CO 342 Graph Theory

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Contents

Chapter 1

Introduction

Definition: Graph

A graph $G = (V, E, i)$ is a 3-tuple where

- V is a finite set of vertices
- E is a finite set of edges with $V \cap E = \emptyset$
- $i : V \times E \to \{0, 1, 2\}$ such that

  $$i(v, e) = \# \text{ of times } e \text{ is incident to } v$$

  such that

  $$\forall e \in E, \sum_{v \in V} i(v, e) = 2$$

Definition: Incident

$v \in V$ and $e \in E$ are incident in G if $i(v, e) \neq 0$.

Definition: Adjacent

$u, v \in V$ are adjacent in G if either

- $i(u, e) = i(v, e) = 1$ if $u \neq v$
- $i(u, e) = 2$

for some $e \in E$.

A graph G is simple if for each pair u, v, at most one edge e is incident to both.

Definition: Walk

An alternating sequence of vertices and edges of G: $v_0, e_1, v_2, e_2, \ldots, e_k, v_k$ such that the ends of edge e_i are v_{i-1} and v_i.
Vertices and edges are not necessarily unique.

Definition: Path

A walk in which the vertices and edges are distinct.

The number of edges is the length of the path/walk.

Definition: Circuit/Cycle

A walk $v_1, e_1, \ldots, e_{k-1}, v_k$ such that $v_1 = v_k$ and $v_1, v_2, \ldots, v_{k-1}$ are distinct.
A loop is a circuit of length 1.

Definition: Degree

The degree of a vertex v is

$$\deg(v) = d(v) = \sum_{e \in E} i(v, e)$$

Definition: Subgraph

A subgraph of $G = (V, E, i)$ is a 3-tuple

$$H = (V', E', i')$$

where $V' \subseteq V, E' \subseteq E$ and i' is the restriction of i to the domain $V' \times E'$.

Definition: Induced Subgraph

If $X \subseteq V$, the subgraph $G[X]$ of G induced by X is the subgraph of G containing exactly the vertices of X and all edges between them.

Chapter 2

Connectivity

Definition: Connected

Vertices u, v are connected if there is uv-walk.
A graph is connected if and only if $V(G) \neq \emptyset$ and every pair of vertices are connected in G. Alternatively, a graph is connected if it has one component.
The empty graph is disconnected.

Proposition

Connectedness is an equivalence relation on vertices of G.

1. Reflexivity: each vertex u is connected to itself.
2. Symmetry: if u is connected to v, then v is connected to u.
3. Transitivity: if u is connected to v and v is connected to w, then u is connected to w.

The equivalence classes of this relation form the vertex sets of each of the components of G. In other words, an equivalence class of this relation is a subset of vertices V' such that every vertex in V' is connected to every other vertex in V' but not connected to any vertex outside of V'.

Proof. Reflexivity holds because u is a walk from u to u for each u.

Symmetry holds because if $u = x_0, e_1, x_2, \ldots, e_k x_k = v$ is a walk from u to v, then $v = x_k, e_k, \ldots, e_1, x_0 = u$ is walk from v to u.

Transitivity holds because if uWv is a walk from u to v and $vW'w$ is a walk from from v to w, then $uWvW'w$ is a walk from u to w.

Lemma

If u and v are connected, there is a path from u to v.

Definition: Components

A component of G is a maximal connected subgraph. Alternatively, an induced subgraph of the form $G[X]$ where X is an equivalence class under connectedness.

Definition: AB-Path

Given sets of vertices A, B in a graph G, an AB-path is a path P from one vertex in A to a vertex in B so that P intersects A only at its first vertex and B only at its last.

Note: if $A \cap B \neq \emptyset$, then every vertex in $A \cap B$ gives an AB-path with no edges.

Definition: aB-Path

For a vertex a and a set of vertices B, an aB-path means an $\{a\}B$-path.

Definition: Separation

A set $X \subseteq V \cup E$ separates A and B in G it there is no AB-path in $G - X$.

Definition: Cut Edge/Bridge

An edge is a cut edge/bridge if there are vertices u, v of G that are not separated by $\emptyset$, but are separated by $\{e\}$.

Definition: Cut Vertex

A cut vertex of G is a vertex v such that there is some pair of vertices a, b not separated by $\emptyset$, but separated by $\{v\}$.

Definition: k-Connected

For $k \geq 1$, G is k-connected if there is no set $X \subseteq V(G)$ with $|X| < k$ such that $G - X$ is disconnected.
There is sometimes a restriction on $|V(G)| > k$.

Note:

- Every graph is 0-connected except the empty graph.
- G is 1-connected if and only if G is connected.
- G is 2-connected if and only if G is connected and has no cut vertex.
- Trees are not 2-connected because trees have leaves, and deleting a neighbour of a leaf disconnects the graph.
- All vertices in a k-connected graph have degree at least k.

Proposition

If G is connected and $A, B \subseteq V(G)$ are nonempty, then there is an AB-path.

Proof. Let $a_0 \in A, b_0 \in B$. Since G is connected, there exists a path P from a_0 to b_0. Let a be the last vertex in $V(P) \cap A$ and b be the first vertex after b in P that is B. Then the maximality in the choice of a and the minimality in the choice of b implies that the subpath aPb is an AB-path.

Proposition

If there is a vertex $x \in V(G)$ that is connected to every other vertex of G, then G is connected.

Proof. If x is connected to every vertex, then given $u, v \in V(G)$, u is connected to x, x is connected to v, so u is connected to v by symmetry and transitivity of connectedness.

Definition: Graph Union

Given two graphs G_1, G_2 (whose vertex sets and edge sets might intersect) and every edge in $E(G_1) \cap E(G_2)$ has the same ends in both graphs, the graph $G_1 \cup G_2$ is the graph with vertices $V(G_1) \cup V(G_2)$ and edges $E(G_1) \cup E(G_2)$, where the ends of each edge e are the same as they are in G_1 and G_2.

Definition: Direct Sum

We write $G_1 \oplus G_2$ to denote the direct sum of G_1, G_2 which is

$$G_1 \cup G_2$$

when $V_1 \cup E_1, V_2 \cup E_2$ are disjoint.

Proposition

If G_1 and G_2 are connected and $V(G_1) \cap V(G_2) \neq \emptyset$, then $G_1 \cup G_2$ is connected.

Proof. Let $x \in V(G_1) \cap V(G_2)$. Since G_1 is connected, every vertex in G_1 is connected in G_1 to x and similarly, every vertex in G_2 is connected in G_2 to x. So every vertex in $G_1 \cup G_2$ is connected to x (because paths in G_1 and G_2 are paths in $G_1 \cup G_2$), so $G_1 \cup G_2$ is connected.

Proposition

If G is a connected graph on n vertices, then there is an ordering $v_1, \ldots, v_n$ of its vertices such that for all $1 \leq i \leq n$, the induced subgraph $G[\{v_1, \ldots, v_i\}]$ is connected.

Proof. Let v_1 be any vertex of G. Let k be maximal such that there exist vertices $v_2, \ldots, v_k$ of G so that, for every $1 \leq i \leq k$, the induced subgraph $G[\{v_1, \ldots, v_i\}]$ is connected.

If $k = n$, then $v_1, \ldots, v_k = v_n$ is the required order. So we may assume that $k < n$, so there exists a vertex $x \notin \{v_1, \dots, v_k\}$.

Since G is connected, there is a $\{v_1, \dots, v_k\}x$-path P in G. Let v_{k+1} be the first vertex of P outside $\{v_1, \dots, v_k\}$. Now the subgraph of G induced by $\{v_1, \dots, v_k, v_{k+1}\}$ is the union of the subgraph induced by $\{v_1, \dots, v_k\}$ and some graph containing v_{k+1} and all its neighbours in $\{v_1, \dots, v_k\}$. Both graphs are connected and since v_{k+1} has a neighbour in $\{v_1, \dots, v_k\}$, they have a vertex in common.

So $\{v_1, \dots, v_k\}$ induces a connected subgraph of G, contradicting the maximality.

Proposition

If G is a connected graph on $n \geq 2$ vertices, then G has a vertex v such that $G - v$ is connected.

Definition: Adding a Path

We say G is obtained from H by adding a path if $G = H \cup P$, for some path P such that $V(P) \cap V(H)$ is exactly the set of the two ends of P and $E(P) \cap E(H) = \emptyset$.

Note: adding a single new edge between two existing vertices is an example of adding a path.

Lemma

If G_1 and G_2 are k-connected graphs, whose union is well-defined and

$$|V(G_1) \cap V(G_2)| \geq k$$

then $G = G_1 \cup G_2$ is k-connected.

Proof. Suppose not. Then there exists $X \subseteq V(G_1 \cup G_2)$ such that $|X| < k$ and $G - X$ is disconnected. Note $(G_1 \cup G_2) - X = (G_1 - X) \cup (G_2 - X)$.

Since $|X| < k$ and each G_i is k-connected, we know that $G_1 - X$ and $G_2 - X$ are connected. Also, since $|X| < k \leq |V(G_1) \cap V(G_2)|$, the graphs $G_1 - X$ and $G_2 - X$ have a vertex in common. So $(G_1 - X) \cup (G_2 - X)$ is connected, a contradiction.

Corollary

If H is 2-connected and G is obtained from H by adding a path, then G is 2-connected.

Proof. Let P be the path with ends u, v. If P has length 1, then P is 2-connected, so $G = H \cup P$ is 2-connected by lemma.

Otherwise, let Q be a path in H from u to v. $P \cup Q$ is a cycle and $G = H \cup (P \cup Q)$. Since $P \cup Q$ is 2-connected, and so is G by lemma.

Theorem (Ear-Decomposition)

For every 2-connected graph G, there are 2-connected subgraphs $G_0, \ldots, G_k$ of G such that

- G_0 is a cycle
- $G_k = G$
- For each $0 \leq i < k$, G_{i+1} is obtained from G_i by adding a path.

Proof. If G has no cycle, then G is a tree and trees are not 2-connected. So G has a cycle G_0.

Let $G_0, G_1, \ldots, G_t$ be a maximal sequence of 2-connected subgraphs of G such that each G_{i+1} is obtained from G_i by adding a path. If $G_t = G$, then we have the required sequence.

If there is an edge $e \in E(G) \setminus E(G_t)$ with both ends in $V(G_t)$, then $G_{t+1} = G_t \cup \{e\}$ is obtained from G_t by adding a path e, and is 2-connected by the lemma, so it contradicts the maximality of t.

Otherwise, since G_t is not a component of G and is not all of G, there is a vertex v of $V(G) \setminus V(G_t)$ having a neighbour $w \in V(G_t)$. Since G is 2-connected, there is a $vV(G_t)$-path P in $G - w$. Let e be the edge from v to w, now $wevP$ is a path intersecting $V(G_t)$ precisely in its two ends, so $G_t \cup wevP$ is obtained from G_t by adding a path, so is 2-connected and contradicts maximality of t.

Proposition

If G is 2-connected, then every pair of vertices of G is contained in a cycle.

Proof. Let $G_0, \ldots, G_k = G$ be an ear-decomposition. This is true for G_0 because G_0 is a cycle.

Suppose it is true for some G_i where $0 \leq i < k$. There are 3 cases: the pair are in G_i, the pair is on the new path added, or one vertex is on the path and one is vertex is in G_i.

Proof is in assignment.

Definition: k-Edge-Connected

Let $k \geq 0$. A graph G is k-edge connected if there is no set $X \subseteq E(G)$ for which $|X| < k$ and $G - X$ is disconnected.

Lemma

If G is k-connected and $|X| \leq k$, then every vertex in X has a neighbour in every connected component H of $G - X$.

Proof. Let $x \in X$ and H be a connected component of $G - X$. Note that $G - (X \setminus \{x\})$ is connected since $|X \setminus \{x\}| < k$. Let P be an xH-path in $G - (X \setminus \{x\})$. Since H is a component of $G - X$ and the penultimate vertex w of P is a vertex of $G - (X \setminus \{x\})$ outside H with a neighbour in H, we must have $w = x$, so x has a neighbour in H.

Definition: Edge Contraction

Given a graph G and $e \in E$ with distinct ends u, v such that e is the only edge from u to v, we write G/e for the graph $((V - \{u, v\}) \cup \{x_{uv}\}, E \setminus \{e\})$ where each edge with no end in $\{u, v\}$ has the same ends as in G and each edge with an end in $\{u, v\}$ has this end replaced by the new vertex x_{uv}.

Proposition

If G is a simple, 3-connected graph with $|V(G)| \geq 4$, then G has an edge e such that G/e is 3-connected.

Proof. Suppose by contradiction that every edge $xy \in E(G)$, the graph G/xy is not 3-connected. Then, G/xy contains a separator S with $|S| \leq 2$.

Since S is not a separator of G, we have that $v_{xy} \in S$ and $|S| = 2$. Let $z \in V(G/xy)$ such that $S = \{v_{xy}, z\}$, then any two vertices separated by $\{v_{xy}, z\}$ in G/xy are separated in G by $T := \{x, y, z\}$.

Fix an edge xy, a vertex z and a component C so that $|C|$ is as small as possible. By the previous lemma, z has a neighbour v in C and by contradiction, C/zv is not 3-connected. So there exists a vertex $w \in V(G)$ such that $\{z, v, w\}$ separates G. As $xy \in E(G)$, $G \setminus \{z, v, w\}$ has a connected component D such that $D \cap \{x, y\} = \emptyset$ (x, y cannot be in different connected components). Then, every neighbour of v in D lies in C (since $v \in C$). By the lemma, v has a neighbour in D. Thus, $D \cap C \neq \emptyset$, and hence $D \subsetneq C$ (since $v \in C \setminus D$). This contradicts the choice of xy, z and C.

Proposition

If G is a simple, 3-connected graph, then there exist $G_0, G_1, \ldots, G_k$ such that $G_0 \cong K_3$, $G_k \cong G$, and G_i is a 3-connected graph with $G_i \cong S_i(G_{i+1}/e)$ for some e, where S_i means remove parallel pairs.

Definition: Internally Disjoint (IDJ)

A collection of uv-paths in a graph G is internally disjoint if no two paths have no vertices or edges in common except for the endpoints u and v.

Definition: uv-Separator

A set $X \subseteq V(G) \setminus \{u, v\}$ is a uv-separator in G if $G - X$ contains no uv-path.

Note that if u and v are adjacent, then no uv-separator exists.

Proposition

If X is a uv-separator of G with $|X| < k$, then there do not exist k internally disjoint uv-paths.

Proof. Suppose there are k IDJ uv-paths $P_1, \ldots, P_k$. Since none of the P_i is a path in $G - X$, each P_i contains a vertex in X. Since $|X| < k$, the pigeonhole principle gives that two of the P_i have a common vertex in X, contradicting IDJ.

Theorem (Menger)

If u, v are non-adjacent in G and every uv-separator in G has size $\geq k$, then G has k internally disjoint uv-paths.

Proof. Suppose not. Let G be a counterexample for which $k + |V(G)| + |E(G)|$ is as small as possible (i.e. every uv-separator in G has size $\geq k$, but there do not exist k IDJ uv-paths in G).

Claim 1: There is no vertex adjacent to both u and v.

Claim 1 *Proof.* Let x be a vertex adjacent to both u and v. If $G - x$ has a uv-separator S with $|S| < k - 1$, then $S \cup \{x\}$ is a uv-separator in G of size $|S| + 1 < k$, contradicting our assumption about G.

Otherwise, since $|V(G - x)| + k - 1 < |V(G)| + k$, induction (i.e. minimality) implies that there are $k - 1$ IDJ paths $P_1, \ldots, P_{k-1}$ in $G - x$. Now, $P_1, \ldots, P_{k-1}, uxv$ are k IDJ paths in G, a contradiction.

Claim 2: Every edge of G is incident with either u or v.

Claim 2 *Proof.* Let e be such an edge with ends x, y. Let S be a smallest uv-separator in $G - e$. Note that $S \cup \{x\}$ and $S \cup \{y\}$ are both uv-separators in G. If $|S| \geq k$, then by applying induction to $G - e$, there are k IDJ uv-paths in $G - e$ and therefore, in G, a contradiction. Therefore, $|S| < k$. Since $S \cup \{x\}$ and $S \cup \{y\}$ are uv-separators in G, we also have $|S \cup \{x\}| \geq k$ and $|S \cup \{y\}| \geq k$. So, $|S| = k - 1$, and $x, y \notin S$.

Since $|S| = k - 1$, there is a uv-path P in $G - S$; since P is not a path of $G - e - S$, we must have $e \in P$, so one of x, y (say x) is connected to u in G and the other is connected to v in G (both via P).

Proof. Let G_u be the graph obtained from G by deleting all vertices connected to v in $G - S - e$, and adding a single vertex v' adjacent to every vertex in $S \cup \{x\}$. Let G_v symmetrically, but swap u and v and x with y.

We have $|V(G_u)| + |E(G_u)| + k < |V(G)| + |E(G)| + k$ and the same for v, so we can apply induction to G_u and G_v.

Subclaim: If T is a uv'-separator in G_u with $|T| < k$, then T is a uv-separator in G.

By the subclaim, there is no uv'-separator in G_u of size $< k$, so by induction, there are k IDJ uv'-paths $P_1, \ldots, P_k$ in G_u. By the same argument, there are k IDJ $u'v$-paths $Q_1, \ldots, Q_k$ in

G_v.

Each P_i has the form $uP_i'w$ where $w \in S \cup \{x\}$, and each Q_j has the form $zQ_j'v$ where $z \in S \cup \{y\}$. Since $k = |S \cup \{x\}| = |S \cup \{y\}|$, we can join P_i and Q_j at the ends where they agree in S and add the edge e to the P_i' ending at x and the Q_j' starting at y, to find k IDJ uv-paths in G of the form uP_ieQ_jv. This is a contradiction. Since G satisfies claims 1 and 2, it is the disjoint union of a star graph at u and a star graph at v, so it is not a counterexample to Menger's Theorem.

$\square$

Theorem (Menger – Version 2)

If u and v are vertices in G and F is a set of edge from u to v, and every uv-separator in $G \setminus F$ has size $\geq k$, then there are $k + |F|$ internally disjoint uv-paths in G.

Proof. In the graph $G \setminus F$, u and v are non-adjacent, so by Menger, there are k IDJ uv-paths in $G \setminus F$. Each edge in F is its own uv-path, so the paths in $G \setminus F$ together with the edges in F give $k + |F|$ IDJ uv-paths in G.

Theorem

If G is a simple graph on $> k$ vertices, then G is k-connected if and only if for every pair u, v of distinct vertices of G, there are k internally disjoint uv-paths.

Proof. ($\implies$) Suppose G is k-connected. Let $u, v \in V(G)$ be distinct. If u, v are non-adjacent and S is a uv-separator, then $G - S$ is disconnected, so $|S| \geq k$ by k-connectedness of G. Thus, every uv-separator has size $\geq k$, so by Menger, G has k IDJ uv-paths.

If u, v are joined by an edge e, then let S be a smallest uv-separator in $G \setminus e$. If $|S| \geq k - 1$, then by Menger 2, there are $(k-1) + 1 = k$ IDJ uv-paths in G, as required.

If $|S| < k - 1$, then let x be a vertex outside $S \cup \{u, v\}$ (exists since $|V(G)| \geq k$). Since u, v are not connected in $(G \setminus e) - S$, x is not connected to both u and v. Suppose WLOG that x is not connected to u in $(G \setminus e) - S$. Therefore, x is not connected to u in $G - (S \cup \{v\})$. But the size of $|S \cup \{v\}| = |S| + 1 < k$, so we have a contradiction to the k-connectedness of G.

($\impliedby$) Suppose G is not k-connected; let S be a set of vertices such that $G - S$ is disconnected and $|S| < k$. Let u, v be vertices in different components of $G - S$; if $P_1, \ldots, P_k$ are IDJ uv-paths, then each must intersect S, but $|S| < k$. This contradicts the pigeonhole principle.

Definition: AB-Separator

A set $X \subseteq V(G)$ such that $G - X$ has no AB-path.

Note: X is allowed to intersect A and/or B. A and B are both examples of an AB-separator.

Proposition

If $A, B \subseteq V(G)$ and S is an AB-separator of size $< k$, then there do not exist k disjoint AB-paths.

Proof. If $P_1, \ldots, P_k$ are disjoint AB-paths, then none is an AB-path in $G - S$, so each P_i contains a vertex in S. But $|S| < k$ and the paths are disjoint, so this contradicts the pigeonhole principle.

Theorem (Menger – Version 3)

If $A, B \subseteq V(G)$ and every AB-separator has size $\geq k$, then there are k disjoint AB-paths.

Definition: aB-Fan

Given a vertex a and a set $B \subseteq V(G)$, an aB-fan is a collection of aB-paths intersecting only at a.

Definition: aB-Separator

A set $X \subseteq V(G) \setminus \{a\}$ such that $G - X$ has no aB-path.

Lemma (Fan Lemma)

If there is no aB-separator of size $< k$, then there is an aB-fan of size k.

Corollary

If G is k-connected and $|V(G)| > k$, then for all $v \in V(G)$ and $X \subseteq V(G) \setminus \{v\}$ with $|X| \geq k$, there is a vX-fan.

Theorem

Let $k \geq 2$. If G is k-connected with $\geq 2k$ vertices, then G has a cycle of length $\geq 2k$.

Proof. Since G is 2-connected, it is a not a tree so it has a cycle. Let C be a longest cycle in G. We may assume C has $< 2k$ vertices, so there is a vertex v outside C. So by fan lemma, there is a vC-fan of size k in G. Let $P_1, \ldots, P_k$ be its paths. Since $k > \frac{1}{2}|V(C)|$, there are two paths P_i, P_j whose endpoints $x_i, x_j \in C$ are joined by an edge e of C. Now, $(C \setminus e)x_iP_ivP_jx_j$ is a cycle longer, then this contradicts the maximality of C.

Chapter 3

Planarity

Definition: Embedding

An embedding of $G = (V, E)$ in $\mathbb{R}^2$ is a function φ such that

- for each vertex v of G, $\varphi(v)$ is a point in $\mathbb{R}^2$, and no two vertices are mapped to same point by φ.
- for each edge e with ends u, v, $\varphi(e)$ is a curve from $\varphi(u)$ to $\varphi(v)$.
- for distinct edges e, f of G, the images of $\varphi(e)$ and $\varphi(f)$ are disjoint (as subsets of $\mathbb{R}^2$), except where e and f intersect at a vertex.
- for all $v \in V, e \in E$, v is in $\varphi(e)$ if v is an end of e.

Definition: Planar Graph

A graph is planar if it has an embedding in $\mathbb{R}^2$, otherwise it is nonplanar.
If φ is an embedding of G in $\mathbb{R}^2$, then we write $\varphi(G)$ for the union of the images of vertices and edges, as subsets of $\mathbb{R}^2$.

Definition: Curve

A continuous, injective function from $[0, 1]$ to $\mathbb{R}^2$ with $\varphi(0) = u$ and $\varphi(1) = v$.

Definition: Interior of a Curve

The image of a curve without endpoints $A(0)$ and $A(1)$.

Definition: Disc

A disc set D of the form $\{(x, y) \in \mathbb{R}^2 : (x - x_0)^2 + (y - y_0)^2 \leq r^2$.

Definition: Open Set in $\mathbb{R}^2$

A set U such that for all $x \in U$, there is a disc D of radius $r > 0$ centered at x, with $D \subseteq U$.

E.g. $\{(x, y) : y > 0\}$.

Definition: Closed Set in $\mathbb{R}^2$

The complement of an open set.

E.g. $\{(x, y) : y \geq 0\}$.

Lemma (Plane Topology Lemmas)

- Points are closed.
- (Images of) curves are closed.
- Discs are closed.
- Finite unions of closed sets are closed.

Theorem (Intermediate Value Theorem)

If $f : [a, b] \to \mathbb{R}$ is continuous with $f(x) \leq M$ and $f(b) \geq M$, then there exists $x \in [a, b]$ such that $f(x) = M$.

Definition: Polygonal

A curve from a to b in $\mathbb{R}^2$ is polygonal if it is a finite union of line segments.
A is polygonal if there exist $0 \leq t_0 \leq t_2 \leq \cdots \leq t_k = 1$ such that $\forall i \in \{1, \ldots, k\}$, the restriction of A to the subinterval $[t_{i-1}, t_i]$ is a straight line segment.

Definition: Connected

Given a set $U \subseteq \mathbb{R}^2$, points $x, y \in U$ are connected in U if either $x = y$ or there is a curve from x to y contained in U.

Definition: Polygonally Connected

Points x, y are polygonally connected in U if $x = y$ or there is a polygonal curve from x to y in U.

Proposition

If U is an open set, then x, y are connected in U if and only if they are polygonally connected in U.

Proof. ($\Longleftarrow$) Obvious.

($\implies$) Suppose, therefore, that u, v are connected. We may assume $u \neq v$. Let $f : [0,1] \to U$ be continuous, injective, with $f(0) = x, f(1) = y$.

Let $S \subseteq \{t \in [0,1] : x \text{ is polygonally connected to } f(t) \text{ in } U\}$. What we want to prove is equivalent to saying $1 \in S$.

Let $t_0 = \sup(S)$, i.e. $t \notin S$ for all $t > t_0$ and for all $t' < t_0$, there is some $t \in S$ with $t' < t \leq t_0$.

Claim 1: $t_0 \in S$.

Claim 1 *Proof.* Clear if $t_0 = 0$, otherwise, let D be a disc centered at $f(t_0)$, contained in U, but not containing $f(0) = x$. By the intermediate value theorem, there is some $0 < t' < t_0$ so that $f(t') \in D$.

Since $t' < t_0$, there exists t'' with $t' < t'' \leq t_0$ such that $t'' \in S$. Assume that $f(t'') \in D$. So there is a point z on the curve in D that is polygonally connected to x. Since $z \in D \subseteq U$, there is a straight line segment contained in U from z to $f(t_0)$, which shows that x is polygonally connected to $f(t_0)$. So $t_0 \in S$.

Claim 2: $t_0 = 1$

Claim 2 *Proof.* Sketch: If $t_0 < 1$, use a similar argument to find t' such that $t_0 < t'$ and $t' \in S$, contradicting $t_0 = \sup(S)$.

Thus, $1 \in S$, as required.

Corollary

Given $U \subseteq \mathbb{R}^2$ open, if x, y connected in U, y, z connected in U, then x, z are connected in U.

Proof. We can glue two polygonal arcs together. We can travel along the arc from x to y and when we first hit the arc from y to z, we switch to that arc.

So connectedness in U is an equivalence relation. Therefore, every open set U has a partition into 'regions' such that x, y are connected in U if and only if they belong to the same region.

Corollary

If G has a planar embedding φ, then it has a planar embedding where all arcs are polygonal.

Proof. Draw discs at each vertex and turn the edges within the discs into radii. Use corollary to make the edges polygonal, one by one.

Definition: Polygon

A polygonal arc, except that we insist on $f(0) = f(1)$ and still injective elsewhere. Informally, a cyclic union of line segments.

Theorem (Jordan Curve Theorem - Polygonal)

If C is a polygon, then $\mathbb{R}^2 \setminus C$ has exactly two regions.

Claim 1: There are ≤ 2 regions in $\mathbb{R}^2 \setminus C$.

Claim 1 ***Proof.*** Let $S_1, \ldots, S_k$ be the line segments in C. For each i, let B_i be the set of points at distance $< \varepsilon$ from a point in S_i, where $\varepsilon > 0$ is chosen small enough so that the B_i only overlap for consecutive i.

Note that:

- For each i, $B_i \setminus C$ has ≤ 2 regions.
- For each $i > 1$, each point in $B_i \setminus C$ is polygonally connected to a point in $B_{i-1} \setminus C$.
- Every point in $\mathbb{R}^2 \setminus C$ is polygonally connected to a point in one of the sets $B_i \setminus C$.

By combining the last two observations, we see that every point in $\mathbb{R}^2 \setminus C$ is connected to a point in some S_i, and therefore (inductively) to a point in $S_1 \setminus C$.

Since $S_1 \setminus C$ has ≤ 2 regions, every point in $\mathbb{R}^2 \setminus C$ lies in one of ≤ 2 regions of $\mathbb{R}^2 \setminus C$.

We need to show $\mathbb{R}^2 \setminus C$ has ≥ 2 regions. Let w be a direction in $\mathbb{R}^2$ that is not parallel to the line between any two vertices of C.

For each $x \in \mathbb{R}^2 \setminus C$, let R_x be the ray in direction w starting at x. Let $n(x)$ be the number of times C intersects R_x, where if C intersects R_x at a vertex y with both segments of C adjacent to y appearing on the same side of R_x, the intersection is not counted.

Claim 2: If x, x' are joined by a line segment in $\mathbb{R}^2 \setminus C$, then $n(x) \equiv n(x') \pmod 2$.

Claim 2 ***Proof.*** By dividing the line segment into subsegments, we may assume that as R_x moves to $R_{x'}$, only one vertex of C is crossed. There are 3 cases:

- V-shape: $n(x') = n(x) - 2$.
- Upside-down v-shape: $n(x') = n(x) + 2$.
- Two line segments going through R_x and $R_{x'}$ with one vertex: $n(x) = n(x')$.

In all cases, $n(x) \equiv n(x') \pmod 2$. Since any two points in the same region of $\mathbb{R}^2 \setminus C$ are polygonally connected if x is in a region f of $\mathbb{R}^2 \setminus C$ and $y \in f$, then there is a polygonal curve from x to y in $\mathbb{R}^2 \setminus C$, so $n(x) \equiv n(y) \pmod 2$ by applying the claim repeatedly.

Let $x \in C$ and D be a disc around x. It is clear that D contains two points $y, z \in \mathbb{R}^2 \setminus C$ with $n(z) = n(y) + 1$. So $n(z) \not\equiv n(y) \pmod 2$, so y, z are in different regions of $\mathbb{R}^2 \setminus C$. So $\mathbb{R}^2 \setminus C$ has ≥ 2 regions, as required.

In fact, for each point $x \in C$ and each disc around x intersects both regions of $\mathbb{R}^2 \setminus C$.

Proof. Since there are ≥ 2 and ≤ 2 regions in $\mathbb{R}^2 \setminus C$, then $\mathbb{R}^2 \setminus C$ has exactly two regions.

Definition: Frontier

Given a set $S \subseteq \mathbb{R}^2$, the frontier of S is the set of points $x \in \mathbb{R}^2$ such that every disc of positive radius centered at x intersects S.

Lemma

If x_1, y_1, x_2, y_2 occur in cyclic order around some polygon C and P is a polygonal curve from x_1 to x_2 with interior of P, $\mathring{P} \subseteq \mathbb{R}^2 \setminus C$, then $\mathbb{R}^2 \setminus (C \cup P)$ has three regions f_0, f_1, f_2 such that f_0 is a region of $\mathbb{R}^2 \setminus C$, and $f_1 \cup f_2 \cup \mathring{P}$ is the other region of $\mathbb{R}^2 \setminus C$, and y_1 is not in the frontier of f_2 and y_2 is not in the frontier of f_1.

Proof. Use polygonal Jordan Curve Theorem.

Proposition

$K_{3,3}$ is nonplanar.

Proof. Suppose that $K_{3,3}$ *is* planar. Let $\{a_1, a_2, a_3\}$, $\{b_1, b_2, b_3\}$ be its bipartition, and let φ be a polygonal embedding. Note that $a_1b_1a_2b_2a_3b_3$ is a cycle in G, so it corresponds to a polygon in $\varphi(G)$ where $\varphi(a_1), \varphi(b_1), \ldots, \varphi(a_3), \varphi(b_3)$ appear in cyclic order.

Let $e_1 = a_1b_2, e_2 = a_2b_3, e_3 = a_3b_1$. Now each $\varphi(e_i)$ is contained in a region of $\mathbb{R}^2 \setminus C$. Otherwise, it would contain an arc from the inside to the outside of C.

Two of the e_i, say e_1, e_2, are contained in the same region of $\mathbb{R}^2 \setminus C$, by pigeonhole principle.

Let f_1, f_2 be the regions of $\mathbb{R}^2 \setminus (C \cup \varphi(e_i))$ for which $f = f_1 \cup f_2 \cup \varphi(e_i)$. Now $\varphi(e_j) \subseteq f = f_1 \cup f_2 \cup \varphi(e_i)$ and $\varphi(e_j)$ does not intersect $\varphi(e_i)$, so $\varphi(e_j)$ is contained in either f_1 or f_2, say f_1.

The ends x_i, y_i of e_i and x_j, y_j of e_j occur in cyclic order x_i, x_j, y_i, y_j around C, so by the lemma above, both x_j, y_j are on the frontier of f_1. This contradicts the lemma.

Definition: Topological Minor

A graph H is a topological minor of a graph G if there is a function ψ such that

- for every vertex v of H, $\psi(v)$ is a vertex of G and if $u \neq v$, then $\psi(u) \neq \psi(v)$.
- for every edge e of H with ends u, v, $\psi(e)$ is a $\psi(u)\psi(v)$-path of G.
- **no** two paths $\psi(e), \psi(e')$ have an internal vertex in common.

Proposition

H is a topological minor of G if and only if some subgraph G' of G is isomorphic to a subdivision of H.

Proposition

If G is planar, and H is a topological minor of G, then H is planar.

Proof. If G has an H-topological minor, then some subdivision H' of H is isomorphic to a subgraph of G, so H' is planar. A planar embedding of H' gives rise to a planar embedding for H.

Corollary

If H is nonplanar and H is a topological minor of G, then G is nonplanar.

Definition: Face (of φ)

A region of the open set $\mathbb{R}^2 \setminus \varphi(G)$.

Proposition

If f is a face of φ, then the frontier of f is a union of some vertices $\varphi(v)$ and some edges $\varphi(e)$.

Definition: Boundary

The boundary of a face f of φ is the subgraph H of G whose vertices and edges form the frontier of f.

i.e. $\varphi(H)$ (points in $\mathbb{R}^2$ used by φ to draw H) is the frontier of f.

Definition: Leaf Edge

An edge incident with a degree 1 vertex.

Lemma

Let φ be a planar embedding of G and e be a leaf edge of G. Then the embedding φ' of $G \setminus e$ given by φ has the same number of faces as φ.

Proof. Sketch: take two points in the face. Go around e with the polygonal arc.

Proposition

If e is in a cycle of G and φ is a planar embedding of G, then e is in the boundary of exactly two faces of G.

Proof. Sketch: Use the polygonal Jordan Curve Theorem.

Proposition

If φ is a planar embedding of G, then φ has exactly one face if and only if G is a forest.

Proof. ($\implies$) Suppose G is not a forest. Then G is a cycle C and each edge in G is in two faces of φ. So φ is more than one face.

($\impliedby$) Suppose that G is a forest. If G has no edges, it clearly has one face. Suppose inductively that G has k edges and that the result holds for all forests with $k-1$ edges.

Let e be a leaf edge of G. Inductively, each embedding of $G \setminus e$ has exactly one face, and by the lemma, $\varphi(G)$ and $\varphi(G \setminus e)$ have the same number of faces, so φ has exactly one face.

Proposition

If e is in a cycle of G and φ is a planar embedding of G, then φ has exactly one more face than the planar embedding of $G \setminus e$.

Proof. Use the polygonal Jordan Curve Theorem (similar to argument that edges in cycles are in two faces).

Theorem (Euler's Formula)

If $G = (V, E)$ is a graph with c components, φ is a planar embedding of G, and F is the set of faces of φ, then

$$|V| - |E| + |F| = 1 + c$$

Proof. Let H be a maximal spanning forest for G. So H consists of a spanning tree H_i for each component G_i of G, and $|E(H_i)| = |V(H_i)| - 1$.

So,

$$|E(H)| = \sum_i |E(H_i)| = \sum_i (|V(H_i)| - 1) = \sum_i |V(H_i)| - c = |V| - c$$

The embedding of H given by φ has one face (H is a forest), $|V|$ vertices, and $|V| - c$ edges. Thus,

$$|V(H)| - |E(H)| + |F(H)| = |V| - (|V| - c) + 1 = 1 + c$$

Let H' be a maximal subgraph of G such that H is a subgraph of H' and $|V| - |E(H')| + |F(H')| = 1 + c$.

If $H' = G$, then G satisfies Euler's formula, as required. Otherwise, G has an edge e outside $E(H')$. Let $H' + e$ be the subgraph of G obtained from H' by adding e. Since e is in a cycle of $H + e$, we know that $|F(H' + e)| = |F(H')| + 1$. Clearly, $|E(H' + e)| = |E(H')| + 1$. So,

$$\begin{aligned} |V| - |E(H'+e)| + |F(H'+e)| &= |V| - |E(H')| - 1 + |F(H')| + 1 \\ &= |V| - |E(H')| + |F(H')| \\ &= 1 + c \end{aligned}$$

So, $H' + e$ contradicts the maximality of H'.

Lemma

If φ is an embedding of a graph G that contains a cycle, then the boundary of every face of G contains a cycle.

Proof. Topological exercise.

Lemma

Each edge in a planar embedding is in ≤ 2 face boundaries.

Proposition

If G is a simple planar graph on ≥ 3 vertices, then $|E(G)| \leq 3|V(G)| - 6$.

Proof. We combine Euler's Formula with an inequality relating the number of edges and the number of faces in the embedding. Let $V = V(G), E = E(G)$. Let F be the set of faces in some planar embedding of G and c is the number of components of G.

If G is a forest, then $|E| \leq |V| - 1 \leq 3|V| - 6$.

Otherwise, every face boundary contains a cycle, so has ≥ 3 edges.

Let $A = \{(e, f) : f \in F, e \text{ is the boundary of } F\}$. Since each e is in the boundary of ≤ 2 faces, we know $|A| \leq 2|E|$. Since each $f \in F$ has ≥ 3 edges in its boundary, we know $|A| \geq 3|F|$. So $3|F| \leq 2|E|$, i.e. $|F| \leq \frac{2}{3}|E|$.

By Euler's Formula,

$$1 + c = |V| - |E| + |F| \leq |V| - |E| + \frac{2}{3}|E| = |V| - \frac{1}{3}|E|$$

So $|E| \leq 3(|V| - 1 - c) \leq 3|V| - 6$ since $c \geq 1$.

Corollary

K_5 is nonplanar.

Proof. $|E| = \binom{5}{2} = 10$ and $3|V| - 6 = 15 - 6 = 9$. So, $|E| \not\leq 3|V| - 6$ so K_5 is nonplanar.

Proposition

If φ is an embedding of a 2-connected graph G, then every face boundary of G is a cycle.

Proof. Induction with ear-decomposition. Adding a path splits one face into two faces bounded by cycles and does not change any other face boundary.

Question: Given a graph G that is known to be planar, can we determine which cycles appear as face boundaries in an embedding of G, without knowing the embedding? No, in general. The problem is the lack of 3-connectedness.

Definition: Non-Separating Cycle

A cycle C of G is non-separating if $G - V(C)$ is connected.

Definition: Chord

An edge that connects two nonadjacent vertices in a cycle.

Definition: Induced Cycle

C is induced in G if there is no edge of $G \setminus E(C)$ with both ends in C (i.e. no chord of C).

Proposition

If φ is an embedding of a 3-connected graph G, then C is a face boundary (facial cycle) of G if and only if C is non-separating and induced in G.

Proof. Assignment.

Lemma

Let G be a planar graph and F be a face boundary in some planar embedding of G. Let G' be the graph obtained from G by adding a vertex v and joining v to each vertex of F. Then φ extends to an embedding of G'.

Proof. Exercise.

Facts About 3-Connected Planar Graphs

- They have a unique embedding in the plane/sphere (up to homeomorphism).
- They have an embedding in the plane where all edges are straight line segments and all faces are convex polygons.
- They are exactly the skeletons of polyhedra.

Theorem

If φ is an embedding of a 3-connected graph G, then the face boundaries of φ are exactly the non-separating induced cycles of G.

Theorem

$K_{3,3}$ and K_5 are nonplanar.

Theorem (Kuratowski – Version 2)

G is planar if and only if neither $K_{3,3}$ nor K_5 is a topological minor of G.

Definition: Minor ($\leq$)

A graph H is a minor of a graph G, denoted $H \leq G$, if H can be obtained from a subgraph G' of G (by deleting vertices or edges) by a sequence of edge contractions.

Note:

- H is a minor of G if and only if H is obtained from G by vertex deletions, edge deletions, and edge contractions.
- H is a minor of G if and only if there is a 'model' of H in G (vertices of H correspond to disjoint connected subgraphs of G, edge of H correspond to edges of G between subgraphs).

Proposition

If G is planar and $H \leq G$, then H is planar.

Proof. Since subgraphs of planar graphs are planar, it is enough to show that contracting a single edge in a planar graph keeps the graph planar. Consider a region that is equal to all points on the edge $e = \{u, v\}$, contract e by creating the new vertex x_{uv} in the middle of e and draw all neighbours of u and v to x_{uv}

Corollary

If G has $K_{3,3}$ or K_5 as a minor, G is nonplanar.

Proposition

If G has H as a topological minor, then G has H has a minor.

Proof. For each edge e of H, let P_e be the corresponding path of G. Let G' be the subgraph of G that is the union of all P_e. Now H is obtained from G' by contracting all but one edge in each path P_e.

Lemma

For every edge or vertex of a planar graph G and every disc D in the plane, there is an embedding of G contained in D such that the edge or vertex is in the boundary of the outer face of φ.

Theorem (Kuratowski)

G is planar if and only if it contains neither $K_{3,3}$ nor K_5 as a minor.

Proof. $(\Longleftarrow)$ Suppose for a contradiction that G has no $K_{3,3}$ or K_5 minor, but is nonplanar. Choose G so that $|V(G)| + |E(G)|$ is as small as possible.

Claim 1: G is connected.

Claim 1 ***Proof.*** Suppose not; let $G_1, \ldots, G_k$ be its components. Since there are ≥ 2 components, we have $|V(G_i)| + |E(G_i)| < |V(G)| + |V(G)|$, but the G_i are subgraphs of G, so none of the G_i have $K_{3,3}$ or K_5 as a minor. Therefore, by the minimality in the choice of G, all the G_i are planar.

We can combine planar embeddings of G_i to make a planar embedding of G, giving a contradiction.

Proof. To continue, we use, but not prove) the following: for any embedding φ of G and any edge e (or vertex v) of G, and any open disc $D \subseteq \mathbb{R}^2$, there is an embedding φ' of G such that $\varphi'(G) \subseteq D$ and e (or v) is contained in the boundary of the unbounded face of φ'.

Claim 2: G is 2-connected.

Claim 2 *Proof.* If not, then G has a cut vertex x. Let G_1, G_2 be proper subgraphs of G such that $G = G_1 \cup G_2$ and $V(G_1) \cap V(G_2) = \{x\}$. Since both G_i are smaller than G and have no $K_{3,3}$ or K_5-minor, both are planar.

Consider embeddings of G_1, G_2 in disjoint discs D_1, D_2 in the plane, where x is embedded by both in the boundary of the outer face. We can find an arc between the two copies of x in the resulting drawing to get an embedding of a graph G' such that $G'/e \cong G$ for some edge e. Since G' is planar and $G \cong G'/e$, G is also planar, a contradiction.

Claim 3: G is 3-connected.

Claim 3 *Proof.* Suppose not. There are vertices $x, y \in V(G)$ and subgraphs G_1, G_2 of G such that $G = G_1 \cup G_2$ and $V(G_1) \cap V(G_2) = \{x, y\}$. By a similar argument to the previous claim, G_1, G_2 are planar.

Let G_1', G_2' be obtained from G_1, G_2 respectively by adding a new edge f from x to y (choose a vertex w of $G_2 - \{x, y\}$ and take a $w\{x, y\}$-fan to get this path, by applying the Fan lemma to G). Since G is 2-connected, there is an xy-path P with ≥ 2 edges in G_2. Now G_1' is obtained from the subgraph $G_1 \cup P$ by contracting all but one edge of P. Since $K_{3,3}, K_5 \not\leq G$ and G_1' is a minor of G with fewer vertices and $K_{3,3}, K_5 \not\leq G'$, G_1' is planar. Similarly G_2' is planar. Now consider embeddings of G_1', G_2' in disjoint discs in $\mathbb{R}^2$ where e is on the outer face.

We can now combine these embeddings and use connectedness in the unbounded face to obtain a planar embedding of the following graph. So G' is planar, so $G = G' \setminus \{e_1, e_2\} / \{f_1, f_2\}$ is planar, a contradiction.

For every $e \in E(G)$, G/e and $G \setminus e$ are planar, because they are minors of G, so they have no $K_{3,3}$ or K_5-minor, and they are 'smaller' than G, so are not counterexamples.

Claim 4: G is simple.

Claim 4 *Proof.* If not, delete an edge e parallel to some other edge, draw $G \setminus e$ and add e back to the embedding.

Note that since G is nonplanar, $|V(G)| \geq 4$.

By a lemma, G has an edge $e = xy$ such that G/e is 3-connected. We also know that G/e is planar. Let u be the vertex of G/e corresponding to e and consider a planar embedding φ of G/e. Let v be the new contracted vertex, then $(G/e) - v$ is a 2-connected planar graph, so every face boundary is a cycle. Now v is embedded in some face of $(G/e) - v$ whose boundary is a cycle C, and all neighbours of v in G/e lie in C.

Lemma

Given X, Y of vertices in a cycle C, either

(i) there exist $x, x' \in X$ and $y, y' \in Y$ such that y, y' are in different components of $C - \{x, x'\}$ (x, y, x', y' in cyclic order) $\implies$ $K_{3,3}$-minor.

(ii) $|X \cap Y| \geq 3 \implies K_5$-minor.

(iii) there are edge-disjoint paths P_X, P_Y of C such that either $E(P_X) \cap E(P_Y) = \emptyset$, $P_X \cup P_Y = C$, and $X \subseteq V(P_X), Y \subseteq V(P_Y)$.

Lemma *Proof.* We may assume by symmetry that $|X| \leq |Y|$. If $|X| \leq 1$, choose P_X to be a path with one edge f containing all vertices in X and choose P_Y to be $C - f$. Then (iii) holds.

So $|X| \geq 2$. If $Y \setminus X = \emptyset$, then $X = Y$. Suppose this holds. If $|X| = |Y| = 2$, then let $\{a, b\} = X = Y$. Then choose P_X and P_Y to be the two distinct ab-paths in C. Now (iii) holds.

Otherwise, $|X| = |Y| \geq 3$, so $|X| \cap |Y| = |X| \geq 3$, so (ii) holds.

So we may assume that there exists $b \in Y \setminus X$. Since $b \notin X$, C is 2-connected and $|X| \geq 2$, there is a bX-fan in C of size 2. Let P_1, P_2 be the paths in this fan. Let $P_Y = P_1 \cup P_2$; since P_1, P_2 form a fan, P_Y has no internal vertices in X.

Let P_X be the other path in C between the ends of P_Y. Since P_Y has no internal vertices in X, we know $X \subseteq V(P_X)$. If $Y \subseteq V(P_Y)$, then (iii) holds.

Otherwise, there is some $b \in Y$ in a different component of $C \setminus \{\text{ends of } P_X\}$ from b, so (i) holds.

Proof. Let $X = \{\text{neighbours of } x \text{ in } C\}$ and $Y = \{\text{neighbours of } y \text{ in } C\}$. We now apply the lemma to X, Y, C. If (ii) holds, then x, y has three common neighbours $a, b, c \in C$. Now, the vertices a, b, c, x, y are the terminals of a topological K_5-minor of G. Therefore, G has a K_5-minor, a contradiction.

If (i) holds, then there exist a, b, a', b' in that order around C such that a, a' are neighbours of x and b, b' are neighbours of y. Now x, y, a, b, a', b' are the terminals of a topological $K_{3,3}$ minor of G. Therefore, G has a $K_{3,3}$-minor, a contradiction.

Suppose, therefore, that (iii) holds. We use the fact that for any polygon $C \subseteq \mathbb{R}^2$ with vertices in cyclic order $a_1, \ldots, a_t$ and any x in the interior of C, we can find arcs $A_1, \ldots, A_t$ from x to the a_i, intersecting only at x, and leaving x in the same cyclic order as the a_i occur around C.

Fact *Proof.* Inductively draw the arcs one by one.

Proof. Using the lemma, construct a planar embedding of G as follows:

- Take the embedding of $G/e - u$ we were considering.

- Add u back and use the lemma to construct arcs from u to all vertices in $X \cup Y$.
- Let D be a small disc centered at u. Within D, split u into two vertices x, y and use straight line segments to alter the embedding of G/e to an embedding of G.

This contradicts the nonplanarity of G. $\square$

The topological minor version of Kuratowski's Theorem is equivalent to the minor version because of the following fact.

Proposition

For a graph G, $K_{3,3}$ or K_5 is a minor of $G \iff K_{3,3}$ or K_5 is a topological minor of G.

This follows from 3 statements.

1. For all H, if H is a topological minor of G, then H is a minor of G.
2. For all H of maximum degree at most 3, if H is a minor of G, then H is a topological minor of G.
3. If G has K_5 a minor, then it has K_5 or $K_{3,3}$ as a topological minor.

Thus, version 1 and version 2 of Kuratowski's Theorem are equivalent.

Note: One can adapt our version of Kuratowski's Theorem to show that every planar graph can be drawn with all edges as straight line segments.

Theorem (Kuratowski – Alternative)

$K_{3,3}$ and K_5 are the excluded minors for planarity.

Theorem

$K_{3,3}$ and K_5 are the unique minor-minimal nonplanar graphs.

Theorem

G is toroidal if and only if G does not have *some graphs* as minors.

Theorem

G is linkedless-embeddable in $\mathbb{R}^3$ if and only if G does not contain *unknown list* as minors.

Theorem (Graph Minors)

A theorem stated like Kuratowski's Theorem for excluded minors.

Chapter 4

Matchings

Definition: Matching

A set $M \subseteq E(G)$ so that no two share an end.

Definition: Vertex Cover

A set $W \subseteq V(G)$ so that every edge of G has an end in W.

Observation: If M is a matching of G and W is a vertex cover of G, then $|M| \leq |W|$. This is because each edge in M has an end in W and no two have a common end.

Corollary

If M is a matching and W is a vertex cover such that $|M| = |W|$, then W contains exactly one end of each edge in M, and no other vertices.

Corollary

If $\nu(G)$ is the size of a maximum matching of G and $\tau(G)$ is the size of a minimum cover of G, then $\nu(G) \leq \tau(G)$.

Proposition

In an even cycle on $2n$ vertices, $\nu(G) = \tau(G) = n$.

Proof. Every other edge and every other vertex are a matching and a cover respectively, each of size n.

Proposition

In a path on n vertices, $\nu(G) = \tau(G) = \left\lfloor \frac{n}{2} \right\rfloor$.

Proof. If $V(P) = \{v_1, \ldots, v_n\}$, then $\left\{v_2, v_4, \ldots, v_{2\left\lfloor \frac{n}{2} \right\rfloor}\right\}$ is a vertex cover and $\{v_1v_2, v_3v_4, \ldots\}$

is a matching. Both have size $\left\lfloor \frac{n}{2} \right\rfloor$.

Note: The statement $\nu(G) = \tau(G)$ fails for odd cycles, because $\nu(C_{2n+1}) = n+1, \tau(C_{2n+1}) = n$.

Proposition

If $G_1, \ldots, G_k$ are the components of G, then $\nu(G) = \sum_{i=1}^{k} \nu(G_i)$ and $\tau(G) = \sum_{i=1}^{k} \tau(G_i)$.

Theorem (König 1931)

If G is a bipartite graph, then $\nu(G) = \tau(G)$.

Proof. (Rizzi 1999) We need to show that $\tau(G) \leq \nu(G)$ for bipartite G. Let G be a counterexample on as few edges as possible.

Claim: G has a vertex vertex of degree ≥ 3.

Claim *Proof.* If not, then every component is a path or a cycle, so König's theorem holds for each component, so holds for G since τ and ν are additive over components.

Let u be a vertex of degree ≥ 3, and v be a neighbour of u. We split into cases, depending on whether $\nu(G - v) = \nu(G)$. If $\nu(G - v) \leq \nu(G) - 1$, then let W_0 be a minimum vertex cover of $G - v$. Since $G - v$ is not a counterexample, we have $|W_0| = \nu(G - v) \leq \nu(G) - 1$. Since W_0 is a vertex cover of $G - v$, $W_0 \cup \{v\}$ is a vertex cover of G. So $\tau(G) \leq |W_0 \cup \{v\}| \leq (\nu(G) - 1) + 1 = \nu(G)$. This contradicts that G is a counterexample.

Otherwise, $\nu(G - v) = \nu(G)$. In other words, each maximum matching of $G - v$ is also a maximum matching of G. Let M be a maximum matching of both $G - v$ and G.

Since $\deg(u) \geq 3$, there is an edge f incident with u but not v, such that $f \notin M$. So $\nu(G - f) \geq |M| = \nu(G) \geq \nu(G - f)$ implying $\nu(G - f) = |M|$.

Since $|E(G - f)| < |E(G)|$, we know that $\tau(G - f) = \nu(G - f) = |M|$. Let W be a vertex cover of $G - f$ with $|W| = |M|$. We know that W contains exactly one end of each edge in M and nothing else.

In particular, $v \notin W$. Since W is a vertex cover, it contains at least one end of the edge uv, so we must have $u \in W$. By choice of W, W contains an end of every edge in $G - f$ and since $u \in W$, W also contains an end of f. Therefore, W is a vertex cover of G.

So, $\tau(G) \leq |W| = |M| = \nu(G)$, which contradicts $\tau(G) \geq \nu(G)$. □

König's theorem can be thought of in different ways for a bipartite graph G:

- Either G has a t-edge matching or there is a good reason it does not, a vertex cover of size of $< t$.
- There is a maximum matching M of G, together with a vertex cover W of G that

'proves' there is no larger matching.

Theorem (Petersen 1891)

Every bridgeless 3-regular graph has a perfect matching.

(Conjecture 1970)

There exists $\beta > 0$ such that every bridgeless 3-regular graph G has $\geq (1+\beta)^{|V(G)|}$ perfect matchings.

Theorem (Esperet/King/Kardos/Kral/Norin)

The conjecture is true for $\beta = 0.0001$.

Zero-Star Problem

Given an $n \times n$ matrix where some entries are given to be zero, can you fill in the other entries so that the matrix has nonzero determinant?

Proof. If there exist $a_1, \ldots, a_n$ permutations of $\{1, \ldots, n\}$ such that entries i, a_i are allowed to be nonzero, then the answer is yes. Encoding the matrix a bipartite graph with both sides of size n, we see that if the graph has a perfect matching, then the answer is yes.

By König's theorem, if there is no perfect matching, then there is a vertex cover in G of size $< n$. □

Idea: If G has a 'small' set of vertices whose deletion gives a graph with a 'large' number of odd components, then matchings in G cannot be too big.

Definition: M-Saturated

Given a matching M of G, the vertices of G that are an end of an edge in M are M-saturated vertices.

Definition: M-Exposed/Unsaturated

Vertices not M-saturated.

We say M saturates its saturated vertices and avoids its unsaturated vertices.

Definition: $oc(G)$

The number of components in G with an odd number of vertices.

Proposition

If M is a matching of G and X is a set of vertices of G, then there are at least $oc(G-X) - |X|$ M-unsaturated vertices in G.

Proof. Let $\mathcal{C}$ be the set of odd components of $G - X$ that contain a vertex that is matched by M to a vertex in X. Since no two edge of M have the same end in X, there are at most $|X|$ edges of M from X to $V - X$, so at most $|X|$ components of $G - X$ contain a vertex matched by M to a vertex in X. Therefore, $|\mathcal{C}| \leq |X|$, so there are at least $oc(G - X) - |\mathcal{C}| \geq oc(G - X) - |X|$ odd components of $G - X$ that contain no vertex matched to anything in X.

For each such component H, no edge of M has exactly one end in H, so the number of saturated vertices in H is even. Since H is odd, it must contain at ≥ 1 unsaturated vertex. There are $\geq oc(G - X) - |X|$ different H, so G has this many M-unsaturated vertices.

Corollary

If there is a set X such that $oc(G - X) > |X|$, then G has no perfect matching.

Proof. By the bound, every matching avoids at least $oc(G-X)-|X| > 0$ vertices, so there is no perfect matching.

Corollary

If $X \subseteq V(G)$, then $\nu(G) \leq \frac{1}{2}(|V(G)| - oc(G - X) + |X|)$.

Proof. Let M be a matching of G. Then there are at least $oc(G-X)-|X|$ M-unsaturated vertices. So there are at most $|V(G)| - oc(G - X) + |X|$ saturated vertices. Therefore, $|M| \leq \frac{1}{2}(|V(G)| - oc(G - X) + |X|)$. This holds for all M, so we get the bound on ν.

Corollary

If $X \subseteq V(G)$ and M is a matching of size $\geq \frac{1}{2}(|V(G)|-oc(G-X)+|X|)$, then equality holds if

- every odd component of $G - X$ has a matching M saturating all but one vertex.
- exactly $|X|$ odd components of $G - X$ contain a vertex matched to a vertex of X.
- every even component of $G - X$ has a perfect matching.

Corollary

$$\nu(G) \leq \min_{X \subseteq V(G)} \frac{1}{2}(|V(G)| - oc(G - X) + |X|)$$

Theorem (Tutte-Berge Formula)

$$\nu(G) = \min_{X \subseteq V(G)} \frac{1}{2}(|V(G)| - oc(G - X) + |X|)$$

Proof. Idea: try to understand the matching number of graphs with the property that deleting any one vertex does not change the matching number.

Definition: Hypomatchable/Factor-Critical

A graph H is hypomatchable if H is connected and $\nu(H-v) = \nu(H)$ for all $v \in V(H)$.

Proposition

If H is a hypomatchable graph, then $V(H)$ is odd and $\nu(H) = \frac{1}{2}(|V(H)|-1)$, i.e. H has a matching saturating all but one vertex.

Proof. Define a relation $\sim$ on $V(H)$ by $u \sim v$ if and only if $u = v$ or $\nu(H - \{u, v\}) < \nu(H)$.

Lemma

$\sim$ is symmetric and reflexive.

Lemma

$\sim$ is transitive.

Proof. Prove if $u \sim v, v \sim w$, then $u \sim w$. Suppose $u \sim v, v \sim w, u \not\sim w$.

Since H is hypomatchable, there is a maximum matching M_v that avoids v. Since $u \not\sim w$, we have $\nu(G - \{u, w\}) = \nu(G)$ so there is a maximum matching M_{uw} avoiding both u and w.

We analyze the structure of M_{uw} and M_v to find either a larger matching (contradicting maximality) or a maximum matching avoiding v and one of u and w (contradicting $u \sim v$ or $v \sim w$).

Since no matching avoids v and one of u and w, we must have that M_v saturates u and w. Similarly, M_{uw} saturates v.

Consider the subgraph H_0 of H with $V(H_0) = V(H)$ and $E(H_0) = E(M_{uv}) \triangle E(M_v) = (E(M_{uw}) \cup E(M_v)) \setminus (E(M_{uw}) \cap E(M_v))$ (symmetric difference). Since no vertex is incident with two edges in the same matching, the paths must alternate between edges in M_{uw} and edges in M_v.

If some path component of H_0 has an odd number of edges, then it contains more edges in one matching than in the other. Let M be the matching in the path with fewer edges than the other and the larger matching be M'. Replacing the edges in $M \cap E(P)$ with the edges in $M' \cap E(P)$ gives a larger matching than M in H, contradicting the maximality of M. So every path of H_0 has an even number of edges.

Since M_{uw} saturates v but not u or w, and M_v saturates u and w but not v, each of u, v, w has degree 1 in H_0. Since every component of H_0 is a path or a cycle, each of u, v, w is an end of a path component of H_0. So there is some path component P of H_0 having one end $x \in \{u, v, w\}$ and whose other end is not in $\{u, v, w\}$.

Let $M \in \{M_{uv}, M_v\}$ that saturates x and M' be the other matching. Taking M, removing the edges in $M \cap E(P)$ and adding back the edges in $M' \cap E(P)$ gives matching that saturates strictly fewer vertices in $\{u, v, w\}$ than M does. In all cases ($x = u, x = v, x = w$), this contradicts either $u \sim v$ or $v \sim w$.

Lemma

If u, v are adjacent in H, then $u \sim v$.

Proof. Since uv is an edge, $\nu(H) \geq \nu(H - \{u, v\}) + 1$ because for each matching M of $H - \{u, v\}$, $M \cup \{(u, v)\}$ is a matching of H. Therefore, $\nu(H) > \nu(H - \{u, v\})$.

Proof. (Proposition) We show that $u \sim v$ for all $u, v \in V(H)$. Let $u = x_1, x_2, \ldots, x_k = v$ be a uv-path. Let i be maximal such that $u \sim x_i$. If $i = k$, we have the result. We know i makes sense, since $u \sim u = x_1$.

If $i < k$, then we know $u \sim x_i$ and $x_i \sim x_{i+1}$ because x_i, x_{i+1} are adjacent. So $u \sim x_{i+1}$ by transitivity, which contradicts the maximality of i.

Now consider a maximum matching M of H. Since $\nu(H - x) = \nu(H)$ for all $x \in V(H)$, M cannot be a perfect matching. So $2|M| \leq V(H) - 1$.

If there are two M-unsaturated vertices u, v, then $\nu(H - \{u, v\}) \geq |M| = \nu(H)$, this contradicts $u \sim v$. Therefore, $2|M| \geq |V(H)| - 1$.

Therefore, $\nu(H) = |M| = \frac{1}{2}(|V(H)| - 1)$ and $|V(H)|$ is odd.

Corollary

If H is any graph such that $\nu(H - x) = \nu(H)$ for all $x \in V(H)$, then every component of H is odd, and $\nu(H) = \frac{1}{2}(|V(H)| - oc(H))$.

Proof. Let $H_1, \ldots, H_k$ be the components of H. We argue that each H_t is hypomatchable; let $x \in V(H_k)$. We have

$$\begin{aligned} \nu(H) = \nu(H - x) &= \sum_{\substack{i=1 \\ i \neq t}}^{k} \nu(H_i) + \nu(H_t - x) \\ &= \sum_{i=1}^{k} \nu(H_i) + \nu(H_t - x) - \nu(H_t) \\ &= \nu(H) + \nu(H_t - x) - \nu(H_t) \end{aligned}$$

So $\nu(H_t - x) = \nu(H_t)$. So H_t is hypomatchable. Each H_i has an odd number of vertices and

$\nu(H_i) = \frac{1}{2}(|V(H_i)| - 1)$, thus

$$\begin{aligned}
\nu(H) &= \sum_{i=1}^{k} \nu(H_i) \\
&= \sum_{i=1}^{k} \frac{1}{2}(|V(H_i)| - 1) \\
&= \frac{1}{2}\sum_{i=1}^{k} |V(H_i)| - \frac{1}{2}k \\
&= \frac{1}{2}(|V(H)| - oc(H))
\end{aligned}$$

Theorem (Tutte)

If $oc(G - X) \leq |X|$ for all $X \subseteq V(G)$, then G has a perfect matching.