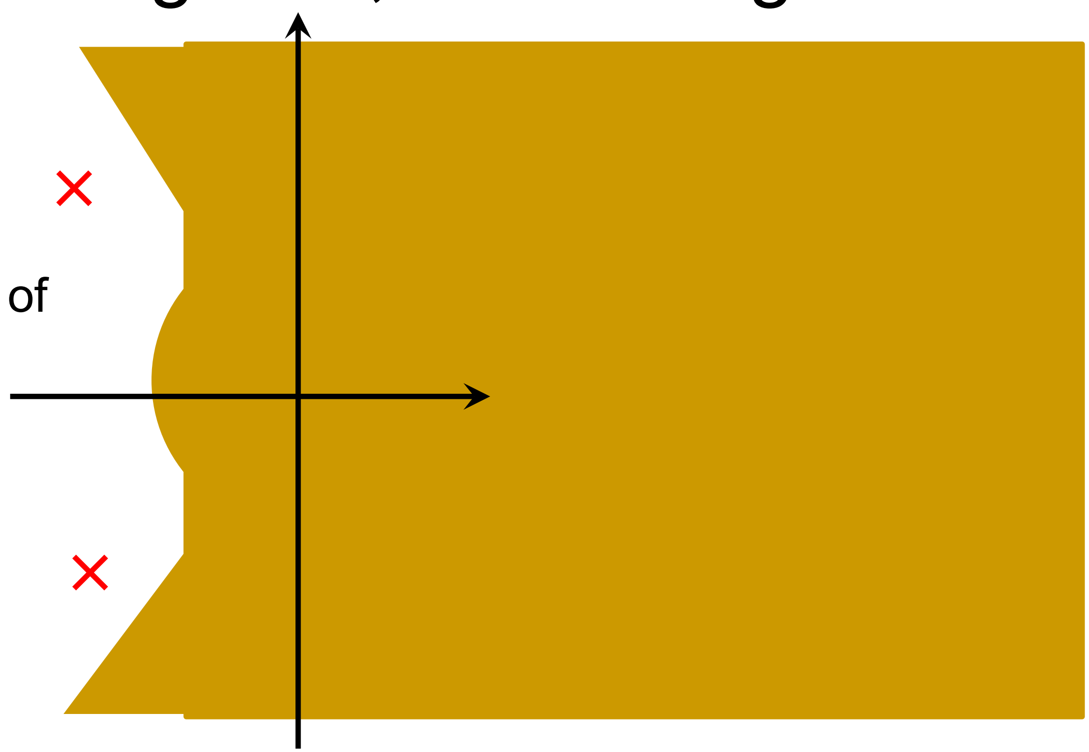
Lecture 12

Root locus

Motivation
(performance via "dominant poles" design)

- Given a combined requirement on overshoot, rise time and settling time, we could get:

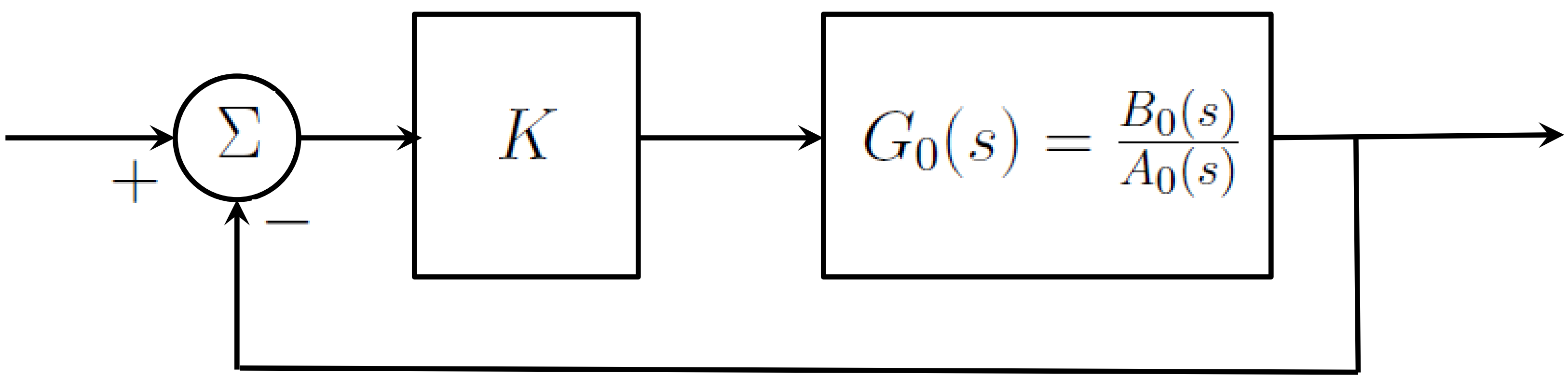
If we place dominant poles of the closed loop in the white region of the complex plane, the system would have the desired transients in step response



Brown colour represents a "forbidden region".

Motivation:
(simplest proportional controller)

- Consider the closed-loop system, where K is a parameter (i.e. "proportional" controller)



Design question: Can we select K so that the poles are in the desired region?

Motivation:

- Transfer function of the closed loop system:

$$G(s) = \frac{KG_0(s)}{1+KG_0(s)} = \frac{K\frac{B_0(s)}{A_0(s)}}{1+K\frac{B_0(s)}{A_0(s)}}$$

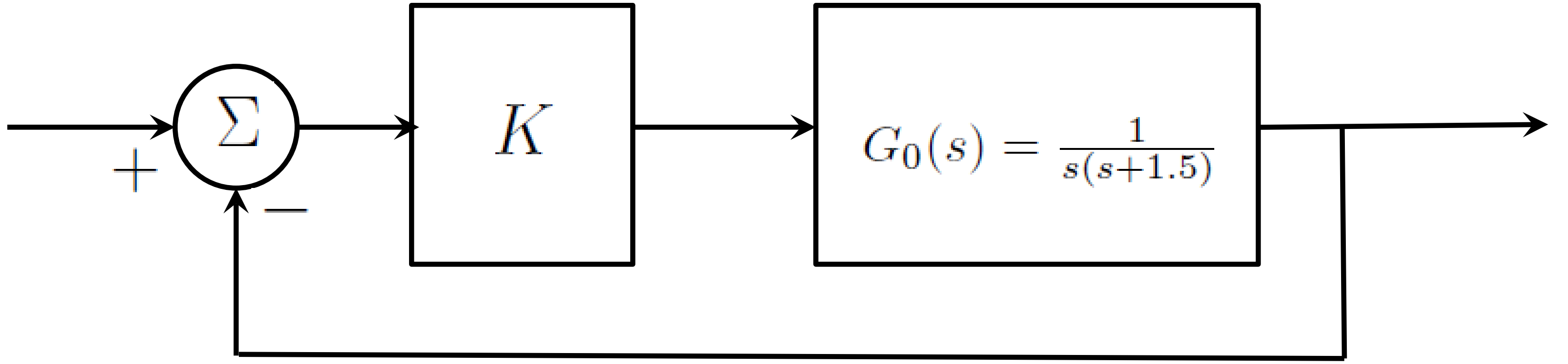
- Locations of closed loop poles depend on the parameter K:

$$1 + K\frac{B_0(s)}{A_0(s)} = 0 \qquad \Leftrightarrow \qquad A_0(s) + KB_0(s) = 0$$

Motivation

(proportional controller)

- Plot the poles for this system for K>0

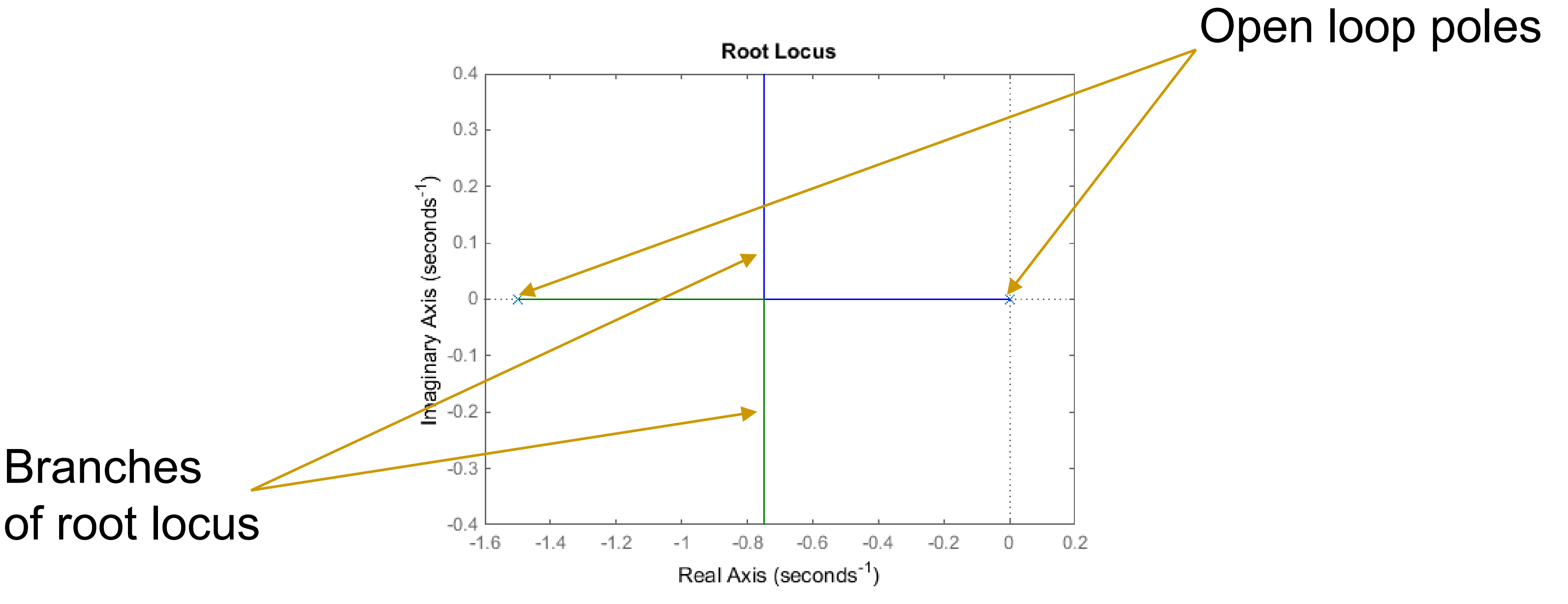


for all possible values of K>0. We have:

$$s^2 + 1.5s + K = 0 \qquad \Longrightarrow \qquad s_{1,2} = \frac{-1.5 \pm \sqrt{1.5^2 - 4K}}{2}$$

Motivation

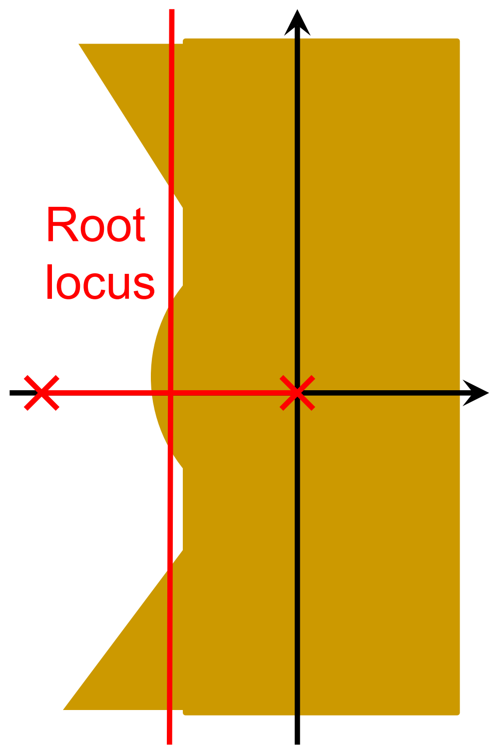
- We obtain a plot in complex plane



- This is a “root locus” for the given system.

A typical design question:

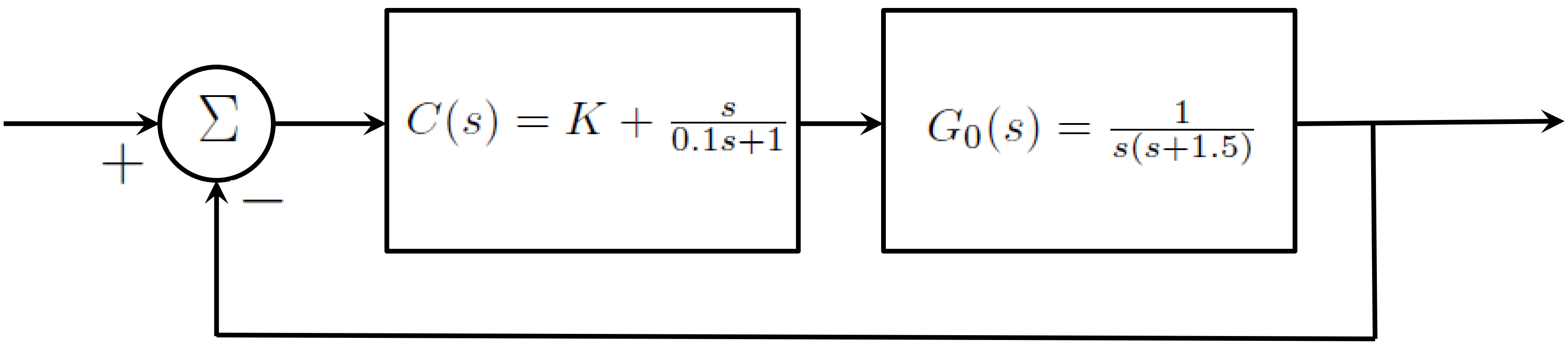
- Can all poles be "placed" into the desired region of complex plane by choosing K?



Complex domain
(pole locations)

Motivation:

arbitrary parameter in the controller



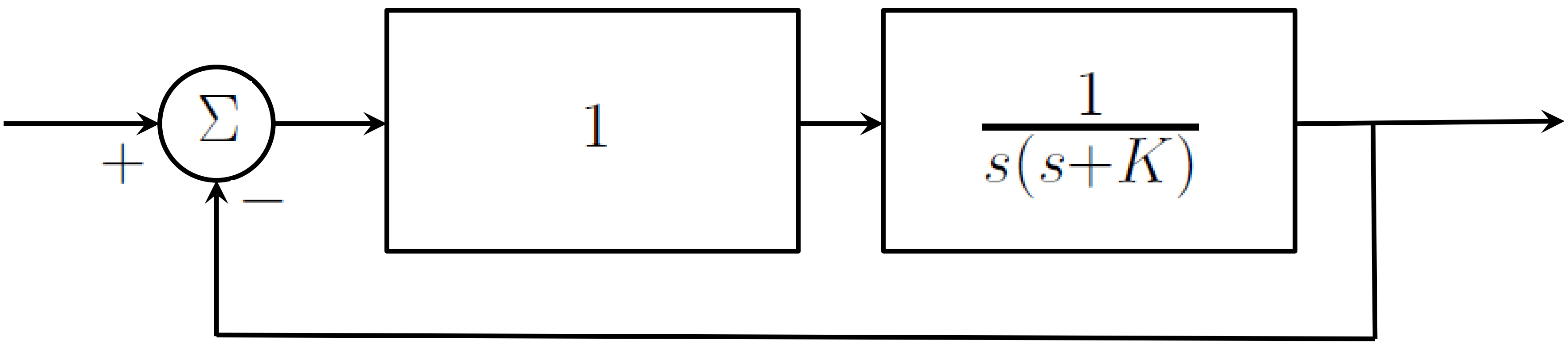
Characteristic equation is:

$$\underbrace{0.1s^3 + 1.15s^2 + 2.5s}_{D(s)} + K\underbrace{(0.1s+1)}_{M(s)} = 0$$

$$1 + K\frac{M(s)}{D(s)} = 0$$

We will consider this example in detail later.

Motivation:
robustness analysis



Characteristic equation is:

$$s^2 + Ks + 1 = \underbrace{(s^2+1)}_{D(s)} + K\underbrace{s}_{M(s)} = 0$$

$$1 + K\frac{M(s)}{D(s)} = 0$$

Problem formulation:

- Plot in the complex s plane the locations of all roots of the equation

$$1 + K \cdot F(s) = 0 \quad \text{where} \quad F(s) = \frac{M(s)}{D(s)} = \frac{\prod_{k=1}^{m}(s - \beta_k)}{\prod_{k=1}^{n}(s - \alpha_k)}$$

as K varies from 0 to infinity.

- This plot is called the (positive) “root locus”.

Phase and magnitude conditions

- Note that if a point s_0 in the complex plane lays on the root locus, it has to satisfy

$$1 + KF(s_0) = 0 \qquad \Leftrightarrow \qquad KF(s_0) = -1$$

which implies that these conditions hold:

magnitude condition: $|K \cdot F(s_0)| = 1$

phase condition: $\angle K \cdot F(s_0) = (2l + 1)\pi \quad \text{for } l = 0, \pm 1, \pm 2, \ldots$

Phase condition

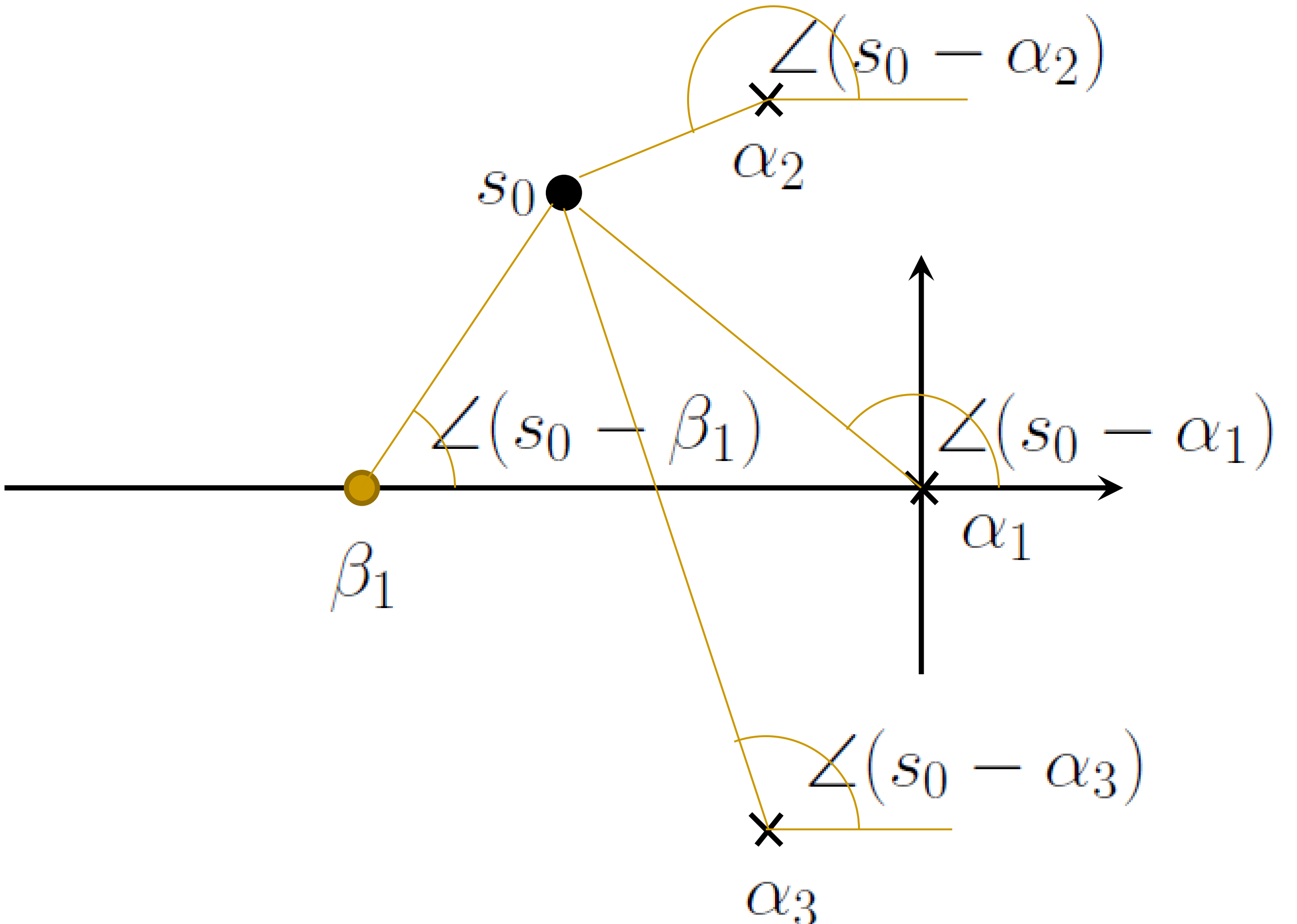
- Since K is positive, the phase depends only on poles and zeros of F(s). In other words, for any point s_0 on the root locus, we have:

$$(2l+1)\pi = \angle F(s_0) = \sum_{k=1}^{m} \angle(s_0 - \beta_k) - \sum_{k=1}^{n} \angle(s_0 - \alpha_k) \text{ for } l = 0, \pm 1, \pm 2, \ldots$$

where

$$F(s) = \frac{M(s)}{D(s)} = \frac{\prod_{k=1}^{m}(s - \beta_k)}{\prod_{k=1}^{n}(s - \alpha_k)}$$

Graphical interpretation



$\angle(s_0 - \beta_1) - \angle(s_0 - \alpha_1) - \angle(s_0 - \alpha_2) - \angle(s_0 - \alpha_3) =$
$(2l + 1)\pi, l = 0, \pm 1, \pm 2, \ldots$

Calculate K for a point on root locus

- The root locus is parameterized with the gain K>0.
- If we want to calculate the value of K that corresponds to a specific point on the root locus, we can use the gain condition:

$$K = \frac{1}{|F(s_0)|}$$

Sketching root locus via Matlab

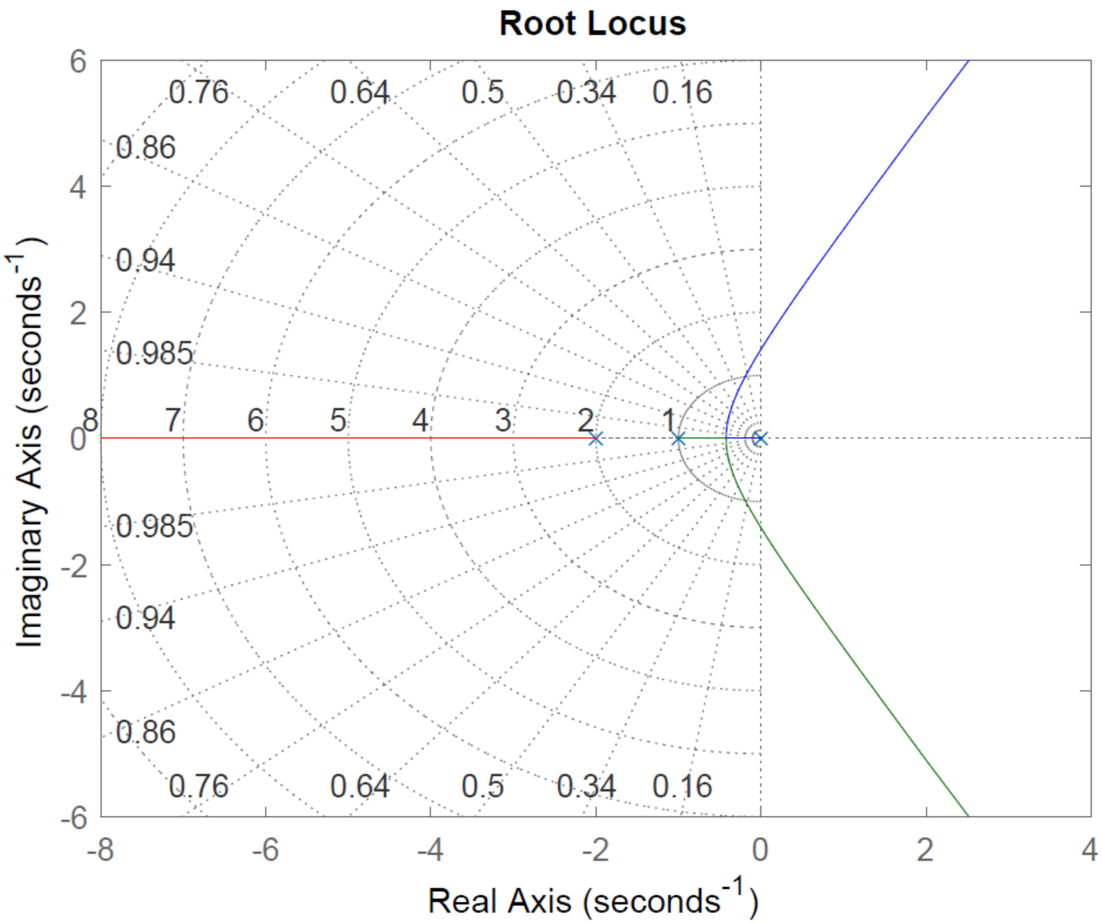
- User can define a transfer function and then use the command “rlocus” to plot its root locus.
- To understand Matlab plot, it is useful to learn how to sketch root locus by hand.
- It is useful to use the command “sgrid” to get a grid of lines with constant damping and constant natural frequencies.

Root locus via Matlab

$$F(s) = \frac{1}{s^3+3s^2+2s} = \frac{1}{s(s+1)(s+2)}$$

>> rlocus([1],[1 3 2 0])
>> sgrid

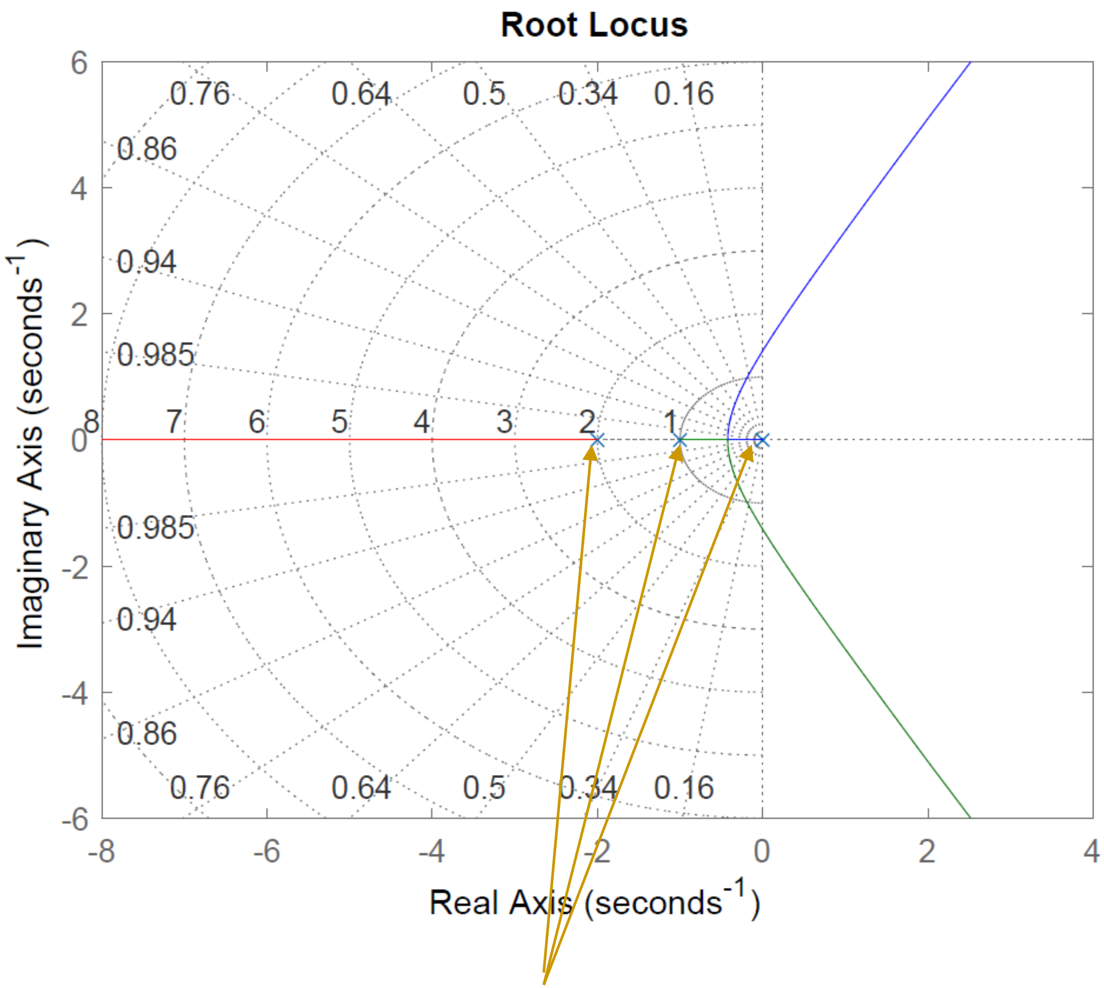
Lines of constant damping and constant natural frequency are ploted using "sgrid"



Main features of root locus

- Number of branches
- Open loop poles (starting points for K=0)
- Open loop zeros (limiting points for K infinity)
- Parts of real line that belong to root locus
- Asymptotes
- Breakaway point (branches intersect)
- Intersections with imaginary axis
- Angles of departure or arrival at poles/zeroes

Open loop poles/zeroes

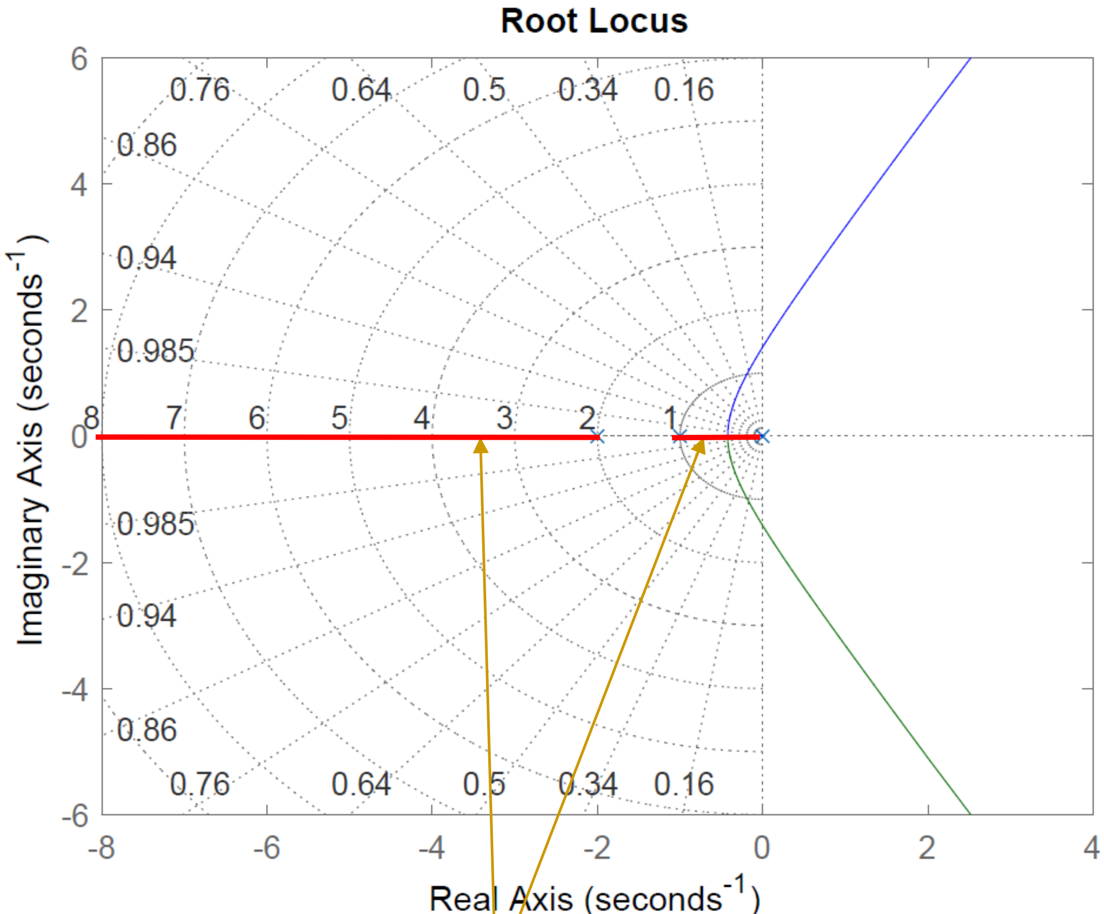


3 open loop poles (crosses)
No open loop zeroes (circles)
3 branches
(blue, green, red)

Open loop poles are points where branches of root locus start from (small K).

Open loop zeroes are points where some branches of root locus converge to (large K).

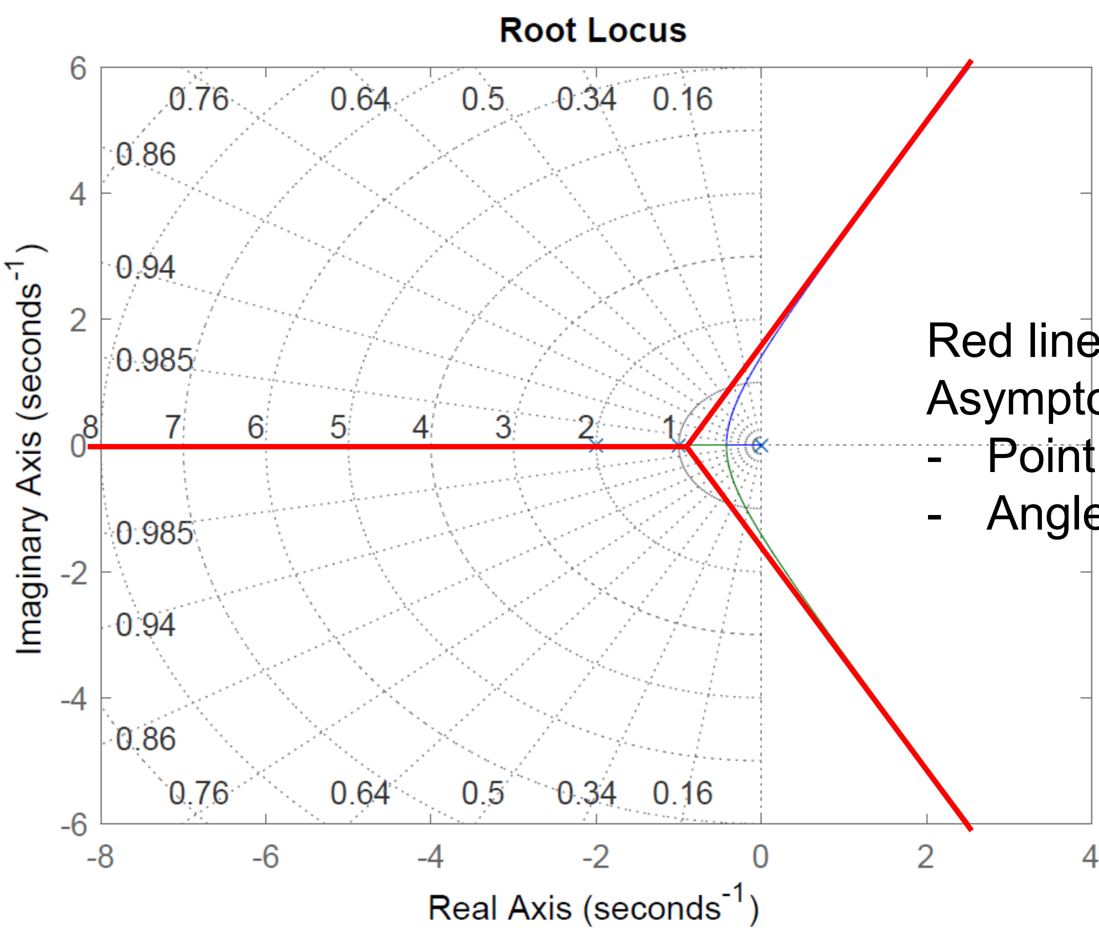
Parts of real axis that belong to locus



Thick red line denotes parts of real axis that belong to root locus

These parts of real axis belong to root locus.

Asymptotes of root locus

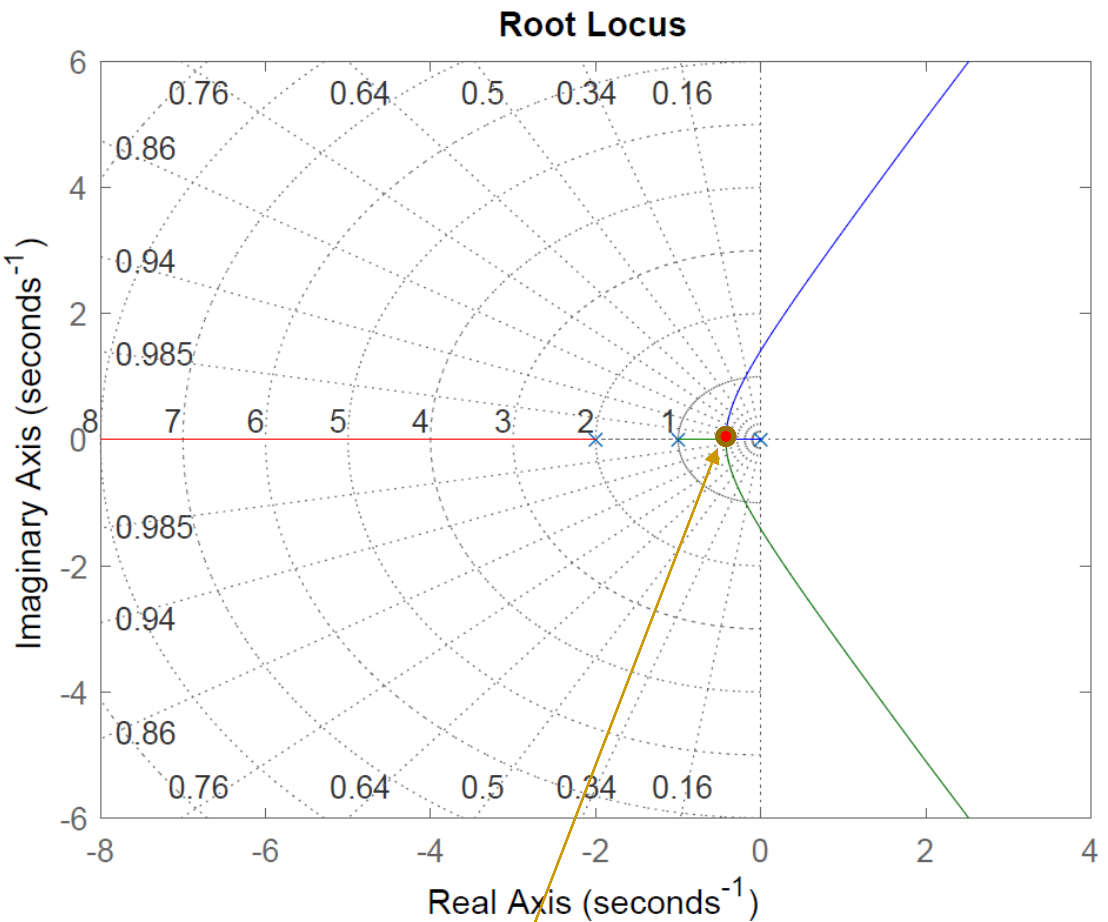


Red lines denote 3 asymptotes

Asymptotes are determined by:

- Point where they intersect the real axis
- Angle with the positive real axis

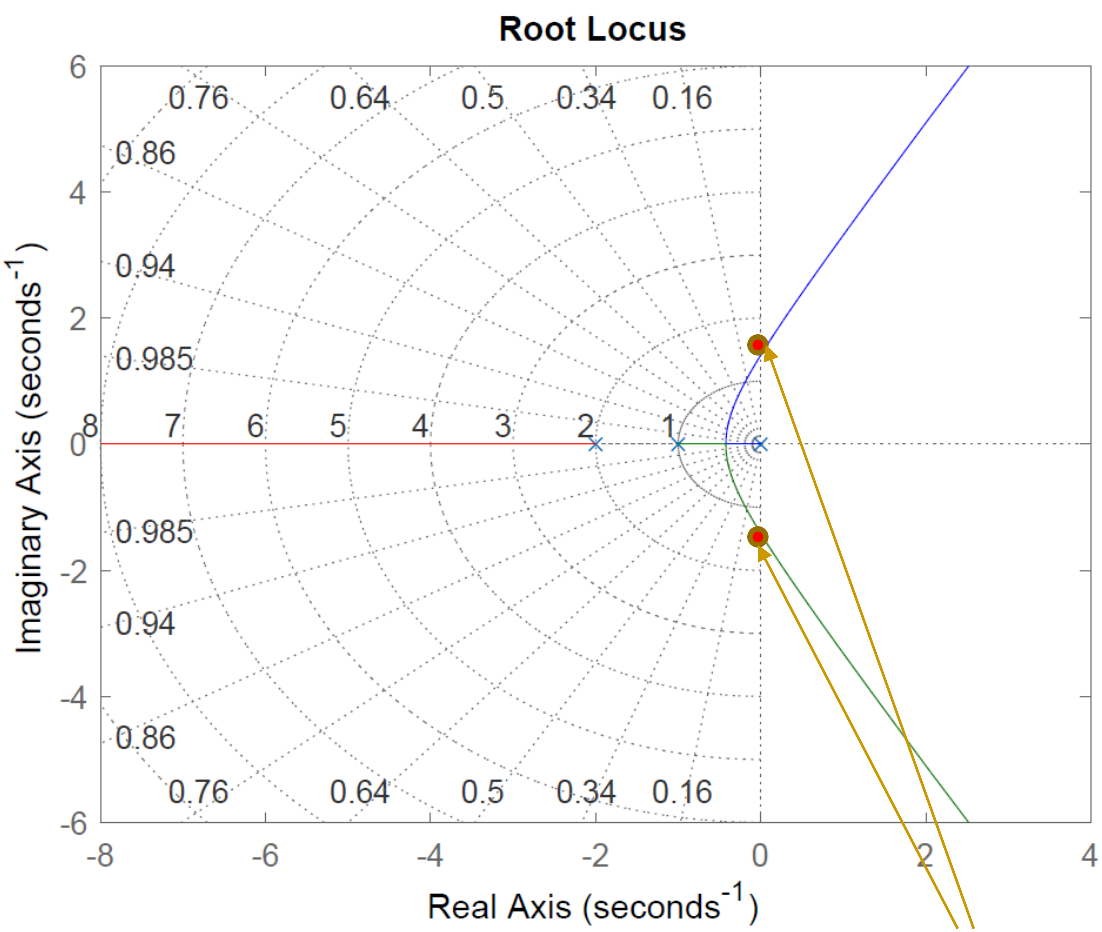
Breakaway point



Sometimes we have repeated complex poles. This is where several branches intersect.

Breakaway point where two branches cross on the real axis – repeated poles.

Intersections with imaginary axis

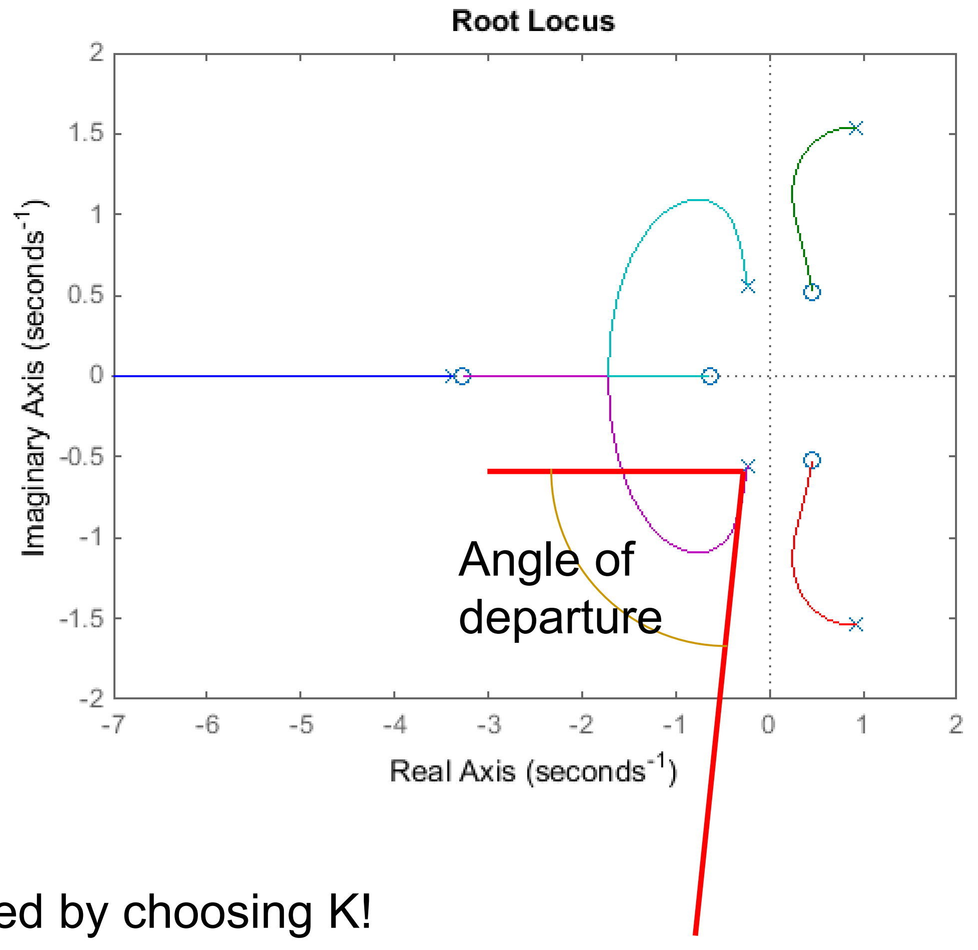


Intersections with imaginary axis.

Angles of departure/arrival

rlocus([1 3 -1 0 1],[1 2 -2 10 4 4])

$$F(s) = \frac{s^4+3s^3-s^2+1}{s^5+2s^4-2s^3+10s^2+4s+4}$$



This system can not be stabilized by choosing K!