

AM 260, Winter 2021

Note on the basic discretization setup

1. Overview

We are interested in solving a linear advection PDE given as

$$u_t + au_x = 0, \quad (1)$$

where a is a constant advection velocity.

2. Initial and boundary conditions

We impose an initial condition at $t = 0$:

$$\text{IC: } u(x, 0) = u_0(x) \quad (2)$$

and a boundary condition on a bounded domain $x_a \leq x \leq x_b$:

$$\text{BC: } u(x_a, t) = g_a(t) \text{ and } u(x_b, t) = g_b(t), \text{ for } t > 0, \quad (3)$$

where g_a and g_b are known functions.

3. Discretization in space and time

Let us take the discretization of interior cells with which we have a spatial resolution of N and a temporal resolution of M :

$$x_i = x_a + (i - \frac{1}{2})\Delta x, \quad i = 1, \dots, N + 4, \quad (4)$$

$$t^n = n\Delta t, \quad n = 0, \dots, M. \quad (5)$$

The N interior points are defined with $i = 3, \dots, N + 2$, whereas the four points with $i = 1, 2$ and $i = N + 3, N + 4$ are the guardcell points (see below). Notice that the cell interface-centered grid points are written using the ‘half-integer’ indices:

$$x_{i+\frac{1}{2}} = x_i + \frac{\Delta x}{2}. \quad (6)$$

4. Imposing Boundary Conditions via guard-cells (or ghost-cells)

We introduce the so-called ‘guard-cells’ or ‘ghost-cells’ (simply GCs) on each end, having two extra layers of GC points on each side of boundaries,

$$x_1 = x_a - \frac{3\Delta x}{2}, \quad (7)$$

$$x_2 = x_a - \frac{\Delta x}{2}, \quad (8)$$

$$x_{N+3} = x_b + \frac{\Delta x}{2}, \quad (9)$$

$$x_{N+4} = x_b + \frac{3\Delta x}{2}. \quad (10)$$

With these two extra layers of GC points (i.e., two GC points on each end), the differential equation is discretized with numerical approximations $U_i^n \approx u(x_i, t^n)$, which are spatially differenced and temporally evolved on the N interior points. The boundary conditions, on the other hand, are explicitly imposed through the four GC points, simply

$$U_1^n = g_a(t^n), \quad (11)$$

$$U_2^n = g_a(t^n), \quad (12)$$

$$U_{N+3}^n = g_b(t^n), \quad (13)$$

$$U_{N+4}^n = g_b(t^n). \quad (14)$$

Of particularly useful boundary conditions are periodic condition and outflow condition, which respectively are given as

$$U_1^n = U_{N+1}^n, \quad (15)$$

$$U_2^n = U_{N+2}^n, \quad (16)$$

$$U_{N+3}^n = U_3^n, \quad (17)$$

$$U_{N+4}^n = U_4^n, \quad (18)$$

for periodic condition, and

$$U_1^n = U_3^n, \quad (19)$$

$$U_2^n = U_3^n, \quad (20)$$

$$U_{N+3}^n = U_{N+2}^n, \quad (21)$$

$$U_{N+4}^n = U_{N+2}^n, \quad (22)$$

for outflow condition.

5. The CFL condition

As studied, the CFL condition provides a necessary condition for choosing the length of Δt depending on the PDE under consideration. The CFL condition amounts to say that, if we let C_a to be the CFL number that satisfies $0 < C_a \leq 1$, a timestep Δt needs to satisfy

$$0 < \Delta t \leq C_a \frac{\Delta x}{|a|} \quad (23)$$

for a numerical method for advection to be stable.

It is important to note that the CFL condition is only a *necessary* condition for stability (and hence convergence). It is not always *sufficient* to guarantee stability, which means that a numerical method satisfying the CFL condition could still become unstable.