ELEN90055 Control Systems
Workshop 1

Characteristics of first- and second-order systems;
Time-domain interpretation of transfer function poles and zeros

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1 Introduction

This workshop introduces several MATLAB functions and analysis techniques frequently employed in control system analysis and design. These tools will be useful for the design projects in subsequent workshops.

Many systems in practice either: (i) can be modelled or approximated by relatively simple first- and second-order linear time-invariant (LTI) systems; or (ii) exhibit LTI dynamics characterised by *dominant* (i.e. slow) pole(s), which lie closer to the stability boundary (i.e. imaginary axis) than the rest[1]. In particular, situation (ii) often arises in control system synthesis, when a higher-order feedback system is designed to behave as a first- or second-order system with desirable properties and performance. Such a design scheme is called dominant pole placement. In this workshop, both time- and frequency-domain responses of commonly encountered first- and second-order stable continuous-time LTI systems are simulated and their characteristics studied. A case study on a vehicle steering system is presented at the end to illustrate the theory.

Note: Exercises 2, 3, and 8 should be completed before your scheduled workshop.

2 MATLAB Simulation

Exercise 1. *: Reproduce all the plots in Table 1. You are free to choose the values of the parameters for the purpose of generating the plots. This freedom is to be explored to gain an understanding of how varying the parameters affects the response.*

The following MATLAB functions may be useful for Exercise 1:

1. `tf`: creates system transfer function model;

2. `step`: plots the *step response* of an LTI model; and

3. `bode`: plots the *frequency response* of an LTI model as a function of frequency on a log-log scale for the magnitude and a log-linear scale for the phase.

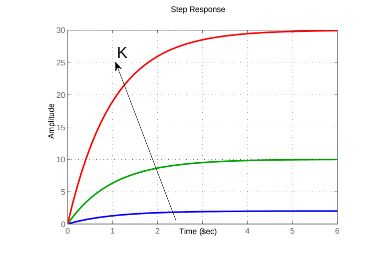
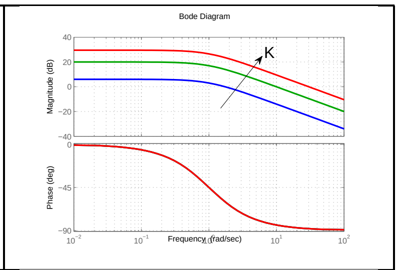
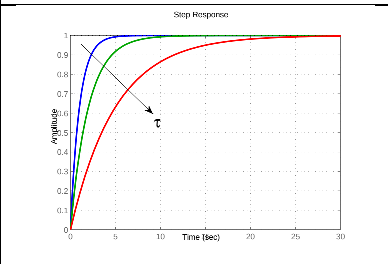
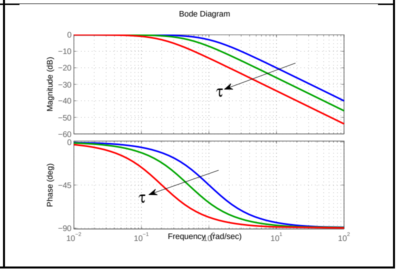
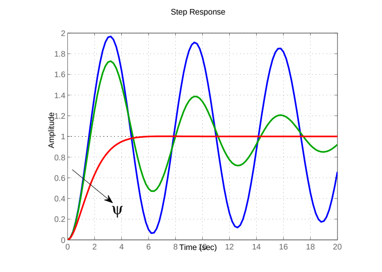
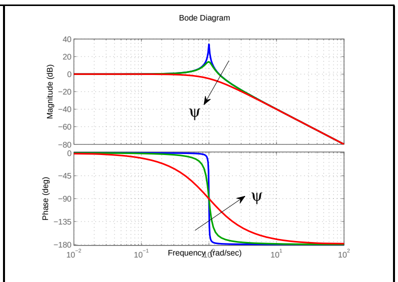
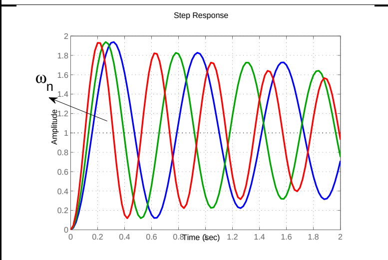
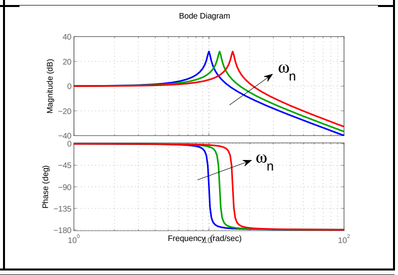
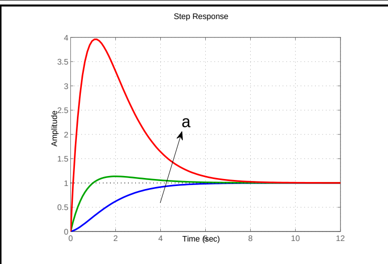
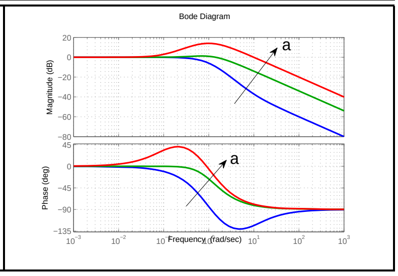
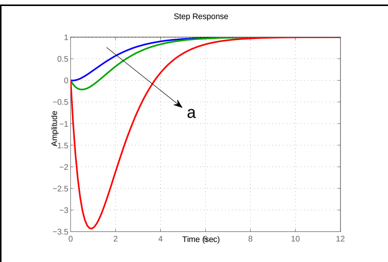
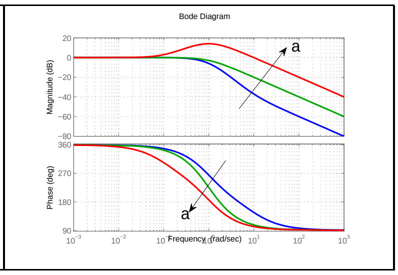
System	Parameter	Unit step response	Frequency response
$\frac{K}{\tau s+1}$	$K > 0$		
	$\tau > 0$		
$\frac{\omega_n^2}{s^2+2\psi\omega_n s+\omega_n^2}$	$0 < \psi < 1$		
	$\omega_n > 0$		
$\frac{as+1}{(s+1)^2}$	$a > 0$		
$\frac{-as+1}{(s+1)^2}$	$a > 0$		

Table 1: System models and the effects of parameter variation

For detailed and comprehensive information regarding MATLAB functions and the syntax for using them, consult the Product Help or type `help` followed by the name of the function on the command line.

You are encouraged to write your code in an M-file by appropriately making use of iteration statements, particularly *for* loops, and the MATLAB's cell array data structure. For instance, to define a size-2 array of transfer functions called `sys`, one may type

```
>> sys{1} = tf( ... );
>> sys{2} = tf( ... );
```

Note that curly brackets are used in the definition instead of the round ones for elementary arrays and matrices. To access the $i - th$ component of `sys`, type `sys{i}`.

3 Performance analysis

Exercise 2. *Table 1 shows simulated unit step responses for LTI systems with four different transfer functions. Derive parametrized expressions for each of these unit step responses, as a function of time. Consult any text book for help if needed.*

To get you started, the derivation for the first system is shown below:

$$y(t) = \mathcal{L}^{-1}\left(\frac{K}{s(\tau s+1)}\right) = \mathcal{L}^{-1}\left(\frac{K}{s} - \frac{K\tau}{\tau s+1}\right) = K(1 - e^{-\frac{t}{\tau}}),$$

where y denotes the unit step response and $\mathcal{L}$ the Laplace transform. The derivation for the second system are a bit involved; see [GGS01, Section 4.8.1] if help is needed.

Recall from Part II of the lecture slides or [GGS01, Section 4.7] the definitions of the performance indicators of a *stable* LTI system based on its step response, namely the steady-state value y_∞, rise time t_r, settling time t_s, overshoot M_p, and undershoot M_u. You may define the deviation δ from the steady-state value y_∞ to be 5% of y_∞. Mathematically, the settling time is related to δ and y_∞ by

$$y_\infty - \delta \leq |y(t)| \leq y_\infty + \delta, \forall t \geq t_s.$$

Exercise 3. *Using the step response expressions obtained in Exercise 2, analytically determine, when possible, y_∞, t_r, t_s, M_p, and M_u for the first two systems system in Table 1. Compare your calculations with the plots.*

The first two simulations in Table 1 reveal a number of fundamental properties of proportional gains and the poles of stable transfer functions. In particular, the magnitude response of $H(s) := \frac{K}{(\tau s+1)}$ is one of a typical low-pass filter.

Exercise 4. *For the first system $\frac{K}{(\tau s+1)}$, the parameters K and τ are respectively called the* d.c. gain *and the* time constant. *Investigate the relationship between*

[1] Recall that any scalar rational transfer function can be decomposed into a sum of partial fractions and that the transients associated with dominant poles decay more slowly than those with the remaining poles.

1. *the value of K and the steady-state value y_∞;*

2. *the value of τ and the rise time t_r.*

Analytically determine the bandwidth of $\frac{K}{(\tau s+1)}$. Subsequently, investigate the relationship between the bandwidth of the system and the rise time of its step response.

Exercise 5. *For the second system transfer function $\frac{\omega_n^2}{s^2+2\psi\omega_n s+\omega_n^2}$, the parameter ψ is the* damping *and ω_n the* undamped natural frequency*. Explain the effect of varying each of the parameters on the system's step response. Relate these to the expression derived in Exercise 2.*

The last two systems in Table 1 serve to demonstrate the effect that zeros have on the response of a stable transfer function. In particular, $s = -a^{-1}$ is a left half-plane (LHP) zero for the third system $\frac{as+1}{(s+1)^2}$. By applying Lemma 4.3 and the approximation preceding (4.8.38) in [GGS01], one obtains the following structural constraint on the system's step response:

$$M_p \geq \frac{a}{t_s}, \text{ for } a \gg 1 \tag{1}$$

Exercise 6. *Using your simulation results and (1), explain the effect of a slow (i.e. close to the imaginary axis) LHP zero on the system's transient performance.*

Notice that $s = a^{-1}$ is right half-plane (RHP) zero for the fourth system $\frac{-as+1}{(s+1)^2}$. Application of [GGS01, Lemma 4.2] yields

$$M_u \geq \frac{1-\delta}{e^{t_s/a}-1}. \tag{2}$$

Exercise 7. *Using your simulation results and (2), explain the effect of a RHP zero on the system's transient performance.*

4 A case study

In this section, we examine an instructive physical example taken from [AM08, Examples 2.8, 5.12 and 9.10]. Consider the vehicle steering system illustrated in Figure 1, in which x, y, and θ denote respectively the Cartesian-coordinates and orientation of the vehicle's centre of mass located at a distance a from the rear wheel, b denotes the wheel base, α the angle between the velocity vector v and the vehicle's length axis, and δ the steering angle of the front wheel. Dynamics of this kind can be found in many real-world examples in one form or another, including the steering wheel on a car and the front wheel of a bicycle, to name a couple.

We are interested in a dynamical model for a vehicle steering system. In terms of the schematic in Figure 1, we have that α and δ are related by:

$$\alpha(\delta) = \arctan\left(\frac{a\tan\delta}{b}\right).$$

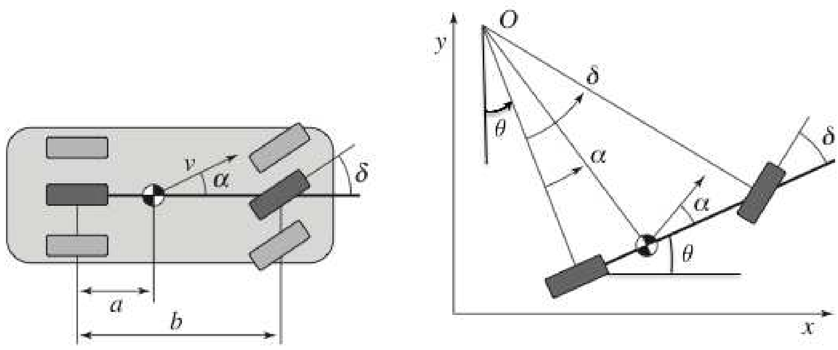


Figure 1: Vehicle steering dynamics. Sourced and modified from [AM08, Figure 2.16].

Let the velocity of the rear wheel be v_0. Note that $v = v_0/\cos(\alpha)$. The nonlinear equations describing the dynamics of the centre of mass can be derived as

$$\dot{\theta} = \frac{v_0}{b/\tan\delta}; \quad \dot{x} = v\cos(\alpha(\delta)+\theta); \quad \dot{y} = v\sin(\alpha(\delta)+\theta). \tag{3}$$

We simplify the model by considering only the influence of the steering angle δ on the lateral (y-component) velocity, i.e. ignoring the x-component dynamics in (3).

Exercise 8. *Show that $(\theta, \delta) = (0, 0)$ is a possible equilibrium for the simplified model and derive a corresponding linearisation. Note that for the specified equilibrium the incremental signals are the same as the original signals.*

Your answer to Exercise 8 should give rise to the following transfer function from $\Delta(s) = \mathcal{L}[\delta](s)$ to $Z(s) = \mathcal{L}[\dot{y}](s)$:

$$G(s) = \frac{av_0 s + v_0^2}{bs}.$$

Note that G has a zero at $s = -v_0/a$. In forward driving ($v_0 > 0$) the zero lies in the LHP, but it is in the RHP when driving in reverse ($v_0 < 0$).

Exercise 9. *Using MATLAB, plot the step response of the vehicle steering transfer function*

$$\hat{G}(s) = \frac{av_0 s + v_0^2}{bs(Ts+1)},$$

where the extra pole with the time constant T is added to approximately account for the dynamics in the steering system. For your simulation, set $a = b = 1$, $T = 0.1$, and $v_0 = 1$ for forward driving and $v_0 = -1$ for reverse driving. Analyse your results and relate them to the physical steering system.

5 Summary of workshop objectives

By the end of this workshop you should be able to:

- Apply MATLAB command-line functions for simulating LTI system behaviour in time and frequency domains;
- Analyse analytically and graphically the behaviour of LTI systems;
- Understand the effects of transfer function poles and zeros on system performance;
- Appreciate that the models studied in this workshop arise commonly in real life;
- Approximate the dynamics of nonlinear models with transfer functions via linearisation around equilibrium points.

References

[AM08] K. J. Astrom and R. M. Murray. Feedback systems: An introduction for scientists and engineers. `www.cds.caltech.edu/~murray/amwiki/index.php/Main_Page`, 2008.

[GGS01] G. Goodwin, S. Graebe, and M. Salgado. *Control System Design*. Prentice-Hall, Upper Saddle River, NJ, 2001.