Assignment 5 – Part 1

p286 Set 5.2: 9, 27, 35 , Set 5.3 : 10, 18, 23.b

9. For all integers $n \geq 3$,

$$4^3 + 4^4 + 4^5 + \cdots + 4^n = \frac{4(4^n - 16)}{3}.$$

9.) Let $P(n)$ be $4^3 + 4^4 + 4^5 + \cdots + 4^n = \frac{4(4^n - 16)}{3}$

[We must show that $P(n)$ is true for all integers $n \geq 3$.

Show that P(3) is true:

$$4^3 = \frac{4(4^3 - 16)}{3}$$

$$64 = \frac{4(64 - 16)}{3}$$

$$64 = \frac{4(48)}{3}$$

$$64 = 64$$

Hence $P(3)$ is true.

Show that for all integers $k \geq 3$, if $P(k)$ is true then $P(k+1)$ is also true:

Suppose that $P(k)$ is true for a particular but arbitrarily chosen integer $k \geq 0$.

That is, suppose that

$$4^3 + 4^4 + 4^5 + \cdots + 4^k = \frac{4(4^k - 16)}{3}$$

[We must show that $P(k+1)$ is true. That is:]

$$4^3 + 4^4 + 4^5 + \cdots + 4^{k+1} = \frac{4(4^{k+1} - 16)}{3}$$

[We will show that the left-hand side of $P(k+1)$ equals the right-hand side.]

$$4^3 + 4^4 + 4^5 + \cdots + 4^{k+1} = 4^3 + 4^4 + 4^5 + \cdots + 4^k + 4^{k+1}$$

$$= \frac{4(4^k - 16)}{3} + 4^{k+1}$$

$$= \frac{4(4^k - 16)}{3} + \frac{3 \cdot 4^{k+1}}{3}$$

$$= \frac{4^{k+1} - 64 + 3 \cdot 4^{k+1}}{3}$$

$$= \frac{4 \cdot 4^{k+1} - 64}{3}$$

$$= \frac{4(4^{k+1} - 16)}{3}$$

which is the right-hand side of $P(k+1)$ [as was to be shown.]

27. $5^3 + 5^4 + 5^5 + \cdots + 5^k$, where k is any integer with $k \geq 3$.

27.) $= 5^3 \cdot (1 + 5 + 5^2 + \cdots + 5^{k-3})$

$= 125 \cdot \left(\frac{5^{k-2} - 1}{4}\right)$

35.) The problem with this proof is that "starting from a statement and deducing a true conclusion does not prove that the statement is true. A true conclusion can also be deduced from a false statement."[Taken from textbook]

Set 5.3 : 10, 18, 23.b

10. $n^3 - 7n + 3$ is divisible by 3, for each integer $n \geq 0$.

10.) Let $P(n)$ *be* $n^3 - 7n + 3$ is divisible by 3, for each integer $n \geq 0$.

Base case $P(0)$:

$P(0) = 0^3 - 7(0) + 3 = 3$ **which is divisible by 3.**

Therefore, $P(0)$ is true.

Suppose that $P(k)$ is true for any particular but arbitrarily chosen integer $k \geq 0$.

$P(k) = k^3 - 7k + 3$ is divisible by 3.

By definition of divisibility, this means that

$k^3 - 7k + 3 = 3r$ for some integer r.

[We must show that $P(k+1)$ is also divisible by 3.]

$(k+1)^3 - 7(k+1) + 3$ is divisible by 3

By algebra

$(k+1)^3 - 7(k+1) + 3 = (k+1)(k^2 + 2k + 1) - 7k - 7 + 3$

$= k^3 + 2k^2 + k + k^2 + 2k + 1 - 7k - 7 + 3$

$= k^3 + 3k^2 - 4k - 3$

$= (k^3 - 7k + 3) + 3k^2 + 3k - 6$

By substitution:

$= 3r + 3(k^2 + k - 2)$

$= 3(r + k^2 + k - 2)$

Thus, we have proved that $P(k+1)$ is divisible by 3. Since we have proved the basis step and the inductive step, we conclude that the proposition is true.

18. $5^n + 9 < 6^n$, for all integers $n \geq 2$.

18.) Let $P(n)$ be $5^n + 9 < 6^n$, for all integers $n \geq 2$.

Base Case $P(2)$:

$5^2 + 9 < 6^2$

$25 + 9 < 36$

$34 < 36$

Therefore, $P(2)$ is true.

Suppose that $P(k)$ is true for any particular but arbitrarily chosen integer $k \geq 2$

$5^k + 9 < 6^k$

[We must show that $P(k+1)$ is also true.]

$5^{k+1} + 9 < 6^{k+1}$

$5 \cdot 5^k + 9 < 6 \cdot 6^k$

$5^k + 9 < \frac{6}{5} \cdot 6^k - \frac{9}{5} + 9$

By substitution:

$5^k + 9 < 6^k$

Thus, we have proven that $P(k+1)$ is true. Since we have proved the basis step and the inductive step, we conclude that the proposition is true.

23. a. $n^3 > 2n + 1$, for all integers $n \geq 2$.
b. $n! > n^2$, for all integers $n \geq 4$.

23.b.) Let $P(n)$ be $n! > n^2$ for all integers $n \geq 4$.

Base Case $P(4)$:

$4! > 4^2$

$4 * 3 * 2 * 1 > 16$

$24 > 16$

Therefore, $P(4)$ is true.

Suppose that $P(k)$ is true for any particular but arbitrarily chosen integer $k \geq 4$

$k! > k^2$

[We must show that $P(k+1)$ is also true.]

$(k+1)! > (k+1)^2$

$(k+1) * k! > (k+1)^2$

$k! > k + 1$ **Dividing both sides by k+1 does not change inequality sign because $k \geq 4$**

If $k! > k^2$, then $k!$ is certainly greater than $k + 1$

Thus, we have proven that $P(k+1)$ is true. Since we have proved the basis step and the inductive step, we conclude that the proposition is true.