

CO 450/650 Combinatorial Optimization

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Contents

I	Introduction	5
1	Introduction	6
2	Linear Programming	7
2.1	Farkas' Lemma	7
2.2	Duality	8
2.3	Complementary Slackness	9
3	Graph Theory	10
3.1	Undirected Graphs	10
3.2	Directed Graphs	12
4	Complexity Classes	13
II	Polyhedral Combinatorics	14
5	Integrality of Polyhedra	15
5.1	Convex Hull	15
5.2	Polytopes	16
5.3	Integral Polytopes	18
5.4	Total Unimodularity	19
5.5	Separation and Optimization	21
5.6	Total Dual Integrality	22

III	Optimal Trees and Paths	23
6	Minimum Spanning Trees	24
6.1	Problem	24
6.2	Kruskal's Algorithm	24
6.3	Linear Programming	25
6.3.1	Complementary Slackness Conditions	26
6.4	Spanning Tree Polytope	27
7	Shortest Paths	28
7.1	Ford's Algorithm	29
7.2	Linear Programming	29
IV	Network Flows	31
8	Maximum Flow	32
8.1	Problem	32
8.2	Augmenting Path Algorithm	32
8.2.1	Shortest Augmenting Paths	33
8.3	Linear Programming	34
V	Matchings	38
9	Matchings	39
9.1	Bipartite Matching	40
9.2	Alternating Paths	40
9.3	Matching LP	41
9.4	Tutte-Berge Formula	42
9.5	Maximum Matching	45
9.6	Perfect Matching	45
9.6.1	Alternating Trees	45

9.6.2	Bipartite Perfect Matching Algorithm	46
9.6.3	Blossom Algorithm for Perfect Matching	47
9.7	Blossom Algorithm for Maximum Matching	50
10	Weighted Matchings	51
10.1	Minimum-Weight Perfect Matching	51
10.2	Minimum-Weight Perfect Matching in Bipartite Graphs	52
10.3	Minimum-Weight Perfect Matching in General Graphs	52
11	T-Joins and Postman Problems	57
11.1	Postman Problem	57
11.2	T -Joins	58
11.3	Optimal T -Join Algorithm	59
11.4	T -Join LP	59
12	General Matching	61
12.1	General Matching LP	62
VI	Matroids	63
13	Matroid Theory	64
VII	Traveling Salesman Problem	65
14	The Traveling Salesman Problem	66

List of Algorithms

1	Kruskal's Algorithm for MST	25
2	Ford's Algorithm	29
3	Ford-Fulkerson Algorithm	33
4	Bipartite Perfect Matching Algorithm	47
5	Blossom Algorithm for Perfect Matching	49
6	Blossom Algorithm for Maximum Matching	50
7	Bipartite Minimum-Weight Perfect Matching Algorithm	52
8	Blossom Algorithm for Minimum-Weight Perfect Matching	56
9	Optimal T -Join Algorithm	59
10	Greedy Algorithm	64

Part I

Introduction

Chapter 1

Introduction

Definition: Combinatorial Optimization

A subfield of mathematical optimization which involves searching for an optimal object in a finite collection of objects.

Typically, the collection has a concise representation, while the number of objects is large. Objects include graphs, networks, and matroids.

The main tool in combinatorial optimization is linear programming duality.

Chapter 2

Linear Programming

Definition: Linear Programming

The problem of finding a vector x that maximizes a given linear function $c^T x$, where x ranges over all vectors satisfying a given system $Ax \leq b$ of linear inequalities.

2.1 Farkas' Lemma

Lemma (Farkas' Lemma for Inequalities)

The system $Ax \leq b$ has a solution x if and only if there is no vector y satisfying $y \geq 0$, $y^T A = 0$, and $y^T b < 0$.

Proof. Suppose $Ax \leq b$ has a solution \bar{x} and suppose there exists a vector $\bar{y} \geq 0$ satisfying $\bar{y}^T A = 0$ and $\bar{y}^T b < 0$. Then we obtain the contradiction

$$0 > \bar{y}^T b \geq \bar{y}^T (A\bar{x}) = (\bar{y}^T A)\bar{x} = 0$$

Now suppose that $Ax \leq b$ has no solution. If A has only one column, then the result is easy. Otherwise, apply Fourier-Motzkin elimination to obtain a system $A'x' \leq b'$ with one less variable. Since $A'x' \leq b'$ also has no solution, we can assume by induction that there exists a vector $y' \geq 0$ satisfying $y'^T A' = 0$ and $y'^T b' < 0$. Now since each inequality in $A'x' \leq b'$ is the sum of positive multiples of inequalities in $Ax \leq b$, we can use y' to construct a vector y satisfying the conditions in the theorem. \square

Lemma (Farkas' Lemma)

The system $Ax = b$ has a nonnegative solution if and only if there is no vector y satisfying $y^T A \geq 0$ and $y^T b < 0$.

Proof. Define

$$A' = \begin{bmatrix} A \\ -A \\ -I \end{bmatrix}, b' = \begin{bmatrix} b \\ -b \\ 0 \end{bmatrix}$$

Then $Ax = b$ has a nonnegative solution x if and only if $A'x' \leq b'$ has a solution x' . Applying Farkas' Lemma for Inequalities to $A'x' \leq b'$ gives the result. \square

Corollary

Suppose the system $Ax \leq b$ has at least one solution. Then every solution x of $Ax \leq b$ satisfies $c^T x \leq \delta$ if and only if there exists a vector $y \geq 0$ such that $y^T A = c$ and $y^T b \leq \delta$.

2.2 Duality

Consider the LP:

$$\begin{array}{ll} \max & c^T x \\ \text{s.t.} & Ax \leq b \end{array}$$

and dual LP

$$\begin{array}{ll} \min & y^T b \\ \text{s.t.} & y^T A = c^T \\ & y \geq 0 \end{array}$$

Theorem (Weak Duality)

Let A be an $m \times n$ matrix, $b \in \mathbb{R}^m$, $c \in \mathbb{R}^n$. Suppose that \bar{x} is a feasible solution to $Ax \leq b$ and \bar{y} is a feasible solution to $y \geq 0, y^T A = c^T$. Then

$$c^T \bar{x} \leq \bar{y}^T b$$

Proof.

$$c^T \bar{x} = (\bar{y}^T A) \bar{x} = \bar{y}^T (A \bar{x}) \leq \bar{y}^T b$$

\square

Theorem (Duality Theorem)

Let A be an $m \times n$ matrix, $b \in \mathbb{R}^m$, $c \in \mathbb{R}^n$, then

$$\max\{c^T x : Ax \leq b\} = \min\{y^T b : y^T A = c^T, y \geq 0\}$$

provided that both sets are nonempty.

Corollary

Let A be an $m \times n$ matrix, $b \in \mathbb{R}^m$, $c \in \mathbb{R}^n$, then

$$\max\{c^T x : Ax \leq b, x \geq 0\} = \min\{y^T b : y^T A \geq c^T\}$$

provided that both sets are nonempty.

2.3 Complementary Slackness

Consider the LP and dual LP

$$\max\{c^T x : Ax \leq b, x \geq 0\} = \min\{y^T b : y^T A \geq c^T, y \geq 0\}$$

Definition: Complementary Slackness Conditions

Suppose \bar{x}, \bar{y} are feasible solutions to the primal and dual.

- If a component of $\bar{x} > 0$, then the corresponding inequality in $y^T A \geq c^T$ is satisfied by \bar{y} with equality, i.e.

$$(\bar{y}^T A - c^T)\bar{x} = 0$$

- If a component of $\bar{y} > 0$, then the corresponding inequality in $Ax \leq b$ is satisfied by \bar{x} with equality, i.e.

$$\bar{y}^T(b - A\bar{x}) = 0$$

Theorem (Complementary Slackness Theorem)

Let x^* be a feasible solution of $\max\{c^T x : Ax \leq b, x \geq 0\}$ and let y^* be a feasible solution of $\min\{y^T b : y^T A \geq c^T, y \geq 0\}$. Then the following are equivalent:

- \bar{x}, \bar{y} are optimal solutions.
- $c^T \bar{x} = \bar{y}^T b$
- The complementary slackness conditions hold.

Proof. ((a) \iff (b)) By Duality Theorem.

((b) \iff (c)) By Weak Duality, we have $c^T \bar{x} \leq \bar{y}^T A\bar{x} \leq \bar{y}^T b$. So,

$$\begin{aligned} c^T \bar{x} = \bar{y}^T b &\iff \bar{y}^T A\bar{x} = c^T \bar{x} \text{ and } \bar{y}^T b = \bar{y}^T A\bar{x} \\ &\iff (\bar{y}^T A - c^T)\bar{x} = 0 \text{ and } \bar{y}^T(b - A\bar{x}) = 0 \end{aligned}$$

□

Chapter 3

Graph Theory

3.1 Undirected Graphs

Definition: Graph

A graph $G = (V, E)$ is a set of vertices/nodes V and a set of edges E . We define $n = |V|$ and $m = |E|$.

Definition: Degree

The degree of a vertex v of a graph G is the number of edges incident with v , denoted $\deg_G(v)$.

Definition: Subgraph

H is a subgraph of G if $E(H) \subseteq E(G)$ and $V(H) \subseteq V(G)$.

Definition: Spanning Subgraph

H is spanning if $V(H) = V(G)$.

Definition: Path

A sequence $P = v_0, e_1, v_1, \dots, e_k, v_k$ where $v_0, \dots, v_k \in V(G)$, $e_1, \dots, e_k \in E(G)$, and $e_i = v_{i-1}v_i$.

We call P a v_0v_1 -path. The length of P is the number of edges in P .

Definition: Simple Path

A path $v_0, e_1, v_1, \dots, e_k, v_k$ where all v_i are distinct.

Definition: Edge-Simple Path

A path $v_0, e_1, v_1, \dots, e_k, v_k$ where all e_i are distinct.

Definition: Closed Path

A path $v_0, e_1, v_1, \dots, e_k, v_k$ where $v_0 = v_k$.

Definition: Circuit/Cycle

An edge-simple, closed path where v_0, \dots, v_{k-1} are distinct.

Definition: Connected

A graph is connected if every pair of vertices is joined by a path.

Theorem

A graph G is connected if and only if there is no set $A \subseteq V$ where $\emptyset \neq A \neq V$ with $\delta(A) = \emptyset$.

Definition: Connected Component

A maximal connected subgraph.

Definition: Cut Vertex

A vertex v of a connected graph G where $G - v$ is not connected.

Definition: Forest

A graph with no circuits.

Definition: Tree

A connected forest.

Definition: Cut

Let $R \subseteq V$, then

$$\delta(R) = \{vw : vw \in E, v \in R, w \notin R\}$$

Definition: rs -Cut

A cut for which $r \in R, s \notin R$.

3.2 Directed Graphs

Definition: Digraph

A digraph $G = (V, E)$ is a set of vertices/nodes V and a set of edges E , sometimes called arcs, where each $e \in E$ has two ends, one called the head $h(e)$ and the other called the tail $t(e)$.

Definition: Forward Arc

In a path $v_0, e_1, v_1, \dots, e_k, v_k$, $e_i \in P$ is called forward if $t(e_i) = v_{i-1}$ and $h(e_i) = v_i$.

Definition: Backward Arc

In a path $v_0, e_1, v_1, \dots, e_k, v_k$, $e_i \in P$ is called backward if $t(e_i) = v_i$ and $h(e_i) = v_{i-1}$.

Definition: Dipath

If all arcs in a path P are forward, then P is a dipath.

Definition: Dicircuit

A dipath that is a circuit.

Definition: Directed Spanning Tree

A directed spanning tree rooted at r is a spanning tree that contains a dipath from r to each $v \in V$.

Chapter 4

Complexity Classes

Definition: Decision Problem

A problem with a yes-no answer.

Definition: \mathcal{P}

Decision problems that can be solved in polynomial time.

Definition: \mathcal{NP}

Decision problems in which we can certify the answer is yes in polynomial time.

Definition: $\text{co-}\mathcal{NP}$

Decision problems in which we can certify the answer is no in polynomial time.

A good characterization means the problem is in $\mathcal{NP} \cap \text{co-}\mathcal{NP}$.

Definition: \mathcal{NP} -Hard

A problem X is \mathcal{NP} -hard if every other problem Y in \mathcal{NP} can be reduced to X .

S. Cook (1971) proved that the satisfiability problem (SAT) is \mathcal{NP} -hard. R. Karp (1972) used Cook's result to show 21 well-known combinatorial optimization problems are also \mathcal{NP} -hard.

To show that the traveling salesman problem (TSP) is \mathcal{NP} -hard, we show that any example of SAT can be formulated as a TSP, of size polynomial in the size of SAT. Then, since Cook shows SAT is \mathcal{NP} -hard, TSP is also \mathcal{NP} -hard.

Part II

Polyhedral Combinatorics

Chapter 5

Integrality of Polyhedra

5.1 Convex Hull

Definition: Convex Combination

$x = \lambda_1 v_1 + \cdots + \lambda_k v_k$ for some vectors v_1, \dots, v_k and nonnegative scalars $\lambda_1, \dots, \lambda_k$ such that $\lambda_1 + \cdots + \lambda_k = 1$.

Definition: Convex Hull

The convex hull of a finite set S , denoted $\text{conv.hull}(S)$, is the set of all vectors that can be written as a convex combination of S .

It is also defined as the smallest convex set containing S .

Proposition

Let $S \subseteq \mathbb{R}^n$ be a finite set and let $w \in \mathbb{R}^n$. Then

$$\max / \min \{w^T x : x \in S\} = \max / \min \{w^T x : x \in \text{conv.hull}(S)\}$$

Theorem (Minkowski)

If S is finite, then $\text{conv.hull}(S)$ is a polyhedron.

$$\begin{aligned} \max \{w^T x : x \in S\} &= \max \{w^T x : x \in \text{conv.hull}(S)\} \\ &= \max \{w^T x : Ax \leq b\} \\ &= \min \{y^T b : y^T A = w^T, y \geq 0\} \end{aligned}$$

So we can use LP duality to attack combinatorial problems. If we understand $Ax \leq b$, then the problem is in $\text{co-}\mathcal{NP}$. Thus, if we have an algorithm to produce the inequalities in $Ax \leq b$ (separation), then the problem is in \mathcal{P} (Ellipsoid method).

5.2 Polytopes

Definition: Polyhedron

A set of the form $\{x : Ax \leq b\}$.

In combinatorial optimization, we typically have $x \geq 0$ as a constraint, so we have polyhedra of the form $\{x : Ax \leq b, x \geq 0\}$.

Definition: Polytope

A polyhedron $P \subseteq \mathbb{R}^n$ is a polytope if there exists $\ell, u \in \mathbb{R}^n$ such that $\ell \leq x \leq u$ for all $x \in P$.

Definition: Convex Set

Let P be a polyhedron, $x_1, x_2 \in P$, and $0 \leq \lambda \leq 1$. If $\lambda x_1 + (1 - \lambda)x_2 \in P$, then P is a convex set.

Definition: Valid Inequality

An inequality $w^T x \leq t$ is valid for a polyhedron P if $P \subseteq \{x : w^T x \leq t\}$.

Definition: Hyperplane

The solution set of $w^T x = t$ where $w \neq 0$.

Definition: Supporting Hyperplane

With respect to a polyhedron P , a hyperplane is supporting if $w^T x \leq t$ is valid for P and $P \cap \{x : w^T x = t\} \neq \emptyset$.

Definition: Face

The intersection of a polyhedron with one of its supporting hyperplanes.

The null set and the polyhedron itself is a face.

Definition: Proper Face

Faces which are not the null set or the polyhedron itself.

Proposition

A nonempty set $F \subseteq P = \{x : Ax \leq b\}$ is a face of P if and only if for some subsystem $A^\circ x \leq b^\circ$ of $Ax \leq b$, we have $F = \{x \in P : A^\circ x = b^\circ\}$.

Proof. (\implies) Suppose F is a face of P . Then there exists a valid inequality $w^T x \leq t$ such that $F = \{x \in P : w^T x = t\}$.

Consider the LP problem $\max\{w^T x : Ax \leq b\}$. The set of optimal solutions is precisely F . Now let y^* be an optimal solution to the dual problem $\min\{y^T b : y^T A = w, y \geq 0\}$ and let $A^\circ x \leq b^\circ$ be those inequalities $a_i^T x \leq b_i$ whose corresponding dual variable y_i^* is positive. By complementary slackness, we have $F = \{x : Ax \leq b, A^\circ x = b^\circ\}$ as required.

(\Leftarrow) Conversely, if $F = \{x \in P : A^\circ x = b^\circ\}$ for some subsystem $A^\circ x \leq b^\circ$ of $Ax \leq b$, then add the inequalities $A^\circ x \leq b^\circ$ to obtain an inequality $w^T x \leq t$. Every $x \in F$ satisfies $w^T x = t$ and every $x \in P \setminus F$ satisfies $w^T x < t$ as required. \square

Proposition

Let F be a minimal nonempty face of $P = \{x : Ax \leq b\}$. Then $F = \{x : A^\circ x = b^\circ\}$ for some subsystem $A^\circ x \leq b^\circ$ of $Ax \leq b$.
Moreover, the rank of the matrix A° is equal to the rank of A .

Definition: Vertex/Extreme Point

A vector $x \in P$ is called a vertex/extreme point if $\{x\}$ is a face of P .
Equivalently, $x \in P$ is a vertex/extreme point if x cannot be written as $\frac{1}{2}x_1 + \frac{1}{2}x_2$ for points $x_1, x_2 \in P, x_1 \neq x_2$.

Note: Not all polyhedra have vertices, but if $P \subseteq \mathbb{R}_+^n$, then P has vertices.

LP Fact

If a polyhedron P has vertices, then the set of optimal LP solutions contains at least one vertex of P .
Moreover, if all vertices of P are integral, then the LP always has an integral optimal solution.

Definition: Pointed Polyhedron

A polyhedron P is pointed if it has at least one vertex.

$\{(x_1, x_2) \in \mathbb{R}^2 : x_1 \geq 0\}$ is a polyhedron with no vertex.

Proposition

If a polyhedron P is pointed, then every minimal nonempty face of P is a vertex.

Proposition

Let $P = \{x : Ax \leq b\}$ and $v \in P$. Then v is a vertex of P if and only if v cannot be written as a convex combination of vectors in $P \setminus \{v\}$.

Theorem

A polytope is equal to the convex hull of its vertices.

Proof. Let P be a nonempty polytope. Since P is bounded, P must be pointed. Let

v_1, \dots, v_k be the vertices of P . Clearly, $\text{conv.hull}(\{v_1, \dots, v_k\}) \subseteq P$. So suppose there exists

$$u \in P \setminus \text{conv.hull}(\{v_1, \dots, v_k\})$$

Then by proposition, there exists an inequality $w^T x \leq t$ that separates u from

$$\text{conv.hull}(\{v_1, \dots, v_k\})$$

Let $t^* = \max\{w^T x : x \in P\}$ and consider the face $F = \{x \in P : w^T x = t^*\}$. Since $u \in P$, we have $t^* > t$. So F contains no vertex of P , a contradiction. \square

Theorem

A set P is a polytope if and only if there exists a finite set V such that P is the convex hull of V .

5.3 Integral Polytopes

Definition: Rational Polyhedron

A polyhedron that can be defined by rational linear systems.

Definition: Integral Polyhedron

A rational polyhedron where every nonempty face contains an integral vector.

Definition: Pointed Integral Polyhedron

A pointed rational polyhedron is integral if and only if all its vertices are integral.

Theorem

A rational polytope P is integral if and only if for all integral vectors w , the optimal value of $\max\{w^T x : x \in P\}$ is an integer.

Proof. To prove sufficiency, suppose that for all integral vectors w , the optimal value of $\max\{w^T x : x \in P\}$ is an integer. Let $v = (v_1, \dots, v_n)^T$ be a vertex of P and let w be an integral vector such that v is the unique optimal solution to $\max\{w^T x : x \in P\}$. By multiplying w by a large positive integer if necessary, we may assume $w^T v > w^T u + u_1 - v_1$ for all vertices u of P other than v . This implies that if we let $\bar{w} = (w_1 + 1, w_2, \dots, w_n)^T$, then v is an optimal solution to $\max\{\bar{w}^T x : x \in P\}$. So $\bar{w}^T v = w^T v + v_1$. But, by assumption, $w^T v$ and $\bar{w}^T v$ are integers. Thus, v_1 is an integer. We can repeat this for each component of v , so v must be integral. \square

5.4 Total Unimodularity

Proposition

Let A be an integral, nonsingular, $m \times n$ matrix. Then $A^{-1}b$ is integral for every integral vector $b \in \mathbb{R}^m$ if and only if $\det(A) = 1$ or -1 .

Proof. (\Leftarrow) Suppose $\det(A) = \pm 1$. By Cramer's Rule, we know that A^{-1} is integral, which implies $A^{-1}b$ is integral for every integral b .

(\Rightarrow) Conversely, suppose $A^{-1}b$ is integral for all integral vectors b . Then, in particular, $A^{-1}e_i$ is integral for all $i = 1, \dots, m$. This means that A^{-1} is integral. So $\det(A)$ and $\det(A^{-1})$ are both integers. But, $\det(A) \cdot \det(A^{-1}) = 1$, this implies $\det(A) = \pm 1$. \square

Definition: Unimodular

A matrix A of full row rank is unimodular if A is integral and each basis of A has determinant ± 1 .

Theorem (Veinott & Dantzig 1968)

Let A be an integral $m \times n$ matrix of full row rank. Then the polyhedron defined by $Ax = b, x \geq 0$ is integral for every integral vector $b \in \mathbb{R}^m$ if and only if A is unimodular.

Proof. (\Leftarrow) Suppose A is unimodular. Let $b \in \mathbb{R}^m$ be an integral vector and let \bar{x} be a vertex of $\{x : Ax = b, x \geq 0\}$. The nonnegativity constraints implies the polyhedron has vertices. Then there are n linearly independent constraints satisfied by \bar{x} with inequality. It follows that the columns of A corresponding to the nonzero components of \bar{x} are linearly independent. Extending these columns to a basis B of A , we have the nonzero components of \bar{x} are contained in the integral vector $B^{-1}b$. So \bar{x} is integral.

(\Rightarrow) Conversely, suppose $\{x : Ax = b, x \geq 0\}$ is integral for all integral vectors b . Let B be a basis of A and let v be an integral vector in \mathbb{R}^m . By previous proposition, it suffices to show that $B^{-1}v$ is integral. Let y be an integral vector such that $y + B^{-1}v \geq 0$ and let $b = B(y + B^{-1}v)$. Note b is integral. Furthermore, by adding zero components to the vector $y + B^{-1}v$, we can obtain a vector $z \in \mathbb{R}^n$ such that $Az = b$. Then, z is a vertex of $\{x : Ax = b, x \geq 0\}$, since z is a polyhedron and satisfies n linearly independent constraints with equality: the m equations $Ax = b$ and the $n - m$ equations $x_i = 0$ for the columns i outside B . So z is integral, and thus, $B^{-1}v$ is integral. \square

Definition: Totally Unimodular (TU)

A matrix is totally unimodular if all of its square submatrices have determinant 0, 1, or -1 .

It is easy to see that A is totally unimodular if and only if $\begin{bmatrix} A & I \end{bmatrix}$ is unimodular where $I \in \mathbb{R}^{m \times m}$.

Theorem (Hoffman-Kruskal)

Let A be an $m \times n$ integral matrix. Then the polyhedron defined by $Ax \leq b, x \geq 0$ is integral for every integral vector $b \in \mathbb{R}^m$ if and only if A is totally unimodular.

Proof. Applying the linear programming trick of adding slack variables, we have that for any integral b , the polyhedron $\{x : Ax \leq b, x \geq 0\}$ is integral if and only if the polyhedron $\{z : \begin{bmatrix} A & I \end{bmatrix} z = b, z \geq 0\}$ is integral. So the result follows from previous theorem. \square

Theorem

Let A be an $m \times n$ totally unimodular matrix and let $b \in \mathbb{R}^m$ be an integral vector. Then the polyhedron defined by $Ax \leq b$ is integral.

Proof. Let F be a minimal face of $\{x : Ax \leq b\}$. Then, by proposition, $F = \{x : A^\circ x = b^\circ\}$ for some subsystem $A^\circ x \leq b^\circ$ of $Ax \leq b$, with A° having full row rank. By reordering the columns, if necessary, we may write A° as $\begin{bmatrix} B & N \end{bmatrix}$ where B is a basis of A° . It follows

$$\bar{x} = \begin{bmatrix} B^{-1}b^\circ \\ 0 \end{bmatrix}$$

is an integral vector in F . \square

Theorem

Let A be a $0, \pm 1$ valued matrix where each column has at most one $+1$ and at most -1 . Then A is totally unimodular.

Proof. Let N be a $k \times k$ submatrix of A . If $k = 1$, then $\det(N)$ is either 0 or ± 1 . So we may suppose that $k \geq 2$ and proceed by induction on k . If N has a column having at most one nonzero, then expanding the determinant along this column, we have that $\det(N)$ is either 0 or ± 1 , by induction. On the other hand, if every column of N has both a $+1$ and a -1 , then the sum of the rows of N is 0 and hence $\det(N) = 0$. \square

Let $D = (V, E)$ be a digraph and let A be its incidence matrix. Then A is totally unimodular.

Definition: Network Matrix

Let $T = (V, E')$ be a spanning tree of D and define the matrix M having rows indexed by E' and columns indexed by E , where $e = (u, v) \in E$ and $e' \in E'$.

$$M_{e',e} = \begin{cases} +1 & \text{if } uv\text{-path in } T \text{ uses } e' \text{ in forward direction} \\ -1 & \text{if } uv\text{-path in } T \text{ uses } e' \text{ in backward direction} \\ 0 & \text{if } uv\text{-path in } T \text{ does not use } e' \end{cases}$$

Theorem (Tutte 1965)

Network matrices are totally unimodular.

Proposition

A is totally unimodular if and only if A^T is totally unimodular.

5.5 Separation and Optimization

Recall that the plan for polyhedral combinatorics is to formulate the problem as optimizing over a finite set of vectors S , find a linear description of $\text{conv.hull}(S)$, and apply the Duality Theorem of Linear Programming. This gives us a min-max relation for the combinatorial problem.

Separation Problem

Given a bounded rational polyhedron $P \subseteq \mathbb{R}^n$ and a rational vector $v \in \mathbb{R}^n$, either conclude that $v \in P$ or, if not, find a rational vector $w \in \mathbb{R}^n$ such that $w^T x < w^T v$ for all $x \in P$.

Optimization Problem

Given a bounded rational polyhedron $P \subseteq \mathbb{R}^n$ and a rational objective vector $w \in \mathbb{R}^n$, either find $x^* \in P$ that maximizes $w^T x$ over all $x \in P$ or conclude that P is empty.

Definition: Classes of Polyhedra

$\mathcal{P} = \{P_t : t \in \mathcal{O}\}$ where \mathcal{O} is some collection of objects and for each $t \in \mathcal{O}$, P_t is a bounded rational polyhedron.

E.g. \mathcal{O} is the collection of all graphs and P_t is the perfect matching polytope for the graph t .

Definition: Proper Class

For each object $t \in \mathcal{P}$, we can compute in polynomial time (with respect to size of t) positive integers n_t and s_t such that $P_t \subseteq \mathbb{R}^{n_t}$. and such that P_t can be described by a linear system where each inequality has size at most s_t .

Definition: Polynomially Solvable

A separation/optimization problem is polynomially solvable over the class \mathcal{P} if there exists a polynomial time algorithm to solve the problem.

Theorem (Separation \equiv Optimization)

For any proper class of polyhedra, the optimization problem is polynomially solvable if and only if the separation problem is polynomially solvable.

5.6 Total Dual Integrality

Definition: Totally Dual Integral

A rational linear system $Ax \leq b$ is totally dual integral if the minimum of

$$\max\{w^T x : Ax \leq b\} = \min\{y^T b : y^T A = w^T, y \geq 0\}$$

can be achieved by an integral vector y for each integral w for which the optima exist.

Theorem (Hoffman 1974)

Let $Ax \leq b$ be a totally dual integral system such that $P = \{x : Ax \leq b\}$ is a rational polytope and b is integral. Then P is an integral polytope.

Proof. Since b is integral, the duality equation implies $\max\{w^T x : x \in P\}$ is an integer for all integral vectors w . Thus, by theorem for integral polytopes, P is integral. \square

Theorem

Let P be a rational polyhedron. Then there exists a totally dual integral system $Ax \leq b$, with A integral, such that $P = \{x : Ax \leq b\}$. Furthermore, if P is a integral polyhedron, then b can be chosen to be integral.

Part III

Optimal Trees and Paths

Chapter 6

Minimum Spanning Trees

6.1 Problem

Definition: Spanning Tree

A subgraph $T \subseteq G$ where $V(T) = V(G)$, T is connected, and T is acyclic.

Lemma

An edge $e = uv$ of G is an edge of a circuit of G if and only if there is a path in $G \setminus e$ from u to v .

Minimum Spanning Tree Problem (MST)

Given a connected graph G and a real cost c_e for each $e \in E$, find a minimum cost spanning tree of G .

Lemma

A spanning connected subgraph of G is a spanning tree if and only if it has exactly $n - 1$ edges.

6.2 Kruskal's Algorithm

Theorem

Kruskal's algorithm finds a MST.

This is a polynomial time algorithm and is very fast in practice for sparse graphs. We can maintain F with a union-find data structure.

Algorithm 1 Kruskal's Algorithm for MST

```
1: Sort  $E$  to  $\{e_1, \dots, e_m\}$  so that  $c_{e_1} \leq \dots \leq c_{e_m}$ 
2:  $F = \emptyset, H = (V, F)$ 
3: for  $i = 1$  to  $m$  do
4:   if ends of  $e_i$  are in different components of  $H$  then
5:      $F \leftarrow F \cup \{e_i\}$ 
6: return  $H$ 
```

6.3 Linear Programming

Definition: $\kappa : E \rightarrow \mathbb{N}$

For $A \subseteq E$, $\kappa(A)$ is the number of components in the subgraph (V, A) of G .

The maximum number of tree edges in A is $|V| - \kappa(A)$.

We can formulate the MST problem as an ILP.

$$\begin{aligned} \min \quad & c^T x \\ \text{s.t.} \quad & \sum (x_e : e \in A) \leq |V| - \kappa(A), \quad \forall A \subsetneq E \\ & \sum (x_e : e \in E) = |V| - 1 \\ & x_e \in \{0, 1\}, \quad \forall e \in E \end{aligned}$$

We can relax the integer program to get the following linear program.

Definition: MST LP

$$\begin{aligned} \min \quad & c^T x \\ \text{s.t.} \quad & x(A) \leq |V| - \kappa(A), \quad \forall A \subsetneq E \\ & x(E) = |V| - 1 \\ & x_e \geq 0, \quad \forall e \in E \end{aligned}$$

We replace the minimization with a maximization in the primal to write the dual.

Definition: MST Dual LP

$$\begin{aligned} \min \quad & \sum ((|V| - \kappa(A))y_A : A \subseteq E) \\ \text{s.t.} \quad & \sum (y_A : e \in A) \geq -c_e, \quad \forall e \in E \\ & y_A \geq 0, \quad \forall A \subsetneq E \end{aligned}$$

6.3.1 Complementary Slackness Conditions

Let T be a tree found by Kruskal's algorithm. Define the characteristic vector of T

$$x_e^0 = \begin{cases} 1 & \text{if } e \in E(T) \\ 0 & \text{if } e \notin E(T) \end{cases}$$

Definition: MST Complementary Slackness Conditions

- (i) For all $e \in E$, if $x_e^0 > 0$, then $\sum(y_A^0 : e \in A) = -c_e$.
- (ii) For all $A \subsetneq E$, if $y_A^0 > 0$, then $\sum(x_e^0 : e \in A) = |V| - \kappa(A)$.

Theorem (Edmonds 1971)

Let x^0 be the characteristic vector of an MST with respect to costs c_e . Then x^0 is an optimal solution to the MST LP.

Proof. We show that x^0 is optimal for the LP and x^0 is the characteristic vector generated by Kruskal's algorithm.

Let e_1, \dots, e_m be the order in which Kruskal's algorithm considers the edges. Let $R_i = \{e_1, \dots, e_i\}$ for $1 \leq i \leq m$. Let y^0 be the dual solution.

- $y_A^0 = 0$ if A is not one of the R_i 's.
- $y_{R_i}^0 = c_{e_{i+1}} - c_{e_i}$ for $1 \leq i \leq m-1$.
- $y_{R_m}^0 = -c_{e_m}$

It follows from the ordering of the edges, $y_A^0 \geq 0$ for $A \neq E$. Now consider the first constraint, then where $e = e_i$, we have

$$\sum(y_A^0 : e \in A) = \sum_{j=i}^m y_{R_j}^0 = \sum_{j=i}^{m-1} (c_{e_{j+1}} - c_{e_j}) - c_{e_m} = -c_{e_i} = -c_e$$

All of the inequalities hold with equality. So y^0 is a feasible dual solution and complementary slackness condition (i) holds.

Now suppose $y_A^0 > 0$ for some $A \subsetneq E$. Thus, $A = R_i$ for some i . Consider the constraint

$$\sum(x_e^0 : e \in R_i) \leq |V| - \kappa(R_i)$$

If this does not hold with equality, then there is some edge of R_i having ends in two different components of $(V, R_i \cap T)$ and this would have been added to T by Kruskal's algorithm. So (x^0, y^0) satisfy the complementary slackness conditions, which means they are optimal solutions to their LPs. Therefore, T is a MST. \square

6.4 Spanning Tree Polytope

Definition: Spanning Tree Polytope

$\text{conv.hull}\{x^H : H \text{ is a spanning tree}\}$ where

$$x_e^H = \begin{cases} 1 & \text{if } e \in E(H) \\ 0 & \text{if } e \notin E(H) \end{cases}$$

Theorem

The spanning tree polytope is the solution set to the following linear system:

$$\begin{aligned} \sum(x_e : e \in A) &\leq |V| - \kappa(A), \quad \forall A \subsetneq E \\ \sum(x_e : e \in E) &= |V| - 1 \\ x_e &\geq 0, \quad \forall e \in E \end{aligned}$$

Proof. Let P be the solution set of the linear system. We showed that for any edge costs $(c_e : e \in E)$, the LP

$$\max\{\sum(-c_e x_e : e \in E) : x \in P\}$$

has an integral optimal solution. So every vertex of P is integral. \square

Chapter 7

Shortest Paths

Shortest Path Problem

Given a digraph G , a vertex $r \in V$, and a real cost vector $(c_e : e \in E)$, find for each $v \in V$, a minimum-cost dipath from r to v .

Note: You can provide a solution to the shortest path problem for r by giving a directed spanning tree rooted at r .

Proof. For each $v \in V \setminus \{r\}$, all shortest paths have at most one arc having head v_j , since the only such arc we need is the last arc of one min-cost rv -dipath.

So the union of the arc sets of all the shortest paths has exactly $|V| - 1$ arcs and thus is a tree. \square

Important Case

If $c_e \geq 0$ for all $e \in E$, then this problem is handled by Dijkstra's algorithm, which starts at r and grows the tree vertex by vertex.

However, the Hamiltonian dipath problem (does G have a simple dipath P with $V(P) = V(G)$) is \mathcal{NP} -hard.

When G has negative-cost dicircuits, this is a problem, since there is no shortest path as we can loop around the dicircuit an infinite amount of times. There do exist polynomial time algorithms that either finds a shortest path or detect a negative-cost dicircuit.

Definition: Feasible Potential

$y = (y_v : v \in V)$ is a feasible potential if it satisfies $y_v + c_{vw} \geq y_w$ for all $vw \in E$.

Proposition

Let y be a feasible potential and let P be an rs -dipath. Then $c(P) \geq y_s - y_r$.

Proof. Suppose that P is $v_0, e_1, v_1, \dots, e_k, v_k$ where $v_0 = r$ and $v_k = s$. Then

$$c(P) = \sum_{i=1}^k c_{e_i} \geq \sum_{i=1}^k (y_{v_i} - y_{v_{i-1}}) = y_{v_k} - y_{v_0} = y_s - y_r$$

□

So a potential y provides a stopping rule. A dipath P and a potential y with $c(P) = y_s - y_r$ implies P is optimal.

7.1 Ford's Algorithm

Definition: Incorrect

Given vertex values $(y_v : v \in V)$, the edge vw is incorrect if $y_v + c_{vw} < y_w$.

To correct vw , we set $y_w = y_v + c_{vw}$ and predecessor(w) = v .

Algorithm 2 Ford's Algorithm

- 1: $y_r = 0, y_v = \infty$ for all $v \in V \setminus \{r\}$
 - 2: predecessor(r) = \emptyset , predecessor(v) = -1 for all $v \in V \setminus \{r\}$
 - 3: **while** y is not a feasible potential **do**
 - 4: Find an incorrect arc vw and correct vw
-

Theorem

If there are no negative-cost dicircuits, then Ford's algorithm terminates in a finite number of steps.

At termination, for each $v \in V$, the predecessors define a shortest rv -dipath of cost y_v .

Specialized versions like Ford-Bellman run in polynomial time.

7.2 Linear Programming

Definition: Shortest Path LP

$$\begin{array}{ll} \max & y_s - y_r \\ \text{s.t.} & y_w - y_v \leq c_{vw}, \forall vw \in E \end{array}$$

Definition: Shortest Path Dual LP

$$\begin{aligned}
& \min \quad \sum (c_e x_e : e \in E) \\
& \text{s.t.} \quad \sum (x_{wv} : wv \in E) - \sum (x_{vw} : vw \in E) = \begin{cases} 0 & \text{if } v \in V \setminus \{r, s\} \\ -1 & \text{if } v = r \\ 1 & \text{if } v = s \end{cases} \\
& \quad \quad x_{vw} \geq 0, \forall vw \in E
\end{aligned}$$

Any rs -dipath is a solution to the dual LP, so if the dual LP has an optimal solution, then it has an optimal solution that is an rs -dipath.

The constraint matrix for the LP is totally unimodular.

Theorem

Let G be a digraph, $r, s \in V$, and $c \in \mathbb{R}^E$. If there exists a minimum-cost dipath from r to v for every $v \in V$, then

$$\min\{c(P) : P \text{ an } rs\text{-dipath}\} = \max\{y_s : y \text{ a feasible potential}\}$$

The vertices of the polyhedron defined by the dual LP constraints are the vectors x^P of simple dipaths.

$$x_e^P = \begin{cases} 1 & \text{if } e \in E(P) \\ 0 & \text{if } e \notin E(P) \end{cases}$$

Note: This is *not* the convex hull of simple dipaths.

Since the matrix is totally unimodular, we could add $x_{vw} \leq 1$ for all $vw \in E$, but this will not give simple dipaths.

Part IV

Network Flows

Chapter 8

Maximum Flow

8.1 Problem

Definition: Net Flow/Excess

$$f_x(v) = x(\delta(\bar{v})) - x(\delta(v)) = \sum(x_{wv} : w \in V, wv \in E) - \sum(x_{vw} : w \in V, vw \in E)$$

Definition: rs -Flow

A vector x that satisfies $f_x(v) = 0$ for all $v \in V$.

Definition: Value of rs -Flow

$$f_x(s)$$

Maximum Flow Problem

Given a digraph $G = (V, E)$, with source r and sink s , find an rs -flow of maximum value.

8.2 Augmenting Path Algorithm

Definition: Augmenting Path

An rs -path P is x -augmenting if for all forward arcs e we have $x_e < u_e$, and for all reverse arcs e we have $x_e > 0$.

Algorithm 3 Ford-Fulkerson Algorithm

```
1:  $x = 0$ 
2: while there is an  $x$ -augmenting path  $P$  do
3:    $\varepsilon_1 = \min(u_e - x_e : e \text{ forward in } P)$ 
4:    $\varepsilon_2 = \min(x_e : e \text{ reverse in } P)$ 
5:    $\varepsilon = \min(\varepsilon_1, \varepsilon_2)$  //  $x$ -width of  $P$ 
6:   if  $\varepsilon = \infty$  then
7:     No maximum flow
8: return  $x$  is maximum flow, set  $R$  of vertices reachable by an  $x$ -augmenting path from  $r$ 
    is minimum cut
```

Definition: Auxiliary Digraph

$G(x)$, depending on G, u, x , where $V(G(x)) = V$ and $vw \in E(G(x))$ if and only if $vw \in E$ and $x_{vw} < u_{vw}$ or $wv \in E$ and $x_{wv} > 0$.

rs -dipaths in $G(x)$ corresponding to x -augmenting paths in G . Each iteration of Ford-Fulkerson can be performed in $O(m)$ time, using breadth-first search.

Theorem

If u is integral and the maximum flow value is $K < \infty$, then the maximum flow algorithm terminates after at most K augmentations.

8.2.1 Shortest Augmenting Paths

Theorem (Dinitz 1970, Edmonds & Karp 1972)

If each augmentation of the augmenting path algorithm on a shortest augmenting path, then there are at most nm augmentations.

Corollary

The augmenting path algorithm with breadth-first search solves the maximum flow problem in time $O(nm^2)$.

Let $d_x(v, w)$ be the least length of a vw -dipath in $G(x)$. $d_x(v, w) = \infty$ if no vw -dipath exists.

Consider a typical augmentation from flow x to flow x' determined by the augmenting path P having vertex-sequence v_0, \dots, v_k .

Lemma

For each $v \in V$, $d_{x'}(r, v) \geq d_x(r, v)$ and $d_{x'}(v, s) \geq d_x(v, s)$.

Proof. Suppose that there exists a vertex v such that $d_{x'}(r, v) < d_x(r, v)$ and choose such v so that $d_{x'}(r, v)$ is as small as possible. Clearly, $d_{x'}(r, v) > 0$. Let P' be a rv -dipath in $G(x')$

of length $d_{x'}(r, v)$ and let w be the second-last vertex of P' . Then

$$d_x(r, v) > d_{x'}(r, v) = d_{x'}(r, w) + 1 \geq d_x(r, w) + 1$$

It follows that wv is an arc of $G(x')$, but not of $G(x)$, otherwise $d_x(r, v) \leq d_x(r, w) + 1$, so $w = v_i$ and $v = v_{i-1}$ for some i . But, this implies that $i - 1 > i + 1$, a contradiction. The second statement is similar. \square

Definition: $\tilde{E}(x)$

$$\tilde{E}(x) = \{e \in E : e \text{ is an arc of a shortest } x\text{-augmenting path}\}$$

Lemma

If $d_{x'}(r, s) = d_x(r, s)$, then $\tilde{E}(x') \subsetneq \tilde{E}(x)$.

Proof. Let $k = d_x(r, s)$ and suppose that $e \in \tilde{E}(x')$. Then e induces an arc vw of $G(x')$ and $d_{x'}(r, v) = i - 1$, $d_{x'}(ws) = k - i$ for some i . Therefore, $d_x(r, v) + d_x(w, s) \leq k - 1$ by previous lemma. Now suppose that $e \notin \tilde{E}(x)$, then $x_e \neq x'_e$, so e is an arc of P , a contradiction. This proves $\tilde{E}(x') \subseteq \tilde{E}(x)$.

There is an arc e of P such that e is forward and $x'_e = u_e$ or e is reverse and $x'_e = 0$. Therefore, any x' -augmenting path using e must use it in the opposite direction from P , so its length, for some i , will be at least $i + k - i + 1 + 1 = k + 23$, so $e \notin \tilde{E}(x')$. \square

Proof. (Dinitz, Edmonds, Karp) It follows from previous lemma that there can be at most m augmentations per stage. Since there are at most $n - 1$ stages, there are at most nm augmentations in all. \square

8.3 Linear Programming

Definition: Maximum Flow LP

$$\begin{aligned} \max \quad & f_x(s) \\ \text{s.t.} \quad & f_x(v) = 0, \forall v \in V \setminus \{r, s\} \\ & 0 \leq x_e \leq u_e, \forall e \in E \end{aligned}$$

We give a different LP approach to this problem.

Definition: Minimum Cut LP

$$\begin{aligned} \min \quad & \sum (u_e y_e : e \in E) \\ \text{s.t.} \quad & \sum (y_e : e \in E(P)) \geq 1, \forall \text{ } rs\text{-simple dipaths } P \\ & y_e \geq 0, \forall e \in E \end{aligned}$$

Every rs -cut $\delta(R)$ gives a feasible solution

$$y_e^R = \begin{cases} 1 & \text{if } e \in \delta(R) \\ 0 & \text{if } e \notin \delta(R) \end{cases}$$

Definition: Maximum Flow LP (Dual Minimum Cut LP)

$$\begin{aligned} \max \quad & \sum (w_P : P \text{ a simple } rs\text{-dipath}) \\ \text{s.t.} \quad & \sum (w_P : e \in E(P)) \leq u_e, \forall e \in E \\ & w_P \geq 0, \forall \text{ simple dipaths } P \end{aligned}$$

Let x be a max flow. We want to find a simple rs -dipath P such that $x_e > 0$ for each $e \in E(P)$. Set $w_P = \min\{x_e : e \in E(P)\}$ and let $x_e = x_e - w_P$ for all $e \in E(P)$. We repeat until $\sum (x_{rv} : rv \in E) = 0$.

$\sum (w_P : P \text{ a simple } rs\text{-dipath})$ is equal to the original value of the flow. Therefore, the max flow equals the min cut which implies the two LPs have integral optimal solutions if u_e is integral for all $e \in E$.

Proposition

There exists a family (P_1, \dots, P_k) of rs -dipaths such that $|\{i : e \in P_i\}| \leq u_e$ for all $e \in E$ if and only if there exists an integral feasible rs -flow of value k .

Proof. (\implies) We have seen family of dipaths determines a corresponding flow.

(\impliedby) Let x be a flow. We assume that x is acyclic, that is, there is no dicircuit C , each of whose arcs e has $x_e > 0$. If a dicircuit does exist, we can decrease x_e by 1 on all arcs of C . The new x remains feasible of value k .

If $k \geq 1$, we can find an arc vs with $x_{vs} \geq 1$. Then, if $v \neq r$, it follows that there is an arc wv with $x_{wv} \geq 1$ by the constraint $f_x(v) = 0$. If $w \neq r$, then the argument can be repeated producing distinct vertices, since x is acyclic, so we get a simple rs -dipath P_k on each arc e with $x_e \geq 1$. We can decrease x_e by 1 for each $e \in P_k$. The new x is an integral feasible flow of value $k - 1$, and the process is repeated. \square

Definition: Path Flow

A vector $x \in \mathbb{R}^E$ such that for some rs -dipath P and some $\alpha \in \mathbb{R}$, $x_e = \alpha$ for each $e \in P$ and $x_e = 0$ for every other arc of G .

Definition: Circuit Flow

A vector $x \in \mathbb{R}^E$ such that for some rs -dicircuit C and some $\alpha \in \mathbb{R}$, $x_e = \alpha$ for each $e \in C$ and $x_e = 0$ for every other arc of G .

Proposition

Every rs -flow of nonnegative value is the sum of at most m flows, each of which is a path flow or a circuit flow.

Proposition

For any rs -cut $\delta(R)$ and any rs -flow x , we have

$$f_x(s) = x(\delta(R)) - x(\delta(\bar{R}))$$

Proof. We add the equations $f_x(v) = 0$ for all $v \in \bar{R} \setminus \{s\}$ as well as the identity $f_x(s) = f_x(s)$. The right hand side sums to $f_x(s)$.

For any arc vw with $v, w \in R$, x_{vw} occurs in none of the equations, so it does not occur in the sum. If $v, w \in \bar{R}$, then x_{vw} occurs in the equation for v with a coefficient of -1 , and in the equation for w with a coefficient of $+1$, so it has a coefficient of 0 in the sum. If $v \in R, w \notin R$, then x_{vw} occurs in the equation for w with a coefficient of 1 , and so has coefficient 1 in the sum. If $v \notin R, w \in R$, then x_{vw} occurs in the sum with a coefficient of -1 . So, the left hand side sums to $x(\delta(R)) - x(\delta(\bar{R}))$, as required. \square

Corollary

For any feasible rs -flow x and any rs -cut $\delta(R)$,

$$f_x(s) \leq u(\delta(R))$$

Proof. Using previous proposition, since $x(\delta(R)) \leq u(\delta(R))$ and $x(\delta(\bar{R})) \geq 0$. \square

Theorem (Max-Flow Min-Cut)

If there is a maximum rs -flow, then

$$\max\{f_x(s) : x \text{ is a feasible } rs\text{-flow}\} = \min\{u(\delta(R)) : \delta(R) \text{ is an } rs\text{-cut}\}$$

Proof. By previous corollary, we need only show that there exists a feasible flow x and a cut $\delta(R)$ such that $f_x(s) = u(\delta(R))$. Let x be a flow of maximum value. Let $R = \{v \in V : \text{there exists an } x\text{-augmenting } rv\text{-path}\}$. Clearly $r \in R$ and $s \notin R$, since there can be no x -augmenting path.

For every arc $vw \in \delta(R)$, we must have $x_{vw} = u_{vw}$, since otherwise adding vw to the x -augmenting rv -path would yield such a path to w , but $w \notin R$. Similar, for every arc $vw \in \delta(\bar{R})$, we have $x_{vw} = 0$. Then by proposition, $f_x(s) = x(\delta(R)) - x(\delta(\bar{R})) = u(\delta(R))$. \square

Theorem

A feasible flow x is maximum if and only if there is not x -augmenting path.

Proof. (\implies) If x is maximum, there is no x -augmenting path.

(\Leftarrow) If there is no x -augmenting path, then the construction of the proof of Max-Flow Min-Cut yields a cut $\delta(R)$ with $f_x(s) = u(\delta(R))$, so x is maximum, by corollary. \square

Theorem

If u is integral and there exists a maximum flow, then there exists a maximum flow that is integral.

Proof. Choose an integral flow x of maximum value. If there is an x -augmenting path, then since x and u are integral, the new flow can be chosen integral, contradicting the choice of x . Hence there is no x -augmenting path, so x is a maximum flow, by previous theorem. \square

Corollary

If x is a feasible rs -flow and $\delta(R)$ is an rs -cut, then x is maximum and $\delta(R)$ is minimum if and only if $x_e = u_e$ for all $e \in \delta(R)$ and $x_e = 0$ for all $e \in \delta(\overline{R})$.

Proof. Combine Max-Flow Min-Cut theorem with the proof of corollary. \square

Part V

Matchings

Chapter 9

Matchings

Definition: Matching

A set $M \subseteq E$ such that no vertex of G is incident with more than one edge in M .

Definition: M -Covered

A vertex v is covered by M if some edge of M is incident with v .

Definition: M -Exposed

A vertex v is exposed if v is not M -covered.

The number of vertices covered by M is $2|M|$ and number of M -exposed vertices is $|V| - 2|M|$.

Definition: Maximum Matching

A matching of maximum cardinality, denoted $\nu(G)$.

Definition: Deficiency

The minimum number of exposed vertices for any matching of G , denoted by $\text{def}(G)$.

Note $\text{def}(G) = |V| - 2\nu(G)$.

Definition: Perfect Matching

A matching that covers all vertices.

9.1 Bipartite Matching

Definition: Bipartite

$G = (V, E)$ is bipartite if $V = V_1 \cup V_2$, where V_1, V_2 disjoint and every edge has one end in V_1 and the other end in V_2 .

Definition: Vertex Cover

A set $C \subseteq V$ such that every edge has at least one in C .

Lemma

If M is a matching and C is a cover, then $|M| \leq |C|$.

Proof. Every $e \in M$ has at least one end in C . No vertex in C meets more than one edge in M . \square

Definition: Minimum Cover

A cover of minimum cardinality, denoted $\tau(G)$.

Theorem (König)

If G is bipartite, $\nu(G) = \tau(G)$.

Proof. We note that $\nu(G) \leq \nu^*(G)$ and $\tau(G) \geq \tau^*(G)$. By using LP duality and the matching LP (*Matching LP*), we show that $\nu(G) = \nu^*(G)$. We also have the matching LP in the form of $Mx^+ = (1, \dots, 1)^T$. Since M is totally unimodular, then M^T is also totally unimodular. So the dual LP has all integral vertices, implying $\tau(G) = \tau^*(G)$. So,

$$\nu(G) = \nu^*(G) = \tau^*(G) = \tau(G)$$

\square

9.2 Alternating Paths

Definition: M -Alternating

A path P is M -alternating if its edges are alternately in and not in M .

Definition: M -Augmenting

An M -alternating path P is M -augmenting if the ends of P are distinct and are both M -exposed.

Definition: Symmetric Difference

For sets S and T , let $S\Delta T$ denote the symmetric difference, which is defined as

$$S\Delta T = (S \cup T) \setminus (S \cap T)$$

Let a path P be an M -augmenting path. Then we can obtain a larger matching $M' = M\Delta E(P)$ with $|M'| = |M| + 1$.

Theorem (Petersen 1891, Berge 1957)

A matching M in a graph G is maximum if and only if there is no M -augmenting path.

Proof. (\implies) Suppose there exists an M -augmenting path P joining v and w . Then $N = M\Delta E(P)$ is a matching that covers all vertices covered by M , plus v and w . So, M is not maximum.

(\impliedby) Conversely, suppose that M is not maximum and some other matching N satisfies $|N| > |M|$. Let $J = N\Delta M$. Each vertex of G is incident with at most two edges of J , so J is the edge set of some vertex disjoint paths and circuits of G . For each such path or circuit, the edges alternately belong to M or N . Therefore, all circuits are even and contain the same number of edges of M and N . Since $|N| > |M|$, there must be at least one path with more edges of N than M . This path is an M -augmenting path. \square

9.3 Matching LP

Definition: Matching LP

P is the set of solutions to

$$\begin{aligned} x(\delta(v)) &\leq 1, \forall v \in V \\ x_e &\geq 0, \forall e \in E \end{aligned}$$

Let \bar{x} be a vertex of P . We show that \bar{x} is integral, which implies that $M = \{e \in E : \bar{x}_e = 1\}$ is a matching and $\nu(G) = \nu^*(G)$.

Recall that for a polyhedron $P = \{x : Ax \leq b\} \subseteq \mathbb{R}^n$, $\bar{x} \in P$ is a vertex if and only if \bar{x} is the unique solution to $A'x = b'$ for some subset of n inequalities $A'x \leq b'$ from $Ax \leq b$.

For our matching P , let $E^+ := \{e : \bar{x}_e > 0\}$ and $E^0 := \{e : \bar{x}_e = 0\}$. We write $\bar{x} = (\bar{x}^+, \bar{x}^0)$ split by (E^+, E^0) .

Since \bar{x} is a vertex, there exists $V^+ \subseteq V$ such that \bar{x} is the unique solution to

$$\begin{aligned} \sum (x_e : e \in \delta(v) \cap E^+) &= 1, \forall v \in V^+ \\ x_e &= 0, \forall e \in E^0 \end{aligned}$$

Restricting to E^+ , we can write the system of equations as

$$Mx^+ = (1, \dots, 1)^T$$

By Cramer's Rule, the solution to the system is $(\bar{x}_1^+, \dots, \bar{x}_k^+)$, where

$$\bar{x}_j^+ = \frac{\det(M^j)}{\det(M)}$$

with M^j obtained from M by replacing the j th column by $(1, \dots, 1)^T$.

Claim: $\det(M) = 1$ or $\det(M) = -1$.

This gives that \bar{x}_j^+ is integer for all j , so \bar{x} is integer. Thus, $\nu(G) = \nu^*(G)$.

Lemma

Let $G = (V, E)$ be a bipartite graph. Let A be the $|V| \times |E|$ matrix $[A_{ve}]$ with

$$A_{ve} = \begin{cases} 1 & \text{if } e \in \delta(v) \\ 0 & \text{if } e \notin \delta(v) \end{cases}$$

then A is totally unimodular.

Proof. By induction of the number of rows k of the submatrix B of A . If B is 1×1 , then this is obvious.

Suppose it is true for $k = 1, \dots, t-1$ and let B be a $t \times t$ submatrix of A .

1. If B has a column of all 0's, then $\det(B) = 0$.
2. If a column of B has exactly one 1, then we compute $\det(B)$ by expanding on that column and use induction.
3. Otherwise, every column of B has exactly two 1's.

We can partition the rows of B into W_1 and W_2 , so that every column has exactly one 1 in W_1 and exactly one 1 in W_2 (W_1 are vertices in V_1 , W_2 in V_2 from G being bipartite).

Now multiplying each row in W_1 by 1 and each row in W_2 by -1 and summing, we get the row vector of all 0's. So $\det(B) = 0$. \square

9.4 Tutte-Berge Formula

Let A be a subset of the vertices which $G - A$ has k components H_1, \dots, H_k having an odd number of vertices. Let M be a matching of G . For each i , either H_i has an M -exposed vertex or M contains an edge having just one end in $V(H_i)$. All such edges have their other

ends in A and since M is a matching, all these ends must be distinct. Therefore, there can be at most $|A|$ edges and so the number of M -exposed vertices is at least $k - |A|$.

Definition: $\text{oc}(H)$

The number of odd components of a graph H .

Thus, for any $A \subseteq V$,

$$\nu(G) \leq \frac{1}{2}(|V| - \text{oc}(G - A) + |A|)$$

If A is a cover of G , then there are $|V| - |A|$ odd components of $G - A$ (each is a single vertex), so the right hand side reduces to $|A|$. This bound is at least as strong as that provided by covers.

Theorem (Tutte-Berge Formula)

For a graph $G = (V, E)$, we have

$$\max\{|M| : M \text{ a matching}\} = \min \left\{ \frac{1}{2}(|V| - \text{oc}(G - A) + |A|) : A \subseteq V \right\}$$

Theorem (Tutte's Matching Theorem 1947)

A graph $G = (V, E)$ has a perfect matching if and only if for all $A \subseteq V$, $\text{oc}(G - A) \leq |A|$.

Definition: Shrink

Let C be an odd circuit in G . Define $G' = G \times C$ as the subgraph obtained from G by shrinking C ; G' has vertex set $(V - V(C)) \cup \{C\}$ and edge set $E \setminus \gamma(V(C))$.

Proposition

Let C be an odd circuit of G , let $G' = G \times C$, and let M' be a matching of G' . Then here is a matching M of G such that $M \subseteq M' \cup E(C)$ and the number of M -exposed vertices of G is the same as the number of M' -exposed vertices of G' .

Proof. Choose a vertex $w \in V(C)$ as follows. If C is covered by $e \in M'$, then choose w to be the vertex in $V(C)$ that is an end of e , and otherwise, choose w arbitrarily. Deleting w from C results in a subgraph having a perfect matching M'' . Take $M = M' \cup M''$. M has the required properties. \square

The previous proposition gives the inequality

$$\nu(G) \geq \nu(G \times C) + \frac{|V(C)| - 1}{2}$$

or equivalently,

$$\text{def}(G) \leq \text{def}(G \times C)$$

Definition: Tight Odd Circuit

An odd circuit C is tight if $\nu(G) = \nu(G \times C) + \frac{|V(C)|-1}{2}$.

Definition: Inessential

A vertex v of G is inessential if there is a maximum matching of G that does not cover v .

Definition: Essential

A vertex not inessential.

Let A be a set that satisfies the Tutte-Berge formula. Let $v \in A$ and consider $G' = G - v$. Then, $G' - (A \setminus \{v\})$ has the same odd components as $G - A$, so $\nu(G') < \nu(G)$, i.e. every $v \in A$ is essential.

Lemma

Let $G = (V, E)$ be a graph and let $vw \in E$. If v, w are both inessential, then there is a tight odd circuit C using vw . Moreover, C is an inessential vertex of $G \times C$.

Definition: Gallai-Edmonds Partition

Let B be the set of inessential vertices of $G = (V, E)$, C be the set of vertices not in B but adjacent to at least one element of B , and D be $V \setminus (B \cup C)$, then (B, C, D) is the Gallai-Edmonds Partition of G .

Proposition

Let (B, C, D) be the Gallai-Edmonds Partition for G .
 C is a minimizer in the Tutte-Berge formula.

Proposition

Let (B, C, D) be the Gallai-Edmonds Partition for G .
 For every maximum matching M and every vertex $v \in C$, there is an edge $vw \in M$ with $w \in B$.

Proposition

Let (B, C, D) be the Gallai-Edmonds Partition for G .
 Every maximum matching contains a perfect matching of $G[D]$.

9.5 Maximum Matching

Maximum Matching Problem

Given a graph G , find a maximum matching of G .

Definition: Maximum Matching ILP

$$\begin{aligned} \max \quad & \sum (x_e : e \in E) \\ \text{s.t.} \quad & x(\delta(v)) \leq 1, \forall v \in V \\ & x_e \geq 0, \forall e \in E \\ & x_e \text{ integer}, \forall e \in E \end{aligned}$$

Definition: Maximum Matching LP Relaxation

$$\begin{aligned} \max \quad & \sum (x_e : e \in E) \\ \text{s.t.} \quad & x(\delta(v)) \leq 1, \forall v \in V \\ & x_e \geq 0, \forall e \in E \end{aligned}$$

Definition: Minimum Cover Dual LP

$$\begin{aligned} \min \quad & \sum (y_v : v \in V) \\ \text{s.t.} \quad & y_u + y_v \geq 1, \forall e = (u, v) \in E \\ & y_v \geq 0, \forall v \in V \end{aligned}$$

Let M be a matching and C be a cover, then

$$x_e^M = \begin{cases} 1 & \text{if } e \in M \\ 0 & \text{if } e \notin M \end{cases}, y_v^C = \begin{cases} 1 & \text{if } v \in C \\ 0 & \text{if } v \notin C \end{cases}$$

So, $\nu(G) \leq \nu^*(G)$ and $\tau(G) \geq \tau^*(G)$, and by LP duality, we have

$$\nu(G) \leq \nu^*(G) = \tau^*(G) \leq \tau(G)$$

9.6 Perfect Matching

9.6.1 Alternating Trees

Suppose we have a matching M of G and a fixed M -exposed vertex r of G . We can iteratively build up sets A, B of vertices such that each vertex in A is the other end of an odd-length M -alternating path beginning at r , and each vertex in B is the other end of an even-length M -alternating path beginning at r .

Begin with $A = \emptyset, B = \{r\}$, and use the rule: if $vw \in E, v \in B, w \notin A \cup B, wz \in M$, then add w to A , z to B . The set $A \cup B$ and edges in the construction form a tree T rooted at r .

Definition: Alternating Tree

A tree T such that

- every vertex of T other than r is covered by an edge of $M \cap E(T)$;
- for every vertex v of T , the path in T from v to r is M -alternating.

We let the vertex sets at odd and even distances from the root as $A(T)$ and $B(T)$ respectively. Note that $|B(T)| = |A(T)| + 1$ since all other vertices other than r come in matched pairs, one in $A(T)$ and one in $B(T)$.

Using vw to Extend T

Input: A matching M' of a graph G' , an M' -alternating tree T , and an edge vw of G' such that $v \in B(T)$, $w \notin V(T)$, and w is M' -covered.

Algorithm: Let wz be the edge in M' covering w (but z is not a vertex of T). Replace T by the tree having edge set $E(T) \cup \{vw, wz\}$.

Use vw to Augment M'

Input: A matching M' of a graph G' , an M' -alternating tree T of G' with root r , and an edge vw of G' such that $v \in B(T)$, $w \notin V(T)$, and w is M' -exposed.

Algorithm: Let P be the path obtained by attaching vw to the path from r to v in T . Replace M' by $M' \Delta E(P)$.

Definition: Frustrated

An M -alternating tree T in a graph G is frustrated if every edge of G has one end in $B(T)$ and the other end in $A(T)$.

Proposition

Suppose that G has a matching M and an M -alternating tree T that is frustrated. Then G has no perfect matching.

Proof. Clearly, every element of $B(T)$ is a single-vertex odd component of $G \setminus A(T)$. Since $|A(T)| < |B(T)|$, then G has no perfect matching. \square

9.6.2 Bipartite Perfect Matching Algorithm

The following algorithm is to obtain a perfect matching in a bipartite graph based on alternating trees or outputs no perfect matching.

Algorithm 4 Bipartite Perfect Matching Algorithm

```
1:  $M = \emptyset$ 
2: Choose an  $M$ -exposed vertex  $r$ 
3:  $T = (\{r\}, \emptyset)$ 
4: while there exists  $vw \in E$  with  $v \in B(T), w \notin V(T)$  do
5:   if  $w$  is  $M$ -exposed then
6:     Use  $vw$  to augment  $M$ 
7:     if there is no  $M$ -exposed vertex in  $G$  then
8:       return Perfect matching  $M$ 
9:     else
10:       $T = (\{r\}, \emptyset)$ , where  $r$  is  $M$ -exposed
11:   else
12:     Use  $vw$  to extend  $T$ 
13: return  $G$  has no perfect matching
```

Proposition

Suppose that G is bipartite, M is a matching of G , and T is an M -alternating tree such that no edge of G joins a vertex in $B(T)$ to a vertex not in $V(T)$. Then T is frustrated, and hence G has no perfect matching.

Proof. We show that every edge having an end in $B(T)$ has an end in $A(T)$. From the hypothesis, the only possible exception would be an edge joining two vertices in $B(T)$. But this edge, together with the paths joining them to the root of T , would form a closed path of odd length, which contradicts G being bipartite. Hence T is frustrated, and so by previous proposition, G has no perfect matching. \square

9.6.3 Blossom Algorithm for Perfect Matching

Definition: Derived Graph

A graph G' obtained from G by a sequence of odd-circuit shrinkings.

Definition: Original Vertex

A vertex in the derived graph G' that is in G .

Definition: Pseudonode

A vertex in the derived graph G' not in G .

Definition: $S(v)$

Given a vertex v of G' , there corresponds a set $S(v)$ of vertices of G , where

$$S(v) = \begin{cases} v & \text{if } v \in V(G) \\ \bigcup_{w \in V(C)} S(w) & \text{if } v = C \text{ is a pseudonode} \end{cases}$$

Proposition

Let G' be a derived graph of G , M' be a matching of G' , and T be an M' -alternating tree of G' such that no element of $A(T)$ is a pseudonode. If T is frustrated, then G has no perfect matching.

Proof. When deleting $A(T)$ from G , we get a component with vertex set $S(v)$ for each $v \in B(T)$. Therefore, $\text{oc}(G \setminus A(T)) > |A(T)|$, so G has no perfect matching by Tutte's theorem. \square

Definition: Blossom

Let $v, w \in B(T)$ and $vw \in E(G)$. The odd circuit in $T + vw$ is a blossom.

When we shrink a blossom, we get a pseudonode and the new graph is a derived graph.

Use vw to Shrink and Update M' and T

Input: A matching M' of a graph G' , an M' -alternating tree T , and an edge vw of G' such that $vw \in B(T)$.

Algorithm: Let C the circuit formed by vw with the vw -path in T . Replace G' with $G' \times C$, M' by $M' \setminus E(C)$, and T by the tree in G' having edge set $E(T) \setminus E(C)$.

Proposition

After application of the shrinking subroutine, M' is a matching of G' , T is an M' -alternating tree of G' , and $C \in B(T)$.

Algorithm 5 Blossom Algorithm for Perfect Matching

```
1: Input: Graph  $G$  and matching  $M$  of  $G$ 
2:  $M' = M$ 
3:  $G' = G$ 
4: Choose an  $M'$ -exposed vertex  $r$  of  $G'$ 
5:  $T = (\{r\}, \emptyset)$ 
6: while there exists  $vw \in E'$  with  $v \in B(T), w \notin A(T)$  do
7:   Case:  $w$  is  $M'$ -exposed
8:     Use  $vw$  to augment  $M'$ 
9:     Extend  $M'$  to a matching  $M$  of  $G$ 
10:    Replace  $M'$  by  $M$ ,  $G'$  by  $G$ 
11:    if there is no  $M'$ -exposed vertex in  $G'$  then
12:      return Perfect matching  $M'$ 
13:    else
14:       $T = (\{r\}, \emptyset)$ , where  $r$  is  $M'$ -exposed
15:    Case:  $w \notin V(T)$ ,  $w$  is  $M'$ -covered
16:      Use  $vw$  to extend  $T$ 
17:    Case:  $w \in B(T)$ 
18:      Use  $vw$  to shrink and update  $M'$  and  $T$ 
19: return  $G', M', T$ ,  $G$  has no perfect matching
```

Theorem

The Blossom Algorithm terminates after $O(n)$ augmentations, $O(n^2)$ shrinking steps, and $O(n^2)$ tree-extension steps.

Moreover, it determines correctly whether G has a perfect matching.

9.7 Blossom Algorithm for Maximum Matching

We can extend the Blossom algorithm for perfect matchings to maximum matchings.

Algorithm 6 Blossom Algorithm for Maximum Matching

```

1: Input: Graph  $G$  and matching  $M$  of  $G$ 
2:  $M' = M, G' = G, \mathcal{T} = \emptyset$ 
3: Choose an  $M'$ -exposed vertex  $r$  of  $G'$ 
4:  $T = (\{r\}, \emptyset)$ 
5: while there exists  $vw \in E'$  with  $v \in B(T), w \notin A(T)$  do
6:   Case:  $w$  is  $M'$ -exposed
7:     Use  $vw$  to augment  $M'$ 
8:     Extend  $M'$  to a matching  $M$  of  $G$ 
9:     Replace  $M'$  by  $M$ ,  $G'$  by  $G$ 
10:    if there is no  $M'$ -exposed vertex in  $G'$  then
11:      return Perfect matching  $M'$ 
12:    else
13:       $T = (\{r\}, \emptyset)$ , where  $r$  is  $M'$ -exposed
14:    Case:  $w \notin V(T)$ ,  $w$  is  $M'$ -covered
15:      Use  $vw$  to extend  $T$ 
16:    Case:  $w \in B(T)$ 
17:      Use  $vw$  to shrink and update  $M'$  and  $T$ 
18:  $\mathcal{T} = \mathcal{T} \cup \{T\}, G' = G \setminus V(T), M' = M \setminus E(T)$ 
19: if there exists an  $M'$ -exposed vertex then
20:   Go to line 5
21: Restore the matching  $M$ 
22: return  $M$ 

```

Theorem

The Blossom Algorithm can be implemented to run in time $O(nm \log n)$.

Chapter 10

Weighted Matchings

10.1 Minimum-Weight Perfect Matching

Definition: Minimum-Weight Perfect Matching ILP

$$\begin{aligned} \min \quad & \sum (c_e x_e : e \in E) \\ \text{s.t.} \quad & x(\delta(v)) = 1, \forall v \in V \\ & x_e \geq 0, \forall e \in E \\ & x_e \text{ integer}, \forall e \in E \end{aligned}$$

Definition: Minimum-Weight Perfect Matching LP Relaxation

$$\begin{aligned} \min \quad & \sum (c_e x_e : e \in E) \\ \text{s.t.} \quad & x(\delta(v)) = 1, \forall v \in V \\ & x_e \geq 0, \forall e \in E \end{aligned}$$

Definition: Minimum-Weight Perfect Matching Dual LP

$$\begin{aligned} \max \quad & \sum (y_v : v \in V) \\ \text{s.t.} \quad & y_u + y_v \leq c_e, \forall e = uv \in E \end{aligned}$$

Definition: Complementary Slackness Conditions for Minimum-Weight Perfect Matching

If $x_e > 0$, then $\bar{c}_e = c_e - y_u - y_v = 0$ for all $e \in E$.

10.2 Minimum-Weight Perfect Matching in Bipartite Graphs

Theorem (Birkhoff)

Let G be a bipartite graph and let $c \in \mathbb{R}^E$. Then G has a perfect matching if and only if the Minimum-Weight Perfect Matching LP Relaxation has a feasible solution. Moreover, if G has a perfect matching, then the minimum weight of a perfect matching is equal to the optimal value of the LP relaxation.

Definition: E_-

$$E_- = \{e \in E : \bar{c}_e = 0\}.$$

Algorithm 7 Bipartite Minimum-Weight Perfect Matching Algorithm

```

1: Let  $y$  be a feasible solution to the dual LP
2:  $M$  is a matching of  $G_- = (V, E_-)$ 
3:  $T = (\{r\}, \emptyset)$ , where  $r$  is an  $M$ -exposed vertex of  $G$ 
4: while true do
5:   while there exists  $vw \in E_-$  with  $v \in B(T), w \notin V(T)$  do
6:     if  $w$  is  $M$ -exposed then
7:       Use  $vw$  to augment  $M$ 
8:       if there is no  $M$ -exposed vertex in  $G$  then
9:         return Perfect matching  $M$ 
10:      else
11:         $T = (\{r\}, \emptyset)$ , where  $r$  is  $M$ -exposed
12:      else
13:        Use  $vw$  to extend  $T$ 
14:      if every  $vw \in E$  with  $v \in B(T)$  has  $w \in A(T)$  then
15:        return  $G$  has no perfect matching
16:      else
17:         $\varepsilon = \min\{\bar{c}_{vw} : v \in B(T), w \notin V(T)\}$ 
18:         $y_v = y_v + \varepsilon$  for  $v \in B(T)$ 
19:         $y_v = y_v - \varepsilon$  for  $v \in A(T)$ 

```

10.3 Minimum-Weight Perfect Matching in General Graphs

By Minkowski's theorem, we know that there is a system of inequalities for the convex hull of a set, but we typically do not know what the system is. However, Edmonds founded the matching polytope theory in the 1960s.

The integer program for the minimum-weight perfect matching problem is

$$\begin{aligned}
\min \quad & \sum (w_e x_e : e \in E) \\
\text{s.t.} \quad & x(\delta(v)) = 1, \forall v \in V \\
& x_e \geq 0, \forall e \in E \\
& x_e \text{ integer}, \forall e \in E
\end{aligned}$$

If G is bipartite, then we do not need the integrality constraint. If G is non-bipartite, then there exists a circuit C with $|E(C)|$ odd.

For a general graph, the original minimum-weight perfect matching LP can achieve an optimal solution, which may be a fractional vertex of the polyhedron defined by the LP. This is caused by the odd circuits in the graph.

Definition: Perfect Matching Polytope

$$\mathcal{PM}(G) = \text{conv.hull}(\{x^M : M \text{ a perfect matching}\})$$

Let $U \subseteq V$ with $|U|$ odd and $|U| \geq 3$. For an undirected graph, every perfect matching must contain at least one edge in $\delta(U)$, so we arrive at the blossom inequality constraint.

Definition: Blossom Inequality

For $U \subseteq V$ with $|U|$ odd and $|U| \geq 3$,

$$x(\delta(U)) \geq 1$$

Theorem (Perfect Matching Polytope Theorem – Edmonds)

For any graph $G = (V, E)$, $\mathcal{PM}(G)$ is the solution set of the linear system

$$\begin{aligned}
x(\delta(v)) &= 1, \forall v \in V \\
x(\delta(U)) &\geq 1, \forall U \subseteq V, |U| \text{ odd}, |U| \geq 3 \\
x_e &\geq 0, \forall e \in E
\end{aligned}$$

Proof. (Schrijver) Let Q denote the solution set of the linear system. Clearly $\mathcal{PM}(G) \subseteq Q$. Suppose $Q \not\subseteq \mathcal{PM}(G)$ and let x be a vertex of Q with $x \notin \mathcal{PM}(G)$. Choose this counterexample G such that $|V| + |E|$ is as small as possible.

Claim 1: $0 < x_e < 1$ for all $e \in E$.

Proof. (Claim 1) Otherwise, if $x_e = 0$, then delete e . If $x_e = 1$, then delete e and the ends of e . □

So each vertex of G has degree at least 2, which implies $|E| \geq |V|$.

Claim 2: $|E| > |V|$.

Proof. (Claim 2) If $|E| = |V|$, then G is a circuit and the theorem is true. □

Since x is a vertex of Q , there are $|E|$ constraints of the linear system satisfied as an equation by x . Thus, there exists an odd $U \subseteq V$ with $3 \leq |U| \leq |V| - 3$ and $x(\delta(U)) = 1$.

Let G' be the graph obtained by shrinking U to a single vertex and G'' be obtained by shrinking $V \setminus U$ to a single vertex. Let x' and x'' be obtained by shrinking x to G' and G'' respectively. So x' and x'' satisfy the linear system for G' and G'' . By induction, $x' \in \mathcal{PM}(G')$ and $x'' \in \mathcal{PM}(G'')$.

Since x is rational, x' and x'' are rational convex combinations of perfect matchings in G' and G'' , i.e.

$$x' = \frac{1}{k} \sum_{i=1}^k x^{M'_i}, x'' = \frac{1}{k} \sum_{i=1}^k x^{M''_i}$$

for some k (the common denominator of the multipliers λ'_i and λ''_i in the convex combinations).

For each edge $e \in \delta(U)$, the number of indices i with $e \in M'_i$ is $kx'_e = kx_e = kx''_e$ which is equal to the number of indices i with $e \in M''_i$.

We may assume that for each i , the two matchings M'_i and M''_i have an edge in $\delta(U)$ in common. So $M_i = M'_i \cup M''_i$ is a perfect matching of G and

$$x = \frac{1}{k} \sum_{i=1}^k x^{M_i}$$

and thus $x \in \mathcal{PM}(G)$, a contradiction. □

Definition: Minimum-Weight Perfect Matching LP – Stronger

$$\begin{aligned} \min \quad & \sum (c_e x_e : e \in E) \\ \text{s.t.} \quad & x(\delta(v)) = 1, \forall v \in V \\ & x(\delta(U)) \geq 1, \forall U \subseteq V, |U| \text{ odd}, |U| \geq 3 \\ & x_e \geq 0, \forall e \in E \end{aligned}$$

Definition: Minimum-Weight Perfect Matching Dual LP – Stronger

$$\begin{aligned} \max \quad & \sum (y_v : v \in V) + \sum (Y_U : U \subseteq V, |U| \text{ odd}, |U| \geq 3) \\ \text{s.t.} \quad & y_v + y_w + \sum (Y_U : e \in U \subseteq V, |U| \text{ odd}, |U| \geq 3) \leq c_e, \forall e = vw \in E \\ & Y_U \geq 0, \forall U \subseteq V, |U| \text{ odd}, |U| \geq 3 \end{aligned}$$

Theorem

Let G be a graph and let $c \in \mathbb{R}^E$. Then G has a perfect matching if and only if the Minimum-Weight Perfect Matching LP has a feasible solution.

Moreover, if G has a perfect matching, then the minimum weight of a perfect matching is equal to the optimal value of the LP.

Change y

Input: A derived pair (G', c') , a feasible solution y of stronger dual LP for this pair, a matching M' of G' consisting of equality edges, and an M' -alternating tree T consisting of equality edges in G' .

Algorithm:

1. $\varepsilon_1 = \min(\bar{c}_e : e \text{ joins in } G' \text{ a vertex in } B(T) \text{ to a vertex not in } V(T))$
2. $\varepsilon_2 = \min(\bar{c}_e/2 : e \text{ joins in } G' \text{ two vertices in } B(T))$
3. $\varepsilon_3 = \min(y_v : v \in A(T), v \text{ is a pseudonode of } G')$
4. $\varepsilon = \min(\varepsilon_1, \varepsilon_2, \varepsilon_3)$
5. Replace

$$y_v = \begin{cases} y_v + \varepsilon & \text{if } v \in B(T) \\ y_v - \varepsilon & \text{if } v \in A(T) \\ y_v & \text{otherwise} \end{cases}$$

Expand Odd Pseudonode v and Update M', T, c'

Input: A matching M' consisting of equality edges of a derived graph G' , an M' -alternating tree T consisting of equality edges, and an odd pseudonode v of G' such that $y_v = 0$.

Algorithm: Let f, g be the edges of T incident with v , let C be the circuit that was shrunk to form v , let u, w be the ends of f, g in $V(C)$, and let P be the even-length path in C joining u to w .

Replace G' by the graph obtained by expanding C . Replace M' by the matching obtained by extending M' to a matching of G' . Replace T by the tree having edge set $E(T) \cup E(P)$. For each edge st with $s \in V(C)$ and $t \notin V(C)$, replace c'_{st} by $c'_{st} + y_s$.

Proposition

After the application of the expand subroutine, M' is a matching contained in E_+ , and T is an M' -alternating tree whose edges are all contained in E_+ .

Theorem

The Blossom Algorithm terminates after $O(n)$ augmentation steps and $O(n^2)$ tree-extension, shrinking, expanding, and dual change steps.

Moreover, it returns a minimum-weight perfect matching or determines correctly that G has no perfect matching.

Algorithm 8 Blossom Algorithm for Minimum-Weight Perfect Matching

```
1: Let  $y$  be a feasible solution to the dual LP,  $M'$  a matching of  $G_=_$ ,  $G' = G$ 
2:  $T = (\{r\}, \emptyset)$ , where  $r$  is an  $M'$ -exposed vertex of  $G'$ 
3: while true do
4:   Case: There exists  $e \in E_=_$  whose ends in  $G'$  are  $v \in B(T)$  and an  $M'$ -exposed vertex
      $w \notin V(T)$ 
5:     Use  $vw$  to augment  $M'$ 
6:     if there is no  $M'$ -exposed vertex in  $G'$  then
7:       Extend  $M'$  to a perfect matching  $M$  of  $G$  and return  $M$ 
8:     else
9:        $T = (\{r\}, \emptyset)$ , where  $r$  is  $M'$ -exposed.
10:  Case: There exists  $e \in E_=_$  whose ends in  $G'$  are  $v \in B(T)$  and an  $M'$ -covered vertex
      $w \notin V(T)$ 
11:    Use  $vw$  to extend  $T$ 
12:  Case: There exists  $e \in E_=_$  whose ends in  $G'$  are  $v, w \in B(T)$ 
13:    Use  $vw$  to shrink and update  $M', T, c'$ 
14:  Case: There is a pseudonode  $v \in A(T)$  with  $y_v = 0$ 
15:    Expand  $v$  and update  $M', T, c'$ 
16:  Case: None of the above
17:    if every  $e \in E$  incident in  $G'$  with  $v \in B(T)$  has its other end in  $A(T)$  and  $A(T)$ 
    contains no pseudonode then
18:      Stop,  $G$  has no perfect matching
19:    else
20:      Change  $y$ 
```

Chapter 11

T -Joins and Postman Problems

11.1 Postman Problem

Definition: Postman Tour

A closed path where each edge is traversed at least once.

Definition: Euler Tour

A closed edge-simple path P such that $E(P) = E(G)$.

Note that if a graph G has an Euler tour, then the optimal postman tour is the Euler tour.

Theorem

A connected graph G has an Euler tour if and only if every vertex of G has even degree.

Definition: Postman Set

A set $J \subseteq E$ is a postman set of G if for every $v \in V$, v is incident with an odd number of edges from J if and only if v has odd degree in G .

Postman Problem

Given a graph $G = (V, E)$ and $c \in \mathbb{R}^E$ such that $c \geq 0$, find a postman set J such that $c(J)$ is minimum.

Definition: Postman Problem LP

$$\begin{aligned}
\min \quad & \sum (c_e x_e : e \in E) \\
\text{s.t.} \quad & x(\delta(v)) \equiv |\delta(v)| \pmod{2}, \forall v \in V \\
& x_e \geq 0, \forall e \in E \\
& x_e \text{ integer}, \forall e \in E
\end{aligned}$$

11.2 T -Joins

Definition: T -Join

Let $G = (V, E)$ be a graph and let $T \subseteq V$ such that $|T|$ is even. A T -join is a set $J \subseteq E$ such that

$$|J \cap \delta(v)| \equiv |T \cap \{v\}| \pmod{2}, \forall v \in V$$

In other words, J is a T -join if and only if the odd-degree vertices of the subgraph (V, J) are exactly the elements of T .

Optimal T -Join Problem

Given a graph $G = (V, E)$, a set $T \subseteq V$ such that $|T|$ is even, and a cost vector $c \in \mathbb{R}^E$, find a T -join J of G such that $c(J)$ is minimum.

Examples:

- Postman sets: Let $T = \{v \in V : |\delta(v)| \text{ is odd}\}$. Then the T -joins are precisely the postman sets. Finding an optimal T -join solves the postman problem.
- Even set: Let $T = \emptyset$. Then a T -join is exactly an even set, that is, a set $A \subseteq E$ such that every vertex of (V, A) has even degree. A set is even if and only if it can be decomposed into edge sets of edge-disjoint circuits.
- rs -paths: Let $r, s \in V$ and let $T = \{r, s\}$. Every T -join J contains the edge-set of an rs -path. (**Proof.** If not, the component of the subgraph (V, J) that contains r has only one vertex of odd degree.)

Proposition

Let J' be a T' -join of G . Then J is a T -join of G if and only if $J \Delta J'$ is a $(T \Delta T')$ -join of G .

Proof. It is enough to prove the “only if” part, since the other part can be deduced by applying this one with J replaced by $J \Delta J'$ and T replaced by $T \Delta T'$.

Suppose that J is a T -join and J' is a T' -join. Let $v \in V$. Then $|(J \Delta J') \cap \delta(v)|$ is even if and only if $|J \cap \delta(v)| \equiv |J' \cap \delta(v)| \pmod{2}$, which is true if and only if v is an element of neither or both of T and T' , that is, if and only if $v \notin T \Delta T'$. \square

11.3 Optimal T -Join Algorithm

Proposition

Every minimal T -join is the union of the edge sets of $\frac{|T|}{2}$ edge-disjoint simple paths, which join the vertices in T in pairs.

Proposition

Suppose that $c \geq 0$. Then there is an optimal T -join that is the union of $\frac{|T|}{2}$ edge-disjoint shortest paths joining the vertices of T in pairs.

Algorithm 9 Optimal T -Join Algorithm

- 1: Identify the set N of edges having negative cost and let the set T' of vertices incident with an odd number of edges from N
 - 2: $c = |c|$, $T = T \Delta T'$
 - 3: Find a least-cost uv -path P_{uv} with respect to c for each pair u, v of vertices from T and let $d(u, v)$ be the cost of P_{uv}
 - 4: Form a complete graph $\hat{G} = (T, \hat{E})$ with uv having weight $d(u, v)$ for each $uv \in \hat{E}$
 - 5: Find a minimum-weight perfect matching M in \hat{G}
 - 6: Let J be the symmetric difference of $E(P_{uv})$ for $uv \in M$
 - 7: $J = J \Delta N$
-

11.4 T -Join LP

Definition: T -Odd

A set $S \subseteq V$ is T -odd if $|S \cap T|$ is odd.

Definition: T -Cut

The set $\delta(S)$ where $S \subseteq V$ is T -odd.

Definition: T -Join LP

$$\begin{aligned}
 \min \quad & \sum (c_e x_e : e \in E) \\
 \text{s.t.} \quad & x(D) \geq 1, \quad \forall T\text{-cuts } D \\
 & x_e \geq 0, \quad \forall e \in E
 \end{aligned}$$

Definition: T -Join Dual LP

$$\begin{aligned} \max \quad & \sum (Z_D : D \text{ a } T\text{-cut}) \\ \text{s.t.} \quad & \sum (Z_D : e \in D, D \text{ a } T\text{-cut}) \leq c_e, \forall e \in E \\ & Z_D \geq 0, \forall T\text{-cuts } D \end{aligned}$$

Theorem

If $G = (V, E)$ is a graph, $T \subseteq V$ with $|T|$ even, and $c \in \mathbb{R}^E$ with $c \geq 0$, then the minimum cost of a T -join of G is equal to the optimal value of the T -join LP.

Theorem

Let $G = (V, E)$ be a graph and $c \in \mathbb{Z}^E$. Suppose that every circuit of G has even c -cost. Then the T -join dual LP has an optimal solution that is integral.

Chapter 12

General Matching

Definition: b -Factor

Let $G = (V, E)$ be a graph and let $b \in \mathbb{Z}^E$. A b -factor of G is a set $M \subseteq E$ such that $|M \cap \delta(v)| = b_v$ for each $v \in V$.

A 1-factor is a perfect matching. A 2-factor is a collection of circuits covering all the vertices exactly once. The minimum-weight 2-factor problem is a useful relaxation of the traveling salesman problem.

The problem of deciding whether a bipartite graph has a b -factor is solvable by network flow methods. The problem of finding a minimum-weight b -factor in a bipartite graph can be solved as a minimum-cost flow problem.

Suppose we are given a vector $u \in (\mathbb{Z} \cup \{\infty\})^E$.

Definition: (b, u) -Matching (u -Capacitated Perfect b -Matching)

A vector $x \in \mathbb{Z}^E$ such that

$$x(\delta(v)) = b_v, \quad \forall v \in V$$

$$0 \leq x_e \leq u_e, \quad \forall e \in E$$

12.1 General Matching LP

Definition: Minimum-Weight b -Factor LP

$$\begin{array}{ll}\min & \sum (c_e x_e : e \in E) \\ \text{s.t.} & x(\delta(v)) = b_v, \forall v \in V \\ & x(\delta(S) \setminus F) - x(F) \geq 1 - |F|, \forall S \subseteq V, F \subseteq \delta(S), |F| + b(S) \equiv 1 \pmod{2} \\ & 0 \leq x_e \leq 1, \forall e \in E\end{array}$$

Part VI

Matroids

Chapter 13

Matroid Theory

Recall Kruskal's algorithm to find a maximum-weight spanning tree. A slight variant of the algorithm is to find a maximum-weight forest. Let $\mathcal{I} = \{J \subseteq E : J \text{ is a forest}\}$.

Algorithm 10 Greedy Algorithm

- 1: $J = \emptyset$
 - 2: **while** there exists $e \notin J$ with $c_e > 0$ and $J \cup \{e\} \in \mathcal{I}$ **do**
 - 3: Choose e with c_e maximum
 - 4: $J = J \cup \{e\}$
 - 5: **return** J
-

The family \mathcal{I} of forests of a graph has the property that the Greedy Algorithm finds a maximum-weight independent set. Families like forests for which the Greedy Algorithm always returns an optimal solution are called matroids.

Definition: Matroid

Let S be a finite set (*ground set*) and \mathcal{I} be a family of subsets of S (*independent sets*). $M = (S, \mathcal{I})$ is a matroid if the following axioms are satisfied:

(M0) $\emptyset \in \mathcal{I}$.

(M1) If $J' \subseteq J \in \mathcal{I}$, then $J' \in \mathcal{I}$.

(M2) For every $A \subseteq S$, every maximal independent subset of A has the same cardinality.

Definition: Basis

A maximal independent subset of a set $A \subseteq S$ is a basis of A .

Definition: Rank

The size (which depends only on A by (M2)) of the basis of A , denoted $r(A)$.

Part VII

Traveling Salesman Problem

Chapter 14

The Traveling Salesman Problem

Definition: Tour

A circuit that passes exactly once through each vertex.

This is also known as a Hamiltonian circuit.

Traveling Salesman Problem (TSP)

Given a finite set of points V and a cost c_{uv} of travel between each pair $u, v \in V$, find a tour of minimal cost.

The TSP can be modeled as a graph problem by considering the complete graph. TSP belongs to class of \mathcal{NP} -hard problems.