Assignment 3

CS 431, Optimization: Theory and Algorithms

To be submitted on or before: April 14, 2022

1. (10 marks) Consider the following linear problem:

$$\left.\begin{array}{lrcl} \max & 10x_1 + 10x_2 + 20x_3 + 20x_4 & & \\ \text{s.t.:} & 12x_1 + 8x_2 + 6x_3 + 4x_4 & \leq & 210 \\ & 3x_1 + 6x_2 + 12x_3 + 24x_4 & \leq & 210 \\ & x_1, x_2, x_3, x_4 & \geq & 0 \end{array}\right\} \text{Primal LP}$$

   (a) Write down the dual linear program.

   (b) Sketch the feasible region for the dual program, solve the model graphically and write down the optimal solution. Can you infer the optimal values of some of the primal variables using the optimal values of the dual variables?

   (c) Find the optimal *primal* solution. (Hint: you do not need to carry out the simplex algorithm or write down the optimal tableau).

2. (6 marks) In this question you will have to prove weak duality for a more generic linear program. Consider the following linear program:

$$\left.\begin{array}{lrcl} \min & c_1^T x_1 + c_2^T x_2 + c_3^T x_3 & & \\ \text{s.t.:} & & & \\ & Ax_1 & \geq & b_1 \\ & Bx_2 & = & b_2 \\ & Cx_3 & \leq & b_3 \\ & x_1, x_2, x_3 & \geq & 0 \end{array}\right\} \text{Primal} \qquad (1)$$

   where $x_1, x_2, x_3 \in \mathbb{R}^{n\times 1}$ and $A, B, C \in \mathbb{R}^{m\times n}$.

   (a) Write down the dual linear program.

   (b) Now assume $\bar{x} = \begin{bmatrix} \bar{x}_1 \\ \bar{x}_2 \\ \bar{x}_3 \end{bmatrix}$ is a feasible solution for the primal problem (1) and $\bar{y} = \begin{bmatrix} \bar{y}_1 \\ \bar{y}_2 \\ \bar{y}_3 \end{bmatrix}$ is a feasible solution for the corresponding dual problem. Prove the Weak Duality Theorem for this primal program, i.e., prove that $c^T\bar{x} \geq b^T\bar{y}$, where $c^T = [c_1^T \; c_2^T \; c_3^T]$ and $b^T = [b_1^T \; b_2^T \; b_3^T]$.

Programming Questions

3. (12 marks) A firm LinkingIn is in the business of linking candidates to jobs. There are n candidates (represented by the set C) and m jobs (represented by the set J). Not all candidate-job pairs are compatible. The firm has information of whether candidate i is compatible with job j for every candidate-job pair (i, j). Suppose this info is represented using an undirected graph $G = (V, E)$, where the candidates C as well as jobs J are represented as vertices while the presence of an edge (i, j) denotes that candidate i and job j are compatible with each other. Edges only exist between C and J. Each candidate can take up at most one job and each job can be assigned to at most one candidate. If candidate i takes up a compatible job j, the profit to LinkingIn is p_{ij} (you could think of p_{ij} as the per month salary that i earns from job j and this amount is the reward that LinkingIn gets).

   The following linear program computes the assignment that maximizes the reward to the firm.

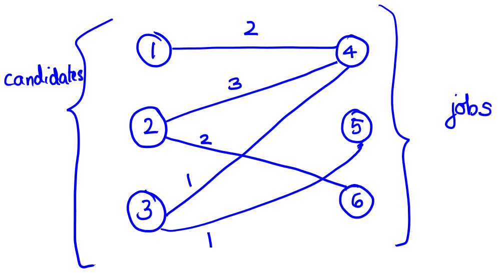
$$\left.\begin{array}{lrcl} \max\limits_{\mathbf{x}} & \sum_{(i,j)\in E} p_{ij}x_{ij} & & \\ \text{s.t} & \sum_{j\in J:(i,j)\in E} x_{ij} & \leq & 1 \,\forall i \in C \\ & \sum_{i\in C:(i,j)\in E} x_{ij} & \leq & 1 \,\forall j \in J \\ & x_{ij} & \geq & 0 \,\forall (i,j) \in E \end{array}\right\} \text{Primal}$$

(a) Write the dual of the above LP.

(b) Consider the following sample instance wherein there are 3 candidates and 3 jobs. The numbers on each edge denote p_{ij}.

$V = \{1, 2, 3, 4, 5, 6\}$

$E = \{(1,4), (2,4), (2,6), (3,4), (3,5)\}$



Implement the primal LP for this instance in Gurobi. Report the maximum possible reward to the company and also the candidate-job assignments that gives the best reward.

(c) Implement the dual LP, report the optimal values of the dual variables and verify strong duality holds.

(d) An alternative is to compute the solution in a greedy manner. This would boil down to forming a list with the p_{ij}s sorted in decreasing order and then traversing the list to make an assignment of a pair if they are still unmatched. What is the greedy solution and the corresponding reward to the firm for the instance in (b)? Is it optimal?

You do not have to implement the greedy solution. You may work out the solution on paper.

(e) *Optional*: It turns out that the primal LP always gives binary valued solutions at optimality. Prove this using the notion of total unimodularity.

4. (12 marks) An investment firm wants to invest at most 40 millions of dollars in 2021 and not more than 20 millions in 2022. Five possible investments have been identified, and each of them can be endorsed with a commitment from 0 to 100% (see Table 1). For example, if investment A is committed at 10% then we will pay 1.1 millions of dollars in 2021 and 0.3 millions in 2022 and we will obtain a net profit of 1.3 millions by the end of 2022.

The objective is to maximize the total cumulative net profit obtained by the end of 2022.

Investments	A	B	C	D	E
2021 quota	11	53	5	5	29
2022 quota	3	6	5	1	34
Net profit	13	16	16	14	39

Table 1: Cost and profit of the investments.

(a) Write a linear programming formulation for the portfolio optimization problem. Use the decision variables x_1, x_2, x_3, x_4, x_5 to indicate the fraction (between 0 and 1) of commitment for each investment.

(b) Implement the formulation in gurobi and compute the optimal portfolio for the investment firm.

(c) Explore the **Pi** attribute in gurobi and use it to compute the values of the dual variables.

(d) Write the dual of this linear program and implement it in gurobi. Report the optimal solution to the dual. Compare it against the result of the **Pi** attribute in part (c).

(e) *Optional:* Use the **Pi** attribute for the dual LP and check if it matches the optimal solution to the original primal LP.

(f) *Optional:* Verify complementary slackness holds for all the primal-dual pairs of LPs.

(g) *Optional:* Write an algebraic formulation of a more general portfolio optimization problem using the following notation for the data:

- N is the number of different investments, M is the number of years;
- a_{ij} is the cost for a 100% commitment in investment $i \in \{1, \ldots, N\}$ for the year $j \in \{1, \ldots, M\}$;
- p_i is the net profit for a 100% commitment in investment $i \in \{1, \ldots, N\}$
- b_j is the maximum amount of million that can be invested in year $j \in \{1, \ldots, M\}$.

The decision variables are:

- x_i is the fraction of commitment for investment $i \in \{1, \ldots, N\}$