Lecture 5

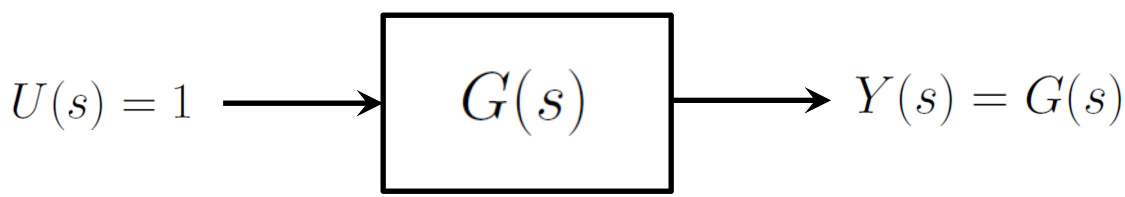
Impulse and Step Responses

DC motor example

Impulse response
⇔ Transfer function

If we apply a unit impulse *δ(t)* as input, output is called the *impulse response.*

Laplace transform of *δ(t)* is 1. So,



- ➔ Transfer function *G* = Laplace transform of impulse response *g!*

Step responses

- Impulses are infinitely large for an infinitely short time. Not often encountered in control.
- In control, for time-domain analysis often of more interest to study the *step response*, i.e. the output when the input is a unit step
- Reveals how system behaves when reference suddenly changes from one constant to another

$$u(t)=1,\ t>0 \Leftrightarrow U(s) = 1/s$$

$$\Rightarrow Y(s) = G(s)/s \Leftrightarrow y(t) = \int_0^t g(\tau)d\tau$$

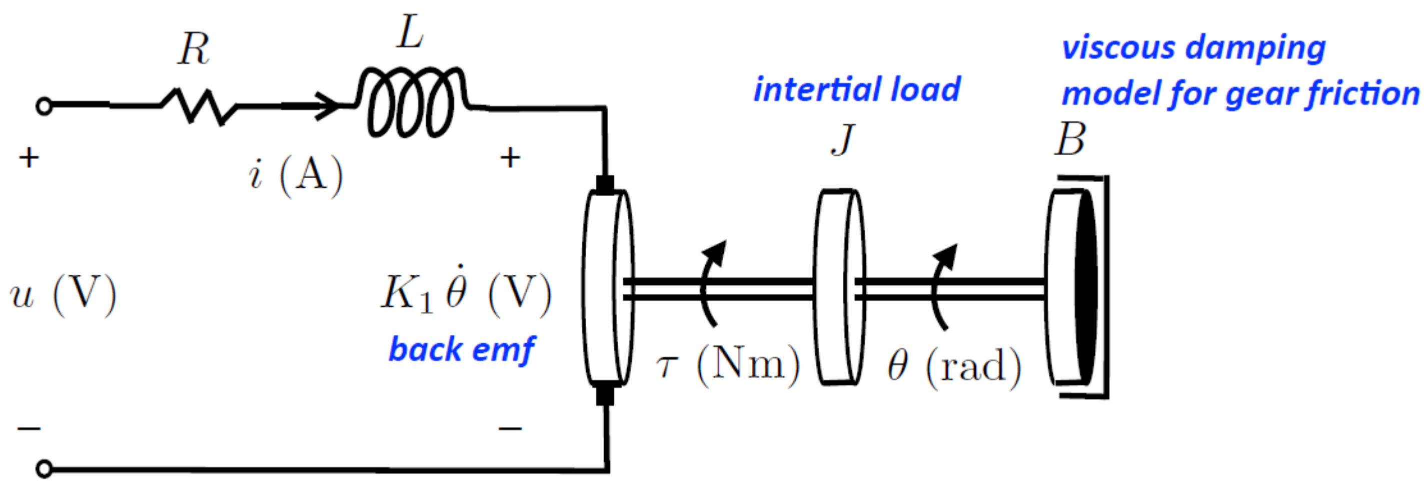
- I.e., impulse response = derivative of step response

Example (DC motor)

- Determine the impulse response of the motor for the input being the supply voltage and the output being the angle of the shaft.

- Determine the step response for the case when the output is the angular velocity of the shaft and the input is the supply voltage.

Example (DC motor)



linearised motor equation: $\tau(t) = K_2 i(t) \quad \overset{\mathcal{L}}{\leftrightarrow} \quad T(s) = K_2 I(s)$

Newton's second law: $J\ddot\theta(t) = \tau(t) - B\dot\theta(t) \quad \overset{\mathcal{L}}{\leftrightarrow} \quad J(s^2\Theta(s) - s\theta(0) - \dot\theta(0))$

$$= T(s) - B(s\Theta(s) - \theta(0))$$

Kirchoff's voltage law: $u(t) = Ri(t) + L\dfrac{di}{dt}(t) + K_1\dot\theta(t)$

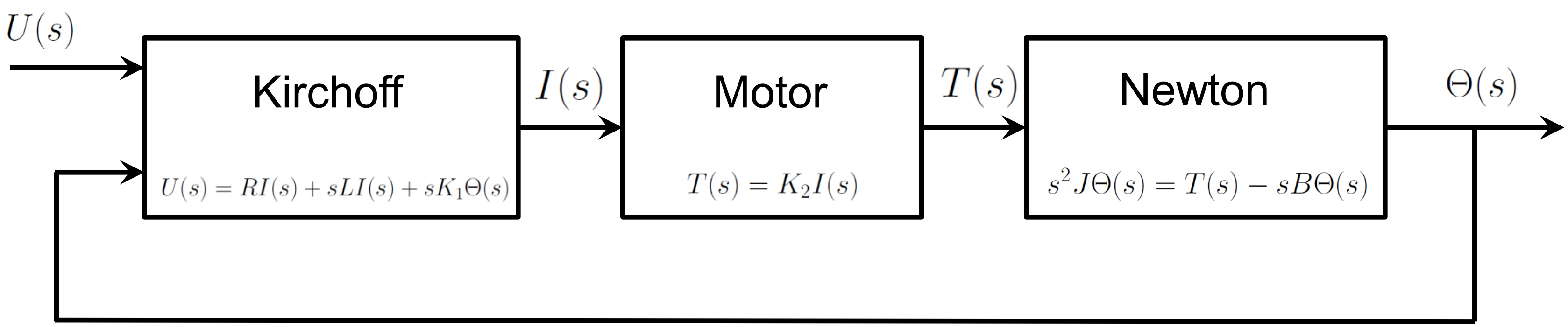
$$\overset{\mathcal{L}}{\leftrightarrow} U(s) = R\,I(s) + L(s\,I(s) - i(0)) + K_1(s\,\Theta(s) - \theta(0))$$

Example

- Assume zero initial conditions:

	Time	**Laplace**
Motor	$\tau(t) = K_2 i(t)$	$T(s) = K_2 I(s)$
Newton	$J\ddot{\theta}(t) = \tau(t) - B\dot{\theta}(t)$	$s^2 J\Theta(s) = T(s) - sB\Theta(s)$
Kirchoff	$u(t) = Ri(t) + L\frac{di}{dt}(t) + K_1\dot{\theta}(t)$	$U(s) = RI(s) + sLI(s) + sK_1\Theta(s)$

Graphical interpretation



Exercise: redraw this diagram so that it represents the block diagram where each block is given by its transfer function.

Example

- Direct calculations yield

$$(s^2 J + sB)\Theta(s) = T(s) = K_2 I(s)$$

$$I(s)(R + sL) = U(s) - sK_1\Theta(s)$$

$$(sJ^2 + sB)\Theta(s) = K_2 \frac{U(s) - sK_1\Theta(s)}{R + sL}$$

$$(s^2 J + sB)(R + sL)\Theta(s) + sK_1K_2\Theta(s) = K_2 U(s)$$

$$\frac{\Theta(s)}{U(s)} = \frac{K_2}{(s^2 J + sB)(R + sL) + sK_1K_2}$$

$$\frac{\Theta(s)}{U(s)} = \frac{K}{s} \cdot \frac{1}{(T_1 s + 1)(T_2 s + 1)}$$

T_1 and T_2 are real valued for $(RJ + BL)^2 > 4(K_1K_2 + RB)LJ$

Example

- We want to rewrite the transfer function as

$$\frac{\Theta(s)}{U(s)} = \frac{K}{s} \cdot \frac{1}{(T_1 s + 1)(T_2 s + 1)} = \frac{A_1}{s} + \frac{A_2}{T_1 s + 1} + \frac{A_3}{T_2 s + 1}$$

- Let us calculate A_1. We multiply by s:

$$K \cdot \frac{1}{(T_1 s + 1)(T_2 s + 1)} = A_1 + s\frac{A_2}{T_1 s + 1} + s\frac{A_3}{T_2 s + 1}$$

and obtain A_1 by setting $s = 0$, which yields

$$K\frac{1}{1 \cdot 1} = A_1 \;\Rightarrow\; A_1 = K$$

Compute A_2 and A_3 in the same way.

Example (summary)

- Transfer function from u (V) to θ (rad)

$$\frac{\Theta(s)}{U(s)} = \frac{K_2}{s\big((s\,L + R)(s\,J + B) + K_1\,K_2\big)} = \frac{K}{s} \cdot \frac{1}{(T_1 s + 1)(T_2 s + 1)}$$

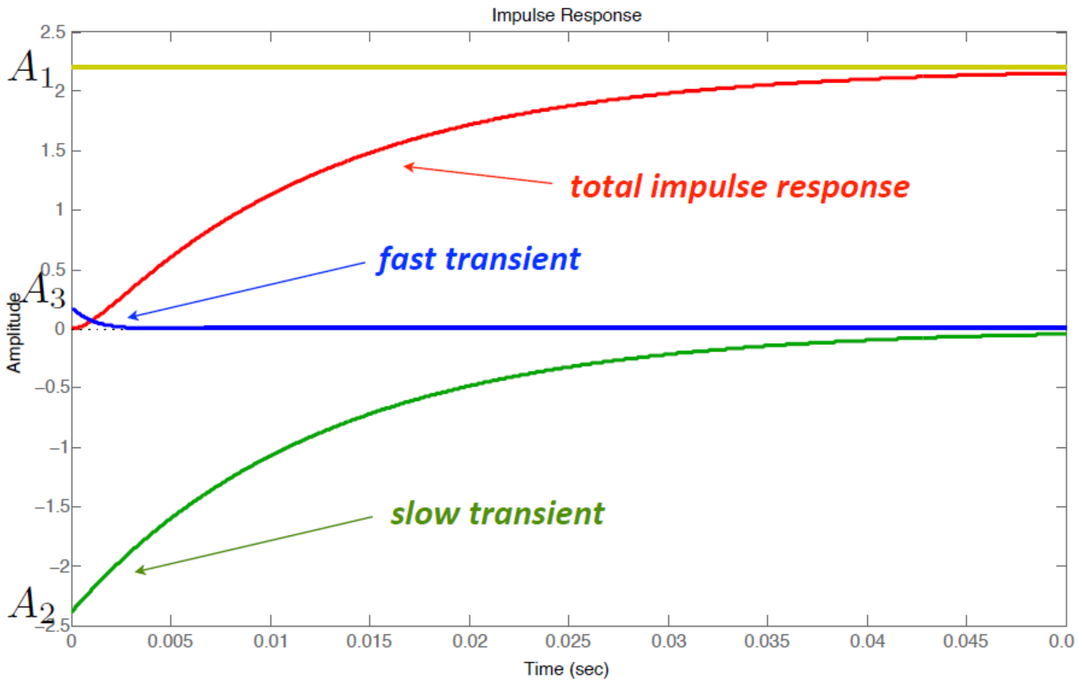
$$\frac{\Theta(s)}{U(s)} = \frac{K}{s} \cdot \frac{1}{(T_1 s + 1)(T_2 s + 1)} = \frac{A_1}{s} + \frac{A_2}{T_1 s + 1} + \frac{A_3}{T_2 s + 1}$$

$$\overset{\mathcal{L}}{\longleftrightarrow} \quad (A_1 + \frac{A_2}{T_1}\, e^{-\frac{t}{T_1}} + \frac{A_3}{T_2}\, e^{-\frac{t}{T_2}})\varsigma(t)$$

$$A_1 = K,\; A_2 = \tfrac{K}{-1/T_1}\tfrac{1}{(-T_2/T_1+1)},\; A_3 = \tfrac{K}{-1/T_2}\tfrac{1}{(-T_1/T_2+1)}$$

Example (impulse response)

Suppose: $K \approx 2.2; \;\; T_1 \approx 12.5$ (ms); $\;\; T_2 = 1$ (ms)




```
K = 2.2;
T1 = 12.5e-3;
T2 = 1e-3;
T = T1*4;

A1 = K;
A2 = (-K*T1)*1/(-T2/T1 + 1);
A3 = (-K*T2)*1/(-T1/T2 + 1);

sys1 = A1*tf(1,[1 0]);
sys2 = A2*tf(1,[T1,1]);
sys3 = A3*tf(1,[T2,1]);

sys = sys1 + sys2 + sys3;

impulse(sys,'r',sys1,'y',sys2,'g', ...
        sys3,'b', [0:0.0001:T])
```

Example (angular velocity step response)

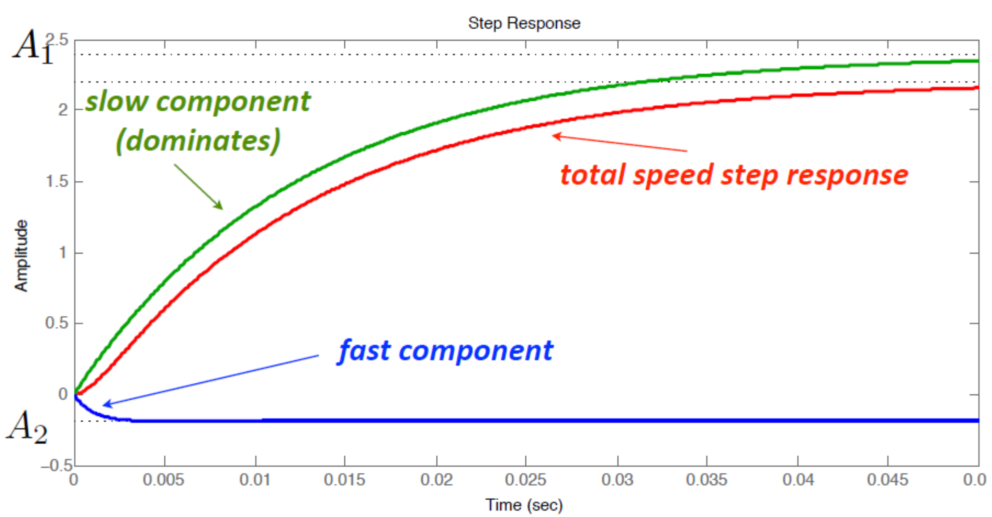
$$\Omega(s) = s\Theta(s) - \theta(0) \overset{\mathcal{L}}{\leftrightarrow} \omega(t) = \dot{\theta}(t) \qquad \frac{\Omega(s)}{U(s)} = \frac{K}{(T_1 s + 1)(T_2 s + 1)}$$

unit step input

For a unit step input: $$\Omega(s) = \frac{K}{(T_1 s + 1)(T_2 s + 1)} \cdot \frac{1}{s}$$

$$= \frac{A_1}{T_1 s + 1} \cdot \frac{1}{s} + \frac{A_2}{T_2 s + 1} \cdot \frac{1}{s}$$

where $A_1 = \dfrac{K}{(-T_2/T_1 + 1)}$ and $A_2 = \dfrac{K}{(-T_1/T_2 + 1)}$




```
K = 2.2;
T1 = 12.5e-3;
T2 = 1e-3;
T = T1*4;

A1 = K/(-T2/T1 + 1);
A2 = K/(-T1/T2 + 1);

sys1 = A1*tf(1,[T1,1]);
sys2 = A2*tf(1,[T2,1]);

sys = sys1 + sys2;

step(sys,'r',sys1,'g', ...
     sys2,'b',[0:0.0001:T])
```