Chapter 2

MULTI-CRITERIA DECISION MAKING METHODS

2.1 BACKGROUND INFORMATION

With the continuing proliferation of decision methods and their variants, it is important to have an understanding of their comparative value. Each of the methods uses numeric techniques to help decision makers choose among a discrete set of alternative decisions. This is achieved on the basis of the impact of the alternatives on certain criteria and thereby on the overall utility of the decision maker(s). The difficulty that always occurs when trying to compare decision methods and choose the best one is that a **paradox** is reached, i.e., **What decision-making method should be used to choose the best decision-making method?** This problem is examined in Chapter 9.

Despite the criticism that multi-dimensional methods have received, some of them are widely used. The weighted sum model (WSM) is the earliest and probably the most widely used method. The weighted product model (WPM) can be considered as a modification of the WSM, and has been proposed in order to overcome some of its weaknesses. The analytic hierarchy process (AHP), as proposed by Saaty [Saaty, 1980 and 1994], is a later development and it has recently become increasingly popular. Professors Belton and Gear [1983] suggested a modification to the AHP (which we will call the *revised AHP*) that appears (as it is demonstrated in later chapters) to be more consistent than the original approach. Some other widely used methods are the ELECTRE and the TOPSIS methods.

In the section that follows these methods are presented in detail. In Chapter 9 the methods are tested in terms of two evaluative criteria. The same chapter also uses the test findings and examines the implication of these findings on the effectiveness of the various decision making approaches.

2.2 DESCRIPTION OF SOME MCDM METHODS

There are three steps in utilizing any decision-making technique involving numerical analysis of alternatives:

1) Determine the relevant criteria and alternatives.
2) Attach numerical measures to the relative importance

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of the criteria and to the impacts of the alternatives on these criteria.

3) *Process the numerical values to determine a ranking of each alternative.*

This section is **only** concerned with the way the WSM, WPM, AHP, revised AHP, ELECTRE, and TOPSIS methods process the numerical data in step 3. The central decision problem examined in this book is described as follows. Given is a set of m alternatives denoted as $A_1, A_2, A_3, \ldots, A_m$ and a set of n decision criteria denoted as $C_1, C_2, C_3, \ldots, C_n$. It is assumed that the decision maker has determined (the absolute or relative) performance value a_{ij} (for $i = 1, 2, 3, \ldots, m$ and $j = 1, 2, 3, \ldots, n$) of each alternative in terms of each criterion. That is, he/she has determined the matrix **A** with the a_{ij} values, along with the criteria weights w_j (i.e., the *decision matrix* as it was defined in Section 1.2). In the next chapter a number of procedures for determining these data is discussed.

Given the a_{ij} and w_j values, then the problem examined in this chapter is how one can rank the alternatives when all the decision criteria are considered simultaneously. In the following sections, the criteria are assumed to represent some kind of profit. That is, the higher the value, the better it is. Next, a number of MCDM models for solving the above problem (i.e., step 3 above) is presented.

2.2.1 The WSM Method

The weighted sum model (WSM) is probably the most commonly used approach, especially in single dimensional problems. If there are m alternatives and n criteria then, the best alternative is the one that satisfies (in the maximization case) the following expression [Fishburn, 1967]:

$$A^*_{WSM\text{-}score} = \max_i \sum_{j=1}^{n} a_{ij} w_j, \quad \textit{for} \quad i = 1, 2, 3, \ldots, m. \tag{2-1}$$

where: $A^*_{WSM\text{-}score}$ is the WSM score of the best alternative, n is the number of decision criteria, a_{ij} is the actual value of the *i-th* alternative in terms of the *j-th* criterion, and w_j is the weight of importance of the *j-th* criterion.

The assumption that governs this model is the **additive utility assumption.** That is, the total value of each alternative is equal to the sum of the products given as (2-1). In single-dimensional cases, where all the units are the same (e.g., dollars, feet, seconds), the WSM can be used without difficulty. Difficulty with this method emerges when it is applied to multi- dimensional MCDM problems. Then, in combining different dimensions, and consequently different units, the additive utility assumption is violated and the result is equivalent to *"adding apples and oranges"*.

Example 2-1:

Suppose that an MCDM problem involves four criteria, *which are expressed in exactly the same unit,* and three alternatives. The relative weights of the four criteria were determined to be: $w_1 = 0.20$, $w_2 = 0.15$, $w_3 = 0.40$, and $w_4 = 0.25$. Also, the performance values of the three alternatives in terms of the four decision criteria are assumed to be as follows:

$$A = \begin{bmatrix} 25 & 20 & 15 & 30 \\ 10 & 30 & 20 & 30 \\ 30 & 10 & 30 & 10 \end{bmatrix}.$$

Therefore, the data for this MCDM problem are summarized in the following decision matrix:

	Criteria			
	C_1	C_2	C_3	C_4
Alts.	(0.20	0.15	0.40	0.25)
A_1	25	20	15	30
A_2	10	30	20	30
A_3	30	10	30	10

When formula (2-1) is applied on the previous data the scores of the three alternatives are:

$$A_{1,\ WSM\text{-}score} = 25\times0.20 + 20\times0.15 + 15\times0.40 + 30\times0.25 = $$
$$= 21.50.$$

Similarly, we get:

$$A_{2,\ WSM\text{-}score} = 22.00,$$

and $A_{3,\ WSM\text{-}score} = 20.00.$

Therefore, the best alternative (in the maximization case) is alternative A_2 (because it has the highest WSM score; 22.00). Moreover, the following ranking is derived: $A_2 > A_1 > A_3$ (where the symbol "$>$" stands for *"better than"*). ■

2.2.2 The WPM Method

The weighted product model (WPM) is very similar to the WSM. The main difference is that instead of addition in the model there is multiplication. Each alternative is compared with the others by multiplying a number of ratios, one for each criterion. Each ratio is raised to the power equivalent to the relative weight of the corresponding criterion. In general, in order to compare two alternatives A_K and A_L, the following product (Bridgman [1922] and Miller and Starr [1969]) has to be calculated:

$$R(A_K/A_L) = \prod_{j=1}^{n} (a_{Kj}/a_{Lj})^{w_j}, \qquad (2\text{-}2)$$

where n is the number of criteria, a_{ij} is the actual value of the *i-th* alternative in terms of the *j-th* criterion, and w_j is the weight of importance of the *j-th* criterion.

If the term $R(A_K / A_L)$ is greater than or equal to one, then it indicates that alternative A_K is more desirable than alternative A_L (in the maximization case). The best alternative is the one that is better than or at least equal to all other alternatives.

The WPM is sometimes called **dimensionless analysis** because its structure eliminates any units of measure. Thus, the WPM can be used in single- and multi-dimensional MCDM. An advantage of the method is that instead of the actual values it can use relative ones. This is true because:

$$\frac{a_{Kj}}{a_{Lj}} = \frac{a_{Kj}/\sum_{i=1}^{n} a_{Ki}}{a_{Lj}/\sum_{i=1}^{n} a_{Li}} = \frac{a'_{Kj}}{a'_{Lj}}. \qquad (2\text{-}3)$$

A relative value a'_{Kj} is calculated using the formula: $a'_{Kj} = a_{Kj}/\sum_{i=1}^{n} a_{Ki}$, where the a_{Kj}'s are the actual values.

Example 2-2:

Consider the problem presented in the previous Example 2-1. However, now the restriction to express all criteria in terms of the same unit is not needed. When the WPM is applied, then the following values are derived:

$$R(A_1/A_2) = (25/10)^{0.20} \times (20/30)^{0.15} \times (15/20)^{0.40} \times (30/30)^{0.25} = $$
$$= 1.007 > 1.$$

Similarly, we also get:

$$R(A_1/A_3) = 1.067 > 1,$$

and $R(A_2/A_3) = 1.059 > 1.$
Therefore, the best alternative is A_1, since it is superior to all the other alternatives. Moreover, the ranking of these alternatives is as follows: $A_1 > A_2 > A_3$. ■

An alternative approach with the WPM method is for the decision maker to use only products without ratios. That is, to use the following variant of formula (2-2):

$$P(A_K) = \prod_{j=1}^{n} \left(a_{Kj}\right)^{w_j}, \tag{2-4}$$

In the previous expression the term $P(A_k)$ denotes the performance value (not a relative one) of alternative A_k when all the criteria are considered under the WPM model. Then, when the previous data are used, exactly the same ranking is derived. Some interesting properties of this method are discussed in Chapter 11, Sections 11.6 and 11.7.

2.2.3 The AHP Method

The analytic hierarchy process (AHP) ([Saaty, 1980 and 1994]) decomposes a complex MCDM problem into a system of hierarchies (more on these hierarchies can be found in [Saaty, 1980]). The final step in the AHP deals with the structure of an $m \times n$ matrix (where m is the number of alternatives and n is the number of criteria). The matrix is constructed by using the relative importances of the alternatives in terms of each criterion. The vector $(a_{i1}, a_{i2}, a_{i3}, \ldots, a_{in})$ for each i is the principal eigenvector of an $n \times n$ reciprocal matrix which is determined by pairwise comparisons of the impact of the m alternatives on the *i-th* criterion (more on this, and other related techniques, is presented in Chapter 3).

The importance of the AHP, its variants, and the use of pairwise comparisons in decision making is best illustrated in the more than 1,000 references cited in [Saaty, 1994]. A number of special issues in refereed journals have been devoted to the AHP and the use of pairwise comparisons in decision making. These issues are: *Socio-Economic Planning Sciences* [Vol. 10, No. 6, 1986]; *Mathematical Modelling* [Vol. 9, No. 3-5, 1987]; *European Journal of Operational Research* [Vol. 48, No. 1, 1990]; and *Mathematical and Computer Modelling* [Vol. 17, No. 4/5, 1993]. Also, four international symposia (called *ISAHP*) have been dedicated on the same topic so far and one such event is now scheduled every two years.

Some evidence is presented in [Saaty, 1980]) (see also Chapters 3 and

4 in this book) which supports the technique of pairwise comparisons for eliciting numerical evaluations of qualitative phenomena from experts and decision makers. However, here we are not concerned with the possible advantages and disadvantages of the pairwise comparison and eigenvector methods for determining the a_{ij} values. Instead, we examine the method used in the AHP to process the a_{ij} values **after** they have been determined. The entry a_{ij}, in the $m \times n$ matrix, represents the relative value of alternative A_i when it is considered in terms of criterion C_j. In the original AHP the sum $\sum_{i=1}^{n} a_{ij}$ is equal to one.

According to the AHP the best alternative (in the maximization case) is indicated by the following relationship (2-5):

$$A^*_{AHP\text{-}score} = \max_i \sum_{j=1}^{n} a_{ij} w_j, \quad for \ \ i = 1, 2, 3, ..., m. \qquad (2\text{-}5)$$

The similarity between the WSM and the AHP is clear. The AHP uses relative values instead of actual ones. Thus, it can be used in single- or multi-dimensional decision making problems.

Example 2-3:

As before, we consider the data used in the previous two examples (note that as in the WPM case the restriction to express all criteria in terms of the same unit is not needed). The AHP uses a series of pairwise comparisons (more on this can be found in Chapter 3) to determine the relative performance of each alternative in terms of each of the decision criteria. In other words, instead of the absolute data, the AHP would use the following relative data:

	Criteria			
	C_1	C_2	C_3	C_4
Alts.	(0.20	0.15	0.40	0.25)
A_1	25/65	20/55	15/65	30/65
A_2	10/65	30/55	20/65	30/65
A_3	30/65	5/55	30/65	5/65

That is, the columns in the decision matrix have been normalized to add up to one. When formula (2-5) is applied on the previous data, the following scores are derived:

$$A_{1,\ AHP\text{-}score} = (25/65)\times 0.20 + (20/55)\times 0.15 + \\ + (15/65)\times 0.40 + (30/65)\times 0.25 =$$

$= 0.34.$

Similarly, we get:

$A_{2,\ AHP\text{-}score} = 0.35,$

and $A_{3,\ AHP\text{-}score} = 0.31.$

Therefore, the best alternative (in the maximization case) is alternative A_2 (because it has the highest AHP score; 0.35). Moreover, the following ranking is derived: $A_2 > A_1 > A_3$. ■

2.2.4 The Revised AHP Method

Belton and Gear [1983] proposed a revised version of the original AHP model. They demonstrated that a ranking inconsistency can occur when the AHP is used. A numerical example was presented that consists of three criteria and three alternatives. In that example (which is also shown next) the indication of the best alternative changes when an identical alternative to one of the nonoptimal alternatives is introduced now creating four alternatives. According to Belton and Gear the root for that inconsistency is the fact that the relative values for each criterion sum up to one. Instead of having the relative values of the alternatives $A_1, A_2, A_3, \ldots, A_m$ sum up to one, they proposed to divide each relative value by the maximum value of the relative values. In particular, they elaborated on the following example.

<u>**Example 2-4**:</u> (from [Belton and Gear, 1983], p. 228)

Suppose that the actual data of a MCDM problem with three alternatives and three criteria are as follows:

	Criteria		
	C_1	C_2	C_3
Alts.	(1/3	1/3	1/3)
A_1	1	9	8
A_2	9	1	9
A_3	1	1	1

In real life problems the decision maker may never know the previous real data. Instead, he/she can use the method of pairwise comparisons (as described in Chapters 3 and 4) to derive the relative data. When the AHP is applied on the previous data, the following decision matrix with the relative data is derived:

	Criteria		
	C_1	C_2	C_3
Alts.	(1/3	1/3	1/3)
A_1	1/11	9/11	8/18
A_2	9/11	1/11	9/18
A_3	1/11	1/11	1/18

Therefore, it can be easily shown that the vector with the final AHP scores is: (0.45, 0.47, 0.08). That is, the three alternatives are ranked as follows: $A_2 > A_1 > A_3$.

Next, we introduce a new alternative, say A_4, which is an **identical copy** of the existing alternative A_2 (i.e., $A_2 \equiv A_4$). Furthermore, it is also assumed that the relative weights of importance of the three criteria remain the same (i.e., 1/3, 1/3, 1/3). When the new alternative A_4 is considered, it can be easily verified that the new decision matrix is as follows:

	Criteria		
	C_1	C_2	C_3
Alts.	(1/3	1/3	1/3)
A_1	1/20	9/12	8/27
A_2	9/20	1/12	9/27
A_3	1/20	1/12	1/27
A_4	9/20	1/12	9/27

Similarly as above, it can be verified that the vector with the final AHP scores is: (0.37, 0.29, 0.06, 0.29). That is, the four alternatives are ranked as follows: $A_1 > A_2 \equiv A_4 > A_3$. Belton and Gear claimed that this result is in logical contradiction with the previous result (in which $A_2 > A_1$).

When the revised AHP is applied on the last data (that is, when the data are normalized by dividing by the largest entry in each column), the following decision matrix is derived:

	Criteria		
	C_1	C_2	C_3
Alts.	(1/3	1/3	1/3)
A_1	1/9	1	8/9
A_2	1	1/9	1
A_3	1/9	1/9	1/9
A_4	1	1/9	1

The vector with the final scores is: (2/3, 19/27, 1/9, 19/27). That is, the four alternatives are ranked as follows: $A_2 \equiv A_4 > A_1 > A_3$. The last ranking is, obviously, the desired one. ■

The revised AHP was sharply criticized by Saaty in [1990]. He claimed that identical alternatives should not be considered in the decision process. However, even earlier Triantaphyllou and Mann in [1989] had demonstrated that similar logical contradictions are possible with the original AHP, as well as with the revised AHP, even when non-identical alternatives are introduced (see also Section 9.4 in Chapter 9). Some discussion on this controversy is also presented in Chapter 11.

Two other MCDM methods are presented next. These methods are of limited acceptance by the scientific and practitioners communities. These are the ELECTRE and TOPSIS methods.

2.2.5 The ELECTRE Method

The ELECTRE (for *Elimination and Choice Translating Reality*; English translation from the French original) method was first introduced in [Benayoun, *et al.*, 1966]. The basic concept of the ELECTRE method is to deal with *"outranking relations"* by using pairwise comparisons among alternatives under each one of the criteria separately. The outranking relationship of the two alternatives A_i and A_j, denoted as $A_i \rightarrow A_j$, describes that even when the *i-th* alternative does not dominate the *j-th* alternative quantitatively, then the decision maker may still take the risk of regarding A_i as almost surely better than A_j [Roy, 1973]. Alternatives are said to be **dominated**, if there is another alternative which excels them in one or more criteria and equals in the remaining criteria.

The ELECTRE method begins with pairwise comparisons of alternatives under each criterion. Using physical or monetary values, denoted as $g_i(A_j)$ and $g_i(A_k)$ of the alternatives A_j and A_k respectively, and by introducing threshold levels for the difference $g_i(A_j) - g_i(A_k)$, the decision maker may declare that he/she is indifferent between the alternatives under consideration, that he/she has a weak or a strict preference for one of the two, or that he/she is unable to express any of these preference relations. Therefore, a set of binary relations of alternatives, the so-called **outranking relations**, may be complete or incomplete. Next, the decision maker is requested to assign weights or importance factors to the criteria in order to express their relative importance.

Through the consecutive assessments of the outranking relations of the alternatives, the ELECTRE method elicits the so-called **concordance index**, defined as the amount of evidence to support the conclusion that alternative A_j outranks, or dominates, alternative A_k, as well as the **discordance index**, the counter-part of the concordance index.

Finally, the ELECTRE method yields a system of binary outranking relations between the alternatives. Because this system is not necessarily complete, the ELECTRE method is sometimes unable to identify the most preferred alternative. It only produces a core of leading alternatives. This method has a clearer view of alternatives by eliminating less favorable ones. This method is especially convenient when there are decision problems that involve a few criteria with a large number of alternatives [Lootsma, 1990].

There are many variants of the ELECTRE method. The organization of the original version of the ELECTRE method is illustrated in the following steps [Benayoun, *et al.*, 1966]:

Step 1: Normalizing the Decision Matrix

This procedure transforms the entries of the decision matrix into dimensionless comparable entries by using the following equation:

$$x_{ij} = \frac{a_{ij}}{\sqrt{\sum_{k=1}^{m} a_{kj}^2}}. \qquad (2\text{-}6)$$

Therefore, the normalized matrix **X** is defined as follows:

$$X = \begin{bmatrix} x_{11} & x_{12} & x_{13} & \cdots & x_{1n} \\ x_{21} & x_{22} & x_{23} & \cdots & x_{2n} \\ \cdot & & & & \cdot \\ \cdot & & & & \cdot \\ \cdot & & & & \cdot \\ x_{m1} & x_{m2} & x_{m3} & \cdots & x_{mn} \end{bmatrix},$$

where m is the number of alternatives, n is the number of criteria, and x_{ij} is the normalized preference measure of the *i-th* alternative in terms of the *j-th* criterion.

Step 2: Weighting the Normalized Decision Matrix

Next, each one of the columns of the previous **X** matrix is multiplied by the associated weight of importance of the corresponding decision criterion. These weights, denoted as $(w_1, w_2, w_3, \ldots, w_n)$, were determined by the decision maker. Therefore, the weighted matrix, denoted as **Y**, is:

$$\mathbf{Y} = \mathbf{X}\,\mathbf{W}, \quad \text{or:}$$

$$\boldsymbol{Y} = \begin{bmatrix} y_{11} & y_{12} & y_{13} & \cdots & y_{1n} \\ y_{21} & y_{22} & y_{23} & \cdots & y_{2n} \\ \cdot & & & & \cdot \\ \cdot & & & & \cdot \\ \cdot & & & & \cdot \\ y_{m1} & y_{m2} & y_{m3} & \cdots & y_{mn} \end{bmatrix} =$$

$$= \begin{bmatrix} w_1x_{11} & w_2x_{12} & w_3x_{13} & \cdots & w_nx_{1n} \\ w_1x_{21} & w_2x_{22} & w_3x_{23} & \cdots & w_nx_{2n} \\ \cdot & & & & \cdot \\ \cdot & & & & \cdot \\ \cdot & & & & \cdot \\ w_1x_{m1} & w_2x_{m2} & w_3x_{m3} & \cdots & w_nx_{mn} \end{bmatrix},$$

where:

$$\boldsymbol{W} = \begin{bmatrix} w_1 & 0 & 0 & \ldots & 0 \\ 0 & w_2 & 0 & \ldots & 0 \\ \cdot & & & & \cdot \\ \cdot & & & & \cdot \\ \cdot & & & & \cdot \\ 0 & 0 & 0 & \ldots & w_n \end{bmatrix}, \quad \textit{and} \quad \sum_{i=1}^{n} w_i = 1.$$

Step 3: Determine the Concordance and Discordance Sets

The **concordance set** C_{kl} of two alternatives A_k and A_l, where $m \geq k, l \geq 1$, is defined as the set of all the criteria for which A_k is preferred to A_l. That is, the following is true:

$$C_{kl} = \{j,\ y_{kj} \geq y_{lj}\}, \quad \text{for } j = 1, 2, 3, \ldots, n.$$

The complementary subset is called the **discordance set** and it is described as follows:

$$D_{kl} = \{j,\ y_{kj} < y_{lj}\}, \quad \text{for } j = 1, 2, 3, \ldots, n.$$

Step 4: Construct the Concordance and Discordance Matrices

The relative value of the elements in the concordance matrix **C** is calculated by means of the concordance index. The **concordance index** c_{kl} is the sum of the weights associated with the criteria contained in the concordance set. That is, the following is true:

$$C_{kl} = \sum_{j \in C_{kl}} w_j, \quad for \quad j = 1, 2, 3, ..., n.$$

The concordance index indicates the relative importance of alternative A_k with respect to alternative A_l. Apparently, $0 \leq c_{kl} \leq 1$. The **concordance matrix** **C** is defined as follows:

$$C = \begin{bmatrix} - & c_{12} & c_{13} & \cdots & c_{1m} \\ c_{21} & - & c_{23} & \cdots & c_{2m} \\ \cdot & & & & \cdot \\ \cdot & & & & \cdot \\ \cdot & & & & \cdot \\ c_{m1} & c_{m2} & c_{m3} & \cdots & - \end{bmatrix},$$

where the entries of matrix **C** are not defined when $k = l$.

The **discordance matrix** **D** expresses the degree that a certain alternative A_k is worse than a competing alternative A_l. The elements d_{kl} of the discordance matrix are defined as follows:

$$d_{kl} = \frac{\max\limits_{j \in D_{kl}} |y_{kj} - y_{lj}|}{\max\limits_{j} |y_{kj} - y_{lj}|}. \tag{2-7}$$

The discordance matrix is defined as follows:

$$D = \begin{bmatrix} - & d_{12} & d_{13} & \cdots & d_{1m} \\ d_{21} & - & d_{23} & \cdots & d_{2m} \\ \cdot & & & & \cdot \\ \cdot & & & & \cdot \\ \cdot & & & & \cdot \\ d_{m1} & d_{m2} & d_{m3} & \cdots & - \end{bmatrix},$$

as with the **C** matrix, the entries of matrix **D** are not defined when $k = l$. Finally, it should be noted here that the previous two $m \times m$ matrices are

not symmetric.

Step 5: Determine the Concordance and Discordance Dominance Matrices

The concordance dominance matrix is constructed by means of a threshold value for the concordance index. For example, alternative A_k will only have a chance to dominate alternative A_l if its corresponding concordance index c_{kl} exceeds at least a certain threshold value $\underline{c}$. That is, this happens if the following condition is true:

$$c_{kl} \geq \underline{c}.$$

The threshold value $\underline{c}$ can be determined as the average concordance index. That is, the following relation could be true:

$$\underline{c} = \frac{1}{m(m-1)} \sum_{\substack{k=1 \\ \text{and } k \neq l}}^{m} \sum_{\substack{l=1 \\ \text{and } l \neq k}}^{m} c_{kl}. \tag{2-8}$$

Based on the threshold value, the elements of the **concordance dominance matrix** **F** are next determined as follows:

$$f_{kl} = 1, \quad \text{if } c_{kl} \geq \underline{c},$$
$$f_{kl} = 0, \quad \text{if } c_{kl} < \underline{c}.$$

Similarly, the **discordance dominance matrix** **G** is defined by using a threshold value $\underline{d}$, where $\underline{d}$ could be defined as follows:

$$\underline{d} = \frac{1}{m(m-1)} \sum_{\substack{k=1 \\ \text{and } k \neq l}}^{m} \sum_{\substack{l=1 \\ \text{and } l \neq k}}^{m} d_{kl}, \tag{2-9}$$

and

$$g_{kl} = 1, \quad \text{if } d_{kl} \geq \underline{d},$$
$$g_{kl} = 0, \quad \text{if } d_{kl} < \underline{d}.$$

Step 6: Determine the Aggregate Dominance Matrix

The elements of the **aggregate dominance matrix** **E** are next defined as follows:

$$e_{kl} = f_{kl} \times g_{kl}. \tag{2-10}$$

Step 7: Eliminate the Less Favorable Alternatives

From the aggregate dominance matrix one can derive a partial preference ordering of the alternatives. If $e_{kl} = 1$, then this means that alternative A_k is preferred to alternative A_l by using both the concordance and discordance criteria.

If any column of the aggregate dominance matrix has at least one element equal to 1, then this column is *"ELECTREally"* dominated by the corresponding row. Therefore, one can simply eliminate any column(s) which have an element equal to one. Then, the best alternative is the one which dominates all other alternatives in this manner.

2.2.6 The TOPSIS Method

TOPSIS (for *the Technique for Order Preference by Similarity to Ideal Solution*) was developed by Yoon and Hwang [1980] as an alternative to the ELECTRE method and can be considered as one of its most widely accepted variants. The basic concept of this method is that the selected alternative should have the shortest distance from the ideal solution and the farthest distance from the negative-ideal solution in some geometrical sense.

The TOPSIS method assumes that each criterion has a tendency of monotonically increasing or decreasing utility. Therefore, it is easy to define the ideal and negative-ideal solutions. The Euclidean distance approach was proposed to evaluate the relative closeness of the alternatives to the ideal solution. Thus, the preference order of the alternatives can be derived by a series of comparisons of these relative distances.

The TOPSIS method evaluates the following decision matrix which refers to m alternatives which are evaluated in terms of n criteria:

$$D = \begin{bmatrix} x_{11} & x_{12} & x_{13} & \cdots & x_{1n} \\ x_{21} & x_{22} & x_{23} & \cdots & x_{2n} \\ \cdot & & & & \cdot \\ \cdot & & & & \cdot \\ \cdot & & & & \cdot \\ x_{m1} & x_{m2} & x_{m3} & \cdots & x_{mn} \end{bmatrix},$$

where x_{ij} denotes the performance measure of the *i-th* alternative in terms of the *j-th* criterion. Next, the steps of the TOPSIS method are presented.

<u>Step 1: Construct the Normalized Decision Matrix</u>

The TOPSIS method first converts the various criteria dimensions into non-dimensional criteria as was the case with the ELECTRE method (i.e., relation (2-6)). An element r_{ij} of the normalized decision matrix **R** is thus calculated as follows:

$$r_{ij} = \frac{x_{ij}}{\sqrt{\sum_{k=1}^{m} x_{kj}^2}}. \tag{2-11}$$

As a remark, it should be stated here that in the ELECTRE and TOPSIS methods the Euclidean distances defined in expressions (2-6) and (2-11), respectively, represent some plausible assumptions. Other alternative distance measures could be used as well, in which case it is possible for one to get different answers for the same problem.

Step 2: Construct the Weighted Normalized Decision Matrix

A set of weights $W = (w_1, w_2, w_3, \ldots, w_n)$, (where: $\Sigma w_i = 1$) defined by the decision maker is next used with the decision matrix to generate the **weighted normalized matrix** **V** as follows:

$$V = \begin{bmatrix} w_1 r_{11} & w_2 r_{12} & w_3 r_{13} & \cdots & w_n r_{1n} \\ w_1 r_{21} & w_2 r_{22} & w_3 r_{23} & \cdots & w_n r_{2n} \\ \cdot & & & & \cdot \\ \cdot & & & & \cdot \\ \cdot & & & & \cdot \\ w_1 r_{m1} & w_2 r_{m2} & w_3 r_{m3} & \cdots & w_n r_{mn} \end{bmatrix}.$$

Step 3: Determine the Ideal and the Negative-Ideal Solutions

The **ideal**, denoted as A^*, and the **negative-ideal**, denoted as A^-, alternatives (solutions) are defined as follows:

$$A^* = \{ (\max_i v_{ij} \mid j \in J), (\min_i v_{ij} \mid j \in J'), i = 1,2,3,\ldots,m\}$$

$$= \{ v_{1*}, v_{2*}, \ldots, v_{n*} \}. \tag{2-12}$$

$$A^- = \{ (\min_i v_{ij} \mid j \in J), (\max_i v_{ij} \mid j \in J'), i = 1,2,3,\ldots,m\}$$

$$= \{ v_{1-}, v_{2-}, \ldots, v_{n-} \}. \tag{2-13}$$

where: $J = \{j = 1, 2, 3, \ldots, n$ and j is associated with benefit criteria$\}$,
$J' = \{j = 1, 2, 3, \ldots, n$ and j is associated with cost/loss criteria$\}$.

The previous two alternatives are fictitious. However, it is reasonable to assume here that for the benefit criteria, the decision maker wants to have a maximum value among the alternatives. For the cost criteria, the decision maker wants to have a minimum value among the alternatives. From the previous definitions it follows that alternative A^* indicates the most preferable alternative or the **ideal solution**. Similarly, alternative A^- indicates the least preferable alternative or the **negative-ideal solution**.

Step 4: Calculate the Separation Measure

The n-dimensional Euclidean distance method is next applied to measure the separation distances of each alternative from the ideal solution and negative-ideal solution. Thus, for the distances from the ideal solution we have:

$$S_{i*} = \sqrt{\sum_{j=1}^{n} (v_{ij} - v_{j*})^2}, \quad \mathit{for} \quad i = 1, 2, 3, ..., m, \qquad (2\text{-}14)$$

where S_{i*} is the distance (in the Euclidean sense) of each alternative from the ideal solution.

Similarly, for the distances from the negative-ideal solution we have:

$$S_{i-} = \sqrt{\sum_{j=1}^{n} (v_{ij} - v_{j-})^2}, \quad \mathit{for} \quad i = 1, 2, 3, ..., m, \qquad (2\text{-}15)$$

where S_{i-} is the distance (in the Euclidean sense) of each alternative from the negative-ideal solution.

Step 5: Calculate the Relative Closeness to the Ideal Solution

The relative closeness of an alternative A_i with respect to the ideal solution A^* is defined as follows:

$$C_{i*} = \frac{S_{i-}}{S_{i*} + S_{i-}},$$

$$\text{where} \quad 1 \geq C_{i*} \geq 0, \quad \mathit{and} \quad i = 1, 2, 3, ..., m. \qquad (2\text{-}16)$$

Apparently, $C_{i*} = 1$, if $A_i = A^*$, and $C_{i-} = 0$, if $A_i = A^-$.

Step 6: Rank the Preference Order

The best (optimal) alternative can now be decided according to the preference rank order of C_{i^*}. Therefore, the best alternative is the one that has the shortest distance to the ideal solution. The previous definition can also be used to demonstrate that any alternative which has the shortest distance from the ideal solution is also guaranteed to have the longest distance from the negative-ideal solution.