Computational Fluid Dynamics

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Contents

1: Assignment 1

Fri 15 Oct 2021 22:42

0.1 Problem 1

$(F2) \Leftrightarrow (F1) \Leftrightarrow (F3) \Leftrightarrow (F4)$

By Reynolds Transport Equation

$$(F2) \Leftrightarrow (F1).$$

By divergence theorem over arbitrary volume. Can be done in either direction

$$(F1) \Leftrightarrow (F3).$$

By vector identity

$$(F3) \Leftrightarrow (F4).$$

0.2 Problem 2

Consider Burgers' equation

a) Multiply the equation by $2u$ and derive a new conservation law.

Let $w(x(t), t) = u^2(x(t), t)$. Then

$$\frac{\mathrm{d}w}{\mathrm{d}t} = \frac{\mathrm{d}u^2}{\mathrm{d}dt} = 2u\left(u_x \frac{\mathrm{d}x}{\mathrm{d}t} + u_t\right).$$

Where $\frac{\mathrm{d}x}{\mathrm{d}t} = u = \sqrt{w}$ (considering only the positive root). So

$$\begin{aligned}
\frac{\mathrm{d}w}{\mathrm{d}t} &= 2uu_t + 2u\left(\frac{u^2}{2}\right)_x = 0 \qquad (2u \cdot \text{Burgers'}) \\
&= w_t + \sqrt{w}u_x = 0
\end{aligned}$$

is the new conservation law in non-conservative form and $\sqrt{w}$ is the characteristic speed, and the flux function $f(w)_x$ satisfies

$$\begin{aligned}
f(w)_x &= \frac{\mathrm{d}f}{\mathrm{d}w}\frac{\partial w}{\partial x} = \sqrt{w}w_x \\
&\Rightarrow \frac{\mathrm{d}f}{\mathrm{d}w} = w \\
&\Rightarrow \boxed{f(w) = \frac{2w^{\frac{3}{2}}}{3}}
\end{aligned}$$

Hence the new conservation law for $w(x(t),t) = u^2(x(t),t)$ is

$$\boxed{w_t + \left(\frac{2w^{\frac{3}{2}}}{3}\right)_x = 0}.$$

b) Show that the original Burgers' equation and the new derived equation have different weak solutions.

Proof. Show these two equations have different shock speeds for the Riemann problem $u_l > u_r$. If u is a weak solution of the Riemann problem then across the curve of discontinuity $x = \xi(t)$, then u must satisfy the condition

$$\frac{f(u_l) - f(u_r)}{u_l - u_r} = \xi'(t) = \sigma.$$

For the Burgers', we have

$$\frac{\frac{u_l^2}{2} - \frac{u_r^2}{2}}{u_l - u_r} = \sigma_1.$$

and for the new derived equation, we have

$$\frac{\frac{2u_l^{\frac{2}{3}}}{3} - \frac{2u_r^{\frac{2}{3}}}{3}}{u_l - u_r} = \sigma_2.$$

Since $\sigma_1 \neq \sigma_2$, these equations have different weak solutions.

□

0.3 Problem 3

Solve Burgers' on $\mathbb{R}$ for small enough $t \leq t_b$ that allows exact piecewise-linear weak solution with the following initial conditions:

$$u(x,0) = g(x) = \begin{cases} 2 & \text{if } |x| < \frac{1}{2} \\ -1 & \text{otherwise} \end{cases}.$$

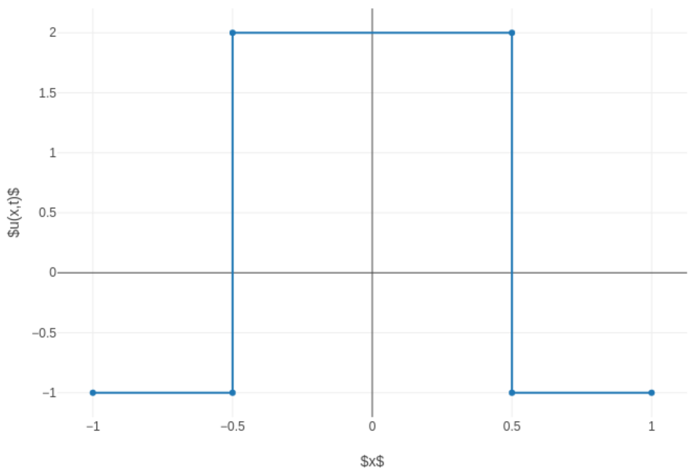

Figure 1: Initial Condition

Considering parameterization $u(x(s), t(s)) \Rightarrow \dfrac{\mathrm{d}u}{\mathrm{d}s} = \dfrac{\mathrm{d}t}{\mathrm{d}s}u_t + \dfrac{\mathrm{d}x}{\mathrm{d}s}u_x$, we get the following characteristic equations

$$\begin{cases} \dfrac{\mathrm{d}t}{\mathrm{d}s} = 1 & \Rightarrow t = s + t_0 \quad \Rightarrow t = s \\ \dfrac{\mathrm{d}u}{\mathrm{d}s} = 0 & \Rightarrow u(s) = u_0 = u(x_0, t_0) = u(x_0, 0) = g(x_0) \\ \dfrac{\mathrm{d}x}{\mathrm{d}s} = u = g(x_0) & \Rightarrow x(s) = g(x_0)s + x_0 \end{cases}.$$

For $t = s$, we have $x(t) = g(x_0)t + x_0$ and $u(x,t) = g(x_0(t))$, so the solution only depends on the initial position x_0. We have 2 cases

Case 1: $x_0 < -\frac{1}{2}$ and $x_0 > \frac{1}{2}$,

$$\begin{aligned} &\Rightarrow g(x_0) = -1 \\ &\Rightarrow x = -t + x_0 \\ &\Rightarrow x_0 = x + t \end{aligned}$$

Lecture 1: Mass Conservation

Mon 04 Oct 2021 11:32

1 Substational Derivative

Let $\vec{u} = \vec{u}(x, y, z, t) =< u(x, y, z, t), v(x, y, z, t), w(x, y, z, t) >$. Then the total derivative

$$d\vec{u} = \frac{\partial \vec{u}}{\partial x}dx + \frac{\partial \vec{u}}{\partial y}dy + \frac{\partial \vec{u}}{\partial z}dz + \frac{\partial \vec{u}}{\partial t}dt.$$

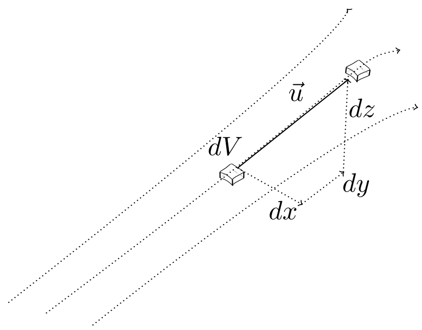

Figure 2: Total derivative

$$\begin{aligned}
\frac{\mathrm{d}\vec{u}}{\mathrm{d}t} &= \frac{\partial \vec{u}}{\partial x}\frac{\mathrm{d}x}{\mathrm{d}t} + \frac{\partial \vec{u}}{\partial y}\frac{\mathrm{d}y}{\mathrm{d}t} + \frac{\partial \vec{u}}{\partial z}\frac{\mathrm{d}z}{\mathrm{d}t} + \frac{\partial \vec{u}}{\partial t} \\
&= \frac{\partial \vec{u}}{\partial x}u + \frac{\partial \vec{u}}{\partial y}v + \frac{\partial \vec{u}}{\partial z}w + \frac{\partial \vec{u}}{\partial t}
\end{aligned}$$

Lagrangian view of a fluid element (substantial derivative) measures the time rate of change of a given quantity (fluid velocity in this case) as it moves from one location to another in both space and time.

$$\boxed{\frac{D\vec{u}}{Dt} := \underbrace{(\vec{u}\cdot\nabla)\vec{u}}_{\text{advection term}} + \underbrace{\frac{\partial \vec{u}}{\partial t}}_{\text{Eulerian view}}}.$$

As an operator,

$$\boxed{\frac{D}{Dt} = \frac{\partial}{\partial t} + (\vec{u}\cdot\nabla)} \tag{1}$$

2 Mass Conservation - continuity equation

2.1 Finite Control Volume (FCV) V fixed in space with fluid moving through it

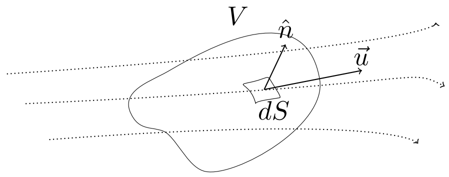

Figure 3: FCV fixed in space

Total mass in volume

$$M_v = \int_V \rho \, dV.$$

mass flux through volume

$$\int_S \rho \vec{u} \cdot \hat{n} \, dS.$$

By conservation, change in mass in volume equals mass flux out

$$\frac{\partial M_v}{\partial t} = -\int_S \rho \vec{u} \cdot \hat{n} \, dS$$
$$\frac{\partial}{\partial t} \int_V \rho \, dV = -\int_S \rho \vec{u} \cdot \hat{n} \, dS$$

This results in the conservative form of the continuity equation in integral form.

$$\boxed{\frac{\partial}{\partial t} \int_V \rho \, dV + \int_S \rho \vec{u} \cdot \hat{n} \, dS = 0} \tag{2}$$

2.2 Finite Control Volume V moving with the fluid

Taking the substantial derivative of the mass M_v of a moving control volume. The change in mass of this control volume is zero as it moves with the fluid.

$$\frac{DM_v}{Dt} = \boxed{\frac{D}{Dt} \int_V \rho \, dV = 0} \tag{3}$$

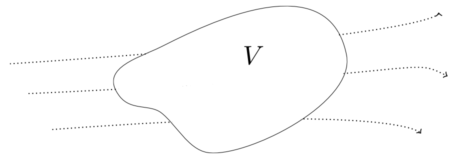

Figure 4: FCV moving with the fluid

This is the non conservative form of the continuity equation in integral form.

2.3 Infinitesimal Fluid Element (IFE) dV fixed in space with fluid moving through it

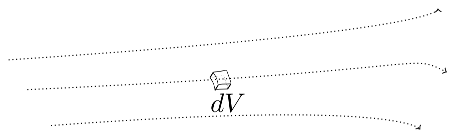

Figure 5: IFE fixed in space

Conservative form in differential form.

2.4 Infinitesimal Fluid Element dV moving along a streamline

Non conservative form in differential form.

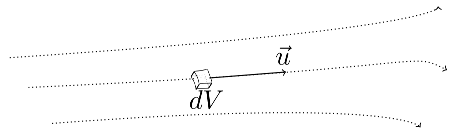

Figure 6: IFE moving along streamline

3 Bringing everything together

From equation **??**, if we assume the volume doesn't change in time, we can bring the derivative inside the integral

$$\int_V \frac{\partial \rho}{\partial t}\, dV + \int_S \rho \vec{u} \cdot \hat{n}\, dS = 0.$$

Then, applying the divergence theorem, we can write the surface integral as a volume integral and combine terms

$$\int_V \frac{\partial \rho}{\partial t}\, dV + \int_V \nabla \cdot \rho \vec{u}\, dV = 0$$

We get an alternative form of FCV fixed in space (F1)

$$\boxed{\int_V \frac{\partial \rho}{\partial t} + \nabla \cdot \rho \vec{u}\, dV = 0} \tag{4}$$

Since this is an arbitrary volume, this implies IFE fixed in space (F3)

$$\boxed{\frac{\partial \rho}{\partial t} + \nabla \cdot \rho \vec{u} = 0} \tag{5}$$

Then applying the vector identity $\nabla \cdot \rho \vec{u} = \vec{u} \cdot \nabla \rho + \rho \nabla \cdot \vec{u}$

$$\int_V \frac{\partial \rho}{\partial t} + \vec{u} \cdot \nabla \rho + \rho \nabla \cdot \vec{u}\, dV = 0.$$

Noting the use of the substantial derivative, we get FCV moving in space (F2)

$$\boxed{\int_V \frac{D\rho}{Dt} + \rho\nabla\cdot\vec{u}\,dV = 0} \tag{6}$$

Since this is an arbitrary volume, we get IFE moving in space (F4)

$$\boxed{\frac{D\rho}{Dt} + \rho\nabla\cdot\vec{u} = 0} \tag{7}$$

So (F1) $\Rightarrow$ (F3) and (F1) $\Rightarrow$ (F2) $\Rightarrow$ (F4)