Lecture 3

Steady-State Equilibria

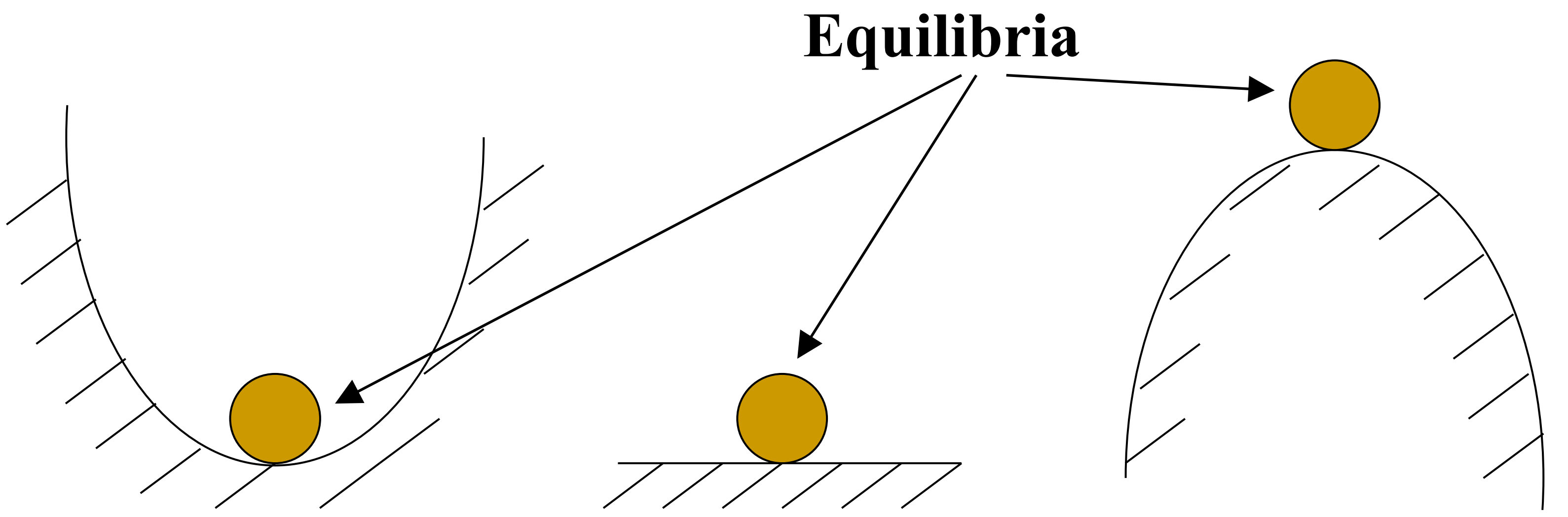
Modeling from first principles

Linearisation

What do we want in control?

- **Stability**: With no disturbances and constant input, output should approach a desired, constant *equilibrium*, even if initial conditions change slightly from nominal values
- **Transient performance**: quick but “graceful” transition from initial value to steady-state.
- **Robustness:** Steady-state behaviour doesn’t change dramatically with uncertainties, disturbances or noise.

What is an equilibrium?



When a system is exactly at an equilibrium point, nothing changes over time.
Can either be stable, marginally stable, or unstable

Why Go for Equilibria?

- Equilibria are “sweet spots”. When system input and output variables start from an equilibrium point, they remain there forever, if no disturbances and noises.
- Nominal operating conditions should be chosen to be equilibrium points.
- In terms of the whole control system, we want to operate near equilibrium points that are stable.

On Unstable Equilibria

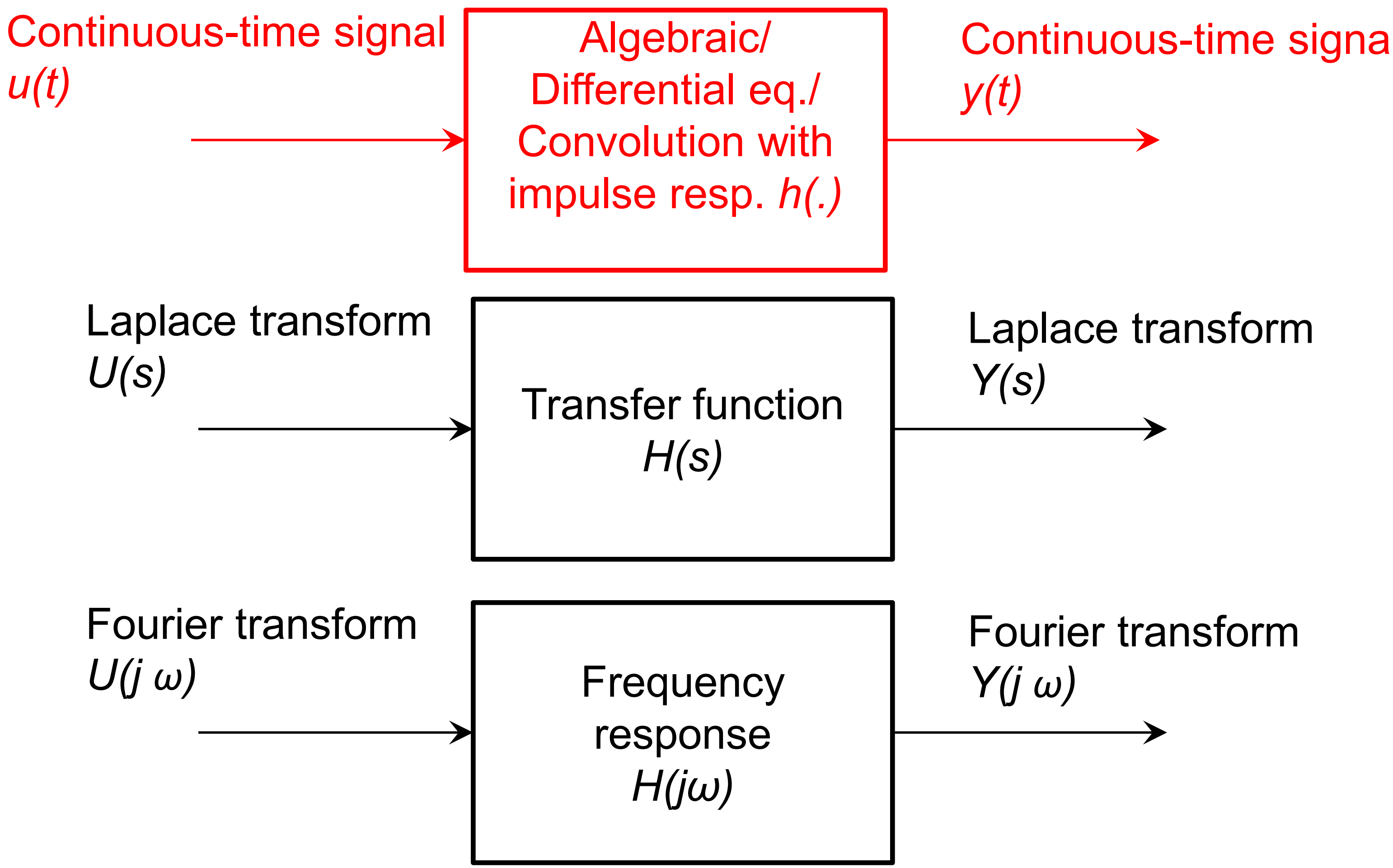
However when we consider just the plant, we sometimes operate at plant equilibria that are unstable.

We render these points stable by interconnecting the plant with a feedback controller.

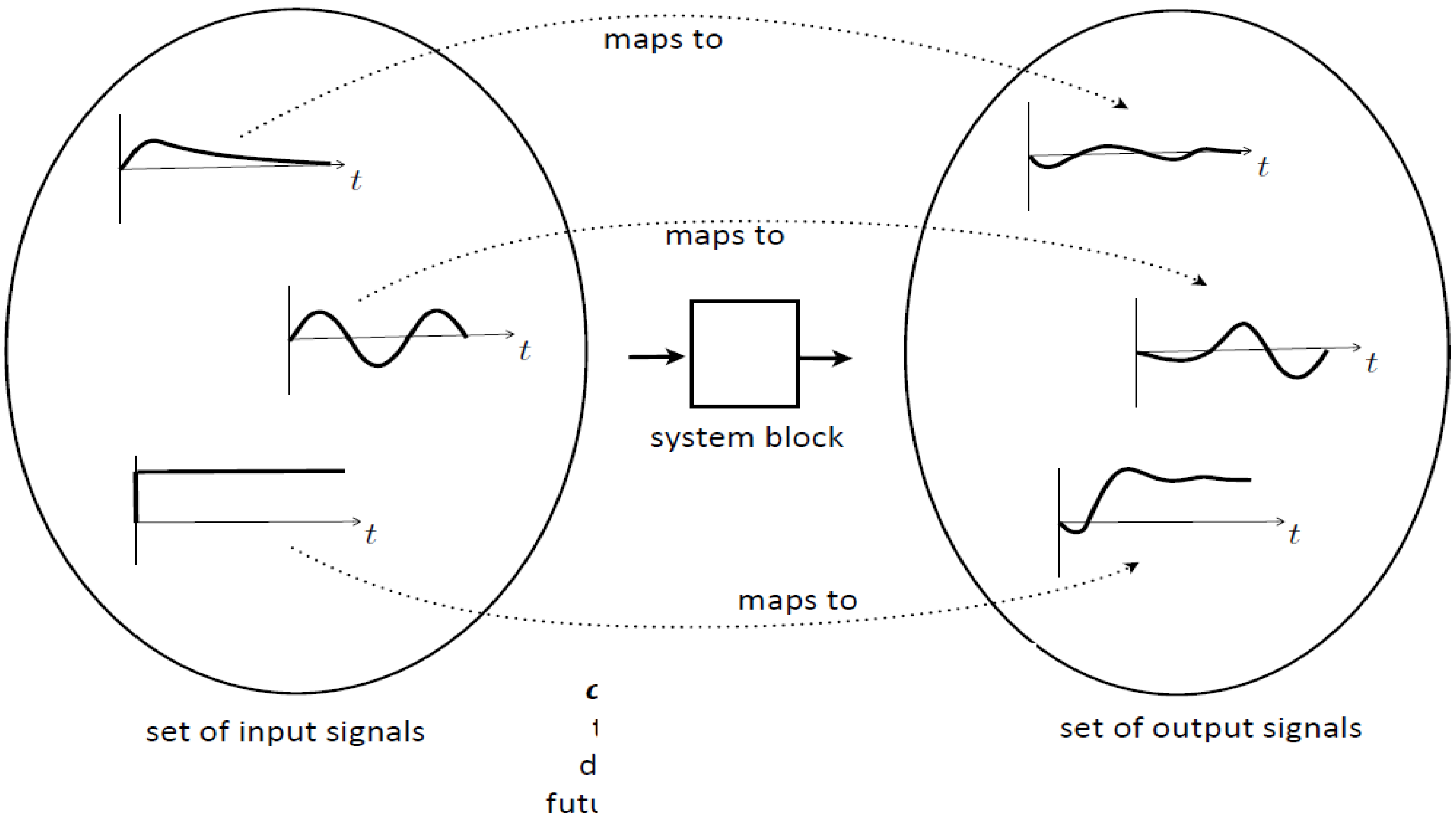
Preview of Techniques

- Stability at equilibrium studied using Laplace transforms, i.e. s-domain, after linearisation
- Transient performance examined in s-domain or frequency domain (Fourier transforms)
- Performance and robustness will be investigated in freq. domain. System freq. response typically needs
  - Good tracking of low freq. references
  - Good damping of disturbances and high freq. noise

System models



System = Mapping between Signal Spaces



How to construct a model

- From first principles (i.e. laws of nature), often followed by
- Linearisation
- Model reduction (e.g. nulling small coefficients)

- Collecting lots of empirical data about input-output signal values over time and then fitting a convenient model (system identification, machine learning)

On models

All models are wrong, but some are useful.

- G. Box (statistician)

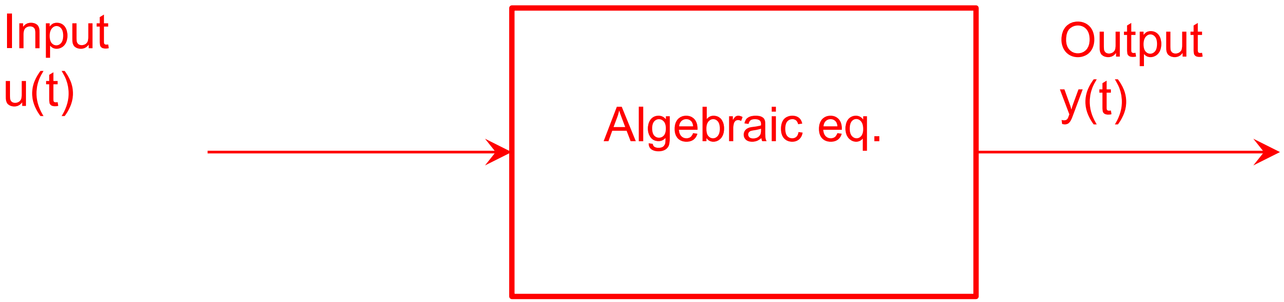
A model should be as simple as possible, but no simpler.

- A. Einstein (…)

Simple models of the plant are often more useful than complex ones, even if they're less accurate:

- Each model parameter easier to estimate and interpret from empirical data
- Well-designed feedback law attenuates the effect of the modelling error

Static models

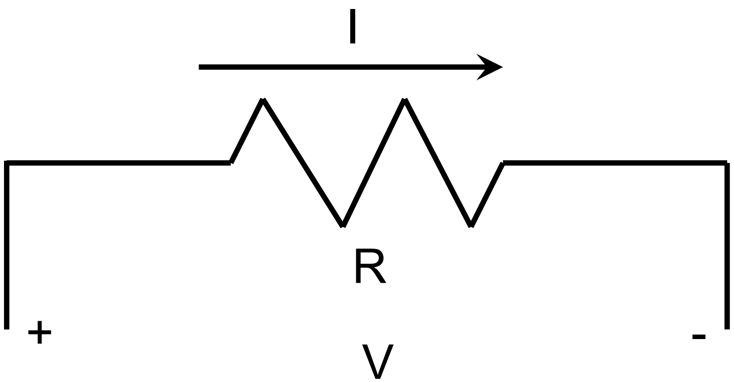


Property: If *u(t)*= *u(t')* then *y(t)* = *y(t')*

Example:

$$y = 10u^2 + 2u$$

Example: Ohm's law for resistor

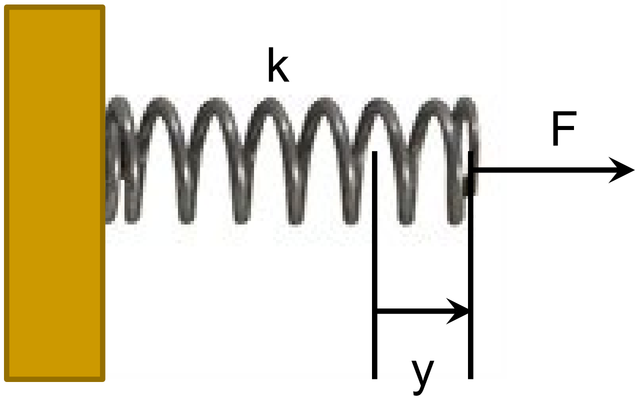


$V = R\, I$

Assumptions:

- Resistor is made of "Ohmic" material
- Temperature is constant
- The current is sufficiently small
- Ageing is ignored, etc.

Example: Hooke's law for spring



$F = k\, y$

Assumptions:

- The deflections are sufficiently small so that the material stays within its linear elasticity region (linearization!).

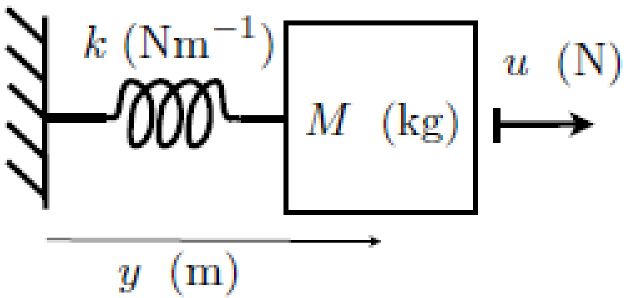
Differential equation models

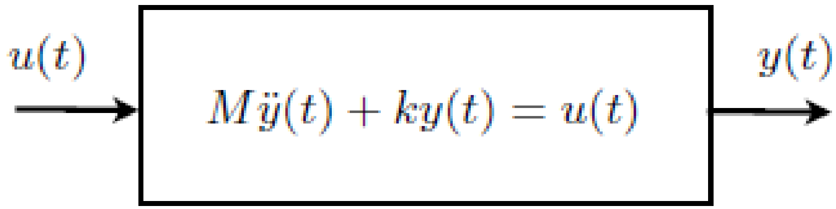
Property : $y(t)$ not necessarily = $y(t')$ when $u(t) = u(t')$.

Example: $$\dot{y} = u$$

$$y(t) = y(0) + \int_0^t u(s)ds$$

Example (mass with spring)

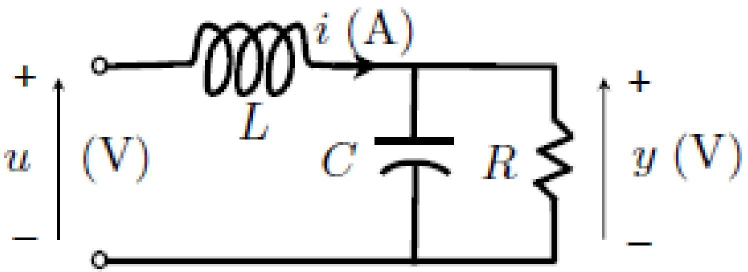


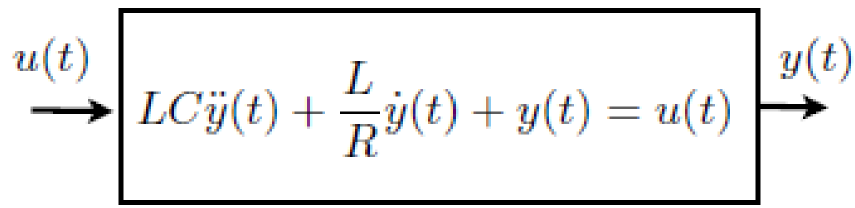


Assumptions:

- Hooke's law
- Newton's second law
- No friction
- No damping

Example (electrical):

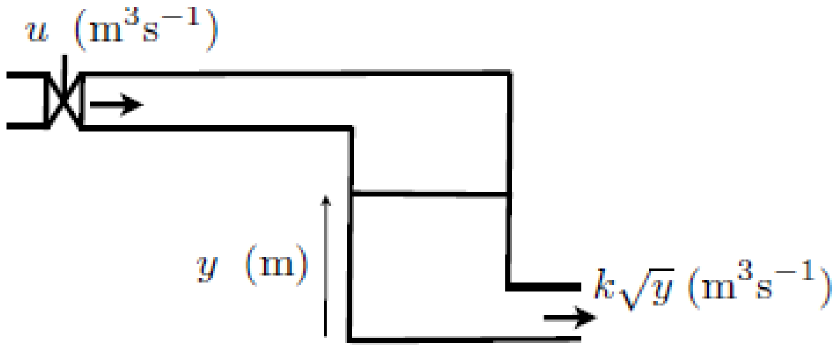


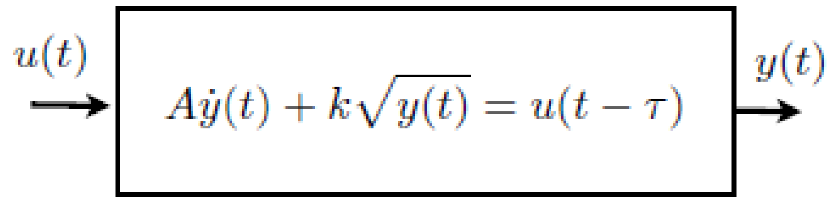


Assumptions:

- Ohm's law
- Ideal capacitor
- Ideal inductor
- Kirchoff's current and voltage laws

Example (fluid flow):



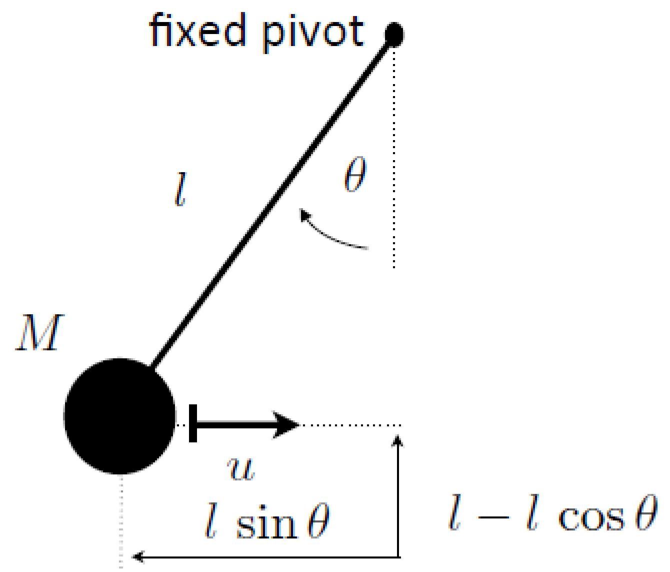


Note the time delay and nonlinearity!

Assumptions:

- Law of conservation of mass
- Constant cross section A
- Atmospheric pressure above water and at the outlet, and so on.

Example (pendulum):

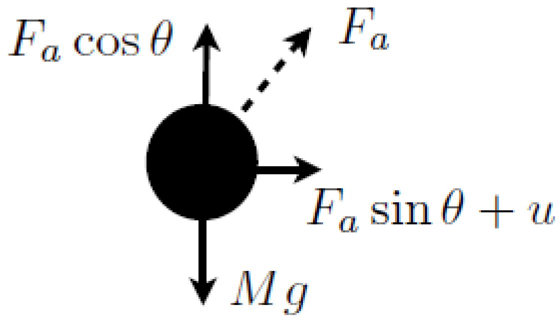


Assumptions:

- Bob is a point mass
- Arm is rigid but massless
- No friction, etc

Example (pendulum)

- Free body diagram:



- Newton's second law (horizontal):

$$-F_a \sin\theta - u = M \frac{d^2}{dt^2}(l\ \sin\theta) = M\,l\,\frac{d}{dt}(\dot{\theta}\ \cos\theta) = M\,l\,(\ddot{\theta}\ \cos\theta - \dot{\theta}^2\ \sin\theta)$$

- Newton's second law (vertical)

$$F_a \cos\theta - M\,g = M \frac{d^2}{dt^2}(l - l\ \cos\theta) = M\,l\,\frac{d}{dt}(\dot{\theta}\ \sin\theta) = M\,l\,(\ddot{\theta}\ \sin\theta + \dot{\theta}^2\ \cos\theta)$$

Example (pendulum)

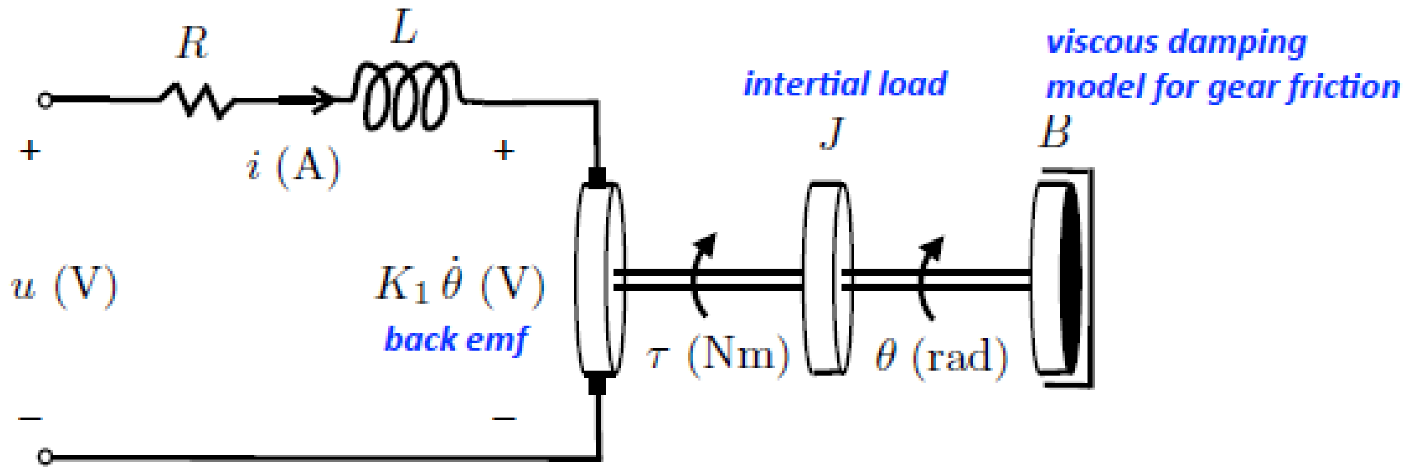
- Eliminating the force in the arm yields:

$$M\,l\,\ddot{\theta} + u\cos\theta + M\,g\,\sin\theta = 0$$

- Note that this model is nonlinear!

- We will revisit this model when we talk about equilibria and linearisation.

Example (DC motor):



(Linearized) Motor: $\tau(t) = K_2 i(t)$

Newton's second law: $J\ddot{\theta}(t) = \tau(t) - B\dot{\theta}(t)$

Kirchoff's voltage law: $u(t) = Ri(t) + L\dfrac{di}{dt}(t) + K_1\dot{\theta}(t)$

Finding Equilibria

- Consider an input-output differential model

$$\ell(\frac{d^n y}{dt^n}(t), \ldots, \frac{dy}{dt}(t), y(t), \frac{d^n u}{dt^n}(t), \ldots, \frac{du}{dt}(t), u(t)) = 0$$

- A signal pair $(u(\cdot),\ y(\cdot))$ is an equilibrium if

$$\frac{d^k y}{dt^k}(t) = 0, \frac{d^k u}{dt^k}(t) = 0 \text{ for } k = 1, \ldots, n \text{ and all times } t$$

- Equilibria can be computed by solving the static equation

$$\ell(0, \ldots, 0, \bar{y}, 0, \ldots, 0, \bar{u}) = 0$$

Example (pendulum)

- Consider the pendulum equation

$$M\,l\,\ddot{\theta} + u\cos\theta + M\,g\,\sin\theta = 0$$

  where we think of θ as the output.
- Equilibria satisfy:

$$\bar{\theta}(t) = \bar{\theta} = const. \Rightarrow \ddot{\bar{\theta}} = 0$$

$$\bar{u}(t) = \bar{u} = const.$$

$$\bar{u}\cos\bar{\theta} + M g \sin\bar{\theta} = 0$$

Example (pendulum)

- If we are interested only in equilibria with

$$\bar{u} = 0$$

then, we have

$$\{\bar{\theta} : \sin \bar{\theta} = 0\} = \{\bar{\theta} : \bar{\theta} = 2k\pi\}$$

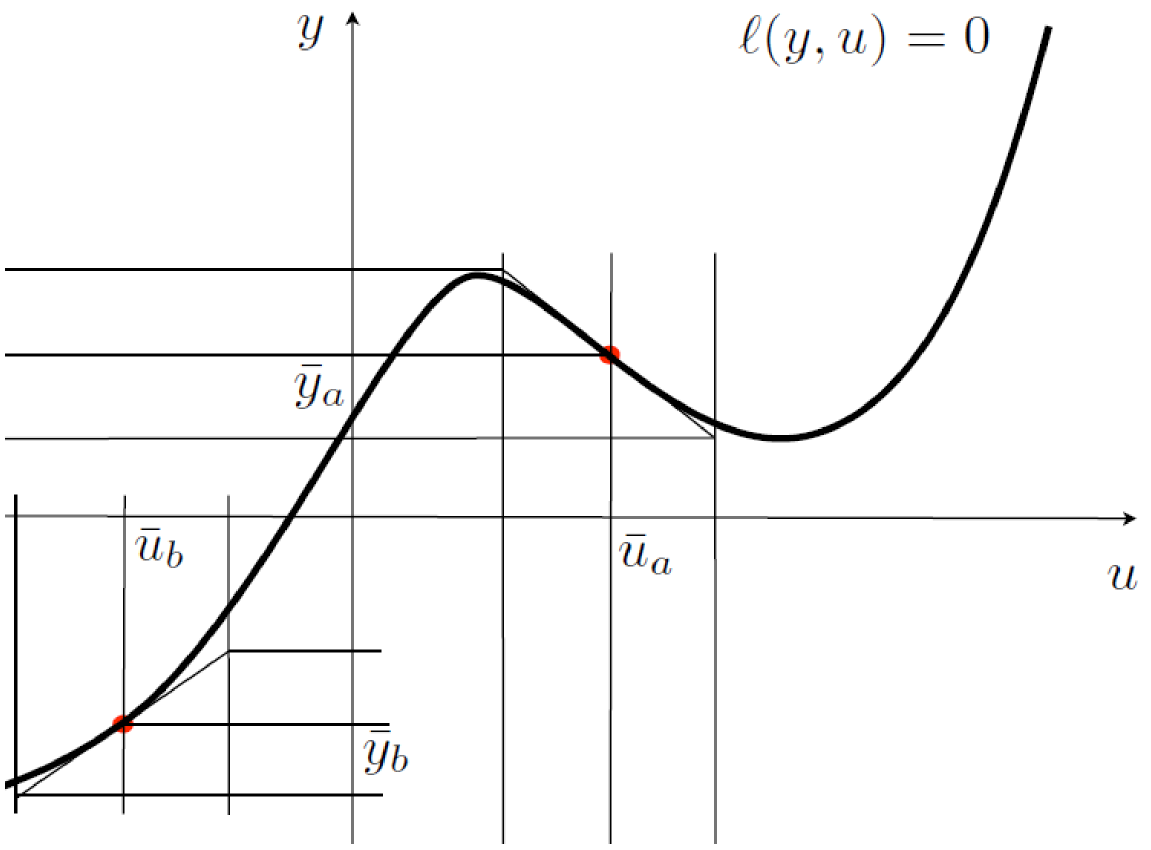
- When mass is down: $(\bar{u}, \bar{\theta}) = (0, 0)$

- When mass is up: $(\bar{u}, \bar{\theta}) = (0, \pi)$

Why Think Linear?

- Often your plant model is already linear time-invariant (LTI).
- But sometimes, you'll get a nonlinear model → Difficult to fully analyse and control
- In the philosophy of starting simple, your first step should be to *linearise* the plant model around an equilibrium of interest.
- If disturbances are not too big, this will often yield satisfactory linear controller designs

Linearising a function



NOTE: linearisation of the same function around different points is different!

Comments:

- Linearizing a nonlinear function means approximating it **around a chosen operating point** by its tangent line
- Yields a good approximation valid near this point
- Linearisation is different around different points, because the tangent lines will be different

Comments:

- Linearizing a nonlinear function means approximating it **around a chosen operating point** by its tangent line
- Yields a good approximation valid near this point
- Linearisation is different around different points, because the tangent lines will be different

Taylor series expansion

- To linearise a function near a point *a,* find its Taylor series

$$f(x) = f(a) + \frac{f'(a)}{1!}(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3 + \dots$$

then ignore the 2nd and higher order terms

- Yields a good approximation for small deviation $\delta_x := x - a$

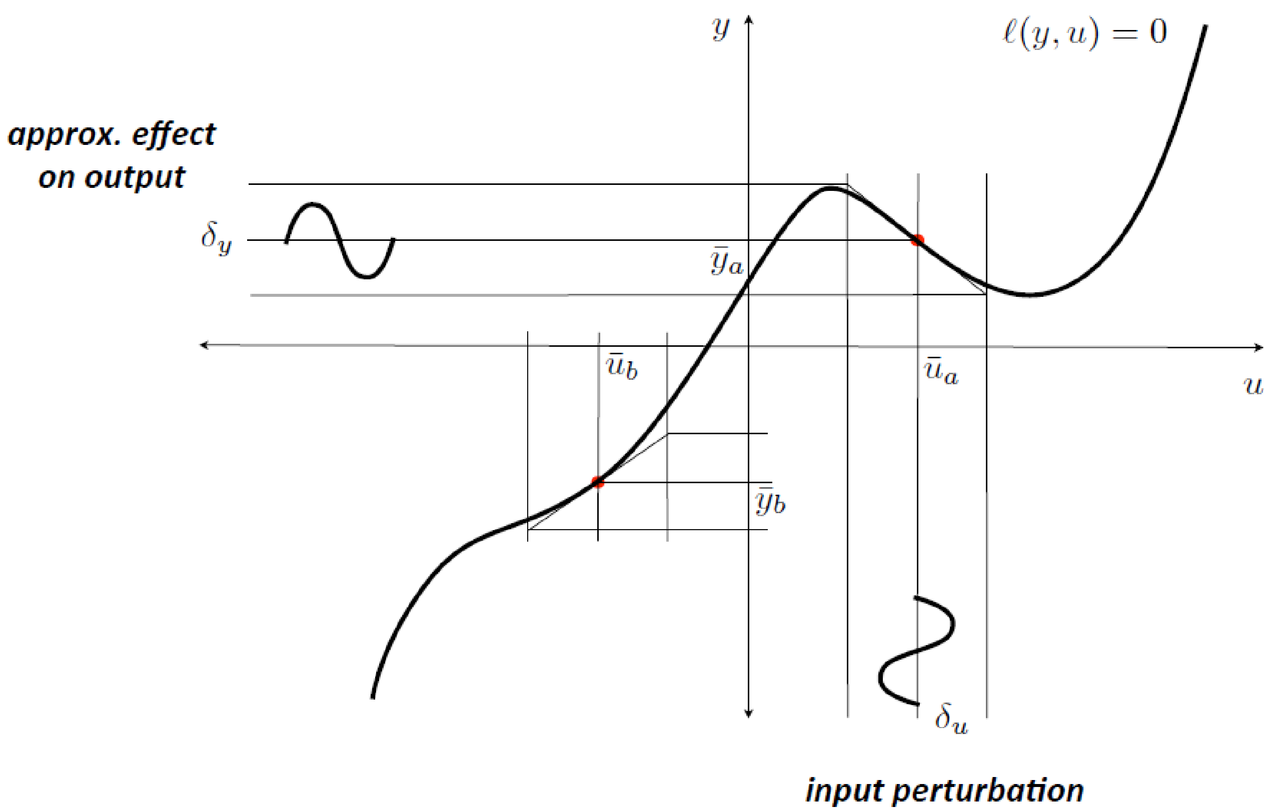
Linearization

- Linearization of this function around *x*=*a* is:

$$f(x) \approx f(a) + f'(a)\underbrace{(x-a)}_{\delta_x}$$

$$f(a) = 0 \;\Rightarrow f(x) \approx f'(a)\delta_x$$

Graphical representation



Linearising a scalar (and static) input-output system

- Suppose output and input signals satisfy *l(y(t),u(t)) = 0*.
- We first select an equilibrium of interest $(\bar{y}, \bar{u})$
- Then find Taylor series around this equilibrium, keeping only 1st order terms

$$\underbrace{\ell(\bar{y}, \bar{u})}_{=0} + \left.\frac{\partial \ell}{\partial y}\right|_{(\bar{y},\bar{u})} \delta_y + \left.\frac{\partial \ell}{\partial u}\right|_{(\bar{y},\bar{u})} \delta_u = 0$$

$$\delta_u := u - \bar{u}; \;\; \delta_y := y - \bar{y}$$

Linearised Scalar Static System

- Linearised model is

$$\delta_y = K\delta_u, \qquad K := -\frac{\left.\frac{\partial \ell}{\partial u}\right|_{(\bar{y},\bar{u})}}{\left.\frac{\partial \ell}{\partial y}\right|_{(\bar{y},\bar{u})}}$$

- Note that linearized model is different for different equilibria.
- Note that the following model is *affine*, not strictly linear:

$$y = Ku + c, \; c \neq 0$$

Note: we assume that the derivative w.r.t. y does not vanish at equilibrium.

Multivariable Linearization

- We consider the multivariate function

$$\ell(y_n, \ldots, y_1, y_0, u_n, \ldots, u_1, u_0) = 0$$

(each subscript denotes time-derivative of that order in the relevant signal)

We find its 1st order Taylor series around

$$y_n = 0, \ldots, y_1 = 0, y_0 = \bar{y}, u_n = 0, \ldots, u_1 = 0, u_0 = \bar{u}$$

$$\begin{aligned}
&\ell_{\text{lin}}(y_n, \ldots, y_1, y_0, u_n, \ldots, u_1, u_0) \\
&\quad \doteq \overbrace{\ell(0, \ldots, 0, \bar{y}, 0, \ldots, 0, \bar{u})}^{\mathbf{0}} + \frac{\partial \ell}{\partial y_n}\Big|_{(0,\ldots,0,\bar{y},0,\ldots,0,\bar{u})}(y_n - 0) + \ldots + \frac{\partial \ell}{\partial y_0}\Big|_{(0,\ldots,0,\bar{y},0,\ldots,0,\bar{u})}(y_0 - \bar{y}) \\
&\quad + \frac{\partial \ell}{\partial u_n}\Big|_{(0,\ldots,0,\bar{y},0,\ldots,0,\bar{u})}(u_n - 0) + \ldots + \frac{\partial \ell}{\partial u_0}\Big|_{(0,\ldots,0,\bar{y},0,\ldots,0,\bar{u})}(u_0 - \bar{u})
\end{aligned}$$

- This is a good approximation for small

$$y_n, \ldots, (y_0 - \bar{y}), u_n, \ldots, (u_0 - \bar{u})$$

- Linearized ODE is a map:

$$\delta_u = (u - \bar{u}) \mapsto \delta_y = (y - \bar{y})$$

that is obtained by noting

$$\frac{d^k \delta_y}{dt^k} = \frac{d^k y}{dt^k}$$
$$\frac{d^k \delta_u}{dt^k} = \frac{d^k u}{dt^k}$$

$$\ell_{\text{lin}}\left(\frac{d^n \delta_y}{dt^n}, \ldots, \frac{d\delta_y}{dt}, (\delta_y + \bar{y}), \frac{d^n \delta_u}{dt^n}, \ldots, \frac{d\delta_u}{dt}, (\delta_u + \bar{u})\right) = 0$$

Example (pendulum)

- From the differential equation, we see

$$\ell(\theta_2, \theta_1, \theta_0, u_0) = Ml\theta_2 + u_0 \cos\theta_0 + Mg\sin\theta_0$$

- We linearise the function around an eq. $(\bar{u}, \bar{\theta})$

$$\begin{aligned}
\ell(\theta_2, \theta_1, \theta_0, u_0) &= Ml\theta_2 + u_0 \cos\theta_0 + Mg\sin\theta_0 \\
&\approx \underbrace{\ell(0, 0, \bar{\theta}, \bar{u})}_{\text{by definition}}^{\mathbf{0}} + \left.\frac{\partial \ell}{\partial \theta_2}\right|_{(0,0,\bar{\theta},\bar{u})} (\theta_2 - 0) + \left.\frac{\partial \ell}{\partial \theta_1}\right|_{(0,0,\bar{\theta},\bar{u})} (\theta_1 - 0) \\
&\quad + \left.\frac{\partial \ell}{\partial \theta_0}\right|_{(0,0,\bar{\theta},\bar{u})} (\theta_0 - \bar{\theta}) + \left.\frac{\partial \ell}{\partial u_0}\right|_{(0,0,\bar{\theta},\bar{u})} (u_0 - \bar{u}) \\
&= 0 + Ml(\theta_2 - 0) + 0(\theta_1 - 0) + (-\bar{u}\sin\bar{\theta} + Mg\cos\bar{\theta})(\theta_0 - \bar{\theta}) + \cos\bar{\theta}\,(u_0 - \bar{u}) \\
&= \ell_{\mathsf{lin}}(\theta_2, \theta_1, \theta_0, u_0)
\end{aligned}$$

Example (pendulum)

- We define incremental input and output

$$\delta_u = u - \bar{u} \qquad \delta_\theta \doteq \theta - \bar{\theta} \qquad \dot{\delta}_\theta = \dot{\theta}, \;\; \ddot{\delta}_\theta = \ddot{\theta}$$

and obtain a linear incremental ODE model:

$$M\, l\, \ddot{\delta}_\theta + (M\, g\, \cos\bar{\theta} - \bar{u}\, \sin\bar{\theta})\, \delta_\theta + \cos\bar{\theta}\, \delta_u = 0$$

- Exercise: what are linearisations around

$(\bar{u}, \bar{\theta}) = (0, 0)$ and $(\bar{u}, \bar{\theta}) = (0, \pi)$

Summary

- Equilibria can be obtained by solving an algebraic equation.
- A system can have multiple equilibria.
- The system can be approximated by a linear system around an equilibrium.
- Linearisation is different around different equilibria.