CO 450/650 Combinatorial Optimization

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List of Algorithms

Part I

Introduction

Chapter 1

Linear Programming

Definition: Linear Programming

The problem of finding a vector x that maximizes a given linear function $c^T x$, where x ranges over all vectors satisfying a given system $Ax \leq b$ of linear inequalities.

1.1 Farkas' Lemma

Lemma (Farkas' Lemma for Inequalities)

The system $Ax \leq b$ has a solution x if and only if there is no vector y satisfying $y \geq 0$, $y^T A = 0$, and $y^T b < 0$.

Proof. Suppose $Ax \leq b$ has a solution $\overline{x}$ and suppose there exists a vector $\overline{y} \geq 0$ satisfying $\overline{y}^T A = 0$ and $\overline{y}^T b < 0$. Then we obtain the contradiction

$$0 > \overline{y}^T b \geq \overline{y}^T (A\overline{x}) = (\overline{y}^T A)\overline{x} = 0$$

Now suppose that $Ax \leq b$ has no solution. If A has only one column, then the result is easy. Otherwise, apply Fourier-Motzkin elimination to obtain a system $A'x' \leq b'$ with one less variable. Since $A'x' \leq b'$ also has no solution, we can assume by induction that there exists a vector $y' \geq 0$ satisfying $y'^T A' = 0$ and $y'^T b' < 0$. Now since each inequality in $A'x' \leq b'$ is the sum of positive multiples of inequalities in $Ax \leq b$, we can use y' to construct a vector y satisfying the conditions in the theorem. $\square$

Lemma (Farkas' Lemma)

The system $Ax = b$ has a nonnegative solution if and only if there is no vector y satisfying $y^T A \geq 0$ and $y^T b < 0$.

Proof. Define

$$A' = \begin{bmatrix} A \\ -A \\ -I \end{bmatrix}, b' = \begin{bmatrix} b \\ -b \\ 0 \end{bmatrix}$$

Then $Ax = b$ has a nonnegative solution x if and only if $A'x' \le b'$ has a solution x'. Applying Farkas' Lemma for Inequalities to $A'x' \le b'$ gives the result. □

Corollary

Suppose the system $Ax \le b$ has at least one solution. Then every solution x of $Ax \le b$ satisfies $c^T x \le \delta$ if and only if there exists a vector $y \ge 0$ such that $y^T A = c$ and $y^T b \le \delta$.

1.2 Duality

Consider the LP:

$$\begin{aligned} \max \quad & c^T x \\ \text{s.t.} \quad & Ax \le b \end{aligned}$$

and dual LP

$$\begin{aligned} \min \quad & y^T b \\ \text{s.t.} \quad & y^T A = c^T \\ & y \ge 0 \end{aligned}$$

Theorem (Weak Duality)

Let A be an $m \times n$ matrix, $b \in \mathbb{R}^m$, $c \in \mathbb{R}^n$. Suppose that $\overline{x}$ is a feasible solution to $Ax \le b$ and $\overline{y}$ is a feasible solution to $y \ge 0, y^T A = c^T$. Then

$$c^T \overline{x} \le \overline{y}^T b$$

Proof.

$$c^T \overline{x} = (\overline{y}^T A)\overline{x} = \overline{y}^T (A\overline{x}) \le \overline{y}^T b$$

□

Theorem (Strong Duality)

Let A be an $m \times n$ matrix, $b \in \mathbb{R}^m$, $c \in \mathbb{R}^n$, then

$$\max\{c^T x : Ax \le b\} = \min\{y^T b : y^T A = c^T, y \ge 0\}$$

provided that both sets are nonempty.

Corollary

Let A be an $m \times n$ matrix, $b \in \mathbb{R}^m$, $c \in \mathbb{R}^n$, then

$$\max\{c^T x : Ax \leq b, x \geq 0\} = \min\{y^T b : y^T A \geq c^T\}$$

provided that both sets are nonempty.

Definition: Complementary Slackness Conditions

For each $i \in \{1, \ldots, m\}$, either $y_i^* = 0$ or $a_i x^* = b_i$.

Theorem (Complementary Slackness Theorem)

Let x^* be a feasible solution of $\max\{c^T x : Ax \leq b\}$ and let y^* be a feasible solution of $\min\{y^T b : y^T A = c^T, y \geq 0\}$. Then x^* and y^* are optimal solutions for the maximum and minimum respectively if and only if the complementary slackness conditions hold.

Chapter 2

Graph Theory

Definition: Graph

A graph $G = (V, E)$ is a set of vertices/nodes V and a set of edges E.
We define $n = |V|$ and $m = |E|$.

Definition: Subgraph

$H = (W, F)$ of $G = (V, E)$ where $W \subseteq V$ and $F \subseteq E$.

Definition: Spanning Subgraph

H is spanning if $V(H) = V(G)$.

Definition: Path

A sequence $P = v_0, e_1, v_1, \dots, e_k, v_k$ where $v_0, \dots, v_k \in V(G)$, $e_1, \dots, e_k \in E(G)$, and $e_i = v_{i-1}v_i$.

We call P a v_0v_1-path. P is called edge-simple if all e_i are distinct and simple if all v_i are distinct.

The length of P is the number of edges in P.

Definition: Circuit/Cycle

An edge-simple closed path.

Definition: Connected

A graph is connected if every pair of vertices is joined by a path.

Definition: Cut Vertex

A vertex v of a connected graph G where $G - v$ is not connected.

Definition: Forest

A graph with no circuits.

Definition: Tree

A connected forest.

Definition: Cut

Let $R \subseteq V$, then

$$\delta(R) = \{vw : vw \in E, v \in R, w \notin R\}$$

Definition: rs-Cut

A cut for which $r \in R, s \notin R$.

Part II

Polyhedral Combinatorics

Chapter 3

Integrality of Polyhedra

3.1 Convex Hull

Definition: Convex Combination

$x = \lambda_1 v_1 + \cdots + \lambda_k v_k$ for some vectors $v_1, \ldots, v_k$ and nonnegative scalars $\lambda_1, \ldots, \lambda_k$ such that $\lambda_1 + \cdots + \lambda_k = 1$.

Definition: Convex Hull

The convex hull of a finite set S, denoted $conv.hull(S)$, is the set of all vectors that can be written as a convex combination of S.

Proposition

Let $S \subseteq \mathbb{R}^n$ be a finite set and let $w \in \mathbb{R}^n$. Then

$$\max\{w^T x : x \in S\} = \max\{w^T x : x \in conv.hull(S)\}$$

3.2 Polytopes

Definition: Polyhedron

The solution set of a finite system of linear inequalities.

Definition: Polytope

A polyhedron $P \subseteq \mathbb{R}^n$ is a polytope if there exists $\ell, u \in \mathbb{R}^n$ such that $\ell \leq x \leq u$ for all $x \in P$.

Definition: Valid Inequality

An inequality $w^T x \leq t$ is valid for a polyhedron P if $P \subseteq \{x : w^T x \leq t\}$.

Definition: Hyperplane

The solution set of $w^T x = t$ where $w \neq 0$.

Definition: Supporting Hyperplane

With respect to a polyhedron P, a hyperplane is supporting if $w^T x \leq t$ is valid for P and $P \cap \{x : w^T x = t\} \neq \emptyset$.

Definition: Face

The intersection of a polyhedron with one of its supporting hyperplanes.

The null set and the polyhedron itself is a face.

Definition: Proper Face

Faces which are not the null set or the polyhedron itself.

Proposition

A nonempty set $F \subseteq P = \{x : Ax \leq b\}$ is a face of P if and only if for some subsystem $A^\circ x \leq b^\circ$ of $Ax \leq b$, we have $F = \{x \in P : A^\circ x = b^\circ\}$.

Proof. ($\Longrightarrow$) Suppose F is a face of P. Then there exists a valid inequality $w^T x \leq t$ such that $F = \{x \in P : w^T x = t\}$.

Consider the LP problem $\max\{w^T x : Ax \leq b\}$. The set of optimal solutions is precisely F. Now let y^* be an optimal solution to the dual problem $\min\{y^T b : y^T A = w, y \geq 0\}$ and let $A^\circ x \leq b^\circ$ be those inequalities $a_i^T x \leq b_i$ whose corresponding dual variable y_i^* is positive. By complementary slackness, we have $F = \{x : Ax \leq b, A^\circ x = b^\circ\}$ as required.

($\Longleftarrow$) Conversely, if $F = \{x \in P : A^\circ x = b^\circ\}$ for some subsystem $A^\circ x \leq b^\circ$ of $Ax \leq b$, then add the inequalities $A^\circ x \leq b^\circ$ to obtain an inequality $w^T x \leq t$. Every $x \in F$ satisfies $w^T x = t$ and every $x \in P \setminus F$ satisfies $w^T x < t$ as required. $\square$

Proposition

Let F be a minimal nonempty face of $P = \{x : Ax \leq b\}$. Then $F = \{x : A^\circ x = b^\circ\}$ for some subsystem $A^\circ x \leq b^\circ$ of $Ax \leq b$.
Moreover, the rank of the matrix A° is equal to the rank of A.

Proposition

If a polyhedron P is pointed, then every minimal nonempty face of P is a vertex.

Proposition

Let $P = \{x : Ax \leq b\}$ and $v \in P$. Then v is a vertex of P if and only if v cannot be written as a convex combination of vectors in $P \setminus \{v\}$.

Theorem

A polytope is equal to the convex hull of its vertices.

Proof. Let P be a nonempty polytope. Since P is bounded, P must be pointed. Let $v_1, \ldots, v_k$ be the vertices of P. Clearly, $conv.hull(\{v_1, \ldots, v_k\}) \subseteq P$. So suppose there exists

$$u \in P \setminus conv.hull(\{v_1, \ldots, v_k\})$$

Then by proposition, there exists an inequality $w^T x \leq t$ that separates u from $conv.hull(\{v_1, \ldots, v_k\})$. Let $t^* = \max\{w^T x : x \in P\}$ and consider the face $F = \{x \in P : w^T x = t^*\}$. Since $u \in P$, we have $t^* > t$. So F contains no vertex of P, a contradiction. $\square$

Theorem

A set P is a polytope if and only if there exists a finite set V such that P is the convex hull of V.

Definition: Vertex

A vector $v \in P$ is called a vertex if $\{v\}$ is a face of P.

Definition: Pointed Polyhedron

A polyhedron P is pointed if it has at least one vertex.

$\{(x_1, x_2) \in \mathbb{R}^2 : x_1 \geq 0\}$ is a polyhedron with no vertex.

3.3 Inegral Polytopes

Definition: Rational Polyhedron

A polyhedron that can be defined by rational linear systems.

Definition: Integral Polyhedron

A rational polyhedron where every nonempty face contains an integral vector.

Definition: Pointed Integral Polyhedron

A pointed rational polyhedron is integral if and only if all its vertices are integral.

Theorem

A rational polytope P is integral if and only if for all integral vectors w, the optimal value of $\max\{w^T x : x \in P\}$ is an integer.

Proof. To prove sufficiency, suppose that for all integral vectors w, the optimal value of $\max\{w^T x : x \in P\}$ is an integer. Let $v = (v_1, \dots, v_n)^T$ be a vertex of P and let w be an integral vector such that v is the unique optimal solution to $\max\{w^T x : x \in P\}$. By multiplying w by a large positive integer if necessary, we may assume $w^T v > w^T u + u_1 - v_1$ for all vertices u of P other than v. This implies that if we let $\overline{w} = (w_1 + 1, w_2, \dots, w_n)^T$, then v is an optimal solution to $\max\{\overline{w}^T x : x \in P\}$. So $\overline{w}^T v = w^T v + v_1$. But, by assumption, $w^T v$ and $\overline{w}^T v$ are integers. Thus, v_1 is an integer. We can repeat this for each component of v, so v must be integral. $\square$

3.4 Total Unimodularity

Proposition

Let A be an integral, nonsingular, $m \times n$ matrix. Then $A^{-1}b$ is integral for every integral vector $b \in \mathbb{R}^m$ if and only if $\det(A) = 1$ or -1.

Proof. $(\Longleftarrow)$ Suppose $\det(A) = \pm 1$. By Cramer's Rule, we know that A^{-1} is integral, which implies $A^{-1}b$ is integral for every integral b.

$(\Longrightarrow)$ Conversely, suppose $A^{-1}b$ is integral for all integral vectors b. Then, in particular, $A^{-1}e_i$ is integral for all $i = 1, \dots, m$. This means that A^{-1} is integral. So $\det(A)$ and $\det(A^{-1})$ are both integers. But, $\det(A) \cdot \det(A^{-1}) = 1$, this implies $\det(A) = \pm 1$. $\square$

Definition: Unimodular

A matrix A of full row rank is unimodular if A is integral and each basis of A has determinant ± 1.

Theorem (Veinott & Dantzig 1968)

Let A be an integral $m \times n$ matrix of full row rank. Then the polyhedron defined by $Ax = b, x \geq 0$ is integral for every integral vector $b \in \mathbb{R}^m$ if and only if A is unimodular.

Proof. $(\Longleftarrow)$ Suppose A is unimodular. Let $b \in \mathbb{R}^m$ be an integral vector and let $\overline{x}$ be a vertex of $\{x : Ax = b, x \geq 0\}$. The nonnegativity constraints implies the polyhedron has vertices. Then there are n linearly independent constraints satisfied by $\overline{x}$ with inequality. It follows that the columns of A corresponding to the nonzero components of $\overline{x}$ are linearly independent. Extending these columns to a basis B of A, we have the nonzero components of $\overline{x}$ are contained in the integral vector $B^{-1}b$. So $\overline{x}$ is integral.

$(\Longrightarrow)$ Conversely, suppose $\{x : Ax = b, x \geq 0\}$ is integral for all integral vectors b. Let B

be a basis of A and let v be an integral vector in $\mathbb{R}^m$. By previous proposition, it suffices to show that $B^{-1}v$ is integral. Let y be an integral vector such that $y + B^{-1}v \geq 0$ and let $b = B(y + B^{-1}v)$. Note b is integral. Furthermore, by adding zero components to the vector $y + B^{-1}v$, we can obtain a vector $z \in \mathbb{R}^n$ such that $Az = b$. Then, z is a vertex of $\{x : Ax = b, x \geq 0\}$, since z is a polyhedron and satisfies n linearly independent constraints with equality: the m equations $Ax = b$ and the $n - m$ equations $x_i = 0$ for the columns i outside B. So z is integral, and thus, $B^{-1}v$ is integral. $\square$

Definition: Totally Unimodular

A matrix is totally unimodular if all of its square submatrices have determinant 0, 1, or -1.

It is easy to see that A is totally unimodular if and only if $\begin{bmatrix} A & I \end{bmatrix}$ is unimodular where $I \in \mathbb{R}^{m \times m}$.

Theorem (Hoffman-Kruskal)

Let A be an $m \times n$ integral matrix. Then the polyhedron defined by $Ax \leq b, x \geq 0$ is integral for every integral vector $b \in \mathbb{R}^m$ if and only if A is totally unimodular.

Proof. Applying the linear programming trick of adding slack variables, we have that for any integral b, the polyhedron $\{x : Ax \leq b, x \geq 0\}$ is integral if and only if the polyhedron $\{z : \begin{bmatrix} A & I \end{bmatrix} z = b, z \geq 0\}$ is integral. So the result follows from previous theorem. $\square$

Theorem

Let A be an $m \times n$ totally unimodular matrix and let $b \in \mathbb{R}^m$ be an integral vector. Then the polyhedron defined by $Ax \leq b$ is integral.

Proof. Let F be a minimal face of $\{x : Ax \leq b\}$. Then, by proposition, $F = \{x : A^\circ x = b^\circ\}$ for some subsystem $A^\circ x \leq b^\circ$ of $Ax \leq b$, with A° having full row rank. By reordering the columns, if necessary, we may write A° as $\begin{bmatrix} B & N \end{bmatrix}$ where B is a basis of A°. It follows

$$\overline{x} = \begin{bmatrix} B^{-1}b^\circ \\ 0 \end{bmatrix}$$

is an integral vector in F. $\square$

Theorem

Let A be a $0, \pm 1$ valued matrix where each column has at most one $+1$ and at most -1. Then A is totally unimodular.

Proof. Let N be a $k \times k$ submatrix of A. If $k = 1$, then $\det(N)$ is either 0 or ± 1. So we may suppose that $k \geq 2$ and proceed by induction on k. If N has a column having at most one nonzero, then expanding the determinant along this column, we have that $\det(N)$ is either 0 or ± 1, by induction. On the other hand, if every column of N has both a $+1$ and a -1, then the sum of the rows of N is 0 and hence $\det(N) = 0$. $\square$

Let $D = (V, E)$ be a digraph and let A be its incidence matrix. Then A is totally unimodular.

Definition: Network Matrix

Let $T = (V, E')$ be a spanning tree of D and define the matrix M having rows indexed by E' and columns indexed by E, where $e = (u, v) \in E$ and $e' \in E'$.

$$M_{e',e} = \begin{cases} +1 & \text{if } uv\text{-path in } T \text{ uses } e' \text{ in forward direction} \\ -1 & \text{if } uv\text{-path in } T \text{ uses } e' \text{ in backward direction} \\ 0 & \text{if } uv\text{-path in } T \text{ does not use } e' \end{cases}$$

Theorem (Tutte 1965)

Network matrices are totally unimodular.

3.5 Edmonds' Matching Polytope

For a graph $G = (V, E)$, let $\mathcal{PM}(G) \subseteq \mathbb{R}^E$ denote the set of characteristic vectors of its perfect matchings.

Theorem (Perfect Matching Polytope Theorem)

For any graph $G = (V, E)$, the convex hull of $\mathcal{PM}(G)$ is identical to the set of solutions of the linear system

$$\begin{aligned} x(\delta(v)) &= 1, \ \forall v \in V \\ x(\delta(S)) &\geq 1, \ \forall S \subseteq V, |S| \geq 3 \text{ and odd} \\ x_e &\geq 0, \ \forall e \in E \end{aligned}$$

Theorem (Birkhoff)

Let G be a bipartite graph. Then the convex hull of the perfect matchings of G is defined by

$$\begin{aligned} x(\delta(v)) &= 1, \ \forall v \in V \\ x_e &\geq 0, \ \forall e \in E \end{aligned}$$

Theorem (Fractional Matching Polytope Theorem)

Let G be a graph and let $x \in FPM(G)$. Then x is a vertex of $FPM(G)$ if and only if $x_e \in \left\{0, \frac{1}{2}, 1\right\}$ for all $e \in E$ and the edges e for which $x_e = \frac{1}{2}$ form vertex-disjoint odd circuits.

Proof. (Perfect Matching Polytope Theorem – Schrijver)

Part III

Optimal Trees and Paths

Chapter 4

Minimum Spanning Trees

4.1 Problem

Definition: Spanning Tree

A subgraph $T \subseteq G$ where $V(T) = V(G)$, T is connected, and T is acyclic.

Lemma

An edge $e = uv$ of G is an edge of a circuit of G if and only if there is a path in $G \setminus e$ from u to v.

Minimum Spanning Tree Problem (MST)

Given a connected graph G and a real cost c_e for each $e \in E$, find a minimum cost spanning tree of G.

Lemma

A spanning connected subgraph of G is a spanning tree if and only if it has exactly $n - 1$ edges.

4.2 Algorithm

Theorem

A graph G is connected if and only if there is no set $A \subseteq V$ where $\emptyset \neq A \neq V$ with $\delta(A) = \emptyset$.

Algorithm 1 Kruskal's Algorithm for MST

```
sort E to {e_1, ..., e_m} so that c_{e_1} ≤ ··· ≤ c_{e_m}
H = (V, F), F = ∅
for i = 1 to m do
    if ends of e_i are in different components of H then
        F ← F ∪ {e_i}
return H
```

4.3 Linear Programming Relaxation

Definition: $\kappa : E \to \mathbb{N}$

$\kappa(A)$ is the number of components in the subgraph (V, A) of G.

We can formulate the MST problem as an IP.

$$\begin{aligned} \min \quad & c^T x \\ \text{s.t.} \quad & x(A) \le |V| - \kappa(A),\ \forall A \subset E \\ & x(E) = |V| - 1 \\ & x_e \in \{0, 1\},\ \forall e \in E \end{aligned}$$

We can relax the integer program to get the following linear program.

Definition: MST LP

$$\begin{aligned} \min \quad & c^T x \\ \text{s.t.} \quad & x(A) \le |V| - \kappa(A),\ \forall A \subset E \\ & x(E) = |V| - 1 \\ & x_e \ge 0,\ \forall e \in E \end{aligned}$$

We replace the minimization with a maximization in the primal to write the dual.

Definition: MST Dual LP

$$\begin{aligned} \min \quad & \sum_{A \subseteq E} (|V| - \kappa(A)) y_A \\ \text{s.t.} \quad & \sum (y_A : e \in A) \ge -c_e,\ \forall e \in E \\ & y_A \ge 0,\ \forall A \subset E \end{aligned}$$

Theorem (Edmonds 1971)

Let x^* be the characteristic vector of an MST with respect to costs c_e. Then x^* is an optimal solution of the linear program.

Proof. We show that x^* is optimal for the LP and x^* is the characteristic vector generated by Kruskal's algorithm. y_E is not required to be nonnegative.

Let $e_1, \dots, d_m$ be the order in which Kruskal's algorithm considers the edges. Let $R_i = \{e_1, \dots, e_i\}$ for $1 \le i \le m$. Let y^* be the be the dual solution. We denote $y^*_A = 0$ unless A is one of the R_i, $y^*_{R_i} = c_{e_{i+1}} - c_{e_i}$ for $1 \le i \le m-1$, and $y^*_{R_m} = -c_{e_m}$. It follows from the ordering of the edges, $y^*_A \ge 0$ for $A \ne E$. Now consider the first constraint, then where $e = e_i$, we have

$$\sum_{A:e\in A} y^*_A = \sum_{j=i}^{m} y^*_{R_j} = \sum_{j=i}^{m-1} (c_{e_{i+1}} - c_{e_i}) = -c_{e_i} = -c_e$$

All of the inequalities hold with equality. So the complementary slackness conditions ($x^*_e > 0 \implies \sum_{A:e\in A} y_A = c_e$) are satisfied.

We want to show now that the second constraint also satisfies complementary slackness conditions ($y^*_A > 0 \implies x(A) \le |V| - \kappa(A)$). We know $A = R_i$ for some i. If the primal constraint does not hold with equality for R_i, then there is some edge of R_i whose addition to $E(T) \cap R_i$ would decrease the number of components of $(V, E(T) \cap R_i)$. But this edge would have ends in two different components of $(V, E(T) \cap R_i)$, and therefore would have been added to T by Kruskal's algorithm.

Therefore, x^* and y^* satisfy complementary slackness conditions. So, x^* is an optimal solution to the LP. $\square$

Chapter 5

Shortest Paths

Shortest Path Problem

Given a digraph G, a vertex $r \in V$, and a real cost vector $(c_e : e \in E)$, find for each $v \in V$, a dipath from r to v of least cost.

Let y_v for $v \in V$ be the least cost of a dipath to v, then y s

Definition: Feasible Potential

$y = (y_v : v \in V)$ is a feasible potential if it satisfies $y_v + c_{vw} \geq y_w$ for all $vw \in E$ and $y_r = 0$.

Proposition

Let y be a feasible potential and let P be a dipath from r to v. Then $c(P) \geq y_v$.

Proof. Suppose that P is $v_0, e_1, v_1, \ldots, e_k, v_k$ where $v_0 = r$ and $v_k = v$. Then

$$c(P) = \sum_{i=1}^{k} c_{e_i} \geq \sum_{i=1}^{k} (y_{v_i} - y_{v_{i-1}}) = y_{v_k} - y_{v_0} = y_v$$

□

5.1 Linear Programming

Theorem

Let G be a digraph, $r, s \in V$, and $c \in \mathbb{R}^E$. If there exists a least-cost dipath from r to v for every $v \in V$, then

$$\min\{c(P) : P \text{ an } rs\text{-dipath}\} = \max\{y_s : y \text{ a feasible potential}\}$$

Definition: Shortest Path LP

$$
\begin{aligned}
\min \quad & \sum(c_e x_e : e \in E) \\
\text{s.t.} \quad & \sum(x_{wv} : w \in V, wv \in E) - \sum(x_{vw} : w \in V, vw \in E) = b_v, \ \forall v \in V \\
& x_{vw} \geq 0, \ \forall vw \in E
\end{aligned}
$$

Definition: Shortest Path Dual LP

$$
\begin{aligned}
\max \quad & y_s - y_r \\
\text{s.t.} \quad & y_w - y_v \leq c_{vw}, \ \forall vw \in E
\end{aligned}
$$

Part IV

Network Flows

Chapter 6

Maximum Flow

6.1 Problem

Definition: Net Flow/Excess

$$f_x(v) = x(\delta(\overline{v})) - x(\delta(v)) = \sum(x_{wv} : w \in V, wv \in E) - \sum(x_{vw} : w \in V, vw \in E)$$

Definition: rs-Flow

A vector x that satisfies $f_x(v) = 0$ for all $v \in V$.

Definition: Value of rs-Flow

$$f_x(s)$$

Maximum Flow Problem

Given a digraph $G = (V, E)$, with source r and sink s, find an rs-flow of maximum value.

Proposition

There exists a family $(P_1, \ldots, P_k)$ of rs-dipaths such that $|\{i : e \in P_i\}| \leq u_e$ for all $e \in E$ if and only if there exists an integral feasible rs-flow of value k.

Proof. ($\Longrightarrow$) We have seen family of dipaths determines a corresponding flow.

($\Longleftarrow$) Let x be a flow. We assume that x is acyclic, that is, there is no dicircuit C, each of whose arcs e has $x_e > 0$. If a dicircuit does exist, we can decrease x_e by 1 on all arcs of C. The new x remains feasible of value k.

If $k \geq 1$, we can find an arc vs with $x_{vs} \geq 1$. Then, if $v \neq r$, it follows that there is an arc wv with $x_{wv} \geq 1$ by the constraint $f_x(v) = 0$. If $w \neq r$, then the argument can be repeated producing distinct vertices, since x is acyclic, so we get a simple rs-dipath P_k on each arc e with $x_e \geq 1$. We can decrease x_e by 1 for each $e \in P_k$. The new x is an integral feasible flow of value $k-1$, and the process is repeated. $\square$

6.2 Maximum Flows and Minimum Cuts

Definition: Maximum Flow LP

$$\begin{aligned} \max \quad & f_x(s) \\ \text{s.t.} \quad & f_x(v) = 0, \ \forall v \in V \setminus \{r, s\} \\ & 0 \leq x_e \leq u_e, \ \forall e \in E \end{aligned}$$

Definition: Path Flow

A vector $x \in \mathbb{R}^E$ such that for some rs-dipath P and some $\alpha \in \mathbb{R}$, $x_e = \alpha$ for each $e \in P$ and $x_e = 0$ for every other arc of G.

Definition: Circuit Flow

A vector $x \in \mathbb{R}^E$ such that for some rs-dicircuit C and some $\alpha \in \mathbb{R}$, $x_e = \alpha$ for each $e \in C$ and $x_e = 0$ for every other arc of G.

Proposition

Every rs-flow of nonnegative value is the sum of at most m flows, each of which is a path flow or a circuit flow.

Proposition

For any rs-cut $\delta(R)$ and any rs-flow x, we have

$$f_x(s) = x(\delta(R)) - x(\delta(\overline{R}))$$

Proof. We add the equations $f_x(v) = 0$ for all $v \in \overline{R} \setminus \{s\}$ as well as the identity $f_x(s) = f_x(s)$. The right hand side sums to $f_x(s)$.

For any arc vw with $v, w \in R$, x_{vw} occurs in none of the equations, so it does not occur in the sum. If $v, w \in \overline{R}$, then x_{vw} occurs in the equation for v with a coefficient of -1, and in the equation for w with a coefficient of $+1$, so it has a coefficient of 0 in the sum. If $v \in R, w \notin R$, then x_{vw} occurs in the equation for w with a coefficient of 1, and so has coefficient 1 in the sum. If $v \notin R, w \in R$, then x_{vw} occurs in the sum with a coefficient of -1. So, the left hand side sums to $x(\delta(R)) - x(\delta(\overline{R}))$, as required. $\square$

Corollary

For any feasible rs-flow x and any rs-cut $\delta(R)$,

$$f_x(s) \le u(\delta(R))$$

Proof. Using previous proposition, since $x(\delta(R)) \le u(\delta(R))$ and $x(\delta(\overline{R})) \ge 0$. $\square$

Definition: Incrementing Path

A path is x-incrementing if every forward arc e has $x_e < u_e$ and every reverse arc e has $x_e > 0$.

Definition: Augmenting Path

An rs-path that is x-incrementing.

Theorem Maximum-Flow Minimum-Cut

If there is a maximum rs-flow, then

$$\max\{f_x(s) : x \text{ is a feasible } rs\text{-flow}\} = \min\{u(\delta(R)) : \delta(R) \text{ is an } rs\text{-cut}\}$$

Proof. By previous corollary, we need only show that there exists a feasible flow x and a cut $\delta(R)$ such that $f_x(s) = u(\delta(R))$. Let x be a flow of maximum value. Let $R = \{v \in V :$ there exists an x-incrementing rv-path$\}$. Clearly $r \in R$ and $s \notin R$, since there can be no x-augmenting path.

For every arc $vw \in \delta(R)$, we must have $x_{vw} = u_{vw}$, since otherwise adding vw to the x-incrementing rv-path would yield such a path to w, but $w \notin R$. Similar, for every arc $vw \in \delta(\overline{R})$, we have $x_{vw} = 0$. Then by proposition, $f_x(s) = x(\delta(R)) - x(\delta(\overline{R})) = u(\delta(R))$. $\square$

Theorem

A feasible flow x is maximum if and only if there is not x-augmenting path.

Proof. ($\implies$) If x is maximum, there is no x-augmenting path.

($\impliedby$) If there is no x-augmenting path, then the construction of the proof of Max-Flow Min-Cut yields a cut $\delta(R)$ with $f_x(s) = u(\delta(R))$, so x is maximum, by corollary. $\square$

Theorem

If u is integral and there exists a maximum flow, then there exists a maximum flow that is integral.

Proof. Choose an integral flow x of maximum value. If there is an x-augmenting path, then since x and u are integral, the new flow can be chosen integral, contradicting the choice of x. Hence there is no x-augmenting path, so x is a maximum flow, by previous theorem. $\square$

Corollary

If x is a feasible rs-flow and $\delta(R)$ is an rs-cut, then x is maximum and $\delta(R)$ is minimum if and only if $x_e = u_e$ for all $e \in \delta(R)$ and $x_e = 0$ for all $e \in \delta(\overline{R})$.

Proof. Combine Max-Flow Min-Cut theorem with the proof of corollary. □

6.3 Augmenting Path Algorithm

Algorithm 2 Ford-Fulkerson Algorithm

$x = 0$
while there is an x-augmenting path P **do**
    $\varepsilon_1 = \min(u_e - x_e : e \text{ forward in } P)$
    $\varepsilon_2 = \min(x_e : e \text{ reverse in } P)$
    $\varepsilon = \min(\varepsilon_1, \varepsilon_2)$ // x-width of P
    if $\varepsilon = \infty$ **then**
        no maximum flow
return x is maximum flow, set R of vertices reachable by an x-incrementing path from r is minimum cut

Definition: Auxiliary Digraph

$G(x)$, depending on G, u, x, where $V(G(x)) = V$ and $vw \in E(G(x))$ if and only if $vw \in E$ and $x_{vw} < u_{vw}$ or $wv \in E$ and $x_{wv} > 0$.

rs-dipaths in $G(x)$ corresponding to x-augmenting paths in G. Each iteration of Ford-Fulkerson can be performed in $O(m)$ time, using breadth-first search.

Theorem

If u is integral and the maximum flow value is $K < \infty$, then the maximum flow algorithm terminates after at most K augmentations.

6.3.1 Shortest Augmenting Paths

Theorem (Dinits 1970, Edmonds & Karp 1972)

If each augmentation of the augmenting path algorithm on a shortest augmenting path, then there are at most nm augmentations.

Corollary

The augmenting path algorithm with breadth-first search solves the maximum flow problem in time $O(nm^2)$.

Let $d_x(v,w)$ be the least length of a vw-dipath in $G(x)$. $d_x(v,w)=\infty$ if no vw-dipath exists.

Consider a typical augmentation from flow x to flow x' determined by the augmenting path P having vertex-sequence $v_0,\dots,v_k$.

Lemma

For each $v \in V$, $d_{x'}(r,v) \geq d_x(r,v)$ and $d_{x'}(v,s) \geq d_x(v,s)$.

Proof. Suppose that there exists a vertex v such that $d_{x'}(r,v) < d_x(r,v)$ and choose such v so that $d_{x'}(r,v)$ is as small as possible. Clearly, $d_{x'}(r,v) > 0$. Let P' be a rv-dipath in $G(x')$ of length $d_{x'}(r,v)$ and let w be the second-last vertex of P'. Then

$$d_x(r,v) > d_{x'}(r,v) = d_{x'}(r,w) + 1 \geq d_x(r,w) + 1$$

It follows that wv is an arc of $G(x')$, but not of $G(x)$, otherwise $d_x(r,v) \leq d_x(r,w)+1$, so $w = v_i$ and $v = v_{i-1}$ for some i. But, this implies that $i-1 > i+1$, a contradiction. The second statement is similar. $\square$

Definition: $\tilde{E}(x)$

$$\tilde{E}(x) = \{e \in E : e \text{ is an arc of a shortest } x\text{-augmenting path}\}$$

Lemma

If $d_{x'}(r,s) = d_x(r,s)$, then $\tilde{E}(x') \subsetneq \tilde{E}(x)$.

Proof. Let $k = d_x(r,s)$ and suppose that $e \in \tilde{E}(x')$. Then e induces an arc vw of $G(x')$ and $d_{x'}(r,v) = i-1$, $d_{x'}(ws) = k-i$ for some i. Therefore, $d_x(r,v)+d_x(w,s) \leq k-1$ by previous lemma. Now suppose that $e \notin \tilde{E}(x)$, then $x_e \neq x'_e$, so e is an arc of P, a contradiction. This proves $\tilde{E}(x') \subseteq \tilde{E}(x)$.

There is an arc e of P such that e is forward and $x'_e = u_e$ or e is reverse and $x'_e = 0$. Therefore, any x'-augmenting path using e must use it in the opposite direction from P, so its length, for some i, will be at least $i+k-i+1+1 = k+23$, so $e \notin \tilde{E}(x')$. $\square$

Proof. (Dinits, Edmonds, Karp) It follows from previous lemma that there can be at most m augmentations per stage. Since there are at most $n-1$ stages, there are at most nm augmentations in all.

6.4 Applications

6.4.1 Bipartite Matchings and Vertex Covers

Theorem (König)

For a bipartite graph G,

$$\max\{|M| : M \text{ a matching}\} = \min\{|C| : C \text{ a cover}\}$$

6.4.2 Flow Feasibility

Flow Feasibility Problem

Given a digraph G, $u \in \mathbb{R}_+^E$, and $b \in \mathbb{R}^V$, find, if possible, $x \in \mathbb{R}^E$ such that

$$f_x(v) = b_v, \ \forall v \in V$$

and

$$0 \leq x_e \leq u_e, \ \forall e \in E$$

Theorem (Gale 1957)

There exists a solution to the flow feasibility problem if and only if $b(V) = 0$ and for every $A \subseteq V$, $b(A) \leq u(\delta(\overline{A}))$.
If b and u are integral, then there is an integral solution.

Corollary

Given a digraph G and $b \in \mathbb{R}^V$, there exists $x \in \mathbb{R}^E$ with

$$f_x(v) = b_v, \ \forall v \in V$$

$$x_e \geq 0, \ \forall e \in E$$

if and only if $b(V) = 0$ and for every $A \subseteq V$ with $\delta(\overline{A}) = \emptyset$, we have $b(A) \leq 0$.

Definition: Circulation

A vector $x \in \mathbb{R}^E$ with $f_x(v) = 0$ for all $v \in V$.

Theorem (Hoffman's Circulation Theorem 1960)

Given a digraph G, $\ell \in (\mathbb{R} \cup \{-\infty\})^E$, and $u \in (\mathbb{R} \cup \{\infty\})^E$, with $\ell \leq u$, there is a circulation x with $\ell \leq x \leq u$ if and only if every $A \subseteq V$ satisfies $u(\delta(\overline{A})) \geq \ell(\delta(A))$.

Part V

Matchings

Chapter 7

Matchings

Definition: Matching

A set $M \subseteq E$ such that no vertex of G is incident with more than one edge in M.

Definition: M-Covered

A vertex v is covered by M if some edge of M is incident with v.

Definition: M-Exposed

A vertex v is exposed if v is not M-covered.

The number of vertices covered by M is $2|M|$ and number of M-exposed vertices is $|V| - 2|M|$.

Definition: Maximum Matching

A matching of maximum cardinality, denoted $\nu(G)$.

Definition: Deficiency

The minimum number of exposed vertices for any matching of G, denoted by $\mathrm{def}(G)$.

Note $\mathrm{def}(G) = |V| - 2\nu(G)$.

Definition: Perfect Matching

A matching that covers all vertices.

7.1 Bipartite Matching

Definition: Bipartite

$G = (V, E)$ is bipartite if $V = V_1 \cup V_2$, where V_1, V_2 disjoint and every edge has one end in V_1 and the other end in V_2.

Definition: Cover

A set $C \subseteq V$ such that every edge has at least one in C.

If M is a matching and C is a cover, then $|M| \leq |C|$ since every edge in M meets one vertex in C, but no vertex in C meets two edges in M.

Definition: Minimum Cover

A cover of minimum cardinality, denoted $\tau(G)$.

Theorem (König)

If G is bipartite, $\nu(G) = \tau(G)$.

In general, $\nu(G) \leq \tau(G)$.

7.2 Alternating Paths

Definition: M-Alternating

A path P is M-alternating if its edges are alternately in and not in M.

Definition: M-Augmenting

An M-alternating path P is M-augmenting if the ends of P are distinct and are both M-exposed.

Definition: Symmetric Difference

For sets S and T, let $S\Delta T$ denote the symmetric difference, which is defined as

$$S\Delta T = (S \cup T) \setminus (S \cap T)$$

Let a path P be an M-augmenting path. Then we can obtain a larger matching $M' = M\Delta E(P)$ with $|M'| = |M| + 1$.

Theorem (Augmenting Path Theorem of Matchings – Berge 1957)

A matching M in a graph G is maximum if and only if there is no M-augmenting path.

Proof. ($\Longrightarrow$) Suppose there exists an M-augmenting path P joining v and w. Then $N = M \Delta E(P)$ is a matching that covers all vertices covered by M, plus v and w. So, M is not maximum.

($\Longleftarrow$) Conversely, suppose that M is not maximum and some other matching N satisfies $|N| > |M|$. Let $J = N \Delta M$. Each vertex of G is incident with at most two edges of J, so J is the edge set of some vertex disjoint paths and circuits of G. For each such path or circuit, the edges alternately belong to M or N. Therefore, all circuits are even and contain the same number of edges of M and N. Since $|N| > |M|$, there must be at least one path with more edges of N than M. This path is an M-augmenting path. $\square$

7.3 Tutte-Berge Formula

Definition: Vertex Cover

A set A of vertices such that every edge has at least one end in A.

Let A be a subset of the vertices which $G - A$ has k components $H_1, \dots, H_k$ having an odd number of vertices. Let M be a matching of G. For each i, either H_i has an M-exposed vertex or M contains an edge having just one end in $V(H_i)$. All such edges have their other ends in A and since M is a matching, all these ends must be distinct. Therefore, there can be at most $|A|$ edges and so the number of M-exposed vertices is at least $k - |A|$.

Definition: $\mathrm{oc}(H)$

The number of odd components of a graph H.

Thus, for any $A \subseteq V$,

$$\nu(G) \leq \frac{1}{2}(|V| - \mathrm{oc}(G - A) + |A|)$$

If A is a cover of G, then there are $|V| - |A|$ odd components of $G - A$ (each is a single vertex), so the right hand side reduces to $|A|$. This bound is at least as strong as that provided by covers.

Theorem (Tutte-Berge Formula)

For a graph $G = (V, E)$, we have

$$\max\{|M| : M \text{ a matching}\} = \min\left\{\frac{1}{2}(|V| - \mathrm{oc}(G - A) + |A|) : A \subseteq V\right\}$$

Theorem (Tutte's Matching Theorem 1947)

A graph $G = (V, E)$ has a perfect matching if and only if for every $A \subseteq V$, $\text{oc}(G - A) \leq |A|$.

Definition: Shrink

Let C be an odd circuit in G. Define $G' = G \times C$ as the subgraph obtained from G by shrinking C; G' has vertex set $(V - V(C)) \cup \{C\}$ and edge set $E \setminus \gamma(V(C))$.

Proposition

Let C be an odd circuit of G, let $G' = G \times C$, and let M' be a matching of G'. Then here is a matching M of G such that $M \subseteq M' \cup E(C)$ and the number of M-exposed vertices of G is the same as the number of M'-exposed vertices of G'.

Proof. Choose a vertex $w \in V(C)$ as follows. If C is covered by $e \in M'$, then choose w to be the vertex in $V(C)$ that is an end of e, and otherwise, choose w arbitrarily. Deleting w from C results in a subgraph having a perfect matching M''. Take $M = M' \cup M''$. M has the required properties. $\square$

The previous proposition gives the inequality

$$\nu(G) \geq \nu(G \times C) + \frac{|V(C)| - 1}{2}$$

or equivalently,

$$\text{def}(G) \leq \text{def}(G \times C)$$

Definition: Tight Odd Circuit

An odd circuit C is tight if $\nu(G) = \nu(G \times C) + \frac{|V(C)|-1}{2}$.

Definition: Inessential

A vertex v of G is inessential if there is a maximum matching of G that does not cover v.

Definition: Essential

A vertex not inessential.

Let A be a set that satisfies the Tutte-Berge formula. Let $v \in A$ and consider $G' = G - v$. Then, $G' - (A \setminus \{v\})$ has the same odd components as $G - A$, so $\nu(G') < \nu(G)$, i.e. every $v \in A$ is essential.

Lemma

Let $G = (V, E)$ be a graph and let $vw \in E$. If v, w are both inessential, then there is a tight odd circuit C using vw. Moreover, C is an inessential vertex of $G \times C$.

Chapter 8

Maximum Matching

Maximum Matching Problem

Given a graph G, find a maximum matching of G.

Definition: Maximum Matching IP

$$\begin{aligned} \max \quad & \sum(x_e : e \in E) \\ \text{s.t.} \quad & x(\delta(v)) \leq 1, \ \forall v \in V \\ & x_e \geq 0, \ \forall e \in E \\ & x_e \text{ integer}, \ \forall e \in E \end{aligned}$$

Definition: Maximum Matching LP

$$\begin{aligned} \max \quad & \sum(x_e : e \in E) \\ \text{s.t.} \quad & x(\delta(v)) \leq 1, \ \forall v \in V \\ & x_e \geq 0, \ \forall e \in E \end{aligned}$$

Definition: Minimum Cover Dual LP

$$\begin{aligned} \min \quad & \sum(y_v : v \in V) \\ \text{s.t.} \quad & y_u + y_v \geq 1, \ \forall e = (u, v) \in E \\ & y_v \geq 0, \ \forall v \in V \end{aligned}$$

8.1 Alternating Trees

Suppose we have a matching M of G and a fixed M-exposed vertex r of G. We can iteratively build up sets A, B of vertices such that each vertex in A is the other end of an odd-length

M-alternating path beginning at r, and each vertex in B is the other end of an even-length M-alternating path beginning at r.

Begin with $A = \emptyset, B = \{r\}$, and use the rule: if $vw \in E, v \in B, w \notin A \cup B, wz \in M$, then add w to A, z to B. The set $A \cup B$ and edges in the construction form a tree T rooted at r.

Definition: Alternating Tree

A tree T such that

- every vertex of T other than r is covered by an edge of $M \cap E(T)$;
- for every vertex v of T, the path in T from v to r is M-alternating.

We let the vertex sets at odd and even distances from the root as $A(T)$ and $B(T)$ respectively. Note that $|B(T)| = |A(T)| + 1$ since all other vertices other than r come in matched pairs, one in $A(T)$ and one in $B(T)$.

Chapter 9

T-Joins

Part VI

Matroids

Part VII

Traveling Salesman Problem