Lecture 8

Grandfather of control and telecommunications

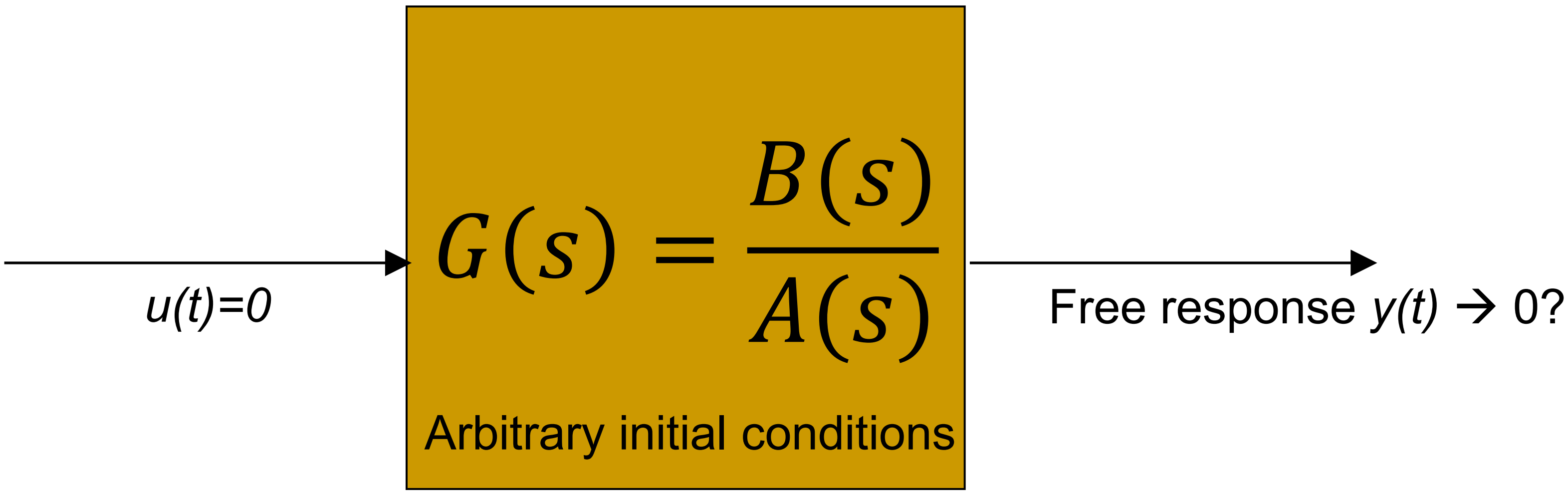
Two Flavours of Stability

Routh-Hurwitz Criterion

Motivation

- Control systems must be designed to meet performance criteria – e.g. step response specs.
- If the control system will be operated over a long period of time, stability is usually also necessary to be able to meet these criteria
- We present two definitions of stability.
- For LTI systems, both conditions turn out to be very closely related

Stability w.r.t. Initial Conditions



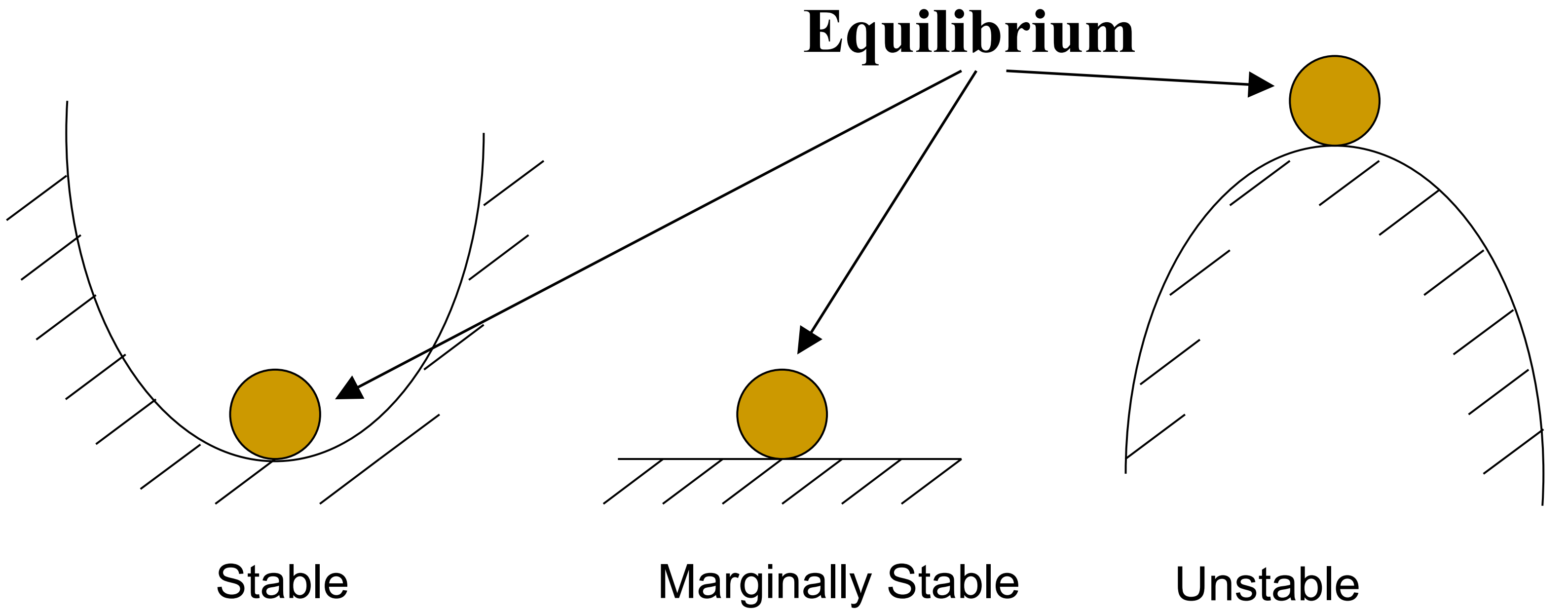
Say *A(s)*, *B(s)* are given polynomials, e.g. from a diff. eq. model

If *A(s)* has order *n*, then *initial conditions* are $\frac{d^j y}{dt^j}(0^-)$, $j = 0, \ldots, n-1$.

Definition (input=zero)

- Stable if output converges to zero for any initial condition.
- Unstable if output diverges for at least some initial conditions
- Marginally stable if output stays bounded from all initial conditions and may converge to zero for some initial conditions.

Visualisation



Free Response

- Free response

$$\sum_{i=0}^{n} a_i \frac{d^i y}{dt^i} = \sum_{i=0}^{m} b_i \frac{d^i u}{dt^i} = 0 \quad , t \geq 0$$

$$\Leftrightarrow \sum_{i=0}^{n} a_i \left(s^i Y(s) - \sum_{j=0}^{i-1} s^j \frac{d^j y}{dt^j}(0^-) \right) = 0$$

$$\Leftrightarrow Y(s) = \frac{\left.\sum_{j=0}^{n-1} s^j \left(\frac{d^j y}{dt^j}(0^-) \sum_{i=j+1}^{n} a_i \right)\right\} = \mathrm{poly}(n-1)}{\left.\sum_{i=0}^{n} a_i s^i \right\} = A(s)}$$

LTI Stability w.r.t. Initial Conditions

- Stable:

  All poles of A(s) ***strictly*** *in the left-half plane (LHP).*

- Marginally stable:

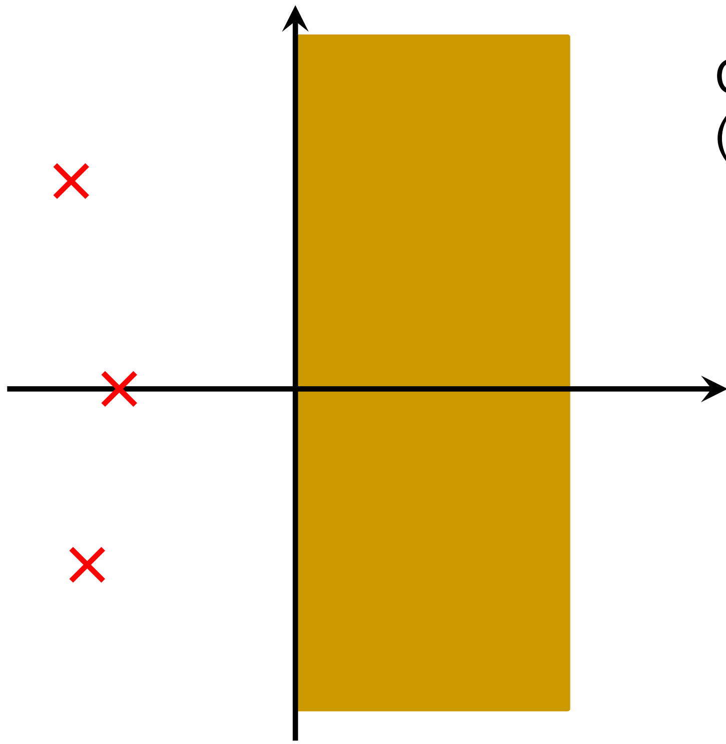
  Non-repeated pole(s) on imaginary axis, all other poles strictly in the LHP

- Unstable:

  Some pole(s) strictly in the right-half plane OR repeated poles on the imaginary axis

You can understand this if you remember how terms in the partial fraction expansion affect the impulse response.

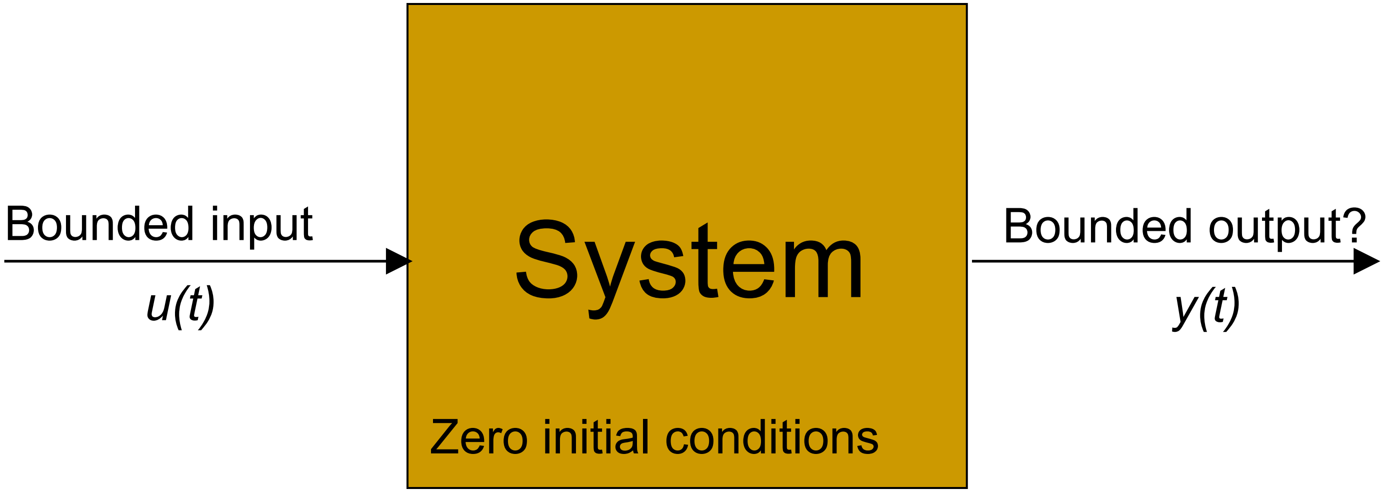
Graphical Interpretation of Stability



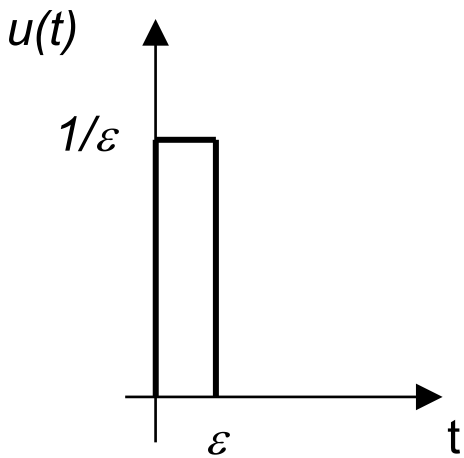
Complex domain
(pole locations)

Brown colour represents a “forbidden (unstable) region”.

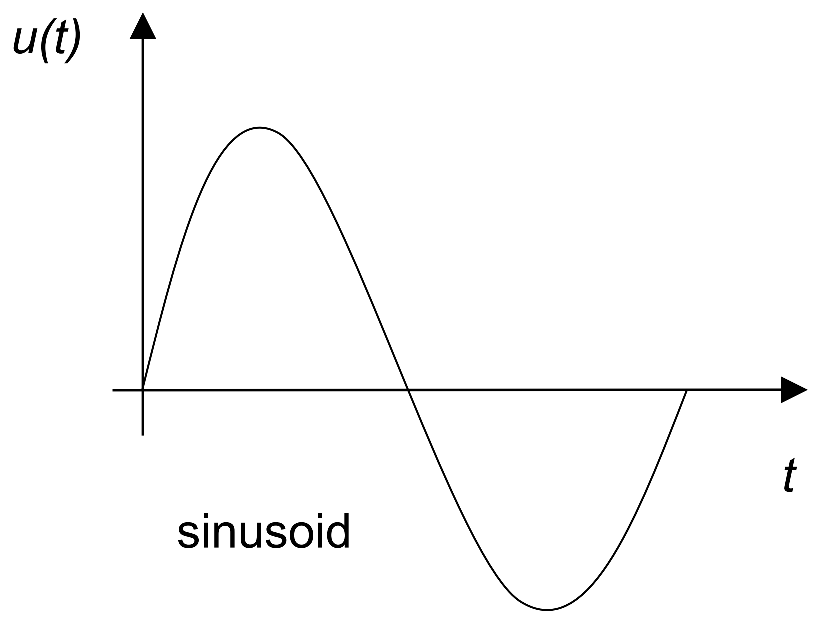
Bounded-Input Bounded-Output (BIBO) Stability



Signals as functions of time



delta function or impulse $\delta(t)$



We measure the size of signal u using: $\bar{u} = \max_{t \geq 0} |u(t)| < \infty$

BIBO Stability Definition (zero initial conditions)

- Formally: For any $\bar{u} > 0$, there exists a finite $\bar{y} > 0$ such that for any input u with $u(t) \leq \bar{u}$,

$$|y(t)| \leq \bar{y}\,, \quad \text{for all } t \geq 0.$$

- Informally: Bounded inputs yield bounded outputs.

BIBO stability for LTI systems

- BIBO stability holds *if and only if (iff)* the impulse response *g* is *absolutely integrable*, i.e.

$$\int_0^\infty |g(t)|dt < \infty$$

- Proof of sufficiency (🡐):

$$|y(t)| = |(g * u)(t)| \leq \int_0^t |g(t-\tau)| \cdot |u(\tau)| d\tau$$

$$= \int_0^t |g(\nu)| \cdot |u(t-\nu)| d\nu \leq \overline{u} \int_0^\infty |g(\nu)| d\nu.$$

BIBO Stability

- Proof of necessity (➔): By contradiction.

Suppose $\int_0^\infty |g(t)|dt = \lim_{T\to\infty} \int_0^T |g(t)|dt = \infty$.

For an arbitrary *T>0*, apply an input

$$u(t) = \begin{cases} \bar{u}.\,\mathrm{sign}(g(T-t)), & 0 \le t \le T, \\ 0, & \text{else.} \end{cases}$$

$$\Rightarrow y(T) = \int_0^T g(t)u(T-t)dt$$

$$= \int_0^T g(t)\bar{u}.\,\mathrm{sign}(g(t))dt$$

$$= \bar{u}.\int_0^T |g(t)|dt \longrightarrow \infty.$$ **QED**

BIBO for LTI Systems with Rational Transfer Functions

- BIBO stability *iff*
  - Transfer function is proper (why?), **and**
  - All poles lie strictly in the LHP (why?).

- If no *unstable pole-zero cancellations* in *B(s)/A(s),*

  BIBO stability ➔ Stability w.r.t. initial conditions

  (but the reverse implication is not true)

- If no unstable pole-zero cancellations, **and** *B(s)/A(s)* is proper

  BIBO stability ⇔ Stability w.r.t. initial conditions

Why pay heed to a pole if cancelled by a zero anyway?

- Pole-zero cancellation said to occur if there is a zero β_k which equals a pole α_j.
- In this case, the factor $(s-\beta_k)$ in the numerator *B(s)* formally cancels out the factor $(s-\alpha_j)$ in the denominator *A(s)*, and both factors disappear from the transfer function *G(s) = B(s)/A(s).*
- So why do we care…?

Why Cancelled Poles Still Count

- A cancelled pole α_j is no problem if initial conditions are zero. But in practice, initial conditions may not be exactly zero.
- Recall *y(t)* = free response (due to nonzero initial conditions) + forced response (due to nonzero input).
- Cancelled pole α_j appears in the free response! Could affect transient performance if too slow.
- If α_j unstable ➔ Trouble as $t \to \infty$

Example of How Pole-Zero Cancellations Arise

Consider a DE system

$\ddot{y} - y = \dot{u} - u,$ where $u(t) = 0, t < 0.$

➔ $(s^2 - 1)Y(s) - (s\dot{y}(0^-) + y(0^-)) = (s-1)U(s)$

⇔ $Y(s) = \frac{1}{s+1}U(s) + \frac{s\dot{y}(0^-) + y(0^-)}{(s+1)(s-1)}$

System is BIBO stable, but not stable w.r.t. initial conditions

Example of System That's Stable w.r.t. Initial Cond's, but isn't BIBO stable

Determining stability for LTI systems

- Consider a transfer function

$$G(s) = \frac{b_m s^m + \cdots + b_1 s + b_0}{a_n s^n + \cdots + a_1 s + a_0} = K \frac{(s-\beta_1)\cdots(s-\beta_m)}{(s-\alpha_1)\cdots(s-\alpha_n)} = K \frac{\prod_{k=1}^{m}(s-\beta_k)}{\prod_{k=1}^{n}(s-\alpha_k)}$$

- Poles are the solutions of either equation:

$$a_n s^n + a_{n-1} s^{n-1} + \ldots + a_1 s + a_0 = 0$$

$$s^n + a'_{n-1} s^{n-1} + \ldots + a'_1 s + a'_0 = (s-\alpha_1)\ldots(s-\alpha_0) = 0$$

The second polynomial is “monic” and often used in tests.

Radical Facts

- A quadratic polynomial

$$as^2 + bs + c \text{ has zeros at } s = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

- Roots of cubic and quartic polynomials can be found in terms of 3rd and 4th roots (*radicals*). But the formulas are impossible to remember
- Galois (1811-32): Polynomial equations of order 5 or higher cannot generally be expressed in terms of radicals!

So how to manually check stability for systems of order>2 ?

A necessary condition for stability

- If a monic polynomial

$$p(s) = s^n + a_{n-1}s^{n-1} + \cdots + a_1 s + a_0$$

has all roots with negative real parts, then all its coefficients are positive; that is

$$a_i > 0 \text{ for } i = 0, 1, \ldots, n$$

- If a coefficient is negative or zero, then some roots have positive or zero real part.
- This is also sufficient condition for first and second order polynomials.

Proof:

- Recall that an arbitrary monic polynomial in *s* with real coefficients can always be written as a product of monic linear and quadratic terms with real coefficients:

$$s + a, \qquad s^2 + bs + c$$

- Both linear and quadratic terms have all roots with strictly negative real parts iff all their coefficients are positive.

Test:

- If some coefficient in the characteristic polynomial is negative or zero, then there are some poles with positive/zero real part.

- If all coefficients in the characteristic polynomial are all positive, then use *Routh-Hurwitz criterion* to check whether there are any poles with positive/zero real part.

Examples

- Consider

for $p(s) = s^2 + ks + (1-k)$

if $[\, p(s_0) = 0 \implies \Re(s_0) < 0 \,]$ *i.e. all zeros have negative real part*

then $0 < k < 1$ *converse also true in this case since $n = 2$*

This polynomial has all positive coefficients but two roots on the imaginary axis:

$$s^3 + s^2 + s + 1$$

has zeros at

$$s = -1, \pm j$$

Routh-Hurwitz test

- For polynomial with positive coefficients:

$$a_n s^n + a_{n-1} s^{n-1} + \ldots + a_1 s + a_0$$

we form the following (Routh) array:

$$\begin{array}{c|cccc}
s^n & \gamma_{0,1} := a_n & \gamma_{0,2} := a_{n-2} & \gamma_{0,3} := a_{n-4} & \cdots \\
s^{n-1} & \gamma_{1,1} := a_{n-1} & \gamma_{1,2} := a_{n-3} & \gamma_{1,3} := a_{n-5} & \cdots \\
s^{n-2} & \gamma_{2,1} & \gamma_{2,2} & \gamma_{2,3} & \cdots \\
\vdots & \vdots & \vdots & \vdots & \vdots \\
s^1 & \gamma_{n-1,1} & \gamma_{n-1,2} & \gamma_{n-1,3} & \cdots \\
s^0 & \gamma_{n,1} & \gamma_{n,2} & \gamma_{n,3} & \cdots
\end{array}$$

$$\gamma_{k,j} := -\frac{1}{\gamma_{k-1,1}} \begin{vmatrix} \gamma_{k-2,1} & \gamma_{k-2,j+1} \\ \gamma_{k-1,1} & \gamma_{k-1,j+1} \end{vmatrix}$$

- Example of computing entries in the third row

$$
\begin{array}{c|cccc}
s^n & \boxed{\begin{matrix}\gamma_{0,1} := a_n \\ \gamma_{1,1} := a_{n-1}\end{matrix}} & \boxed{\begin{matrix}\gamma_{0,2} := a_{n-2} \\ \gamma_{1,2} := a_{n-3}\end{matrix}} & \begin{matrix}\gamma_{0,3} := a_{n-4} \\ \gamma_{1,3} := a_{n-5}\end{matrix} & \begin{matrix}\cdots \\ \cdots\end{matrix} \\
s^{n-1} & & & & \\
s^{n-2} & \boxed{\gamma_{2,1}} & \gamma_{2,2} & \gamma_{2,3} & \cdots \\
\vdots & \vdots & \vdots & \vdots & \vdots \\
s^1 & \gamma_{n-1,1} & \gamma_{n-1,2} & \gamma_{n-1,3} & \cdots \\
s^0 & \gamma_{n,1} & \gamma_{n,2} & \gamma_{n,3} & \cdots
\end{array}
$$

$$
\gamma_{2,1} := -\frac{1}{\gamma_{1,1}} \begin{vmatrix} \gamma_{0,1} & \gamma_{0,2} \\ \gamma_{1,1} & \gamma_{1,2} \end{vmatrix}
$$

- Example of computing entries in the third row

$$
\begin{array}{c|cccc}
s^n & \boxed{\begin{matrix}\gamma_{0,1} := a_n \\ \gamma_{1,1} := a_{n-1}\end{matrix}} & \begin{matrix}\gamma_{0,2} := a_{n-2} \\ \gamma_{1,2} := a_{n-3}\end{matrix} & \boxed{\begin{matrix}\gamma_{0,3} := a_{n-4} \\ \gamma_{1,3} := a_{n-5}\end{matrix}} & \begin{matrix}\cdots \\ \cdots\end{matrix} \\
s^{n-1} & & & & \\
s^{n-2} & \gamma_{2,1} & \boxed{\gamma_{2,2}} & \gamma_{2,3} & \cdots \\
\vdots & \vdots & \vdots & \vdots & \vdots \\
s^1 & \gamma_{n-1,1} & \gamma_{n-1,2} & \gamma_{n-1,3} & \cdots \\
s^0 & \gamma_{n,1} & \gamma_{n,2} & \gamma_{n,3} & \cdots
\end{array}
$$

$$
\gamma_{2,2} := -\frac{1}{\gamma_{1,1}} \begin{vmatrix} \gamma_{0,1} & \gamma_{0,3} \\ \gamma_{1,1} & \gamma_{1,3} \end{vmatrix}
$$

Routh-Hurwitz test

- The polynomial with positive coefficients

$$a_n s^n + a_{n-1} s^{n-1} + \ldots + a_1 s + a_0$$

  has all roots with strictly negative real parts if and only if all elements in the first column of the Routh array do not change their sign.
- No need to solve for the roots!

Example

- For a first order polynomial

$$a_1 s + a_0$$

Routh array is

$$\begin{array}{c|c} s^1 & a_1 \\ s^0 & a_0 \end{array}$$

Example

- For a quadratic polynomial

$$a_2 s^2 + a_1 s + a_0$$

Routh array is

$$\begin{array}{c|cc} s^2 & a_2 & a_0 \\ s^1 & a_1 & 0 \\ s^0 & a_0 = -\frac{a_2 \cdot 0 - a_0 \cdot a_1}{a_1} & 0 \end{array}$$

Example

- For third order polynomials

$$a_3 s^3 + a_2 s^2 + a_1 s + a_0$$

Routh array is

$$\begin{array}{c|cc} s^3 & a_3 & a_1 \\ s^2 & a_2 & a_0 \\ s^1 & -\frac{a_3 \cdot a_0 - a_1 \cdot a_2}{a_2} & 0 \\ s^0 & a_0 & \end{array}$$

Example

- For fourth order polynomials we have:

For $p(s) = \sum_{k=0}^{4} a_k s^k,\ a_4 > 0,$

$$
\begin{array}{c|ccc}
s^4 & a_4 & a_2 & a_0 \\
s^3 & a_3 & a_1 & 0 \\
s^2 & b_0 \doteq -\frac{a_4 a_1 - a_2 a_3}{a_3} & b_1 \doteq -\frac{a_4 \cdot 0 - a_0 a_3}{a_3} = a_0 & 0 \\
s^1 & c \doteq -\frac{a_3 b_1 - a_1 b_0}{b_0} & 0 & \\
s^0 & -\frac{b_0 \cdot 0 - b_1 c}{c} = b_1 & &
\end{array}
$$

Example

- Consider a feedback system with

$$G_0(s) = \frac{1}{s(s+1.5)} \qquad C(s) = K + \frac{s}{0.1s+1}$$

- The characteristic polynomial is

$$0.1s^3 + 1.15s^2 + (0.1K + 2.5)s + K$$

- Find all values of K for which the system is stable.

Example

- We use formulas for 3rd order polynomials:

$$\begin{array}{c|cc} s^3 & 0.1 & (0.1K + 2.5) \\ s^2 & 1.15 & K \\ s^1 & \frac{3K}{230} + 2.5 & 0 \\ s^0 & K & \end{array}$$

- The system is stable if

$$K > 0$$

Example

- Consider the same system.
- For which values of K would all roots of the characteristic polynomial have real parts strictly smaller than -1?
- We modify the polynomial by introducing

$$w = s + 1 \qquad\qquad \Re(w) < 0 \quad \Leftrightarrow \quad \Re(s) < -1$$

$$0.1(w-1)^3 + 1.15(w-1)^2 + (0.1K + 2.5)(w-1) + K$$
$$= 0.1w^3 + 0.85w^2 + (0.1K + 0.5)w + (0.9K - 1.45)$$

Example

- Routh array is

$$
\begin{array}{c|cc}
w^3 & 0.1 & 0.1K+0.5 \\
w^2 & 0.85 & 0.9K-1.45 \\
w^1 & \frac{1}{170}(-K+114) & \\
w^0 & 0.9K-1.45 &
\end{array}
$$

- Hence, we have the conditions:

$$0.9K - 1.45 > 0 \text{ and } \frac{1}{170}(114 - K) > 0$$

or $\frac{29}{18} < K < 114$

Exercise

- Form the Routh array for the following polynomial:

$$s^3 + s^2 + s + 1$$