Chapter 7. Higher order methods for Scalar conservation Laws

linear nonlinear

"spatially high order"

Goal: We want "high-order" methods that have no unphysical oscillations near large gradients

RK3

$n \to n+1$

Rmk

1D

$$
\begin{cases}
\text{FOG} = O(\Delta t + \Delta x) \\
\text{PLM} = O(\Delta t^2 + \Delta x^2) \\
\text{PPM} = O(\Delta t^2 + \Delta x^3) \\
\text{WENO} = O(\Delta t^2 + \Delta x^5)
\end{cases}
$$

"monotone methods"

The Godunov Thm

$\boxed{\text{Monotone methods are at most first-order.}}$

↳ "Only true for **linear** schemes"

→ TVD (total variation diminishing)

↑

→ "limiter"

slope-limiter

"flux-limiter"

Bram van Leer 5 paper

(1973 ~ 1979)

U^{n-1}

Def. A 2-level method

$$U_i^{n+1} = \mathcal{N}(U^n_{i-r+1}, \cdots, U^n_{i+s}), \quad r, s > 0$$

is said to be a TVD scheme

if $TV(U^{n+1}) \le TV(U^n)$, $\forall$

$$TV(U^n) = \sup_N \sum_{j=1}^{N} |U_j^n - U_{j-1}^n|$$

Def. A two-level method is

monotone if $\dfrac{\partial \mathcal{N}}{\partial U_k^n} \geq 0$, $\forall k$

Ex) LF for $u_t + a u_x = 0$ is monotone

$\because$ $U_i^{n+1} = \frac{1}{2}(1+C_a) U_{i-1}^n + \frac{1}{2}(1-C_a) U_{i+1}^n$

$$\rightarrow \frac{\partial N}{\partial U_k} = \begin{cases} \frac{1}{2}(1+C_a) > 0 & k = i-1 \\ 0 & k = i \\ \frac{1}{2}(1-C_a) > 0 & k = i+1 \end{cases}$$

$(0 < C_a \leq 1)$

Ex) LW is NOT monotone;

$$U_i^{n+1} = \frac{1}{2} C_a (1+C_a) U_{i-1}^n + (1-C_a^2) U_i^n - \underbrace{\frac{1}{2} C_a (1-C_a)}_{U_{i+1}^n}$$

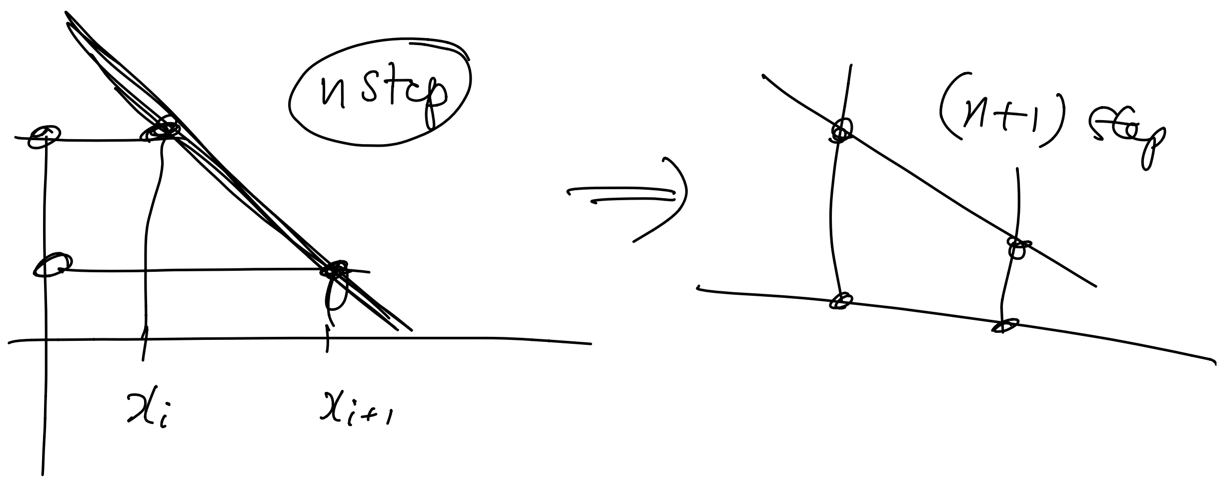
$\because$ $\frac{\partial U_i^{n+1}}{\partial U_{i+1}^n} = -\frac{1}{2} C_a (1-C_a) \leq 0$

$0 < C_a \leq 1$

Def. A two-level method is monotonic preserving scheme (MPS) if

$$U_i^n \ge U_{i+1}^n \quad (\text{or } U_i^n \le U_{i+1}^n), \ \forall i$$

$$\Rightarrow U_i^{n+1} \ge U_{i+1}^{n+1} \quad (\text{or } U_i^{n+1} \le U_{i+1}^{n+1}), \ \forall i.$$



Rmk. Any TVD method is MPS.

Def. TVB (total variation bounded

if $TV(U^n) \le R$

$\forall \; t \le T, \; \forall n$

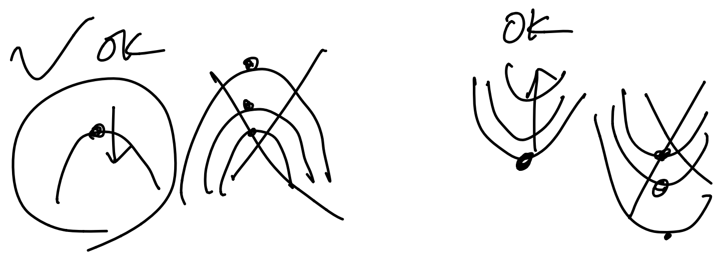
$\quad \| \; n\Delta t$

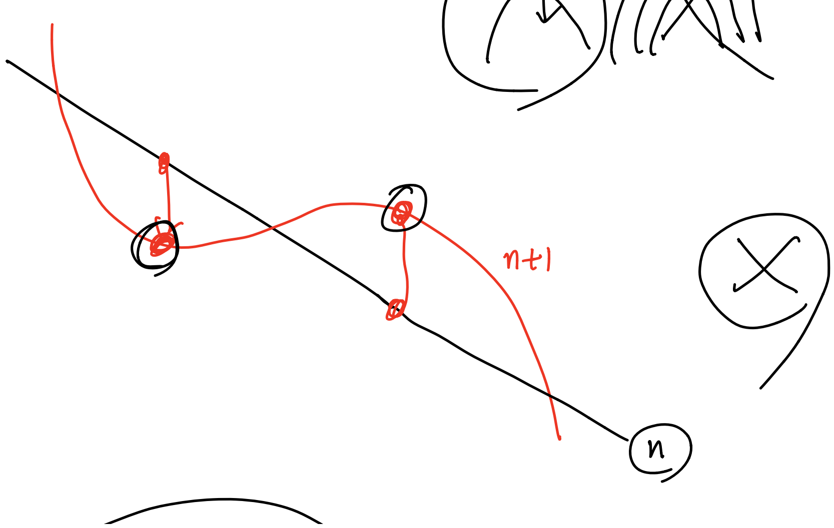
<u>Rmk</u>. If a method is TVB, then

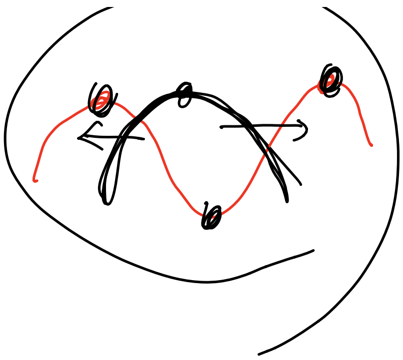
① soln has no new extrema

② " " local min that keeps increasing

③ " " local max that keep decreasing.







→ TVB is the weakest nonlinear stability condition

→ $S_{mon} \subseteq S_{TVD} \subseteq S_{TVB} \subseteq S_{MPS}$