Largest eigenvalue of sparse random graphs

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Introduction

Definition (Random Graph $G(n, p)$)

A discrete probability space composed of all labeled graphs on vertices $[n]$ where each edge (i, j) where $i \neq j$ appears randomly and independently with probability p.

Definition (Almost Surely (a.s.))

A graph property $\mathcal{A}$ holds almost surely in $G(n, p)$ if the probability that $G(n, p)$ has $\mathcal{A}$ approaches 1 as $n \to \infty$.

Previous Work

- For $p(n) \gg \log n$, average degree $\bar{d}$ and maximum degree Δ are asymptotically equal to np. We know $\bar{d} \le \lambda_1 \le \Delta$ so $\lambda_1 = (1 + o(1))np$
- For $p(n) = c$ where c is constant, λ_1 has asymptotically a normal distribution with expectation $(n-1)p + (1-p)$ and variance $2p(1-p)$ (Furedi and Komlos)
- For $p(n) = 1/n$, spectral radius $\max_i\{|\lambda_i|\}$ tends to infinity as $n \to \infty$ (Khorunzhy and Vengerovsky)
- What about **sparse** random graphs (i.e. $p(n) \in O(\log n)$)?

Main Result

Theorem

Let $G = G(n, p)$ be a random graph and let Δ be the max degree of G. Then almost surely the largest eigenvalue of G satisfies

$$\lambda_1(G) = (1 + o(1)) \max\left(\sqrt{\Delta}, np\right)$$

where $o(1)$ tends to 0 as $\max\left(\sqrt{\Delta}, np\right)$ tends to infinity.

Proof Outline

1. Given a random graph $G = G(n, p)$, partition its vertices into edge-disjoint subgraphs G_i that have more structure to work with
2. Find bounds on $\lambda_1(G_i)$ for each of the subgraphs (next section)
3. Since $G = \bigcup_i G_i$, use A4 Q8: $\lambda_1(G) = \sum_i \lambda_1(G_i)$ (or $\lambda_1(G) = \max_i \lambda_1(G_i)$ when vertex-disjoint) to complete the proof

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Maximum Degree

Definition (Δ_p)

$$\Delta_p = \max\{k : \mathbb{E}[\# v \in V(G) : \deg(v) = k] \geq 1\}$$

Lemma (2.1)

Let $G = G(n, p)$ be a random graph. Then

1. *The maximum degree of G almost surely satisfies $\Delta(G) = (1 + o(1))\Delta_p$.*
2. *If $np \to 0$, then almost surely G is a forest.*
3. *If $p \leq \frac{e^{-(\log \log n)^2}}{n}$, then almost surely all connected components of G are size of at most $(1 + o(1))\Delta_p$.*
4. *If $p \leq \frac{\log^{1/2} n}{n}$, then almost surely every vertex of G is contained in at most one cycle of length ≤ 4.*

Lemma 2.1 Proof Sketch

1. Note for $v \in V(G)$, $\deg(v) \sim \text{Binomial}(n-1, p)$. Using Chernoff bounds, one can show the upper asymptotic bound
2. Count
$$\mathbb{E}[\# \text{ cycles length } k] = \binom{n}{k}\frac{(k-1)!}{2}p^k$$
Observe if $np \to 0$, then $\sum_{k=3}^{\infty} \mathbb{E}[\# \text{ cycles length } k] = 0$.
3. This choice of p yields $np \to 0$, so G is a.s. a forest by (ii). Define r.v. Y to be the number of trees $t > (1+o(1))\Delta_p$ vertices. One can show $\mathbb{E}[Y] = \binom{n}{t} t^{t-2}p^{t-1} \le o(1)$ (Cayley's Formula).
4. The expected number of cycles of length $s, t \le 4$ with overlapping vertices is bounded above by
$$O(n^s n^{t-1} p^{s+t}) \le O(\log^4 n/n) \le o(1)$$

Random Set Intersection

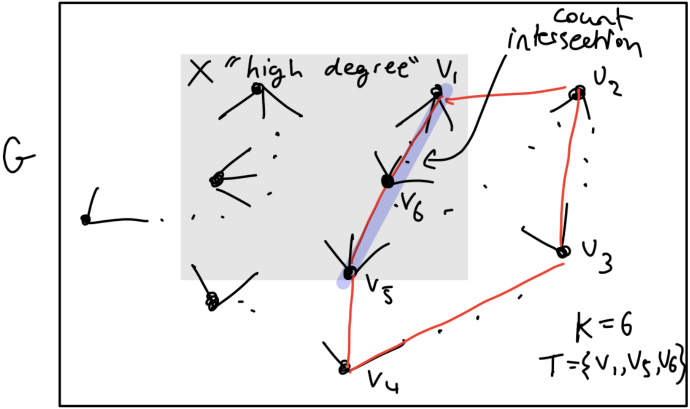
Lemma (2.2)

Let $p \geq \frac{e^{-(\log\log n)^2}}{n}$ *and let* X *be the set of vertices of a random graph* $G = G(n,p)$ *with degree larger than* $np(1 + 1/\log\log n) + \Delta_p^{1/3}$. *Then*

1. *Almost surely every cycle of* G *of length* k *intersects* X *in less than* $k/2$ *vertices.*
2. *Almost surely every vertex in* G *has less than* $\Delta_p^{7/8}$ *neighbors in* X.

Lemma 2.2 Proof Sketch

Idea: (i) says a.s. the intersection is *less* than $k/2$ vertices. Estimate the probability of an intersection of *at least* $k/2$ vertices and show this is bounded by $o(1)$. Same idea for (ii), but omitted for brevity.



Lemma 2.2 Proof Sketch

We will only show the $e^{-(\log\log n)^2/n} \leq p \leq \log^{1/4} n/n$ case. Aside from some counting differences, the $p \geq \log^{1/4} n/n$ case is very similar. Consider an arbitrary set of vertices T where $|T| = t$ and all $v \in T$ have $\deg(v) \geq d$ where $d := \log^{1/3} n/\log\log n$. There are two types of edges from vertices of T:

1. edges in the cut $(T, V(G) - T)$
2. internal edges

One can see that $\sum_{v\in T} \deg(v) = 2|\text{internal edges}| + |\text{cut edges}| \geq dt$. If both $|\text{internal edges}|, |\text{cut edges}| < \frac{dt}{3}$, then we have $dt < dt$: contradiction. So at least one of $|\text{internal edges}| \geq \frac{dt}{3}$ or $|\text{cut edges}| \geq \frac{dt}{3}$.

Lemma 2.2 Proof Sketch

Let's count both cases:

1. cut edges: Binomial$(t(n-t), p)$.

$$\binom{t(n-t)}{dt/3} p^{dt/3} \leq e^{-\Omega(t \log^{1/3} n)}$$

2. internal edges: Binomial$(t(t-1)/2, p)$.

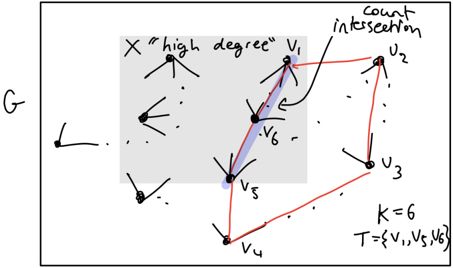
$$\binom{t(t-1)/2}{dt/3} p^{dt/3} \leq e^{-\Omega(t \log^{1/3} n)}$$

Note: our choice of d satisfies $d < np(1 + 1/\log\log n) + \Delta_p^{1/3}$ which is the minimum degree requirement for vertices in set X of the lemma statement. So, the probability all vertices in set X that have degree at least $np(1 + 1/\log\log n) + \Delta_p^{1/3}$ is also at most $e^{-\Omega(t \log^{1/3} n)}$. Also, replacing t with $2t$ doesn't change the argument.

Lemma 2.2 Proof Sketch

Now, let's estimate the probability of a cycle of length k with *at least* $k/2$ vertices be contained in X. This counting is by choosing k vertices for a cycle alongside their edges, then choosing a set T of size $t = \lceil k/2 \rceil$, requiring all elements of T to belong to X.

$$\sum_{k \geq 3} n^k p^k \binom{k}{\lceil k/2 \rceil} e^{-\Omega(\lceil k/2 \rceil \log^{1/3} n)} \leq o(1)$$



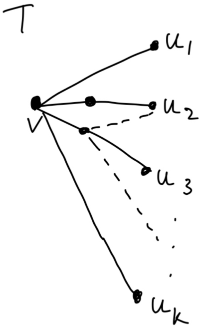
Neighbor Property of a Vertex

Lemma (2.3)

Let $G = G(n, p)$ be a random graph with $\frac{e^{-(\log\log n)^2}}{n} \leq p \leq \frac{\log^{1/2} n}{n}$. Then almost surely G contains no vertex which has at least $\Delta_p^{1/3}$ other vertices of G with degree $\geq \Delta_p^{3/4}$ within distance ≤ 2.

Lemma 2.3 Proof

By contradiction, claim such a vertex v exists. Then consider the subgraph which is a tree rooted at v.



Where $k := \Delta_p^{1/3}$, we see this tree has size t where $k + 1 \leq t \leq 2k + 1$. Each u_i has degree $\geq \Delta_p^{3/4}$ by assumption, so each u_i has $\geq \Delta_p^{3/4} - t \geq \frac{1}{2}\Delta_p^{3/4}$ neighbors outside T.

Lemma 2.3 Proof

Consider the cut $(T, V(G) - T)$ which has at least $\frac{1}{2}\Delta_p^{3/4} \times k = \frac{1}{2}\Delta_p^{13/12}$ edges. Since the number of edges follows Binomial$(t(n-t), p)$: we see

$$\binom{t(n-t)}{\frac{1}{2}\Delta_p^{13/12}} p^{\frac{1}{2}\Delta_p^{13/12}} \leq e^{-\log^{25/24} n}$$

Counting by summing across all possible tree sizes using Cayley's formula:

$$\sum_{k+1 \leq t \leq 2k+1} \binom{n}{t} t^{t-2} p^{t-1} e^{-\log^{25/24} n} \leq o(1)$$

so a.s., no such tree exists.

Maximum Eigenvalue Upper and Lower Bounds

Proposition (3.1)

Let G be a graph on n vertices and m edges. Then

1. $\max(\sqrt{\Delta}, 2m/n) \leq \lambda_1(G) \leq \Delta$
2. *If $E(G) = \cup_i E(G_i)$, then $\lambda_1(G) \leq \sum_i \lambda_1(G_i)$. If in addition G_i's are vertex disjoint, $\lambda_1(G) = \max_i \lambda_1(G_i)$*
3. *If G is a forest, then $\lambda_1(G) \leq \min(2\sqrt{\Delta - 1}, \sqrt{n-1})$. If G is a star on $\Delta + 1$ vertices then $\lambda_1(G) = \sqrt{\Delta}$*
4. *If G is bipartite such that degrees on both sides of bipartition are bounded by Δ_1 and Δ_2 respectively, $\lambda_1(G) \leq \sqrt{\Delta_1 \Delta_2}$*

Prop 3.1 Proof Sketch

1. See Assign 4 Q10
2. See Assign 4 Q8
3. From class, $\text{tr}(A^2) = 2|E(G)| \leq 2(n-1)$. Also $\text{tr}(A^2) = \sum_i \lambda_i^2$ and since G is bipartite, $\lambda_1 = -\lambda_n$ (Perron-Frobenius). So

$$(\lambda_1)^2 + (\lambda_n)^2 \leq 2(n-1) \implies 2\lambda_1^2 \leq 2(n-1) \implies \lambda_1 \leq \sqrt{n-1}$$

   For $\lambda_1 \leq 2\sqrt{\Delta - 1}$, see paper *Bounding the largest eigenvalue of trees in terms of the largest vertex degree*
4. Consider bipartite graph G's adjacency matrix A. Since $(A^2)_{ij}$ counts the number of 2-walks from i to j and G is bipartite, any given 2-walk has at most Δ_1 choices "there" and Δ_2 choices "back" so a maximum row sum is $\Delta_1\Delta_2$, which is an upper bound on the maximum eigenvalue. So

$$\lambda_1(A^2) = \lambda_1^2(G) \leq \Delta_1\Delta_2 \implies \lambda_1 \leq \sqrt{\Delta_1\Delta_2}$$

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Main Result

To recap:

Theorem

Let $G = G(n, p)$ be a random graph and let Δ be the max degree of G. Then almost surely the largest eigenvalue of G satisfies

$$\lambda_1(G) = (1 + o(1)) \max\left(\sqrt{\Delta}, np\right)$$

where $o(1)$ tends to 0 as $\max\left(\sqrt{\Delta}, np\right)$ tends to infinity.

Main Result Proof Outline

To recap:

1. Given a random graph $G = G(n, p)$, partition its vertices into edge-disjoint subgraphs G_i that have more structure to work with
2. Find bounds on $\lambda_1(G_i)$ for each of the subgraphs (using results from the previous section)
3. Since $G = \bigcup_i G_i$, use A4 Q8: $\lambda_1(G) = \sum_i \lambda_1(G_i)$ (or $\lambda_1(G) = \max_i \lambda_1(G_i)$ when vertex-disjoint) to complete the proof

Case 1: "very sparse" $p \le e^{-(\log\log n)^2}/n$

If $p \le e^{-(\log\log n)^2}/n$, G is a disjoint union of trees of size at most $(1+o(1))\Delta_p$ (Lemma 2.1). Then $\lambda_1(G) \le (1+o(1))\sqrt{\Delta_p}$ (Prop 3.1). By Lemma 2.1, $\Delta = (1+o(1))\Delta_p$ a.s, so $\lambda_1(G) \ge (1+o(1))\sqrt{\Delta_p}$ too by Prop 3.1.

Case 2: "not very sparse" $p \geq \log^{1/2} n/n$

One can verify $\Delta_p = o((np)^2)$ here. So, it suffices to prove

$$\lambda_1(G) = (1 + o(1)) \max(\sqrt{\Delta_p}, np) = (1 + o(1))np$$

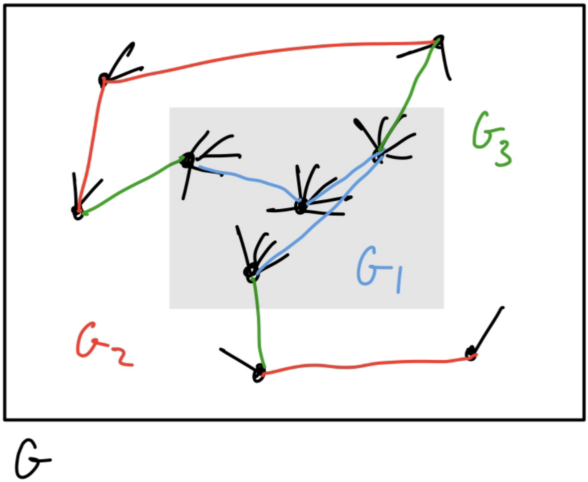
- "$\geq$": Recall edges follow Binomial$(\binom{n}{2}, p)$. Then

$$\mathbb{E}[\#\text{ edges}] = \binom{n}{2} p \approx \frac{n^2}{2} p,$$

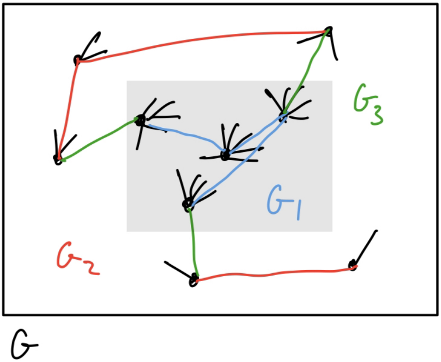
so expected average degree is $\frac{2m}{n} = np$, which is a lower bound (3.1)

Case 2: "not very sparse" $p \geq \log^{1/2} n/n$

- "$\leq$": Split G into subgraphs:



Case 2: "not very sparse" $p \geq \log^{1/2} n/n$



- G_1: induced by vertices v where $\deg(v) > np(1 + 1/\log\log n) + \Delta_p^{1/3}$.
- G_2: induced by vertices $V(G) - V(G_1)$
- G_3: bipartite graph of cut edges in $(V(G_1), V(G) - V(G_1))$

Case 2: "not very sparse" $p \geq \log^{1/2} n/n$

- Because the max degree of G_2 is $np(1 + 1/\log\log n) + \Delta_p^{1/3} = (1 + o(1))np$, then by Prop 3.1: $\lambda_1(G_2) \leq (1 + o(1))np$
- Lemma 2.2: G_1 has no cycles a.s. G_3 is bipartite so cycles must be even length, so half of the cycle's vertices must be in $V(G_1)$, but this is not the case a.s. by Lemma 2.2. So G_1, G_3 a.s. are forests
  - Prop 3.1: $\lambda_1(G_1), \lambda_1(G_3) \leq 2\sqrt{\Delta_p - 1} \leq (2 + o(1))\sqrt{\Delta_p}$

Overall: $\lambda_1(G) \leq \lambda_1(G_1) + \lambda_1(G_2) + \lambda_1(G_3) = (1 + o(1))np$

Case 3: $e^{-(\log\log n)^2/n} \le p \le \log^{1/2} n/n$

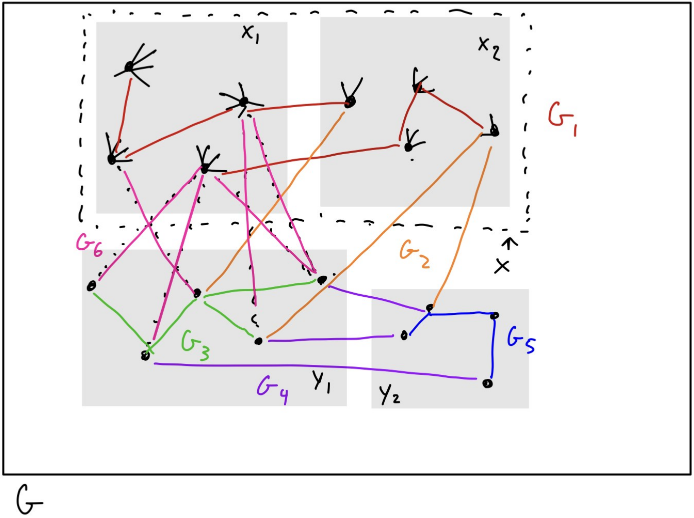
Must show $\lambda_1(G) = (1+o(1))\max(\sqrt{\Delta(G)}, np)$.

- "$\geq$": same as before, prop 3.1 (i) gives us

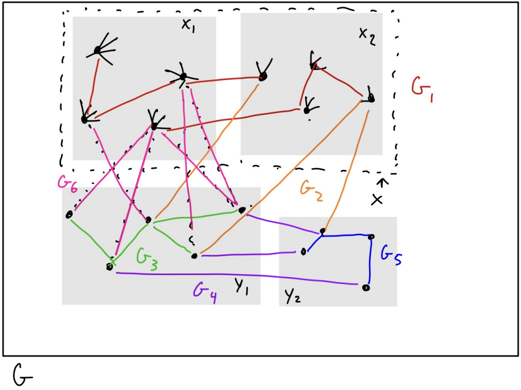
$$\lambda_1(G) \geq (1+o(1))\max\left(\sqrt{\Delta(G)}, np\right)$$

- "$\leq$": Split G into more subgraphs...

Case 3: $e^{-(\log\log n)^2/n} \leq p \leq \log^{1/2} n/n$

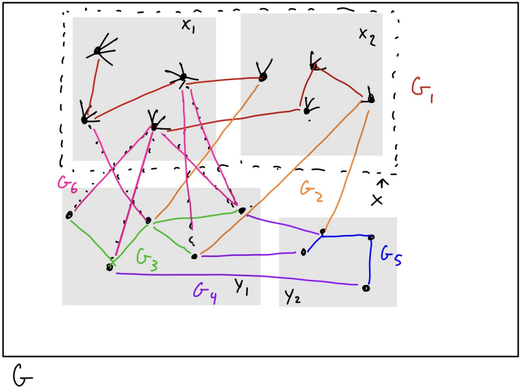


Case 3: $e^{-(\log\log n)^2/n} \leq p \leq \log^{1/2} n/n$



- X_1: $\{v \in V(G) : \deg(v) \geq \Delta_p^{3/4}\}$
- X_2: $\{v \in V(G) : np(1 + 1/\log\log n) + \Delta_p^{1/3} < \deg(v) < \Delta_p^{3/4}\}$
- $X : X_1 \cup X_2$
- Y_1: $V(G) - X$ with ≥ 1 neighbor in X_1, Y_2: $V(G) - X \cup Y_1$

Case 3: $e^{-(\log\log n)^2/n} \le p \le \log^{1/2} n/n$



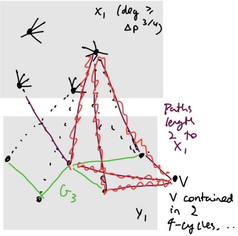
- G_1: induced by X, G_2: bipartite graph between X_2 and $V(G) - X$
- G_3: induced by Y_1, G_4: bipartite graph between Y_1 and Y_2
- G_5: induced by Y_2
- G_6: bipartite graph between X_1 and Y_1

Case 3: $e^{-(\log\log n)^2/n} \leq p \leq \log^{1/2} n/n$

"$\lambda_1(G_1) \leq o(\sqrt{\Delta_p})$": forest by Lemma 2.2. By Prop 3.1,
$\lambda_1(G_1) \leq 2\sqrt{\Delta_p^{7/8}} = o(\sqrt{\Delta_p})$ (max number of neighbours upper bound)
"$\lambda_1(G_2) \leq o(\sqrt{\Delta_p})$": forest by Lemma 2.2. By Prop 3.1,
$\lambda_1(G_2) \leq 2\sqrt{\Delta_p^{3/4}} = o(\sqrt{\Delta_p})$ (max degree upper bound)

$\lambda_1(G_3) \leq o(\sqrt{\Delta_p})$

- G_3 is induced by Y_1. Suppose $v \in V(G) - X$ has $\geq \Delta_p^{1/3} + 1$ neighbors in Y_1.
- Every neighbor of v has a neighbor in X_1 by definition, so there are $\geq \Delta_p^{1/3} + 1$ paths of length 2 from v to X_1.
- By Lemma 2.1, v a.s. is in at most one cycle of length 4 $\implies$ all but at most 1 of the endpoints in X_1 are different.



- v has at least $\Delta_p^{1/3}$ vertices in X_1 within distance 2, but lemma 2.3 says a.s. there is no such vertex. So v has $\leq \Delta_p^{1/3}$ neighbors in Y_1.
- Max degree of G_3 is $\leq \Delta_p^{1/3}$, so $\lambda_1(G_3) \leq \Delta_p^{1/3} = o(\sqrt{\Delta_p})$

$\lambda_1(G_4) \leq o(\sqrt{\Delta_p})$

- G_4 is bipartite graph between Y_1 and Y_2.
- Max degree of $v \in Y_1$ is $\leq np(1 + 1/\log\log n) + \Delta_p^{1/3}$ and from previously, max degree of $v \in Y_2$ is $\leq \Delta_p^{1/3}$.
- By prop 3.1 (iv), $np \leq \log^{1/2} n$, and $\Delta_p = \Omega(\log n/(\log\log n)^2)$ gives us

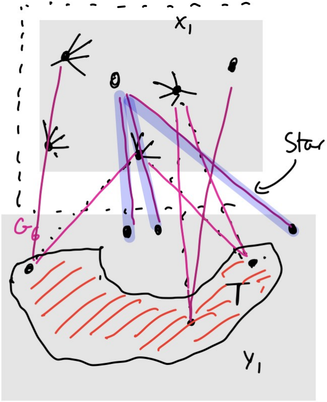
$$\begin{aligned} \lambda_1(G_4) &\leq \sqrt{(np(1 + 1/\log\log n) + \Delta_p^{1/3})(\Delta_p^{1/3})} \\ &\leq \Delta_p^{1/3} + (1 + o(1))\Delta_p^{1/6}\sqrt{np} \\ &= o(\sqrt{\Delta_p}) \end{aligned}$$

$\lambda_1(G_5) \leq (1 + o(1))np + \Delta_p^{1/3}$

- G_5 is the subgraph induced by Y_2. By definition, max degree of G_5 is $\leq (1 + o(1))np + \Delta_p^{1/3}$.
- Thus, $\lambda_1(G_5) \leq (1 + o(1))np + \Delta_p^{1/3}$.

$\lambda_1(G_6) \leq (1+o(1))\sqrt{\Delta_p} + \Delta_p^{1/3}$

- G_6 is the bipartite graph between X_1 and Y_1. Let $T := \{v \in Y_1 : \deg_{G_6}(v) > 1\}$.
- Let $u \in X_1$ with $\geq \Delta_p^{1/3}$ neighbours in T. Every neighbour of u in T has an additional neighbor back in $X_1 \setminus \{u\}$.



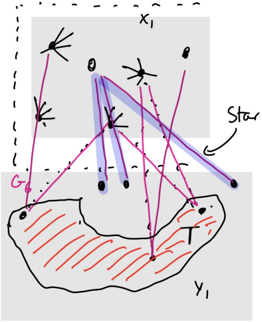
$\lambda_1(G_6) \leq (1+o(1))\sqrt{\Delta_p} + \Delta_p^{1/3}$

- $\geq \Delta_p^{1/3} + 1$ 2-paths from u to $X_1 \setminus \{u\}$. Lemma 2.1 $\implies$ u is in at most 1 cycle of length 4 $\implies$ all but ≤ 1 endpoint of these paths in X_1 are different $\implies$ u has $\geq \Delta_p^{1/3}$ distinct vertices of X_1 in distance 2. Lemma 2.3 implies a.s. there is no vertex u with this property.
- Every vertex in Y_1 has degree at most $\Delta_p^{1/3}$ in G_6 from the bounds on p.

$\lambda_1(G_6) \le (1+o(1))\sqrt{\Delta_p} + \Delta_p^{1/3}$ Continued

- Let H be subgraph of G_6 containing edges from X_1 to T.
- Max degree of H is $\le \Delta_p^{1/3} \implies \lambda_1(H) \le \Delta_p^{1/3}$.
- Max degree of $v \in Y_1 \setminus T$ is 1 and G_6 itself is bipartite, then $G_6 - H$ is union of vertex-disjoint stars. The max degree of each star is max degree of G, so

$$\lambda_1(G_6) \le \lambda_1(H) + \lambda_1(G_6 - H) \le \Delta_p^{1/3} + (1+o(1))\sqrt{\Delta_p}$$



$\lambda_1(G)$

- There are no edges between X_1 and Y_2, so the edges of G are the union of $E(G_i)$ for $i = 1, \ldots, 6$).
- G_5 and G_6 are vertex disjoint, so $\lambda_1(G_5 \cup G_6) = \max(\lambda_1(G_5), \lambda_1(G_6))$.

$$\lambda_1(G_5 \cup G_6) = \max\left((1 + o(1))np + \Delta_p^{1/3}, (1 + o(1))\sqrt{\Delta_p} + \Delta_p^{1/3}\right)$$

$$\begin{aligned}
\lambda_1(G) &= \sum_{i=1}^{4} \lambda(G_i) + \max(\lambda_1(G_5), \lambda_1(G_6)) \\
&= o(\sqrt{\Delta_p}) + \max\left((1 + o(1))np + \Delta_p^{1/3}, (1 + o(1))\sqrt{\Delta_p} + \Delta_p^{1/3}\right) \\
&= (1 + o(1)) \max\left(\sqrt{\Delta_p}, np\right) \\
&= (1 + o(1)) \max\left(\sqrt{\Delta(G)}, np\right)
\end{aligned}$$