Lecture 4

Revision of Laplace Transforms

Algebra of Block Diagrams

Outline

- Motivation
- Complex numbers
- Definition of Laplace Transform
- Properties of Laplace Transform
- Partial fraction expansion
- Conclusions

Why we love Laplace transforms

- Time derivatives, integrals and convolutions become algebraic operations in *s*-domain ➔ Much, much simpler to analyse and design systems – and write them down concisely.
- Different functions in time-domain yield different functions in *s*-domain ➔ No loss of information going from one domain to the other
- Leads to crucial concepts of transfer functions, poles, zeros etc

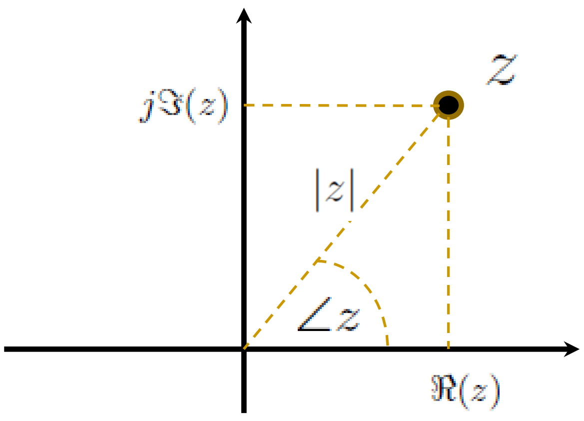
Complex numbers

- Complex numbers consist of real and imaginary parts:

$$
\begin{aligned}
z &= \Re(z) + j\Im(z) \\
&= \left(\sqrt{\Re(z)^2 + \Im(z)^2}\right) e^{j\,\arctan(\Im(z)/\Re(z))} \\
&= |z| e^{j\angle z}
\end{aligned}
$$

phase (in radians!)

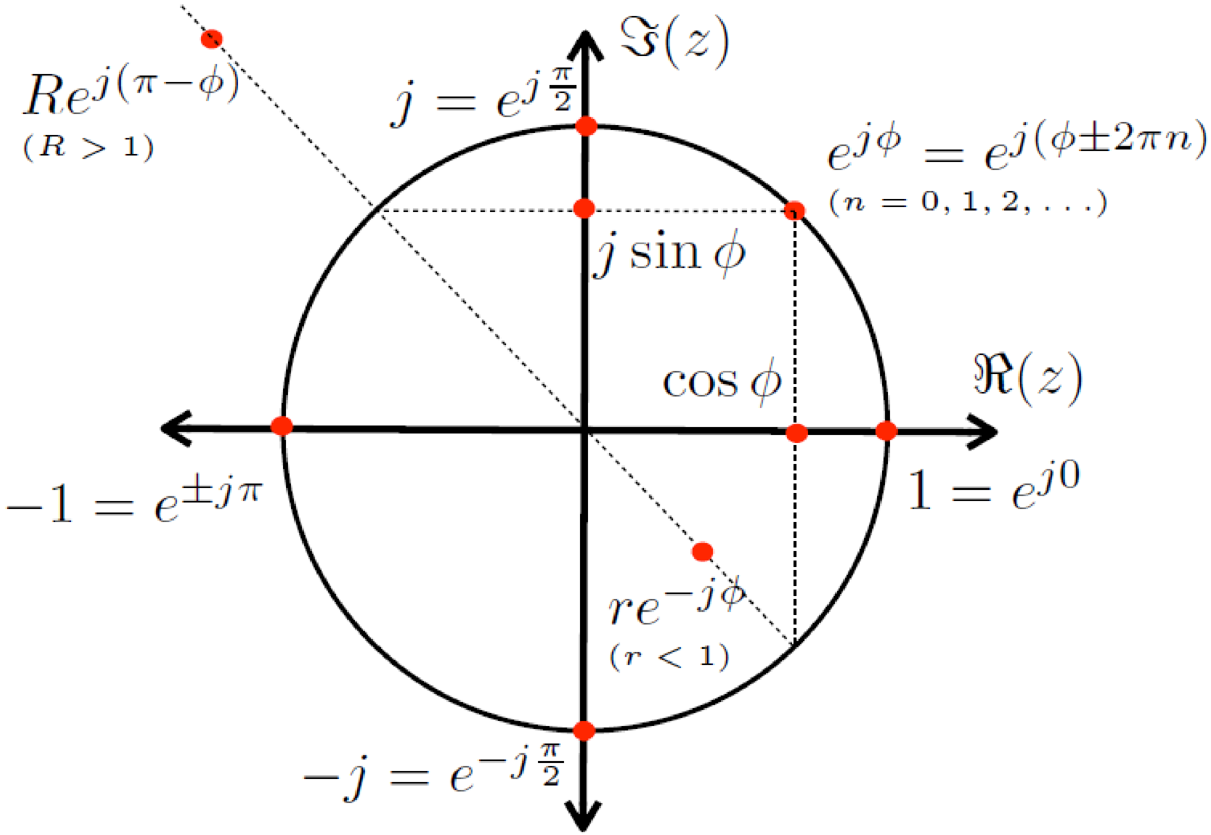
magnitude



$\Re(z) < 0$

$j := \sqrt{-1}$

Complex numbers in complex plane



Operations with complex numbers

- Addition of complex numbers:

$$z_1 + z_2 = (\Re(z_1) + \Re(z_2)) + j(\Im(z_1) + \Im(z_2))$$

- Multiplication of complex numbers:

$$z_1 \cdot z_2 = |z_1||z_2|e^{j(\angle z_1 + \angle z_2)}$$

- Exponential of a complex number:

$$e^z = e^{\Re(z)} \, e^{j\Im(z)} = e^{\Re(z)}(\cos \Im(z) + j \sin \Im(z))$$

Laplace transform

Motivation (possible design steps)

Time domain:
Step 1: Modelling
Step 2: Design specifications

Complex (Laplace) domain.
Step 4: analysis/design

$y(t)$	$Y(s)$
$\sum_{k=0}^{n} a_k \frac{d^k y}{dt^k} = \sum_{k=0}^{n} b_k \frac{d^k u}{dt^k}$	$Y(s) = G(s)U(s)$
Overshoot, Undershoot, Rise time, Setting time, Steady-state error	Locations of poles and zeroes of transfer function in the complex plane
Stability	Locations of poles in the complex plane

Step 3: $\mathcal{L}[y]$

Step 5: $\mathcal{L}^{-1}[Y]$

Step 6: Verification, interpretation

Laplace transform

- Laplace transform for signal $y(t), 0 \leq t < \infty$ is

$$\mathcal{L}\left[y(t)\right] = Y(s) = \int_{0^-}^{\infty} e^{-st} y(t) dt$$

- Inverse Laplace transform of *Y(s)* is

$$\mathcal{L}^{-1}\left[y(s)\right] = y(t) = \frac{1}{2\pi j} \int_{\sigma - j\infty}^{\sigma + j\infty} e^{st} Y(s) ds$$

- Region of convergence is $\Re\{s\} \geq \sigma$ for which $|y(t)| < ke^{\sigma t}; \forall t \geq 0$ where $\sigma \in \mathbb{R}$, $k < \infty$

$f(t) \qquad (t \geq 0)$	$\mathcal{L}[f(t)]$	Region of Convergence
1	$\frac{1}{s}$	$\sigma > 0$
$\delta_D(t)$	1	$\|\sigma\| < \infty$
t	$\frac{1}{s^2}$	$\sigma > 0$
$t^n \qquad n \in \mathbb{Z}^+$	$\frac{n!}{s^{n+1}}$	$\sigma > 0$
$e^{\alpha t} \qquad \alpha \in \mathbb{C}$	$\frac{1}{s-\alpha}$	$\sigma > \Re\{\alpha\}$
$te^{\alpha t} \qquad \alpha \in \mathbb{C}$	$\frac{1}{(s-\alpha)^2}$	$\sigma > \Re\{\alpha\}$
$\cos(\omega_o t)$	$\frac{s}{s^2+\omega_o^2}$	$\sigma > 0$
$\sin(\omega_o t)$	$\frac{\omega_o}{s^2+\omega_o^2}$	$\sigma > 0$

Taken from “Control Systems Design”, Goodwin, Graebe & Salgado

$f(t)$	$\mathcal{L}[f(t)]$	Names
$\sum_{i=1}^{l} a_i f_i(t)$	$\sum_{i=1}^{l} a_i F_i(s)$	Linear combination
$\frac{dy(t)}{dt}$	$sY(s) - y(0^-)$	Derivative Law
$\frac{d^k y(t)}{dt^k}$	$s^k Y(s) - \sum_{i=1}^{k} s^{k-i} \left.\frac{d^{i-1}y(t)}{dt^{i-1}}\right\vert_{t=0^-}$	High order derivative
$\int_{0^-}^{t} y(\tau)d\tau$	$\frac{1}{s}Y(s)$	Integral Law
$y(t-\tau)\mu(t-\tau)$	$e^{-s\tau}Y(s)$	Delay
$ty(t)$	$-\frac{dY(s)}{ds}$	
$t^k y(t)$	$(-1)^k \frac{d^k Y(s)}{ds^k}$	
$\int_{0^-}^{t} f_1(\tau) f_2(t-\tau)d\tau$	$F_1(s)F_2(s)$	Convolution
$\lim_{t\to\infty} y(t)$	$\lim_{s\to 0} sY(s)$	Final Value Theorem
$\lim_{t\to 0^+} y(t)$	$\lim_{s\to\infty} sY(s)$	Initial Value Theorem

Taken from “Control Systems Design”, Goodwin, Graebe & Salgado

Comments:

- It is useful to understand at least a few proofs of items in the tables of Laplace transforms.
- You need to understand all proofs that are given in slides at the end of this lecture.
- It is essential to know how to use Laplace tables/rules in solving problems.

Example:

- Determine the unit step response of the following system, assuming the initial condition y(0)=10:

$$\dot{y} + y = u$$

- Step 1: take Laplace transform of both sides:

$$\underbrace{sY(s) - y(0)}_{\mathcal{L}\{\dot{y}\}} + \underbrace{Y(s)}_{\mathcal{L}\{y\}} = \underbrace{\frac{1}{s}}_{\mathcal{L}\{1\}}$$

(we used the linearity of LT and formulas for the derivative and for the unit step signal)

- Step 2: solve for Y(s) – this involves algebraic calculations!

$$Y(s) = \frac{y(0)}{s+1} + \frac{1}{s(s+1)}$$

- Step 3: rewrite Y(s) noting that y(0)=10 and using "partial fraction expansion":

$$\begin{aligned} Y(s) &= \frac{y(0)}{s+1} + \frac{1}{s(s+1)} \\ &= \frac{10}{s+1} + \frac{1}{s} - \frac{1}{s+1} \\ &= \frac{9}{s+1} + \frac{1}{s} \end{aligned}$$

- Step 4: take the inverse Laplace transform

$$y(t) = \mathcal{L}^{-1}\{\tfrac{9}{s+1}\} + \mathcal{L}^{-1}\{\tfrac{1}{s}\} = 9e^{-t} + 1$$

Example continued

- Suppose that assuming the same input and initial condition, we now want to calculate

$$\lim_{t\to\infty} y(t)$$

- One way to do this is to go through all previous steps, obtain y(t) and then calculate:

$$\lim_{t\to\infty} y(t) = \lim_{t\to\infty} 9e^{-t} + 1 = 1$$

Example continued

- But this is a place where we can use “Final Value Theorem” and calculate the limit directly in complex domain:

$$\lim_{t\to\infty} y(t) = \lim_{s\to 0} sY(s) = \lim_{s\to 0} s\left[\frac{9}{s+1} + \frac{1}{s}\right] = 1$$

- Note that you always need to first verify that the limit exists when using Final Value Theorem (e.g. stability).

Homework

- Consider the system

$$\ddot{y} + 2\dot{y} + 10y = 2u$$

- Assuming zero initial conditions, find:
- Response to unit step signal
- Response to unit impulse signal
- Response to a sine signal with amplitude 2
- Limits as time goes to infinity for all the above

Example revisited

- Recall that in the example in we needed to find the inverse Laplace transform of:

$$\frac{1}{s(s+1)}$$

- This term can not be found in LT tables. We need to transform it to its “partial fraction form”:

$$\frac{1}{s(s+1)} = \frac{1}{s} - \frac{1}{s+1}$$

- Note that in new form, we can find all terms in LT tables. We show how to do this in general.

Partial fraction expansion
(non-repeated poles)

- We can represent strictly proper transfer functions with different poles as follows:

'partial fraction' expansion

$$G(s) = \frac{b_m s^m + \cdots + b_1 s + b_0}{a_n s^n + \cdots + a_1 s + a_0} = K \frac{\prod_{k=1}^{m} (s - \beta_k)}{\prod_{k=1}^{n} (s - \alpha_k)} = \sum_{k=1}^{n} \frac{B_k}{s - \alpha_k}$$

where "residues" are computed as follows

$$B_k = \lim_{s \to \alpha_k} (s - \alpha_k) G(s)$$

- Using LT tables, we have

$$g(t) = \sum_{k=1}^{n} B_k e^{\alpha_k t} \text{ for } t \geq 0$$

Example:

- If we consider

$$G(s) := \frac{1}{s(s+1)} = \frac{B_1}{s} + \frac{B_2}{s+1}$$

its poles are s=0 and s=-1. Using the formulas we have just given, we obtain

$$B_1 = \lim_{s\to 0} sG(s) = 1$$

$$B_2 = \lim_{s\to -1} (s+1)G(s) = \frac{1}{-1}$$

which confirms what we already shown earlier by direct computation.

Complex conjugate poles

- Consider a typical term in FPE:

$$\frac{B_k}{s-\alpha_k}$$

- When the pole is complex, then the residue is too and they appear in conjugate pairs:

$$\frac{B_k}{s-\alpha_k} + \frac{B_k^*}{s-\alpha_k^*} \qquad \begin{aligned} \alpha_k &= \sigma_k + j\omega_k \\ B_k &= |B_k| e^{\phi_k} \end{aligned}$$

(if system impulse response is real-valued).

Complex conjugate pole

- Direct calculations yield:

$$
\begin{aligned}
B_k e^{\alpha_k t} + B_k^* e^{\alpha_k^* t} &= |B_k| e^{j\Phi_k} e^{(\sigma_k + j\omega_k)t} + |B_k| e^{-j\Phi_k} e^{(\sigma_k - j\omega_k)t} \\
&= |B_k| e^{j\Phi_k} e^{\sigma_k t} e^{j\omega_k t} + |B_k| e^{-j\Phi_k} e^{\sigma_k t} e^{-j\omega_k t} \\
&= |B_k| e^{\sigma_k t} \left\{ e^{j\Phi_k + j\omega_k t} + e^{-(j\Phi_k + j\omega_k t)} \right\} \\
& \quad \text{let } \widetilde{\Phi}_k := \Phi_k + \omega_k t \\
&= |B_k| e^{\sigma_k} \left\{ \cos \widetilde{\Phi}_k + j \sin \widetilde{\Phi}_k + \cos \widetilde{\Phi}_k - j \sin \widetilde{\Phi}_k \right\} \\
&= 2 |B_k| e^{\sigma_k t} \cos(\Phi_k + \omega_k t)
\end{aligned}
$$

- In the last line we have a real function!

Partial fraction expansion
(repeated poles)

- Consider now a transfer function with repeated poles that can be written as follows:

$$G(s) = \frac{b(s)}{a(s)} = \frac{b_m s^m + \ldots + b_0}{a_n s^n + \ldots + a_0} = \frac{b(s)}{(s-\alpha_1)^{k_1} \ldots (s-\alpha_l)^{k_l}}$$

$$\alpha_i \neq \alpha_j, \; i \neq j$$

$$m < n$$

- PFE can be written as follows:

$$
\begin{aligned}
G(s) = & \frac{B_{1,k_1}}{(s-\alpha_1)^{k_1}} + \frac{B_{1,k_1-1}}{(s-\alpha_1)^{k_1-1}} + \ldots + \frac{B_{1,1}}{s-\alpha_1} \\
& + \frac{B_{2,k_2}}{(s-\alpha_2)^{k_2}} + \frac{B_{2,k_2-1}}{(s-\alpha_2)^{k_2-1}} + \ldots + \frac{B_{2,1}}{s-\alpha_2} \\
& \vdots \qquad \vdots \qquad \vdots \\
& + \frac{B_{l,k_l}}{(s-\alpha_l)^{k_l}} + \frac{B_{l,k_l-1}}{(s-\alpha_l)^{k_l-1}} + \ldots + \frac{B_{l,1}}{s-\alpha_l}
\end{aligned}
$$

$$
\begin{aligned}
B_{i,k_i} &= \lim_{s\to\alpha_i} (s-\alpha_i)^{k_i} G(s), i = 1, 2, \ldots, l \\
B_{i,k_i-j} &= \lim_{s\to\alpha_i} \frac{1}{j!} \frac{d^j}{ds^j} (s-\alpha_i)^{k_i} G(s), i = 1, 2, \ldots, l, j = 1, \ldots, k_i - 1
\end{aligned}
$$

See Ogata “Modern Control Engineering”

Partial fraction expansion via Matlab

- We can write in Matlab:

```
num=[2 5 3 6]
den=[1 6 11 6]
[r,p,k]=residue(num,den)
r =
   -6.0000
   -4.0000
    3.0000
p =
   -3.0000
   -2.0000
   -1.0000
k =
     2
```

$$G(s) = \frac{2s^3+5s^2+3s+6}{s^3+6s^2+11s+6}$$

$$G(s) = \frac{-6}{s+3} + \frac{-4}{s+2} + \frac{3}{s+1} + 2$$

Example 1 (exponential):

Find $Y_a(s) = \mathcal{L}[y_a](s)$, where $y_a(t) \doteq e^{-at}$ and a is a real constant :

$$Y_a(s) = \int_0^\infty e^{-at} e^{-st}\, dt = \left[\frac{e^{-(s+a)t}}{-(s+a)}\right]_0^\infty = \frac{1}{(s+a)} \quad \text{for } \Re(s) > -a$$

Example 2 (powers):

Find $Y_n(s) = \mathcal{L}[y_n](s)$, where $y_n(t) \doteq \dfrac{t^n}{n!} = \displaystyle\int_0^t y_{n-1}(\tau)d\tau$ and $n \geq 0$ is an integer :

$$Y_n(s) = \int_0^{\infty} \frac{t^n}{n!} e^{-st}\, dt = \left[\frac{t^n}{n!} \frac{e^{-st}}{-s} \right]_0^{\infty} + \int_0^{\infty} \frac{t^{(n-1)}}{(n-1)!} \frac{e^{-st}}{s} dt$$ ***integration by parts***

$$= \frac{1}{s} Y_{n-1}(s) = \frac{1}{s^n} Y_0(s) = \frac{1}{s^{(n+1)}}, \qquad \text{for } \Re(s) > 0$$

Property 1 (integration):

TRANSFORM POST INTEGRATION: With $y_{\text{int}}(t) \doteq \int_0^t y(\tau)d\tau$,

$$\begin{aligned}
Y_{\text{int}}(s) &= \int_0^\infty \left(\int_0^t y(\tau)d\tau\right) e^{-st}dt \\
&= \left[\left(\int_0^t y(\tau)d\tau\right)\frac{e^{-st}}{-s}\right]_0^\infty + \int_0^\infty y(t)\frac{e^{-st}}{s}dt \\
&= \frac{1}{s}Y(s) \qquad (s \in \{z : \Re(z) > 0\} \cap \text{ROC}(Y))
\end{aligned}$$

integration by parts

Property 2 (time shift):

TRANSFORM POST SHIFTING: With $y_\tau(t) = \{y(t-\tau)$ if $t \geq \tau;\ 0$ otherwise$\}$,

$$Y_\tau(s) = \int_\tau^\infty y(t-\tau)e^{-st}\,dt = \int_0^\infty y(\nu)e^{-s(\nu+\tau)}\,d\nu = e^{-s\tau}Y(s) \quad (s \in \mathrm{ROC}(Y))$$

Property 3 (exponential weighting):

TRANSFORM POST EXPONENTIAL WEIGHTING: With $y_z(t) = e^{zt} y(t)$,

$$Y_z(s) = \int_0^\infty y(t) e^{-(s-z)t}\, dt = Y(s-z) \quad (s \in \{\Re(z) + w \,:\, w \in \mathrm{ROC}(Y)\})$$

Property 4 (differentiation):

TRANSFORM POST DIFFERENTIATION:

integration by parts

$$\mathcal{L}[\dot{y}](s) = \int_0^\infty \frac{dy}{dt} e^{-st}\, dt = \left[y(t)e^{-st}\right]_0^\infty + s\int_0^\infty y(t)e^{-st}\, dt;$$

$$= sY(s) - y(0)$$

'initial conditions'

$$\mathcal{L}[\frac{d^n y}{dt^n}](s) = s^n Y(s) - s^{n-1}y(0) - s^{n-2}\dot{y}(0) - \ldots - \frac{d^{n-1}y}{dt^{n-1}}(0)$$

Property 5 (linearity)

LINEARITY OF THE TRANSFORM:

$$\mathcal{L}[a_1y_1 + a_2y_2](s) = a_1Y_1(s) + a_2Y_2(s) \quad \text{(because integration is linear)}$$

Properties 6 & 7 (limits)

With $Y(s) = \mathcal{L}[y](s)$, the following hold whenever the limits exist.

FINAL VALUE: $\lim_{t\to\infty} y(t) = \lim_{\sigma\to 0_+} (s\,Y(s))\Big|_{s=\sigma+j0}$

INITIAL VALUE: $\lim_{t\to 0} y(t) = \lim_{\sigma\to\infty} (s\,Y(s))\Big|_{s=\sigma+j0}$

NOTE: You need to always check that limits exist before applying formulas!
We will see that existence of these limits is related to the notion of "stability".

Proof

Suppose there exist constants $K, a > 0$ and y_∞ such that $|y(t) - y_\infty| < Ke^{-at}$. Then $y_\infty = \lim_{t\to\infty} y(t)$, $Y(s)$ is defined for $\Re(s) > 0$, and

$$\begin{aligned}
\lim_{\sigma\to 0_+} \left| \sigma \int_0^\infty y(t)e^{-\sigma t} dt - y_\infty \right| &= \lim_{s\to 0_+} \sigma \left| \int_0^\infty (y(t) - y_\infty)e^{-\sigma t} dt \right| \\
&\leq \lim_{\sigma\to 0_+} \sigma \int_0^\infty Ke^{-at}e^{-\sigma t} dt \leq \lim_{\sigma\to 0_+} \sigma \frac{K}{\sigma + a} = 0.
\end{aligned}$$

Transfer function

- Consider an input-output model:

$$\sum_{k=0}^{n} a_k \frac{d^k y}{dt^k} = \sum_{k=0}^{n} b_k \frac{d^k u}{dt^k}$$

- Assuming zero initial conditions:

$$\left(\sum_{k=0}^{n} a_k s^k\right) Y(s) = \left(\sum_{k=0}^{n} b_k s^k\right) U(s)$$

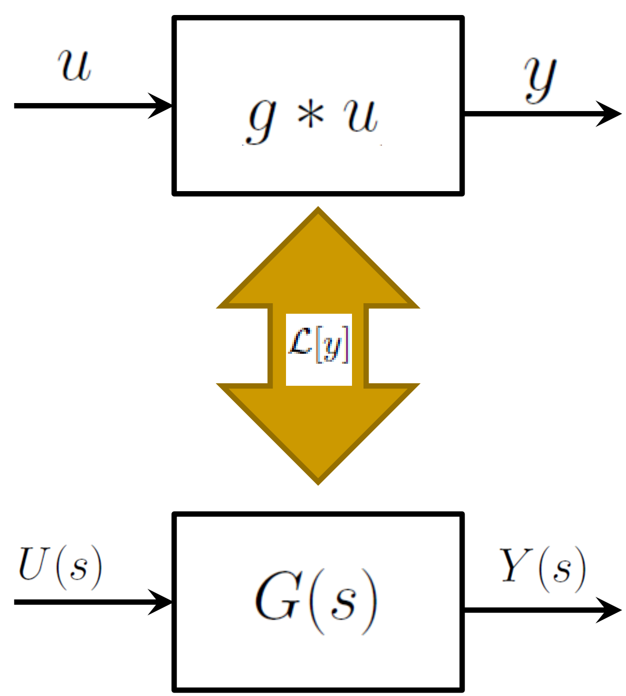
- This yields:

$$Y(s) = G(s)U(s) \text{ where } G(s) = \frac{B(s)}{A(s)}$$

'transfer function'

Property 8 (convolution)

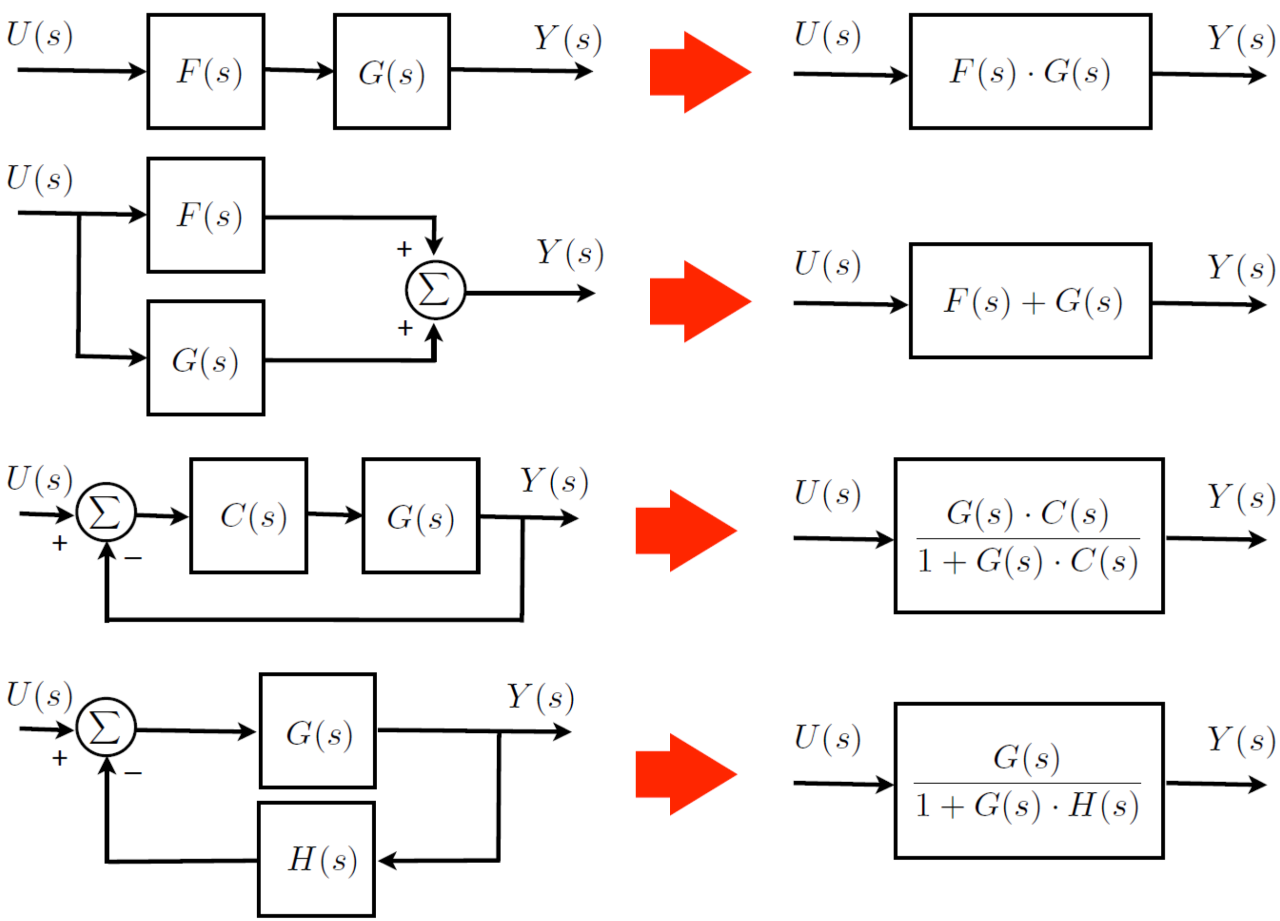
$$\begin{aligned}
\mathcal{L}[g*u](s) &= \int_0^\infty e^{-st}\left\{\int_0^t g(t-\tau)u(\tau)d\tau\right\}dt \\
&= \underbrace{\int_0^\infty \int_0^\infty e^{-st} g(t-\tau)u(\tau)d\tau dt}_{(\text{since } g(t-\tau)=0, t<\tau)} \\
&= \underbrace{\int_0^\infty \int_0^\infty e^{-st} g(t-\tau)u(\tau)dt d\tau}_{\text{Fubini's Theorem}} \\
&= \underbrace{\int_0^\infty \int_{-\tau}^\infty e^{-s(T+\tau)} g(T)u(\tau)dT d\tau}_{T:=t-\tau} \\
&= \underbrace{\int_0^\infty e^{-s\tau}u(\tau)\left\{\int_0^\infty e^{-sT} g(T)dT\right\}d\tau}_{(\text{since } g(t-\tau)=0, t<\tau)} \\
&= \int_0^\infty e^{-sT}g(T)dT \int_0^\infty e^{-s\tau}u(\tau)d\tau \\
&= G(s)U(s)
\end{aligned}$$



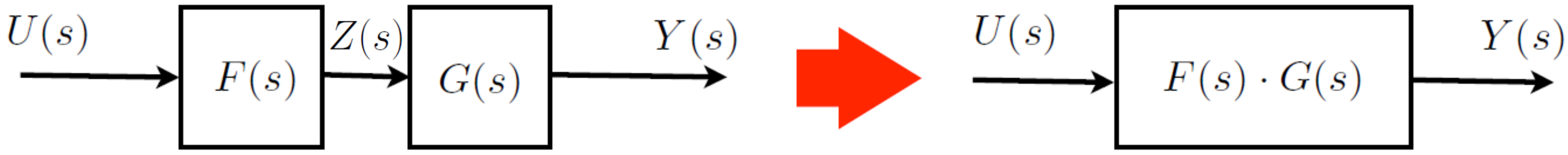
Comments:

- Transfer functions obtained assuming zero initial conditions.
- Transfer function = Laplace transform of system impulse response.
- Often easier to find the transfer function than the impulse response
- Convolution is difficult to do. But if we go to s-domain it becomes multiplication: easy.

Algebra of block diagrams



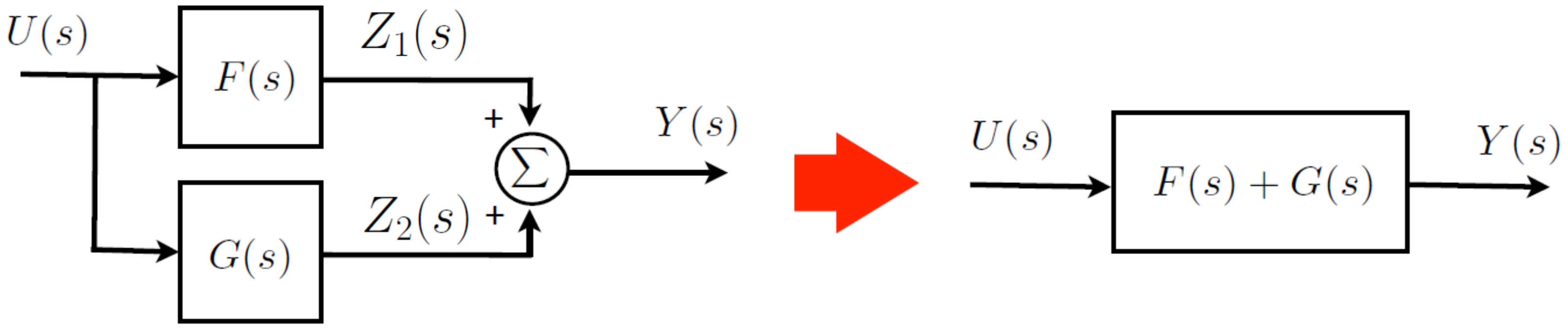
Series connection



$$\frac{Y(s)}{U(s)} = ?$$

$$\frac{Y(s)}{U(s)} = \frac{Z(s)}{U(s)} \frac{Y(s)}{Z(s)} = F(s)G(s)$$

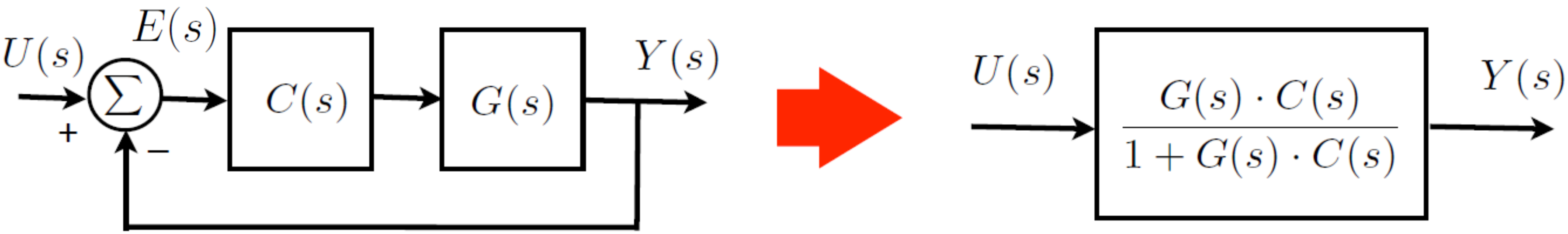
Parallel connection



$$\frac{Y(s)}{U(s)} = ?$$

$$\frac{Y(s)}{U(s)} = \frac{Z_1(s)+Z_2(s)}{U(s)} = \frac{Z_1(s)}{U(s)} + \frac{Z_2(s)}{U(s)} = F(s) + G(s)$$

Unity feedback connection



$$\frac{Y(s)}{U(s)} = ?$$

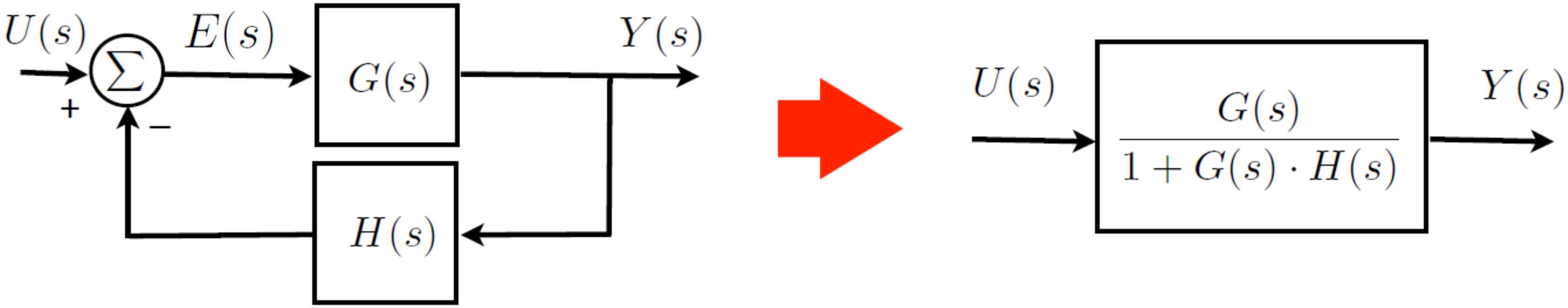
$Y(s) = C(s)G(s)E(s)$ Feedforward transfer function

$E(s) = U(s) - Y(s) = U(s) - C(s)G(s)E(s)$ Comparator

$$E(s) = \frac{U(s)}{1+C(s)G(s)}$$

$$\frac{Y(s)}{U(s)} = \frac{C(s)G(s)}{1+C(s)G(s)}$$

General feedback connection



$$\frac{Y(s)}{U(s)} = ?$$

$$Y(s) = G(s)E(s)$$ Feedforward connection

$$E(s) = U(s) - Y(s) = U(s) - H(s)Y(s)$$ Comparator + feedback

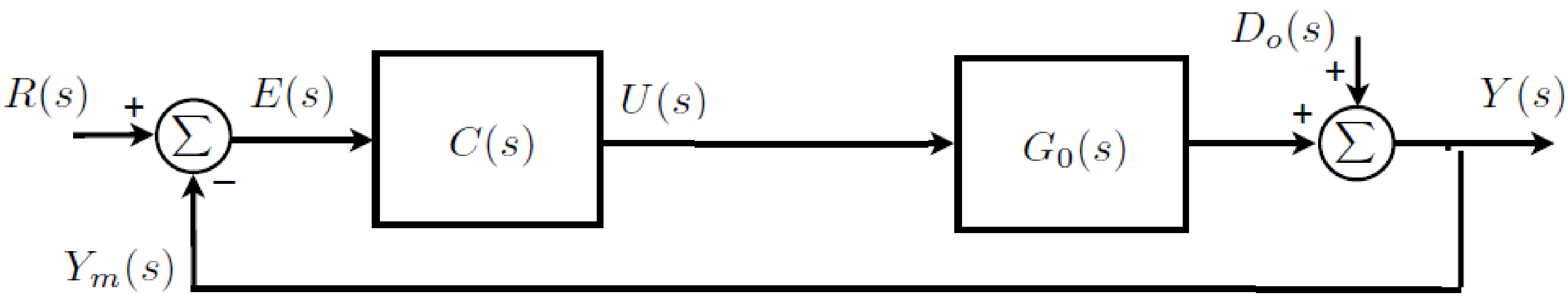
$$E(s) = U(s) - H(s)G(s)E(s)$$

$$E(s) = \frac{U(s)}{1+H(s)G(s)}$$

$$\frac{Y(s)}{U(s)} = \frac{G(s)}{1+H(s)G(s)}$$

Two inputs:

Turn off one input and compute the transfer function in the usual manner for the other input.



$$Y(s) = T_0(s)R(s) + S_0(s)D_o(s)$$

$$T_0(s) = \frac{Y(s)}{R(s)} = \frac{G_0(s)C(s)}{1 + G_0(s)C(s)} \qquad \text{with only } r \neq 0$$

$$S_0(s) = \frac{Y(s)}{D_o(s)} = \frac{1}{1 + G_0(s)C(s)} \qquad \text{with only } d_o \neq 0$$

This is how we will compute “sensitivity functions”.