

# CO 342 Graph Theory

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# Chapter 1

## Introduction

### Definition: Graph

A graph  $G = (V, E, i)$  is a 3-tuple where

- $V$  is a finite set of vertices
- $E$  is a finite set of edges with  $V \cap E = \emptyset$
- $i : V \times E \rightarrow \{0, 1, 2\}$  such that

$$i(v, e) = \# \text{ of times } e \text{ is incident to } v$$

such that

$$\forall e \in E, \sum_{v \in V} i(v, e) = 2$$

### Definition: Incident

$v \in V$  and  $e \in E$  are incident in  $G$  if  $i(v, e) \neq 0$ .

### Definition: Adjacent

$u, v \in V$  are adjacent in  $G$  if either

- $i(u, e) = i(v, e) = 1$  if  $u \neq v$
- $i(u, e) = 2$

for some  $e \in E$ .

A graph  $G$  is simple if for each pair  $u, v$ , at most one edge  $e$  is incident to both.

**Definition: Walk**

An alternating sequence of vertices and edges of  $G$ :  $v_0, e_1, v_1, e_2, \dots, e_k, v_k$  such that the ends of edge  $e_i$  are  $v_{i-1}$  and  $v_i$ .

Vertices and edges are not necessarily unique.

**Definition: Path**

A walk in which the vertices and edges are distinct.

The number of edges is the length of the path/walk.

**Definition: Circuit/Cycle**

A walk  $v_1, e_1, \dots, e_{k-1}, v_k$  such that  $v_1 = v_k$  and  $v_1, v_2, \dots, v_{k-1}$  are distinct.

A loop is a circuit of length 1.

**Definition: Degree**

The degree of a vertex  $v$  is

$$\deg(v) = d(v) = \sum_{e \in E} i(v, e)$$

**Definition: Subgraph**

A subgraph of  $G = (V, E, i)$  is a 3-tuple

$$H = (V', E', i')$$

where  $V' \subseteq V, E' \subseteq E$  and  $i'$  is the restriction of  $i$  to the domain  $V' \times E'$ .

**Definition: Induced Subgraph**

If  $X \subseteq V$ , the subgraph  $G[X]$  of  $G$  induced by  $X$  is the subgraph of  $G$  containing exactly the vertices of  $X$  and all edges between them.

# Chapter 2

## Connectivity

### Definition: Connected

Vertices  $u, v$  are connected if there is  $uv$ -walk.

A graph is connected if and only if  $V(G) \neq \emptyset$  and every pair of vertices are connected in  $G$ . Alternatively, a graph is connected if it has one component.

The empty graph is disconnected.

### Proposition

Connectedness is an equivalence relation on vertices of  $G$ .

1. Reflexivity: each vertex  $u$  is connected to itself.
2. Symmetry: if  $u$  is connected to  $v$ , then  $v$  is connected to  $u$ .
3. Transitivity: if  $u$  is connected to  $v$  and  $v$  is connected to  $w$ , then  $u$  is connected to  $w$ .

The equivalence classes of this relation form the vertex sets of each of the components of  $G$ . In other words, an equivalence class of this relation is a subset of vertices  $V'$  such that every vertex in  $V'$  is connected to every other vertex in  $V'$  but not connected to any vertex outside of  $V'$ .

**Proof.** Reflexivity holds because  $u$  is a walk from  $u$  to  $u$  for each  $u$ .

Symmetry holds because if  $u = x_0, e_1, x_2, \dots, e_k x_k = v$  is a walk from  $u$  to  $v$ , then  $v = x_k, e_k, \dots, e_1, x_0 = u$  is walk from  $v$  to  $u$ .

Transitivity holds because if  $uWv$  is a walk from  $u$  to  $v$  and  $vW'w$  is a walk from from  $v$  to  $w$ , then  $uWvW'w$  is a walk from  $u$  to  $w$ .

### Lemma

If  $u$  and  $v$  are connected, there is a path from  $u$  to  $v$ .

**Definition: Components**

A component of  $G$  is a maximal connected subgraph. Alternatively, an induced subgraph of the form  $G[X]$  where  $X$  is an equivalence class under connectedness.

**Definition:  $AB$ -Path**

Given sets of vertices  $A, B$  in a graph  $G$ , an  $AB$ -path is a path  $P$  from one vertex in  $A$  to a vertex in  $B$  so that  $P$  intersects  $A$  only at its first vertex and  $B$  only at its last.

Note: if  $A \cap B \neq \emptyset$ , then every vertex in  $A \cap B$  gives an  $AB$ -path with no edges.

**Definition:  $aB$ -Path**

For a vertex  $a$  and a set of vertices  $B$ , an  $aB$ -path means an  $\{a\}B$ -path.

**Definition: Separation**

A set  $X \subseteq V \cup E$  separates  $A$  and  $B$  in  $G$  if there is no  $AB$ -path in  $G - X$ .

**Definition: Cut Edge/Bridge**

An edge is a cut edge/bridge if there are vertices  $u, v$  of  $G$  that are not separated by  $\emptyset$ , but are separated by  $\{e\}$ .

**Definition: Cut Vertex**

A cut vertex of  $G$  is a vertex  $v$  such that there is some pair of vertices  $a, b$  not separated by  $\emptyset$ , but separated by  $\{v\}$ .

**Definition:  $k$ -Connected**

For  $k \geq 1$ ,  $G$  is  $k$ -connected if there is no set  $X \subseteq V(G)$  with  $|X| < k$  such that  $G - X$  is disconnected.

There is sometimes a restriction on  $|V(G)| > k$ .

Note:

- Every graph is 0-connected except the empty graph.
- $G$  is 1-connected if and only if  $G$  is connected.
- $G$  is 2-connected if and only if  $G$  is connected and has no cut vertex.
- Trees are not 2-connected because trees have leaves, and deleting a neighbour of a leaf disconnects the graph.
- All vertices in a  $k$ -connected graph have degree at least  $k$ .

### Proposition

If  $G$  is connected and  $A, B \subseteq V(G)$  are nonempty, then there is an  $AB$ -path.

**Proof.** Let  $a_0 \in A, b_0 \in B$ . Since  $G$  is connected, there exists a path  $P$  from  $a_0$  to  $b_0$ . Let  $a$  be the last vertex in  $V(P) \cap A$  and  $b$  be the first vertex after  $a$  in  $P$  that is in  $B$ . Then the maximality in the choice of  $a$  and the minimality in the choice of  $b$  implies that the subpath  $aPb$  is an  $AB$ -path.

### Proposition

If there is a vertex  $x \in V(G)$  that is connected to every other vertex of  $G$ , then  $G$  is connected.

**Proof.** If  $x$  is connected to every vertex, then given  $u, v \in V(G)$ ,  $u$  is connected to  $x$ ,  $x$  is connected to  $v$ , so  $u$  is connected to  $v$  by symmetry and transitivity of connectedness.

### Definition: Graph Union

Given two graphs  $G_1, G_2$  (whose vertex sets and edge sets might intersect) and every edge in  $E(G_1) \cap E(G_2)$  has the same ends in both graphs, the graph  $G_1 \cup G_2$  is the graph with vertices  $V(G_1) \cup V(G_2)$  and edges  $E(G_1) \cup E(G_2)$ , where the ends of each edge  $e$  are the same as they are in  $G_1$  and  $G_2$ .

### Definition: Direct Sum

We write  $G_1 \oplus G_2$  to denote the direct sum of  $G_1, G_2$  which is

$$G_1 \cup G_2$$

when  $V_1 \cup E_1, V_2 \cup E_2$  are disjoint.

### Proposition

If  $G_1$  and  $G_2$  are connected and  $V(G_1) \cap V(G_2) \neq \emptyset$ , then  $G_1 \cup G_2$  is connected.

**Proof.** Let  $x \in V(G_1) \cap V(G_2)$ . Since  $G_1$  is connected, every vertex in  $G_1$  is connected in  $G_1$  to  $x$  and similarly, every vertex in  $G_2$  is connected in  $G_2$  to  $x$ . So every vertex in  $G_1 \cup G_2$  is connected to  $x$  (because paths in  $G_1$  and  $G_2$  are paths in  $G_1 \cup G_2$ ), so  $G_1 \cup G_2$  is connected.

### Proposition

If  $G$  is a connected graph on  $n$  vertices, then there is an ordering  $v_1, \dots, v_n$  of its vertices such that for all  $1 \leq i \leq n$ , the induced subgraph  $G[\{v_1, \dots, v_i\}]$  is connected.

**Proof.** Let  $v_1$  be any vertex of  $G$ . Let  $k$  be maximal such that there exist vertices  $v_2, \dots, v_k$  of  $G$  so that, for every  $1 \leq i \leq k$ , the induced subgraph  $G[\{v_1, \dots, v_i\}]$  is connected.

If  $k = n$ , then  $v_1, \dots, v_k = v_n$  is the required order. So we may assume that  $k < n$ , so there

exists a vertex  $x \notin \{v_1, \dots, v_k\}$ .

Since  $G$  is connected, there is a  $\{v_1, \dots, v_k\}x$ -path  $P$  in  $G$ . Let  $v_{k+1}$  be the first vertex of  $P$  outside  $\{v_1, \dots, v_k\}$ . Now the subgraph of  $G$  induced by  $\{v_1, \dots, v_k, v_{k+1}\}$  is the union of the subgraph induced by  $\{v_1, \dots, v_k\}$  and some graph containing  $v_{k+1}$  and all its neighbours in  $\{v_1, \dots, v_k\}$ . Both graphs are connected and since  $v_{k+1}$  has a neighbour in  $\{v_1, \dots, v_k\}$ , they have a vertex in common.

So  $\{v_1, \dots, v_k\}$  induces a connected subgraph of  $G$ , contradicting the maximality.

### Proposition

If  $G$  is a connected graph on  $n \geq 2$  vertices, then  $G$  has a vertex  $v$  such that  $G - v$  is connected.

### Definition: Adding a Path

We say  $G$  is obtained from  $H$  by adding a path if  $G = H \cup P$ , for some path  $P$  such that  $V(P) \cap V(H)$  is exactly the set of the two ends of  $P$  and  $E(P) \cap E(H) = \emptyset$ .

Note: adding a single new edge between two existing vertices is an example of adding a path.

### Lemma

If  $G_1$  and  $G_2$  are  $k$ -connected graphs, whose union is well-defined and

$$|V(G_1) \cap V(G_2)| \geq k$$

then  $G = G_1 \cup G_2$  is  $k$ -connected.

**Proof.** Suppose not. Then there exists  $X \subseteq V(G_1 \cup G_2)$  such that  $|X| < k$  and  $G - X$  is disconnected. Note  $(G_1 \cup G_2) - X = (G_1 - X) \cup (G_2 - X)$ .

Since  $|X| < k$  and each  $G_i$  is  $k$ -connected, we know that  $G_1 - X$  and  $G_2 - X$  are connected. Also, since  $|X| < k \leq |V(G_1) \cap V(G_2)|$ , the graphs  $G_1 - X$  and  $G_2 - X$  have a vertex in common. So  $(G_1 - X) \cup (G_2 - X)$  is connected, a contradiction.

### Corollary

If  $H$  is 2-connected and  $G$  is obtained from  $H$  by adding a path, then  $G$  is 2-connected.

**Proof.** Let  $P$  be the path with ends  $u, v$ . If  $P$  has length 1, then  $P$  is 2-connected, so  $G = H \cup P$  is 2-connected by lemma.

Otherwise, let  $Q$  be a path in  $H$  from  $u$  to  $v$ .  $P \cup Q$  is a cycle and  $G = H \cup (P \cup Q)$ . Since  $P \cup Q$  is 2-connected, and so is  $G$  by lemma.



### Theorem (Ear-Decomposition)

For every 2-connected graph  $G$ , there are 2-connected subgraphs  $G_0, \dots, G_k$  of  $G$  such that

- $G_0$  is a cycle
- $G_k = G$
- For each  $0 \leq i < k$ ,  $G_{i+1}$  is obtained from  $G_i$  by adding a path.

**Proof.** If  $G$  has no cycle, then  $G$  is a tree and trees are not 2-connected. So  $G$  has a cycle  $G_0$ .

Let  $G_0, G_1, \dots, G_t$  be a maximal sequence of 2-connected subgraphs of  $G$  such that each  $G_{i+1}$  is obtained from  $G_i$  by adding a path. If  $G_t = G$ , then we have the required sequence.

If there is an edge  $e \in E(G) \setminus E(G_t)$  with both ends in  $V(G_t)$ , then  $G_{t+1} = G_t \cup \{e\}$  is obtained from  $G_t$  by adding a path  $e$ , and is 2-connected by the lemma, so it contradicts the maximality of  $t$ .

Otherwise, since  $G_t$  is not a component of  $G$  and is not all of  $G$ , there is a vertex  $v$  of  $V(G) \setminus V(G_t)$  having a neighbour  $w \in V(G_t)$ . Since  $G$  is 2-connected, there is a  $vV(G_t)$ -path  $P$  in  $G - w$ . Let  $e$  be the edge from  $v$  to  $w$ , now  $wevP$  is a path intersecting  $V(G_t)$  precisely in its two ends, so  $G_t \cup wevP$  is obtained from  $G_t$  by adding a path, so is 2-connected and contradicts maximality of  $t$ .

### Proposition

If  $G$  is 2-connected, then every pair of vertices of  $G$  is contained in a cycle.

**Proof.** Let  $G_0, \dots, G_k = G$  be an ear-decomposition. This is true for  $G_0$  because  $G_0$  is a cycle.

Suppose it is true for some  $G_i$  where  $0 \leq i < k$ . There are 3 cases: the pair are in  $G_i$ , the pair is on the new path added, or one vertex is on the path and one is vertex is in  $G_i$ .

Proof is in assignment.

### Definition: $k$ -Edge-Connected

Let  $k \geq 0$ . A graph  $G$  is  $k$ -edge connected if there is no set  $X \subseteq E(G)$  for which  $|X| < k$  and  $G - X$  is disconnected.

### Lemma

If  $G$  is  $k$ -connected and  $|X| \leq k$ , then every vertex in  $X$  has a neighbour in every connected component  $H$  of  $G - X$ .

**Proof.** Let  $x \in X$  and  $H$  be a connected component of  $G - X$ . Note that  $G - (X \setminus \{x\})$  is connected since  $|X \setminus \{x\}| < k$ . Let  $P$  be an  $xH$ -path in  $G - (X \setminus \{x\})$ . Since  $H$  is a

component of  $G - X$  and the penultimate vertex  $w$  of  $P$  is a vertex of  $G - (X \setminus \{x\})$  outside  $H$  with a neighbour in  $H$ , we must have  $w = x$ , so  $x$  has a neighbour in  $H$ .

### Definition: Edge Contraction

Given a graph  $G$  and  $e \in E$  with distinct ends  $u, v$  such that  $e$  is the only edge from  $u$  to  $v$ , we write  $G/e$  for the graph  $((V - \{u, v\}) \cup \{x_{uv}\}, E \setminus \{e\})$  where each edge with no end in  $\{u, v\}$  has the same ends as in  $G$  and each edge with an end in  $\{u, v\}$  has this end replaced by the new vertex  $x_{uv}$ .

### Proposition

If  $G$  is a simple, 3-connected graph with  $|V(G)| \geq 4$ , then  $G$  has an edge  $e$  such that  $G/e$  is 3-connected.

**Proof.** Suppose by contradiction that every edge  $xy \in E(G)$ , the graph  $G/xy$  is not 3-connected. Then,  $G/xy$  contains a separator  $S$  with  $|S| \leq 2$ .

Since  $S$  is not a separator of  $G$ , we have that  $v_{xy} \in S$  and  $|S| = 2$ . Let  $z \in V(G/xy)$  such that  $S = \{v_{xy}, z\}$ , then any two vertices separated by  $\{v_{xy}, z\}$  in  $G/xy$  are separated in  $G$  by  $T := \{x, y, z\}$ .

Fix an edge  $xy$ , a vertex  $z$  and a component  $C$  so that  $|C|$  is as small as possible. By the previous lemma,  $z$  has a neighbour  $v$  in  $C$  and by contradiction,  $C/zv$  is not 3-connected. So there exists a vertex  $w \in V(G)$  such that  $\{z, v, w\}$  separates  $G$ . As  $xy \in E(G)$ ,  $G \setminus \{z, v, w\}$  has a connected component  $D$  such that  $D \cap \{x, y\} = \emptyset$  ( $x, y$  cannot be in different connected components). Then, every neighbour of  $v$  in  $D$  lies in  $C$  (since  $v \in C$ ). By the lemma,  $v$  has a neighbour in  $D$ . Thus,  $D \cap C \neq \emptyset$ , and hence  $D \subsetneq C$  (since  $v \in C \setminus D$ ). This contradicts the choice of  $xy, z$  and  $C$ .

### Proposition

If  $G$  is a simple, 3-connected graph, then there exist  $G_0, G_1, \dots, G_k$  such that  $G_0 \cong K_3$ ,  $G_k \cong G$ , and  $G_i$  is a 3-connected graph with  $G_i \cong S_i(G_{i+1}/e)$  for some  $e$ , where  $S_i$  means remove parallel pairs.

### Definition: Internally Disjoint (IDJ)

A collection of  $uv$ -paths in a graph  $G$  is internally disjoint if no two paths have no vertices or edges in common except for the endpoints  $u$  and  $v$ .

### Definition: $uv$ -Separator

A set  $X \subseteq V(G) \setminus \{u, v\}$  is a  $uv$ -separator in  $G$  if  $G - X$  contains no  $uv$ -path.

Note that if  $u$  and  $v$  are adjacent, then no  $uv$ -separator exists.

### Proposition

If  $X$  is a  $uv$ -separator of  $G$  with  $|X| < k$ , then there do not exist  $k$  internally disjoint  $uv$ -paths.

**Proof.** Suppose there are  $k$  IDJ  $uv$ -paths  $P_1, \dots, P_k$ . Since none of the  $P_i$  is a path in  $G - X$ , each  $P_i$  contains a vertex in  $X$ . Since  $|X| < k$ , the pigeonhole principle gives that two of the  $P_i$  have a common vertex in  $X$ , contradicting IDJ.

### Theorem (Menger)

If  $u, v$  are non-adjacent in  $G$  and every  $uv$ -separator in  $G$  has size  $\geq k$ , then  $G$  has  $k$  internally disjoint  $uv$ -paths.

**Proof.** Suppose not. Let  $G$  be a counterexample for which  $k + |V(G)| + |E(G)|$  is as small as possible (i.e. every  $uv$ -separator in  $G$  has size  $\geq k$ , but there do not exist  $k$  IDJ  $uv$ -paths in  $G$ ).

**Claim 1:** There is no vertex adjacent to both  $u$  and  $v$ .

**Claim 1 Proof.** Let  $x$  be a vertex adjacent to both  $u$  and  $v$ . If  $G - x$  has a  $uv$ -separator  $S$  with  $|S| < k - 1$ , then  $S \cup \{x\}$  is a  $uv$ -separator in  $G$  of size  $|S| + 1 < k$ , contradicting our assumption about  $G$ .

Otherwise, since  $|V(G - x)| + k - 1 < |V(G)| + k$ , induction (i.e. minimality) implies that there are  $k - 1$  IDJ paths  $P_1, \dots, P_{k-1}$  in  $G - x$ . Now,  $P_1, \dots, P_{k-1}, uv$  are  $k$  IDJ paths in  $G$ , a contradiction.

**Claim 2:** Every edge of  $G$  is incident with either  $u$  or  $v$ .

**Claim 2 Proof.** Let  $e$  be such an edge with ends  $x, y$ . Let  $S$  be a smallest  $uv$ -separator in  $G - e$ . Note that  $S \cup \{x\}$  and  $S \cup \{y\}$  are both  $uv$ -separators in  $G$ . If  $|S| \geq k$ , then by applying induction to  $G - e$ , there are  $k$  IDJ  $uv$ -paths in  $G - e$  and therefore, in  $G$ , a contradiction. Therefore,  $|S| < k$ . Since  $S \cup \{x\}$  and  $S \cup \{y\}$  are  $uv$ -separators in  $G$ , we also have  $|S \cup \{x\}| \geq k$  and  $|S \cup \{y\}| \geq k$ . So,  $|S| = k - 1$ , and  $x, y \notin S$ .

Since  $|S| = k - 1$ , there is a  $uv$ -path  $P$  in  $G - S$ ; since  $P$  is not a path of  $G - e - S$ , we must have  $e \in P$ , so one of  $x, y$  (say  $x$ ) is connected to  $u$  in  $G$  and the other is connected to  $v$  in  $G$  (both via  $P$ ).

**Proof.** Let  $G_u$  be the graph obtained from  $G$  by deleting all vertices connected to  $v$  in  $G - S - e$ , and adding a single vertex  $v'$  adjacent to every vertex in  $S \cup \{x\}$ . Let  $G_v$  symmetrically, but swap  $u$  and  $v$  and  $x$  with  $y$ .

We have  $|V(G_u)| + |E(G_u)| + k < |V(G)| + |E(G)| + k$  and the same for  $v$ , so we can apply induction to  $G_u$  and  $G_v$ .

**Subclaim:** If  $T$  is a  $uv'$ -separator in  $G_u$  with  $|T| < k$ , then  $T$  is a  $uv$ -separator in  $G$ .

By the subclaim, there is no  $uv'$ -separator in  $G_u$  of size  $< k$ , so by induction, there are  $k$  IDJ  $uv'$ -paths  $P_1, \dots, P_k$  in  $G_u$ . By the same argument, there are  $k$  IDJ  $u'v$ -paths  $Q_1, \dots, Q_k$  in

$G_v$ .

Each  $P_i$  has the form  $uP'_i w$  where  $w \in S \cup \{x\}$ , and each  $Q_j$  has the form  $zQ'_j v$  where  $z \in S \cup \{y\}$ . Since  $k = |S \cup \{x\}| = |S \cup \{y\}|$ , we can join  $P_i$  and  $Q_j$  at the ends where they agree in  $S$  and add the edge  $e$  to the  $P'_i$  ending at  $x$  and the  $Q'_j$  starting at  $y$ , to find  $k$  IDJ  $uv$ -paths in  $G$  of the form  $uP_i e Q_j v$ . This is a contradiction. Since  $G$  satisfies claims 1 and 2, it is the disjoint union of a star graph at  $u$  and a star graph at  $v$ , so it is not a counterexample to Menger's Theorem. □

### Theorem (Menger – Version 2)

If  $u$  and  $v$  are vertices in  $G$  and  $F$  is a set of edge from  $u$  to  $v$ , and every  $uv$ -separator in  $G \setminus F$  has size  $\geq k$ , then there are  $k + |F|$  internally disjoint  $uv$ -paths in  $G$ .

**Proof.** In the graph  $G \setminus F$ ,  $u$  and  $v$  are non-adjacent, so by Menger, there are  $k$  IDJ  $uv$ -paths in  $G \setminus F$ . Each edge in  $F$  is its own  $uv$ -path, so the paths in  $G \setminus F$  together with the edges in  $F$  give  $k + |F|$  IDJ  $uv$ -paths in  $G$ .

### Theorem

If  $G$  is a simple graph on  $> k$  vertices, then  $G$  is  $k$ -connected if and only if for every pair  $u, v$  of distinct vertices of  $G$ , there are  $k$  internally disjoint  $uv$ -paths.

**Proof.** ( $\implies$ ) Suppose  $G$  is  $k$ -connected. Let  $u, v \in V(G)$  be distinct. If  $u, v$  are non-adjacent and  $S$  is a  $uv$ -separator, then  $G - S$  is disconnected, so  $|S| \geq k$  by  $k$ -connectedness of  $G$ . Thus, every  $uv$ -separator has size  $\geq k$ , so by Menger,  $G$  has  $k$  IDJ  $uv$ -paths.

If  $u, v$  are joined by an edge  $e$ , then let  $S$  be a smallest  $uv$ -separator in  $G \setminus e$ . If  $|S| \geq k - 1$ , then by Menger 2, there are  $(k - 1) + 1 = k$  IDJ  $uv$ -paths in  $G$ , as required.

If  $|S| < k - 1$ , then let  $x$  be a vertex outside  $S \cup \{u, v\}$  (exists since  $|V(G)| \geq k$ ). Since  $u, v$  are not connected in  $(G \setminus e) - S$ ,  $x$  is not connected to both  $u$  and  $v$ . Suppose WLOG that  $x$  is not connected to  $u$  in  $(G \setminus e) - S$ . Therefore,  $x$  is not connected to  $u$  in  $G - (S \cup \{v\})$ . But the size of  $|S \cup \{v\}| = |S| + 1 < k$ , so we have a contradiction to the  $k$ -connectedness of  $G$ .

( $\impliedby$ ) Suppose  $G$  is not  $k$ -connected; let  $S$  be a set of vertices such that  $G - S$  is disconnected and  $|S| < k$ . Let  $u, v$  be vertices in different components of  $G - S$ ; if  $P_1, \dots, P_k$  are IDJ  $uv$ -paths, then each must intersect  $S$ , but  $|S| < k$ . This contradicts the pigeonhole principle.

### Definition: AB-Separator

A set  $X \subseteq V(G)$  such that  $G - X$  has no  $AB$ -path.

Note:  $X$  is allowed to intersect  $A$  and/or  $B$ .  $A$  and  $B$  are both examples of an  $AB$ -separator.

**Proposition**

If  $A, B \subseteq V(G)$  and  $S$  is an  $AB$ -separator of size  $< k$ , then there do not exist  $k$  disjoint  $AB$ -paths.

**Proof.** If  $P_1, \dots, P_k$  are disjoint  $AB$ -paths, then none is an  $AB$ -path in  $G - S$ , so each  $P_i$  contains a vertex in  $S$ . But  $|S| < k$  and the paths are disjoint, so this contradicts the pigeonhole principle.

**Theorem (Menger – Version 3)**

If  $A, B \subseteq V(G)$  and every  $AB$ -separator has size  $\geq k$ , then there are  $k$  disjoint  $AB$ -paths.

**Definition:  $aB$ -Fan**

Given a vertex  $a$  and a set  $B \subseteq V(G)$ , an  $aB$ -fan is a collection of  $aB$ -paths intersecting only at  $a$ .

**Definition:  $aB$ -Separator**

A set  $X \subseteq V(G) \setminus \{a\}$  such that  $G - X$  has no  $aB$ -path.

**Lemma (Fan Lemma)**

If there is no  $aB$ -separator of size  $< k$ , then there is an  $aB$ -fan of size  $k$ .

**Corollary**

If  $G$  is  $k$ -connected and  $|V(G)| > k$ , then for all  $v \in V(G)$  and  $X \subseteq V(G) \setminus \{v\}$  with  $|X| \geq k$ , there is a  $vX$ -fan.

**Theorem**

Let  $k \geq 2$ . If  $G$  is  $k$ -connected with  $\geq 2k$  vertices, then  $G$  has a cycle of length  $\geq 2k$ .

**Proof.** Since  $G$  is 2-connected, it is not a tree so it has a cycle. Let  $C$  be a longest cycle in  $G$ . We may assume  $C$  has  $< 2k$  vertices, so there is a vertex  $v$  outside  $C$ . So by fan lemma, there is a  $vC$ -fan of size  $k$  in  $G$ . Let  $P_1, \dots, P_k$  be its paths. Since  $k > \frac{1}{2}|V(C)|$ , there are two paths  $P_i, P_j$  whose endpoints  $x_i, x_j \in C$  are joined by an edge  $e$  of  $C$ . Now,  $(C \setminus e)x_iP_i v P_j x_j$  is a cycle longer, then this contradicts the maximality of  $C$ .

# Chapter 3

## Planarity

### Definition: Embedding

An embedding of  $G = (V, E)$  in  $\mathbb{R}^2$  is a function  $\varphi$  such that

- for each vertex  $v$  of  $G$ ,  $\varphi(v)$  is a point in  $\mathbb{R}^2$ , and no two vertices are mapped to same point by  $\varphi$ .
- for each edge  $e$  with ends  $u, v$ ,  $\varphi(e)$  is a curve from  $\varphi(u)$  to  $\varphi(v)$ .
- for distinct edges  $e, f$  of  $G$ , the images of  $\varphi(e)$  and  $\varphi(f)$  are disjoint (as subsets of  $\mathbb{R}^2$ ), except where  $e$  and  $f$  intersect at a vertex.
- for all  $v \in V, e \in E$ ,  $v$  is in  $\varphi(e)$  if  $v$  is an end of  $e$ .

### Definition: Planar Graph

A graph is planar if it has an embedding in  $\mathbb{R}^2$ , otherwise it is nonplanar.

If  $\varphi$  is an embedding of  $G$  in  $\mathbb{R}^2$ , then we write  $\varphi(G)$  for the union of the images of vertices and edges, as subsets of  $\mathbb{R}^2$ .

### Definition: Curve

A continuous, injective function from  $[0, 1]$  to  $\mathbb{R}^2$  with  $\varphi(0) = u$  and  $\varphi(1) = v$ .

### Definition: Interior of a Curve

The image of a curve without endpoints  $A(0)$  and  $A(1)$ .

### Definition: Disc

A disc set  $D$  of the form  $\{(x, y) \in \mathbb{R}^2 : (x - x_0)^2 + (y - y_0)^2 \leq r^2\}$ .

**Definition: Open Set in  $\mathbb{R}^2$** 

A set  $U$  such that for all  $x \in U$ , there is a disc  $D$  of radius  $r > 0$  centered at  $x$ , with  $D \subseteq U$ .

E.g.  $\{(x, y) : y > 0\}$ .

**Definition: Closed Set in  $\mathbb{R}^2$** 

The complement of an open set.

E.g.  $\{(x, y) : y \geq 0\}$ .

**Lemma (Plane Topology Lemmas)**

- Points are closed.
- (Images of) curves are closed.
- Discs are closed.
- Finite unions of closed sets are closed.

**Theorem (Intermediate Value Theorem)**

If  $f : [a, b] \rightarrow \mathbb{R}$  is continuous with  $f(a) \leq M$  and  $f(b) \geq M$ , then there exists  $x \in [a, b]$  such that  $f(x) = M$ .

**Definition: Polygonal**

A curve from  $a$  to  $b$  in  $\mathbb{R}^2$  is polygonal if it is a finite union of line segments.  
 $A$  is polygonal if there exist  $0 \leq t_0 \leq t_1 \leq \dots \leq t_k = 1$  such that  $\forall i \in \{1, \dots, k\}$ , the restriction of  $A$  to the subinterval  $[t_{i-1}, t_i]$  is a straight line segment.

**Definition: Connected**

Given a set  $U \subseteq \mathbb{R}^2$ , points  $x, y \in U$  are connected in  $U$  if either  $x = y$  or there is a curve from  $x$  to  $y$  contained in  $U$ .

**Definition: Polygonally Connected**

Points  $x, y$  are polygonally connected in  $U$  if  $x = y$  or there is a polygonal curve from  $x$  to  $y$  in  $U$ .

**Proposition**

If  $U$  is an open set, then  $x, y$  are connected in  $U$  if and only if they are polygonally connected in  $U$ .

**Proof.** ( $\Leftarrow$ ) Obvious.

( $\implies$ ) Suppose, therefore, that  $u, v$  are connected. We may assume  $u \neq v$ . Let  $f : [0, 1] \rightarrow U$  be continuous, injective, with  $f(0) = x, f(1) = y$ .

Let  $S \subseteq \{t \in [0, 1] : x \text{ is polygonally connected to } f(t) \text{ in } U\}$ . What we want to prove is equivalent to saying  $1 \in S$ .

Let  $t_0 = \sup(S)$ , i.e.  $t \notin S$  for all  $t > t_0$  and for all  $t' < t_0$ , there is some  $t \in S$  with  $t' < t \leq t_0$ .

**Claim 1:**  $t_0 \in S$ .

**Claim 1 Proof.** Clear if  $t_0 = 0$ , otherwise, let  $D$  be a disc centered at  $f(t_0)$ , contained in  $U$ , but not containing  $f(0) = x$ . By the intermediate value theorem, there is some  $0 < t' < t_0$  so that  $f(t') \in D$ .

Since  $t' < t_0$ , there exists  $t''$  with  $t' < t'' \leq t_0$  such that  $t'' \in S$ . Assume that  $f(t'') \in D$ . So there is a point  $z$  on the curve in  $D$  that is polygonally connected to  $x$ . Since  $z \in D \subseteq U$ , there is a straight line segment contained in  $U$  from  $z$  to  $f(t_0)$ , which shows that  $x$  is polygonally connected to  $f(t_0)$ . So  $t_0 \in S$ .

**Claim 2:**  $t_0 = 1$

**Claim 2 Proof.** Sketch: If  $t_0 < 1$ , use a similar argument to find  $t'$  such that  $t_0 < t'$  and  $t' \in S$ , contradicting  $t_0 = \sup(S)$ .

Thus,  $1 \in S$ , as required.

#### Corollary

Given  $U \subseteq \mathbb{R}^2$  open, if  $x, y$  connected in  $U$ ,  $y, z$  connected in  $U$ , then  $x, z$  are connected in  $U$ .

**Proof.** We can glue two polygonal arcs together. We can travel along the arc from  $x$  to  $y$  and when we first hit the arc from  $y$  to  $z$ , we switch to that arc.

So connectedness in  $U$  is an equivalence relation. Therefore, every open set  $U$  has a partition into ‘regions’ such that  $x, y$  are connected in  $U$  if and only if they belong to the same region.

#### Corollary

If  $G$  has a planar embedding  $\varphi$ , then it has a planar embedding where all arcs are polygonal.

**Proof.** Draw discs at each vertex and turn the edges within the discs into radii. Use corollary to make the edges polygonal, one by one.

#### Definition: Polygon

A polygonal arc, except that we insist on  $f(0) = f(1)$  and still injective elsewhere. Informally, a cyclic union of line segments.



### Theorem (Jordan Curve Theorem - Polygonal)

If  $C$  is a polygon, then  $\mathbb{R}^2 \setminus C$  has exactly two regions.

**Claim 1:** There are  $\leq 2$  regions in  $\mathbb{R}^2 \setminus C$ .

**Claim 1 Proof.** Let  $S_1, \dots, S_k$  be the line segments in  $C$ . For each  $i$ , let  $B_i$  be the set of points at distance  $< \varepsilon$  from a point in  $S_i$ , where  $\varepsilon > 0$  is chosen small enough so that the  $B_i$  only overlap for consecutive  $i$ .

Note that:

- For each  $i$ ,  $B_i \setminus C$  has  $\leq 2$  regions.
- For each  $i > 1$ , each point in  $B_i \setminus C$  is polygonally connected to a point in  $B_{i-1} \setminus C$ .
- Every point in  $\mathbb{R}^2 \setminus C$  is polygonally connected to a point in one of the sets  $B_i \setminus C$ .

By combining the last two observations, we see that every point in  $\mathbb{R}^2 \setminus C$  is connected to a point in some  $S_i$ , and therefore (inductively) to a point in  $S_1 \setminus C$ .

Since  $S_1 \setminus C$  has  $\leq 2$  regions, every point in  $\mathbb{R}^2 \setminus C$  lies in one of  $\leq 2$  regions of  $\mathbb{R}^2 \setminus C$ .

We need to show  $\mathbb{R}^2 \setminus C$  has  $\geq 2$  regions. Let  $w$  be a direction in  $\mathbb{R}^2$  that is not parallel to the line between any two vertices of  $C$ .

For each  $x \in \mathbb{R}^2 \setminus C$ , let  $R_x$  be the ray in direction  $w$  starting at  $x$ . Let  $n(x)$  be the number of times  $C$  intersects  $R_x$ , where if  $C$  intersects  $R_x$  at a vertex  $y$  with both segments of  $C$  adjacent to  $y$  appearing on the same side of  $R_x$ , the intersection is not counted.

**Claim 2:** If  $x, x'$  are joined by a line segment in  $\mathbb{R}^2 \setminus C$ , then  $n(x) \equiv n(x') \pmod{2}$ .

**Claim 2 Proof.** By dividing the line segment into subsegments, we may assume that as  $R_x$  moves to  $R_{x'}$ , only one vertex of  $C$  is crossed. There are 3 cases:

- V-shape:  $n(x') = n(x) - 2$ .
- Upside-down v-shape:  $n(x') = n(x) + 2$ .
- Two line segments going through  $R_x$  and  $R_{x'}$  with one vertex:  $n(x) = n(x')$ .

In all cases,  $n(x) \equiv n(x') \pmod{2}$ . Since any two points in the same region of  $\mathbb{R}^2 \setminus C$  are polygonally connected if  $x$  is in a region  $f$  of  $\mathbb{R}^2 \setminus C$  and  $y \in f$ , then there is a polygonal curve from  $x$  to  $y$  in  $\mathbb{R}^2 \setminus C$ , so  $n(x) \equiv n(y) \pmod{2}$  by applying the claim repeatedly.

Let  $x \in C$  and  $D$  be a disc around  $x$ . It is clear that  $D$  contains two points  $y, z \in \mathbb{R}^2 \setminus C$  with  $n(z) = n(y) + 1$ . So  $n(z) \not\equiv n(y) \pmod{2}$ , so  $y, z$  are in different regions of  $\mathbb{R}^2 \setminus C$ . So  $\mathbb{R}^2 \setminus C$  has  $\geq 2$  regions, as required.

In fact, for each point  $x \in C$  and each disc around  $x$  intersects both regions of  $\mathbb{R}^2 \setminus C$ .

**Proof.** Since there are  $\geq 2$  and  $\leq 2$  regions in  $\mathbb{R}^2 \setminus C$ , then  $\mathbb{R}^2 \setminus C$  has exactly two regions.

### Definition: Frontier

Given a set  $S \subseteq \mathbb{R}^2$ , the frontier of  $S$  is the set of points  $x \in \mathbb{R}^2$  such that every disc of positive radius centered at  $x$  intersects  $S$ .

### Lemma

If  $x_1, y_1, x_2, y_2$  occur in cyclic order around some polygon  $C$  and  $P$  is a polygonal curve from  $x_1$  to  $x_2$  with interior of  $P$ ,  $\dot{P} \subseteq \mathbb{R}^2 \setminus C$ , then  $\mathbb{R}^2 \setminus (C \cup P)$  has three regions  $f_0, f_1, f_2$  such that  $f_0$  is a region of  $\mathbb{R}^2 \setminus C$ , and  $f_1 \cup f_2 \cup \dot{P}$  is the other region of  $\mathbb{R}^2 \setminus C$ , and  $y_1$  is not in the frontier of  $f_2$  and  $y_2$  is not in the frontier of  $f_1$ .

**Proof.** Use polygonal Jordan Curve Theorem.

### Proposition

$K_{3,3}$  is nonplanar.

**Proof.** Suppose that  $K_{3,3}$  is planar. Let  $\{a_1, a_2, a_3\}, \{b_1, b_2, b_3\}$  be its bipartition, and let  $\varphi$  be a polygonal embedding. Note that  $a_1b_1a_2b_2a_3b_3$  is a cycle in  $G$ , so it corresponds to a polygon in  $\varphi(G)$  where  $\varphi(a_1), \varphi(b_1), \dots, \varphi(a_3), \varphi(b_3)$  appear in cyclic order.

Let  $e_1 = a_1b_2, e_2 = a_2b_3, e_3 = a_3b_1$ . Now each  $\varphi(e_i)$  is contained in a region of  $\mathbb{R}^2 \setminus C$ . Otherwise, it would contain an arc from the inside to the outside of  $C$ .

Two of the  $e_i$ , say  $e_1, e_2$ , are contained in the same region of  $\mathbb{R}^2 \setminus C$ , by pigeonhole principle.

Let  $f_1, f_2$  be the regions of  $\mathbb{R}^2 \setminus (C \cup \varphi(e_i))$  for which  $f = f_1 \cup f_2 \cup \varphi(e_i)$ . Now  $\varphi(e_j) \subseteq f = f_1 \cup f_2 \cup \varphi(e_i)$  and  $\varphi(e_j)$  does not intersect  $\varphi(e_i)$ , so  $\varphi(e_j)$  is contained in either  $f_1$  or  $f_2$ , say  $f_1$ .

The ends  $x_i, y_i$  of  $e_i$  and  $x_j, y_j$  of  $e_j$  occur in cyclic order  $x_i, x_j, y_i, y_j$  around  $C$ , so by the lemma above, both  $x_j, y_j$  are on the frontier of  $f_1$ . This contradicts the lemma.

### Definition: Topological Minor

A graph  $H$  is a topological minor of a graph  $G$  if there is a function  $\psi$  such that

- for every vertex  $v$  of  $H$ ,  $\psi(v)$  is a vertex of  $G$  and if  $u \neq v$ , then  $\psi(u) \neq \psi(v)$ .
- for every edge  $e$  of  $H$  with ends  $u, v$ ,  $\psi(e)$  is a  $\psi(u)\psi(v)$ -path of  $G$ .
- no two paths  $\psi(e), \psi(e')$  have an internal vertex in common.

### Proposition

$H$  is a topological minor of  $G$  if and only if some subgraph  $G'$  of  $G$  is isomorphic to a subdivision of  $H$ .

**Proposition**

If  $G$  is planar, and  $H$  is a topological minor of  $G$ , then  $H$  is planar.

**Proof.** If  $G$  has an  $H$ -topological minor, then some subdivision  $H'$  of  $H$  is isomorphic to a subgraph of  $G$ , so  $H'$  is planar. A planar embedding of  $H'$  gives rise to a planar embedding for  $H$ .

**Corollary**

If  $H$  is nonplanar and  $H$  is a topological minor of  $G$ , then  $G$  is nonplanar.

**Definition: Face (of  $\varphi$ )**

A region of the open set  $\mathbb{R}^2 \setminus \varphi(G)$ .

**Proposition**

If  $f$  is a face of  $\varphi$ , then the frontier of  $f$  is a union of some vertices  $\varphi(v)$  and some edges  $\varphi(e)$ .

**Definition: Boundary**

The boundary of a face  $f$  of  $\varphi$  is the subgraph  $H$  of  $G$  whose vertices and edges form the frontier of  $f$ .

i.e.  $\varphi(H)$  (points in  $\mathbb{R}^2$  used by  $\varphi$  to draw  $H$ ) is the frontier of  $f$ .

**Definition: Leaf Edge**

An edge incident with a degree 1 vertex.

**Lemma**

Let  $\varphi$  be a planar embedding of  $G$  and  $e$  be a leaf edge of  $G$ . Then the embedding  $\varphi'$  of  $G \setminus e$  given by  $\varphi$  has the same number of faces as  $\varphi$ .

**Proof.** Sketch: take two points in the face. Go around  $e$  with the polygonal arc.

**Proposition**

If  $e$  is in a cycle of  $G$  and  $\varphi$  is a planar embedding of  $G$ , then  $e$  is in the boundary of exactly two faces of  $G$ .

**Proof.** Sketch: Use the polygonal Jordan Curve Theorem.

**Proposition**

If  $\varphi$  is a planar embedding of  $G$ , then  $\varphi$  has exactly one face if and only if  $G$  is a forest.

**Proof.** ( $\implies$ ) Suppose  $G$  is not a forest. Then  $G$  is a cycle  $C$  and each edge in  $G$  is in two faces of  $\varphi$ . So  $\varphi$  is more than one face.

( $\impliedby$ ) Suppose that  $G$  is a forest. If  $G$  has no edges, it clearly has one face. Suppose inductively that  $G$  has  $k$  edges and that the result holds for all forests with  $k - 1$  edges.

Let  $e$  be a leaf edge of  $G$ . Inductively, each embedding of  $G \setminus e$  has exactly one face, and by the lemma,  $\varphi(G)$  and  $\varphi(G \setminus e)$  have the same number of faces, so  $\varphi$  has exactly one face.

### Proposition

If  $e$  is in a cycle of  $G$  and  $\varphi$  is a planar embedding of  $G$ , then  $\varphi$  has exactly one more face than the planar embedding of  $G \setminus e$ .

**Proof.** Use the polygonal Jordan Curve Theorem (similar to argument that edges in cycles are in two faces).

### Theorem (Euler's Formula)

If  $G = (V, E)$  is a graph with  $c$  components,  $\varphi$  is a planar embedding of  $G$ , and  $F$  is the set of faces of  $\varphi$ , then

$$|V| - |E| + |F| = 1 + c$$

**Proof.** Let  $H$  be a maximal spanning forest for  $G$ . So  $H$  consists of a spanning tree  $H_i$  for each component  $G_i$  of  $G$ , and  $|E(H_i)| = |V(H_i)| - 1$ .

So,

$$|E(H)| = \sum_i |E(H_i)| = \sum_i (|V(H_i)| - 1) = \sum_i |V(H_i)| - c = |V| - c$$

The embedding of  $H$  given by  $\varphi$  has one face ( $H$  is a forest),  $|V|$  vertices, and  $|V| - c$  edges. Thus,

$$|V(H)| - |E(H)| + |F(H)| = |V| - (|V| - c) + 1 = 1 + c$$

Let  $H'$  be a maximal subgraph of  $G$  such that  $H$  is a subgraph of  $H'$  and  $|V| - |E(H')| + |F(H')| = 1 + c$ .

If  $H' = G$ , then  $G$  satisfies Euler's formula, as required. Otherwise,  $G$  has an edge  $e$  outside  $E(H')$ . Let  $H' + e$  be the subgraph of  $G$  obtained from  $H'$  by adding  $e$ . Since  $e$  is in a cycle of  $H + e$ , we know that  $|F(H' + e)| = |F(H')| + 1$ . Clearly,  $|E(H' + e)| = |E(H')| + 1$ . So,

$$\begin{aligned} |V| - |E(H' + e)| + |F(H' + e)| &= |V| - |E(H')| - 1 + |F(H')| + 1 \\ &= |V| - |E(H')| + |F(H')| \\ &= 1 + c \end{aligned}$$

So,  $H' + e$  contradicts the maximality of  $H'$ .

### Lemma

If  $\varphi$  is an embedding of a graph  $G$  that contains a cycle, then the boundary of every face of  $G$  contains a cycle.

**Proof.** Topological exercise.

#### Lemma

Each edge in a planar embedding is in  $\leq 2$  face boundaries.

#### Proposition

If  $G$  is a simple planar graph on  $\geq 3$  vertices, then  $|E(G)| \leq 3|V(G)| - 6$ .

**Proof.** We combine Euler's Formula with an inequality relating the number of edges and the number of faces in the embedding. Let  $V = V(G)$ ,  $E = E(G)$ . Let  $F$  be the set of faces in some planar embedding of  $G$  and  $c$  is the number of components of  $G$ .

If  $G$  is a forest, then  $|E| \leq |V| - 1 \leq 3|V| - 6$ .

Otherwise, every face boundary contains a cycle, so has  $\geq 3$  edges.

Let  $A = \{(e, f) : f \in F, e \text{ is the boundary of } f\}$ . Since each  $e$  is in the boundary of  $\leq 2$  faces, we know  $|A| \leq 2|E|$ . Since each  $f \in F$  has  $\geq 3$  edges in its boundary, we know  $|A| \geq 3|F|$ . So  $3|F| \leq 2|E|$ , i.e.  $|F| \leq \frac{2}{3}|E|$ .

By Euler's Formula,

$$1 + c = |V| - |E| + |F| \leq |V| - |E| + \frac{2}{3}|E| = |V| - \frac{1}{3}|E|$$

So  $|E| \leq 3(|V| - 1 - c) \leq 3|V| - 6$  since  $c \geq 1$ .

#### Corollary

$K_5$  is nonplanar.

**Proof.**  $|E| = \binom{5}{2} = 10$  and  $3|V| - 6 = 15 - 6 = 9$ . So,  $|E| \not\leq 3|V| - 6$  so  $K_5$  is nonplanar.

#### Proposition

If  $\varphi$  is an embedding of a 2-connected graph  $G$ , then every face boundary of  $G$  is a cycle.

**Proof.** Induction with ear-decomposition. Adding a path splits one face into two faces bounded by cycles and does not change any other face boundary.

Question: Given a graph  $G$  that is known to be planar, can we determine which cycles appear as face boundaries in an embedding of  $G$ , without knowing the embedding? No, in general. The problem is the lack of 3-connectedness.

#### Definition: Non-Separating Cycle

A cycle  $C$  of  $G$  is non-separating if  $G - V(C)$  is connected.

**Definition: Chord**

An edge that connects two nonadjacent vertices in a cycle.

**Definition: Induced Cycle**

$C$  is induced in  $G$  if there is no edge of  $G \setminus E(C)$  with both ends in  $C$  (i.e. no chord of  $C$ ).

**Proposition**

If  $\varphi$  is an embedding of a 3-connected graph  $G$ , then  $C$  is a face boundary (facial cycle) of  $G$  if and only if  $C$  is non-separating and induced in  $G$ .

*Proof.* Assignment.

**Lemma**

Let  $G$  be a planar graph and  $F$  be a face boundary in some planar embedding of  $G$ . Let  $G'$  be the graph obtained from  $G$  by adding a vertex  $v$  and joining  $v$  to each vertex of  $F$ . Then  $\varphi$  extends to an embedding of  $G'$ .

*Proof.* Exercise.

**Facts About 3-Connected Planar Graphs**

- They have a unique embedding in the plane/sphere (up to homeomorphism).
- They have an embedding in the plane where all edges are straight line segments and all faces are convex polygons.
- They are exactly the skeletons of polyhedra.

**Theorem**

If  $\varphi$  is an embedding of a 3-connected graph  $G$ , then the face boundaries of  $\varphi$  are exactly the non-separating induced cycles of  $G$ .

**Theorem**

$K_{3,3}$  and  $K_5$  are nonplanar.

**Theorem (Kuratowski – Version 2)**

$G$  is planar if and only if neither  $K_{3,3}$  nor  $K_5$  is a topological minor of  $G$ .

**Definition: Minor ( $\leq$ )**

A graph  $H$  is a minor of a graph  $G$ , denoted  $H \leq G$ , if  $H$  can be obtained from a subgraph  $G'$  of  $G$  (by deleting vertices or edges) by a sequence of edge contractions.

Note:

- $H$  is a minor of  $G$  if and only if  $H$  is obtained from  $G$  by vertex deletions, edge deletions, and edge contractions.
- $H$  is a minor of  $G$  if and only if there is a ‘model’ of  $H$  in  $G$  (vertices of  $H$  correspond to disjoint connected subgraphs of  $G$ , edge of  $H$  correspond to edges of  $G$  between subgraphs).

### Proposition

If  $G$  is planar and  $H \leq G$ , then  $H$  is planar.

**Proof.** Since subgraphs of planar graphs are planar, it is enough to show that contracting a single edge in a planar graph keeps the graph planar. Consider a region that is equal to all points on the edge  $e = \{u, v\}$ , contract  $e$  by creating the new vertex  $x_{uv}$  in the middle of  $e$  and draw all neighbours of  $u$  and  $v$  to  $x_{uv}$

### Corollary

If  $G$  has  $K_{3,3}$  or  $K_5$  as a minor,  $G$  is nonplanar.

### Proposition

If  $G$  has  $H$  as a topological minor, then  $G$  has  $H$  as a minor.

**Proof.** For each edge  $e$  of  $H$ , let  $P_e$  be the corresponding path of  $G$ . Let  $G'$  be the subgraph of  $G$  that is the union of all  $P_e$ . Now  $H$  is obtained from  $G'$  by contracting all but one edge in each path  $P_e$ .

### Lemma

For every edge or vertex of a planar graph  $G$  and every disc  $D$  in the plane, there is an embedding of  $G$  contained in  $D$  such that the edge or vertex is in the boundary of the outer face of  $\varphi$ .

### Theorem (Kuratowski)

$G$  is planar if and only if it contains neither  $K_{3,3}$  nor  $K_5$  as a minor.

**Proof.** ( $\Leftarrow$ ) Suppose for a contradiction that  $G$  has no  $K_{3,3}$  or  $K_5$  minor, but is nonplanar. Choose  $G$  so that  $|V(G)| + |E(G)|$  is as small as possible.

**Claim 1:**  $G$  is connected.

**Claim 1 Proof.** Suppose not; let  $G_1, \dots, G_k$  be its components. Since there are  $\geq 2$  components, we have  $|V(G_i)| + |E(G_i)| < |V(G)| + |E(G)|$ , but the  $G_i$  are subgraphs of  $G$ , so none of the  $G_i$  have  $K_{3,3}$  or  $K_5$  as a minor. Therefore, by the minimality in the choice of  $G$ , all the  $G_i$  are planar.

We can combine planar embeddings of  $G_i$  to make a planar embedding of  $G$ , giving a contradiction.

**Proof.** To continue, we use, but not prove) the following: for any embedding  $\varphi$  of  $G$  and any edge  $e$  (or vertex  $v$ ) of  $G$ , and any open disc  $D \subseteq \mathbb{R}^2$ , there is an embedding  $\varphi'$  of  $G$  such that  $\varphi'(G) \subseteq D$  and  $e$  (or  $v$ ) is contained in the boundary of the unbounded face of  $\varphi'$ .

**Claim 2:**  $G$  is 2-connected.

**Claim 2 Proof.** If not, then  $G$  has a cut vertex  $x$ . Let  $G_1, G_2$  be proper subgraphs of  $G$  such that  $G = G_1 \cup G_2$  and  $V(G_1) \cap V(G_2) = \{x\}$ . Since both  $G_i$  are smaller than  $G$  and have no  $K_{3,3}$  or  $K_5$ -minor, both are planar.

Consider embeddings of  $G_1, G_2$  in disjoint discs  $D_1, D_2$  in the plane, where  $x$  is embedded by both in the boundary of the outer face. We can find an arc between the two copies of  $x$  in the resulting drawing to get an embedding of a graph  $G'$  such that  $G'/e \cong G$  for some edge  $e$ . Since  $G'$  is planar and  $G \cong G'/e$ ,  $G$  is also planar, a contradiction.

**Claim 3:**  $G$  is 3-connected.

**Claim 3 Proof.** Suppose not. There are vertices  $x, y \in V(G)$  and subgraphs  $G_1, G_2$  of  $G$  such that  $G = G_1 \cup G_2$  and  $V(G_1) \cap V(G_2) = \{x, y\}$ . By a similar argument to the previous claim,  $G_1, G_2$  are planar.

Let  $G'_1, G'_2$  be obtained from  $G_1, G_2$  respectively by adding a new edge  $f$  from  $x$  to  $y$  (choose a vertex  $w$  of  $G_2 - \{x, y\}$  and take a  $w\{x, y\}$ -fan to get this path, by applying the Fan lemma to  $G$ ). Since  $G$  is 2-connected, there is an  $xy$ -path  $P$  with  $\geq 2$  edges in  $G_2$ . Now  $G'_1$  is obtained from the subgraph  $G_1 \cup P$  by contracting all but one edge of  $P$ . Since  $K_{3,3}, K_5 \not\leq G$  and  $G'_1$  is a minor of  $G$  with fewer vertices and  $K_{3,3}, K_5 \not\leq G'$ ,  $G'_1$  is planar. Similarly  $G'_2$  is planar. Now consider embeddings of  $G'_1, G'_2$  in disjoint discs in  $\mathbb{R}^2$  where  $e$  is on the outer face.

We can now combine these embeddings and use connectedness in the unbounded face to obtain a planar embedding of the following graph. So  $G'$  is planar, so  $G = G' \setminus \{e_1, e_2\} / \{f_1, f_2\}$  is planar, a contradiction.

For every  $e \in E(G)$ ,  $G/e$  and  $G \setminus e$  are planar, because they are minors of  $G$ , so they have no  $K_{3,3}$  or  $K_5$ -minor, and they are ‘smaller’ than  $G$ , so are not counterexamples.

**Claim 4:**  $G$  is simple.

**Claim 4 Proof.** If not, delete an edge  $e$  parallel to some other edge, draw  $G \setminus e$  and add  $e$  back to the embedding.

Note that since  $G$  is nonplanar,  $|V(G)| \geq 4$ .

By a lemma,  $G$  has an edge  $e = xy$  such that  $G/e$  is 3-connected. We also know that  $G/e$  is planar. Let  $u$  be the vertex of  $G/e$  corresponding to  $e$  and consider a planar embedding  $\varphi$  of  $G/e$ . Let  $v$  be the new contracted vertex, then  $(G/e) - v$  is a 2-connected planar graph, so every face boundary is a cycle. Now  $v$  is embedded in some face of  $(G/e) - v$  whose boundary is a cycle  $C$ , and all neighbours of  $v$  in  $G/e$  lie in  $C$ .



**Lemma**

Given  $X, Y$  of vertices in a cycle  $C$ , either

- (i) there exist  $x, x' \in X$  and  $y, y' \in Y$  such that  $y, y'$  are in different components of  $C - \{x, x'\}$  ( $x, y, x', y'$  in cyclic order)  $\implies K_{3,3}$ -minor.
- (ii)  $|X \cap Y| \geq 3 \implies K_5$ -minor.
- (iii) there are edge-disjoint paths  $P_X, P_Y$  of  $C$  such that either  $E(P_X) \cap E(P_Y) = \emptyset$ ,  $P_X \cup P_Y = C$ , and  $X \subseteq V(P_X), Y \subseteq V(P_Y)$ .

**Lemma Proof.** We may assume by symmetry that  $|X| \leq |Y|$ . If  $|X| \leq 1$ , choose  $P_X$  to be a path with one edge  $f$  containing all vertices in  $X$  and choose  $P_Y$  to be  $C - f$ . Then (iii) holds.

So  $|X| \geq 2$ . If  $Y \setminus X = \emptyset$ , then  $X = Y$ . Suppose this holds. If  $|X| = |Y| = 2$ , then let  $\{a, b\} = X = Y$ . Then choose  $P_X$  and  $P_Y$  to be the two distinct  $ab$ -paths in  $C$ . Now (iii) holds.

Otherwise,  $|X| = |Y| \geq 3$ , so  $|X| \cap |Y| = |X| \geq 3$ , so (ii) holds.

So we may assume that there exists  $b \in Y \setminus X$ . Since  $b \notin X$ ,  $C$  is 2-connected and  $|X| \geq 2$ , there is a  $bX$ -fan in  $C$  of size 2. Let  $P_1, P_2$  be the paths in this fan. Let  $P_Y = P_1 \cup P_2$ ; since  $P_1, P_2$  form a fan,  $P_Y$  has no internal vertices in  $X$ .

Let  $P_X$  be the other path in  $C$  between the ends of  $P_Y$ . Since  $P_Y$  has no internal vertices in  $X$ , we know  $X \subseteq V(P_X)$ . If  $Y \subseteq V(P_Y)$ , then (iii) holds.

Otherwise, there is some  $b \in Y$  in a different component of  $C \setminus \{\text{ends of } P_X\}$  from  $b$ , so (i) holds.

**Proof.** Let  $X = \{\text{neighbours of } x \text{ in } C\}$  and  $Y = \{\text{neighbours of } y \text{ in } C\}$ . We now apply the lemma to  $X, Y, C$ . If (ii) holds, then  $x, y$  has three common neighbours  $a, b, c \in C$ . Now, the vertices  $a, b, c, x, y$  are the terminals of a topological  $K_5$ -minor of  $G$ . Therefore,  $G$  has a  $K_5$ -minor, a contradiction.

If (i) holds, then there exist  $a, b, a', b'$  in that order around  $C$  such that  $a, a'$  are neighbours of  $x$  and  $b, b'$  are neighbours of  $y$ . Now  $x, y, a, b, a', b'$  are the terminals of a topological  $K_{3,3}$  minor of  $G$ . Therefore,  $G$  has a  $K_{3,3}$ -minor, a contradiction.

Suppose, therefore, that (iii) holds. We use the fact that for any polygon  $C \subseteq \mathbb{R}^2$  with vertices in cyclic order  $a_1, \dots, a_t$  and any  $x$  in the interior of  $C$ , we can find arcs  $A_1, \dots, A_t$  from  $x$  to the  $a_i$ , intersecting only at  $x$ , and leaving  $x$  in the same cyclic order as the  $a_i$  occur around  $C$ .

**Fact Proof.** Inductively draw the arcs one by one.

**Proof.** Using the lemma, construct a planar embedding of  $G$  as follows:

- Take the embedding of  $G/e - u$  we were considering.

- Add  $u$  back and use the lemma to construct arcs from  $u$  to all vertices in  $X \cup Y$ .
- Let  $D$  be a small disc centered at  $u$ . Within  $D$ , split  $u$  into two vertices  $x, y$  and use straight line segments to alter the embedding of  $G/e$  to an embedding of  $G$ .

This contradicts the nonplanarity of  $G$ . □

The topological minor version of Kuratowski's Theorem is equivalent to the minor version because of the following fact.

### Proposition

For a graph  $G$ ,  $K_{3,3}$  or  $K_5$  is a minor of  $G \iff K_{3,3}$  or  $K_5$  is a topological minor of  $G$ .

This follows from 3 statements.

1. For all  $H$ , if  $H$  is a topological minor of  $G$ , then  $H$  is a minor of  $G$ .
2. For all  $H$  of maximum degree at most 3, if  $H$  is a minor of  $G$ , then  $H$  is a topological minor of  $G$ .
3. If  $G$  has  $K_5$  a minor, then it has  $K_5$  or  $K_{3,3}$  as a topological minor.

Thus, version 1 and version 2 of Kuratowski's Theorem are equivalent.

Note: One can adapt our version of Kuratowski's Theorem to show that every planar graph can be drawn with all edges as straight line segments.

### Theorem (Kuratowski – Alternative)

$K_{3,3}$  and  $K_5$  are the excluded minors for planarity.

### Theorem

$K_{3,3}$  and  $K_5$  are the unique minor-minimal nonplanar graphs.

### Theorem

$G$  is toroidal if and only if  $G$  does not have *some graphs* as minors.

### Theorem

$G$  is linkedless-embeddable in  $\mathbb{R}^3$  if and only if  $G$  does not contain *unknown list* as minors.

### Theorem (Graph Minors)

A theorem stated like Kuratowski's Theorem for excluded minors.

# Chapter 4

## Matchings

### Definition: Matching

A set  $M \subseteq E(G)$  so that no two share an end.

### Definition: Vertex Cover

A set  $W \subseteq V(G)$  so that every edge of  $G$  has an end in  $W$ .

Observation: If  $M$  is a matching of  $G$  and  $W$  is a vertex cover of  $G$ , then  $|M| \leq |W|$ . This is because each edge in  $M$  has an end in  $W$  and no two have a common end.

### Corollary

If  $M$  is a matching and  $W$  is a vertex cover such that  $|M| = |W|$ , then  $W$  contains exactly one end of each edge in  $M$ , and no other vertices.

### Corollary

If  $\nu(G)$  is the size of a maximum matching of  $G$  and  $\tau(G)$  is the size of a minimum cover of  $G$ , then  $\nu(G) \leq \tau(G)$ .

### Proposition

In an even cycle on  $2n$  vertices,  $\nu(G) = \tau(G) = n$ .

**Proof.** Every other edge and every other vertex are a matching and a cover respectively, each of size  $n$ .

### Proposition

In a path on  $n$  vertices,  $\nu(G) = \tau(G) = \lfloor \frac{n}{2} \rfloor$ .

**Proof.** If  $V(P) = \{v_1, \dots, v_n\}$ , then  $\{v_2, v_4, \dots, v_{2\lfloor \frac{n}{2} \rfloor}\}$  is a vertex cover and  $\{v_1v_2, v_3v_4, \dots\}$

is a matching. Both have size  $\lfloor \frac{n}{2} \rfloor$ .

Note: The statement  $\nu(G) = \tau(G)$  fails for odd cycles, because  $\nu(C_{2n+1}) = n+1$ ,  $\tau(C_{2n+1}) = n$ .

### Proposition

If  $G_1, \dots, G_k$  are the components of  $G$ , then  $\nu(G) = \sum_{i=1}^k \nu(G_i)$  and  $\tau(G) = \sum_{i=1}^k \tau(G_i)$ .

### Theorem (König 1931)

If  $G$  is a bipartite graph, then  $\nu(G) = \tau(G)$ .

**Proof.** (Rizzi 1999) We need to show that  $\tau(G) \leq \nu(G)$  for bipartite  $G$ . Let  $G$  be a counterexample on as few edges as possible.

**Claim:**  $G$  has a vertex vertex of degree  $\geq 3$ .

**Claim Proof.** If not, then every component is a path or a cycle, so König's theorem holds for each component, so holds for  $G$  since  $\tau$  and  $\nu$  are additive over components.

Let  $u$  be a vertex of degree  $\geq 3$ , and  $v$  be a neighbour of  $u$ . We split into cases, depending on whether  $\nu(G - v) = \nu(G)$ . If  $\nu(G - v) \leq \nu(G) - 1$ , then let  $W_0$  be a minimum vertex cover of  $G - v$ . Since  $G - v$  is not a counterexample, we have  $|W_0| = \nu(G - v) \leq \nu(G) - 1$ . Since  $W_0$  is a vertex cover of  $G - v$ ,  $W_0 \cup \{v\}$  is a vertex cover of  $G$ . So  $\tau(G) \leq |W_0 \cup \{v\}| \leq (\nu(G) - 1) + 1 = \nu(G)$ . This contradicts that  $G$  is a counterexample.

Otherwise,  $\nu(G - v) = \nu(G)$ . In other words, each maximum matching of  $G - v$  is also a maximum matching of  $G$ . Let  $M$  be a maximum matching of both  $G - v$  and  $G$ .

Since  $\deg(u) \geq 3$ , there is an edge  $f$  incident with  $u$  but not  $v$ , such that  $f \notin M$ . So  $\nu(G - f) \geq |M| = \nu(G) \geq \nu(G - f)$  implying  $\nu(G - f) = |M|$ .

Since  $|E(G - f)| < |E(G)|$ , we know that  $\tau(G - f) = \nu(G - f) = |M|$ . Let  $W$  be a vertex cover of  $G - f$  with  $|W| = |M|$ . We know that  $W$  contains exactly one end of each edge in  $M$  and nothing else.

In particular,  $v \notin W$ . Since  $W$  is a vertex cover, it contains at least one end of the edge  $uv$ , so we must have  $u \in W$ . By choice of  $W$ ,  $W$  contains an end of every edge in  $G - f$  and since  $u \in W$ ,  $W$  also contains an end of  $f$ . Therefore,  $W$  is a vertex cover of  $G$ .

So,  $\tau(G) \leq |W| = |M| = \nu(G)$ , which contradicts  $\tau(G) \geq \nu(G)$ .  $\square$

König's theorem can be thought of in different ways for a bipartite graph  $G$ :

- Either  $G$  has a  $t$ -edge matching or there is a good reason it does not, a vertex cover of size of  $< t$ .
- There is a maximum matching  $M$  of  $G$ , together with a vertex cover  $W$  of  $G$  that

‘proves’ there is no larger matching.

**(Conjecture 1970)**

There exists  $\beta > 0$  such that every bridgeless 3-regular graph  $G$  has  $\geq (1 + \beta)^{|V(G)|}$  perfect matchings.

**Theorem (Esperet, King, Kardos, Kral, Norin)**

The conjecture is true for  $\beta = 0.0001$ .

**Zero-Star Problem**

Given an  $n \times n$  matrix where some entries are given to be zero, can you fill in the other entries so that the matrix has nonzero determinant?

**Proof.** If there exist  $a_1, \dots, a_n$  permutations of  $\{1, \dots, n\}$  such that entries  $i, a_i$  are allowed to be nonzero, then the answer is yes. Encoding the matrix as a bipartite graph with both sides of size  $n$ , we see that if the graph has a perfect matching, then the answer is yes.

By König’s theorem, if there is no perfect matching, then there is a vertex cover in  $G$  of size  $< n$ .  $\square$

Idea: If  $G$  has a ‘small’ set of vertices whose deletion gives a graph with a ‘large’ number of odd components, then matchings in  $G$  cannot be too big.

**Definition:  $M$ -Saturated**

Given a matching  $M$  of  $G$ , the vertices of  $G$  that are an end of an edge in  $M$  are  $M$ -saturated vertices.

**Definition:  $M$ -Exposed/Unsaturated**

Vertices not  $M$ -saturated.

We say  $M$  saturates its saturated vertices and avoids its unsaturated vertices.

**Definition:  $oc(G)$**

The number of components in  $G$  with an odd number of vertices.

**Proposition**

If  $M$  is a matching of  $G$  and  $X$  is a set of vertices of  $G$ , then there are at least  $oc(G - X) - |X|$   $M$ -unsaturated vertices in  $G$ .

**Proof.** Let  $\mathcal{C}$  be the set of odd components of  $G - X$  that contain a vertex that is matched by  $M$  to a vertex in  $X$ . Since no two edges of  $M$  have the same end in  $X$ , there are at most  $|X|$  edges of  $M$  from  $X$  to  $V - X$ , so at most  $|X|$  components of  $G - X$  contain a vertex matched by  $M$  to a vertex in  $X$ . Therefore,  $|\mathcal{C}| \leq |X|$ , so there are at least

$oc(G - X) - |\mathcal{C}| \geq oc(G - X) - |X|$  odd components of  $G - X$  that contain no vertex matched to anything in  $X$ .

For each such component  $H$ , no edge of  $M$  has exactly one end in  $H$ , so the number of saturated vertices in  $H$  is even. Since  $H$  is odd, it must contain at  $\geq 1$  unsaturated vertex. There are  $\geq oc(G - X) - |X|$  different  $H$ , so  $G$  has this many  $M$ -unsaturated vertices.

### Corollary

If there is a set  $X$  such that  $oc(G - X) > |X|$ , then  $G$  has no perfect matching.

**Proof.** By the bound, every matching avoids at least  $oc(G - X) - |X| > 0$  vertices, so there is no perfect matching.

### Corollary

If  $X \subseteq V(G)$ , then  $\nu(G) \leq \frac{1}{2}(|V(G)| - oc(G - X) + |X|)$ .

**Proof.** Let  $M$  be a matching of  $G$ . Then there are at least  $oc(G - X) - |X|$   $M$ -unsaturated vertices. So there are at most  $|V(G)| - oc(G - X) + |X|$  saturated vertices. Therefore,  $|M| \leq \frac{1}{2}(|V(G)| - oc(G - X) + |X|)$ . This holds for all  $M$ , so we get the bound on  $\nu$ .

### Corollary

If  $X \subseteq V(G)$  and  $M$  is a matching of size  $\geq \frac{1}{2}(|V(G)| - oc(G - X) + |X|)$ , then equality holds if

- every odd component of  $G - X$  has a matching  $M$  saturating all but one vertex.
- exactly  $|X|$  odd components of  $G - X$  contain a vertex matched to a vertex of  $X$ .
- every even component of  $G - X$  has a perfect matching.

### Corollary

$$\nu(G) \leq \min_{X \subseteq V(G)} \frac{1}{2}(|V(G)| - oc(G - X) + |X|)$$

### Theorem (Tutte-Berge Formula)

$$\nu(G) = \min_{X \subseteq V(G)} \frac{1}{2}(|V(G)| - oc(G - X) + |X|)$$

**Proof.** Idea: try to understand the matching number of graphs with the property that deleting any one vertex does not change the matching number.

**Definition: Hypomatchable/Factor-Critical**

A graph  $H$  is hypomatchable if  $H$  is connected and  $\nu(H - v) = \nu(H)$  for all  $v \in V(H)$ .

**Proposition**

If  $H$  is a hypomatchable graph, then  $|V(H)|$  is odd and  $\nu(H) = \frac{1}{2}(|V(H)| - 1)$ , i.e.  $H$  has a matching saturating all but one vertex.

**Proof.** Define a relation  $\sim$  on  $V(H)$  by  $u \sim v$  if and only if  $u = v$  or  $\nu(H - \{u, v\}) < \nu(H)$ .

**Lemma**

$\sim$  is symmetric and reflexive.

**Lemma**

$\sim$  is transitive.

**Proof.** Prove if  $u \sim v, v \sim w$ , then  $u \sim w$ . Suppose  $u \sim v, v \sim w, u \not\sim w$ .

Since  $H$  is hypomatchable, there is a maximum matching  $M_v$  that avoids  $v$ . Since  $u \not\sim w$ , we have  $\nu(G - \{u, w\}) = \nu(G)$  so there is a maximum matching  $M_{uw}$  avoiding both  $u$  and  $w$ .

We analyze the structure of  $M_{uw}$  and  $M_v$  to find either a larger matching (contradicting maximality) or a maximum matching avoiding  $v$  and one of  $u$  and  $w$  (contradicting  $u \sim v$  or  $v \sim w$ ).

Since no matching avoids  $v$  and one of  $u$  and  $w$ , we must have that  $M_v$  saturates  $u$  and  $w$ . Similarly,  $M_{uw}$  saturates  $v$ .

Consider the subgraph  $H_0$  of  $H$  with  $V(H_0) = V(H)$  and  $E(H_0) = E(M_{uw}) \triangle E(M_v) = (E(M_{uw}) \cup E(M_v)) \setminus (E(M_{uw}) \cap E(M_v))$  (symmetric difference). Since no vertex is incident with two edges in the same matching, the paths must alternate between edges in  $M_{uw}$  and edges in  $M_v$ .

If some path component of  $H_0$  has an odd number of edges, then it contains more edges in one matching than in the other. Let  $M$  be the matching in the path with fewer edges than the other and the larger matching be  $M'$ . Replacing the edges in  $M \cap E(P)$  with the edges in  $M' \cap E(P)$  gives a larger matching than  $M$  in  $H$ , contradicting the maximality of  $M$ . So every path of  $H_0$  has an even number of edges.

Since  $M_{uw}$  saturates  $v$  but not  $u$  or  $w$ , and  $M_v$  saturates  $u$  and  $w$  but not  $v$ , each of  $u, v, w$  has degree 1 in  $H_0$ . Since every component of  $H_0$  is a path or a cycle, each of  $u, v, w$  is an end of a path component of  $H_0$ . So there is some path component  $P$  of  $H_0$  having one end  $x \in \{u, v, w\}$  and whose other end is not in  $\{u, v, w\}$ .

Let  $M \in \{M_{uw}, M_v\}$  that saturates  $x$  and  $M'$  be the other matching. Taking  $M$ , removing the edges in  $M \cap E(P)$  and adding back the edges in  $M' \cap E(P)$  gives matching that saturates strictly fewer vertices in  $\{u, v, w\}$  than  $M$  does. In all cases ( $x = u, x = v, x = w$ ), this

contradicts either  $u \sim v$  or  $v \sim w$ .

### Lemma

If  $u, v$  are adjacent in  $H$ , then  $u \sim v$ .

**Proof.** Since  $uv$  is an edge,  $\nu(H) \geq \nu(H - \{u, v\}) + 1$  because for each matching  $M$  of  $H - \{u, v\}$ ,  $M \cup \{(u, v)\}$  is a matching of  $H$ . Therefore,  $\nu(H) > \nu(H - \{u, v\})$ .

**Proof.** (Proposition) We show that  $u \sim v$  for all  $u, v \in V(H)$ . Let  $u = x_1, x_2, \dots, x_k = v$  be a  $uv$ -path. Let  $i$  be maximal such that  $u \sim x_i$ . If  $i = k$ , we have the result. We know  $i$  makes sense, since  $u \sim u = x_1$ .

If  $i < k$ , then we know  $u \sim x_i$  and  $x_i \sim x_{i+1}$  because  $x_i, x_{i+1}$  are adjacent. So  $u \sim x_{i+1}$  by transitivity, which contradicts the maximality of  $i$ .

Now consider a maximum matching  $M$  of  $H$ . Since  $\nu(H - x) = \nu(H)$  for all  $x \in V(H)$ ,  $M$  cannot be a perfect matching. So  $2|M| \leq V(H) - 1$ .

If there are two  $M$ -unsaturated vertices  $u, v$ , then  $\nu(H - \{u, v\}) \geq |M| = \nu(H)$ , this contradicts  $u \sim v$ . Therefore,  $2|M| \geq |V(H)| - 1$ .

Therefore,  $\nu(H) = |M| = \frac{1}{2}(|V(H)| - 1)$  and  $|V(H)|$  is odd.

### Corollary

If  $H$  is any graph such that  $\nu(H - x) = \nu(H)$  for all  $x \in V(H)$ , then every component of  $H$  is odd, and  $\nu(H) = \frac{1}{2}(|V(H)| - oc(H))$ .

**Proof.** Let  $H_1, \dots, H_k$  be the components of  $H$ . We argue that each  $H_t$  is hypomatchable; let  $x \in V(H_k)$ . We have

$$\begin{aligned} \nu(H) = \nu(H - x) &= \sum_{\substack{i=1 \\ i \neq t}}^k \nu(H_i) + \nu(H_t - x) \\ &= \sum_{i=1}^k \nu(H_i) + \nu(H_t - x) - \nu(H_t) \\ &= \nu(H) + \nu(H_t - x) - \nu(H_t) \end{aligned}$$

So  $\nu(H_t - x) = \nu(H_t)$ . So  $H_t$  is hypomatchable. Each  $H_i$  has an odd number of vertices and



$\nu(H_i) = \frac{1}{2}(|V(H_i)| - 1)$ , thus

$$\begin{aligned}\nu(H) &= \sum_{i=1}^k \nu(H_i) \\ &= \sum_{i=1}^k \frac{1}{2}(|V(H_i)| - 1) \\ &= \frac{1}{2} \sum_{i=1}^k |V(H_i)| - \frac{1}{2}k \\ &= \frac{1}{2}(|V(H)| - oc(H))\end{aligned}$$

**Proof.** (Tutte-Berge) Let  $G$  be a graph and let  $X \subseteq V(G)$  be maximal such that  $\nu(G-X) = \nu(G) - |X|$  (this is well-defined because  $X = \emptyset$  satisfies this condition). If there is a vertex  $u$  in  $G-X$  such that  $\nu(G-X-u) < \nu(G-X)$ , then  $\nu(G-(X \cup \{u\})) < \nu(G-X) = \nu(G) - |X|$ , so  $\nu(G-(X \cup \{u\})) \leq \nu(G) - |X \cup \{u\}|$ . But also  $\nu(G-(X \cup \{u\})) \geq \nu(G-X) - 1 = \nu(G) - |X \cup \{u\}|$  (because deleting one vertex drops  $\nu$  by  $\leq 1$ ). So,  $\nu(G-(X \cup \{u\})) = \nu(G) - |X \cup \{u\}|$ , which contradicts the fact that  $X$  is maximal.

Therefore, we have that  $\nu((G-X)-u) = \nu(G-X)$  for all  $u \in V(G-X)$ . By the proposition, we have  $\nu(G-X) = \frac{1}{2}(|V(G-X)| - oc(G-X))$ .

So  $\nu(G) = \nu(G-X) + |X| = \frac{1}{2}(|V(G-X)| - oc(G-X)) + |X| = \frac{1}{2}(|V(G)| + |X| - oc(G-X))$ . We proved earlier that  $\nu(G) \leq \frac{1}{2}(|V(G)| + |Y| - oc(G-Y))$  for all  $Y$ . We have found some particular  $X$  where equality holds. Therefore,

$$\nu(G) = \min_{X \subseteq V(G)} \frac{1}{2}(|V(G)| + |X| - oc(G-X))$$

□

### Definition: Berge Witness

A Berge Witness is a set  $X_0 \subseteq V(G)$  such that  $\nu(G) = \frac{1}{2}(|V(G)| + |X_0| - oc(G-X_0))$ .

Our proof showed that if  $X$  is a maximal set such that  $\nu(G-X) = \nu(G) - |X|$ , then  $X$  is a Berge witness.

### Theorem (Tutte)

$G$  has a perfect matching if and only if  $oc(G-X) \leq |X|$  for all  $X \subseteq V(G)$ .

**Proof.**  $G$  has a perfect matching if and only if

$$\frac{1}{2}|V(G)| = \nu(G) = \min_{X \subseteq V(G)} \frac{1}{2}(|V(G)| + |X| - oc(G-X))$$

if and only if

$$\frac{1}{2}|V(G)| = \frac{1}{2}|V(G)| + \min_{X \subseteq V(G)} \frac{1}{2}(|X| - oc(G-X))$$

if and only if

$$\min_{X \subseteq V(G)} \frac{1}{2}(|X| - oc(G - X)) = 0$$

Since  $oc(G - X) \leq |X|$  for all  $X$ , then this minimum is clearly equal to zero (it is always nonnegative, but is also  $\leq 0$ , by choosing  $X = \emptyset$ ).

Conversely, if the minimum is zero, then clearly  $|X| - oc(G - X) \geq \min_{X \subseteq V(G)} (|X| - oc(G - X)) = 0$  for all  $X$ , so  $|X| \geq oc(G - X)$ .  $\square$

### Theorem (Petersen 1891)

If  $G$  is a 3-regular graph with no cut edge, then  $G$  has a perfect matching.

**Proof.** By Tutte's theorem, we need to show that  $oc(G - X) \leq |X|$  for all  $X \subseteq V(G)$ .

Suppose  $G$  is connected and that this fails, so there is some  $X \subseteq V(G)$  such that  $G - X$  has more than  $|X|$  odd components.

**Claim:** For each odd component  $H$  of  $G - X$ , there are at least 3 edges from  $X$  to  $H$ .

**Claim Proof.** If there is only one edge  $e$  from  $X$  to  $H$ , then  $H$  is a component of  $G - e$ , so  $e$  is a cut edge, a contradiction.

If there are two edges  $e, f$  from  $X$  to  $H$ , then  $H$  is a component of  $G - \{e, f\}$ , and in  $G - \{e, f\}$ , the degree of the vertices in  $H$  sum to  $3|V(H)| - 2$ , which is odd, contradicting handshaking theorem.

So the number of edges from  $G - X$  to  $X$  is  $\geq 3(oc(G - X)) > 3|X|$  by choice of  $X$ . But since  $G$  is 3-regular, the number of edges from  $G - X$  to  $X$  is  $\leq 3|X|$ , contradiction.  $\square$

### Proposition

For  $n \equiv 2 \pmod{4}$ , there is an  $\left(\frac{n}{2} - 1\right)$ -regular graph with no perfect matching.

**Proof.**  $K_{n/2} \oplus K_{n/2}$ .

### Proposition

For  $n$  odd, there is a graph with minimum degree  $\frac{n-1}{2}$  and no perfect matching.

**Proof.** Bipartite graph with bipartition size  $\frac{n-1}{2}$  and  $\frac{n+1}{2}$ .

### Proposition (Folklore)

If  $n$  is even and every vertex of  $G$  has degree  $\geq \frac{1}{2}|V(G)|$ , then  $G$  has a perfect matching.

**Proof.** Let  $M$  be a maximal matching. Since  $|V(M)|$  is even, if  $M$  is not a perfect matching, then there exist  $u, v \notin V(M)$ . If there is some  $xy \in M$  such that there are  $\geq 3$  edges from  $u, v$  to  $x, y$ , make  $M$  bigger. For each  $e \in M$ , there is  $\leq 2$  edges from  $u, v$  to  $e$ . So the number of edges from  $u, v$  to  $M$  is  $\leq 2|M|$ . But  $\deg(u), \deg(v) \geq \frac{n}{2}$ , so there are at least

$\geq n$  edges from  $u, v$  to  $M$ .

$$2|M| \geq \text{number of edges from } u, v \text{ to } M \geq n$$

thus,  $|M| \geq \frac{n}{2}$ , so  $M$  is perfect.