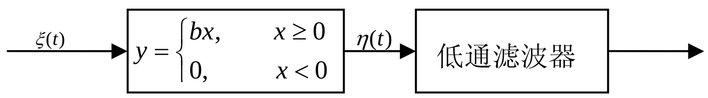
高斯随机过程通过非线性系统（续 1）

高斯随机过程通过半波整流器的研究

半波整流非线性函数关系：$y = \begin{cases} bx, & x \ge 0 \\ 0, & x < 0 \end{cases}$



1. 输入是窄带平稳实高斯随机过程

输入随机过程的概率密度

$$f_{\xi;t}(x) = \frac{1}{\sqrt{2\pi\sigma_\xi^2}} \exp\left(-\frac{x^2}{2\sigma_\xi^2}\right)$$

输出随机过程的概率密度

$$f_{\eta;t}(y) = \frac{1}{2}\delta(y_t) + \frac{1}{\sqrt{2\pi b^2\sigma_\xi^2}} \exp\left(-\frac{y^2}{2b^2\sigma_\xi^2}\right) \cdot U(y_t)$$

各阶矩、方差

偶数阶矩，考虑到输入窄带平稳实高斯随机过程的概率密度函数是偶函数，

$$E\left[\eta^{2m}\right] = \frac{1}{2} b^{2m} E\left[\xi^{2m}\right] = \frac{1}{2} b^{2m}\sigma^{2m}(2m-1)\cdots 3\cdot 1$$

奇数阶矩

$$E\left[\eta^{2m+1}\right] = \frac{m!}{\sqrt{2\pi}} 2^{2m} b^{2m+1}\sigma_\xi^{2m+1}(2m-1)\cdots 5\cdot 3\cdot 1$$

均值

$$E\left[\eta(t)\right] = \frac{1}{\sqrt{2\pi}} b\sigma_\xi$$

方差

$$D\left[\eta(t)\right] = E\left[\eta^2(t)\right] - \left\{E\left[\eta(t)\right]\right\}^2$$

$$= \frac{1}{2} b^2\sigma_\xi^2 - \left(\frac{1}{\sqrt{2\pi}} b\sigma_\xi\right)^2 = \frac{1}{2} b^2\sigma_\xi^2\left(1 - 1/\pi\right)$$

相关函数

$$R_{\eta\eta}(\tau)=R_{\eta\eta}(t_1,t_2)$$

$$=\frac{b^2}{2\pi\sigma_\xi^2\sqrt{(1-\rho^2(\tau))}}\int_0^\infty\int_0^\infty x_1x_2\cdot\exp\left(-\frac{x_1^2+x_2^2-2\rho(\tau)x_1x_2}{2\sigma_\xi^2(1-\rho^2(\tau))}\right)dx_1dx_2$$

其中，$\rho(\tau)=\frac{E\{x_{t_1}x_{t_2}\}}{\sigma_\xi^2}$，

利用典型的积分变换 1，得：

$$R_{\eta\eta}(\tau)\approx\frac{1}{2\pi}b^2\sigma_\xi^2+\frac{1}{4}b^2R_{\xi\xi}(\tau)+\frac{b^2}{4\pi\sigma_\xi^2}R^2{}_{\xi\xi}(\tau)$$

$$R_{\xi\xi}(\tau)=\sigma_\xi^2\rho(\tau)$$

功率谱

$$P_{\eta\eta}(f)=\int_{-\infty}^{\infty}R_{\eta\eta}(\tau)e^{j2\pi f\tau}d\tau$$

$$=\frac{1}{2\pi}b^2\sigma_\xi^2\delta(f)+\frac{1}{4}b^2P_{\xi\xi}(f)+\frac{b^2}{4\pi\sigma_\xi^2}\int_{-\infty}^{\infty}P_{\xi\xi}(f')P_{\xi\xi}(f-f')df'$$

2. 输入信号是矩形带通窄带实平稳随机过程

输入的功率谱密度：

$$P_{\xi\xi}(f)=\begin{cases}N_0/2, & f_0-\frac{\Delta f}{2}<f<f_0+\frac{\Delta f}{2}\\ 0, & \text{otherwise}\end{cases}$$

$$\sigma_\xi{}^2=\Delta f\cdot N_0$$

非线性器件输出信号的功率谱密度：

直流分量：

$$\frac{1}{2\pi}b^2\sigma_\xi^2\delta(f)=\frac{1}{2\pi}b^2(\Delta f\cdot N_0)\delta(f)$$

低频分量 $0\le|f|\le\Delta f$

$$\frac{b^2}{4\pi\sigma_\xi^2}\left(\frac{N_0}{2}\right)^2 2\Delta f\left(1-\frac{|f|}{\Delta f}\right)=\frac{b^2}{4\pi}\left(\frac{N_0}{2}\right)\left(1-\frac{|f|}{\Delta f}\right)$$

带通信号分量 $f_c-\Delta f/2\le|f|\le f_c+\Delta f/2$

$$\frac{1}{4}b^2\frac{N_0}{2}$$

二倍频分量 $2f_c-\Delta f\le|f|\le 2f_c+\Delta f$

$$\frac{b^2}{4\pi\sigma_\xi^2}\left(\frac{N_0}{2}\right)^2\Delta f\left(1-\frac{|f|}{\Delta f}\right)=\frac{b^2}{8\pi}\left(\frac{N_0}{2}\right)\left(1-\frac{|f|}{\Delta f}\right)$$

低通滤波器输出信号的功率谱密度：

直流分量：

$$\frac{1}{2\pi}b^2\sigma_\xi^2\delta(f)=\frac{1}{2\pi}b^2(\Delta f\cdot N_0)\delta(f)$$

低频分量 $0\le|f|\le\Delta f$

$$\frac{b^2}{4\pi\sigma_\xi^2}\left(\frac{N_0}{2}\right)^2 2\Delta f\left(1-\frac{|f|}{\Delta f}\right)=\frac{b^2}{4\pi}\left(\frac{N_0}{2}\right)\left(1-\frac{|f|}{\Delta f}\right)$$

典型的坐标变换 1

原积分：

$$R_{\eta\eta}(\tau)=R_{\eta\eta}(t_1,t_2)$$
$$=\frac{b^2}{2\pi\sigma_\xi^2\sqrt{(1-\rho^2(\tau))}}\int_0^\infty\int_0^\infty x_1x_2\cdot\exp\left(-\frac{x_1^2+x_2^2-2\rho(\tau)x_1x_2}{2\sigma_\xi^2(1-\rho^2(\tau))}\right)dx_1dx_2$$

其中，$\rho(\tau)=E[x(t_1)x(t_2)]/\sigma_\xi^2$

变换

$$u=\frac{x_1}{\sqrt{2\sigma_\xi^2(1-\rho^2(\tau))}}$$

$$v=\frac{x_2}{\sqrt{2\sigma_\xi^2(1-\rho^2(\tau))}}$$

$$\left|\frac{\partial(x_1,x_2)}{\partial(u,v)}\right|=2\sigma_\xi^2(1-\rho^2(\tau))$$

积分

$$\begin{aligned} R_{\eta\eta}(\tau) &= R_{\eta\eta}(t_1,t_2) \\ &= \frac{b^2}{2\pi\sigma_\xi^2(1-\rho^2(\tau))^{1/2}}\int_0^\infty\int_0^\infty x_1x_2\cdot\exp\left(-\frac{x_1^2+x_2^2-2\rho(\tau)x_1x_2}{2\sigma_\xi^2(1-\rho^2(\tau))}\right)dx_1dx_2 \\ &= \frac{2b^2\sigma_\xi^2(1-\rho^2(\tau))^{3/2}}{\pi}\int_0^\infty\int_0^\infty uv\cdot\exp\left[-\left(u^2+v^2-2\rho(\tau)uv\right)\right]dudv \end{aligned}$$

典型的坐标变换 2

积分之间的关系：

$$I=\int_0^\infty\int_0^\infty \exp\left[-\left(u^2+v^2-2wuv\right)\right]dudv$$

$$\frac{dI}{dw}=2\int_0^\infty\int_0^\infty uv\cdot\exp\left[-\left(u^2+v^2-2wuv\right)\right]dudv$$

$$\int_0^\infty\int_0^\infty uv\cdot\exp\left[-\left(u^2+v^2-2wuv\right)\right]dudv=\frac{1}{2}\frac{dI}{dw}$$

典型的坐标变换 3

原积分：

$$I=\int_0^\infty\int_0^\infty \exp\left[-\left(u^2+v^2-2wuv\right)\right]dudv$$

积分变换：

从u,v平面到r,θ平面，参数$\cos\alpha=w\le 1,\quad \alpha\in(0,\pi)$

$$u=\frac{r\cos\left(\frac{\alpha}{2}+\theta\right)}{\sin\alpha},\qquad v=\frac{r\cos\left(\frac{\alpha}{2}-\theta\right)}{\sin\alpha}$$

注意到下列关系：

$$u=0,\quad \frac{\alpha}{2}+\theta=\frac{\pi}{2},\quad \theta=\left(\frac{\pi}{2}-\frac{\alpha}{2}\right)$$

$$v=0,\quad \frac{\alpha}{2}-\theta=\frac{\pi}{2},\quad \theta=-\left(\frac{\pi}{2}-\frac{\alpha}{2}\right)$$

$$\begin{vmatrix} \dfrac{\partial u}{\partial r} & \dfrac{\partial v}{\partial r} \\ \dfrac{\partial u}{\partial \theta} & \dfrac{\partial v}{\partial \theta} \end{vmatrix} = \begin{vmatrix} \dfrac{\cos\left(\dfrac{\alpha}{2}+\theta\right)}{\sin\alpha} & \dfrac{r\cos\left(\dfrac{\alpha}{2}-\theta\right)}{\sin\alpha} \\ -\dfrac{r\sin\left(\dfrac{\alpha}{2}+\theta\right)}{\sin\alpha} & \dfrac{r\sin\left(\dfrac{\alpha}{2}-\theta\right)}{\sin\alpha} \end{vmatrix} = \frac{r}{\sin\alpha}$$

$$\begin{aligned}
&u^2+v^2-2\rho(\tau)uv \\
&= r^2\frac{\cos^2\left(\dfrac{\alpha}{2}+\theta\right)}{\sin^2\alpha}+r^2\frac{\cos^2\left(\dfrac{\alpha}{2}-\theta\right)}{\sin^2\alpha}-2r^2\cos\alpha\frac{\cos\left(\dfrac{\alpha}{2}+\theta\right)}{\sin\alpha}\frac{\cos\left(\dfrac{\alpha}{2}-\theta\right)}{\sin\alpha} \\
&= \frac{r^2}{2\sin^2\alpha}\left(1+\cos(\alpha+2\theta)+1+\cos(\alpha-2\theta)-2\cos\alpha\cos\alpha-2\cos\alpha\cos 2\theta\right) \\
&= \frac{r^2}{2\sin^2\alpha}\left(1+\cos(\alpha+2\theta)+1+\cos(\alpha-2\theta)\right. \\
&\qquad\qquad \left.-\cos 2\alpha-1-\cos(\alpha+2\theta)-\cos(\alpha-2\theta)\right) \\
&= r^2
\end{aligned}$$

原积分：

$$\begin{aligned}
I &= \int_0^\infty\int_0^\infty \exp\left[-\left(u^2+v^2-2wuv\right)\right]dudv \\
&= \int_0^\infty\int_{-\frac{\pi-\alpha}{2}}^{\frac{\pi-\alpha}{2}} \frac{r}{\sin\alpha}\exp -r^2 dr d\theta \\
&= \frac{\pi-\alpha}{2\sin\alpha} = \frac{\pi/2+\sin^{-1}w}{2\sqrt{1-w^2}}
\end{aligned}$$

$$\cos\alpha = w$$

$$\alpha = \pi/2-\sin^{-1}w$$

原积分：

$$\begin{aligned}
\frac{dI}{dw} &= \frac{d}{dw}\frac{\pi/2+\sin^{-1}w}{2\sqrt{1-w^2}} \\
&= \frac{\sqrt{1-w^2}\dfrac{1}{\sqrt{1-w^2}}+\left(\pi/2+\sin^{-1}w\right)\dfrac{w}{\sqrt{1-w^2}}}{2\left(1-w^2\right)} \\
&= \frac{1}{2\left(1-w^2\right)}+\frac{w}{2\left(1-w^2\right)^{3/2}}\left(\pi/2+\sin^{-1}w\right)
\end{aligned}$$

原积分：

$$\int_0^\infty\int_0^\infty uv\cdot\exp\left[-\left(u^2+v^2-2wuv\right)\right]dudv=\frac{1}{2}\frac{dI}{dw}$$
$$=\frac{1}{4\left(1-w^2\right)}+\frac{w}{4\left(1-w^2\right)^{3/2}}\left(\pi/2+\sin^{-1}w\right)$$

原积分：

$$R_{\eta\eta}(\tau)=R_{\eta\eta}(t_1,t_2)$$
$$=\frac{b^2}{2\pi\sigma_\xi^2(1-\rho^2(\tau))}\int_0^\infty\int_0^\infty x_1x_2\cdot\exp\left(-\frac{x_1^2+x_2^2-2\rho(\tau)x_1x_2}{2\sigma_\xi^2(1-\rho^2(\tau))}\right)dx_1dx_2$$
$$=\frac{1}{2\pi}b^2\sigma_\xi^2\cdot\left[\left(1-\rho(\tau)\right)^{1/2}+\rho(\tau)\left(\pi/2+\sin^{-1}\rho(\tau)\right)\right]$$
$$=\frac{1}{2\pi}b^2\sigma_\xi^2\cdot\left[\left(1-\frac{1}{2}\rho^2(\tau)-\frac{1}{2\cdot4}\rho^4(\tau)-\frac{1}{2\cdot4\cdot6}\rho^6(\tau)-\cdots\right)\right.$$
$$\left.+\rho(\tau)\left(\frac{\pi}{2}+\rho(\tau)+\frac{1}{2\cdot3}\rho^3(\tau)+\frac{1\cdot3}{2\cdot4\cdot5}\rho^5(\tau)\right)\right]$$
$$=\frac{1}{2\pi}b^2\sigma_\xi^2\cdot\left[1+\frac{\pi}{2}\rho(\tau)+\frac{1}{2}\rho^2(\tau)+\frac{1}{24}\rho^4(\tau)+\frac{1}{80}\rho^8(\tau)+\cdots\right]$$

由于$\left|\rho(\tau)\right|\le1$

$$R_{\eta\eta}(\tau)=\frac{1}{2\pi}b^2\sigma_\xi^2+\frac{1}{4}b^2R_{\xi\xi}(\tau)+\frac{b^2}{4\pi\sigma_\xi^2}R^2{}_{\xi\xi}(\tau)$$
$$R_{\xi\xi}(\tau)=\sigma_\xi^2\rho(\tau)$$