CO 450/650 Combinatorial Optimization

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List of Algorithms

Chapter 1

Introduction

Combinatorial optimization deals with problems in which we want to search for an optimal object in a finite set. Typically the set has a concise representation, but the number of objects is large.

Definition: Graph

A graph $G = (V, E)$ is a set of vertices/nodes V and a set of edges E.
We define $n = |V|$ and $m = |E|$.

Definition: Subgraph

$H = (W, F)$ of $G = (V, E)$ where $W \subseteq V$ and $F \subseteq E$.

Definition: Spanning Subgraph

H is spanning if $V(H) = V(G)$.

Definition: Path

A sequence $P = v_0, e_1, v_1, \ldots, e_k, v_k$ where $v_0, \ldots, v_k \in V(G)$, $e_1, \ldots, e_k \in E(G)$, and $e_i = v_{i-1}v_i$.

We call P a v_0v_1-path. P is called edge-simple if all e_i are distinct and simple if all v_i are distinct.

The length of P is the number of edges in P.

Definition: Circuit/Cycle

An edge-simple closed path.

Definition: Connected

A graph is connected if every pair of vertices is joined by a path.

Definition: Cut Vertex

A node v of a connected graph G where $G - v$ is not connected.

Definition: Forest

A graph with no circuits.

Definition: Tree

A connected forest.

Definition: Cut

Let $R \subseteq V$, then

$$\delta(R) = \{vw : vw \in E, v \in R, w \notin R\}$$

Definition: rs-Cut

A cut for which $r \in R, s \notin R$.

Part I

Minimum Spanning Trees

Chapter 2

Minimum Spanning Trees

2.1 Problem

Definition: Spanning Tree

A subgraph $T \subseteq G$ where $V(T) = V(G)$, T is connected, and T is acyclic.

Lemma

An edge $e = uv$ of G is an edge of a circuit of G if and only if there is a path in $G \setminus e$ from u to v.

Minimum Spanning Tree Problem (MST)

Given a connected graph G and a real cost c_e for each $e \in E$, find a minimum cost spanning tree of G.

Lemma

A spanning connected subgraph of G is a spanning tree if and only if it has exactly $n-1$ edges.

2.2 Algorithm

Theorem

A graph G is connected if and only if there is no set $A \subseteq V$ where $\emptyset \neq A \neq V$ with $\delta(A) = \emptyset$.

Algorithm 1 Kruskal's Algorithm for MST

sort E to $\{e_1, \dots, e_m\}$ so that $c_{e_1} \le \dots \le c_{e_m}$
$H = (V, F), F = \emptyset$
for $i = 1$ to m **do**
    if ends of e_i are in different components of H **then**
        $F \leftarrow F \cup \{e_i\}$
return H

2.3 Linear Programming Relaxation

Definition: $\kappa : E \to \mathbb{N}$

$\kappa(A)$

We can formulate the MST problem as an IP.

$$\begin{aligned} \min \quad & c^T x \\ \text{s.t.} \quad & x(A) \le |V| - \kappa(A), \ \forall A \subset E \\ & x(E) = |V| - 1 \\ & x_e \in \{0, 1\}, \ \forall e \in E \end{aligned}$$

We can relax the integer program to get the following linear program.

Definition: MST Linear Program

$$\begin{aligned} \min \quad & c^T x \\ \text{s.t.} \quad & x(A) \le |V| - \kappa(A), \ \forall A \subset E \\ & x(E) = |V| - 1 \\ & x_e \ge 0, \ \forall e \in E \end{aligned}$$

We replace the minimization with a maximization in the primal to write the dual.

Definition: MST Dual Linear Program

$$\begin{aligned} \min \quad & \sum_{A \subseteq E} (|V| - \kappa(A)) y_A \\ \text{s.t.} \quad & \sum_{A : e \in A} y_A \ge -c_e, \ \forall e \in E \\ & y_A \ge 0, \ \forall A \subset E \end{aligned}$$

Theorem (Edmonds 1971)

Let x^* be the characteristic vector of an MST with respect to costs c_e. Then x^* is an optimal solution of the linear program.

Proof. We show that x^* is optimal for the LP and x^* is the characteristic vector generated by Kruskal's algorithm. y_E is not required to be nonnegative.

Let $e_1, \ldots, d_m$ be the order in which Kruskal's algorithm considers the edges. Let $R_i = \{e_1, \ldots, e_i\}$ for $1 \leq i \leq m$. Let y^* be the be the dual solution. We denote $y^*_A = 0$ unless A is one of the R_i, $y^*_{R_i} = c_{e_{i+1}} - c_{e_i}$ for $1 \leq i \leq m-1$, and $y^*_{R_m} = -c_{e_m}$. It follows from the ordering of the edges, $y^*_A \geq 0$ for $A \neq E$. Now consider the first constraint, then where $e = e_i$, we have

$$\sum_{A:e\in A} y^*_A = \sum_{j=i}^{m} y^*_{R_j} = \sum_{j=i}^{m-1} (c_{e_{i+1}} - c_{e_i}) = -c_{e_i} = -c_e$$

All of the inequalities hold with equality. So the complementary slackness conditions ($x^*_e > 0 \implies \sum_{A:e\in A} y_A = c_e$) are satisfied.

We want to show now that the second constraint also satisfies complementary slackness conditions ($y^*_A > 0 \implies x(A) \leq |V| - \kappa(A)$). We know $A = R_i$ for some i. If the primal constraint does not hold with equality for R_i, then there is some edge of R_i whose addition to $E(T) \cap R_i$ would decrease the number of components of $(V, E(T) \cap R_i)$. But this edge would have ends in two different components of $(V, E(T) \cap R_i)$, and therefore would have been added to T by Kruskal's algorithm.

Therefore, x^* and y^* satisfy complementary slackness conditions. So, x^* is an optimal solution to the LP. $\square$

Part II

Network Flows

Chapter 3

Maximum Flow

3.1 Problem

Definition: Net Flow/Excess

$$f_x(v) = x(\delta(\overline{v})) - x(\delta(v)) = \sum_{w \in V, wv \in E} x_{wv} - \sum_{w \in V, vw \in E} x_{vw}$$

Definition: rs-Flow

A vector x that satisfies $f_x(v) = 0$ for all $v \in V$.

Definition: Value of rs-Flow

$$f_x(s)$$

Maximum Flow Problem

Given a digraph $G = (V, E)$, with source r and sink s, find an rs-flow of maximum value.

Proposition

There exists a family $(P_1, \ldots, P_k)$ of rs-dipaths such that $|\{i : e \in P_i\}| \leq u_e$ for all $e \in E$ if and only if there exists an integral feasible rs-flow of value k.

Proof. ($\Longrightarrow$) We have seen family of dipaths determines a corresponding flow.

($\Longleftarrow$) Let x be a flow. We assume that x is acylic, that is, there is no dicircuit C, each of whose arcs e has $x_e > 0$. If a dicircuit does exist, we can decrease x_e by 1 on all arcs of C. The new x remains feasible of value k.

If $k \geq 1$, we can find an arc vs with $x_{vs} \geq 1$. Then, if $v \neq r$, it follows that there is an arc

wv with $x_{wv} \geq 1$ by the constraint $f_x(v) = 0$. If $w \neq r$, then the argument can be repeated producing distinct vertices, since x is acyclic, so we get a simple rs-dipath P_k on each arc e with $x_e \geq 1$. We can decrease x_e by 1 for each $e \in P_k$. The new x is an integral feasible flow of value $k - 1$, and the process is repeated. $\square$

3.2 Maximum Flows and Minimum Cuts

Definition: Maximum Flow Linear Program

$$\begin{aligned} \max \quad & f_x(s) \\ \text{s.t.} \quad & f_x(v) = 0, \ \forall v \in V \setminus \{r, s\} \\ & 0 \leq x_e \leq u_e, \ \forall e \in E \end{aligned}$$

Definition: Path Flow

A vector $x \in \mathbb{R}^E$ such that for some rs-dipath P and some $\alpha \in \mathbb{R}$, $x_e = \alpha$ for each $e \in P$ and $x_e = 0$ for every other arc of G.

Definition: Circuit Flow

A vector $x \in \mathbb{R}^E$ such that for some rs-dicircuit C and some $\alpha \in \mathbb{R}$, $x_e = \alpha$ for each $e \in C$ and $x_e = 0$ for every other arc of G.

Proposition

Every rs-flow of nonnegative value is the sum of at most m flows, each of which is a path flow or a circuit flow.

Proposition

For any rs-cut $\delta(R)$ and any rs-flow x, we have

$$f_x(s) = x(\delta(R)) - x(\delta(\overline{R}))$$

Proof. We add the equations $f_x(v) = 0$ for all $v \in \overline{R} \setminus \{s\}$ as well as the identity $f_x(s) = f_x(s)$. The right hand side sums to $f_x(s)$.

For any arc vw with $v, w \in R$, x_{vw} occurs in none of the equations, so it does not occur in the sum. If $v, w \in \overline{R}$, then x_{vw} occurs in the equation for v with a coefficient of -1, and in the equation for w with a coefficient of $+1$, so it has a coefficient of 0 in the sum. If $v \in R, w \notin R$, then x_{vw} occurs in the equation for w with a coefficient of 1, and so has coefficient 1 in the sum. If $v \notin R, w \in R$, then x_{vw} occurs in the sum with a coefficient of -1. So, the left hand side sums to $x(\delta(R)) - x(\delta(\overline{R}))$, as required. $\square$

Corollary

For any feasible rs-flow x and any rs-cut $\delta(R)$,

$$f_x(s) \leq u(\delta(R))$$

Proof. Using previous proposition, since $x(\delta(R)) \leq u(\delta(R))$ and $x(\delta(\overline{R})) \geq 0$. $\square$

Definition: Incrementing Path

A path is x-incrementing if every forward arc e has $x_e < u_e$ and every reverse arc e has $x_e > 0$.

Definition: Augmenting Path

An rs-path that is x-incrementing.

Theorem Maximum-Flow Minimum-Cut

If there is a maximum rs-flow, then

$$\max\{f_x(s) : x \text{ is a feasible } rs\text{-flow}\} = \min\{u(\delta(R)) : \delta(R) \text{ is an } rs\text{-cut}\}$$

Proof. By previous corollary, we need only show that there exists a feasible flow x and a cut $\delta(R)$ such that $f_x(s) = u(\delta(R))$. Let x be a flow of maximum value. Let $R = \{v \in V :$ there exists an x-incrementing rv-path$\}$. Clearly $r \in R$ and $s \notin R$, since there can be no x-augmenting path.

For every arc $vw \in \delta(R)$, we must have $x_{vw} = u_{vw}$, since otherwise adding vw to the x-incrementing rv-path would yield such a path to w, but $w \notin R$. Similar, for every arc $vw \in \delta(\overline{R})$, we have $x_{vw} = 0$. Then by proposition, $f_x(s) = x(\delta(R)) - x(\delta(\overline{R})) = u(\delta(R))$. $\square$

Theorem

A feasible flow x is maximum if and only if there is not x-augmenting path.

Proof. ($\implies$) If x is maximum, there is no x-augmenting path.

($\impliedby$) If there is no x-augmenting path, then the construction of the proof of Max-Flow Min-Cut yields a cut $\delta(R)$ with $f_x(s) = u(\delta(R))$, so x is maximum, by corollary. $\square$

Theorem

If u is integral and there exists a maximum flow, then there exists a maximum flow that is integral.

Proof. Choose an integral flow x of maximum value. If there is an x-augmenting path, then since x and u are integral, the new flow can be chosen integral, contradicting the choice of x. Hence there is no x-augmenting path, so x is a maximum flow, by previous theorem. $\square$

Corollary

If x is a feasible rs-flow and $\delta(R)$ is an rs-cut, then x is maximum and $\delta(R)$ is minimum if and only if $x_e = u_e$ for all $e \in \delta(R)$ and $x_e = 0$ for all $e \in \delta(\overline{R})$.

Proof. Combine Max-Flow Min-Cut theorem with the proof of corollary. □

3.3 Augmenting Path Algorithm

Ford-Fulkerson Algorithm

$x = 0$
while there is an x-augmenting path P **do**
  $\varepsilon_1 = \min(u_e - x_e : e \text{ forward in } P)$
  $\varepsilon_2 = \min(x_e : e \text{ reverse in } P)$
  $\varepsilon = \min(\varepsilon_1, \varepsilon_2)$ // x-width of P
  if $\varepsilon = \infty$ **then**
    no maximum flow
return x is maximum flow, set R of vertices reachable by an x-incrementing path from r is minimum cut

Definition: Auxiliary Digraph

$G(x)$, depending on G, u, x, where $V(G(x)) = V$ and $vw \in E(G(x))$ if and only if $vw \in E$ and $x_{vw} < u_{vw}$ or $wv \in E$ and $x_{wv} > 0$.

rs-dipaths in $G(x)$ corresponding to x-augmenting paths in G. Each iteration of Ford-Fulkerson can be performed in $O(m)$ time, using breadth-first search.

Theorem

If u is integral and the maximum flow value is $K < \infty$, then the maximum flow algorithm terminates after at most K augmentations.

3.3.1 Shortest Augmenting Paths

Theorem (Dinits 1970, Edmonds & Karp 1972)

If each augmentation of the augmenting path algorithm on a shortest augmenting path, then there are at most nm augmentations.

Corollary

The augmenting path algorithm with breadth-first search solves the maximum flow problem in time $O(nm^2)$.

Let $d_x(v,w)$ be the least length of a vw-dipath in $G(x)$. $d_x(v,w)=\infty$ if no vw-dipath exists.

Consider a typical augmentation from flow x to flow x' determined by the augmenting path P having vertex-sequence $v_0,\dots,v_k$.

Lemma

For each $v \in V$, $d_{x'}(r,v) \geq d_x(r,v)$ and $d_{x'}(v,s) \geq d_x(v,s)$.

Proof. Suppose that there exists a vertex v such that $d_{x'}(r,v) < d_x(r,v)$ and choose such v so that $d_{x'}(r,v)$ is as small as possible. Clearly, $d_{x'}(r,v) > 0$. Let P' be a rv-dipath in $G(x')$ of length $d_{x'}(r,v)$ and let w be the second-last vertex of P'. Then

$$d_x(r,v) > d_{x'}(r,v) = d_{x'}(r,w) + 1 \geq d_x(r,w) + 1$$

It follows that wv is an arc of $G(x')$, but not of $G(x)$, otherwise $d_x(r,v) \leq d_x(r,w)+1$, so $w = v_i$ and $v = v_{i-1}$ for some i. But, this implies that $i-1 > i+1$, a contradiction. The second statement is similar. □

Definition: $\tilde{E}(x)$

$$\tilde{E}(x) = \{e \in E : e \text{ is an arc of a shortest } x\text{-augmenting path}\}$$

Lemma

If $d_{x'}(r,s) = d_x(r,s)$, then $\tilde{E}(x') \subsetneq \tilde{E}(x)$.

Proof. Let $k = d_x(r,s)$ and suppose that $e \in \tilde{E}(x')$. Then e induces an arc vw of $G(x')$ and $d_{x'}(r,v) = i-1$, $d_{x'}(ws) = k-i$ for some i. Therefore, $d_x(r,v)+d_x(w,s) \leq k-1$ by previous lemma. Now suppose that $e \notin \tilde{E}(x)$, then $x_e \neq x'_e$, so e is an arc of P, a contradiction. This proves $\tilde{E}(x') \subseteq \tilde{E}(x)$.

There is an arc e of P such that e is forward and $x'_e = u_e$ or e is reverse and $x'_e = 0$. Therefore, any x'-augmenting path using e must use it in the opposite direction from P, so its length, for some i, will be at least $i+k-i+1+1 = k+23$, so $e \notin \tilde{E}(x')$. □

Proof. (Dinits, Edmonds, Karp) It follows from previous lemma that there can be at most m augmentations per stage. Since there are at most $n-1$ stages, there are at most nm augmentations in all.

3.4 Applications

3.4.1 Bipartite Matchings and Vertex Covers

Theorem (König)

For a bipartite graph G,

$$\max\{|M| : M \text{ a matching}\} = \min\{|C| : C \text{ a cover}\}$$

3.4.2 Flow Feasibility

Flow Feasibility Problem

Given a digraph G, $u \in \mathbb{R}^E_+$, and $b \in \mathbb{R}^V$, find, if possible, $x \in \mathbb{R}^E$ such that

$$f_x(v) = b_v, \ \forall v \in V$$

and

$$0 \leq x_e \leq u_e, \ \forall e \in E$$

Theorem (Gale 1957)

There exists a solution to the flow feasibility problem if and only if $b(V) = 0$ and for every $A \subseteq V$, $b(A) \leq u(\delta(\overline{A}))$.
If b and u are integral, then there is an integral solution.

Corollary

Given a digraph G and $b \in \mathbb{R}^V$, there exists $x \in \mathbb{R}^E$ with

$$f_x(v) = b_v, \ \forall v \in V$$

$$x_e \geq 0, \ \forall e \in E$$

if and only if $b(V) = 0$ and for every $A \subseteq V$ with $\delta(\overline{A}) = \emptyset$, we have $b(A) \leq 0$.

Definition: Circulation

A vector $x \in \mathbb{R}^E$ with $f_x(v) = 0$ for all $v \in V$.

Theorem (Hoffman's Circulation Theorem 1960)

Given a digraph G, $\ell \in (\mathbb{R} \cup \{-\infty\})^E$, and $u \in (\mathbb{R} \cup \{\infty\})^E$, with $\ell \leq u$, there is a circulation x with $\ell \leq x \leq u$ if and only if every $A \subseteq V$ satisfies $u(\delta(\overline{A})) \geq \ell(\delta(A))$.

Part III

Matchings

Chapter 4

Matchings

Definition: Matching

A set $M \subseteq E$ such that no vertex of G is incident with more than one edge in M.

Definition: M-Covered

A vertex v is covered by M if some edge of M is incident with v.

Definition: M-Exposed

A vertex v is exposed if v is not M-covered.

The number of vertices covered by M is $2|M|$ and number of M-exposed vertices is $|V| - 2|M|$.

Definition: Maximum Matching

A matching of maximum cardinality, denoted by $\nu(G)$.

Definition: Deficiency

The minimum number of exposed vertices for any matching of G, denoted by $\operatorname{def}(G)$.

Note $\operatorname{def}(G) = |V| - 2\nu(G)$.

Definition: Perfect Matching

A matching that covers all vertices.

4.1 Alternating Paths

Definition: M-Alternating

A path P is M-alternating if its edges are alternately in and not in M.

Definition: M-Augmenting

An M-alternating path P is M-augmenting if the ends of P are distinct and are both M-exposed.

Definition: Symmetric Difference

For sets S and T, let $S \triangle T$ denote the symmetric difference, which is defined as

$$S\Delta T = (S \cup T) \setminus (S \cap T)$$

Theorem (Augmenting Path Theorem of Matchings – Berge 1957)

A matching M in a graph G is maximum if and only if there is M-augmenting path.

Proof. ($\Longrightarrow$) Suppose there exists an M-augmenting path P joining v and w. Then $N = M\Delta E(P)$ is a matching that covers all vertices covered by M, plus v and w. So, M is not maximum.

($\Longleftarrow$) Conversely, suppose that M is not maximum and some other matching N satisfies $|N| > |M|$. Let $J = N\Delta M$. Each vertex of G is incident with at most two edges of J, so J is the edge set of some vertex disjoint paths and circuits of G. For each such path or circuit, the edges alternately belong to M or N. Therefore, all circuits are even and contain the same number of edges of M and N. Since $|N| > |M|$, there must be at least one path with more edges of N than M. This path is an M-augmenting path. $\square$

4.2 Tutte-Berge Formula

Definition: Vertex Cover

A set A of vertices such that every edge has at least one end in A.

Let A be a subset of the vertices which $G - A$ has k components $H_1, \ldots, H_k$ having an odd number of vertices. Let M be a matching of G. For each i, either H_i has an M-exposed vertex or M contains an edge having just one end in $V(H_i)$. All such edges have their other ends in A and since M is a matching, all these ends must be distinct. Therefore, there can be at most $|A|$ edges and so the number of M-exposed vertices is at least $k - |A|$.

Definition: $\mathrm{oc}(H)$

The number of odd components of a graph H.

Thus, for any $A \subseteq V$,

$$\nu(G) \leq \frac{1}{2}(|V| - \mathrm{oc}(G-A) + |A|)$$

If A is a cover of G, then there are $|V|-|A|$ odd components of $G-A$ (each is a single vertex), so the right hand side reduces to $|A|$. This bound is at least as strong as that provided by covers.

Theorem (Tutte-Berge Formula)

For a graph $G = (V, E)$, we have

$$\max\{|M| : M \text{ a matching}\} = \min\{\frac{1}{2}(|V| - \mathrm{oc}(G-A) + |A|) : A \subseteq V\}$$

Theorem (Tutte's Matching Theorem 1947)

A graph $G = (V, E)$ has a perfect matching if and only if for every $A \subseteq V$, $\mathrm{oc}(G-A) \leq |A|$.

Part IV

T-Joins

Part V

Traveling Salesman Problem

Part VI

Matroids