\} = \left\{0|\frac{1}{2}\right\} = \frac{1}{4}$$

You can prove for all Hackenbush games g, $\forall g^L < g$ and $\forall g^R > g$, then g is a number.

You can increase the number of red lines to get the sequence $1, \frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \ldots$.

Theorem 8.8

For all $n \geq 1$,

$$\frac{1}{2^n} = \left\{0|\frac{1}{2^{n-1}}\right\}$$