Assignment 3 – Part 1

Set 4.6 - 12, 16, 28 (Prove by Contradiction), Set 4.7 - 8, 16.c (Prove by Contradiction)

Set 4.6 - 12, 16, 28 (Prove by Contradiction)

12.) If a and b are rational numbers, b≠0, and r is an irrational number, then a + br is irrational.

Proof (by contradiction):

Assume for the sake of contradiction *a* and *b* are rational numbers, $b \neq 0$, *r* is an irrational number, and $a + br$ is rational. [We must deduce a contradiction.] By definition of rational, $a = \frac{c}{d}$, $b = \frac{e}{f}$, and $a + br = \frac{g}{h}$ for some integers *c, d, e, f, g,* and *h* with $d \neq 0, e \neq 0, f \neq 0, \text{and } h \neq 0$. By substitution and algebra,

$$\frac{c}{d} + \frac{e}{f}(r) = \frac{g}{h}$$

$$\frac{e}{f}(r) = \frac{g}{h} - \frac{c}{d}$$

$$\frac{e}{f}(r) = \frac{gd-ch}{dh}$$

$$r = \frac{f(gd-ch)}{deh}$$

Now $f(gd - ch)$ and deh are both integers since products and differences of integers are integers, $deh \neq 0$ by the zero-product property. Hence, *r* is the quotient of the two integers $f(gd - ch)$ and deh with $deh \neq 0$. Thus, by definition of rational, *r* is rational, which contradicts the supposition that *r* is irrational. [Hence the supposition is false and the theorem is true.]

16.) For all odd integers a, b, and c, if z is a solution of $ax^2 + bx + c = 0$ then z is irrational. (In the proof, use the properties of even and odd integers that are listed in Example 4.2.3.p195)

Proof (by contradiction):

Assume for the sake of contradiction a, b, and c are odd numbers, z is a solution of $ax^2 + bx + c = 0$, and z is rational. [We must deduce a contradiction.] By definition of rational, $z = \frac{e}{f}$ for some integers e and f with $f \neq 0$. We may assume that e and f have no common factor because if they did, $z = \frac{e\prime}{f\prime}$. Because e and f have no common factor, they are not both even. By substitution,

$$a\left(\frac{e}{f}\right)^2 + b\left(\frac{e}{f}\right) + c = 0$$

$$\frac{ae^2}{f^2} + \frac{be}{f} + c = 0$$

$$ae^2 + bef + cf^2 = 0$$ **Multiply whole equation by f^2**

Case 1 (assuming e is even and f is odd)

Assume e if even and f is odd. The expressions ae^2 and bef will be even because any product of an even and odd integer is even. cf^2 will be odd because the product of any two odd integers is odd. Adding $ae^2, bef, and\ cf^2$ will result in an odd number because the sum of any odd and even integer is odd. This is a contradiction because 0 is not odd.

Case 2 (assuming f is even and e is odd)

Assume e if odd and f is even. The expressions $bef\ and\ cf^2$ will be even because any product of an even and odd integer is even. ae^2 will be odd because the product of any two odd integers is odd. Adding $ae^2, bef, and\ cf^2$ will result in an odd number because the sum of any odd and even integer is odd. This is a contradiction because 0 is not odd.

Case 3 (assuming e is odd and f is odd)
Assume e and f are odd. $ae^2, bef, and\ cf^2$ will be odd because any product of odd integers is odd. The sum of any 2 odd integers is even, and the sum of an even integer with an odd integer is odd. Thus, adding $ae^2, bef, and\ cf^2$ will result in an odd number. Again, this is a contradiction because 0 is not odd.

This leaves the case in which e and f are both even, which was supposed to be impossible. [Hence the supposition is false and the theorem is true.]

28.) For all integers m and n, if mn is even then m is even or n is even.

Proof (by contradiction):

Assume for the sake of contradiction, mn is even, m is odd, and n is odd. By definition in Example 4.2.3, an odd integer times another odd integer is odd. Therefore, mn must be odd, which contradicts our supposition that mn is even. [Hence the supposition is false and the theorem is true.]

Set 4.7 - 8, 16.c (Prove by Contradiction) p.212

8.) The difference of any two irrational numbers is irrational. **FALSE.**

Proof (by counterexample):

$$\sqrt{2} - \sqrt{2} = 0$$

0 is not rational because it can be written in a rational form ($\frac{0}{1}$).

16.c.) Prove that √3 is irrational.

Proof (by contradiction):

Assume for the sake of contradiction $\sqrt{3}$ is rational. Then there are integers *m* and *n* with no common factors and $n \neq 0$ such that

$$\sqrt{3} = \frac{m}{n}$$

$$3 = \frac{m^2}{n^2}$$

$$m^2 = 3n^2$$

Thus m^2 is divisible by 3, and so, by 16.b., m is also divisible by 3. By definition of divisibility, $m = 3k$ for some integer k, and so

$$m^2 = 9k^2$$

$$3n^2 = 9k^2$$

$$n^2 = 3k^2$$

Hence n^2 is divisible by 3, and so, by 16.b., *n* is also divisible by 3. Consequently, *m* and *n* are both divisible by 3, which contradicts our assumption that *m* and *n* have no common factor. [Hence the supposition is false and the theorem is true.]