# Assignment 1

CS 431 Optimization: Theory and Algorithms
Course Instructor: Dr. Divya Padmanabhan

To be submitted by: Feb 15, 2022

1. Consider the system of equations and let $\mathbf{A}$ denote the associate matrix of co-efficients. (7 marks)

$$x + 2y + 3z + w = 5$$
$$3x + 6y + 9z + 3w = 15$$
$$x + y + 2z + w = 3$$

   (a) Find a basis $\mathcal{B}_1$ for the row space of $\mathbf{A}$. What is its dimension?
   (b) Find a basis $\mathcal{B}_2$ for the null space of $\mathbf{A}$. What is its dimension?
   (c) Which are the vectors spanned by $\mathcal{B}_1 \cup \mathcal{B}_2$?
   (d) Find all solutions to the system of equations.

2. Let $A \in \mathbb{R}^{m \times n}$ and $A$ has rank $r$. Suppose $m \leq n$. Suppose there exist vectors $b \in \mathbb{R}^m$ such that $Ax = b$ has no solution. Say true or false with reason for each of the statements. (4 marks)

   (a) $A^\top y = 0$ has a non-zero solution $y$.
   (b) $m \leq r \leq n$
   (c) $r < m$, $r < n$
   (d) $Ax = 0$ has only one solution namely $x = 0$.

3. For what range of numbers $a$ and $b$ are the following matrices positive definite? (2 marks)

$$\text{(a) } \mathbf{A} = \begin{bmatrix} a & 2 & 4 \\ 2 & a & 2 \\ 4 & 2 & a \end{bmatrix} \quad \text{(b) } \mathbf{A} = \begin{bmatrix} 1 & 2 & 4 \\ 2 & b & 8 \\ 4 & 8 & 7 \end{bmatrix}$$

4. Consider the functions $f(x_1, x_2) = \frac{x_1^2 + x_2^3}{2}$ and the vector valued function $g(s,t) = \begin{bmatrix} 4s + 3t \\ 2s + t \end{bmatrix}$. (2 marks)

   (a) Use chain rule to compute $\partial f(g(s,t))/\partial s$ and $\partial f(g(s,t))/\partial t$.
   (b) Alternatively compute the function $h(s,t) = f(g(s,t))$ explicitly and then compute $\partial h(s,t)/\partial s$, $\partial h(s,t)/\partial t$.

5. Find the limit points of the following sets. Report if the sets are open. Report if the sets are closed. Provide explanations for your answer. (6 marks)

   (a) $\Omega_1 = \{(x_1, 0) \in \mathbb{R}^2 : 2 < x_1 < 3\}$
   (b) $\Omega_2 = \{x \in \mathbb{R} : x = 1 \text{ or } 2 \leq x \leq 3\}$.
   (c) $\Omega_3 = \{(x_1, 0) \in \mathbb{R}^2 : x_1 \in \Omega_2\}$

6. Explain why the function $f(x)$ has a discontinuity at $x = 0$. (1 mark)

$$f(x) = \begin{cases} 1 - x, & \text{if } x < 0, \\ x^2 + 5, & \text{if } x \geq 0, \end{cases}$$

**Programming Questions**

*You are required to submit your working code as well. Please name your files with the prefix <your full name> _ <QuestionNo>. For Questions 9 and 10, you need to use Gurobi solver. For Questions 7 and 8, you may use either MATLAB or Python.*

7. Let $f(x) = x^5 + 5e^{-3x}$. Compute the zero order, first order and second order Taylor series approximations for the function around $x_0 = 1$ and $x_0 = 2$. Plot these functions along with $f(x)$. You need to generate two sets of plots - one each for the $x_0$ values. (4 marks)

8. The Rosenbrock function is a commonly used function for testing the performance of optimization algorithms and is computed as follows, (4 marks)

$$f(x_1, x_2) = (1 - x_1)^2 + 100(x_2 - x_1^2)^2$$

   (a) Write down the gradient and Hessian of $f(x_1, x_2)$.

   (b) Generate a single plot showing the contours $f(x_1, x_2) = \alpha$ for $\alpha \in \{0.5, 2, 5, 10, 50, 100, 200, 400, 800\}$. You may use the range $-2 \leq x_1 \leq 2, -2 \leq x_2 \leq 2$. Also mark the gradient vectors on this plot.

9. You have recently moved to a different city for work and are living with a room-mate. There are several household chores that need to be done but both of you have a hectic schedule at work. Both of you have decided to completely taking on a given chore (rather than divide each chore between both). Each of you have different speeds of performing various chores, the time taken by each of you (in hours per week) is given in the table below. Each of you does not want to perform more than two chores. At the same time all chores need to be completed and the total time taken needs to be minimized. (10 marks)

| | Shopping | Cooking | Cleaning | Laundry |
|---|---|---|---|---|
| You | 4.5 | 7.8 | 3.6 | 2.9 |
| Room-mate | 4.9 | 7.2 | 4.3 | 3.1 |

   (a) Formulate this as an optimization problem where the decision variables take binary values. Implement the formulation in Gurobi and compute the best way to assign chores.

   (b) If you perform chores in this manner, how long would you spend collectively on the chores in a week? How long would each of you spend?

   (c) Suppose instead of minimizing the total time spent, you want the difference between the times spent by both of you to be minimized. How will you formulate this?

   (d) Implement the formulation in (c) and report the best way to assign chores from this formulation. What is the difference in the times you both need to spend now? What is the total time spent?

10. Recall the problem discussed in class, where we wanted to send commodities from a source node $S$ to a sink node $T$ on the graph $G = (V, E)$ with link capacities $c_{ij}$. Two equivalent formulations for this problem are provided (we will see why these are equivalent later in the course). Implement both for an input graph and capacities (your formulation needs to be written so as to work on any input, please include comments in your code to describe the input format your code expects). (10 marks)

$$\max_{v,\mathbf{x}} \quad v$$

$$\text{s.t} \quad \sum_{j:(i,j)\in E} x_{ij} - \sum_{j:(j,i)\in E} x_{ji} = \begin{cases} v, & \text{if } i = s, \\ -v, & \text{if } i = t, \\ 0, & \text{otherwise,} \end{cases}$$

$$0 \leq x_{ij} \leq c_{ij}, \quad \text{for } (i,j) \in E.$$

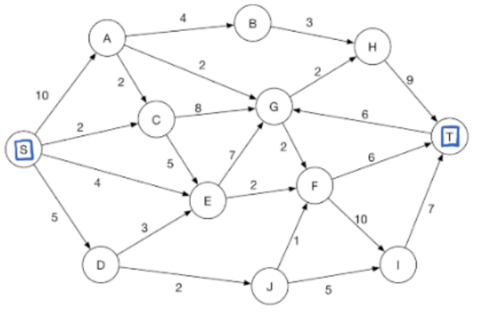
Decision variables are $v$ and $\mathbf{x}$. $v$ can take any sign.

$$\min_{\mathbf{u},\mathbf{y}} \quad \textstyle\sum_{(i,j)\in E} c_{ij} y_{ij}$$

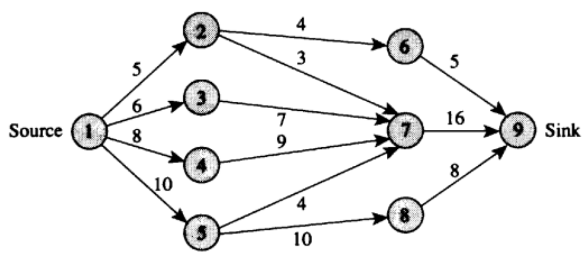
$$\begin{aligned} \text{s.t} \quad & u_i - u_j + y_{ij} \geq 0 \text{ for } (i,j) \in E \\ & -u_s + u_t = 1 \\ & y_{ij} \geq 0 \text{ for } (i,j) \in E \end{aligned}$$

Decision variables: $u_i$ for $i \in V$ and $y_{ij}$ for $(i,j) \in E$.

Report the optimal solutions $\mathbf{x}^*, v^*, \mathbf{u}^*, \mathbf{y}^*$ obtained as well as the optimal objective values of both formulations for the following two graphs:




(a)




(b)