

CO 450/650 Combinatorial Optimization

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Part I

Introduction

Chapter 1

Introduction

Definition: Combinatorial Optimization

A subfield of mathematical optimization which involves searching for an optimal object in a finite collection of objects.

Typically, the collection has a concise representation, while the number of objects is large. Objects include graphs, networks, and matroids.

The main tool in combinatorial optimization is linear programming duality.

Chapter 2

Linear Programming

Definition: Linear Programming

The problem of finding a vector x that maximizes a given linear function $c^T x$, where x ranges over all vectors satisfying a given system $Ax \leq b$ of linear inequalities.

2.1 Farkas' Lemma

Lemma (Farkas' Lemma for Inequalities)

The system $Ax \leq b$ has a solution x if and only if there is no vector y satisfying $y \geq 0$, $y^T A = 0$, and $y^T b < 0$.

Proof. Suppose $Ax \leq b$ has a solution \bar{x} and suppose there exists a vector $\bar{y} \geq 0$ satisfying $\bar{y}^T A = 0$ and $\bar{y}^T b < 0$. Then we obtain the contradiction

$$0 > \bar{y}^T b \geq \bar{y}^T (A\bar{x}) = (\bar{y}^T A)\bar{x} = 0$$

Now suppose that $Ax \leq b$ has no solution. If A has only one column, then the result is easy. Otherwise, apply Fourier-Motzkin elimination to obtain a system $A'x' \leq b'$ with one less variable. Since $A'x' \leq b'$ also has no solution, we can assume by induction that there exists a vector $y' \geq 0$ satisfying $y'^T A' = 0$ and $y'^T b' < 0$. Now since each inequality in $A'x' \leq b'$ is the sum of positive multiples of inequalities in $Ax \leq b$, we can use y' to construct a vector y satisfying the conditions in the theorem. \square

Lemma (Farkas' Lemma)

The system $Ax = b$ has a nonnegative solution if and only if there is no vector y satisfying $y^T A \geq 0$ and $y^T b < 0$.

Proof. Define

$$A' = \begin{bmatrix} A \\ -A \\ -I \end{bmatrix}, b' = \begin{bmatrix} b \\ -b \\ 0 \end{bmatrix}$$

Then $Ax = b$ has a nonnegative solution x if and only if $A'x' \leq b'$ has a solution x' . Applying Farkas' Lemma for Inequalities to $A'x' \leq b'$ gives the result. \square

Corollary

Suppose the system $Ax \leq b$ has at least one solution. Then every solution x of $Ax \leq b$ satisfies $c^T x \leq \delta$ if and only if there exists a vector $y \geq 0$ such that $y^T A = c$ and $y^T b \leq \delta$.

2.2 Duality

Consider the LP:

$$\begin{array}{ll} \max & c^T x \\ \text{s.t.} & Ax \leq b \end{array}$$

and dual LP

$$\begin{array}{ll} \min & y^T b \\ \text{s.t.} & y^T A = c^T \\ & y \geq 0 \end{array}$$

Theorem (Weak Duality)

Let A be an $m \times n$ matrix, $b \in \mathbb{R}^m$, $c \in \mathbb{R}^n$. Suppose that \bar{x} is a feasible solution to $Ax \leq b$ and \bar{y} is a feasible solution to $y \geq 0, y^T A = c^T$. Then

$$c^T \bar{x} \leq \bar{y}^T b$$

Proof.

$$c^T \bar{x} = (\bar{y}^T A) \bar{x} = \bar{y}^T (A \bar{x}) \leq \bar{y}^T b$$

\square

Theorem (Strong Duality)

Let A be an $m \times n$ matrix, $b \in \mathbb{R}^m$, $c \in \mathbb{R}^n$, then

$$\max\{c^T x : Ax \leq b\} = \min\{y^T b : y^T A = c^T, y \geq 0\}$$

provided that both sets are nonempty.

Corollary

Let A be an $m \times n$ matrix, $b \in \mathbb{R}^m$, $c \in \mathbb{R}^n$, then

$$\max\{c^T x : Ax \leq b, x \geq 0\} = \min\{y^T b : y^T A \geq c^T\}$$

provided that both sets are nonempty.

Definition: Complementary Slackness Conditions

For each $i \in \{1, \dots, m\}$, either $y_i^* = 0$ or $a_i x^* = b_i$.

Theorem (Complementary Slackness Theorem)

Let x^* be a feasible solution of $\max\{c^T x : Ax \leq b\}$ and let y^* be a feasible solution of $\min\{y^T b : y^T A = c^T, y \geq 0\}$. Then x^* and y^* are optimal solutions for the maximum and minimum respectively if and only if the complementary slackness conditions hold.

Chapter 3

Graph Theory

Definition: Graph

A graph $G = (V, E)$ is a set of vertices/nodes V and a set of edges E . We define $n = |V|$ and $m = |E|$.

Definition: Subgraph

$H = (W, F)$ of $G = (V, E)$ where $W \subseteq V$ and $F \subseteq E$.

Definition: Spanning Subgraph

H is spanning if $V(H) = V(G)$.

Definition: Path

A sequence $P = v_0, e_1, v_1, \dots, e_k, v_k$ where $v_0, \dots, v_k \in V(G)$, $e_1, \dots, e_k \in E(G)$, and $e_i = v_{i-1}v_i$.

We call P a v_0v_1 -path. P is called edge-simple if all e_i are distinct and simple if all v_i are distinct.

The length of P is the number of edges in P .

Definition: Circuit/Cycle

An edge-simple closed path.

Definition: Connected

A graph is connected if every pair of vertices is joined by a path.

Definition: Cut Vertex

A vertex v of a connected graph G where $G - v$ is not connected.

Definition: Forest

A graph with no circuits.

Definition: Tree

A connected forest.

Definition: Cut

Let $R \subseteq V$, then

$$\delta(R) = \{vw : vw \in E, v \in R, w \notin R\}$$

Definition: rs -Cut

A cut for which $r \in R, s \notin R$.

Chapter 4

Complexity Theory

Definition: Decision Problem

A problem with a yes-no answer.

Definition: \mathcal{P}

Decision problems that can be solved in polynomial time.

Definition: \mathcal{NP}

Decision problems in which we can certify the answer is yes in polynomial time.

Definition: $\text{co-}\mathcal{NP}$

Decision problems in which we can certify the answer is no in polynomial time.

A good characterization means the problem is in $\mathcal{NP} \cap \text{co-}\mathcal{NP}$.

Definition: \mathcal{NP} -Hard

A problem X is \mathcal{NP} -hard if every other problem Y in \mathcal{NP} can be reduced to X .

S. Cook (1971) proved that the satisfiability problem (SAT) is \mathcal{NP} -hard. R. Karp (1972) used Cook's result to show 21 well-known combinatorial optimization problems are also \mathcal{NP} -hard.

To show that the traveling salesman problem (TSP) is \mathcal{NP} -hard, we show that any example of SAT can be formulated as a TSP, of size polynomial in the size of SAT. Then, since Cook shows SAT is \mathcal{NP} -hard, TSP is also \mathcal{NP} -hard.

Part II

Polyhedral Combinatorics

Chapter 5

Integrality of Polyhedra

5.1 Convex Hull

Definition: Convex Combination

$x = \lambda_1 v_1 + \cdots + \lambda_k v_k$ for some vectors v_1, \dots, v_k and nonnegative scalars $\lambda_1, \dots, \lambda_k$ such that $\lambda_1 + \cdots + \lambda_k = 1$.

Definition: Convex Hull

The convex hull of a finite set S , denoted $\text{conv.hull}(S)$, is the set of all vectors that can be written as a convex combination of S .

It is also defined as the smallest convex set containing S .

Proposition

Let $S \subseteq \mathbb{R}^n$ be a finite set and let $w \in \mathbb{R}^n$. Then

$$\max / \min \{w^T x : x \in S\} = \max / \min \{w^T x : x \in \text{conv.hull}(S)\}$$

Theorem (Minkowski)

If S is finite, then $\text{conv.hull}(S)$ is a polyhedron.

$$\begin{aligned} \max(w^T x : x \in S) &= \max(w^T x : x \in \text{conv.hull}(S)) \\ &= \max(w^T x : Ax \leq b) \\ &= \min(y^T b : y^T A = w^T, y \geq 0) \end{aligned}$$

So we can use LP duality to attack combinatorial problems. If we understand $Ax \leq b$, then the problem is in $\text{co-}\mathcal{NP}$. Thus, if we have an algorithm to produce the inequalities in $Ax \leq b$ (separation), then the problem is in \mathcal{P} (Ellipsoid method).

5.2 Polytopes

Definition: Polyhedron

A set of the form $\{x : Ax \leq b\}$.

In combinatorial optimization, we typically have $x \geq 0$ as a constraint, so we have polyhedra of the form $\{x : Ax \leq b, x \geq 0\}$.

Definition: Polytope

A polyhedron $P \subseteq \mathbb{R}^n$ is a polytope if there exists $\ell, u \in \mathbb{R}^n$ such that $\ell \leq x \leq u$ for all $x \in P$.

Definition: Convex Set

Let P be a polyhedron, $x_1, x_2 \in P$, and $0 \leq \lambda \leq 1$. If $\lambda x_1 + (1 - \lambda)x_2 \in P$, then P is a convex set.

Definition: Valid Inequality

An inequality $w^T x \leq t$ is valid for a polyhedron P if $P \subseteq \{x : w^T x \leq t\}$.

Definition: Hyperplane

The solution set of $w^T x = t$ where $w \neq 0$.

Definition: Supporting Hyperplane

With respect to a polyhedron P , a hyperplane is supporting if $w^T x \leq t$ is valid for P and $P \cap \{x : w^T x = t\} \neq \emptyset$.

Definition: Face

The intersection of a polyhedron with one of its supporting hyperplanes.

The null set and the polyhedron itself is a face.

Definition: Proper Face

Faces which are not the null set or the polyhedron itself.

Proposition

A nonempty set $F \subseteq P = \{x : Ax \leq b\}$ is a face of P if and only if for some subsystem $A^\circ x \leq b^\circ$ of $Ax \leq b$, we have $F = \{x \in P : A^\circ x = b^\circ\}$.

Proof. (\implies) Suppose F is a face of P . Then there exists a valid inequality $w^T x \leq t$ such that $F = \{x \in P : w^T x = t\}$.

Consider the LP problem $\max\{w^T x : Ax \leq b\}$. The set of optimal solutions is precisely F . Now let y^* be an optimal solution to the dual problem $\min\{y^T b : y^T A = w, y \geq 0\}$ and let $A^\circ x \leq b^\circ$ be those inequalities $a_i^T x \leq b_i$ whose corresponding dual variable y_i^* is positive. By complementary slackness, we have $F = \{x : Ax \leq b, A^\circ x = b^\circ\}$ as required.

(\Leftarrow) Conversely, if $F = \{x \in P : A^\circ x = b^\circ\}$ for some subsystem $A^\circ x \leq b^\circ$ of $Ax \leq b$, then add the inequalities $A^\circ x \leq b^\circ$ to obtain an inequality $w^T x \leq t$. Every $x \in F$ satisfies $w^T x = t$ and every $x \in P \setminus F$ satisfies $w^T x < t$ as required. \square

Proposition

Let F be a minimal nonempty face of $P = \{x : Ax \leq b\}$. Then $F = \{x : A^\circ x = b^\circ\}$ for some subsystem $A^\circ x \leq b^\circ$ of $Ax \leq b$. Moreover, the rank of the matrix A° is equal to the rank of A .

Definition: Vertex/Extreme Point

A vector $x \in P$ is called a vertex/extreme point if $\{x\}$ is a face of P . Equivalently, $x \in P$ is a vertex/extreme point if x cannot be written as $\frac{1}{2}x_1 + \frac{1}{2}x_2$ for points $x_1, x_2 \in P, x_1 \neq x_2$.

Note: Not all polyhedra have vertices, but if $P \subseteq \mathbb{R}_+^n$, then P has vertices.

LP Fact

If a polyhedron P has vertices, then the set of optimal LP solutions contains at least one vertex of P . Moreover, if all vertices of P are integral, then the LP always has an integral optimal solution.

Definition: Pointed Polyhedron

A polyhedron P is pointed if it has at least one vertex.

$\{(x_1, x_2) \in \mathbb{R}^2 : x_1 \geq 0\}$ is a polyhedron with no vertex.

Proposition

If a polyhedron P is pointed, then every minimal nonempty face of P is a vertex.

Proposition

Let $P = \{x : Ax \leq b\}$ and $v \in P$. Then v is a vertex of P if and only if v cannot be written as a convex combination of vectors in $P \setminus \{v\}$.

Theorem

A polytope is equal to the convex hull of its vertices.

Proof. Let P be a nonempty polytope. Since P is bounded, P must be pointed. Let

v_1, \dots, v_k be the vertices of P . Clearly, $\text{conv.hull}(\{v_1, \dots, v_k\}) \subseteq P$. So suppose there exists

$$u \in P \setminus \text{conv.hull}(\{v_1, \dots, v_k\})$$

Then by proposition, there exists an inequality $w^T x \leq t$ that separates u from

$$\text{conv.hull}(\{v_1, \dots, v_k\})$$

Let $t^* = \max\{w^T x : x \in P\}$ and consider the face $F = \{x \in P : w^T x = t^*\}$. Since $u \in P$, we have $t^* > t$. So F contains no vertex of P , a contradiction. \square

Theorem

A set P is a polytope if and only if there exists a finite set V such that P is the convex hull of V .

5.3 Integral Polytopes

Definition: Rational Polyhedron

A polyhedron that can be defined by rational linear systems.

Definition: Integral Polyhedron

A rational polyhedron where every nonempty face contains an integral vector.

Definition: Pointed Integral Polyhedron

A pointed rational polyhedron is integral if and only if all its vertices are integral.

Theorem

A rational polytope P is integral if and only if for all integral vectors w , the optimal value of $\max\{w^T x : x \in P\}$ is an integer.

Proof. To prove sufficiency, suppose that for all integral vectors w , the optimal value of $\max\{w^T x : x \in P\}$ is an integer. Let $v = (v_1, \dots, v_n)^T$ be a vertex of P and let w be an integral vector such that v is the unique optimal solution to $\max\{w^T x : x \in P\}$. By multiplying w by a large positive integer if necessary, we may assume $w^T v > w^T u + u_1 - v_1$ for all vertices u of P other than v . This implies that if we let $\bar{w} = (w_1 + 1, w_2, \dots, w_n)^T$, then v is an optimal solution to $\max\{\bar{w}^T x : x \in P\}$. So $\bar{w}^T v = w^T v + v_1$. But, by assumption, $w^T v$ and $\bar{w}^T v$ are integers. Thus, v_1 is an integer. We can repeat this for each component of v , so v must be integral. \square

5.4 Total Unimodularity

Proposition

Let A be an integral, nonsingular, $m \times n$ matrix. Then $A^{-1}b$ is integral for every integral vector $b \in \mathbb{R}^m$ if and only if $\det(A) = 1$ or -1 .

Proof. (\Leftarrow) Suppose $\det(A) = \pm 1$. By Cramer's Rule, we know that A^{-1} is integral, which implies $A^{-1}b$ is integral for every integral b .

(\Rightarrow) Conversely, suppose $A^{-1}b$ is integral for all integral vectors b . Then, in particular, $A^{-1}e_i$ is integral for all $i = 1, \dots, m$. This means that A^{-1} is integral. So $\det(A)$ and $\det(A^{-1})$ are both integers. But, $\det(A) \cdot \det(A^{-1}) = 1$, this implies $\det(A) = \pm 1$. \square

Definition: Unimodular

A matrix A of full row rank is unimodular if A is integral and each basis of A has determinant ± 1 .

Theorem (Veinott & Dantzig 1968)

Let A be an integral $m \times n$ matrix of full row rank. Then the polyhedron defined by $Ax = b, x \geq 0$ is integral for every integral vector $b \in \mathbb{R}^m$ if and only if A is unimodular.

Proof. (\Leftarrow) Suppose A is unimodular. Let $b \in \mathbb{R}^m$ be an integral vector and let \bar{x} be a vertex of $\{x : Ax = b, x \geq 0\}$. The nonnegativity constraints implies the polyhedron has vertices. Then there are n linearly independent constraints satisfied by \bar{x} with inequality. It follows that the columns of A corresponding to the nonzero components of \bar{x} are linearly independent. Extending these columns to a basis B of A , we have the nonzero components of \bar{x} are contained in the integral vector $B^{-1}b$. So \bar{x} is integral.

(\Rightarrow) Conversely, suppose $\{x : Ax = b, x \geq 0\}$ is integral for all integral vectors b . Let B be a basis of A and let v be an integral vector in \mathbb{R}^m . By previous proposition, it suffices to show that $B^{-1}v$ is integral. Let y be an integral vector such that $y + B^{-1}v \geq 0$ and let $b = B(y + B^{-1}v)$. Note b is integral. Furthermore, by adding zero components to the vector $y + B^{-1}v$, we can obtain a vector $z \in \mathbb{R}^n$ such that $Az = b$. Then, z is a vertex of $\{x : Ax = b, x \geq 0\}$, since z is a polyhedron and satisfies n linearly independent constraints with equality: the m equations $Ax = b$ and the $n - m$ equations $x_i = 0$ for the columns i outside B . So z is integral, and thus, $B^{-1}v$ is integral. \square

Definition: Totally Unimodular (TU)

A matrix is totally unimodular if all of its square submatrices have determinant 0, 1, or -1 .

It is easy to see that A is totally unimodular if and only if $\begin{bmatrix} A & I \end{bmatrix}$ is unimodular where $I \in \mathbb{R}^{m \times m}$.

Theorem (Hoffman-Kruskal)

Let A be an $m \times n$ integral matrix. Then the polyhedron defined by $Ax \leq b, x \geq 0$ is integral for every integral vector $b \in \mathbb{R}^m$ if and only if A is totally unimodular.

Proof. Applying the linear programming trick of adding slack variables, we have that for any integral b , the polyhedron $\{x : Ax \leq b, x \geq 0\}$ is integral if and only if the polyhedron $\{z : \begin{bmatrix} A & I \end{bmatrix} z = b, z \geq 0\}$ is integral. So the result follows from previous theorem. \square

Theorem

Let A be an $m \times n$ totally unimodular matrix and let $b \in \mathbb{R}^m$ be an integral vector. Then the polyhedron defined by $Ax \leq b$ is integral.

Proof. Let F be a minimal face of $\{x : Ax \leq b\}$. Then, by proposition, $F = \{x : A^\circ x = b^\circ\}$ for some subsystem $A^\circ x \leq b^\circ$ of $Ax \leq b$, with A° having full row rank. By reordering the columns, if necessary, we may write A° as $\begin{bmatrix} B & N \end{bmatrix}$ where B is a basis of A° . It follows

$$\bar{x} = \begin{bmatrix} B^{-1}b^\circ \\ 0 \end{bmatrix}$$

is an integral vector in F . \square

Theorem

Let A be a $0, \pm 1$ valued matrix where each column has at most one $+1$ and at most -1 . Then A is totally unimodular.

Proof. Let N be a $k \times k$ submatrix of A . If $k = 1$, then $\det(N)$ is either 0 or ± 1 . So we may suppose that $k \geq 2$ and proceed by induction on k . If N has a column having at most one nonzero, then expanding the determinant along this column, we have that $\det(N)$ is either 0 or ± 1 , by induction. On the other hand, if every column of N has both a $+1$ and a -1 , then the sum of the rows of N is 0 and hence $\det(N) = 0$. \square

Let $D = (V, E)$ be a digraph and let A be its incidence matrix. Then A is totally unimodular.

Definition: Network Matrix

Let $T = (V, E')$ be a spanning tree of D and define the matrix M having rows indexed by E' and columns indexed by E , where $e = (u, v) \in E$ and $e' \in E'$.

$$M_{e',e} = \begin{cases} +1 & \text{if } uv\text{-path in } T \text{ uses } e' \text{ in forward direction} \\ -1 & \text{if } uv\text{-path in } T \text{ uses } e' \text{ in backward direction} \\ 0 & \text{if } uv\text{-path in } T \text{ does not use } e' \end{cases}$$

Theorem (Tutte 1965)

Network matrices are totally unimodular.

Proposition

A is totally unimodular if and only if A^T is totally unimodular.

5.5 Edmonds' Matching Polytope

For a graph $G = (V, E)$, let $\mathcal{PM}(G) \subseteq \mathbb{R}^E$ denote the set of characteristic vectors of its perfect matchings.

Theorem (Perfect Matching Polytope Theorem)

For any graph $G = (V, E)$, the convex hull of $\mathcal{PM}(G)$ is identical to the set of solutions of the linear system

$$\begin{aligned}x(\delta(v)) &= 1, \quad \forall v \in V \\x(\delta(S)) &\geq 1, \quad \forall S \subseteq V, |S| \geq 3 \text{ and odd} \\x_e &\geq 0, \quad \forall e \in E\end{aligned}$$

Theorem (Birkhoff)

Let G be a bipartite graph. Then the convex hull of the perfect matchings of G is defined by

$$\begin{aligned}x(\delta(v)) &= 1, \quad \forall v \in V \\x_e &\geq 0, \quad \forall e \in E\end{aligned}$$

Theorem (Fractional Matching Polytope Theorem)

Let G be a graph and let $x \in FPM(G)$. Then x is a vertex of $FPM(G)$ if and only if $x_e \in \{0, \frac{1}{2}, 1\}$ for all $e \in E$ and the edges e for which $x_e = \frac{1}{2}$ form vertex-disjoint odd circuits.

Proof. (Perfect Matching Polytope Theorem – Schrijver)

Part III

Optimal Trees and Paths

Chapter 6

Minimum Spanning Trees

6.1 Problem

Definition: Spanning Tree

A subgraph $T \subseteq G$ where $V(T) = V(G)$, T is connected, and T is acyclic.

Lemma

An edge $e = uv$ of G is an edge of a circuit of G if and only if there is a path in $G \setminus e$ from u to v .

Minimum Spanning Tree Problem (MST)

Given a connected graph G and a real cost c_e for each $e \in E$, find a minimum cost spanning tree of G .

Lemma

A spanning connected subgraph of G is a spanning tree if and only if it has exactly $n - 1$ edges.

6.2 Algorithm

Theorem

A graph G is connected if and only if there is no set $A \subseteq V$ where $\emptyset \neq A \neq V$ with $\delta(A) = \emptyset$.

Algorithm 1 Kruskal's Algorithm for MST

```
1: sort  $E$  to  $\{e_1, \dots, e_m\}$  so that  $c_{e_1} \leq \dots \leq c_{e_m}$ 
2:  $H = (V, F), F = \emptyset$ 
3: for  $i = 1$  to  $m$  do
4:   if ends of  $e_i$  are in different components of  $H$  then
5:      $F \leftarrow F \cup \{e_i\}$ 
6: return  $H$ 
```

6.3 Linear Programming Relaxation

Definition: $\kappa : E \rightarrow \mathbb{N}$

$\kappa(A)$ is the number of components in the subgraph (V, A) of G .

We can formulate the MST problem as an IP.

$$\begin{aligned} \min \quad & c^T x \\ \text{s.t.} \quad & x(A) \leq |V| - \kappa(A), \quad \forall A \subset E \\ & x(E) = |V| - 1 \\ & x_e \in \{0, 1\}, \quad \forall e \in E \end{aligned}$$

We can relax the integer program to get the following linear program.

Definition: MST LP

$$\begin{aligned} \min \quad & c^T x \\ \text{s.t.} \quad & x(A) \leq |V| - \kappa(A), \quad \forall A \subset E \\ & x(E) = |V| - 1 \\ & x_e \geq 0, \quad \forall e \in E \end{aligned}$$

We replace the minimization with a maximization in the primal to write the dual.

Definition: MST Dual LP

$$\begin{aligned} \min \quad & \sum_{A \subseteq E} (|V| - \kappa(A)) y_A \\ \text{s.t.} \quad & \sum (y_A : e \in A) \geq -c_e, \quad \forall e \in E \\ & y_A \geq 0, \quad \forall A \subset E \end{aligned}$$

Theorem (Edmonds 1971)

Let x^* be the characteristic vector of an MST with respect to costs c_e . Then x^* is an optimal solution of the linear program.

Proof. We show that x^* is optimal for the LP and x^* is the characteristic vector generated by Kruskal's algorithm. y_E is not required to be nonnegative.

Let e_1, \dots, e_m be the order in which Kruskal's algorithm considers the edges. Let $R_i = \{e_1, \dots, e_i\}$ for $1 \leq i \leq m$. Let y^* be the dual solution. We denote $y_A^* = 0$ unless A is one of the R_i , $y_{R_i}^* = c_{e_{i+1}} - c_{e_i}$ for $1 \leq i \leq m-1$, and $y_{R_m}^* = -c_{e_m}$. It follows from the ordering of the edges, $y_A^* \geq 0$ for $A \neq E$. Now consider the first constraint, then where $e = e_i$, we have

$$\sum_{A: e \in A} y_A^* = \sum_{j=i}^m y_{R_j}^* = \sum_{j=i}^{m-1} (c_{e_{j+1}} - c_{e_j}) = -c_{e_i} = -c_e$$

All of the inequalities hold with equality. So the complementary slackness conditions ($x_e^* > 0 \implies \sum_{A: e \in A} y_A^* = c_e$) are satisfied.

We want to show now that the second constraint also satisfies complementary slackness conditions ($y_A^* > 0 \implies x(A) \leq |V| - \kappa(A)$). We know $A = R_i$ for some i . If the primal constraint does not hold with equality for R_i , then there is some edge of R_i whose addition to $E(T) \cap R_i$ would decrease the number of components of $(V, E(T) \cap R_i)$. But this edge would have ends in two different components of $(V, E(T) \cap R_i)$, and therefore would have been added to T by Kruskal's algorithm.

Therefore, x^* and y^* satisfy complementary slackness conditions. So, x^* is an optimal solution to the LP. \square

Chapter 7

Shortest Paths

Shortest Path Problem

Given a digraph G , a vertex $r \in V$, and a real cost vector $(c_e : e \in E)$, find for each $v \in V$, a dipath from r to v of least cost.

Let y_v for $v \in V$ be the least cost of a dipath to v , then y s

Definition: Feasible Potential

$y = (y_v : v \in V)$ is a feasible potential if it satisfies $y_v + c_{vw} \geq y_w$ for all $vw \in E$ and $y_r = 0$.

Proposition

Let y be a feasible potential and let P be a dipath from r to v . Then $c(P) \geq y_v$.

Proof. Suppose that P is $v_0, e_1, v_1, \dots, e_k, v_k$ where $v_0 = r$ and $v_k = v$. Then

$$c(P) = \sum_{i=1}^k c_{e_i} \geq \sum_{i=1}^k (y_{v_i} - y_{v_{i-1}}) = y_{v_k} - y_{v_0} = y_v$$

□

7.1 Linear Programming

Theorem

Let G be a digraph, $r, s \in V$, and $c \in \mathbb{R}^E$. If there exists a least-cost dipath from r to v for every $v \in V$, then

$$\min\{c(P) : P \text{ an } rs\text{-dipath}\} = \max\{y_s : y \text{ a feasible potential}\}$$

Definition: Shortest Path LP

$$\begin{array}{ll}\min & \sum (c_e x_e : e \in E) \\ \text{s.t.} & \sum (x_{vw} : w \in V, vw \in E) - \sum (x_{vw} : w \in V, vw \in E) = b_v, \forall v \in V \\ & x_{vw} \geq 0, \forall vw \in E\end{array}$$

Definition: Shortest Path Dual LP

$$\begin{array}{ll}\max & y_s - y_r \\ \text{s.t.} & y_w - y_v \leq c_{vw}, \forall vw \in E\end{array}$$

Part IV

Network Flows

Chapter 8

Maximum Flow

8.1 Problem

Definition: Net Flow/Excess

$$f_x(v) = x(\delta(\bar{v})) - x(\delta(v)) = \sum(x_{wv} : w \in V, wv \in E) - \sum(x_{vw} : w \in V, vw \in E)$$

Definition: rs -Flow

A vector x that satisfies $f_x(v) = 0$ for all $v \in V$.

Definition: Value of rs -Flow

$$f_x(s)$$

Maximum Flow Problem

Given a digraph $G = (V, E)$, with source r and sink s , find an rs -flow of maximum value.

Proposition

There exists a family (P_1, \dots, P_k) of rs -dipaths such that $|\{i : e \in P_i\}| \leq u_e$ for all $e \in E$ if and only if there exists an integral feasible rs -flow of value k .

Proof. (\implies) We have seen family of dipaths determines a corresponding flow.

(\impliedby) Let x be a flow. We assume that x is acyclic, that is, there is no dicircuit C , each of whose arcs e has $x_e > 0$. If a dicircuit does exist, we can decrease x_e by 1 on all arcs of C . The new x remains feasible of value k .

If $k \geq 1$, we can find an arc vs with $x_{vs} \geq 1$. Then, if $v \neq r$, it follows that there is an arc wv with $x_{wv} \geq 1$ by the constraint $f_x(v) = 0$. If $w \neq r$, then the argument can be repeated

producing distinct vertices, since x is acyclic, so we get a simple rs -dipath P_k on each arc e with $x_e \geq 1$. We can decrease x_e by 1 for each $e \in P_k$. The new x is an integral feasible flow of value $k - 1$, and the process is repeated. \square

8.2 Maximum Flows and Minimum Cuts

Definition: Maximum Flow LP

$$\begin{aligned} \max \quad & f_x(s) \\ \text{s.t.} \quad & f_x(v) = 0, \forall v \in V \setminus \{r, s\} \\ & 0 \leq x_e \leq u_e, \forall e \in E \end{aligned}$$

Definition: Path Flow

A vector $x \in \mathbb{R}^E$ such that for some rs -dipath P and some $\alpha \in \mathbb{R}$, $x_e = \alpha$ for each $e \in P$ and $x_e = 0$ for every other arc of G .

Definition: Circuit Flow

A vector $x \in \mathbb{R}^E$ such that for some rs -dicircuit C and some $\alpha \in \mathbb{R}$, $x_e = \alpha$ for each $e \in C$ and $x_e = 0$ for every other arc of G .

Proposition

Every rs -flow of nonnegative value is the sum of at most m flows, each of which is a path flow or a circuit flow.

Proposition

For any rs -cut $\delta(R)$ and any rs -flow x , we have

$$f_x(s) = x(\delta(R)) - x(\delta(\overline{R}))$$

Proof. We add the equations $f_x(v) = 0$ for all $v \in \overline{R} \setminus \{s\}$ as well as the identity $f_x(s) = f_x(s)$. The right hand side sums to $f_x(s)$.

For any arc vw with $v, w \in R$, x_{vw} occurs in none of the equations, so it does not occur in the sum. If $v, w \in \overline{R}$, then x_{vw} occurs in the equation for v with a coefficient of -1 , and in the equation for w with a coefficient of $+1$, so it has a coefficient of 0 in the sum. If $v \in R, w \notin R$, then x_{vw} occurs in the equation for w with a coefficient of 1 , and so has coefficient 1 in the sum. If $v \notin R, w \in R$, then x_{vw} occurs in the sum with a coefficient of -1 . So, the left hand side sums to $x(\delta(R)) - x(\delta(\overline{R}))$, as required. \square

Corollary

For any feasible rs -flow x and any rs -cut $\delta(R)$,

$$f_x(s) \leq u(\delta(R))$$

Proof. Using previous proposition, since $x(\delta(R)) \leq u(\delta(R))$ and $x(\delta(\bar{R})) \geq 0$. \square

Definition: Incrementing Path

A path is x -incrementing if every forward arc e has $x_e < u_e$ and every reverse arc e has $x_e > 0$.

Definition: Augmenting Path

An rs -path that is x -incrementing.

Theorem Maximum-Flow Minimum-Cut

If there is a maximum rs -flow, then

$$\max\{f_x(s) : x \text{ is a feasible } rs\text{-flow}\} = \min\{u(\delta(R)) : \delta(R) \text{ is an } rs\text{-cut}\}$$

Proof. By previous corollary, we need only show that there exists a feasible flow x and a cut $\delta(R)$ such that $f_x(s) = u(\delta(R))$. Let x be a flow of maximum value. Let $R = \{v \in V : \text{there exists an } x\text{-incrementing } rv\text{-path}\}$. Clearly $r \in R$ and $s \notin R$, since there can be no x -augmenting path.

For every arc $vw \in \delta(R)$, we must have $x_{vw} = u_{vw}$, since otherwise adding vw to the x -incrementing rv -path would yield such a path to w , but $w \notin R$. Similar, for every arc $vw \in \delta(\bar{R})$, we have $x_{vw} = 0$. Then by proposition, $f_x(s) = x(\delta(R)) - x(\delta(\bar{R})) = u(\delta(R))$. \square

Theorem

A feasible flow x is maximum if and only if there is not x -augmenting path.

Proof. (\implies) If x is maximum, there is no x -augmenting path.

(\impliedby) If there is no x -augmenting path, then the construction of the proof of Max-Flow Min-Cut yields a cut $\delta(R)$ with $f_x(s) = u(\delta(R))$, so x is maximum, by corollary. \square

Theorem

If u is integral and there exists a maximum flow, then there exists a maximum flow that is integral.

Proof. Choose an integral flow x of maximum value. If there is an x -augmenting path, then since x and u are integral, the new flow can be chosen integral, contradicting the choice of x . Hence there is no x -augmenting path, so x is a maximum flow, by previous theorem. \square

Corollary

If x is a feasible rs -flow and $\delta(R)$ is an rs -cut, then x is maximum and $\delta(R)$ is minimum if and only if $x_e = u_e$ for all $e \in \delta(R)$ and $x_e = 0$ for all $e \in \delta(\bar{R})$.

Proof. Combine Max-Flow Min-Cut theorem with the proof of corollary. \square

8.3 Augmenting Path Algorithm

Algorithm 2 Ford-Fulkerson Algorithm

```

1:  $x = 0$ 
2: while there is an  $x$ -augmenting path  $P$  do
3:    $\varepsilon_1 = \min(u_e - x_e : e \text{ forward in } P)$ 
4:    $\varepsilon_2 = \min(x_e : e \text{ reverse in } P)$ 
5:    $\varepsilon = \min(\varepsilon_1, \varepsilon_2)$  //  $x$ -width of  $P$ 
6:   if  $\varepsilon = \infty$  then
7:     no maximum flow
8: return  $x$  is maximum flow, set  $R$  of vertices reachable by an  $x$ -incrementing path from
    $r$  is minimum cut

```

Definition: Auxiliary Digraph

$G(x)$, depending on G, u, x , where $V(G(x)) = V$ and $vw \in E(G(x))$ if and only if $vw \in E$ and $x_{vw} < u_{vw}$ or $wv \in E$ and $x_{wv} > 0$.

rs -dipaths in $G(x)$ corresponding to x -augmenting paths in G . Each iteration of Ford-Fulkerson can be performed in $O(m)$ time, using breadth-first search.

Theorem

If u is integral and the maximum flow value is $K < \infty$, then the maximum flow algorithm terminates after at most K augmentations.

8.3.1 Shortest Augmenting Paths

Theorem (Dinitz 1970, Edmonds & Karp 1972)

If each augmentation of the augmenting path algorithm on a shortest augmenting path, then there are at most nm augmentations.

Corollary

The augmenting path algorithm with breadth-first search solves the maximum flow problem in time $O(nm^2)$.

Let $d_x(v, w)$ be the least length of a vw -dipath in $G(x)$. $d_x(v, w) = \infty$ if no vw -dipath exists.

Consider a typical augmentation from flow x to flow x' determined by the augmenting path P having vertex-sequence v_0, \dots, v_k .

Lemma

For each $v \in V$, $d_{x'}(r, v) \geq d_x(r, v)$ and $d_{x'}(v, s) \geq d_x(v, s)$.

Proof. Suppose that there exists a vertex v such that $d_{x'}(r, v) < d_x(r, v)$ and choose such v so that $d_{x'}(r, v)$ is as small as possible. Clearly, $d_{x'}(r, v) > 0$. Let P' be a rv -dipath in $G(x')$ of length $d_{x'}(r, v)$ and let w be the second-last vertex of P' . Then

$$d_x(r, v) > d_{x'}(r, v) = d_{x'}(r, w) + 1 \geq d_x(r, w) + 1$$

It follows that wv is an arc of $G(x')$, but not of $G(x)$, otherwise $d_x(r, v) \leq d_x(r, w) + 1$, so $w = v_i$ and $v = v_{i-1}$ for some i . But, this implies that $i - 1 > i + 1$, a contradiction. The second statement is similar. \square

Definition: $\tilde{E}(x)$

$$\tilde{E}(x) = \{e \in E : e \text{ is an arc of a shortest } x\text{-augmenting path}\}$$

Lemma

If $d_{x'}(r, s) = d_x(r, s)$, then $\tilde{E}(x') \subsetneq \tilde{E}(x)$.

Proof. Let $k = d_x(r, s)$ and suppose that $e \in \tilde{E}(x')$. Then e induces an arc vw of $G(x')$ and $d_{x'}(r, v) = i - 1$, $d_{x'}(ws) = k - i$ for some i . Therefore, $d_x(r, v) + d_x(w, s) \leq k - 1$ by previous lemma. Now suppose that $e \notin \tilde{E}(x)$, then $x_e \neq x'_e$, so e is an arc of P , a contradiction. This proves $\tilde{E}(x') \subseteq \tilde{E}(x)$.

There is an arc e of P such that e is forward and $x'_e = u_e$ or e is reverse and $x'_e = 0$. Therefore, any x' -augmenting path using e must use it in the opposite direction from P , so its length, for some i , will be at least $i + k - i + 1 + 1 = k + 23$, so $e \notin \tilde{E}(x')$. \square

Proof. (Dinitz, Edmonds, Karp) It follows from previous lemma that there can be at most m augmentations per stage. Since there are at most $n - 1$ stages, there are at most nm augmentations in all.

8.4 Applications

8.4.1 Flow Feasibility

Flow Feasibility Problem

Given a digraph G , $u \in \mathbb{R}_+^E$, and $b \in \mathbb{R}^V$, find, if possible, $x \in \mathbb{R}^E$ such that

$$f_x(v) = b_v, \quad \forall v \in V$$

and

$$0 \leq x_e \leq u_e, \quad \forall e \in E$$

Theorem (Gale 1957)

There exists a solution to the flow feasibility problem if and only if $b(V) = 0$ and for every $A \subseteq V$, $b(A) \leq u(\delta(\overline{A}))$.

If b and u are integral, then there is an integral solution.

Corollary

Given a digraph G and $b \in \mathbb{R}^V$, there exists $x \in \mathbb{R}^E$ with

$$f_x(v) = b_v, \quad \forall v \in V$$

$$x_e \geq 0, \quad \forall e \in E$$

if and only if $b(V) = 0$ and for every $A \subseteq V$ with $\delta(\overline{A}) = \emptyset$, we have $b(A) \leq 0$.

Definition: Circulation

A vector $x \in \mathbb{R}^E$ with $f_x(v) = 0$ for all $v \in V$.

Theorem (Hoffman's Circulation Theorem 1960)

Given a digraph G , $\ell \in (\mathbb{R} \cup \{-\infty\})^E$, and $u \in (\mathbb{R} \cup \{\infty\})^E$, with $\ell \leq u$, there is a circulation x with $\ell \leq x \leq u$ if and only if every $A \subseteq V$ satisfies $u(\delta(\overline{A})) \geq \ell(\delta(A))$.

Part V

Matchings

Chapter 9

Matchings

Definition: Matching

A set $M \subseteq E$ such that no vertex of G is incident with more than one edge in M .

Definition: M -Covered

A vertex v is covered by M if some edge of M is incident with v .

Definition: M -Exposed

A vertex v is exposed if v is not M -covered.

The number of vertices covered by M is $2|M|$ and number of M -exposed vertices is $|V| - 2|M|$.

Definition: Maximum Matching

A matching of maximum cardinality, denoted $\nu(G)$.

Definition: Deficiency

The minimum number of exposed vertices for any matching of G , denoted by $\text{def}(G)$.

Note $\text{def}(G) = |V| - 2\nu(G)$.

Definition: Perfect Matching

A matching that covers all vertices.

9.1 Bipartite Matching

Definition: Bipartite

$G = (V, E)$ is bipartite if $V = V_1 \cup V_2$, where V_1, V_2 disjoint and every edge has one end in V_1 and the other end in V_2 .

Definition: Cover

A set $C \subseteq V$ such that every edge has at least one in C .

Lemma

If M is a matching and C is a cover, then $|M| \leq |C|$.

Proof. Every $e \in M$ has at least one end in C . No vertex in C meets more than one edge in M . \square

Definition: Minimum Cover

A cover of minimum cardinality, denoted $\tau(G)$.

Theorem (König)

If G is bipartite, $\nu(G) = \tau(G)$.

Proof. We note that $\nu(G) \leq \nu^*(G)$ and $\tau(G) \geq \tau^*(G)$. By using LP duality and the matching LP (*Matching LP*), we show that $\nu(G) = \nu^*(G)$. We also have the matching LP in the form of $Mx^+ = (1, \dots, 1)^T$. Since M is totally unimodular, then M^T is also totally unimodular. So the dual LP has all integral vertices, implying $\tau(G) = \tau^*(G)$. So,

$$\nu(G) = \nu^*(G) = \tau^*(G) = \tau(G)$$

\square

9.2 Alternating Paths

Definition: M -Alternating

A path P is M -alternating if its edges are alternately in and not in M .

Definition: M -Augmenting

An M -alternating path P is M -augmenting if the ends of P are distinct and are both M -exposed.

Definition: Symmetric Difference

For sets S and T , let $S\Delta T$ denote the symmetric difference, which is defined as

$$S\Delta T = (S \cup T) \setminus (S \cap T)$$

Let a path P be an M -augmenting path. Then we can obtain a larger matching $M' = M\Delta E(P)$ with $|M'| = |M| + 1$.

Theorem (Petersen 1891, Berge 1957)

A matching M in a graph G is maximum if and only if there is no M -augmenting path.

Proof. (\implies) Suppose there exists an M -augmenting path P joining v and w . Then $N = M\Delta E(P)$ is a matching that covers all vertices covered by M , plus v and w . So, M is not maximum.

(\impliedby) Conversely, suppose that M is not maximum and some other matching N satisfies $|N| > |M|$. Let $J = N\Delta M$. Each vertex of G is incident with at most two edges of J , so J is the edge set of some vertex disjoint paths and circuits of G . For each such path or circuit, the edges alternately belong to M or N . Therefore, all circuits are even and contain the same number of edges of M and N . Since $|N| > |M|$, there must be at least one path with more edges of N than M . This path is an M -augmenting path. \square

9.3 Matching LP

Definition: Matching LP

P is the set of solutions to

$$\begin{aligned} x(\delta(v)) &\leq 1, \forall v \in V \\ x_e &\geq 0, \forall e \in E \end{aligned}$$

Let \bar{x} be a vertex of P . We show that \bar{x} is integral, which implies that $M = \{e \in E : \bar{x}_e = 1\}$ is a matching and $\nu(G) = \nu^*(G)$.

Recall that for a polyhedron $P = \{x : Ax \leq b\} \subseteq \mathbb{R}^n$, $\bar{x} \in P$ is a vertex if and only if \bar{x} is the unique solution to $A'x = b'$ for some subset of n inequalities $A'x \leq b'$ from $Ax \leq b$.

For our matching P , let $E^+ := \{e : \bar{x}_e > 0\}$ and $E^0 := \{e : \bar{x}_e = 0\}$. We write $\bar{x} = (\bar{x}^+, \bar{x}^0)$ split by (E^+, E^0) .

Since \bar{x} is a vertex, there exists $V^+ \subseteq V$ such that \bar{x} is the unique solution to

$$\begin{aligned} \sum (x_e : e \in \delta(v) \cap E^+) &= 1, \forall v \in V^+ \\ x_e &= 0, \forall e \in E^0 \end{aligned}$$

Restricting to E^+ , we can write the system of equations as

$$Mx^+ = (1, \dots, 1)^T$$

By Cramer's Rule, the solution to the system is $(\bar{x}_1^+, \dots, \bar{x}_k^+)$, where

$$\bar{x}_j^+ = \frac{\det(M^j)}{\det(M)}$$

with M^j obtained from M by replacing the j th column by $(1, \dots, 1)^T$.

Claim: $\det(M) = 1$ or $\det(M) = -1$.

This gives that \bar{x}_j^+ is integer for all j , so \bar{x} is integer. Thus, $\nu(G) = \nu^*(G)$.

Lemma

Let $G = (V, E)$ be a bipartite graph. Let A be the $|V| \times |E|$ matrix $[A_{ve}]$ with

$$A_{ve} = \begin{cases} 1 & \text{if } e \in \delta(v) \\ 0 & \text{if } e \notin \delta(v) \end{cases}$$

then A is totally unimodular.

Proof. By induction on the number of rows k of the submatrix B of A . If B is 1×1 , then this is obvious.

Suppose it is true for $k = 1, \dots, t-1$ and let B be a $t \times t$ submatrix of A .

1. If B has a column of all 0's, then $\det(B) = 0$.
2. If a column of B has exactly one 1, then we compute $\det(B)$ by expanding on that column and use induction.
3. Otherwise, every column of B has exactly two 1's.

We can partition the rows of B into W_1 and W_2 , so that every column has exactly one 1 in W_1 and exactly one 1 in W_2 (W_1 are vertices in V_1 , W_2 in V_2 from G being bipartite).

Now multiplying each row in W_1 by 1 and each row in W_2 by -1 and summing, we get the row vector of all 0's. So $\det(B) = 0$. \square

9.4 Tutte-Berge Formula

Definition: Vertex Cover

A set A of vertices such that every edge has at least one end in A .

Let A be a subset of the vertices which $G - A$ has k components H_1, \dots, H_k having an odd number of vertices. Let M be a matching of G . For each i , either H_i has an M -exposed vertex or M contains an edge having just one end in $V(H_i)$. All such edges have their other ends in A and since M is a matching, all these ends must be distinct. Therefore, there can be at most $|A|$ edges and so the number of M -exposed vertices is at least $k - |A|$.

Definition: $\text{oc}(H)$

The number of odd components of a graph H .

Thus, for any $A \subseteq V$,

$$\nu(G) \leq \frac{1}{2}(|V| - \text{oc}(G - A) + |A|)$$

If A is a cover of G , then there are $|V| - |A|$ odd components of $G - A$ (each is a single vertex), so the right hand side reduces to $|A|$. This bound is at least as strong as that provided by covers.

Theorem (Tutte-Berge Formula)

For a graph $G = (V, E)$, we have

$$\max\{|M| : M \text{ a matching}\} = \min \left\{ \frac{1}{2}(|V| - \text{oc}(G - A) + |A|) : A \subseteq V \right\}$$

Theorem (Tutte's Matching Theorem 1947)

A graph $G = (V, E)$ has a perfect matching if and only if for every $A \subseteq V$, $\text{oc}(G - A) \leq |A|$.

Definition: Shrink

Let C be an odd circuit in G . Define $G' = G \times C$ as the subgraph obtained from G by shrinking C ; G' has vertex set $(V - V(C)) \cup \{C\}$ and edge set $E \setminus \gamma(V(C))$.

Proposition

Let C be an odd circuit of G , let $G' = G \times C$, and let M' be a matching of G' . Then here is a matching M of G such that $M \subseteq M' \cup E(C)$ and the number of M -exposed vertices of G is the same as the number of M' -exposed vertices of G' .

Proof. Choose a vertex $w \in V(C)$ as follows. If C is covered by $e \in M'$, then choose w to be the vertex in $V(C)$ that is an end of e , and otherwise, choose w arbitrarily. Deleting w from C results in a subgraph having a perfect matching M'' . Take $M = M' \cup M''$. M has the required properties. \square

The previous proposition gives the inequality

$$\nu(G) \geq \nu(G \times C) + \frac{|V(C)| - 1}{2}$$

or equivalently,

$$\text{def}(G) \leq \text{def}(G \times C)$$

Definition: Tight Odd Circuit

An odd circuit C is tight if $\nu(G) = \nu(G \times C) + \frac{|V(C)|-1}{2}$.

Definition: Inessential

A vertex v of G is inessential if there is a maximum matching of G that does not cover v .

Definition: Essential

A vertex not inessential.

Let A be a set that satisfies the Tutte-Berge formula. Let $v \in A$ and consider $G' = G - v$. Then, $G' - (A \setminus \{v\})$ has the same odd components as $G - A$, so $\nu(G') < \nu(G)$, i.e. every $v \in A$ is essential.

Lemma

Let $G = (V, E)$ be a graph and let $vw \in E$. If v, w are both inessential, then there is a tight odd circuit C using vw . Moreover, C is an inessential vertex of $G \times C$.

Definition: Gallai-Edmonds Partition

Let B be the set of inessential vertices of $G = (V, E)$, C be the set of vertices not in B but adjacent to at least one element of B , and D be $V \setminus (B \cup C)$, then (B, C, D) is the Gallai-Edmonds Partition of G .

Proposition

Let (B, C, D) be the Gallai-Edmonds Partition for G .
 C is a minimizer in the Tutte-Berge formula.

Proposition

Let (B, C, D) be the Gallai-Edmonds Partition for G .
 For every maximum matching M and every vertex $v \in C$, there is an edge $vw \in M$ with $w \in B$.

Proposition

Let (B, C, D) be the Gallai-Edmonds Partition for G .
 Every maximum matching contains a perfect matching of $G[D]$.

9.5 Maximum Matching

Maximum Matching Problem

Given a graph G , find a maximum matching of G .

Definition: Maximum Matching ILP

$$\begin{aligned} \max \quad & \sum (x_e : e \in E) \\ \text{s.t.} \quad & x(\delta(v)) \leq 1, \forall v \in V \\ & x_e \geq 0, \forall e \in E \\ & x_e \text{ integer}, \forall e \in E \end{aligned}$$

Definition: Maximum Matching LP Relaxation

$$\begin{aligned} \max \quad & \sum (x_e : e \in E) \\ \text{s.t.} \quad & x(\delta(v)) \leq 1, \forall v \in V \\ & x_e \geq 0, \forall e \in E \end{aligned}$$

Definition: Minimum Cover Dual LP

$$\begin{aligned} \min \quad & \sum (y_v : v \in V) \\ \text{s.t.} \quad & y_u + y_v \geq 1, \forall e = (u, v) \in E \\ & y_v \geq 0, \forall v \in V \end{aligned}$$

Let M be a matching and C be a cover, then

$$x_e^M = \begin{cases} 1 & \text{if } e \in M \\ 0 & \text{if } e \notin M \end{cases}, y_v^C = \begin{cases} 1 & \text{if } v \in C \\ 0 & \text{if } v \notin C \end{cases}$$

So, $\nu(G) \leq \nu^*(G)$ and $\tau(G) \geq \tau^*(G)$, and by LP duality, we have

$$\nu(G) \leq \nu^*(G) = \tau^*(G) \leq \tau(G)$$

9.6 Perfect Matching

9.6.1 Alternating Trees

Suppose we have a matching M of G and a fixed M -exposed vertex r of G . We can iteratively build up sets A, B of vertices such that each vertex in A is the other end of an odd-length M -alternating path beginning at r , and each vertex in B is the other end of an even-length M -alternating path beginning at r .

Begin with $A = \emptyset, B = \{r\}$, and use the rule: if $vw \in E, v \in B, w \notin A \cup B, wz \in M$, then add w to A , z to B . The set $A \cup B$ and edges in the construction form a tree T rooted at r .

Definition: Alternating Tree

A tree T such that

- every vertex of T other than r is covered by an edge of $M \cap E(T)$;
- for every vertex v of T , the path in T from v to r is M -alternating.

We let the vertex sets at odd and even distances from the root as $A(T)$ and $B(T)$ respectively. Note that $|B(T)| = |A(T)| + 1$ since all other vertices other than r come in matched pairs, one in $A(T)$ and one in $B(T)$.

Using vw to Extend T

Input: A matching M' of a graph G' , an M' -alternating tree T , and an edge vw of G' such that $v \in B(T)$, $w \notin V(T)$, and w is M' -covered.

Algorithm: Let wz be the edge in M' covering w (but z is not a vertex of T). Replace T by the tree having edge set $E(T) \cup \{vw, wz\}$.

Use vw to Augment M'

Input: A matching M' of a graph G' , an M' -alternating tree T of G' with root r , and an edge vw of G' such that $v \in B(T)$, $w \notin V(T)$, and w is M' -exposed.

Algorithm: Let P be the path obtained by attaching vw to the path from r to v in T . Replace M' by $M' \Delta E(P)$.

Definition: Frustrated

An M -alternating tree T in a graph G is frustrated if every edge of G has one end in $B(T)$ and the other end in $A(T)$.

Proposition

Suppose that G has a matching M and an M -alternating tree T that is frustrated. Then G has no perfect matching.

Proof. Clearly, every element of $B(T)$ is a single-vertex odd component of $G \setminus A(T)$. Since $|A(T)| < |B(T)|$, then G has no perfect matching. \square

9.6.2 Bipartite Perfect Matching Algorithm

The following algorithm is to obtain a perfect matching in a bipartite graph based on alternating trees or outputs no perfect matching.

Algorithm 3 Bipartite Perfect Matching Algorithm

```
1:  $M = \emptyset$ 
2: Choose an  $M$ -exposed vertex  $r$ 
3:  $T = (\{r\}, \emptyset)$ 
4: while there exists  $vw \in E$  with  $v \in B(T), w \notin V(T)$  do
5:   if  $w$  is  $M$ -exposed then
6:     Use  $vw$  to augment  $M$ 
7:     if there is no  $M$ -exposed vertex in  $G$  then
8:       return Perfect matching  $M$ 
9:     else
10:       $T = (\{r\}, \emptyset)$ , where  $r$  is  $M$ -exposed
11:   else
12:     Use  $vw$  to extend  $T$ 
13: return  $G$  has no perfect matching
```

Proposition

Suppose that G is bipartite, M is a matching of G , and T is an M -alternating tree such that no edge of G joins a vertex in $B(T)$ to a vertex not in $V(T)$. Then T is frustrated, and hence G has no perfect matching.

Proof. We show that every edge having an end in $B(T)$ has an end in $A(T)$. From the hypothesis, the only possible exception would be an edge joining two vertices in $B(T)$. But this edge, together with the paths joining them to the root of T , would form a closed path of odd length, which contradicts G being bipartite. Hence T is frustrated, and so by previous proposition, G has no perfect matching. \square

9.6.3 Blossom Algorithm for Perfect Matching

Definition: Derived Graph

A graph G' obtained from G by a sequence of odd-circuit shrinkings.

Definition: Original Vertex

A vertex in the derived graph G' that is in G .

Definition: Pseudonode

A vertex in the derived graph G' not in G .

Definition: $S(v)$

Given a vertex v of G' , there corresponds a set $S(v)$ of vertices of G , where

$$S(v) = \begin{cases} v & \text{if } v \in V(G) \\ \bigcup_{w \in V(C)} S(w) & \text{if } v = C \text{ is a pseudonode} \end{cases}$$

Proposition

Let G' be a derived graph of G , M' be a matching of G' , and T be an M' -alternating tree of G' such that no element of $A(T)$ is a pseudonode. If T is frustrated, then G has no perfect matching.

Proof. When deleting $A(T)$ from G , we get a component with vertex set $S(v)$ for each $v \in B(T)$. Therefore, $\text{oc}(G \setminus A(T)) > |A(T)|$, so G has no perfect matching by Tutte's theorem. \square

Definition: Blossom

Let $v, w \in B(T)$ and $vw \in E(G)$. The odd circuit in $T + vw$ is a blossom.

When we shrink a blossom, we get a pseudonode and the new graph is a derived graph.

Use vw to Shrink and Update M' and T

Input: A matching M' of a graph G' , an M' -alternating tree T , and an edge vw of G' such that $vw \in B(T)$.

Algorithm: Let C the circuit formed by vw with the vw -path in T . Replace G' with $G' \times C$, M' by $M' \setminus E(C)$, and T by the tree in G' having edge set $E(T) \setminus E(C)$.

Proposition

After application of the shrinking subroutine, M' is a matching of G' , T is an M' -alternating tree of G' , and $C \in B(T)$.

Algorithm 4 Blossom Algorithm for Perfect Matching

```
1: Input: Graph  $G$  and matching  $M$  of  $G$ 
2:  $M' = M$ 
3:  $G' = G$ 
4: Choose an  $M'$ -exposed vertex  $r$  of  $G'$ 
5:  $T = (\{r\}, \emptyset)$ 
6: while there exists  $vw \in E'$  with  $v \in B(T), w \notin A(T)$  do
7:   Case:  $w$  is  $M'$ -exposed
8:     Use  $vw$  to augment  $M'$ 
9:     Extend  $M'$  to a matching  $M$  of  $G$ 
10:    Replace  $M'$  by  $M$ ,  $G'$  by  $G$ 
11:    if there is no  $M'$ -exposed vertex in  $G'$  then
12:      return Perfect matching  $M'$ 
13:    else
14:       $T = (\{r\}, \emptyset)$ , where  $r$  is  $M'$ -exposed
15:    Case:  $w \notin V(T)$ ,  $w$  is  $M'$ -covered
16:      Use  $vw$  to extend  $T$ 
17:    Case:  $w \in B(T)$ 
18:      Use  $vw$  to shrink and update  $M'$  and  $T$ 
19: return  $G', M', T$ ,  $G$  has no perfect matching
```

Theorem

The Blossom Algorithm terminates after $O(n)$ augmentations, $O(n^2)$ shrinking steps, and $O(n^2)$ tree-extension steps.

Moreover, it determines correctly whether G has a perfect matching.

9.7 Blossom Algorithm for Maximum Matching

We can extend the Blossom algorithm for perfect matchings to maximum matchings.

Algorithm 5 Blossom Algorithm for Maximum Matching

```
1: Input: Graph  $G$  and matching  $M$  of  $G$ 
2:  $M' = M, G' = G, \mathcal{T} = \emptyset$ 
3: Choose an  $M'$ -exposed vertex  $r$  of  $G'$ 
4:  $T = (\{r\}, \emptyset)$ 
5: while there exists  $vw \in E'$  with  $v \in B(T), w \notin A(T)$  do
6:   Case:  $w$  is  $M'$ -exposed
7:     Use  $vw$  to augment  $M'$ 
8:     Extend  $M'$  to a matching  $M$  of  $G$ 
9:     Replace  $M'$  by  $M$ ,  $G'$  by  $G$ 
10:    if there is no  $M'$ -exposed vertex in  $G'$  then
11:      return Perfect matching  $M'$ 
12:    else
13:       $T = (\{r\}, \emptyset)$ , where  $r$  is  $M'$ -exposed
14:    Case:  $w \notin V(T)$ ,  $w$  is  $M'$ -covered
15:      Use  $vw$  to extend  $T$ 
16:    Case:  $w \in B(T)$ 
17:      Use  $vw$  to shrink and update  $M'$  and  $T$ 
18:  $\mathcal{T} = \mathcal{T} \cup \{T\}, G' = G \setminus V(T), M' = M \setminus E(T)$ 
19: if there exists an  $M'$ -exposed vertex then
20:   Go to line 5
21: Restore the matching  $M$ 
22: return  $M$ 
```

Theorem

The Blossom Algorithm can be implemented to run in time $O(nm \log n)$.

Chapter 10

Weighted Matchings

10.1 Minimum-Weight Perfect Matching

Definition: Minimum-Weight Perfect Matching ILP

$$\begin{array}{ll}\min & \sum (c_e x_e : e \in E) \\ \text{s.t.} & x(\delta(v)) = 1, \forall v \in V \\ & x_e \in \{0, 1\}, \forall e \in E\end{array}$$

Definition: Minimum-Weight Perfect Matching LP Relaxation

$$\begin{array}{ll}\min & \sum (c_e x_e : e \in E) \\ \text{s.t.} & x(\delta(v)) = 1, \forall v \in V \\ & x_e \geq 0, \forall e \in E\end{array}$$

Definition: Minimum-Weight Perfect Matching Dual LP

$$\begin{array}{ll}\max & \sum (y_v : v \in V) \\ \text{s.t.} & y_u + y_v \leq c_e, \forall e = uv \in E\end{array}$$

Definition: Complementary Slackness Conditions for Minimum-Weight Perfect Matching

If $x_e > 0$, then $\bar{c}_e = c_e - y_u - y_v = 0$ for all $e \in E$.

10.2 Minimum-Weight Perfect Matching in Bipartite Graphs

Theorem (Birkhoff)

Let G be a bipartite graph and let $c \in \mathbb{R}^E$. Then G has a perfect matching if and only if the Minimum-Weight Perfect Matching LP Relaxation has a feasible solution. Moreover, if G has a perfect matching, then the minimum weight of a perfect matching is equal to the optimal value of the LP relaxation.

Definition: $E_=$

$$E_ = \{e \in E : \bar{c}_e = 0\}.$$

Algorithm 6 Bipartite Minimum-Weight Perfect Matching Algorithm

```

1: Let  $y$  be a feasible solution to the dual LP
2:  $M$  is a matching of  $G_ = (V, E_ )$ 
3:  $T = (\{r\}, \emptyset)$ , where  $r$  is an  $M$ -exposed vertex of  $G$ 
4: while true do
5:   while there exists  $vw \in E_$  with  $v \in B(T), w \notin V(T)$  do
6:     if  $w$  is  $M$ -exposed then
7:       Use  $vw$  to augment  $M$ 
8:       if there is no  $M$ -exposed vertex in  $G$  then
9:         return Perfect matching  $M$ 
10:      else
11:         $T = (\{r\}, \emptyset)$ , where  $r$  is  $M$ -exposed
12:      else
13:        Use  $vw$  to extend  $T$ 
14:      if every  $vw \in E$  with  $v \in B(T)$  has  $w \in A(T)$  then
15:        return  $G$  has no perfect matching
16:      else
17:         $\varepsilon = \min\{\bar{c}_{vw} : v \in B(T), w \notin V(T)\}$ 
18:         $y_v = y_v + \varepsilon$  for  $v \in B(T)$ 
19:         $y_v = y_v - \varepsilon$  for  $v \in A(T)$ 

```

10.3 Minimum-Weight Perfect Matching in General Graphs

Definition: Odd Cut

A set of the form $\delta(S)$ where S is an odd-cardinality set of vertices.

Definition: Blossom Inequality

If x is the characteristic vector of a perfect matching, then for every odd cut D of G ,

$$x(D) \geq 1$$

Let \mathcal{C} denote the set of all odd cuts of G that are *not* of the form $\delta(v)$ for some vertex v .

Definition: Minimum-Weight Perfect Matching LP – Stronger

$$\begin{aligned} \min \quad & \sum (c_e x_e : e \in E) \\ \text{s.t.} \quad & x(\delta(v)) = 1, \forall v \in V \\ & x(D) \geq 1, \forall D \in \mathcal{C} \\ & x_e \geq 0, \forall e \in E \end{aligned}$$

Definition: Minimum-Weight Perfect Matching Dual LP – Stronger

$$\begin{aligned} \max \quad & \sum (y_v : v \in V) + \sum (Y_D : D \in \mathcal{C}) \\ \text{s.t.} \quad & y_v + y_w + \sum (Y_D : e \in D \in \mathcal{C}) \leq c_e, \forall e = vw \in E \\ & Y_D \geq 0, \forall D \in \mathcal{C} \end{aligned}$$

Theorem

Let G be a graph and let $c \in \mathbb{R}^E$. Then G has a perfect matching if and only if the Minimum-Weight Perfect Matching LP has a feasible solution.

Moreover, if G has a perfect matching, then the minimum weight of a perfect matching is equal to the optimal value of the LP.

Chapter 11

T-Joins

Part VI

Matroids

Part VII

Traveling Salesman Problem