Godunov Method for FVM

$$U_i^n \approx \underbrace{\frac{1}{\Delta x}\int_{I_i} u(x, t^n)\, dx}$$ FVM

$u(x_i, t^n)$ FDM

$$\boxed{u_t + a u_x = 0}$$

$$u_t + \left(\frac{u^2}{2}\right)_x = 0$$

$$U_i^{n+1} = U_i^n - \frac{a\Delta t}{\Delta x}\left[U_i^n - U_{i-1}^n\right]$$

→ 1^{st} order upwind a>0

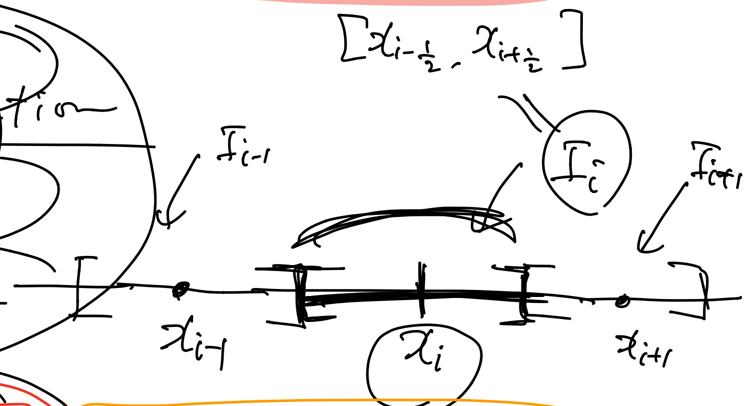
→ The Godunov method (1959)

for "nonlinear Euler equs."

Reconstruction

Evolution

Average



Step 1 : Reconstruct a piecewise poly $\tilde{u}(x,t^n)$ $\forall x \in I_i$

$\tilde{u}(x_i,t^n)$

$$U_i^n = \frac{1}{\Delta x}\int_{I_i} u(x,t^n)\,dx$$

Ex for piecewise constant :

$$\tilde{u}(x,t^n) = U_i^n \quad \forall x \in I_i$$

pointwise quantity

Volume or integral quantity

Step 2 : Evolve the hyperbolic eqn

$u_t + a u_x = 0$

either exactly or approximately with the initial data $\tilde{u}(x, t^n)$ to obtain $\tilde{u}(x, t^{n+1})$

Step 3 Average $\tilde{u}(x, t^{n+1})$ over each cell I_i to get a new average data

$$v_i^{n+1} = \frac{1}{\Delta x} \int_{I_i} \tilde{u}(x, t^{n+1})\, dx.$$

Repeat Step 1 — Step 3 until

$$t = t_{max} = N \Delta t.$$

Rmk. The Godunov's method is the 1st successful conservative method

CIR method (1952)

Courant Isaacson Rees

$$\frac{df(u)}{du} = \lambda(u)$$

advection speed

CIR

(i) if $f'(U_i^n) > 0$ a

$$U_i^{n+1} = U_i^n - \frac{f'(U_i^n)\Delta t}{\Delta x}\left[U_i^n - U_{i-1}^n\right]$$

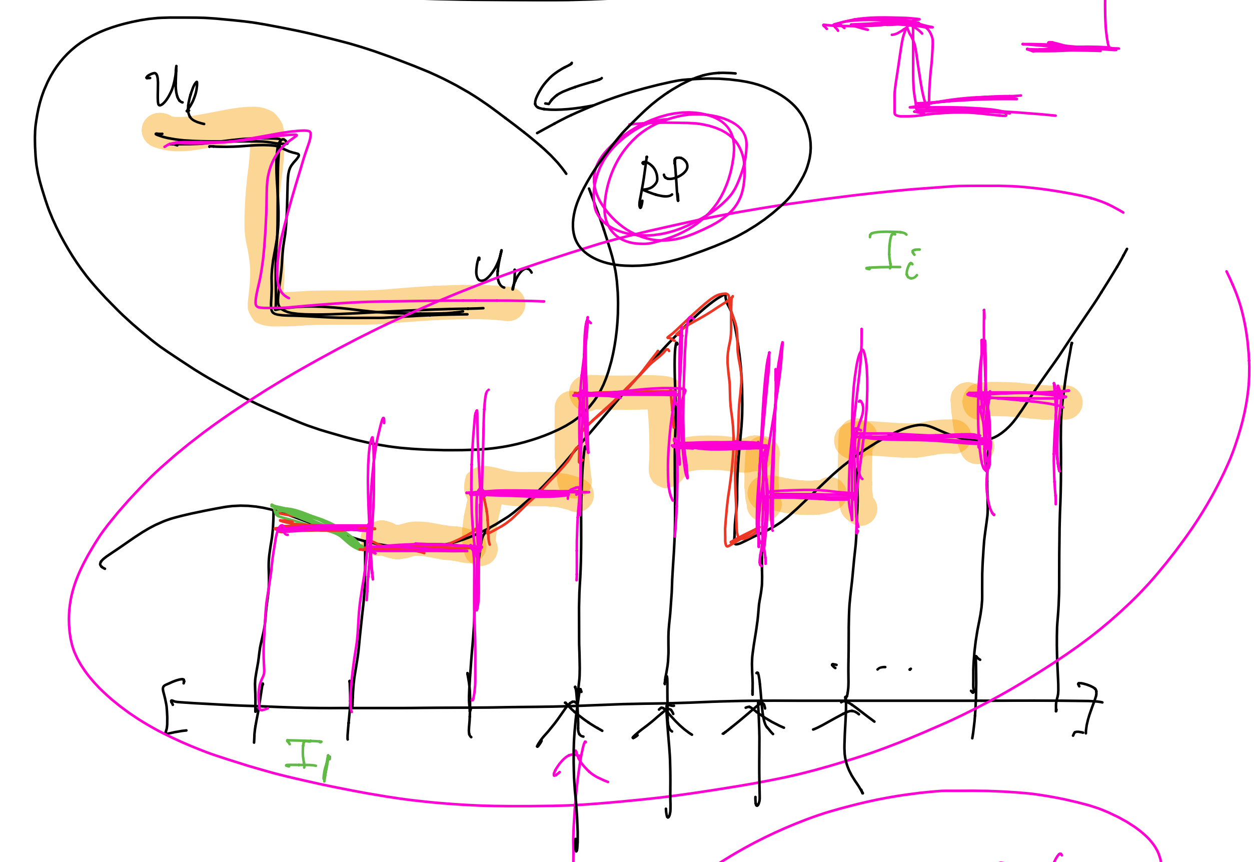
$$= \left[\left(1 - \frac{f'(U_i^n)\Delta t}{\Delta x}\right)U_i^n + \left(\frac{f'(U_i^n)\Delta t}{\Delta x}\right)U_{i-1}^n\right.$$

(ii) if $f'(U_i^n) < 0$

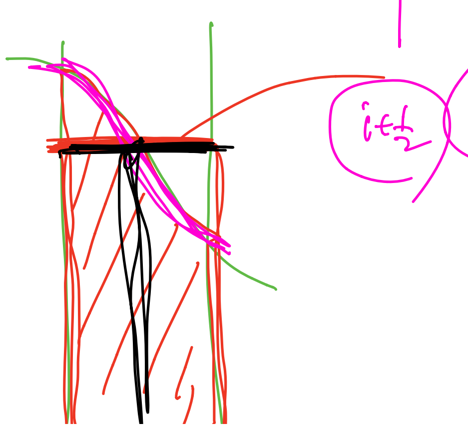
$$U_i^{n+1} = U_i^n - \frac{f'(U_i^n)\Delta t}{\Delta x}\left[U_{i+1}^n - U_i^n\right]$$

$$= \left[\left(1 + \frac{f'(U_i^n)\Delta t}{\Delta x}\right)U_i^n + \left(-\frac{f'(U_i^n)\Delta t}{\Delta x}\right)U_{i+1}^n\right.$$

$$u_t + f'(u)\, u_x = 0$$



local RPs



$F^n_{i+\frac{1}{2}}$

$$\int_a^b f(x)\,dx = f\left(\frac{a+b}{2}\right)$$

midpt rule

$U_i^1 \quad U_i^2 \cdots$

$$\begin{cases} \text{PDE: } u_t + f(u)_x = 0 \\ \text{IC: } u(x,0) = u_0(x) = \begin{cases} U_i^0 & x < x_{i+\frac{1}{2}} \\ U_{i+1}^0 & x > x_{i+\frac{1}{2}} \end{cases} \end{cases}$$

at each i.

Godunov method

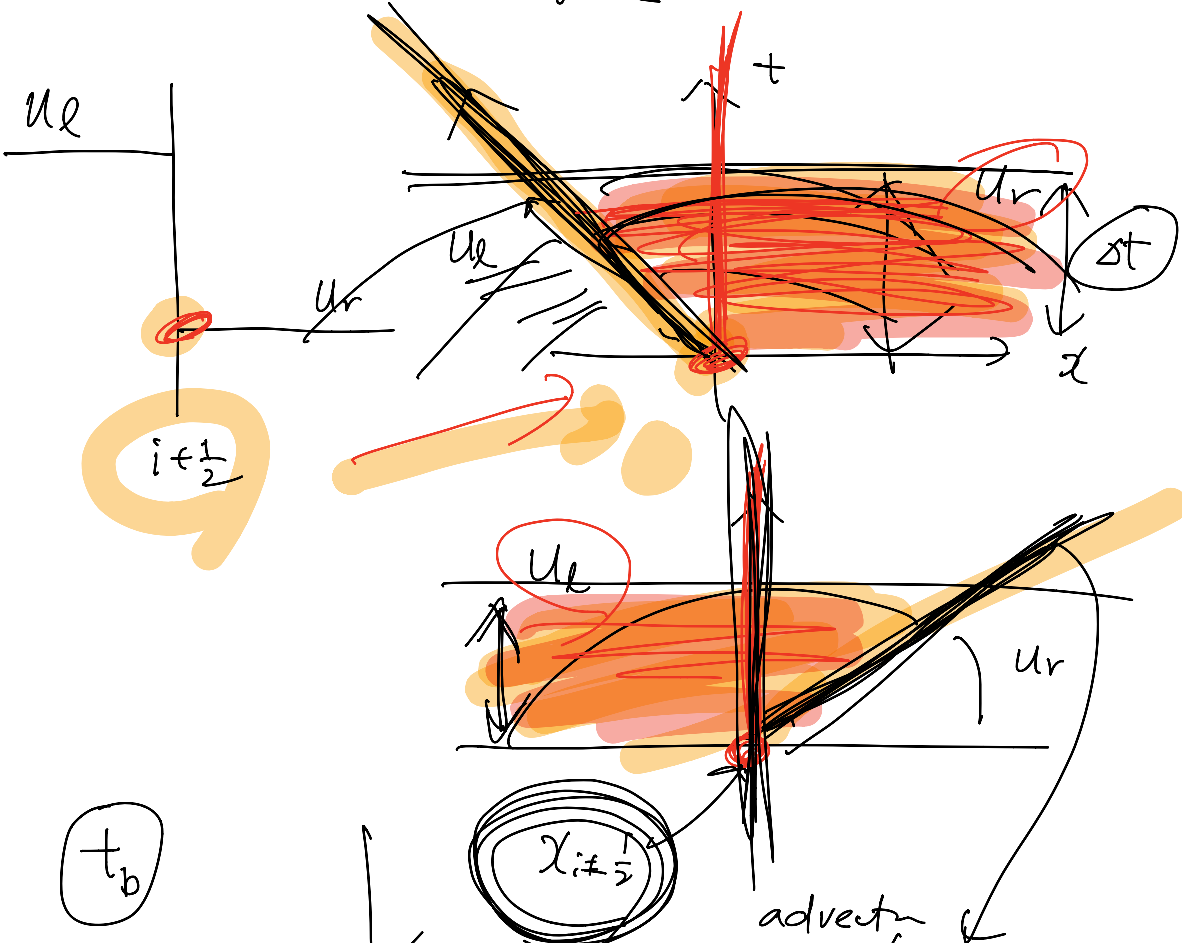
(i) considering the conservative integral form

(ii) computing upwind stable Godunov fluxes

Using piecewise polys $\tilde{u}(x,t)$ on I_i, $x \in I_i$, we define

$$F^{n,\,God}_{i\pm\frac{1}{2}} \stackrel{def}{=} \frac{1}{\Delta t}\int_0^{\Delta t} f\left(\tilde{u}\left(x_{i\pm\frac{1}{2}}, t\right)\right) dt$$

→ time averaged flux at $i\pm\frac{1}{2}$



advectn velocity

$$x = ct + x_{i\pm\frac{1}{2}}$$

→ let $\tilde{u}^*_{i\pm\frac{1}{2}}$ be the constant solns of RP :

$$\begin{cases} RP\left(U_{i-1}^{n}, U_{i}^{n}\right) \leftarrow i-\frac{1}{2} \\[1em] RP\left(U_{i}^{n}, U_{i+1}^{n}\right) \leftarrow i+\frac{1}{2} \end{cases}$$

$$\to F^{n,\,God}_{i\pm\frac{1}{2}} = \frac{1}{\Delta t}\int_{0}^{\Delta t} f\Big(\underbrace{\tilde{u}\left(x_{i\pm\frac{1}{2}}, t\right)}_{\text{constant over } [0,\,\Delta t]}\Big)\, dt$$

$$= \frac{1}{\Delta t}\int_{0}^{\Delta t} \underbrace{f\left(\tilde{u}^{*}_{i\pm\frac{1}{2}}\right)}_{\text{const.}}\, dt$$

$$= f\left(\tilde{u}^{*}_{i\pm\frac{1}{2}}\right)$$

<u>Rmk</u>. The Godunov flux $F^{n,\,God}_{i+\frac{1}{2}}$ is <u>consistent</u> with f

if $U^{n}_{i-1} = U^{n}_{i} = U^{n}_{i+1} = \bar{u}$

then $\tilde{u}^*_{i\pm\frac{1}{2}} = \bar{u}$,

therefore $F^{n,\,God}_{i\pm\frac{1}{2}} = f(\bar{u})$ $\boxtimes$

Ex For constant coeff advection,

$u_t + a u_x = 0,$ $\leftarrow$ $f(u) = au$

$$F^{n,\,God}_{i+\frac{1}{2}} = f(\tilde{u}^*_{i+\frac{1}{2}}) \leftarrow$$

I_i I_{i+1} $i+\frac{1}{2}$

$$= f(RP(U^n_i, U^n_{i+1}))$$

$$= \begin{cases} f(U^n_i) = aU^n_i, & a>0 \\ f(U^n_{i+1}) = aU^n_{i+1}, & a<0 \end{cases}$$

$\Rightarrow$ CIR $U^n_i \to U^{n+1}_i$

$$u_t + a u_x = 0$$

$$u_t + \left(\frac{u^2}{2}\right)_x = 0$$

(i) Shock soln ; $U_i^n > U_{i+1}^n$

$x = x_{i+\frac{1}{2}}$

$$F_{i+\frac{1}{2}}^{n,God} = f(\tilde{u}_{i+\frac{1}{2}}^*) = \begin{cases} f(U_i^n), & S_{i+\frac{1}{2}} \geq \frac{x - x_{i+\frac{1}{2}}}{t} \\ f(U_{i+1}^n), & S_{i+\frac{1}{2}} < \frac{x - x_{i+\frac{1}{2}}}{t} \end{cases}$$

$$S_{i+\frac{1}{2}} = \frac{[f]}{[u]} = \frac{1}{2}(U_i^n + U_{i+1}^n)$$

(ii) $U_i^n \leq U_{i+1}^n$ Rarefaction

$$F_{i+\frac{1}{2}}^{n,God} = f(\tilde{u}_{i+\frac{1}{2}}^*)$$

$$= \begin{cases} f(U_i^n), & \dfrac{x - x_{i+\frac{1}{2}}}{t} \le U_i^n \\ f\left(\dfrac{x - x_{i+\frac{1}{2}}}{t}\right), & U_i^n < \dfrac{x - x_{i+\frac{1}{2}}}{t} \le U_{i+1}^n \\ f(U_{i+1}^n), & \dfrac{x - x_{i+\frac{1}{2}}}{t} > U_{i+1}^n \end{cases}$$

(i) $U_i^n > U_{i+1}^n$;

$$F_{i+\frac{1}{2}}^{n,God} = \begin{cases} f(U_i^n) & S_{i+\frac{1}{2}} \geq 0 \\ f(U_{i+1}^n) & S_{i+\frac{1}{2}} < 0 \end{cases}$$

(ii) $U_i^n \le U_{i+1}^n$;

$f(U^n)$ $\quad 0 < U^n_i$

$$F^{n,God}_{i+\frac{1}{2}} = \begin{cases} f(U_i^n), & 0 \leq U_i^n \\ f(0), & U_i^n < 0 \leq U_{i+1}^n \\ f(U_{i+1}^n), & 0 > U_{i+1}^n \end{cases}$$