Lecture 6

Poles & Zeros

Effect of poles on impulse responses

1st and 2nd order systems

Motivation

- It is very important for analysis/design to understand how poles/zeros affect impulse/step responses.
- Step response design specifications can be related to locations of poles and zeros

E.g. zero steady-state error tracking a step function with unity feedback → Open-loop pole at s=0

- It is possible to get a model of an LTI system by measuring its step response

Poles and zeros

- Consider a proper transfer function $n_r = n - m \geq 0$

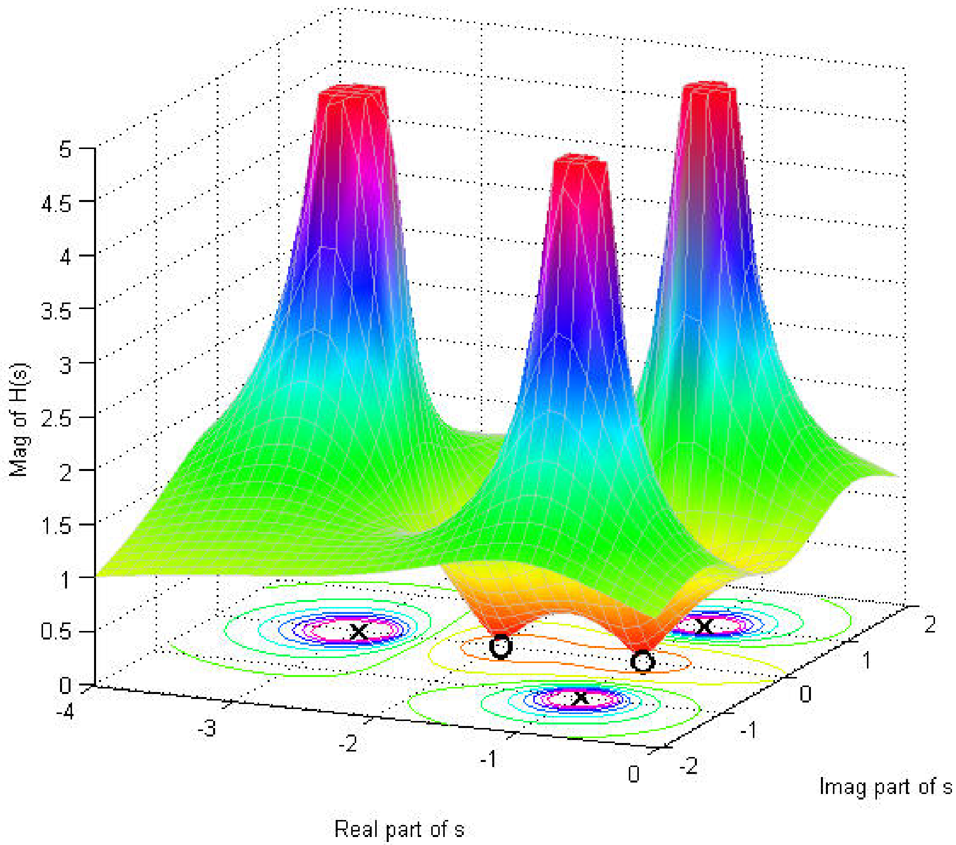
$$G(s) = \frac{b_m s^m + \cdots + b_1 s + b_0}{a_n s^n + \cdots + a_1 s + a_0} = K \frac{(s-\beta_1)\cdots(s-\beta_m)}{(s-\alpha_1)\cdots(s-\alpha_n)} = K \frac{\prod_{k=1}^{m}(s-\beta_k)}{\prod_{k=1}^{n}(s-\alpha_k)}$$

- If a_k, b_k are real, then α_k, β_k are either real or complex conjugate of the form

$$\sigma_k \pm j\omega_k$$

- β_k are the zeros of the transfer function.
- α_k are the poles of the transfer function.

Visualising Poles and Zeros in s-Domain

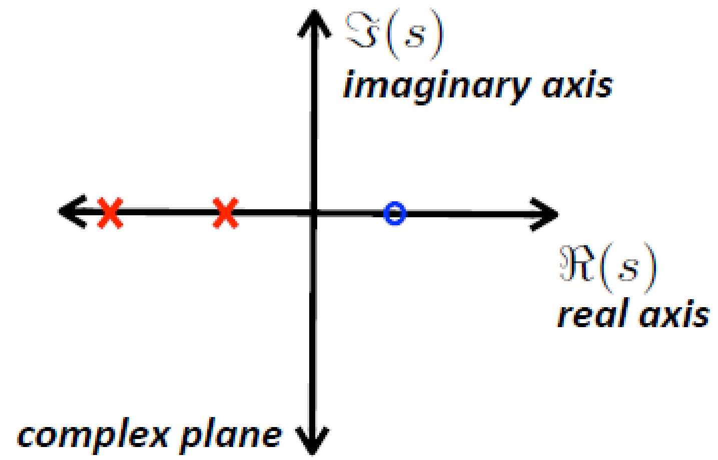


Source: Swarthmore College, US

Examples

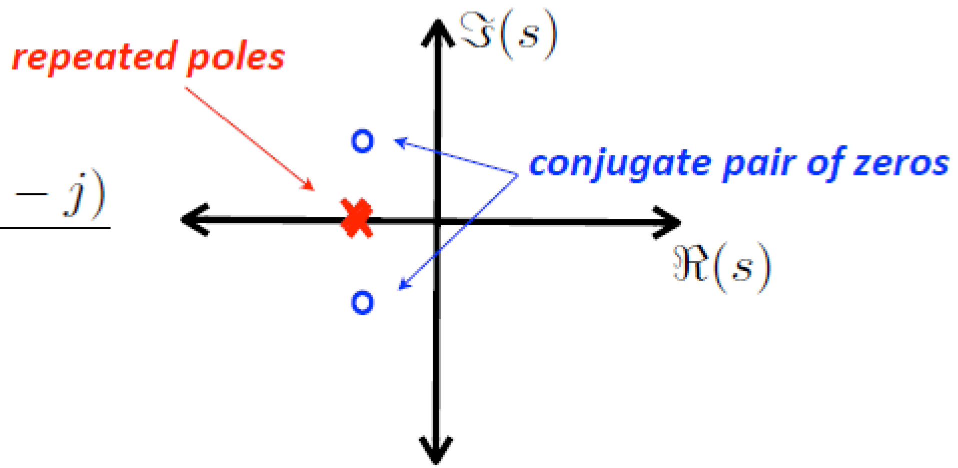
- Example 1:

$$\frac{s-1}{s^2+3s+2} = \frac{s-1}{(s+1)(s+2)}$$



- Example 2:

$$\frac{s^2+2s+2}{s^3+3s^2+3s+1} = \frac{(s+1+j)(s+1-j)}{(s+1)^3}$$



Effect of poles (and zeros) on transient response

- Qualitative- oscillatory or smooth; stable or unstable
- Quantitative – rate of decay/explosion, oscillation frequency, rise time, settling time,

Partial fractional expansion:

- We can write any rational function using its partial fractional expansion (different poles):

'partial fraction' expansion

$$G(s) = \frac{b_m s^m + \cdots + b_1 s + b_0}{a_n s^n + \cdots + a_1 s + a_0} = K \frac{\prod_{k=1}^{m}(s-\beta_k)}{\prod_{k=1}^{n}(s-\alpha_k)} = \sum_{k=1}^{n} \frac{B_k}{s-\alpha_k}$$

or for repeated poles:

$$\begin{aligned} G(s) \; = \; & \frac{B_{1,k_1}}{(s-\alpha_1)^{k_1}} + \frac{B_{1,k_1-1}}{(s-\alpha_1)^{k_1-1}} + \ldots + \frac{B_{1,1}}{s-\alpha_1} \\ & + \frac{B_{2,k_2}}{(s-\alpha_2)^{k_2}} + \frac{B_{2,k_2-1}}{(s-\alpha_2)^{k_2-1}} + \ldots + \frac{B_{2,1}}{s-\alpha_2} \\ & \vdots \qquad \vdots \qquad \vdots \\ & + \frac{B_{l,k_l}}{(s-\alpha_l)^{k_l}} + \frac{B_{l,k_l-1}}{(s-\alpha_l)^{k_l-1}} + \ldots + \frac{B_{l,1}}{s-\alpha_l} \end{aligned}$$

Solutions in time domain

- Taking inverse Laplace transform of the previous expressions leads to a set of possible time-domain functions that arise:

- Polynomial functions
- Exponentials in time
- Sinusoidal functions
- Their products

Impulse response of LTI systems

The impulse response of a real rational transfer function with strictly positive relative degree (i.e. strictly proper) is a linear combination of terms of the form: *if proper but not strictly, the impulse response also includes impulsive terms*

$e^{\alpha t}$ for each isolated real pole $s = \alpha$; $t^n e^{\alpha t}$ for each n-times repeat

$e^{\sigma t}\cos(\omega t + \phi)$ for each conjugate pair of complex poles $s = \sigma \pm j\omega$; and

$e^{\sigma t} t^n \cos(\omega t + \phi)$ for each n-times repeated complex conjugate pair $s = \sigma \pm j\omega$

Each such term is called a *mode* of the system.

- The growth rate *σ*, freq. *ω* and powers *n* depend only each corresponding pole
- But the linear coefficients in front of each mode, and the phase *ϕ*, also depend on other poles and zeros

Example:

- Consider the transfer function:

$$G(s) = \frac{s^2+1}{s(s-1)^3}$$

- Its impulse response is obtained via its partial fraction expansion ($\mathcal{L}\{\frac{t^{n-1}}{(n-1)!}e^{at}\} = \frac{1}{(s-a)^n}$):

$$G(s) = \frac{A}{s-1} + \frac{B}{(s-1)^2} + \frac{C}{(s-1)^3} + \frac{D}{s}$$

$$G(s) = \frac{1}{s-1} + \frac{2}{(s-1)^3} - \frac{1}{s}$$

$$\Longrightarrow \; g(t) = \mathcal{L}^{-1}\{G(s)\} = e^t + t^2e^t - 1, t \geq 0$$

Comments:

- When we have repeated poles, we get terms of this form in partial fraction expansion:

$$1/(s-\alpha)^{n+1}$$

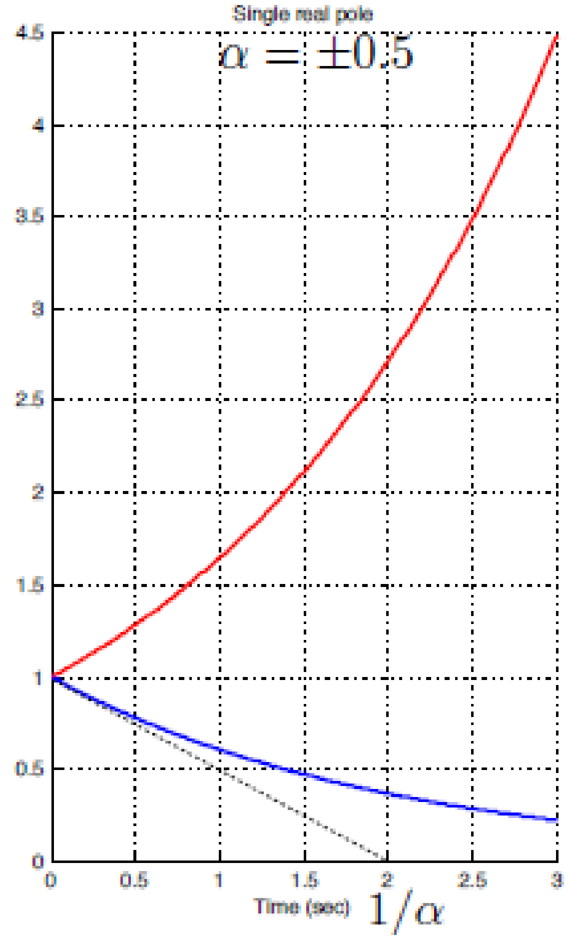
- They lead to response terms of the form

$$t^n e^{\alpha t} \quad \text{and} \quad e^{\sigma t} t^n \cos(\omega t + \phi)$$

- Note that response is still dominated by exponential terms provided $\Re(\alpha) \neq 0$.

1^{st} and 2^{nd} order systems

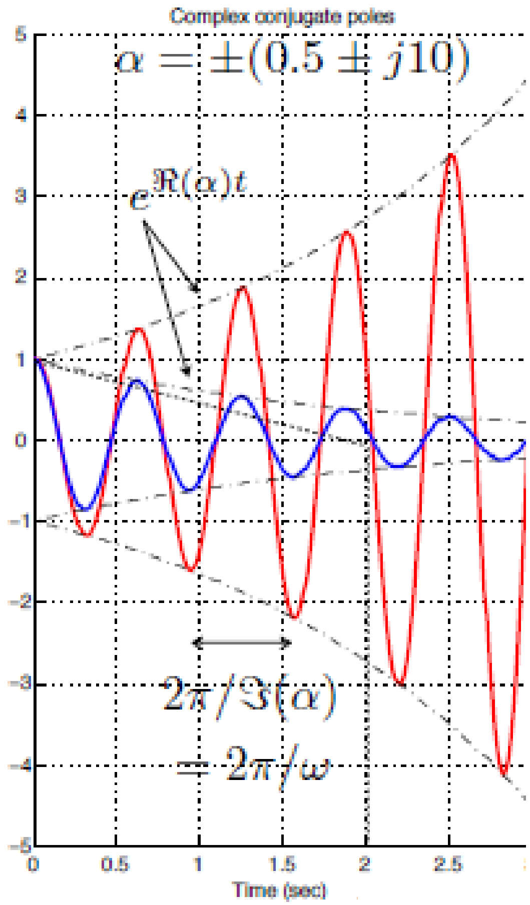
Impulse response terms with real pole



$$G(s) = \frac{1}{s-\alpha} \implies g(t) = \mathcal{L}^{-1}\{\frac{1}{s-\alpha}\} = e^{\alpha t}$$

'time-constants'

Impulse response with complex pole



'time-constants' $1/\Re(\alpha) = 1/\sigma$

When $\alpha = \sigma + j\omega$ we only consider

$\Re(e^{\alpha t}) = \Re(e^{\sigma t} e^{j\omega t}) = e^{\sigma t}\cos(\omega t)$

because there is a conjugate pole that contributes to cancel the imaginary part and reinforce the real part as the corresponding residue is also conjugate ...

note that the angle of the residue affects the sinusoidal term

$\Re(|B| e^{j\phi} e^{\alpha t}) = |B| e^{\sigma t}\cos(\omega t + \phi)$

E.g. recall: $\cos(\omega t) = \frac{1}{2}\left(e^{-j\omega t} + e^{j\omega t}\right)$

2nd order systems

- Consider a generic 2nd order system:

$$\ddot{y}(t) + 2\psi\omega_n\dot{y}(t) + \omega_n^2 y(t) = \omega_n^2 u(t) \quad \overset{\mathcal{L}}{\leftrightarrow} \quad G(s) = \frac{Y(s)}{U(s)} = \frac{\omega_n^2}{s^2 + 2\psi\omega_n s + \omega_n^2}$$

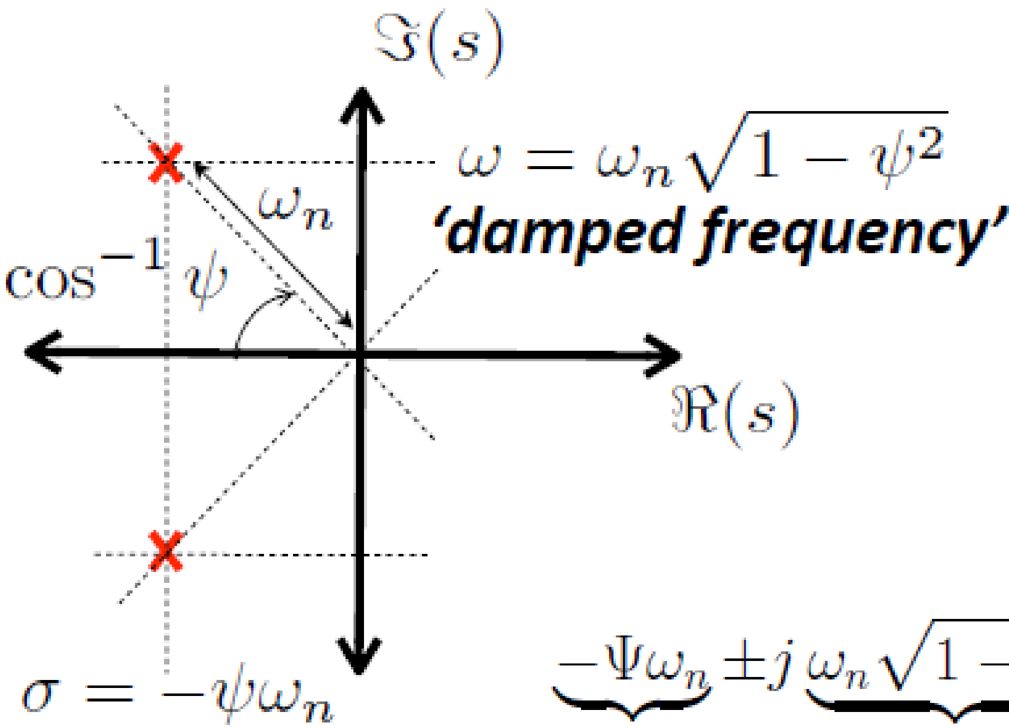
- Suppose $0 < \psi < 1$ then the poles are

$$\frac{-2\psi\omega_n \pm \sqrt{(2\psi\omega_n)^2 - 4\omega_n^2}}{2} = \underbrace{-\Psi\omega_n}_{\sigma} \pm j\,\underbrace{\omega_n\sqrt{1-\Psi^2}}_{\omega}$$

Terminology

- ω_n natural (undamped) frequency
- ω damped frequency (imaginary part)
- Ψ damping factor
- σ real part

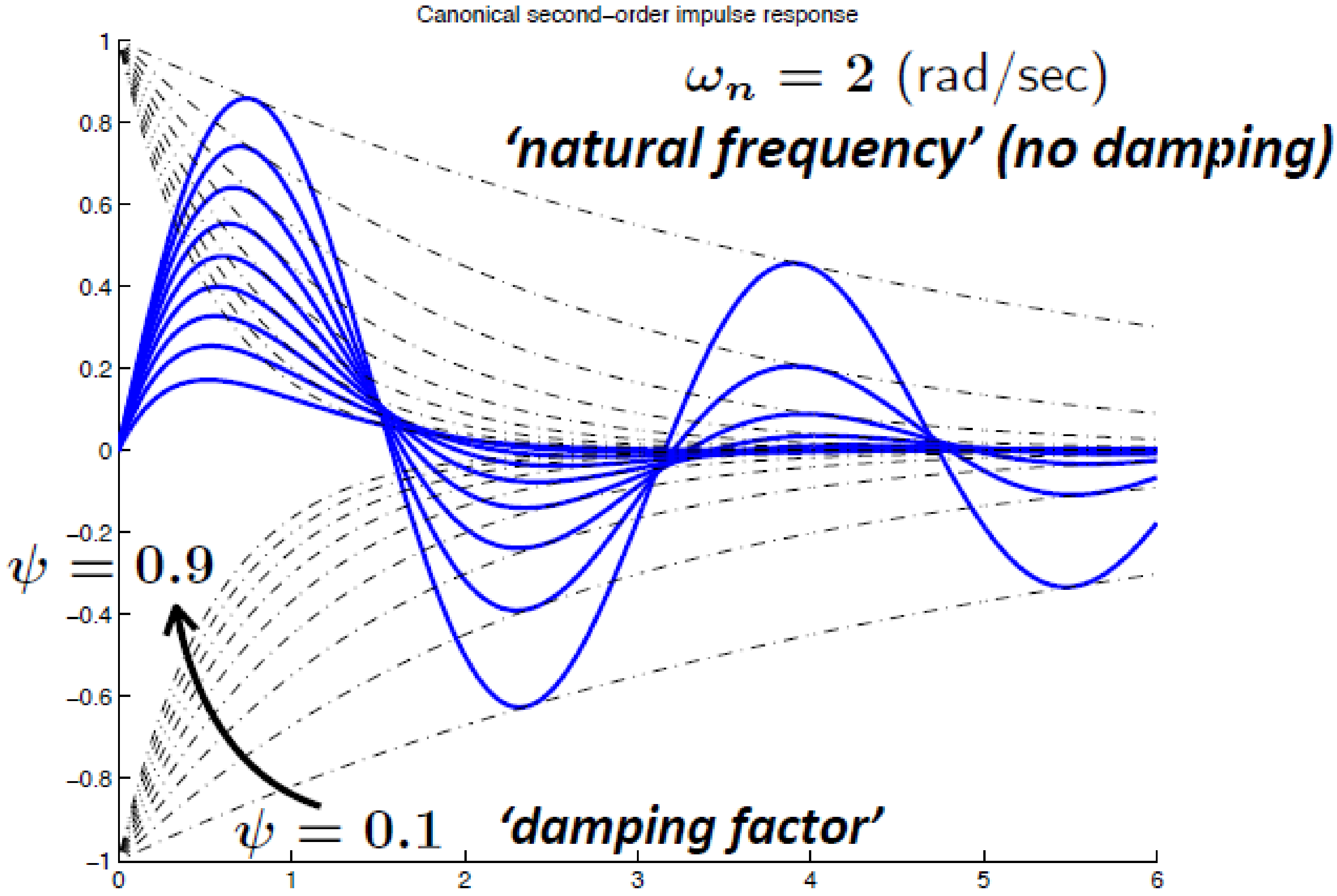
Graphic representation of poles



$$\underbrace{-\Psi\omega_n}_{\sigma} \pm j\, \underbrace{\omega_n\sqrt{1-\Psi^2}}_{\omega}$$

$$\sqrt{\sigma^2+\omega^2} = \sqrt{\Psi^2\omega_n^2+\omega_n^2(1-\Psi^2)} = \omega_n$$

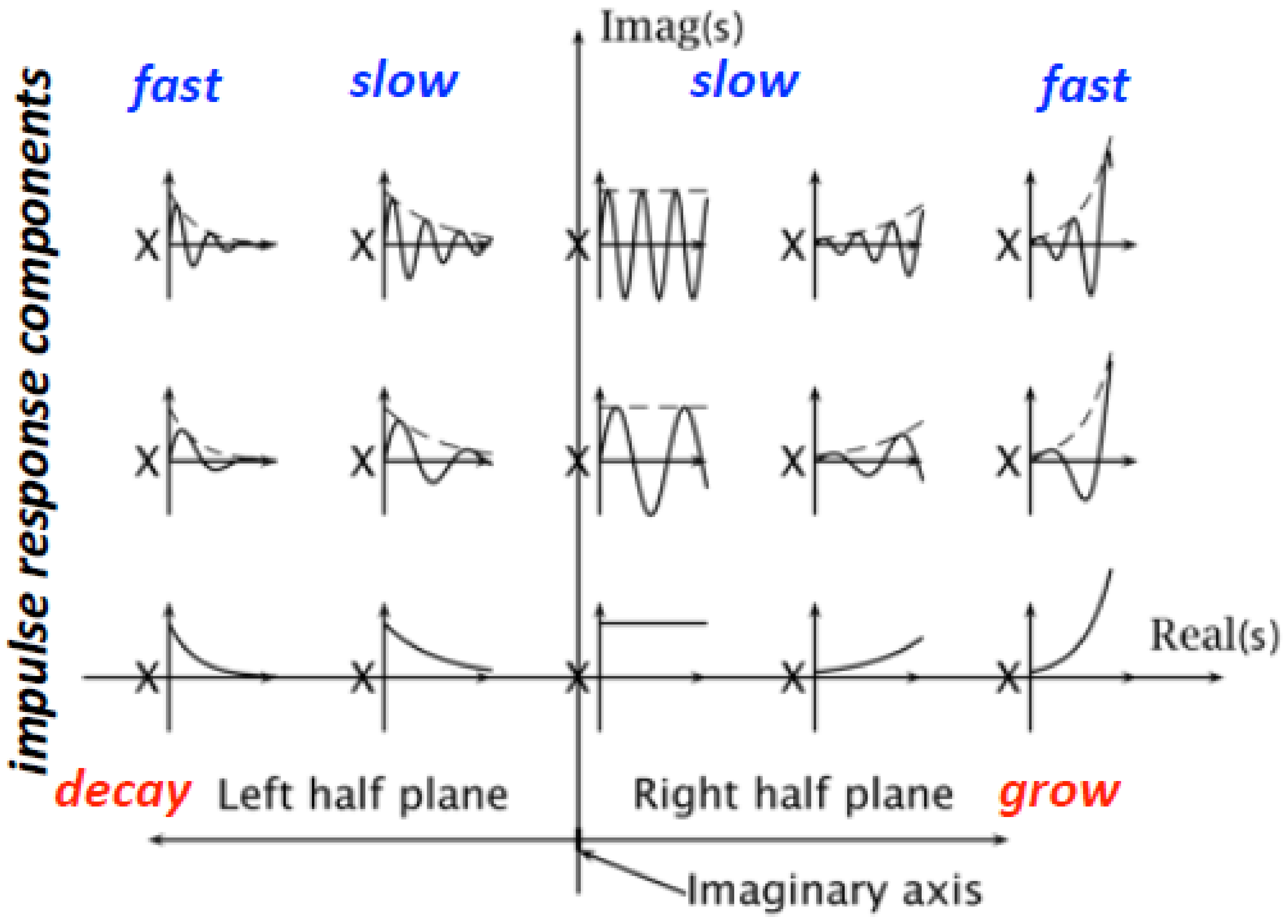
Response of 2nd order systems



Important relationships:

- Time constant increases as damping decreases.
- Peak of oscillation increases as damping decreases.
- Frequency of oscillation increases when damping decreases.

General systems – summary:



courtesy gv@eng.cam.ac.uk

Summary

- response due to poles with +ve real part *grows* and these are said to be **'unstable'**
- response due to poles with -ve real part *decays* and these are said to be **'stable'**
- response due to imaginary axis poles is bounded if not repeated, else response grows
- a stable response is ***dominated*** by **'slowest'** poles (i.e. closest to imaginary axis)