Ex PLM flux $\boxed{u_t + a u_x = 0}$,

$f(u) = au$

$$\rightarrow F_{i+\frac{1}{2}}^{PLM,\, n+\frac{1}{2}} = \begin{cases} f(\tilde{U}^L), & a>0 \\ f(\tilde{U}^R), & a<0 \end{cases}$$

$$= \begin{cases} a\,\tilde{U}^L_{i+\frac{1}{2}}, & a>0 \\ a\,\tilde{U}^R_{i+\frac{1}{2}}, & a<0 \end{cases}$$

$$C_a = \frac{a\Delta t}{\Delta x}$$

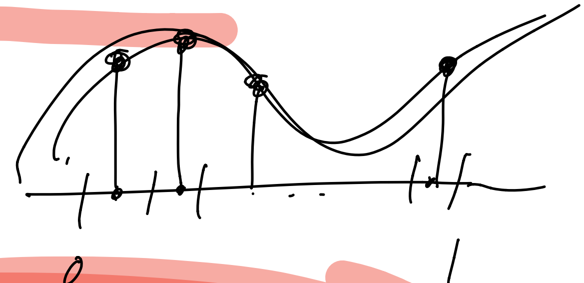
$$= \begin{cases} a\left(U_i^n + \frac{\Delta x}{2}\Delta_i^n\left(1 - C_a\right)\right), & a>0 \\ a\left(U_{i+1}^n - \frac{\Delta x}{2}\Delta_{i+1}^n\left(1 + C_a\right)\right), & a<0 \end{cases}$$

Rmk $\Delta t = \frac{C_a \Delta x}{|a|}$ $\quad 0 < a \le 1$

Rmk $u_t + u u_x = 0$

$$\Rightarrow \Delta t = \frac{C_a \Delta x}{|u|} \qquad u = u(x,t)$$

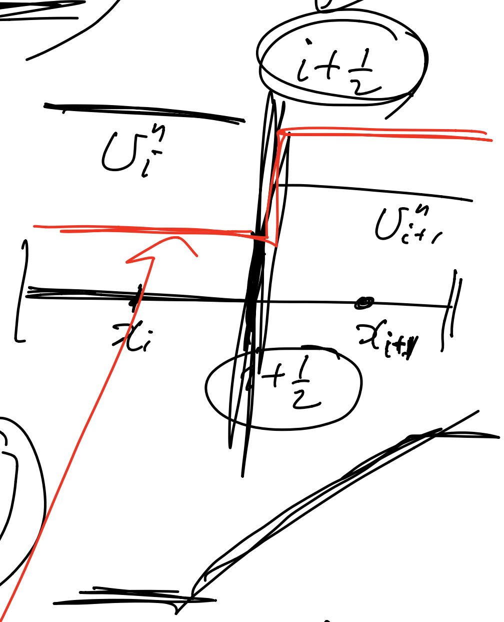
U_i^n, $i = 1, \dots, N$



$$\Rightarrow \Delta t = \frac{C_a \Delta x}{\max\limits_{1 \le i \le N} |U_i^n|}$$

PLM for Burgers' Eqn (i)

(i) $U_i^n \geq U_{i+1}^n$ (shock)



2nd order in time

$F^{PLM,\, n+\frac{1}{2}}_{i+\frac{1}{2}}$

$f(u) = \frac{u^2}{2}$

$||$

$$
\begin{cases}
f(\tilde{U}^L_{i+\frac{1}{2}}) = \frac{1}{2}\left(U_i^n + \frac{\Delta x}{2}\Delta_i^n (1-C_a)\right)^2, & S_{i+\frac{1}{2}} > 0 \\
f(\tilde{U}^R_{i+\frac{1}{2}}) = \frac{1}{2}\left(U_{i+1}^n - \frac{\Delta x}{2}\Delta_{i+1}^n (1+C_a)\right)^2, & S_{i+\frac{1}{2}} \leq 0
\end{cases}
$$

$$S_{i+\frac{1}{2}} = \frac{[f]}{[u]} = \frac{1}{2}\left(U_i^n + U_{i+1}^n\right)$$

(ii) $U_i^n < U_{i+1}^n$ (rarefaction)

$f(\tilde{U}^L_{i+1})$

$$F^{PLM, n+\frac{1}{2}}_{i+\frac{1}{2}} = \begin{cases} \frac{1}{2}\left[U^n_i + \frac{\Delta x}{2}\Delta^n_i(1-Ca)\right]^2, & 0 \le U^n_i \\ 0 & U^n_i < 0 \le U^n_{i+1} \\ f(\tilde{U}^k_{i+\frac{1}{2}}). & 0 > U^n_{i+1} \end{cases}$$

j, k

$$\frac{1}{2}\left(U^n_{i+1} - \frac{\Delta x}{2}\Delta^n_{i+1}(1+Ca)\right)^2$$

CFL Condition

$$\lambda_{max} = \max_i \{ S_{i+\frac{1}{2}} \}$$

$$\Delta t = \frac{Ca\Delta x}{\lambda_{max}}$$

do $i = 1, \cdots, N$

$t = i\Delta t$

end do

while $t < t_{max}$

$t = t_{max}$

$n = 1$; $\forall i$

$$U_i^{n+1} = U_i^n - \frac{\Delta t}{\Delta x}\left[F_{i+\frac{1}{2}}^{n+\frac{1}{2}, PLM} - F_{i-\frac{1}{2}}^{n+\frac{1}{2}, PLM} \right]$$

Chapter 8 FVM for the Euler's Eqns.

nonlinear eqns

$$U_t + F(U)_x = 0$$

(i) local linearizations of the nonlinear flux Jacobian matrix

$$\left(\frac{\partial F}{\partial U}\right)_{n \times n} \qquad n = 3 \qquad 1D$$

$$U = \begin{pmatrix} \rho \\ \rho u \\ \rho E \end{pmatrix} \begin{matrix} u_1 \\ u_2 \\ u_3 \end{matrix}$$

conservative variable

$A_{n \times n}$

$$U_t + \frac{\partial F}{\partial U} U_x = 0$$

$$u_t + a u_x = 0$$

$$U_t + A U_x = 0$$

(ii) diagonalize $A = R D R^{-1}$

$$D = \begin{bmatrix} \lambda_1 & & 0 \\ & \lambda_2 & \\ 0 & & \lambda_3 \end{bmatrix}$$

R

R-L

$A \cong D$

$$\begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix}_t + \begin{bmatrix} \lambda_1 & & \\ & \lambda_2 & \\ & & \lambda_3 \end{bmatrix} \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix}_x = 0$$

(iii) decouple the system

$$(u_1)_t + \lambda_1 (u_1)_x = 0$$

$$\vdots$$

$$(u_3)_t + \lambda_3 (u_3)_x = 0$$

(iv) Use the linear theory for linear scalar equs.

$$F(U) = \begin{pmatrix} \rho u \\ \rho u^2 + p \\ u(\rho E + p) \end{pmatrix}$$

$$\rho_t + (\rho u)_x = 0$$

$$(\rho u)_t + (\rho u^2 + p)_x = 0$$

$$(\rho E)_t + (u(\rho E + p))_x = 0$$

Riemann solvers (RS) B E

approximate RS

- Roe
- HLL
- HLLC
- LLF

HLLD for MHD

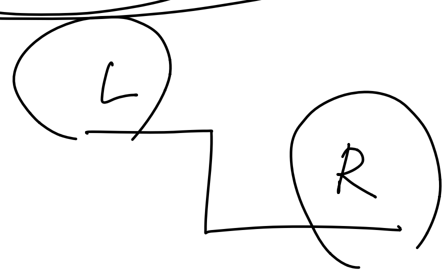
+ two-shock, two-rarefactn

Osher's RS

"Exact" RS

iterative RS → hybrid RS

1D



Nonlinear Hyperbolic Systems

$$\begin{cases} U_t + F(U)_x = 0 \\ U(x,0) = U_0(x) \end{cases}$$

$$\frac{\partial F}{\partial U} U_x \qquad A$$

$$\Downarrow$$

$$A(x_{i+\frac{1}{2}}) = \tilde{A}$$

$$A(x_0) = \tilde{A}$$

$$(*)\begin{cases} U_t + \tilde{A} U_x = 0 \\ U(x,0) = U_0(x) \end{cases}$$

$$F(U) = \tilde{A} U$$

$$U = \begin{pmatrix} \rho \\ \rho u \\ \rho v \\ \rho w \\ \rho E \end{pmatrix} \qquad F(U) = \begin{pmatrix} \rho u \\ \rho u^2 + p \\ \rho u v \\ \rho u w \\ u(\rho E + p) \end{pmatrix}$$

Conservative Vector.

flux Vector

$$G(U) = \begin{pmatrix} \rho v \\ \rho v u \\ \rho v^2 + p \\ \rho v w \\ v(\rho E + p) \end{pmatrix} \qquad \underline{H(U)} \quad \| \quad \vdots$$

<u>Def.</u> The system of conservation law $\circledast$ is called <u>hyperbolic</u> if

A is diagonalizable with real eigenvalues

$$\underline{\lambda_1 \le \lambda_2 \le \cdots \le \lambda_m} \quad \text{s.t.}$$

we can decompose

$$\tilde{A} = R \Lambda R^{-1}, \quad = L$$

$$R = [\, r_1 \mid r_2 \mid \cdots \mid r_m \,] \qquad \Lambda = \begin{bmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_m \end{bmatrix}$$

$$l_i \cdot r_j = \delta_{ij}$$

$$\begin{bmatrix} l_1 \\ l_2 \\ \vdots \\ l_m \end{bmatrix}$$

<u>Def</u>. strictly hyperbolic if

$$\lambda_1 \underset{\neq}{<} \lambda_2 \underset{\neq}{\leq} \cdots \underset{\neq}{\leq} \lambda_m$$

Decoupled system ; $\tilde{A} = \underline{R}\,\underline{\Lambda}\,\underline{R^{-1}}$

$U_t + \tilde{A} U_x = 0$; linear system

$\rightarrow \quad R^{-1} U_t + \Lambda R^{-1} U_x = 0$

$\rightarrow \quad (LU)_t + \Lambda (LU)_x = 0$

$\rightarrow \quad W = LU \qquad \begin{pmatrix} \lambda_1 & & \\ & \lambda_2 & \\ & & \lambda_3 \end{pmatrix} \qquad W = \begin{pmatrix} w_1 \\ w_2 \\ w_3 \end{pmatrix}$

conservative

$$W_t + \Lambda W_x = 0$$

variables

characteristic variables

$$\frac{\partial w_k}{\partial t} + \lambda_k \frac{\partial w_k}{\partial x} = 0, \quad k = 1, \cdots, m$$

(**)

a decoupled system of m indep. scalar eqns.

$$w_k(x,t) = w_k(x - \lambda_k t, 0)$$

$$U = R W$$