① Algorithm of the second-order piecewise linear method (PLM)

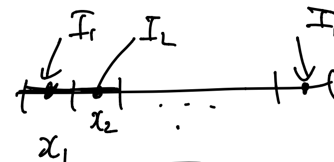
REA

**Step 1** Reconstruction

① 1st order: $\tilde{u}(x, t^n) = U_i^n$ (FOG)

$x \in I_i$

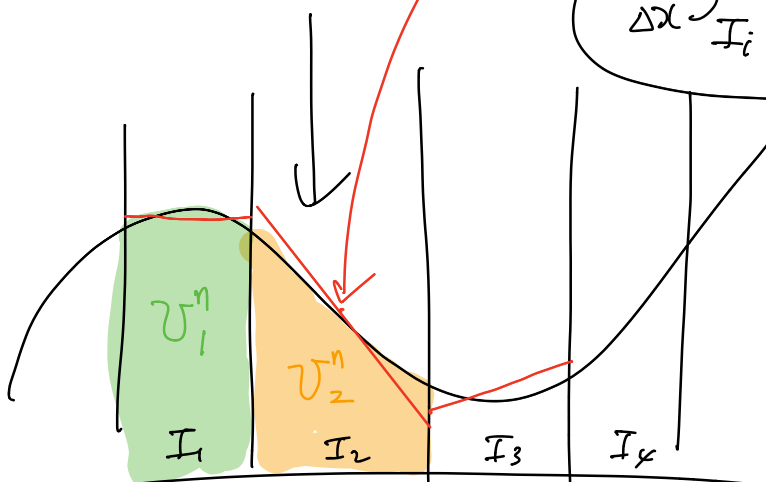
$I_i = [x_{i-\frac{1}{2}}, x_{i+\frac{1}{2}}]$




$$U_i^n = \frac{1}{\Delta x}\int_{I_i} u(x,t^n)\,dx$$

$\Delta x = x_i - x_{i-1}$

uniform grid




$I_i$ @ $x_i$

② 2nd order: $\tilde{u}(x, t^n) = U_i^n + \Delta_i^n (x - x_i)$

$x \in I_i$

$\tilde{u}(x_i) = U_i^n$

$\rightarrow \Delta_i^n$ : slope on each $I_i$ @ $x_i$

$$\Delta_i^n = \begin{cases} \dfrac{U_{i+1}^n - U_{i-1}^n}{2\Delta x} & \text{: centered slope} \\[2ex] \dfrac{U_i^n - U_{i-1}^n}{\Delta x} & \text{: upwind slope } (a>0) \\[2ex] \dfrac{U_{i+1}^n - U_i^n}{\Delta x} & \text{: downwind slope } (a>0) \end{cases}$$

$u_t + a u_x = 0$

$a>0$

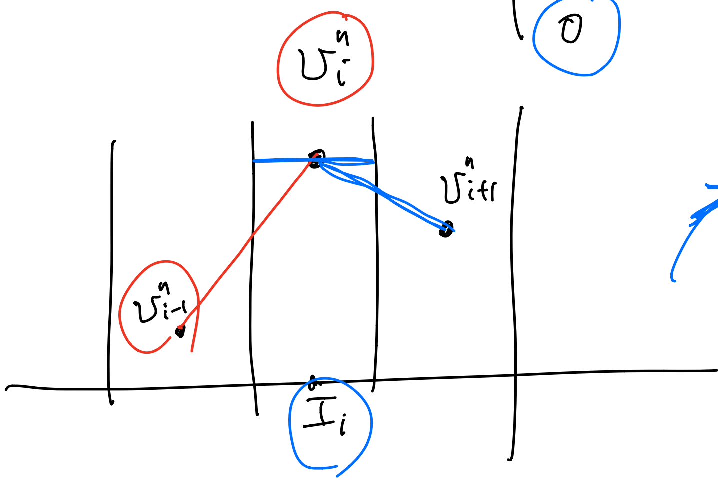
"linear slopes"

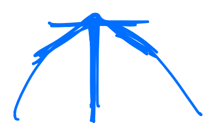
$\rightarrow$ TVD slope "limiters" : $I_i$

$$\Delta_i^n = \begin{cases} \text{minmod}\left( \dfrac{U_i^n - U_{i-1}^n}{\Delta x}, \dfrac{U_{i+1}^n - U_i^n}{\Delta x} \right) \\[2ex] \text{minmod}\left( \dfrac{U_{i+1}^n - U_{i-1}^n}{2\Delta x}, \dfrac{2(U_{i+1}^n - U_i^n)}{\Delta x}, \dfrac{2(U_i^n - U_{i-1}^n)}{\Delta x} \right) \\[2ex] \text{vanLeer}\left( \dfrac{U_i^n - U_{i-1}^n}{\Delta x}, \dfrac{U_{i+1}^n - U_i^n}{\Delta x} \right) \end{cases}$$

MC

$$\boxed{\mathrm{minmod}(a,b)} = \begin{cases} a, & |a| < |b| \;\&\; ab>0 \\ b, & |a| > |b| \;\&\; ab>0 \\ 0 & ab < 0 \end{cases}$$

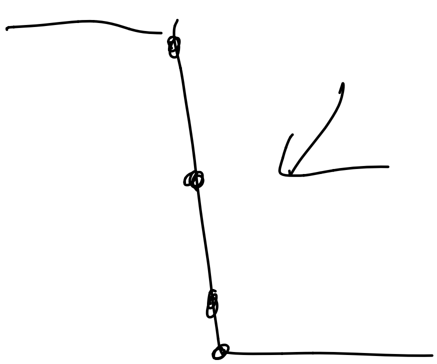


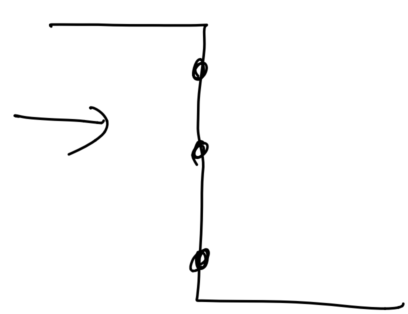



- $\underline{\mathrm{minmod}(a,b,c)} = \mathrm{minmod}\left(\mathrm{minmod}(a,b), c\right)$

$$= \mathrm{minmod}\left(\mathrm{minmod}(b,c), a\right)$$

$$= \mathrm{minmod}\left(\mathrm{minmod}(c,a), b\right)$$





$$\text{vanLeer}(a,b) = \begin{cases} 0 & ab \le 0 \\ \dfrac{2ab}{a+b} & ab > 0 \end{cases}$$

harmonic mean

minmod $\le$ vanLeer $\le$ MC

diffusive $\longleftarrow$ $\longrightarrow$ sharp

---

The linear reconstruction satisfies :

$$\tilde{u}(x_i, t^n) = U_i^n = \frac{1}{\Delta x} \int_{I_i} \tilde{u}(x, t^n)\, dx$$

$\longrightarrow$ This is important in developing FVM

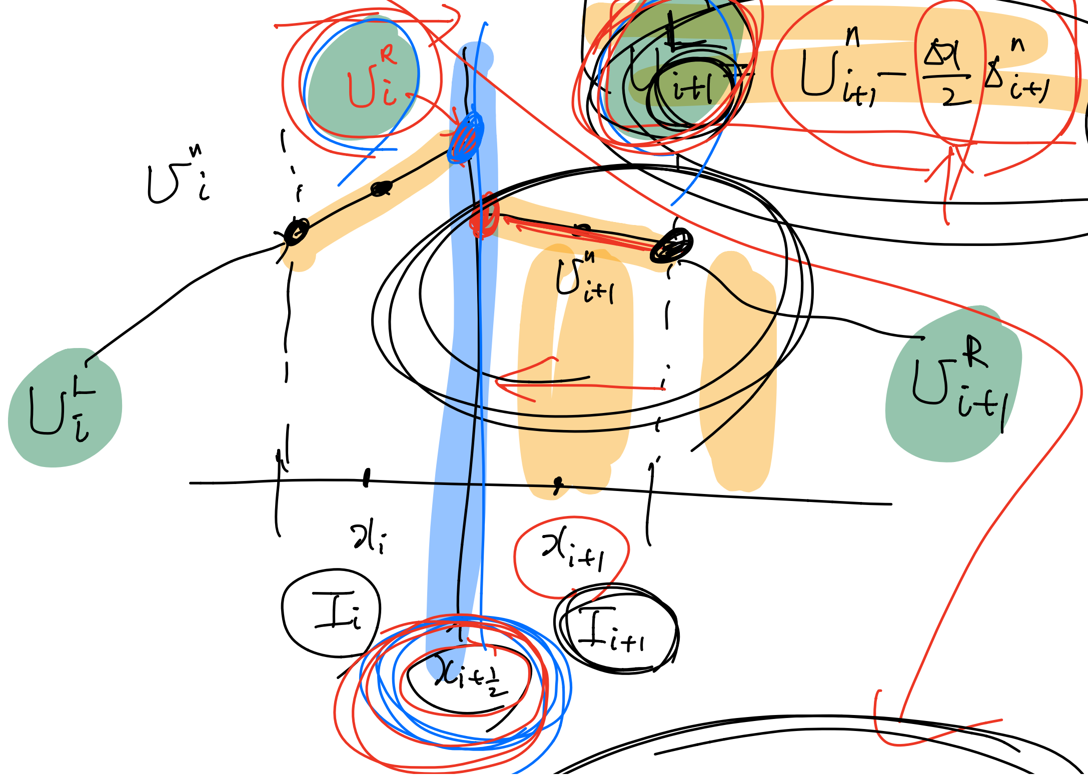
$$\rightarrow \begin{cases} \text{FOG} : & \tilde{u}(x,t^n) = c_0 \,, \quad x \in I_i \\ \text{PLM} : & \tilde{u}(x,t^n) = c_0 + c_1(x-x_i) \,, \quad x \in I_i \\ \text{PPM} : & \tilde{u}(x,t^n) = c_0 + c_1(x-x_i) + c_2(x-x_i)^2 \,, \quad x \in I_i \end{cases}$$

$U_i^n$

$$\tilde{u}(x_i,t^n) = U_i^n$$

$$\text{PLM} : \quad \tilde{u}(x,t^n) = U_i^n + \Delta_i^n (x - x_i) \,, \quad x \in I_i$$

$$\rightarrow \tilde{u}(x_{i+\frac{1}{2}}, t^n) = \begin{cases} U_i^R = U_i^n + \frac{\Delta x}{2}\Delta_i^n & a>0 \\ U_{i+1}^L = U_{i+1}^n - \frac{\Delta x}{2}\Delta_{i+1}^n & a<0 \end{cases}$$



$$\tilde{u}(x, t^n) = U^n_{i+1} + \Delta^n_{i+1}(x - x_{i+1})$$

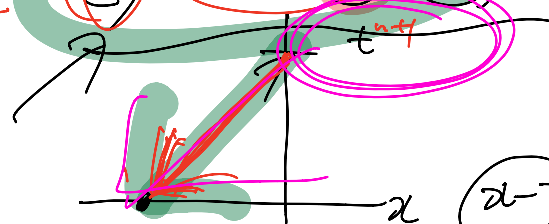
$x \in I_{i+1}$

## Step 2 Evolve (valid for linear scalar advection)

$(u_t + a u_x = 0)$

For $a > 0$,

$$\tilde{u}(x, t^{n+1}) = \tilde{u}(x - a\Delta t, t^n), \quad x \in I_i$$



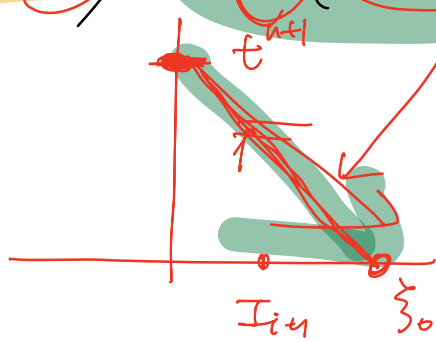

$a > 0$

$x - t$

For $a < 0$,

$$\tilde{u}(x, t^{n+1}) = \tilde{u}(x - a\Delta t, t^n), \quad x \in I_{i+1}$$

$\xi_0$




## Step 3 Average

$u_t + a u_x = 0$

$\tilde{u}(x, t)$

$$t \in [t^n, t^{n+1}]$$

$a > 0$ ;

$$\tilde{u}(x_{i+\frac{1}{2}}, t) = \tilde{u}(x_{i+\frac{1}{2}} - a(t-t^n), t^n)$$

$$= U_i^n + \Delta_i^n (x_{i+\frac{1}{2}} - a(t-t^n) - x_i)$$

$$= U_i^n + \Delta_i^n \left( \frac{\Delta x}{2} - a(t-t^n) \right)$$

$a < 0$ ;

$$\tilde{u}(x_{i+\frac{1}{2}}, t) = \tilde{u}(x_{i+\frac{1}{2}} - a(t-t^n), t^n)$$

$$= U_{i+1}^n + \Delta_{i+1}^n (x_{i+\frac{1}{2}} - a(t-t^n) - x_{i+1})$$

$$= U_{i+1}^n - \Delta_{i+1}^n \left( \frac{\Delta x}{2} + a(t-t^n) \right),$$

$\therefore$

$$\tilde{u}(x_{i+\frac{1}{2}}, t) = \begin{cases} \tilde{U}_L = U_i^n + \Delta_i^n \left( \frac{\Delta x}{2} - a(t-t^n) \right) & a > 0 \\ \tilde{u}_R = U_{i+1}^n - \Delta_{i+1}^n \left( \frac{\Delta x}{2} + a(t-t^n) \right) & a < 0 \end{cases}$$

$$C_a = \frac{a\Delta t}{\Delta x}$$

$$\tilde{u}(x_{i+\frac{1}{2}}, t^{n+\frac{1}{2}}) = \begin{cases} \tilde{u}_L = U_i^n + \frac{\Delta x}{2}\Delta_i^n(1 - C_a) & a>0 \\ \tilde{u}_R = U_{i+\frac{1}{2}}^n - \frac{\Delta x}{2}\Delta_{i+1}^n(1 + C_a) & a<0 \end{cases}$$

$$F_{i+\frac{1}{2}}^{PLM} = \frac{1}{\Delta x}\int_{t^n}^{t^{n+1}} f(\tilde{u}(x_{i+\frac{1}{2}}, t))\,dt$$

$$= f(\tilde{u}(x_{i+\frac{1}{2}}, t^{n+\frac{1}{2}}))$$

mid-pt rule

$$= \begin{cases} f(\tilde{u}^L), & a>0 \\ f(\tilde{u}^R), & a<0 \end{cases}$$

$\blacksquare$