Modified Eqn

(i) PDE $\longrightarrow$ DE (difference eqn)

(ii) DE $\longrightarrow$ PDE $\longleftarrow$ modified eqn

$\Rightarrow$ { dissipation errors (first-order methods)
dispersion errors (second-order methods)

(1) Dissipation errors in first-order methods

Pick the upwind method ($u_t + a u_x = 0$, $a>0$)

$\rightarrow$ FTBS ;

$$\rightarrow \boxed{U_i^{n+1} = U_i^n - \frac{a\Delta t}{\Delta x}\left[U_i^n - U_{i-1}^n\right]}$$

$$\rightarrow u_i^{n+1} = u_i^n - \frac{a\Delta t}{\Delta x}\left[u_i^n - u_{i-1}^n\right] \quad \cdots (*)$$

where $u_i^n = u(x_i, t^n)$: exact soln

$= u(x, t+\Delta t)$

$\rightarrow$ Dropping (i,n) ;

$$u_{i-1}^n = u(x-\Delta x, t)$$

→ (*) becomes

$$u(x, t+\Delta t) = u(x,t) - \frac{a\Delta t}{\Delta x}\left[u(x,t) - u(x-\Delta x, t)\right]$$

→ Taylor expansion ;

$$\cancel{u(x,t)} + \Delta t\, u_t(x,t) + \frac{\Delta t^2}{2} u_{tt}(x,t) + O(\Delta t^3)$$

$$= \cancel{u(x,t)} - \frac{a\Delta t}{\Delta x}\left[\cancel{u(x,t)} - \{\cancel{u(x,t)} - \Delta x\, u_x(x,t) + \frac{\Delta x^2}{2} u_{xx}(x,t) + O(\Delta x^3)\}\right]$$

$O(\Delta t + \Delta x)$

$$\rightarrow u_t + a u_x = \frac{1}{2}\left(a u_{xx} \Delta x - u_{tt} \Delta t\right) - \left(\frac{1}{6}\right)\left(a u_{xxx} \Delta x^2 + u_{ttt} \Delta t^2\right) + O\left(\Delta t^3 + \Delta x^3\right)$$

modified eqn

→ If keeping $O(\Delta t + \Delta x)$ term ;

Our modified eqn ;

$u_{tt} \to u_{xx}$

$$\boxed{u_t + a u_x = \frac{1}{2}\left(a u_{xx} \Delta x - u_{tt} \Delta t\right)} \quad (**)$$

$$(u_t)_t = (-a u_x)_t + \frac{1}{2}\left(a \Delta x\, u_{xxt} - \Delta t\, u_{ttt}\right)$$

$$u_{xt} = (u_t)_x = -a u_{xx} + \frac{1}{2}\left(a \Delta x\, u_{xxx} - \Delta t\, u_{ttx}\right)$$

$$\Rightarrow u_{tt} = a^2 u_{xx} + O(\Delta t + \Delta x)$$

$(***)$

$$\Rightarrow (**) ; \; u_t + a u_x = \frac{1}{2}\left(a u_{xx} \Delta x - \Delta t\, a^2 u_{xx}\right) + O(\Delta t^2 + \Delta x^2) + \cancel{\Delta t \Delta x}$$

$0 < C_a \leq 1$

C_a

$$\boxed{\Rightarrow u_t + a u_x = \frac{a \Delta x}{2}\left[1 - \frac{a \Delta t}{\Delta x}\right] u_{xx} + O(\Delta t^2 + \Delta x^2)}$$

K

$$\Rightarrow \boxed{u_t + a u_x = K u_{xx} + O(\Delta t^2 + \Delta x^2)}$$

parabolic PDE or

(advection - diffusion)

$k \geq 0$ ✓ (

$$K = \begin{cases} \dfrac{a\Delta x}{2}[1 - Ca] & \text{for upwind} \\[2ex] \dfrac{\Delta x^2}{2\Delta t}[1 - Ca^2] & \text{for LF} \end{cases}$$

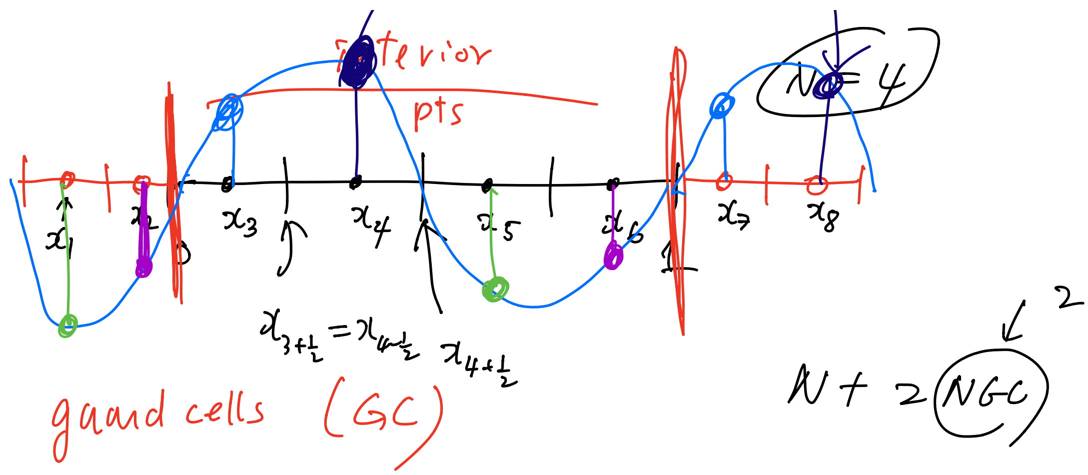
$$= \frac{1}{Ca}\frac{\Delta x}{2}[1 - Ca^2]$$

$\rightarrow K \rightarrow 0$ as $\Delta x \rightarrow 0$

(vanishing viscosity)

Ex $Ca = 0.8$, $a = 1$, $\Delta x = 1$, $\Delta t = 0.8$

$$K = \begin{cases} 0.1 & \text{upwind} \\ 0.1152 & \text{LF} \Leftarrow \end{cases}$$



Dispersion errors in second-order methods

$u_t + a u_x = 0$

Consider LW;

$$u_t + a u_x = \frac{a \Delta x^2}{6}\left[C_a^2 - 1 \right] u_{xxx} \qquad (*)$$

$= \mu$

→ Fourier Transform of $(*)$;

(i) $f(x,t) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \hat{f}(\xi, t)\, e^{I \xi x}\, d\xi$

(ii) $\hat{f}(\xi, t) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} f(x,t)\, e^{-I \xi x}\, dx$

$$(I = \sqrt{-1})$$

$$(\hat{u})_t$$

$$\to \begin{cases} u_t = \dfrac{1}{\sqrt{2\pi}} \displaystyle\int_{\mathbb{R}} \widehat{(u_t)}(\xi, t)\, e^{I\xi x}\, d\xi \\ u_x = \dfrac{1}{2\pi} \displaystyle\int_{\mathbb{R}} \widehat{(u_x)}(x, t)\, e^{I\xi x}\, dx \end{cases}$$

$$I\xi\, \hat{u}$$

$$\to \widehat{(u_t)} + \widehat{(a u_x)} = \widehat{(\mu u_{xxx})}$$

$$\iff (\hat{u})_t + aI\xi\, \hat{u} = \mu (I\xi)^3\, \hat{u}$$

$$\Rightarrow (\hat{u})_t = -I \underbrace{(a\xi + \mu\xi^3)}_{\omega}\, \hat{u} = -I\omega\, \hat{u}$$

$$\Rightarrow \hat{u}(\xi, t) = e^{-I\omega t}\, \hat{\eta}(\xi), \quad \hat{\eta} = \hat{u}(\xi, 0)$$

$$\Rightarrow u(x, t) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \hat{u}(\xi, t)\, e^{I\xi x}\, d\xi$$

$$= \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \hat{\eta}(\xi)\, e^{-I\omega t} e^{I\xi x}\, d\xi$$

Solu to our modified eqn

$$= \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \hat{\eta}(\xi)\, e^{I\xi\left(x - \frac{\omega}{\xi} t\right)}\, d\xi$$

$\rightarrow \frac{\omega}{\xi}$: speed at which the oscillating wave propagates

$\rightarrow$ phase velocity

$$\rightarrow C_p(\xi) = \frac{\omega(\xi)}{\xi} = \frac{1}{\xi}\left(a\xi + \mu\xi^3\right)$$

$$= a + \mu\xi^2$$

$\rightarrow$ C_p varies with ξ

$C_p \approx a$ when $\xi \ll 1$

$C_p \neq a$ when $\xi > 1$

→ group velocity $C_g \equiv \dfrac{d\omega}{d\xi}$

$= a + 3\mu\xi^2$

→ $\mu = \begin{cases} \dfrac{a\Delta x^2}{6}(C_a^2 - 1) & LW \\ \dfrac{a\Delta x^2}{6}(C_a^2 - 3C_a + 2) & BW \end{cases}$

$(0 < C_a \leq 1 \Longrightarrow$

$\mu \leq 0$ LW

$\mu \geq 0$ BW

→ For $a > 0$,

$C_g = \begin{cases} a + 3\mu\xi^2 < a & \text{for } LW \\ a + \mu\xi^2 > a & \text{for } BW \end{cases}$

Rmk BW & Fromm's methods

$\longrightarrow 0 \le Ca \le 2 \qquad a > 0$

$-2 \le Ca \le 0 \qquad a < 0$

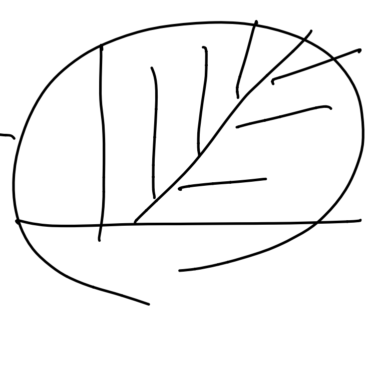
Chapter 6. Computing discont. solns of "non-linear" conservation laws.

Issues

① nonlinearly unstable

② converge to a non-weak soln

③ converge to a wrong weak soln



$u_l > u_r$

$$\boxed{u_t + \left(\frac{u^2}{2}\right)_x = 0} \qquad f(u) = \frac{u^2}{2}$$

?

$$(u^2)_t + \left(\frac{2u^3}{3}\right)_x = 0 \rightarrow w = u^2$$

$f(u)$

$$\boxed{w_t + \left(\frac{2}{3} w^{3/2}\right)_x = 0}$$

$$u u_t \rightarrow \frac{1}{2}(u^2)_t$$

$$\frac{\partial}{\partial x}\left(\frac{u^2}{2}\right)$$

Ex Original Burgers' Eqn : $\underbrace{u_t + \left(\frac{u^2}{2}\right)_x = 0}_{\text{"Conservative"}}$

$$u_t + u u_x = 0 \qquad u \frac{\partial u}{\partial x}$$

"non-conservative"