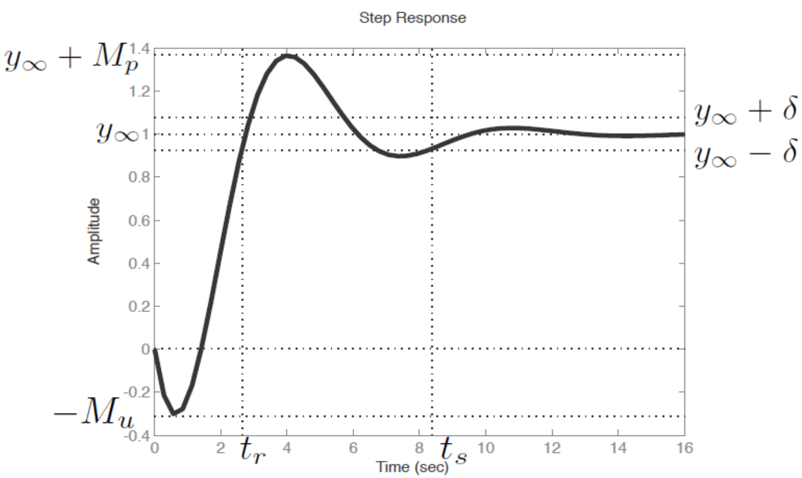
Lecture 7

Location, location, location…
Relating pole/zero positions to step response specifications

Step Response Specifications



y_∞	Steady-state value
M_u	Undershoot
M_p	Overshoot
t_r	Raise time
t_s	Settling time

Canonical 2nd Order Lowpass System

$$G(s) = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$

$$\sigma = \zeta\omega_n$$

$$\omega_d = \omega_n\sqrt{1 - \zeta^2}$$

- Assume stability and underdamping ($0 < \zeta < 1$), i.e. the poles have nonzero imaginary parts and negative real parts.
- We will see that overshoot, rise time and settling time specs will impose constraints on the permitted pole locations

Partial Fraction

- Denote step response by y(t)

$$Y(s)=\frac{G(s)}{s}=\frac{\omega_n^2}{s\left(s^2+2\zeta\omega_n s+\omega_n^2\right)}=\frac{\omega_n^2}{s\left((s+\sigma)^2+\omega_d^2\right)}$$

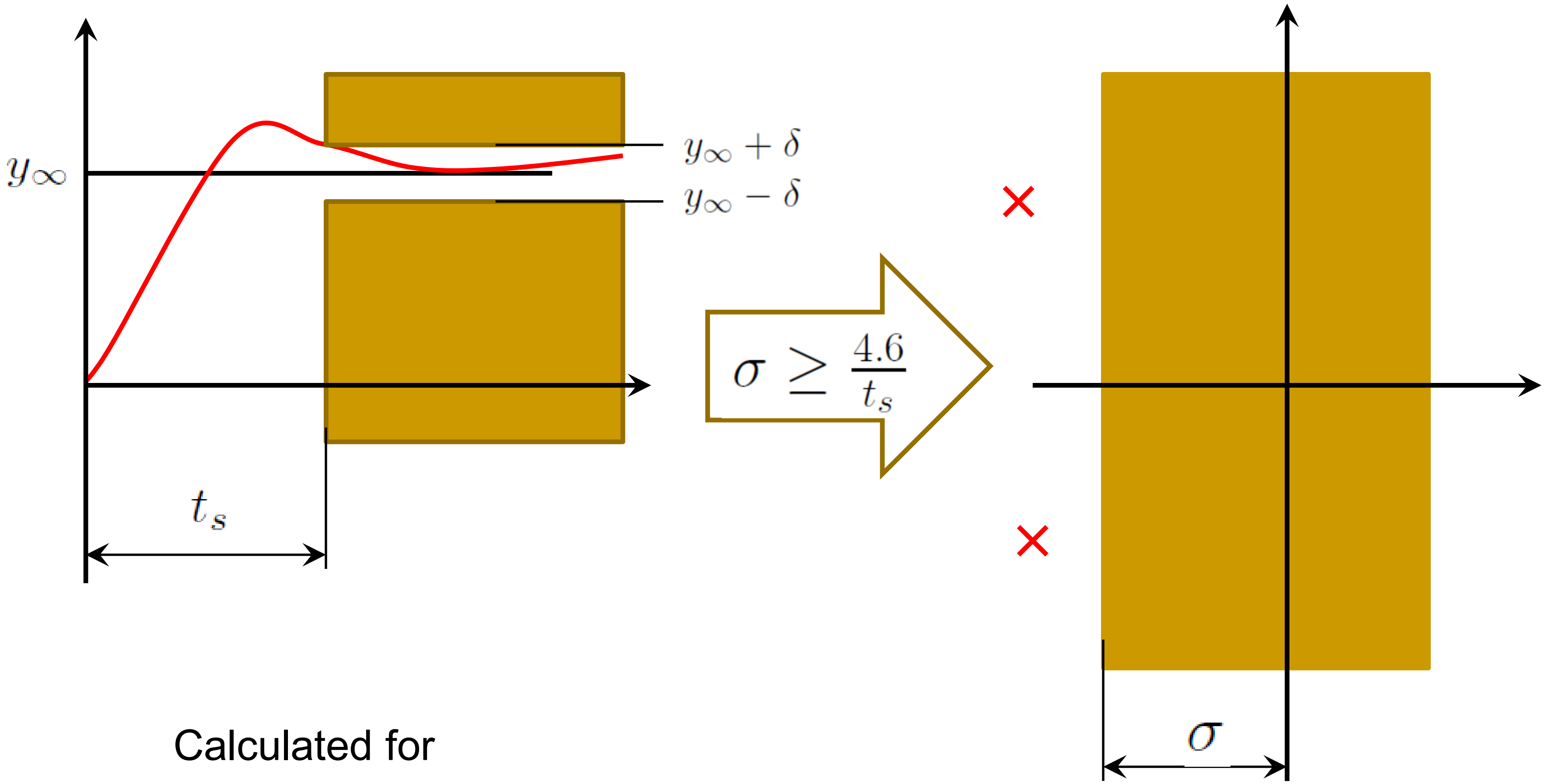
$$=\frac{1}{s}-\frac{2\zeta\omega_n}{(s^2+2\zeta\omega_n s+\omega_n^2)}-\frac{s}{(s^2+2\zeta\omega_n s+\omega_n^2)}$$

Table look-up:

$$y(t)$$

$$=1-\frac{\zeta}{\sqrt{1-\zeta^2}}e^{-\sigma t}\sin(\omega_d t)-e^{-\sigma t}\cos(\omega_d t)$$

Settling time



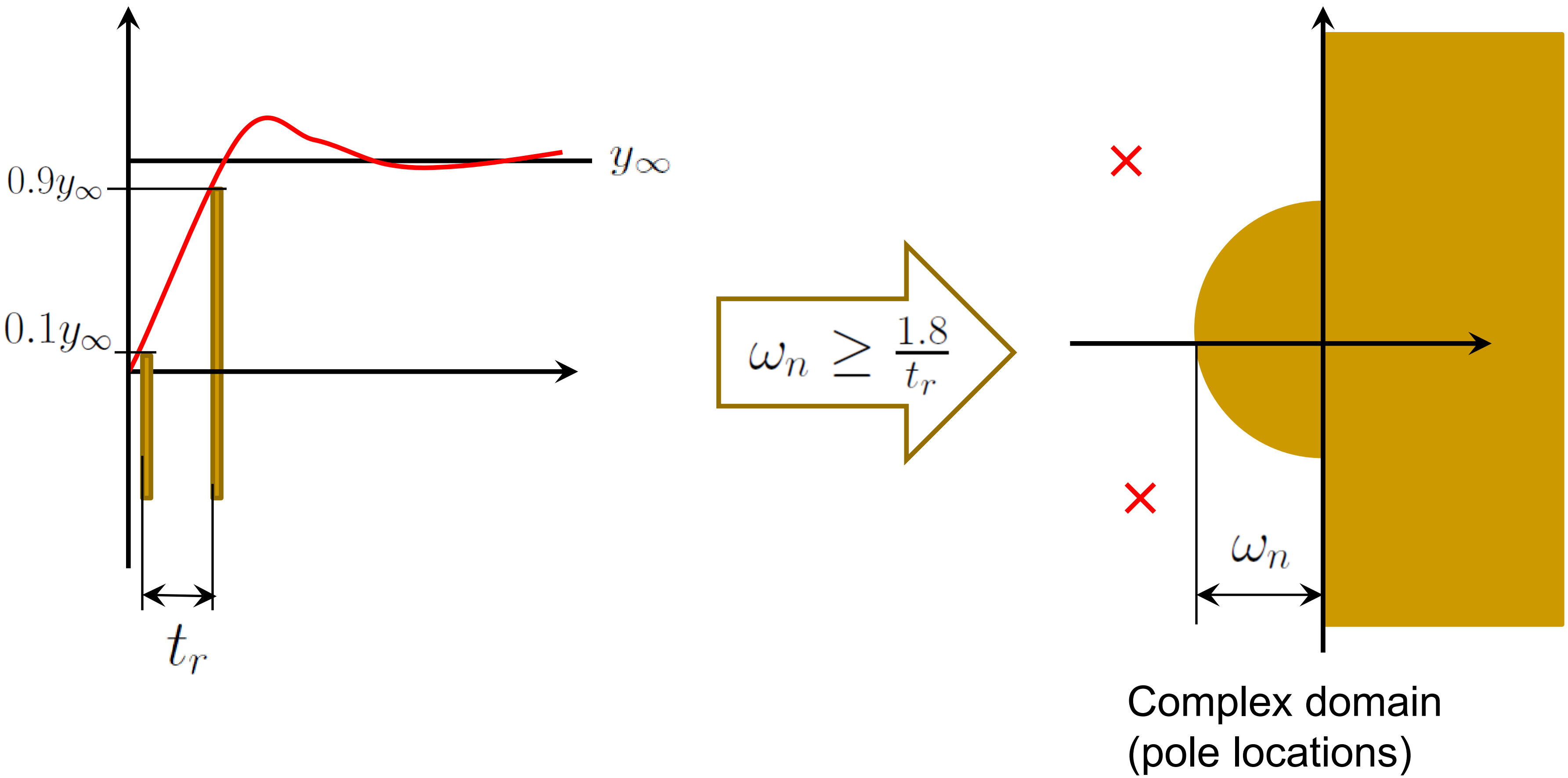
Calculated for

$$\delta = 0.01 y_\infty$$

Complex domain
(pole locations)

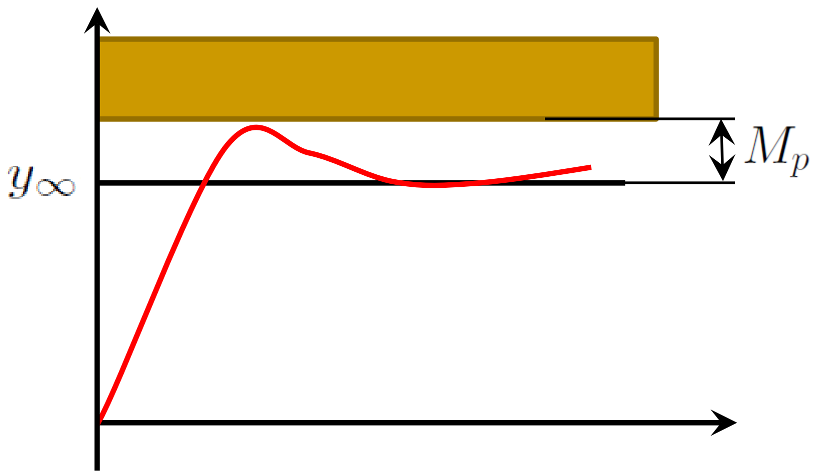
Brown colour represents a “forbidden region”.

Rise time

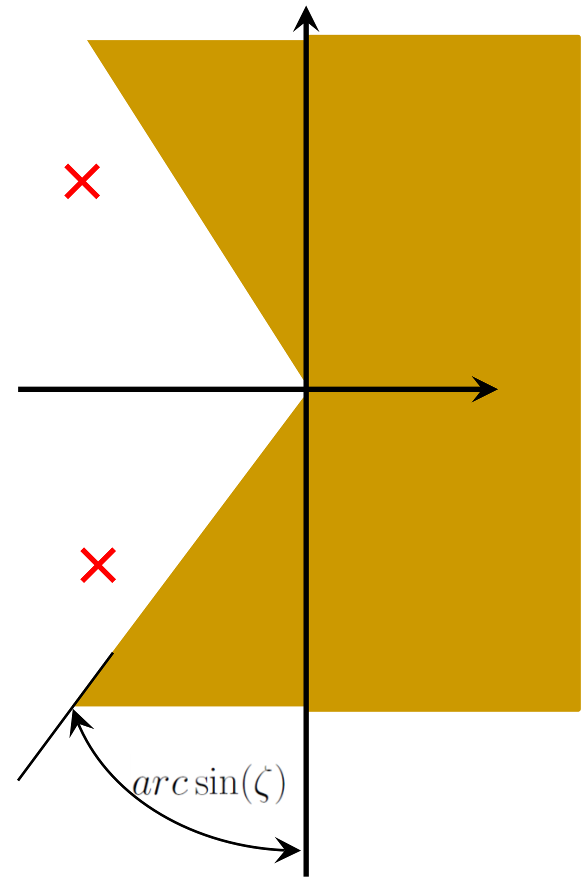


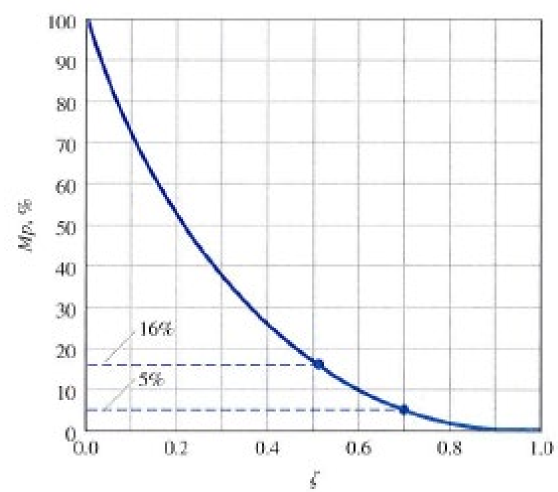
Brown colour represents a “forbidden region”.

Overshoot



$$\zeta \geq \zeta(M_p)$$



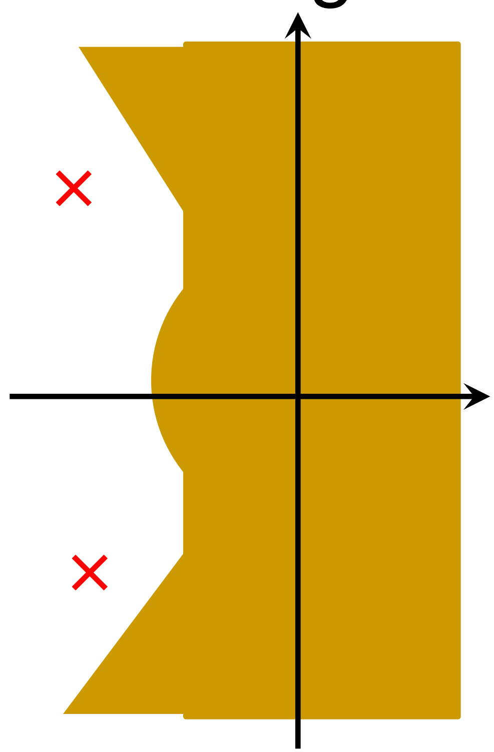


Complex domain
(pole locations)

Taken from Franklin, Powel & Emami-Naeini “Feedback control of dynamic systems”.

Summary

- Given a combined requirement on overshoot, rise time and settling time, we could get:



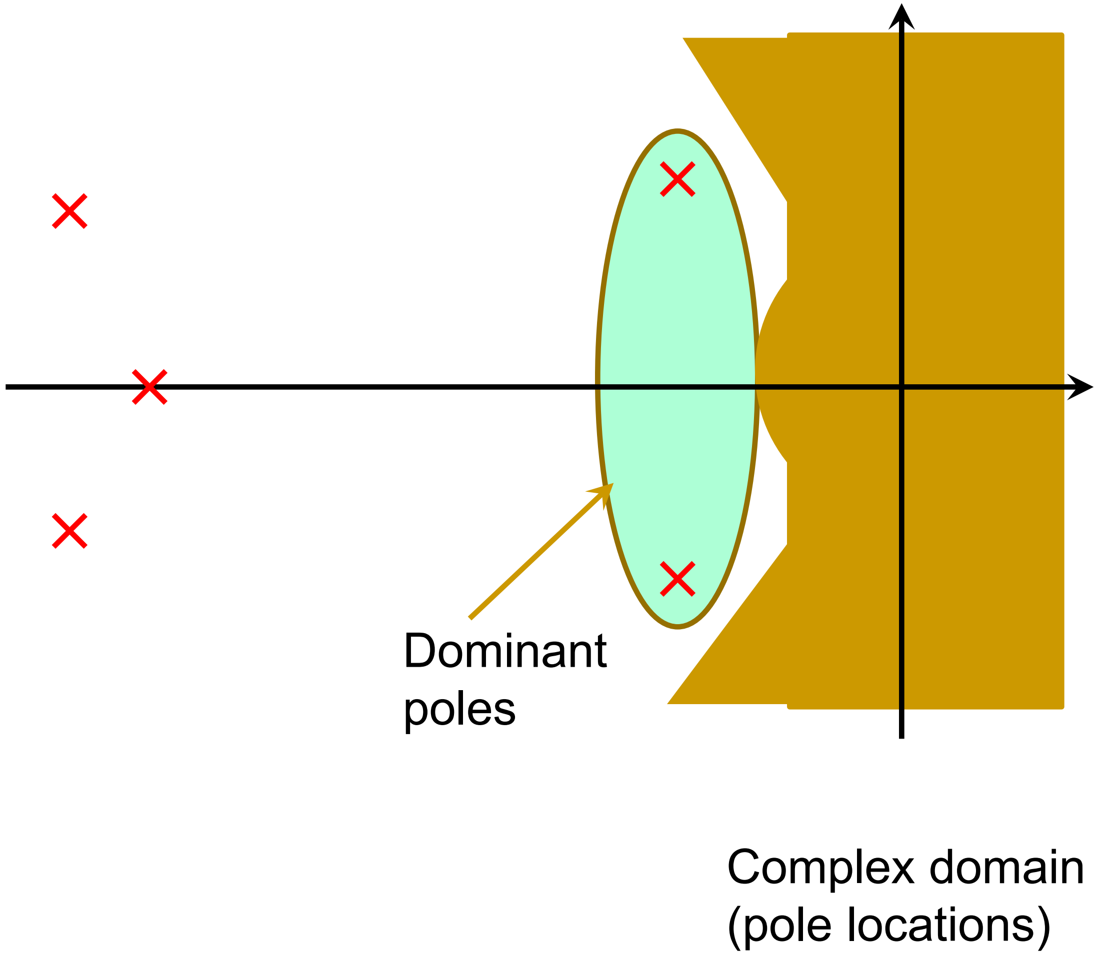
Complex domain
(pole locations)

Brown colour represents a “forbidden region”.

From 2^{nd} to Higher Order

- Formulas for canonical 2^{nd} order systems provide a “rule of thumb” for designing higher order lowpass systems
- They work well if there are two complex conjugate, stable, “dominant” poles that satisfy the required constraints. (Dominant means all other poles are far to the left, i.e. their modes decay much more rapidly)
- The formulas also work for proper stable systems with zeros all much larger than the dominant poles.

Graphical illustration



Approximating by 1st Order System

- By designing a single, real-valued dominant pole, we could even use a 1st order approximation. Even simpler!
- However, we then lose some useful degrees of design freedom.
- E.g. in a 1st order step response, the rise time is always about the same as the settling time.
- But for a 2nd order system, we could potentially place the poles so that the rise time is much shorter ➔ a controller that's more responsive, at the expense of overshoot.
- Higher order systems also have superior noise rejection properties (more after we introduce freq. response)

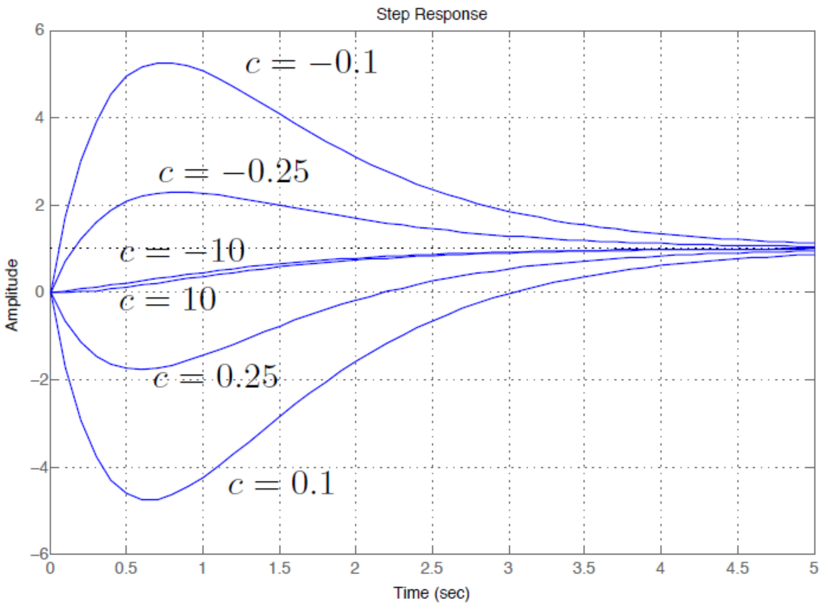
Fundamental trade-offs

- Sometimes, a step-response specification may not be achievable, due to a clash with another spec.
- In other words, there may be fundamental limitations, or trade-offs, in performance.
- We look next at a couple of the trade-offs caused by zeros

Example (real unstable zero)

- Consider this system with zero at $s = c$

$$G(s) = \frac{-(s-c)}{c(s+1)(0.5s+1)}$$



Overshoot: when zero is negative and small relative to the dominant pole at -1.

Undershoot: when zero is positive.

Slow zeros seem to have biggest impact.

Fundamental limitation:

(undershoot vs. settling time with unstable, real zero)

Suppose $G(s)$ is a stable transfer function, with $G(0) = 1$ and $G(c) = 0$ for a real valued $c > 0$, such that the step response $y(t)$ satisfies

'settling time' - see slide 19

$$1 - \delta \leq |y(t)| \leq 1 + \delta \quad \text{for all } t \geq t_s \qquad (\delta \text{ small} \sim 0.01)$$

Then the step response exhibits undershoot at the level of

$$M_u \geq \frac{1-\delta}{e^{ct_s} - 1} \approx \frac{1}{ct_s} \text{ for small } ct_s \text{ and } \delta$$

Proof:

Define $v(t) = 1 - y(t)$ so that $\max_{t \geq 0} v(t) = 1 + M_u > 0$ and $|v(t)| \leq \delta$ for $t \geq t_s$.

We have $V(s) = \big(1 - G(s)\big)\frac{1}{s}$ and $V(c) = \frac{1}{c} = \int_0^\infty v(t) e^{-ct}\, dt.$

okay because
$c \in$ ROC(V)
$= \{s : \Re(s) > 0\}$

As such,

$$\frac{1}{c} = V(c) = \int_0^{t_s} v(t) e^{-ct}\, dt + \int_{t_s}^\infty v(t) e^{-ct}\, dt$$

$$\leq (1 + M_u) \int_0^{t_s} e^{-ct}\, dt + \delta \int_{t_s}^\infty e^{-ct}\, dt$$

$$= (1 + M_u) \frac{1 - e^{-ct_s}}{c} + \delta \frac{e^{-ct_s}}{c}.$$

Fundamental limitation:

(overshoot vs. settling time with slow, stable real zero)

Suppose $G(s)$ is a stable transfer function, with $G(0) = 1$, $G(c) = 0$ for a real valued $c < 0$ and a dominant pole having real part $-p < 0$, such that

(i) $\eta \doteq \left|\frac{c}{p}\right| << 1$ and (ii) the unit step response $y(t)$ satisfies

$$|1 - y(t)| < Ke^{-pt} \text{ for } t \geq t_s \text{ and some } K > 0,$$

where t_s is the settling time for a suitable δ.

Then the step response has a percentage **overshoot**

$$M_p \geq \frac{1}{e^{-c\, t_s} - 1}\left(1 - \frac{K\eta}{1 - \eta}\right).$$