# Lecture 11

## Sensitivity transfer functions

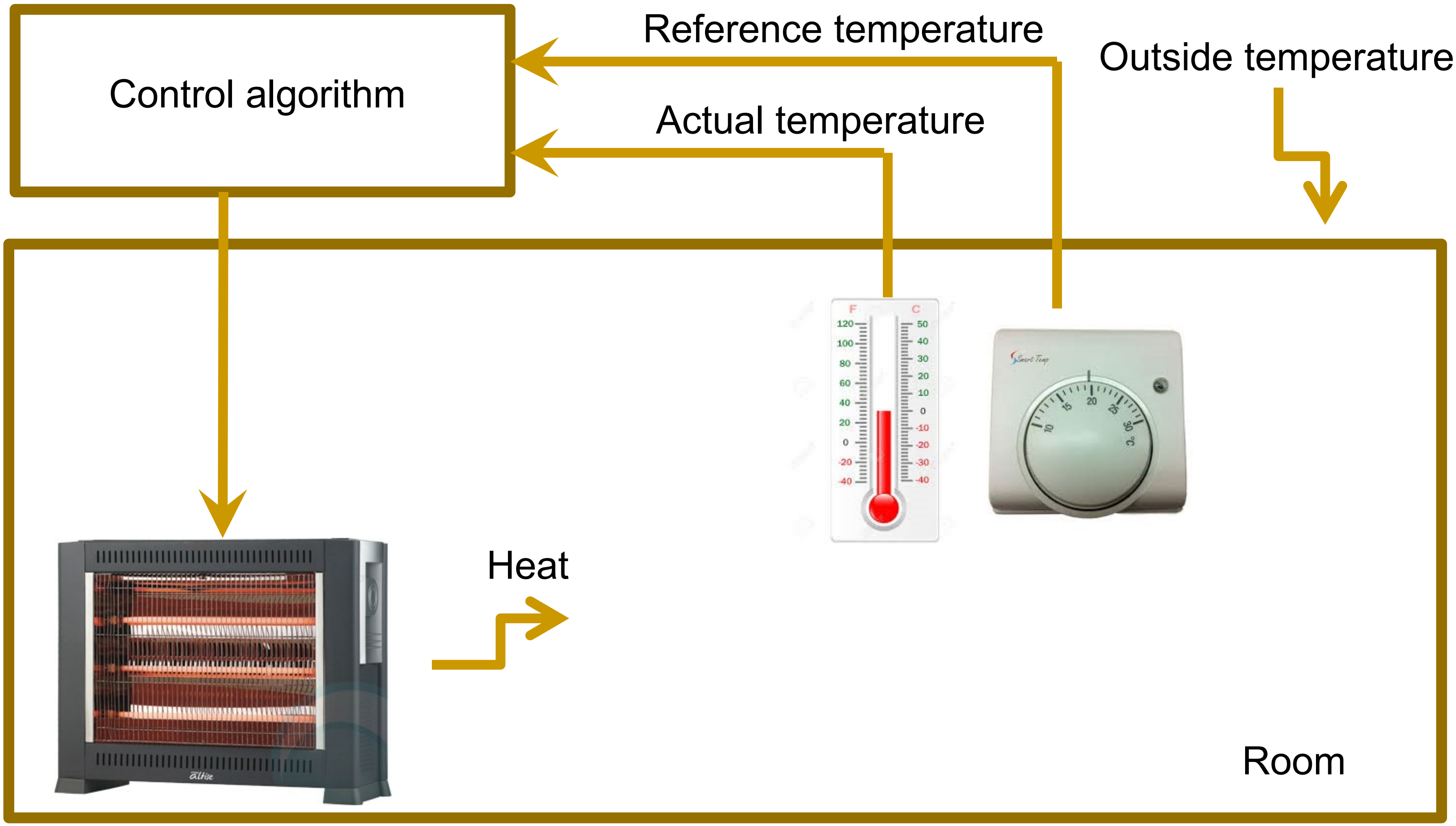
# Outline

- Motivation
- Examples
- Sensitivity transfer functions
- Steady state errors
- Conclusions

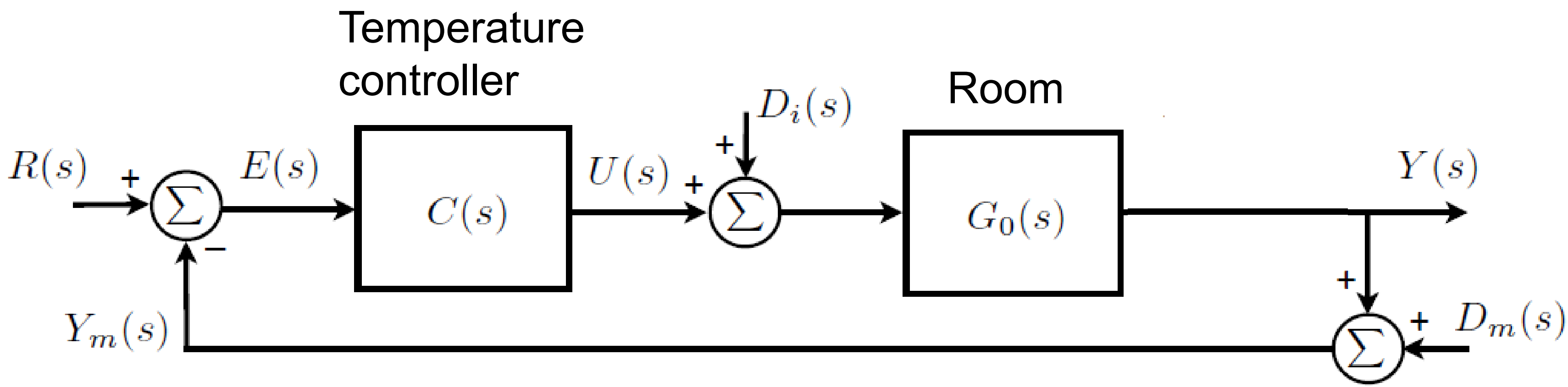
# Motivation

- Any control system typically has a number of different inputs (reference inputs, different disturbances)
- We want output to be sensitive to the reference, but also insensitive to disturbances entering the loop.
- Effects of references and disturbances on signals in the loop captured by a gang of four *sensitivity transfer functions*.

# Example (room temperature control)

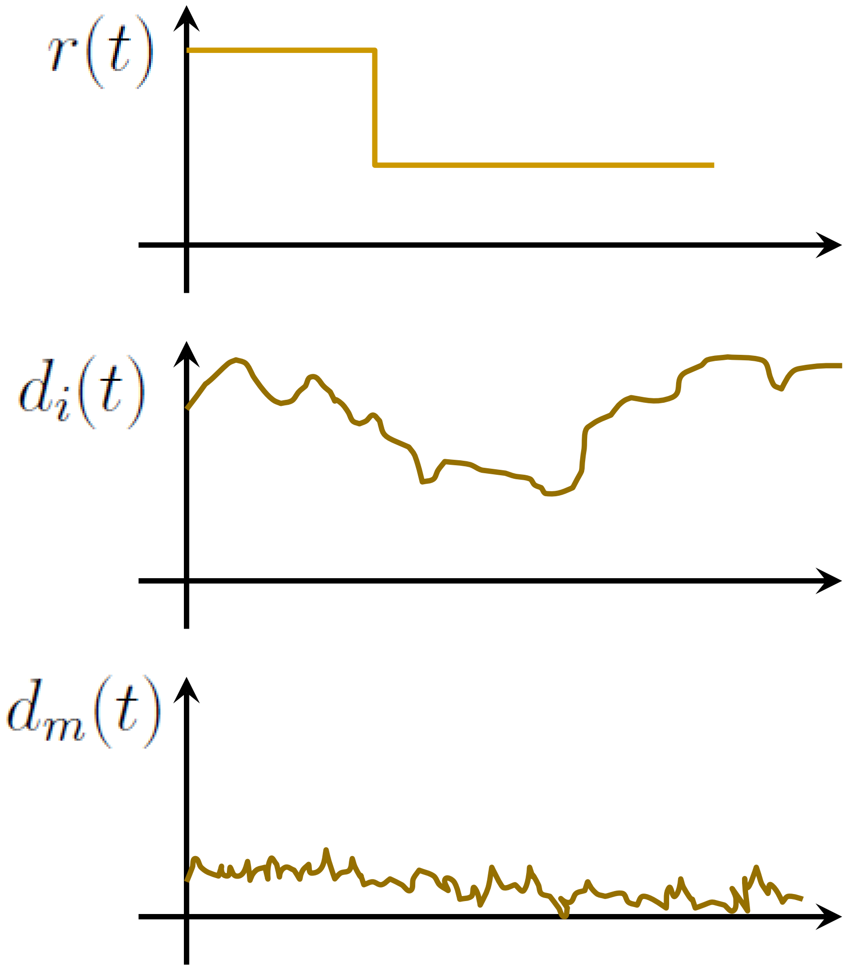


# Example (room temperature)




- Reference signal = desired temperature $R(s)$
- Input disturbance = heat dissipation (outside temperature, opening doors & windows) $D_i(s)$
- Measurement noise $D_m(s)$

# Example (room temperature)

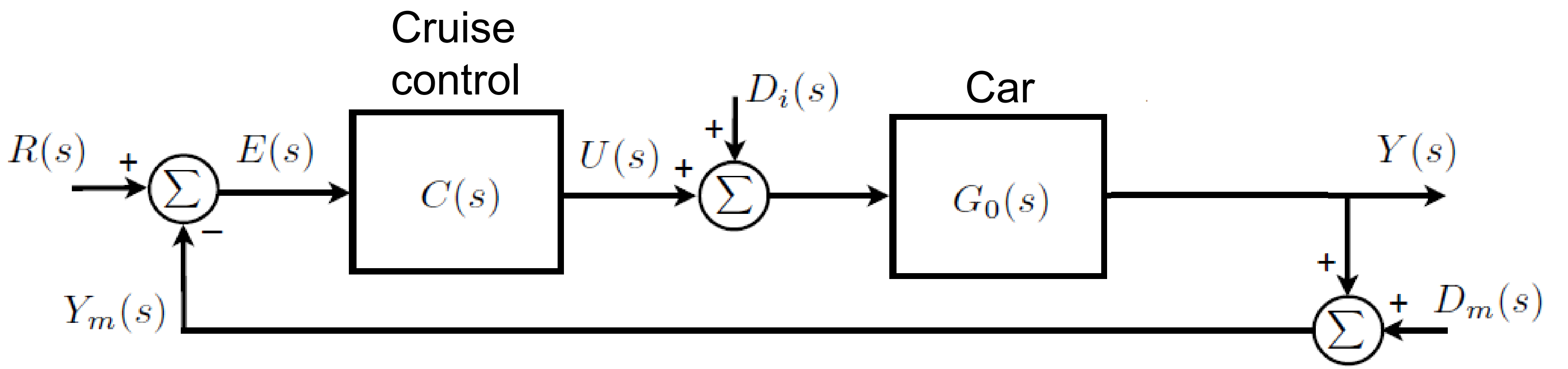



Reference is typically a piecewise constant signal.

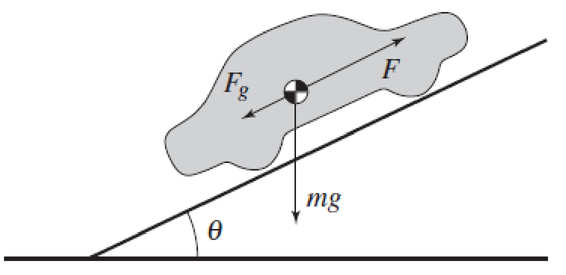
Input disturbance is typically a slowly varying large signal (low frequency, large amplitude).

Measurement noise is typically a fast varying small signal (high frequency, small amplitude).

# Example (cruise control)

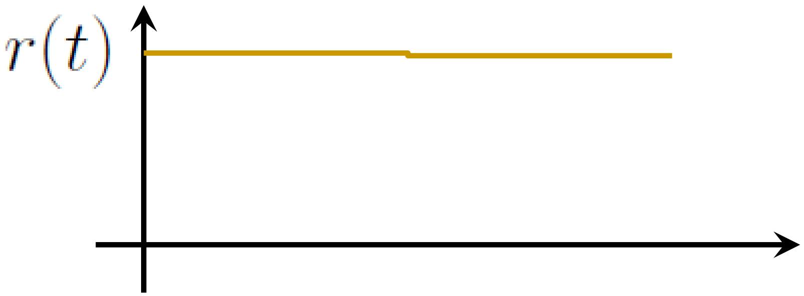



- Reference signal = desired velocity $R(s)$
- Input disturbance = slope of the road $D_i(s)$
- Measurement noise $D_m(s)$

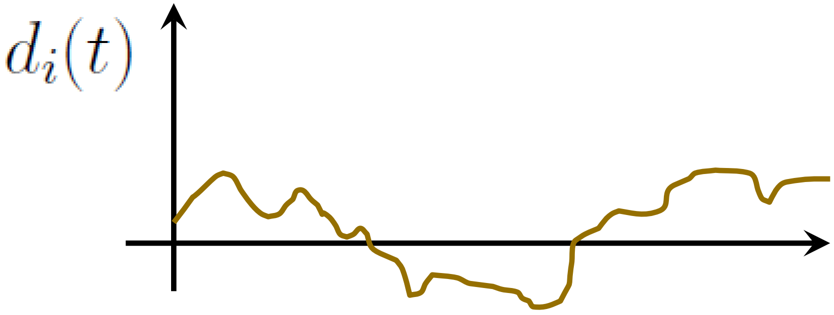



Taken from Astrom and Murray, Feedback systems.

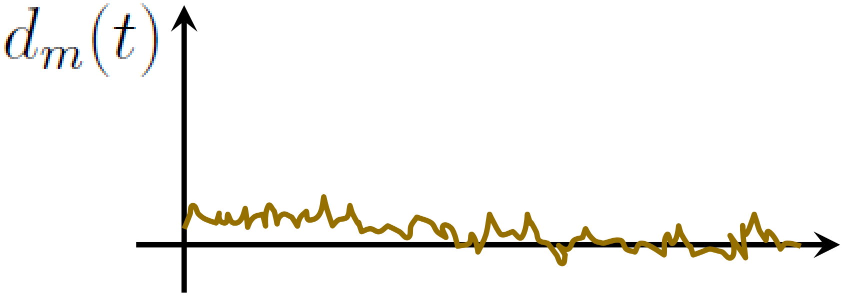
# Example (cruise control)




Reference is typically a constant signal.




Input disturbance is typically a slowly varying large signal (low frequency, large amplitude).




Measurement noise is typically a fast varying small signal (high frequency, small amplitude).

# Comments

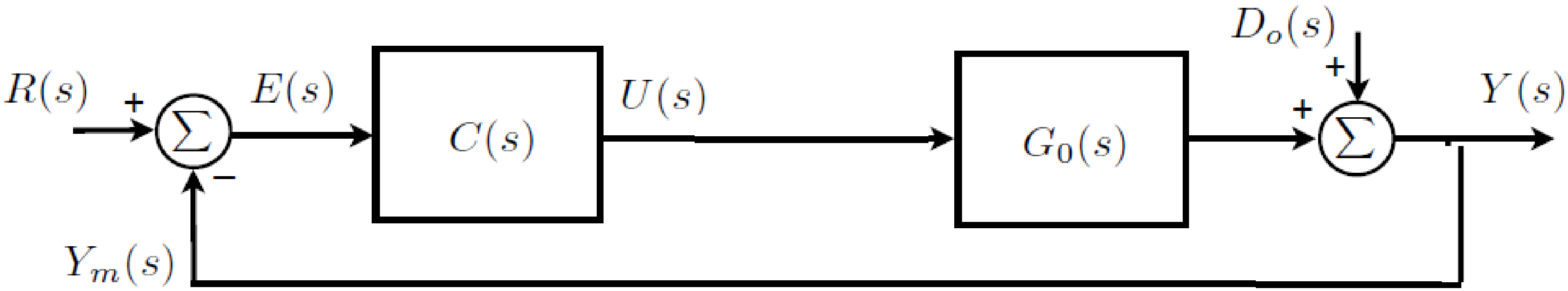
- Examples suggest that different control systems share similar structure and inputs possess similar properties.

- We now consider a general closed loop and analyse the effect of various inputs on the output.

# Sensitivity transfer functions

# Recall:

Turn off one input and compute the transfer function in the usual manner for the other input.



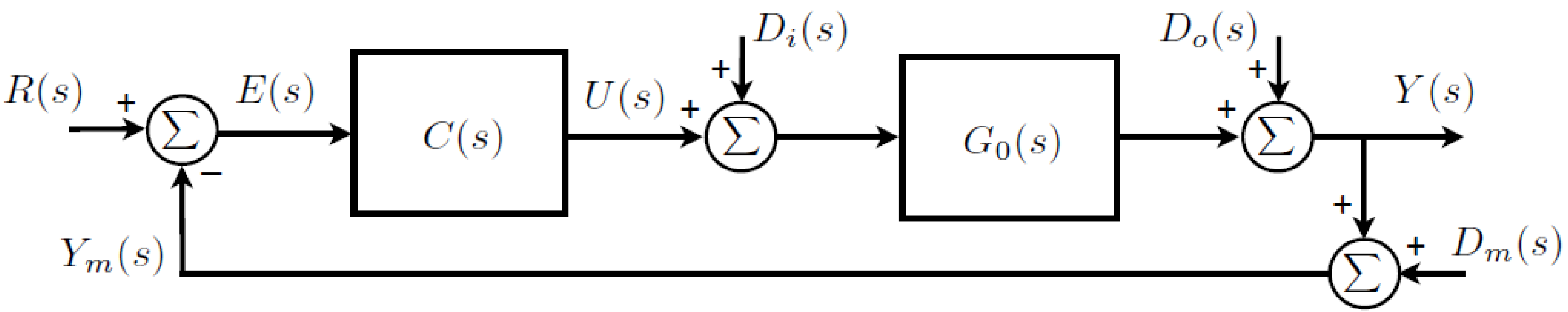

$$Y(s) = T_0(s)R(s) + S_0(s)D_o(s)$$

$$T_0(s) = \frac{Y(s)}{R(s)} = \frac{G_0(s)C(s)}{1 + G_0(s)C(s)} \qquad \text{with only } r \neq 0$$

$$S_0(s) = \frac{Y(s)}{D_o(s)} = \frac{1}{1 + G_0(s)C(s)} \qquad \text{with only } d_o \neq 0$$

This is how we will compute “sensitivity functions”.

# Typical block diagram




$G_0(s) = \dfrac{B_0(s)}{A_0(s)}$ $\boldsymbol{nominal}$ plant transfer function,

$C(s) = \dfrac{P(s)}{L(s)}$ controller transfer function

$\{B_0(s),\ A_0(s)\}$ and $\{P(s), L(s)\}$ coprime (no common roots) polynomial pairs

$D_i(s),\ D_o(s), D_m(s)$ plant input, plant output and measurement noise uncertainty

$R(s), U(s), Y(s)$ reference, control, and plant output signals

# Coprime polynomials

- Transfer function with coprime polynomials $A_0(s), B_0(s)$

$$G_0(s) = \frac{B_0(s)}{A_0(s)} = \frac{s-3}{(s+1)(s+2)} = \frac{s-3}{s^2+3s+2}$$

- Transfer function with polynomials $A_0(s), B_0(s)$ that are NOT coprime

$$G_0(s) = \frac{B_0(s)}{A_0(s)} = \frac{s+1}{(s+1)(s+2)} = \frac{s+1}{s^2+3s+2}$$

# Possible performance specifications:

- It would be good if the output follows the reference (despite disturbances); i.e. the error should be very sensitive to the reference input.

- It would be good if disturbance not to affect the error for any value of reference; i.e. the error should be insensitive to the disturbance input.

# Transfer functions:

- From the block diagram we have

$$Y(s) = D_0(s) + G_0(s)\Big(U(s) + D_i(s)\Big) \qquad (1)$$

$$U(s) = C(s)\Big(R(s) - D_m(s) - Y(s)\Big) \qquad (2)$$

- Next we will show that there exist “sensitivity” transfer functions so that we can write:

$$\begin{aligned} Y(s) &= T_0(s)R(s) + S_0(s)D_o(s) + S_{i0}D_i(s) - T_0(s)D_m(s) \\ U(s) &= S_{u0}(s)\left(R(s) - D_m(s) - D_o(s) - G_0(s)D_i(s)\right) \end{aligned}$$

# Transfer functions with Y as output

- We substitute (2) into (1) – i.e. eliminate U:

$$Y = D_0 + G_0C(R - D_m - Y) + G_0D_i$$

$$\underbrace{Y} = D_0 + G_0CR - G_0CD_m \underbrace{-G_0CY} + G_0D_i$$

$$Y(1 + G_0C) = G_0CR + D_0 + G_0D_i - G_0CD_m$$

$$Y(s) = \underbrace{\frac{G_0(s)C(s)}{1 + G_0(s)C(s)}}_{T_0(s)} R(s) + \underbrace{\frac{1}{1 + G_0(s)C(s)}}_{S_0(s)} D_0(s) +$$

$$\underbrace{\frac{G_0(s)}{1 + G_0(s)C(s)}}_{S_{i0}(s)} D_i(s) - \underbrace{\frac{G_0(s)C(s)}{1 + G_0(s)C(s)}}_{T_0(s)} D_m(s)$$

# Transfer functions with U as output

- We substitute (1) into (2) – i.e. eliminate Y. Please do this as an exercise! We obtain:

- $U(s) = \boxed{\frac{C(s)}{1+C(s)G_0(s)}}\big(R(s) - D_m(s) - D_o(s)\big) - \frac{C(s)G_0(s)}{1+C(s)G_0(s)} D_i(s)$

- NB: Another approach to getting *Y(s)* or *U(s)*

1. Suppress all inputs except one
2. Redraw the loop in unity or general feedback form. Use the corresponding transfer function to get *Y(s)* or *U(s)* in terms of that input.
3. Use linear superposition to sum up the contributions from all inputs.

# "Sensitivity" transfer functions:

- For nominal plant $G_0(s) = \frac{B_0(s)}{A_0(s)}$ and controller $C(s) = \frac{P(s)}{L(s)}$ we can define:

$$T_0(s) \doteq \frac{G_0(s)C(s)}{1 + G_0(s)C(s)} = \frac{B_0(s)P(s)}{A_0(s)L(s) + B_0(s)P(s)}$$ complementary sensitivity

$$S_0(s) \doteq \frac{1}{1 + G_0(s)C(s)} = \frac{A_0(s)L(s)}{A_0(s)L(s) + B_0(s)P(s)}$$ (output) sensitivity

$$S_{i0}(s) \doteq \frac{G_0(s)}{1 + G_0(s)C(s)} = \frac{B_0(s)L(s)}{A_0(s)L(s) + B_0(s)P(s)}$$ input-disturbance sensitivity

$$S_{u0}(s) \doteq \frac{C(s)}{1 + G_0(s)C(s)} = \frac{A_0(s)P(s)}{A_0(s)L(s) + B_0(s)P(s)}$$ control sensitivity

# Algebraic relationships:

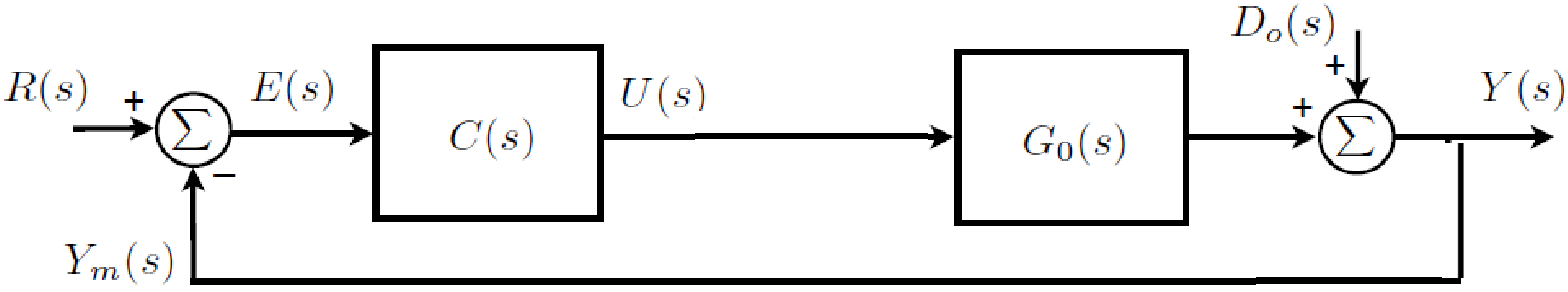
- Sensitivities are algebraically related:

$$S_0(s) + T_0(s) = 1, \quad S_{i0}(s) = G_0(s)S_0(s) = T_0(s)/C(s),$$

$$S_{u0}(s) = S_0(s)C(s) = T_0(s)/G_0(s)$$

# Special case: two inputs

# Two inputs:




$$Y(s) = T_0(s)R(s) + S_0(s)D_o(s)$$

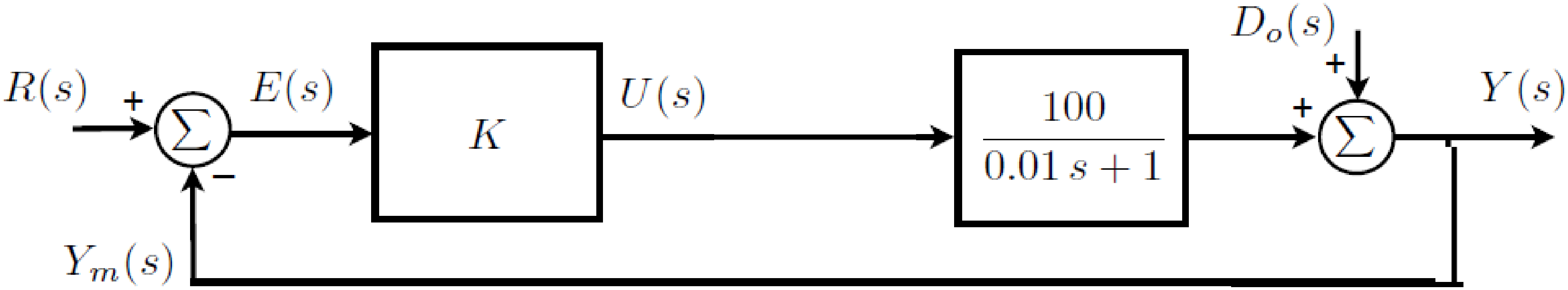
$$T_0(s) = \frac{Y(s)}{R(s)} = \frac{G_0(s)C(s)}{1 + G_0(s)C(s)} \qquad \text{with only } r \neq 0$$

$$S_0(s) = \frac{Y(s)}{D_o(s)} = \frac{1}{1 + G_0(s)C(s)} \qquad \text{with only } d_o \neq 0$$

# Important constraint:

- *C(s)* is the only degree of freedom for "shaping" the two transfer functions.
- If we shape one transfer function (e.g. for good tracking), this fixes the other (e.g. for disturbance rejection).
- This may lead to a trade-off between:
  - Good tracking;
  - Good disturbance rejection.

# Example:




$$G_0(s) = \frac{100}{0.01\,s+1} \qquad C(s) = K$$

$$Y(s) = T_0(s)R(s) + S_0(s)D_o(s)$$

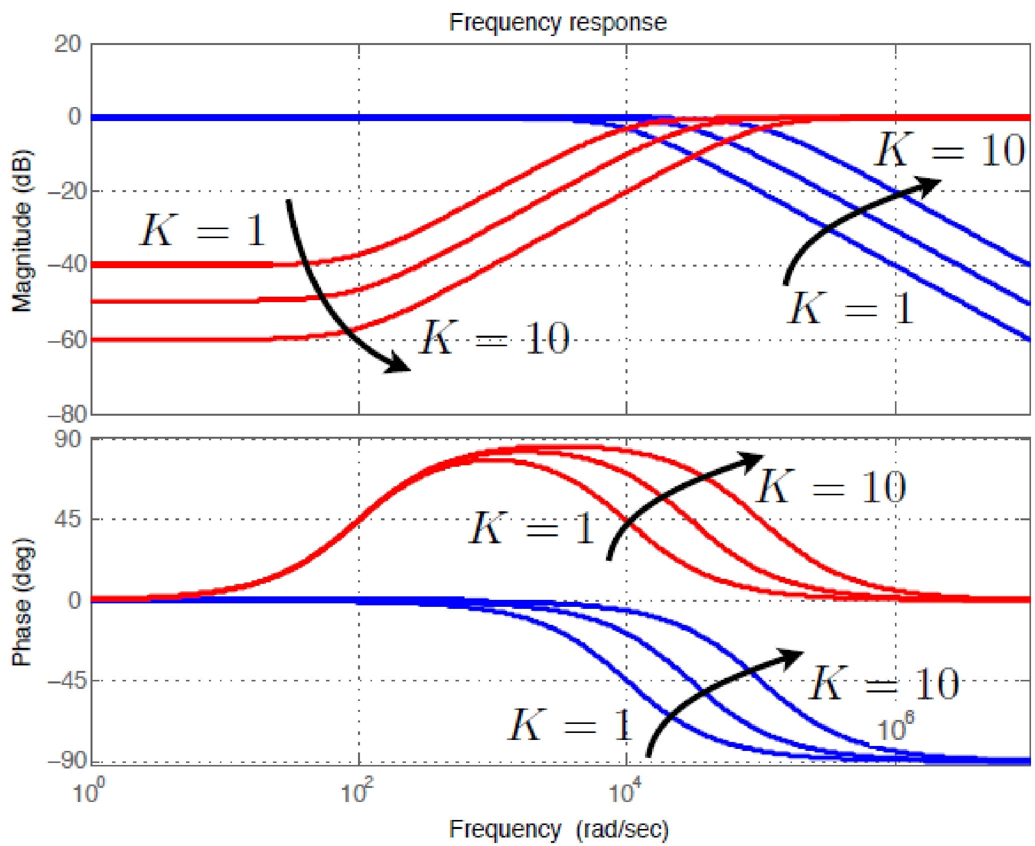
$$S_0(s) = \frac{1}{1+G_0(s)C(s)} = \frac{1}{1+\frac{100K}{0.01s+1}} = \frac{0.01s+1}{0.01s+1+100K}$$

$$T_0(s) = \frac{G_0(s)C(s)}{1+G_0(s)C(s)} = \frac{\frac{100K}{0.01s+1}}{1+\frac{100K}{0.01s+1}} = \frac{100K}{0.01s+1+100K}$$

# Bode diagram

$$\frac{1}{1+G_0C}$$

increasing the controller gain $K$ *reduces* the 'sensitivity' of the controlled output to *low* frequency disturbances at the plant output

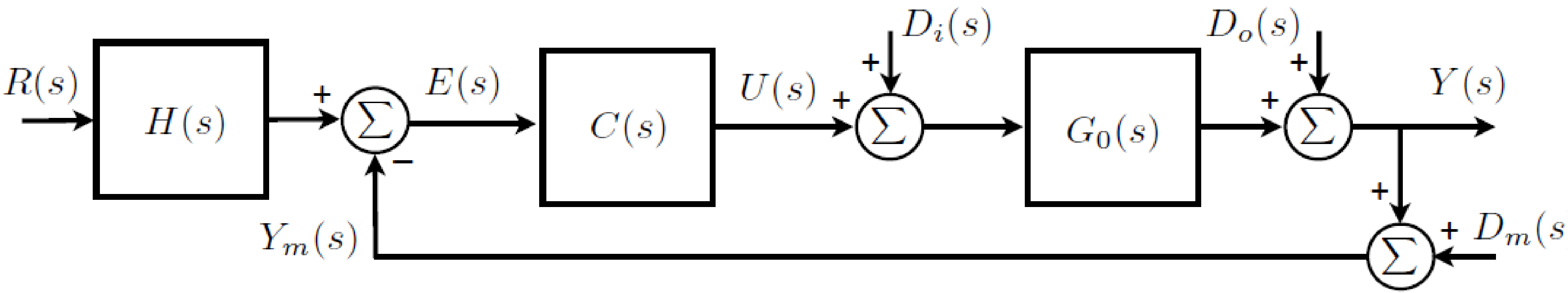



$$\frac{G_0C}{1+G_0C}$$

increasing the controller gain $K$ *increases* the range of frequencies (i.e. oscillating signals) that can be tracked by the output without attenuation or lag in steady-state

# Two degree of freedom controllers

# Input-output relationships




$H(s)$ is a *stable* transfer function (reference filter)

provides a second degree-of-freedom for 'shaping' reference response

$$Y(s) = \frac{G_0(s)C(s)H(s)}{1 + G_0(s)C(s)}R(s) + \frac{1}{1 + G_0(s)C(s)}D_o(s) + \frac{G_0(s)}{1 + G_0(s)C(s)}D_i(s) - \frac{G_0(s)C(s)}{1 + G_0(s)C(s)}D_m(s)$$

can only 'shape' one of these closed-loop transfer functions using $C(s)$ ... setting one determines the other two!!!

# Disturbance-input relationships

- We are also interested in how disturbances affect the inputs so it is sometimes of interest to consider:

$$U(s) = \frac{C(s)H(s)}{1+G_0(s)C(s)}R(s) - \frac{C(s)}{1+G_0(s)C(s)}D_o(s)$$
$$- \frac{G_0(s)C(s)}{1+G_0(s)C(s)}D_i(s) - \frac{C(s)}{1+G_0(s)C(s)}D_m(s)$$

# Input and output relationships

- We can rewrite the original relationships as

$$Y(s) = T_0(s)H(s)R(s) + S_0(s)D_o(s) + S_{i0}(s)D_i(s) - T_0(s)D_m(s)$$
$$U(s) = S_{u0}\Big(H(s)R(s) - D_m(s) - D_o(s) - G_0(s)D_i(s)\Big)$$

- We can obtain sensitivities in the same manner as before using the block diagram. Please do this as an exercise.

# Conclusions

- We have introduced one and two degree of freedom controllers and have introduced different sensitivity functions.
- Sensitivities are algebraically related! This typically leads to trade-offs in design.
- We will study in more detail these sensitivities in the next lecture.