

An $O(\log n)$ -Approximation Algorithm for (p, q) -Flexible Graph Connectivity via Independent Rounding

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Introduction

Problem: (p, q) -Flexible Graph Connectivity (FGC)

Given a graph $G = (V, E)$, where $E = \mathcal{S} \cup \mathcal{U}$ is partitioned into safe and unsafe edges, edge costs $c_e \in \mathbb{R}_{\geq 0}$ for all $e \in E$, find a minimum-cost $F \subseteq E$ such that for any $F' \subseteq \mathcal{U}$ with $|F'| \leq q$, the subgraph $(V, F \setminus F')$ is p -edge-connected.

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Closely related problem:

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Given an unweighted graph, find the smallest k -edge-connected subgraph that spans all vertices.

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Special cases:

- $(1, 1)$ -FGC \rightarrow 2-ECSS, tree augmentation problem, forest augmentation problem.
- $(1, 0)$ -FGC \rightarrow minimum spanning tree.

Previous Work

- [AHM20]: Adjiashvili et al. introduced (p, q) -FGC.
 - $(1, 1)$ -FGC: 2.527-approximation
 - $(1, q)$ -FGC: $\left(\frac{2}{q+1} + \nu_{q+1} \cdot \frac{q}{q+1}\right)$ -approximation

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- [BCHI21, BCHI24]
 - $(p, 1)$ -FGC: 4-approximation
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 - (p, q) -FGC: $O(q \log n)$ -approximation

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- [CJ23]
 - $(p, 2)$ -FGC, $(p, 3)$ -FGC, $(2p, 4)$ -FGC: $O(p)$ -approximation
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- [BCKS24]
 - $(1, q)$ -FGC: 7-approximation

Main Result

Theorem 1.1

Let $\mathcal{I} = (G = (V, E), \mathcal{S}, \mathcal{U}, c, p, q)$ be an instance of (p, q) -FGC. There is a randomized algorithm that outputs, with probability $\geq \frac{1}{3}$, a feasible solution of cost $\leq 200 \log n \cdot \text{OPT}$. The runtime is polynomial in n, p, q , and the input encoding size.

Overview of algorithm and idea:

- Randomized rounding of LP relaxation.
- Knapsack cover inequalities.
- Efficient separation oracle.

Characterizing Feasible Solutions

Proposition 1.2 (Feasibility)

An edge set $F \subseteq E$ is feasible for (p, q) -FGC if and only if for every $\emptyset \neq R \subsetneq V$, $F \cap \delta(R)$ contains p safe edges or $p + q$ (safe or unsafe) edges.

Boyd et al. gives a connection between (p, q) -FGC and Cap- k -ECSS.

Definition: Cap- k -ECSS

Given a graph $G = (V, E)$, edge-costs c_e for all $e \in E$, integer edge-capacities u_e for all $e \in E$, and a global edge-connectivity parameter $k \in \mathbb{Z}_{\geq 1}$, find a minimum-cost $F \subseteq E$ such that for every $\emptyset \neq R \subsetneq V$, $u(\delta_F(R)) \geq k$.

- Feasible solution to (p, q) -FGC \rightarrow feasible in Cap- k -ECSS with parameters $k = p(p + q)$, $u_e = p + q$, $e \in \mathcal{S}$, and $u_e = p$, $e \in \mathcal{U}$.
- Covering constraints for Cap- k -ECSS are valid for (p, q) -FGC, so $\text{OPT}(\text{Cap-}k\text{-ECSS}) \leq \text{OPT}$.

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Valid inequalities:

$$(p + q) \cdot x(\delta(R) \cap \mathcal{S}) + p \cdot x(\delta(R) \cap \mathcal{U}) \geq p(p + q), \forall \emptyset \neq R \subsetneq V \quad (1)$$

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In [CJ23], they add a robustness constraint to get a valid IP:

$$x(\delta(R) \setminus F') \geq p, \forall \emptyset \neq R \subsetneq V, \forall F' \subseteq \delta(R) \cap \mathcal{U}, |F'| \leq q \quad (2)$$

- Separation oracle runs in $n^{O(q)}$ running time (check all unsafe edges subsets).
- Instead use knapsack cover inequalities.

Knapsack Cover Inequality

Lemma 1.3

$F \subseteq E$ is a feasible solution to (p, q) -FGC if and only if for every nonempty $R \subsetneq V$ and every partition $\delta_F(R) = J \cup K$, the follow holds:

$$(p - |J \cap S|)^+ \cdot |K| + (q - |J \cap U|)^+ \cdot |K \cap S| \geq (p - |J \cap S|)^+ (p + q - |J|)^+ \quad (3)$$

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By feasibility, either $|\delta_F(R)| \geq p + q$ or $|\delta_F(R) \cap S| \geq p$. Recall $K = \delta_F(R) \setminus J$, so we have

- (a) $|K| \geq p + q - |J|$, or

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- (a) $|K| \geq p + q - |J|$, or
- (b) $|K \cap S| \geq p - |J \cap S|$

In the first case, ((3)) holds. For the second case, we can write the inequality

$$(p - |J \cap S|) + (q - |J \cap U|)^+ \geq p + q - |J|$$

So

$$\text{LHS} \geq ((p - |J \cap S|) + (q - |J \cap U|)^+) \cdot |K \cap S| \geq (p + q - |J|)(p - |J \cap S|) \geq \text{RHS}$$

Proof Continued

(\Leftarrow) Suppose F is infeasible. There must be a nontrivial R such that

- $|\delta_F(R)| < p + q$, AND
- $|\delta_F(R) \cap \mathcal{S}| < p$.

Take $J = \delta_F(R)$ and $K = \emptyset$. LHS = 0 and RHS > 0. This violates ((3)).



Checking Feasibility

Lemma 1.4 [BCHI24]

For any $F \subseteq E$, we can efficiently check the feasibility of F for the given instance of (p, q) -FGC.

Checking Feasibility

Lemma 1.5 [BCHI24]

For any $F \subseteq E$, we can efficiently check the feasibility of F for the given instance of (p, q) -FGC.

Theorem 1.5 [Kar93]

For any $\alpha \geq 1$, any capacitated network has at most $O(n^{2\alpha})$ α -approximate minimum cuts.

LP Relaxation of (p, q) -FGC

For each edge $e \in E$, we have a decision variable $x_e \in \{0, 1\}$ that denotes inclusion of e in F . We relax integrality constraints to get the LP relaxation:

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subject to
$$(p - |J \cap S|)^+ \cdot x(K) + (q - |J \cap U|)^+ \cdot x(K \cap S) \\ \geq (p - |J \cap S|)^+ (p + q - |J|)^+, \quad (5)$$

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$$0 \leq x_e \leq 1, \quad \forall e \in E \quad (6)$$

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Let OPT_{LP} be the optimum value of this LP.

Lemma 2.1

$$\text{OPT}_{LP} \leq \text{OPT}.$$

Note for $J = \emptyset$, (5) becomes

$$p \cdot x(\delta(R)) + q \cdot x(\delta(R) \cap S) \geq p(p + q)$$

Capacitated Graph

We build capacitated graph $H_x = (V, E)$, with capacities $u_x : E \rightarrow \mathbb{R}_{\geq 0}$, where

$$u_x(e) := (p + q)x_e, \quad \forall e \in \mathcal{S}$$

and

$$u_x(e) := px_e, \quad \forall e \in \mathcal{U}$$

Constraint (reposted)

$$(p - |J \cap \mathcal{S}|)^+ \cdot x(K) + (q - |J \cap \mathcal{U}|)^+ \cdot x(K \cap S) \geq (p - |J \cap \mathcal{S}|)^+ (p + q - |J|)^+ \quad (5)$$

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Lemma 2.2 (Feasibility of x)

Let $x \in [0, 1]^E$ and $H_x = (G, u_x)$. We have the following:

- (1) If the minimum cut $u_x(\delta(R))$ in H_x is $< p(p + q)$, then x is infeasible (for this R and $J = \emptyset$).
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Proof. For (1), constraint (5) becomes $u(\delta(R)) < p(p + q)$ for $J = \emptyset$. This is a violated inequality.

Proof Continued

For (2), suppose there is an R with $u_x(\delta(R)) \geq 2p(p + q)$ and let $J \subseteq \delta(R)$ be arbitrary. Constraint (5) is satisfied when $|J \cap S| \geq p$ or $|J| \geq p + q$.

Proof Continued

For (2), suppose there is an R with $u_x(\delta(R)) \geq 2p(p+q)$ and let $J \subseteq \delta(R)$ be arbitrary. Constraint (5) is satisfied when $|J \cap S| \geq p$ or $|J| \geq p+q$. Assume otherwise, i.e. $|J \cap S| < p$ and $|J| < p+q$. RHS is now nonnegative, i.e.

$$\text{RHS} = (p - |J \cap S|)(p + q - |J|)$$

By definition of u_x , we have

$$u_x(\delta(R)) = (p+q)x(\delta(R) \cap S) + px(\delta(R) \cap U)$$

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By $u_x(\delta(R)) \geq 2p(p+q)$ and an averaging argument,

- $x(\delta(R) \cap S) \geq p$, or
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Assume first case:

$$\begin{aligned}\text{LHS of (5)} &= (p - |J \cap S|)^+ \cdot x(\delta(R) \setminus J) + (q - |J \cap U|)^+ \cdot x((\delta(R) \setminus J) \cap S) \\ &\geq x((\delta(R) \setminus J) \cap S)(p + q - |J|) \\ &= [x(\delta(R) \cap S) - x(J \cap S)](p + q - |J|) \\ &\geq (p - |J \cap S|)(p + q - |J|) \\ &= \text{RHS of (5)}\end{aligned}$$

Proof Continued

For the second case:

$$\begin{aligned}\text{LHS of (5)} &\geq (p - |J \cap S|) \cdot x(\delta(R) \setminus J) \\&\geq (p - |J \cap S|)(x(\delta(R)) - |J|) \\&\geq (p - |J \cap S|)(p + q - |J|) \\&= \text{RHS of (5)}\end{aligned}$$



Knapsack Cover Inequality Checking

Lemma 2.3

Let $x \in [0, 1]^E$ and nontrivial $R \subseteq V$. Let L_s and L_u be ordered lists of safe and unsafe edges in $\delta(R)$ in nonincreasing order of their x_e values, respectively. For nonnegative $a \in \{0, 1, \dots, \min(p - 1, |L_s|)\}$ and $b \in \{0, 1, \dots, \min(p + q - 1, |L_u|)\}$, let $J_{a,b}$ be the edge-sets of the first a edges in L_s and first b edges in L_u . Let $\mathcal{J} = \bigcup_{a,b} J_{a,b}$. If x satisfies (5) for every choice of $J \in \mathcal{J}$, then x satisfies the same constraint for every choice of $J \subseteq \delta(R)$.

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Proof. Consider the numerical quantities in constraint (5):

$$\alpha_1 := (p - |J \cap S|)^+, \alpha_2 := (q - |J \cap U|)^+, \alpha_3 := (p - |J \cap S|)^+(p + q - |J|)^+$$

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$$\alpha_1 := (p - |J \cap S|)^+, \alpha_2 := (q - |J \cap U|)^+, \alpha_3 := (p - |J \cap S|)^+(p + q - |J|)^+$$

Since $a := |J \cap S|$ and $b := |J \cap U|$ determine these values, we can group the possible choices of $a \in \{0, 1, \dots, \min(p - 1, |\delta(R) \cap S|)\}$ and $b \in \{0, 1, \dots, \min(p + q - 1, |\delta(R) \cap U|)\}$.

If the RHS is 0, the constraint is trivially satisfied.

Knapsack Cover Inequality Checking

Lemma 2.3

Let $x \in [0, 1]^E$ and nontrivial $R \subseteq V$. Let L_s and L_u be ordered lists of safe and unsafe edges in $\delta(R)$ in nonincreasing order of their x_e values, respectively. For nonnegative $a \in \{0, 1, \dots, \min(p - 1, |L_s|)\}$ and $b \in \{0, 1, \dots, \min(p + q - 1, |L_u|)\}$, let $J_{a,b}$ be the edge-sets of the first a edges in L_s and first b edges in L_u . Let $\mathcal{J} = \bigcup_{a,b} J_{a,b}$. If x satisfies (5) for every choice of $J \in \mathcal{J}$, then x satisfies the same constraint for every choice of $J \subseteq \delta(R)$.

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If the RHS is 0, the constraint is trivially satisfied. Fix a and b . Group variables by safe and unsafe edges, then (5) can be re-written as

$$(\alpha_1 + \alpha_2) \sum_{e \in \delta_S(R) \setminus J} x_e + \alpha_1 \sum_{e \in \delta_U(R) \setminus J} x_e \geq \alpha_3$$

Separation Oracle

The set $J_{a,b}$ minimizes the LHS of the inequality (since $J_{a,b}$ are the biggest x_e values by L_s and L_u). Thus, if (5) is satisfied for $J_{a,b}$, then it is satisfied for all J . And if the constraint is satisfied for all a, b , then every choice of $J \subseteq \delta(R)$ is satisfied.



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Using both lemmas, we can design a separation oracle for the LP.

Lemma 2.4

Let $x \in [0, 1]^E$. In time polynomial in n, p, q , we can determine whether x is feasible, or find a violated inequality.

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Let $x \in [0, 1]^E$. In time polynomial in n, p, q , we can determine whether x is feasible, or find a violated inequality.

Proof. Let $H_x = (G, u_x)$. We can compute a minimum cut $(R^*, \overline{R^*})$ in H_x efficiently. Let $\lambda := u_x(\delta(R^*))$. If $\lambda < p(p + q)$, we get a violated inequality $(R = R^*, J = \emptyset)$ by using Lemma 2.2.

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Suppose $\lambda \geq p(p+q)$. By second part of Lemma 2.2, an R with $u_x(\delta(R)) \geq 2p(p+q)$ cannot give a violated inequality for any choice of $J \subseteq \delta(R)$. Thus, we focus on when $\lambda < 2p(p+q)$. Define \mathcal{R} as the set of vertices whose cut capacity is $< 2p(p+q)$. $|\mathcal{R}| = O(|V|^4)$ by Theorem 1.5, since $\lambda \geq p(p+q)$. Using [NNI97], we can efficiently enumerate \mathcal{R} .

Proof Continued

Fix $R \in \mathcal{R}$ and let L_s and L_u be the nonincreasing list of safe and unsafe edges by x_e . By Lemma 2.3, if there is a violation in (5) for some J , then there is a violation for some $J_{a,b}$. Thus, we find either a violated inequality for this R , or we have a certificate that there are no violations for any J .



Proof Continued

Fix $R \in \mathcal{R}$ and let L_s and L_u be the nonincreasing list of safe and unsafe edges by x_e . By Lemma 2.3, if there is a violation in (5) for some J , then there is a violation for some $J_{a,b}$. Thus, we find either a violated inequality for this R , or we have a certificate that there are no violations for any J . ■

Theorem 2.5

Suppose that we are given a feasible instance of (p, q) -FGC. We can efficiently compute a vector x that is feasible for the LP relaxation of (p, q) -FGC and also satisfies

$$c^T x := \sum_{e \in E} c_e x_e \leq \text{OPT}$$

Independent Rounding Algorithm

Algorithm: Independent Rounding For (p, q) -FGC

1. Solve LP relaxation for solution x , using separation oracle and Ellipsoid method.
2. $y_e = \min(1, 100 \log n \cdot x_e)$.
3. $F = \{e \in E : \text{choose } e \text{ with probability } y_e\}$.

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Theorem 3.1

With probability at least $1/3$, F is feasible and has cost at most $200 \log n \cdot c^T x$.

We prove this by union-bound over all cuts in the graph. Note by Chernoff bound, we have

$$\Pr[Z < \mu/3] \leq e^{-\mu/5}$$

where Z is the sum of binary random variables and $\mu = E[Z]$.

Bad Event

Define $A := \{e \in E : y_e = 1\}$ as the edges that are always in F . For any $e \notin A$,

$$y_e = 100 \log n \cdot x_e < 1$$

Define B_R as the indicator random variable for the event

$$\{|\delta_F(R) \cap S| < p \text{ and } |\delta_F(R)| < p + q\}$$

$B_R = 1$ whenever $\delta(R)$ certifies infeasibility of F .

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If $|\delta_A(R) \cap S| \geq p$ or $|\delta_A(R)| \geq p + q$, then $\Pr[B_R = 1] = 0$.

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Thus, we focus on $|\delta_A(R) \cap S| < p$ and $|\delta_A(R)| < p + q$. Since x is feasible, it satisfies (5) for this R and all $J \subseteq \delta(R)$. Rewriting the constraint as sum of safe and unsafe edges:

$$\begin{aligned} ((p - |J \cap S| + (q - |J \cap U|)^+) \cdot x(\delta(K \cap S)) + (p - |J \cap S|) \cdot x(K \cap U) \\ \geq (p - |J \cap S|)(p + q - |J|) \quad (7) \end{aligned}$$

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It is natural to take $J = \delta_A(R)$ and $K = \delta(R) \setminus J$. $y_e < 1$ for all $e \in K$.

Low Probability of B_R

Let Y_e denote whether $e \in F$ for $e \in K$. Then

$$E[Y_e] = \Pr[e \in F] = y_e = 100 \log n \cdot x_e$$

Define $Z_s = \sum_{e \in K \cap S} Y_e$ and $Z_u = \sum_{e \in K \cap U} Y_e$. Note $E[Z_s] = y(K \cap S)$ and $E[Z_u] = y(K \cap U)$.

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$$\Pr[B_R = 1] \leq n^{-10}$$

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Proof. If R satisfies $|\delta_A(R) \cap S| \geq p$ or $|\delta_A(R)| \geq p + q$, then this is trivial. By an averaging argument of (7), we have either:

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1. $((p - |J \cap S|) + (q - |J \cap U|)^+) \cdot x(\delta(K \cap S)) \geq \text{RHS}/2$. Since,

- $y(K \cap S) = 100 \log n \cdot x(K \cap S)$, and
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So,

$$x(K \cap S) \geq (p - |J \cap S|)/2 \implies y(K \cap S) \geq 50 \log n(p - |J \cap S|) =: \mu_1$$

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2. Similarly,

$$y(K \cap U) \geq 50 \log n(p + q - |J|) =: \mu_2$$

Low Probability of B_R

Thus, since $p - |J \cap \mathcal{S}| = p - |\delta_A(R) \cap \mathcal{S}| \geq 1$ and
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Low Probability of B_R

Thus, since $p - |J \cap \mathcal{S}| = p - |\delta_A(R) \cap \mathcal{S}| \geq 1$ and
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1. $E[Z_s] = y(K \cap \mathcal{S}) \geq 50 \log n(p - |J \cap \mathcal{S}|) =: \mu_1$

$$\begin{aligned}\Pr[B_R = 1] &\leq \Pr[|\delta_F(R) \cap \mathcal{S}| < p] \\&= \Pr[|F \cap K \cap \mathcal{S}| < p - |J \cap \mathcal{S}|] \\&= \Pr[Z_s < p - |J \cap \mathcal{S}|] \leq \Pr[Z_s < \mu_1/3] \\&\leq e^{-10 \log n(p - |J \cap \mathcal{S}|)} \\&\leq e^{-10 \log n} = n^{-10}\end{aligned}$$

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Thus, since $p - |J \cap \mathcal{S}| = p - |\delta_A(R) \cap \mathcal{S}| \geq 1$ and $p + q - |J| = p + q - |\delta_A(R)| \geq 1$, we have

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Stronger Bound for Large u_x Cuts

Lemma 3.4

Suppose $u_x(\delta(R)) \geq \ell p(p + q)$ holds for $\ell \geq 4$, then

$$\Pr[B_R = 1] \leq n^{-5\ell}$$

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Suppose $u_x(\delta(R)) \geq \ell p(p + q)$ holds for $\ell \geq 4$, then

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Proof. Assume R does not satisfy B_R event, otherwise trivial. By definition of u_x ,

$$u_x(\delta(R)) = (p + q) \cdot x(\delta(R) \cap \mathcal{S}) + p \cdot x(\delta(R) \cap \mathcal{U}) \geq \ell p(p + q)$$

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By an averaging argument, we have

- $x(\delta(R) \cap \mathcal{S}) = x((J \cup K) \cap \mathcal{S}) \geq \ell p/2$, or
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By nontrivial assumption, $x(J \cap \mathcal{S}) \leq |J \cap \mathcal{S}| < p$ and $x(J \cap \mathcal{U}) \leq x(J) \leq |J| < p+q$. So

- $x(K \cap \mathcal{S}) \geq \ell p/4$, or
- $x(K \cap \mathcal{U}) \geq \ell(p+q)/4$.

Stronger Bound for Large u_x Cuts

1. $y(K \cap \mathcal{S}) \geq 100 \log n \cdot x(K \cap \mathcal{S}) \geq 25\ell p \log n =: \mu_1$

$$\begin{aligned}\Pr[B_R = 1] &\leq \Pr[|\delta_F(R) \cap \mathcal{S}| < p] \\&= \Pr[|F \cap K \cap \mathcal{S}| < p - |J \cap \mathcal{S}|] \\&\leq \Pr[|F \cap K \cap \mathcal{S}| < p] \\&\leq \Pr[Z_s < \mu_1/3] \\&\leq e^{-5\ell p \log n} \\&\leq n^{-5\ell}\end{aligned}$$

2. $y(K \cap \mathcal{U}) \geq 100 \log n \cdot x(K \cap \mathcal{U}) \geq 25\ell(p+q) \log n =: \mu_2$

$$\begin{aligned}\Pr[B_R = 1] &\leq \Pr[|\delta_F(R)| < p + q] \\&\leq \Pr[|F \cap K \cap \mathcal{U}| < p + q] \\&\leq \Pr[Z_u < \mu_2/3] \\&\leq e^{-5\ell(p+q) \log n} \\&\leq n^{-5\ell}\end{aligned}$$



Proof of Main Theorem

Proof of Theorem 3.1. First,

$$\begin{aligned} E[c(F)] &= \sum_{e \in E} c_e E[Y_e] \leq \sum_{e \in E} c_e (100 \log n \cdot x_e) = 100 \log n \sum_{e \in E} c_e x_e \\ &\leq 100 \log n \cdot \text{OPT} \end{aligned}$$

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$$\Pr[c(F) > 200 \log n \cdot \text{OPT}] \leq \frac{E[c(F)]}{200 \log n \cdot \text{OPT}} = \frac{1}{2}$$

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Now we show infeasible F happens with probability $\leq 1/6$. For F to be infeasible, $B_R = 1$ for at least one nontrivial R . Partition all cuts of H_x by their size, i.e.

$$C_{<4} \cup C_5 \cup C_6 \cup \dots$$

where $C_{<4}$ is all R such that $u_x(\delta(R)) \in [p(p+q), 4p(p+q))$ and C_ℓ has all R where $u_x(\delta(R)) \in [\ell p(p+q), (\ell+1)p(p+q))$.

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Note $C_\ell = \emptyset$ when $\ell > |E|$ since the capacity of each edge is $\leq p+q$, so largest capacity $\leq |E|(p+q)$.

Main Theorem Proof Continued

By Karger, $|C_{<4}| = O(n^8)$ since minimum cut is $\geq p(p+q)$. The probability of infeasibility (bad event) is $\Pr[B_R = 1] \leq n^{-10}$ for any R .

Main Theorem Proof Continued

By Karger, $|C_{<4}| = O(n^8)$ since minimum cut is $\geq p(p+q)$. The probability of infeasibility (bad event) is $\Pr[B_R = 1] \leq n^{-10}$ for any R . Over all R ,

$$\sum_{R \in C_{<4}} \Pr[B_R = 1] \leq O(n^8) \cdot n^{-10} = O(n^{-2})$$

Main Theorem Proof Continued

By Karger, $|C_{<4}| = O(n^8)$ since minimum cut is $\geq p(p+q)$. The probability of infeasibility (bad event) is $\Pr[B_R = 1] \leq n^{-10}$ for any R . Over all R ,

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By Karger, $|C_\ell| \leq O(n^{2\ell+2})$ for any $\ell \geq 4$. A bad event happening has probability $\Pr[B_R = 1] \leq n^{-5\ell}$.

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Thus, over all cuts in H_x ,

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Thus, over all cuts in H_x ,

$$\sum_R \Pr[B_R = 1] = \sum_{R \in C_{<4}} \Pr[B_R = 1] + \sum_{\ell \geq 4} \left(\sum_{R \in C_\ell} \Pr[B_R = 1] \right)$$

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$$\begin{aligned} \sum_R \Pr[B_R = 1] &= \sum_{R \in C_{<4}} \Pr[B_R = 1] + \sum_{\ell \geq 4} \left(\sum_{R \in C_\ell} \Pr[B_R = 1] \right) \\ &\leq O(n^{-2}) + \sum_{\ell \geq 4} O(n^{-2\ell}) \leq \frac{1}{6} \end{aligned}$$

for large n .

Main Theorem Proof Continued

Let E_1 be the event that $c(F) \leq 200 \log n(c^T x)$ and E_2 be the event of F being a feasible solution. We just showed

$$\Pr[\bar{E}_1] \leq \frac{1}{2}, \quad \Pr[\bar{E}_2] \leq \frac{1}{6}$$

Main Theorem Proof Continued

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Thus, the probability of F being a feasible solution and cost at most $200 \log n \cdot \text{OPT}$ is

$$\begin{aligned}\Pr[E_1 \wedge E_2] &= 1 - \Pr[\bar{E}_1 \vee \bar{E}_2] \\ &\geq 1 - \left(\Pr[\bar{E}_1] + \Pr[\bar{E}_2] \right) \text{ (union-bound)} \\ &\geq 1 - \left(\frac{1}{2} + \frac{1}{6} \right) \\ &= 1 - \frac{2}{3} \\ &= \frac{1}{3}\end{aligned}$$

Main Theorem Proof Continued

Let E_1 be the event that $c(F) \leq 200 \log n (c^T x)$ and E_2 be the event of F being a feasible solution. We just showed

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Thus, with probability at least $1/3$, the solution F outputted is feasible and has cost at most $200 \log n \cdot \text{OPT}$.

Conclusion

This paper gives first $O(\log n)$ -approximation for general (p, q) -FGC.

It remains open to obtain an $f(p, q)$ -approximation for some function f . The integrality gap is also open for the knapsack cover inequality LP relaxation, whether it can be bounded as $f(p, q)$ or some constant.

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General survivable network design has an iterative rounding method. However, demand function is not skew supermodular, so we cannot do an uncrossing argument. The constraints also depend on J , so the coefficients are not uniform.

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