CS 365 Models of Computation (Advanced)

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Chapter 1

Introduction

1.1 Two Simplifying Restrictions

Definition: Computational Problem

A task where for each possible input to the problem, there is one or more valid outputs that is to be produced.

This however is too broad, so we impose two simplifying restrictions.

Simplifying Restriction 1

We only consider problems whose inputs are binary strings.

The binary alphabet is $\{0, 1\}$ and the set of strings of length n is denoted $\{0, 1\}^n$. The unique string in $\{0, 1\}^0$ is the empty string, denoted ε.

We write $\{0, 1\}^* = \bigcup_{n \geq 0} \{0, 1\}^n$ to denote the set of all possible binary strings.

Proposition

For every finite set $\mathcal{X}$ with k elements, there is a one-to-one encoding function $h : \mathcal{X} \to \{0, 1\}^{\lceil \log k \rceil}$.

Proof. Fix any ordering $a_1, \ldots, a_k$ of the elements of $\mathcal{X}$. Then define the encoding function h that maps a_i to the string that gives the binary representation of i. ■

Simplifying Restriction 2

We only consider decision problems, where there is exactly one valid output for each input, and this output is in $\{0, 1\}$.

1.2 Functions and Languages

A decision problem where all inputs are binary strings of length n can be described a Boolean function

$$f : \{0,1\}^n \to \{0,1\}$$

where for each $x \in \{0,1\}^n$, the value $f(x)$ represents the valid output for input x.

We do not want to restrict to just length n binary strings. So problems can be represented by a family of Boolean functions $\{f_n\}_{n\geq 0}$.

Definition: Language

A language is $L \subseteq \{0,1\}^*$.

A language L is equivalent to the family of functions $\{f_n\}$ if for every $x \in \{0,1\}^*$ of length n,

$$x \in L \Longleftrightarrow f_n(x) = 1$$

1.3 Cardinality of Languages

Definition: Finite Set

A set S is finite if there is a one-to-one mapping between the elements of S and the elements in the set $\{1, 2, \ldots, n\}$ for some $n \geq 0$.

Definition: Infinite Set

A set not finite.

Definition: Countable Set

A set S is countable if there is a one-to-one mapping between the elements of S and the set of natural numbers $\mathbb{N}$.

Definition: Uncountable Set

A set not countable.

The set of binary strings $\{0,1\}^n$ is finite. The set $\{0,1\}^*$ is infinite.

Proposition

The set $\{0,1\}^*$ is countable.

Proof. Consider the mapping $h : \{0,1\}^* \to \mathbb{N}$ where for each $x \in \{0,1\}^*$, we define $h(x)$ to be the natural number with binary representation $1x$, where we use $1x$ to denote string concatenation. The mapping h is one-to-one. ∎

Theorem

The set of all languages is uncountable.

Proof. This proof is an example of a diagonalization argument.

Assume for contradiction that the set of all languages is countable. Then we can list the set of languages in some order $L_1, L_2, \ldots$.

We can build a table whose columns are labelled by the strings in $\{0,1\}^*$ in lexicographical order and rows labelled by the languages $L_1, L_2, \ldots$ in the order we defined. For each cell (L_k, x) in the table, enter a 1 in the cell if $x \in L_k$ and 0 otherwise.

Consider now the language D that we obtain by look at the diagonal entries of this table and using their negation to determine if the corresponding string is in D. Namely, if x is the kth string in the lexicographical ordering of $\{0,1\}^*$, then $x \in D$ if and only if $x \notin L_k$.

D is a language so by our assumption, there is a value $n \in \mathbb{N}$ such that $D = L_n$ is the nth language in our list. Let x denote the nth string in the lexicographical order of $\{0,1\}^*$. But then $x \in D$ holds if and only if $x \notin L_n$, so $D \neq L_n$. We have arrived at our contradiction, so the set of all languages must be uncountable. ■

1.4 First Uncomputability Result

Proposition

There exist a language L for which there is no program that accepts each input $x \in \{0,1\}^*$ if and only if $x \in L$.

Proof. Assume for contradiction that for every language, there is a program that accepts exactly the set of strings in that language. Then there is a map from the set of all languages to the set of all programs. But every program can be represented as a binary string. So there is a mapping from the set of all languages to $\{0,1\}^*$. But since $\{0,1\}^*$ is countable, there is a mapping from the set of all languages to $\mathbb{N}$, contradicting the previous theorem. ■

Chapter 2

Turing Machines

We want a definition of a computer than can capture any computer, no matter how complicated. Our goal is to identify an explicit language that cannot be computed by algorithms over any machine model.

2.1 Definition

Consider an infinite tape split into squares. It has a finite number of states. Each square contains exactly one symbol. There is a tape head that points to over one of the squares. The head is allowed to move left or right and each state has a set of rules.

Definition: Deterministic 1-Tape Turing Machine

An abstract machine described by the triple

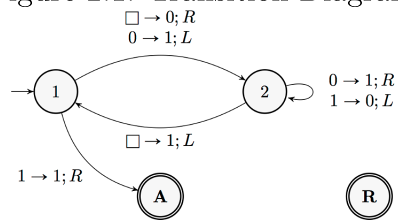
$$M = (m, k, \delta)$$

with $m, k \geq 1$ where

- $Q = \{1, 2, \ldots, m\}$ is the set of internal states,
- $\Gamma = \{\square, 0, 1, 2, \ldots, k\}$ is the tape alphabet, and
- $\delta\colon \mathbf{Q} \times \Gamma \to (\mathbf{Q} \cup \{\mathbf{A}, \mathbf{R}\}) \times \Gamma \times \{L, R\}$ is the transition function.

The state $\mathbf{1}$ is the initial state of the Turing machine M. $\mathbf{A}$ and $\mathbf{R}$ are the accept and reject states, respectively.

Figure 2.1: Transition Diagram



Definition: Configuration

A string $w\mathbf{q}y$ where

- $\mathbf{q} \in \mathbf{Q} \cup \{\mathbf{A}, \mathbf{R}\}$ represents the current state of the machine,
- $wy \in \Gamma^*$ is the current string on the tape, and
- the position of the tape head is on the first symbol of y.

Two configurations are equivalent when they are identical up to blank symbols at the beginning of w or at the end of y. In other words,

$$w\mathbf{q}y = \square w\mathbf{q}y = w\mathbf{q}y\square$$

Definition: Yields

For any strings $w, y \in \Gamma^*$, symbols $a, b, c \in \Gamma$, and states $\mathbf{q} \in \Sigma$ and $\mathbf{r} \in \Sigma \cup \{\mathbf{A}, \mathbf{R}\}$, the configuration $wa\mathbf{q}by$ of the Turing machine M yields the configuration $w\mathbf{r}acy$, denoted

$$wa\mathbf{q}by \vdash w\mathbf{r}acy$$

when $\delta(\mathbf{q}, b) = (\mathbf{r}, c, L)$. Similarly,

$$wa\mathbf{q}by \vdash wac\mathbf{r}y$$

when $\delta(\mathbf{q}, b) = (\mathbf{r}, c, R)$.

A configuration can *derive* another configuration in 0, 1, or more steps.

Definition: Accepts

A Turing machine M accepts $x \in \{0, 1\}^*$ if $\mathbf{1}x$ derives an accepting configuration $w\mathbf{A}y$.

Definition: Rejects

A Turing machine M rejects $x \in \{0, 1\}^*$ if $\mathbf{1}x$ derives a rejecting configuration $w\mathbf{R}y$.

Definition: Halts

A Turing machine M halts on x if it accepts or rejects x.

Definition: Decides

A Turing machine M decides the language $L \subseteq \{0,1\}^*$ if it accepts every $x \in L$ and rejects every $x \notin L$.

Definition: Recognize

A Turing machine M recognizes L if M accepts every $x \in L$ and M rejects or does not halt on $x \notin L$.

Definition: Decidable

A language $L \subseteq \{0,1\}^*$ if and only if there is a Turing machine that decides L.

2.2 Universal Turing Machine

Proposition

There is an encoding that maps each Turing machine M to a binary string $\langle M \rangle \in \{0,1\}^*$.

Proof. Consider the Turing machine $M = (m, k, \delta)$. We find a mapping $\{0, 1, +\}$ to the string $\langle m \rangle + \langle k \rangle + \langle \delta(1,0) \rangle + \cdots$. The positive integers m and k can be encoded by taking their binary representation. The transition function δ can be represented as a table of $m \cdot (k+2)$ entries (one for each internal state-tape symbol pair). Each of these entries can be encoded as a binary string. We can combine all these elements into a single binary representation to obtain the encoding of M. ■

Theorem

There is a Universal Turing Machine U such that for every Turing machine M and every input $x \in \{0,1\}^n$, when the input to U is the string $\langle M \rangle x$, then U simulates the execution of M on input x.

Proof. First, U turns x into the initial configuration of M by having the string $\mathbf{1}x$. Then

- Read the current state $\mathbf{q}$ and the symbol a at the tap head in M's current configuration.
- Go back to the encoding $\langle M \rangle$ of M to read the entry $\delta(\mathbf{q}, a)$ of its transition table.
- Update the configuration appropriately by overwriting the symbol a at the tap head position, moving the tape head left or right, and updating the current state of the machine.

■

2.3 Church-Turing Thesis

Church-Turing Thesis

Any decision problem that can be solved by any computer that respects the laws of physics corresponds to a language that can be decided by a Turing machine.

Proposition

Every language $L \subseteq \{0,1\}^*$ that can be computed using counter machines can also be decided by a Turing machine.

Chapter 3

Recursion Theorem

3.1 Building Blocks

Proposition

For every string $s \in \{0,1\}^*$, there exists a Turing machine P_s that on input $x \in \{0,1\}^*$ writes the string sx on the tape and then accepts.

Proof. When $s = a_1 \cdots a_n$ we can simply let P_s be the simple Turing machine that repeatedly moves left and overwrites then n blank symbols to the left of x with $a_n, \ldots, a_1$ in that order. ■

Proposition

There is a Turing machine F that on input $s \in \{0,1\}^*$, replaces s with $\langle P_s \rangle$ on the tape tape then accepts.

Proof. Given s as input, it is straightforward to determine the transition function for the corresponding Turing machine P_s as defined above. We can then write its encoding on the tape. ■

Definition: Concatenation of Turing Machines

The concatenation of Turing machines A and B is the Turing machine AB that on every input, first runs A on that input, then runs B on what is on the tape when A halts.

$\langle A \rangle \langle B \rangle \neq \langle AB \rangle$.

Proposition

There is a Turing machine C that on input $\langle A \rangle \langle B \rangle$ for any two Turing machines A and B, replaces that input with $\langle AB \rangle$ on the tape and then accepts.

Proof. The Turing machine AB has the same initial state as A. When A halts, the machine AB instead transitions to the initial state of B. When B halts, AB does and accepts if and only if B did. Therefore, the transition function for AB is easy to determine when we have the transition functions for both A and B and we can easily encode it to generate the desired output. ■

3.2 The Recursion Theorem

Theorem (Recursion Theorem)

For every Turing machine M, there exists a Turing machine Q_M that on every input $x \in \{0,1\}^*$ simulates M on the input $\langle Q_M \rangle\, x$.

Proof. Define a Turing machine R that on input $\langle N \rangle\, x$ for any Turing machine N and any binary string x replaces that input with the string

$$\left\langle P_{\langle N \rangle} N \right\rangle x$$

on the tape and halts. R exists because

1. Call F on $\langle N \rangle$ to get $\left\langle P_{\langle N \rangle} \right\rangle$.
2. Call C on $\left\langle P_{\langle N \rangle} \right\rangle \langle N \rangle$ to get $\left\langle P_{\langle N \rangle} N \right\rangle$.
3. Keep x to the right of the string.

Now, we define Q_M to be the Turing machine

$$Q_M = P_{\langle RM \rangle} RM$$

When we run Q_M on the input x, we obtain

$$x \overset{P_{\langle RM \rangle}}{\to} \langle RM \rangle\, x \overset{R}{\to} \left\langle P_{\langle RM \rangle} RM \right\rangle x = \langle Q_M \rangle\, x \overset{M}{\to} M(\langle Q_M \rangle\, x)$$

where we write $M(\langle Q_M \rangle\, x)$ to denote the output of M on input $\langle Q_M \rangle\, x$.

Corollary

There is a Turing machine Q that on input ε prints out $\langle Q \rangle$ on the tape and then halts.

Proof. Take M that does nothing. Q_M will now run M on $\langle Q_M \rangle\, x$. When $x = \varepsilon$, Q_M ends up writing its own description on the tape and halts. ■

3.3 Application To Undecidability

Corollary

Without loss of generality, we can always assume that a Turing machine has access to its encoding as well as its usual input x on the tape.

Proof. Design a Turing machine M that assumes the input is the form $\langle N \rangle \, x$ for any Turing machine N and is correct when $\langle N \rangle$ is its own description.

Then Q_M runs M on $\langle Q_M \rangle \, x$. ■

Theorem

The language

$$A_{TM} = \{\langle M \rangle \, x : M \text{ accepts } x\}$$

is undecidable.

Proof. Assume T decides A_{TM}. Let D be the Turing machine that

1. Obtains its own description $\langle D \rangle$ using the Recursion Theorem.
2. Run T on input $\langle D \rangle \, x$.
3. Do the opposite of T; reject if T accepts, and accept if T rejects.

By construction, D accepts x if and only if T does not accept $\langle D \rangle \, x$. This contradicts that T decides A_{TM}. ■

Chapter 4

Undecidability

4.1 More Undecidable Languages

Theorem

The language

$$\text{Halt}_{TM} = \{\langle M \rangle\, x : M \text{ halts on input } x\}$$

is undecidable.

Proof. Assume on the contrary that Halt_{TM} is decidable by Turing machine T. Consider the machine M that on input x does the following:

1. Obtain its own encoding $\langle M \rangle$ using the Recursion Theorem.
2. Run T on input $\langle M \rangle\, x$.
3. If T accepts, run forever in an infinite loop; otherwise, halt and accept.

By construction, M halts on x if and only if T does not accept $\langle M \rangle\, x$. This contradicts that T decides Halt_{TM}. ■

Definition: $L(M)$

The language recognized by M.

$$L(M) = \{x \in \{0,1\}^* : M \text{ accepts } x\}$$

Theorem

The language

$$\text{Empty}_{TM} = \{\langle M \rangle : L(M) = \emptyset\}$$

is undecidable.

Proof. Assume on the contrary that Empty_{TM} is decidable by Turing machine T. Consider the machine M that on input x does the following:

1. Obtain its own encoding $\langle M \rangle$ using the Recursion Theorem.
2. Run T on input $\langle M \rangle$.
3. Accept if T accepts; otherwise reject.

By this construction, $L(M) = \{0,1\}^*$ when T accepts $\langle M \rangle$ and $L(M) = \emptyset$ when T rejects. This contradicts the claim that T decides Empty_{TM}. ■

We can extend this theorem to show that it is impossible to decide any non-trivial property of languages of Turing machines.

Theorem (Rice's Theorem)

Let P be a subset of all languages over $\{0,1\}^n$ such that

1. There exists a Turing machine M_1 for which $L(M_1) \in P$, and
2. There exists a Turing machine M_2 for which $L(M_2) \notin P$.

Then the language

$$L_P = \{\langle M \rangle : L(M) \in P\}$$

is undecidable.

Proof. Assume on the contrary that L_P is decidable by Turing machine T. Consider the machine M that on input x does the following:

1. Obtain its own encoding $\langle M \rangle$ using the Recursion Theorem.
2. Run T on input $\langle M \rangle$.
3. If T accepts, simulate M_2 on x; otherwise simulate M_1 on x.

By this construction, $L(M) = L(M_2) \notin P$ when T accepts $\langle M \rangle$ and $L(M) = L(M_1) \in P$ when T rejects. This contradicts the claim that T decides L_P. ■

4.2 Reductions

Theorem

The language

$$A^{\varepsilon}_{TM} = \{\langle M \rangle : M \text{ accepts } \varepsilon\}$$

is undecidable.

Proof. Assume on the contrary that Turing machine T decides A_{TM}^{ε}.

Define a Turing machine A that takes input $\langle M \rangle \, x$. Let M' be a Turing machine that first writes x on the tape and then copies the behaviour of M. From the encoding of M and the string x, A can determine the encoding of M'. So it can call T on $\langle M' \rangle$ to determine whether M' accepts ε or not.

Since M' accepts ε if and only if M accepts x, then A decides the language A_{TM}, a contradiction. ■

Definition: k Busy Beaver Number

Maximum number BB_k of steps that a Turing machine with k states can complete before halting on a tape that is initially empty.

Theorem

The language

$$B = \{\langle k \rangle \, \langle n \rangle : BB_k \leq n\}$$

is undecidable.

Proof. Assume on the contrary that there is a Turing machine T that decides B.

Define a Turing machine A that takes input $\langle M \rangle \, x$. As a first step, A determines the number k of states in M. (Done from encoding of M) Then by calling T with input $\langle k \rangle \, \langle n \rangle$ for $n = 1, 2, \ldots$ until T accepts, A can determine the value of BB_k.

Now A can simulate up to BB_k steps of computation of M on input ε. Specifically, it can do that by copying the behaviour of the Universal Turing Machine with an additional twist: a counter that is incremented after each simulation step and that interrupts the simulation when it reaches the value BB_k. If M accepts or rejects during the simulation, A does the same. Otherwise, at the end of BB_k steps of simulation, A halts and rejects.

A decides the language A_{TM}^{ε}. That is because if M accepts or rejects ε, then A does the same. And if M runs for more than BB_k steps, then by definition of the Busy beaver numbers, it must run forever, which means that it does not accept ε. ■

4.3 Recognizability

Every language that is decidable is also recognizable. The converse statement is false.

Proposition

The undecidable language A_{TM} is recognizable.

Proof. Consider the Universal Turing Machine U. On input $\langle M \rangle \, x$, it simulates M on x and accepts if and only if M accepts x. Therefore, U recognizes A_{TM}. ■

Theorem

If a language L and its complement $\overline{L} = \{0,1\}^* \setminus L$ are both recognizable, then L and $\overline{L}$ are both decidable.

Proof. Assume that T_1 and T_2 recognize L and $\overline{L}$, respectively. Let M be a Turing machine that simulates both T_1 and T_2 in parallel. Specifically, it dedicates separate portions of the tape for the simulation of both T_1 and T_2 and interleaves their simulations by performing one step of computation of each of them at a time. The simulation is completed when either T_1 or T_2 accepts. If T_1 is the machine that accepts, M also accepts. Otherwise, if T_2 accepts, then M rejects.

When $x \in L$, then T_1 is guaranteed to accept x after a finite number of steps and T_2 is guaranteed to not accept, so M correctly accepts x. Similarly, when $x \in \overline{L}$, then T_2 is guaranteed to accept after a finite number of steps and T_1 will not accept so M correctly rejects x. Therefore, M decides L and the machine M' obtained by switching the accept and reject labels in M decides $\overline{L}$. ■

Corollary

The language $\overline{A_{TM}}$ is unrecognizable.

Proof. We have see that A_{TM} is recognizable. If $\overline{A_{TM}}$ was also recognizable, then by previous theorem, A_{TM} would be decidable, which is a contradiction. ■