Lecture 9

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Frequency response

Bode diagrams

Motivation

- Frequency domain analysis and design are instrumental in achieving appropriate performance and robustness of a closed loop system.
- Frequency response can be experimentally obtained.
- Nyquist plots and Bode plots are essential tools in control systems design.

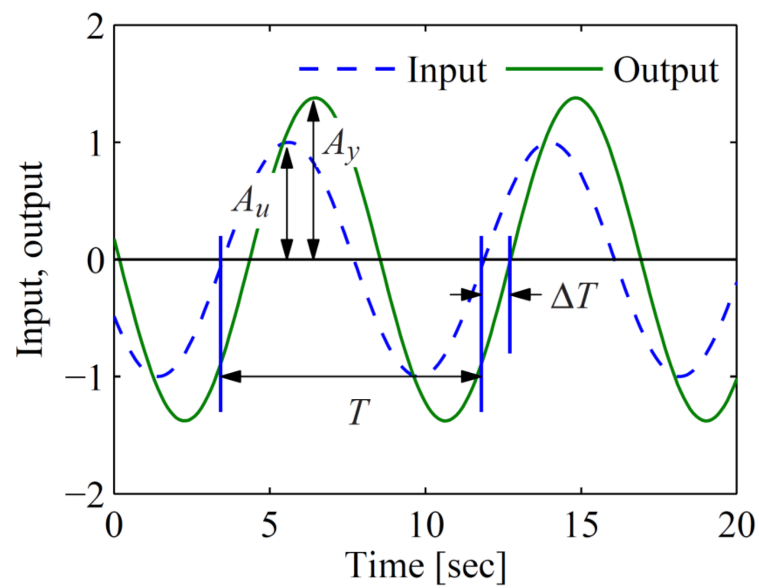
Steady state periodic response

System performance often characterised in terms of response to important input types.

- Impulse response= output when a short, sharp shock is applied
- Step response= output when input is suddenly changed
- Frequency response: characterises *steady-state* output of system when input is a sinusoid of frequency ω

Sinusoidal steady-state response

$u = A_u \cos(\omega t)$ → Causal, stable, LTI system *G(s)* → $y_{ss} = A_y \cos(\omega t + \theta)$



Gain: $\dfrac{A_y}{A_u}$

Phase: $\theta = -\dfrac{2\pi \Delta T}{T}$

Claim: Gain = $|G(j\omega)|$, Phase = $\angle G(j\omega)$

Taken from Astrom and Murray, Feedback systems

Proof of Claim (assume BIBO stable)

Let $u'(t) = e^{j\omega t}$

$y'(t) =$

$\int_0^t g(v) e^{j\omega(t-v)} dv = e^{j\omega t} \int_0^t g(v) e^{-j\omega v} dv$

$\Leftrightarrow y'(t) e^{-j\omega t} = \int_0^t g(v) e^{-j\omega v} dv$

$\Rightarrow \left| y'(t) e^{-j\omega t} - G(j\omega) \right| = \left| -\int_t^\infty g(v) e^{-j\omega v} dv \right|$

$\leq \int_t^\infty |g(v)| dv \to 0$

$\Leftrightarrow y'(t) = G(j\omega)\, e^{j\omega t}$ + (transient → 0 as t → ∞)

Proof (continued)

- So if $u(t) = \cos \omega t = \Re\left[e^{j\omega t}\right]$,

$$y(t) = \int_0^t g(v)\Re\left[e^{j\omega(t-v)}\right] \; dv$$

$$= \Re\left[\int_0^t g(v)e^{j\omega(t-v)}dv\right]$$

$$= \Re[y'(t)]$$

$$= \Re\left[G(j\omega)\, e^{j\omega t}\right] + \text{transient}$$

$$= |G(j\omega)| \cos\left(\omega t + \angle G(j\omega)\right) + \text{transient}$$

Example

- Consider the following example: $G(s) = \dfrac{4}{2s+1}$ with $u(t) = \sin(5t) \overset{\mathcal{L}}{\leftrightarrow} \dfrac{5}{s^2+25}$

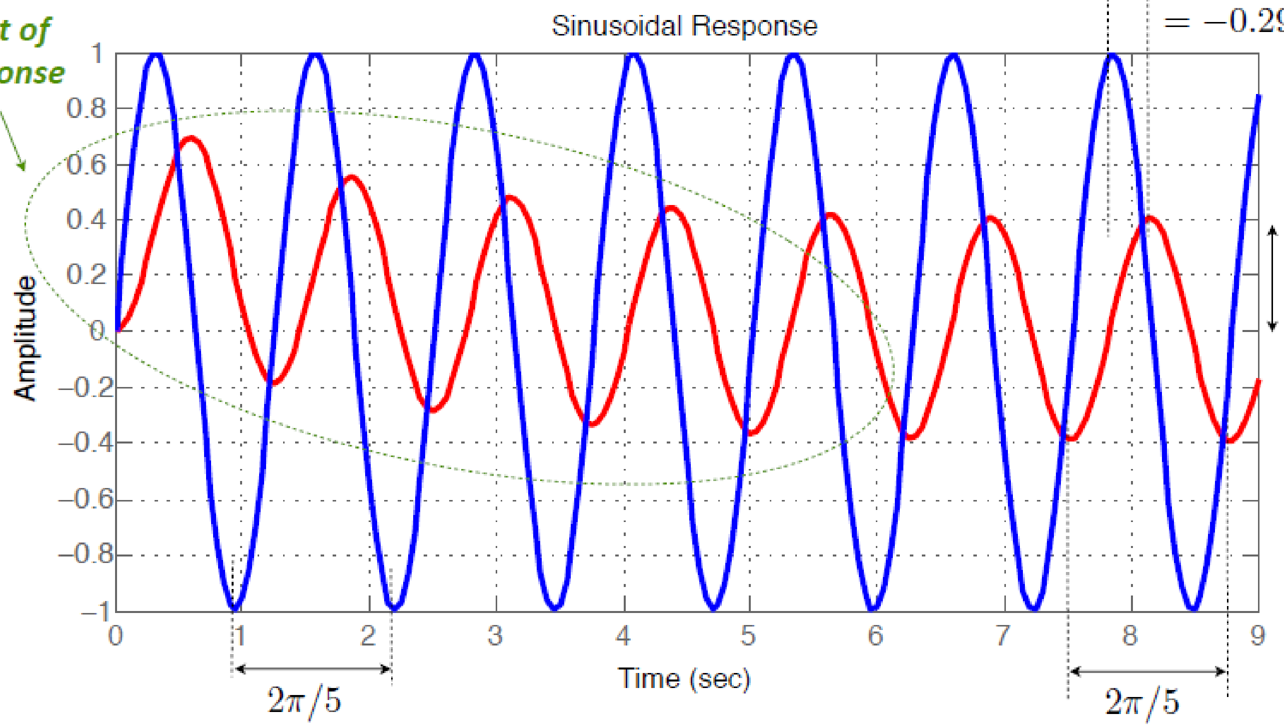
$$y(t) = \mathcal{L}^{-1}\left[\frac{4}{2s+1}\cdot\frac{5}{s^2+25}\right](t) = y_{\mathrm{tr}}(t) + |G(j5)|\sin(5t + \angle G(j5))$$

transient decays to 0 as $t \to \infty$

$$\frac{\angle G(j5)}{5} = 0.2 \cdot \arctan\frac{\Im(G(j5))}{\Re(G(j5))}$$

$= -0.294$ ***since this is negative the output is said to LAG the input***

decaying influence of transient part of response



$$|G(j5)| = \frac{4}{\sqrt{10^2+1}}$$

Steady state response:

- Note that in steady state we obtain

$$y_{ss}(t) = G(j\omega)e^{j\omega t} = |G(j\omega)|e^{j(\omega t + \angle G(j\omega))}$$

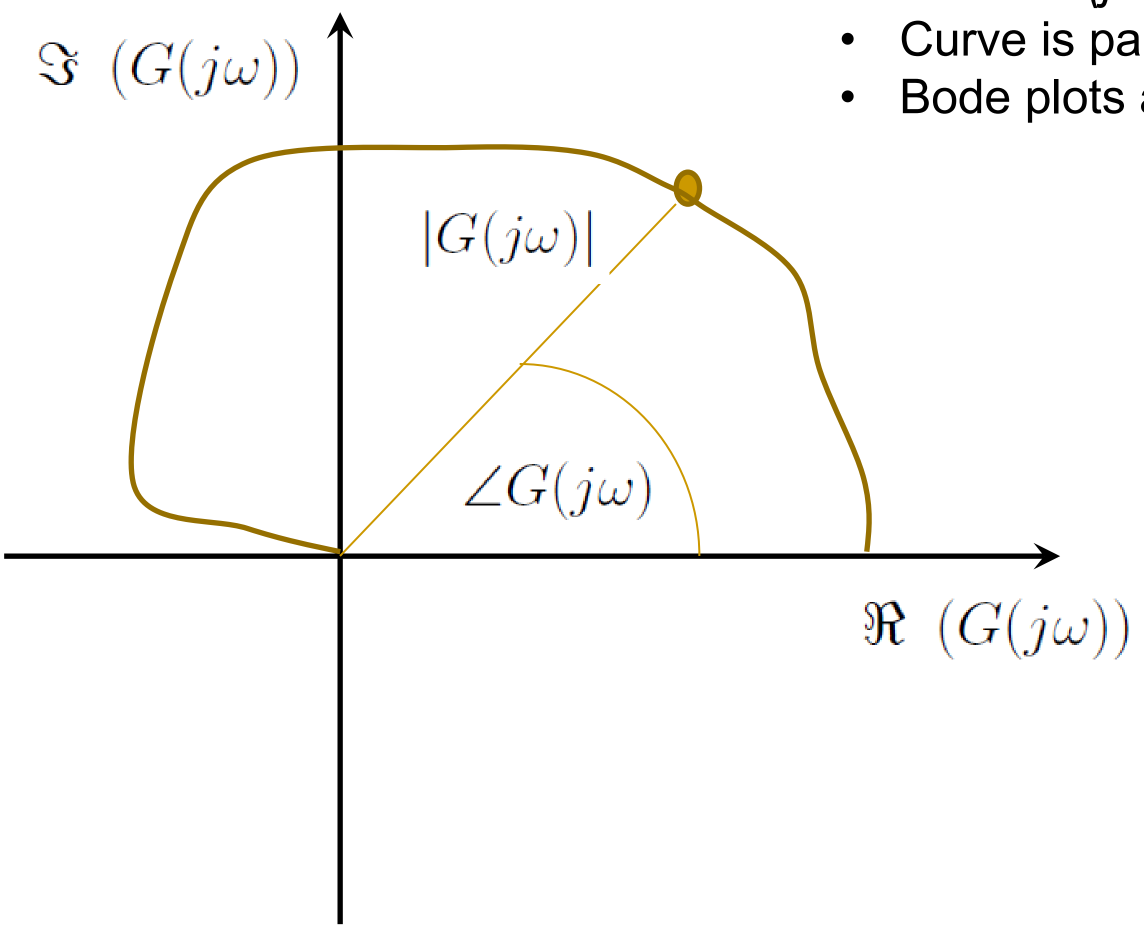
'frequency response'

- The steady state output is periodic of the same frequency ω but
  - scaled in magnitude by $|G(j\omega)|$ and
  - shifted in phase by $\angle G(j\omega)$

Frequency Response for Some Unstable Systems

- We can also formally define a frequency response for non-BIBO systems, provided there are no poles with positive real part.
- Just set $s = j\omega$ in the transfer function. E.g.
  - $H(s) = s^{\pm n}$ has freq. response $(j\omega)^{\pm n}$
  - $H(s) = \frac{1}{s^2+\omega_0^2}$ has freq. response $\frac{1}{(j\omega)^2+\omega_0^2}$
- (Made rigorous by adding a small real part -ϵ to any poles on imaginary axis; multiplying *H(s)* by $\frac{1}{(s+\epsilon)^n}$ if relative degree *n>0*; then letting $\epsilon \to 0$.)

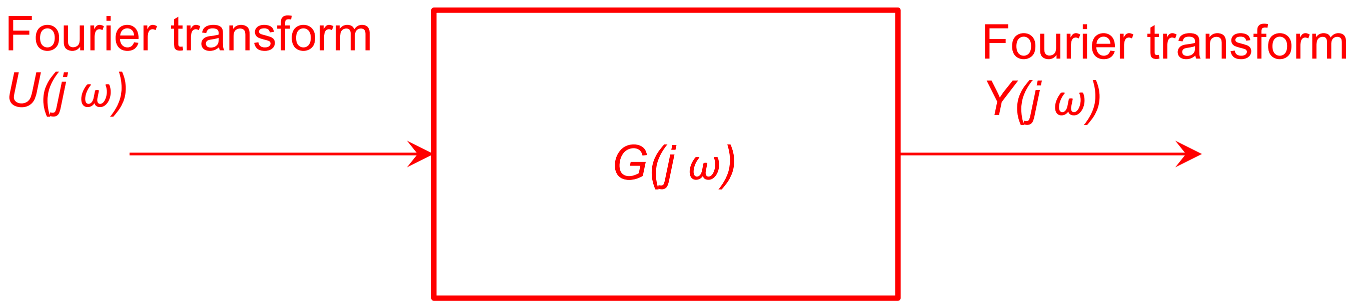
Nyquist plot



- Plot of *G(jω)* in complex plane
- Curve is parameterised by frequency.
- Bode plots are more convenient.

Link to Fourier Transforms

$$Y(j\omega) = G(j\omega)U(j\omega)$$



$$u(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{j\omega t} U(j\omega) d\omega$$

- Any input signal can be regarded as a "sum" of terms containing $e^{j\omega t}$ for $\omega \in (-\infty, +\infty)$

Bode diagrams

- We represent the frequency response

$$H(j\omega) = |H(j\omega)| e^{j\angle H(j\omega)}$$

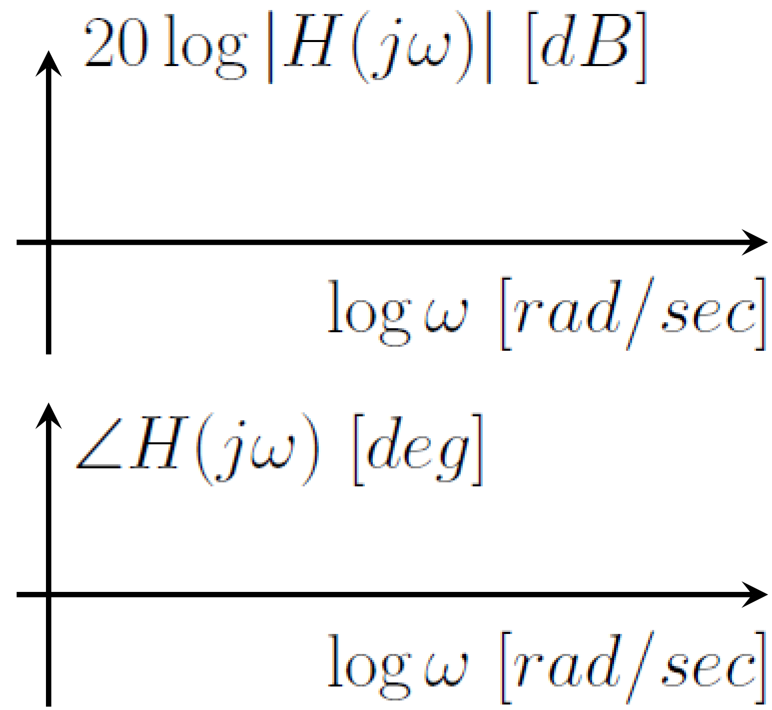
in a “convenient” form using two plots.

$\log|H|$ gives additive property.

Multiplication with 10 gives dB .

$\log|H|^2 = 2\log|H|$

$\log\omega$ gives more compact diagrams.



Why? (additive property)

- Note if we have

$$A = ae^{j\theta},\ |A| = a,\ \angle A = \theta\ ;\quad B = be^{j\phi}, |B| = b,\ \angle B = \phi$$

then, $A \cdot B = |A||B|e^{j(\theta+\phi)} = abe^{j(\theta+\phi)}$

$$\log|A| \cdot |B| = \log a \cdot b = \log|A| + \log|B|$$

$$\begin{aligned} \angle A \cdot B &= \angle abe^{j(\theta+\phi)} \\ &= \theta + \phi \\ &= \angle A + \angle B \end{aligned}$$

Bode diagrams

- Consider a transfer function:

$$H(s) = K\frac{\prod_{i=1}^{m}(\beta_i s + 1)}{s^k \prod_{i=1}^{n}(\alpha_i s + 1)}$$

- Then, we can write

$$20\log|H(j\omega)| = 20\log(|K|) - 20k\log|\omega| + \sum_{i=1}^{m} 20\log|\beta_i j\omega + 1| - \sum_{i=1}^{n} 20\log|\alpha_i j\omega + 1|$$

$$\angle(H(j\omega)) = \angle(K) - k\frac{\pi}{2} + \sum_{i=1}^{m}\angle(\beta_i j\omega + 1) - \sum_{i=1}^{n}\angle(\alpha_i j\omega + 1)$$

Bode diagrams

- If we obtain Bode diagrams (gain and phase) for some basic functions, then Bode diagrams of an arbitrary transfer function is obtained by “adding” gain plots and phase plots of the individual building blocks.
- Bode diagrams are invaluable in “loop shaping” design techniques.
- We will plot Bode diagrams for a number of simple transfer functions.