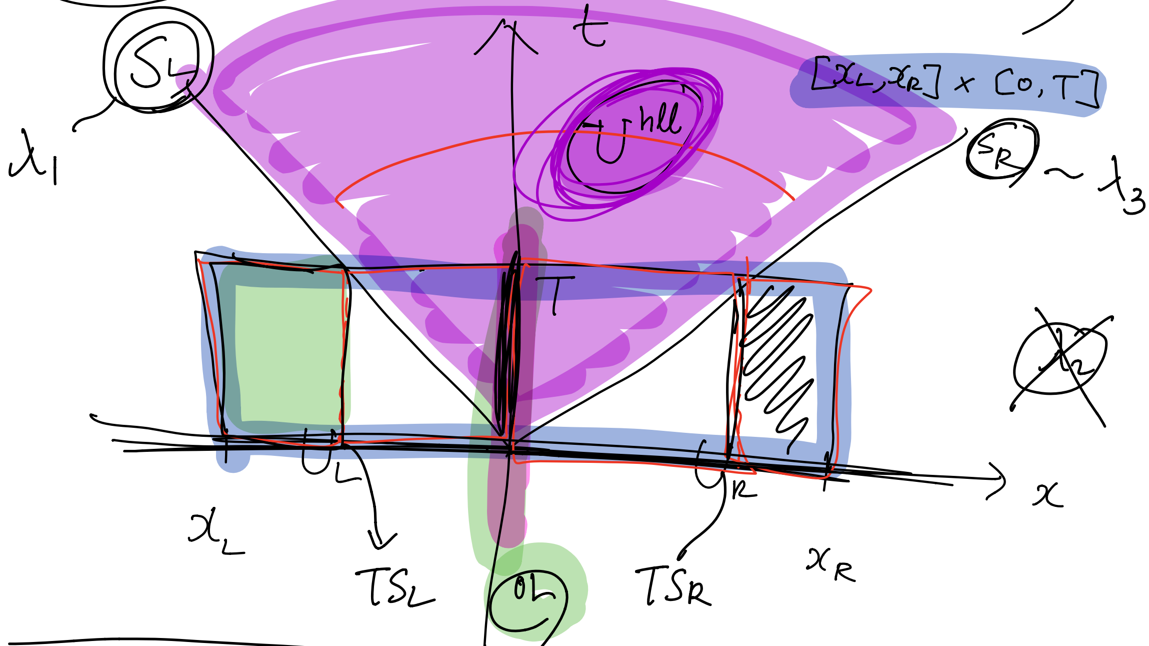
$U^{hll} \rightarrow F^{hll} \neq F(U^{hll})$

HLL (Harten – Lax – van Leer)



$$\boxed{U_t + F(U)_x = 0}$$ governing eqn.

integral form over $[x_L, x_R] \times [0, T]$

$$\int_{x_L}^{x_R} U(x,T)\,dx - \int_{x_L}^{x_R} U(x,0)\,dx$$

$$= \int_0^T F(\overbrace{U(x_L,t)}^{=U_L})\,dt - \int_0^T F(\overbrace{U(x_R,t)}^{U_R})\,dt$$

$$\equiv F_L = F(U_L) \qquad \equiv F_R = F(U_R)$$

$$\int_{x_L}^{x_R} U(x,T)\,dx = x_R U_R - x_L U_L + T(F_L - F_R) \quad (*)$$

$$\rightarrow \text{LHS} = \int_{x_L}^{x_R} U(x,T)\,dx$$

$$= \int_{x_L}^{TS_L} U(x,T)\,dx + \int_{TS_L}^{TS_R} U(x,T)\,dx + \int_{TS_R}^{x_R} U(x,T)\,dx$$

$$= (TS_L - x_L)U_L + \int_{TS_L}^{TS_R} U(x,T)\,dx + (x_R - TS_R)U_R$$

$$\rightarrow \boxed{\int_{TS_L}^{TS_R} U(x,T)\,dx = T(S_R U_R - S_L U_L + F_L - F_R)}$$

$$\rightarrow \frac{1}{T(S_R - S_L)}\int_{TS_L}^{TS_R} U(x,T)\,dx = U^{hll}$$

$$= \frac{S_R U_R - S_L U_L + F_L - F_R}{(S_R - S_L)}$$

Consider the integral over $[x_L, 0] \times [0, T]$:

$$\to \int_{x_L}^{0} U(x,T)\,dx - \int_{x_L}^{0} U(x,0)\,dx = \int_0^T \underbrace{F(U_L)}_{F_L}\,dt - \int_0^T \underbrace{F(U_{0L})}_{F_{0L}}\,dt$$

flux along the t-axis

$$= T(F_L - F_{0L})$$

$$\int_{x_L}^{TS_L} \big(\underbrace{U(x,T)}_{U_L} - \underbrace{U(x,0)}_{U_L}\big)\,dx + \int_{TS_L}^{0} \big(U(x,T) - \underbrace{U(x,0)}_{U_L}\big)\,dx$$

$$= \int_{TS_L}^{0} U(x,T)\,dx + TS_L U_L$$

$[0, x_R] \times [0, T]$

$$\to \boxed{\int_{TS_L}^{0} U(x,T)\,dx = -TS_L U_L + T(F_L - F_{0L})} \quad (**)$$

$$\to \boxed{F_{0L} = F_L - S_L U_L - \frac{1}{T}\int_{TS_L}^{0} U(x,T)\,dx} \quad (***)$$

U^{hll}

$$\rightarrow \boxed{F_{0R} = F_R - S_R U_R + \frac{1}{T}\int_0^{TS_R} \underbrace{U(x,T)}_{U^{hll}}\, dx}$$

$$\rightarrow \int_{TS_L}^{TS_R} U(x,T)\, dx = T\left(F_L - F_R - S_L U_L + S_R U_R\right)$$

$$\rightarrow U(x,t) = \begin{cases} U_L & x/t \le S_L \\ U^{hll} & S_L < x/t < S_R \\ U_R & x/t \ge S_R \end{cases}$$

$$\rightarrow F_{i+\frac{1}{2}} = F_{i+\frac{1}{2}} = \begin{cases} F_L & 0 \le S_L \\ F^{hll} & S_L \le 0 \le S_R \\ F_R & 0 \ge S_R \end{cases}$$

$$F_{0L} = F_L - S_L U_L - \frac{1}{T}\int_{TS_L}^{0} U^{hll}\, dx$$

$$= F_L - S_L U_L + \cancel{\frac{1}{T}}\, \cancel{T} S_L U^{hll}$$

$$= F_L + S_L (U^{hll} - U_L)$$

$$F_{0R} = F_R + S_R (U^{hll} - U_R)$$

$$\| \quad F^{hll}$$

F^{Roe}

$$\rightarrow F^{hll} = F_L + S_L \left(\frac{S_R U_R - S_L U_L + F_L - F_R}{S_R - S_L} - U_L \right)$$

$$F = \begin{pmatrix} \rho u \\ \rho u^2 + p \\ u(\rho E + p) \end{pmatrix} = \cdots$$

$$= \frac{S_R F_L - S_L F_R + S_L S_R (U_R - U_L)}{S_R - S_L}$$

$F_{L,R} = F(U_{L,R})$

Rmk. ① $F^{hll} \neq F(U^{hll})$

$$\lambda_{1,L} \qquad \lambda_{1,R}$$

② $S_L = \min \{ \overbrace{U_L - C_{s,L}}^{=\lambda_{1,L}}, \overbrace{U_R - C_{s,R}}^{=\lambda_{1,R}} \}$

flow velocity

$$U_{L,R} = \begin{pmatrix} \rho_{L,R} \\ \rho_{L,R} u_{L,R} \\ \rho_{L,R} E_{L,R} \end{pmatrix}$$

$$c_{s,L} = \sqrt{\frac{\gamma p_L}{\rho_L}}$$

$$c_{s,R} = \sqrt{\frac{\gamma p_R}{\rho_R}}$$

$$S_R = \max\{\underbrace{u_L + c_{s,L}}_{\lambda_{3,L}},\ \underbrace{u_R + c_{s,R}}_{\lambda_{3,R}}\}$$

λ_2

$\lambda_1, \lambda_2, \lambda_3$

Roe Riemann Solver. (P. Roe, 1981)

Roe's approach: $\underline{A(U)} \to A(\underline{U_{avg}})$

$$U_{avg} = U_{avg}(U_L, U_R) = \begin{cases} \cancel{\frac{1}{2}(U_L + U_R)} \\ \text{Roe avg} \end{cases}$$

$$\to U_t + \bar{A} U_x = 0$$

$$U(x,0) = \begin{cases} U_L, & x>0 \\ U_R, & x<0. \end{cases}$$

Def

$$\bar{A} = A(U_{avg})$$

$$F(U)_x = A(U)\,U_x$$

$$\bar{F}(U) \neq F(U)$$

$$\Downarrow$$

$$\bar{F}(U)_x = \bar{A}\,U_x$$

Three conditions on $\bar{A}$

① Hyperbolicity: $\bar{A}$ has real eig. vals $\bar{\lambda}_1 \le \bar{\lambda}_2 \le \bar{\lambda}_3$.

② Consistency with exact Jacobian for constant flow;

$$\bar{A}(U,U) = A(U)$$

~~3~~ conservation across jump condition;

$$F(U_R) - F(U_L) = \bar{A}(U_R - U_L)$$

Dropping "—";

char. variables

$$U_L = \sum_{i=1}^{3} \alpha_i r_i$$

$$W = L U$$

$$U_R = \sum_{i=1}^{3} \beta_i r_i$$

$$l_k = (l_{k,1}, l_{k,2}\ l_{k,3})^T$$

$$\alpha_k = l_k \cdot U_L$$

$$= l_{k,1} \rho_L + l_{k,2} (\rho u)_L + l_{k,3} (\rho E)_L$$

$$\beta_k = l_{k,1} \rho_R + l_{k,2} (\rho u)_R + l_{k,3} (\rho E)_R$$

$$\Rightarrow \Delta U = \sum_{i=1}^{3} (\beta_i - \alpha_i) r_i$$

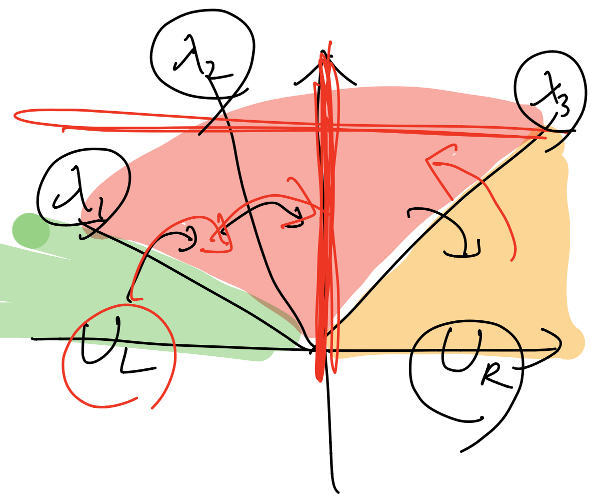
$$= \sum_{i=1}^{3} l_k \cdot (U_R - U_L) r_i$$

From the left state U_L :

$$U_{i+\frac{1}{2}}(0)$$

$$\|$$

$$U_L + \sum_{\lambda_i \le 0} l_i \cdot (U_R - U_L) r_i$$



$$\|$$

$$U_R - \sum_{\lambda_i \ge 0} l_i \cdot (U_R - U_L) r_i$$

$$\bar{F} = \bar{A} U$$

$$F^{Roe}_{i+\frac{1}{2}} = ?$$

<u>Idea</u> : $F^{Roe}_{i+\frac{1}{2}} = \bar{A}\, U_{i+\frac{1}{2}}(0)$

ex

$$F^{Roe}_{i+\frac{1}{2}} = \bar{A}\,U_{i+\frac{1}{2}}(0)$$

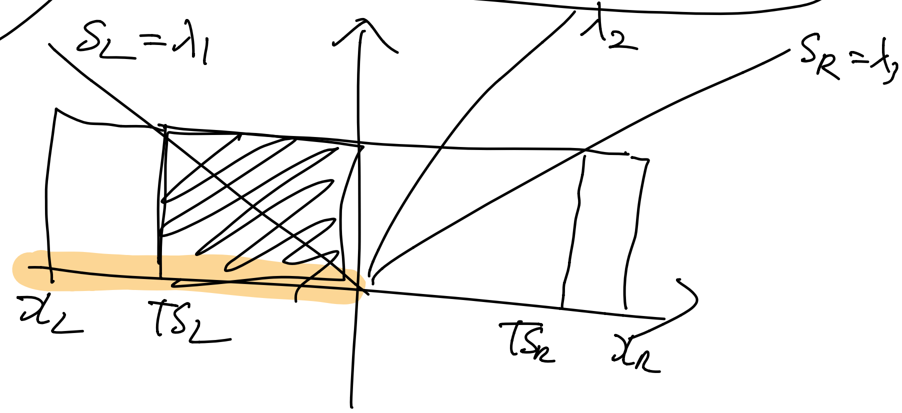
$$= \bar{A}\, U_L$$

$$= \bar{F}(U_L)$$

$$\ne F(U_L) = F_L$$

$U_{i+\frac{1}{2}}(x,t)$

$[TS_L, 0] \times [0, T]$



$$U_t + \bar{F}(U)_x = 0$$

U_L

$$\int_{TS_L}^{0} U_{i+\frac{1}{2}}(x,T)\,dx - \int_{TS_L}^{0} U_{i+\frac{1}{2}}(x,0)\,dx$$

$U_{i+\frac{1}{2}}(0)$

$$= \int_0^T \bar{F}(U_L)\,dt - \int_0^T \bar{F}(U_{0L})\,dt$$

$$\int_{TS_L}^{0} U_{i+\frac{1}{2}}(x,T)\,dx = -TS_L U_L + T\left(\bar{F}(U_L) - \bar{F}(U_{i+\frac{1}{2}}(0))\right)$$

$[0, TS_R] \times [0, T]$

② $$\int_0^{TS_R} U_{i+\frac{1}{2}}(x,T)\,dx = TS_R U_R + T\left(\bar{F}(U_{i+\frac{1}{2}}(0)) - \bar{F}(U_R)\right)$$

③

$$\rightarrow \boxed{\begin{aligned} F_{0L} &= F_L - S_L U_L - \frac{1}{T}\int_{TS_L}^{0} U_{i+\frac{1}{2}}(x,T)\,dx \\ F_{0R} &= F_R - S_R U_R + \frac{1}{T}\int_{0}^{TS_R} U_{i+\frac{1}{2}}(x,T)\,dx \end{aligned}}$$

④

$$\rightarrow F_{0L} = F_L - S_L U_L - \frac{1}{T}\left(\cdot \quad -- \right)$$

$$= F_L - \bar{F}(U_L) + \bar{F}(U_{i+\frac{1}{2}}(0))$$

$$\equiv \bar{F}_L$$

$$F_{0R} = F_R - \bar{F}_R - \bar{F}(U_{i+\frac{1}{2}}(0))$$

$\rightarrow$ Now $\bar{F} = \bar{A} U$

$$\bar{A} r_k = \lambda_k r_k$$

$$\rightarrow F_{0L} = F_L - \cancel{\bar{A} U_L} + \bar{A}\left[\cancel{U_L} + \sum_{\lambda_k \le 0} l_k \cdot (U_R - U_L) r_k\right]$$

$$= F_L + \sum_{\lambda_k \le 0} l_k \cdot (U_R - U_L) r_k$$

$$\|$$

$$F_{0R} = F_R - \sum_{\lambda_k \ge 0} l_k \cdot (U_R - U_L) r_k$$

$$\|$$

$$F^{Roe}_{i+\frac{1}{2}} = \frac{1}{2}\left(F_{0L} + F_{0R}\right)$$

$$= \frac{1}{2}\left(F_R + F_L\right) - \frac{1}{2}\sum_{k=1}^{3} l_k \cdot (U_R - U_L) |\lambda_k| r_k$$

$$\bar{A} = A(U_{avg}) = A\left(\frac{1}{2}(U_L + U_R)\right)$$

Roe avgs

$$F(U_R) - F(U_L) = \bar{A}(U_{avg})(U_R - U_L)$$

$$\rho_{avg} = \sqrt{\rho_L \rho_R}$$

$$u_{avg} = \frac{\sqrt{\rho_L}\, u_L + \sqrt{\rho_R}\, u_R}{\sqrt{\rho_L} + \sqrt{\rho_R}}$$

$$H_{avg} = \frac{\sqrt{\rho_L}\, H_L + \sqrt{\rho_R}\, H_R}{\sqrt{\rho_L} + \sqrt{\rho_R}}$$

$$V_{avg}^2 = u_{avg}^2 + v_{avg}^2 + w_{avg}^2$$

$$\rightarrow \quad C_{avg} = \left[(\gamma - 1)\left(H_{avg} - \frac{1}{2} V_{avg}^2\right)\right]^{\frac{1}{2}}$$

enthalpy

$$\rightarrow \quad H_{L,R} = (e + p/\rho)_{L,R}$$

Chapter 9. $\underbrace{\text{FOG, PLM, PPM}}$, WENO