Combinatorial Game Theory

Keven Qiu
Instructor: Ryan Hayward

1 Introduction

Definition 1.1: Combinatorial Game Theory (CGT)

The study of finite 2-player, alternating turns, deterministic rules/moves, perfect information games.

Definition 1.2: Normal Play Games

Whoever cannot move loses.

We can represent combinatorial games with a tree of all continuations. This is a directed acyclic graph where each node contains the game state and the player to move.

Games are played with **L**eft and **R**ight. In the tree, solid lines represent L moving and dotted represents R moving.

Definition 1.3: ptm

Player-to-move.

Definition 1.4: optm

Opponent-to-move.

Definition 1.5: WLD Game

A combinatorial game that can result in a win, loss, or draw.

Theorem 1.6 (Zermelo)

In a Win-Loss-Draw game, one of the following holds:

- ptm (first player) has a winning strategy.
- optm (second player) has a winning strategy.
- ptm and optm each have a draw strategy (a non-losing strategy).

Proof. Consider a Tree of all Continuations (ToaC). We prove this by induction. Assume the theorem holds for all continuations from the root $g_1, \ldots, g_t$. We label each node of the each subtree with $+$, $-$ or 0. There are three cases:

Case 1 At least 1 of the g_i is labeled $-$, say g_j. $g_j = -$ means the player-to-move at g_j loses. Thus, if you are at g and you move to g_j, then opponent is the player-to-move at g_j and loses. You force a win, so ptm wins.

Cbse 2 Assume not case 1 and that some $g_j = 0$. All g_i's are 0 or +. You move to the $g_j = 0$ and therefore guarantee a ptm-draw. If opponent moves first, then they can either move to a + and we win, or they move to 0 and forces a draw. So optm-draw under optimal play.

Ccse 3 Every $g_i = +$. Every move ptm makes goes to +, so next player can always win. Thus, optm-win.

■

1.1 Outcome Classes

Definition 1.7: $\mathcal{Y}$

Left wins, no matter if they are 1st or 2nd player.

Definition 1.8: $\mathcal{R}$

Right wins, no matter if they are 1st or 2nd player.

Definition 1.9: $\mathcal{P}$

The 2nd player wins ($\oslash$).

Definition 1.10: $\mathcal{N}$

The 1st player wins.

2 Clobber

Clobber is a normal play combinatorial game. Each player either controls X or O and each turn they can capture an adjacent opponent symbol. The player who cannot move loses.

Linear clobber is when the game is played on a path graph. We can represent this in options notation:

- $x = \{|\} = \oslash$. The game tree is a single root node.
- $xo = \{\oslash|\oslash\} = *$.
- $xox = \{\oslash, \oslash|\oslash, \oslash\} = \{\oslash, \oslash\} = *$.
- $xxo = \{\oslash|*\} = \uparrow$.
- $oox = \{*|\oslash\} = \downarrow$.
- $xoxo = \{\oslash, *, xxo|\oslash, *, oox\}$.

3 Hex

A combinatorial game played on a grid of hexagons. The goal is for one player to connect their colored sides using their same symbol/rock.

Hex has no draws.

Theorem 3.1

For $n \times n$ hex, the first player always wins.

Proof. We prove by contradiction. Let Left make the first play of the game, and assume that Right has a winning strategy. With Left's first move she puts a stone on any cell, a placement called "extra". With this move Left becomes Second, and she henceforth follows the winning strategy that is available to Right. At some point, if this strategy calls for Left to place a stone where the extra sits; then she will simply make another arbitrary placement. Thus Left can win in contradiction to the hypotheses. ∎

Definition 3.2: Equivalent Games

Two games G, H are equivalent if they have the same canonical form.

Theorem 3.3 (Alternate Equivalent Games)

Two combinatorial games G, H are equivalent iff for all games X, $\text{oc}(G + X) = \text{oc}(H + X)$ (outcome class).

4 Go

Go is played on a graph, typically a grid. It is a combinatorial game where the winning condition is more stones and territory occupied. A legal move must be played on an empty cell and the block (a connected component) must have at least 1 liberty/free space.

5 Nim

The game of Nim involves piles of items and on each turn, a player picks at least 1 item from a pile. The last person to pick items is the winner.

Theorem 5.1 (Bouton)

There is a winning move iff the XOR sum is nonzero.

Consider the piles 6, 3, 1. In binary this is $110, 011, 001$. The XOR sum is 100, thus there is a winning move for ptm.

There is a legal move from each pile whose leftmost bit matches the leftmost 1 bit of XOR sum. For the example, we add 100 to 110 to get 010, so we pick 4 from the first pile of 6.