AM 260, Winter 2021
Homework 2

Posted on Tue, Feb 2, 2021
Due 11:59 pm, Mon, Feb 15, 2021

Submit your homework to your Git repository by 11:59 pm

- You are recommended to use LaTex or MS-words like text editors for homework. A scanned copy of a handwritten solutions will still be accepted on condition that your handwriting is clean and well-organized, and your scanned copy is fully readable.

Part 1: Theory Problems

Problem 1. Consider the Lax-Friedrichs (LF) method for solving the scalar advection $u_t + f(u)_x = 0$ with $f(u) = au$, where $a > 0$ or $a < 0$,

$$U_i^{n+1} = \frac{1}{2}\Big(U_{i+1}^n + U_{i-1}^n\Big) - \frac{\Delta t}{2\Delta x}\Big(f(U_{i+1}^n) - f(U_{i-1}^n)\Big). \tag{1}$$

(a) Show that the LF method is consistent and stable for $|C_a| \le 1$, where $C_a = \frac{a\Delta t}{\Delta x}$.

(b) Show that the LF method is $\mathcal{O}(\Delta t + \Delta x)$.

(c) Rewrite the LF method in the conservative form,

$$U_i^{n+1} = U_i^n - \frac{\Delta t}{\Delta x}\Big(\hat{f}_{i+1/2}^n - \hat{f}_{i-1/2}^n\Big), \tag{2}$$

that is to say, find the expressions for $\hat{f}_{i\pm1/2}^n$ as functions of U_k^n and the original flux $f(U_k^n)$, $k = -1, 0, 1$.

Problem 2. Consider the Lax-Wendroff (LW) method for solving the scalar advection $u_t + au_x = 0$ with $f(u) = au$, where $a > 0$ or $a < 0$, and $C_a = \frac{a\Delta t}{\Delta x}$,

$$U_i^{n+1} = U_i^n - \frac{C_a}{2}\Big(U_{i+1}^n - U_{i-1}^n\Big) + \frac{C_a^2}{2}\Big(U_{i+1}^n - 2U_i^n + U_{i-1}^n\Big). \tag{3}$$

(a) Show that the LW method is consistent and stable if $|C_a| \le 1$.

(b) Show that the LW method is $\mathcal{O}(\Delta t^2 + \Delta x^2)$.

Problem 3. Use the von Neumann analysis to show that the forward in time centered in space scheme (FTCS) for the advection $u_t + au_x = 0$ with $a > 0$ or $a < 0$,

$$U_j^{n+1} = U_j^n - \frac{a\Delta t}{2\Delta x}\left(U_{j+1}^n - U_{j-1}^n\right) \tag{4}$$

is unconditionally unstable (i.e., unstable for any choices of $\Delta t > 0$).

Part 2: Coding Problems

Use Fortran 90 or C to implement the following schemes. A template MATLAB code for the upwind method to solve the linear scalar advection $u_t + au_x = 0$ is available as an example from the Homework 2 website. Study this MATLAB code first. To learn the basic discretization strategies, take a look at the separate document, "Note on the basic discretization setup" from the Homework 2 website.

Problem 4. Implement the LF method in Eqn. (1) to numerically solve the sinusoidal advection problem

$$u_t + au_x = 0, \quad a = 1, \tag{5}$$

with an IC: $u(x, 0) = \sin(2\pi x)$, on $x \in [0, 1]$. Use the periodic boundary condition on both ends at $x = 0$ and $x = 1$.

Run your code on two different grid resolutions of $N = 32, 64, 128$ with CFL numbers of 0.8, 1.0, and 1.2. Show your plots at $t = t_{cycle1}$ at all two grid resolutions, where t_{cycle1} is the time the sinusoidal wave returns to the initial position (Hint: You can easily find t_{cycle1} analytically first). Describe your findings and compare the LF results with the first-order upwind method provided in the MATLAB code.

Problem 5. Repeat the comparison study in Problem 4 on $[-1, 1]$ using a discontinuous initial condition,

$$u(x, 0) = \begin{cases} 1 \text{ for } |x| < 1/3, \\ \\ 0 \text{ for } 1/3 < |x| \le 1. \end{cases} \tag{6}$$

As before, use the periodic boundary condition on both ends at $x = -1$ and $x = 1$. Use the same sets of grid resolutions, the CFL numbers, and $t = t_{cycle1}$ as in Problem 4.

Problem 6. Repeat Problem 4 using the LW method in Eqn. (3).

Problem 7. Repeat Problem 5 using the LW method in Eqn. (3).