PPM

$$P_i(x) = c_0 + c_1(x-x_i) + c_2(x-x_i)^2$$
$$x \in I_i.$$

PPM, 1984. ← Colella & Woodward

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$\bar{q}_i^n$ $\quad$ q_i^n

→ primitive fn or antiderivative fn

$F(x)$ of $f(x)$

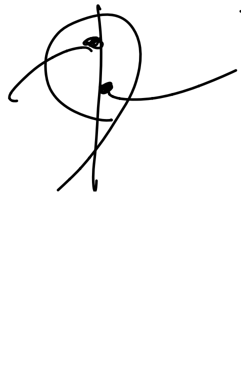
$$f(x) = \frac{dF}{dx} \quad \text{or} \quad F(x) = \int_{x_{i-\frac{1}{2}}}^{x} f(\xi)\, d\xi$$

→ reconstruction $\bar{q}_i^n$ ← input given

→ $P_i(x)$ ⟶ $q^n_{i\pm\frac{1}{2}}$ (output) → $q^{n+\frac{1}{2}}_{i\pm\frac{1}{2}}$

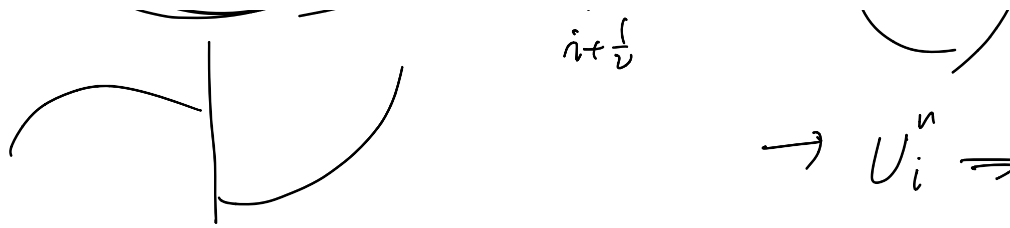
Cond 1

Cond 2



characteristic tracing

→ $F^{n+\frac{1}{2}}_{i\pm\frac{1}{2}}$



$\rightarrow U_i^n \Rightarrow U_i^{n+1}$

$\boxed{1}$ Parabolic profile → for each prim. var.

$$P_i(x) = \underline{C_0 + C_1(x - x_i) + C_2(x - x_i)^2}$$

$\underline{\rho,\ u,\ p}$ $P_{i:\underline{m}}$

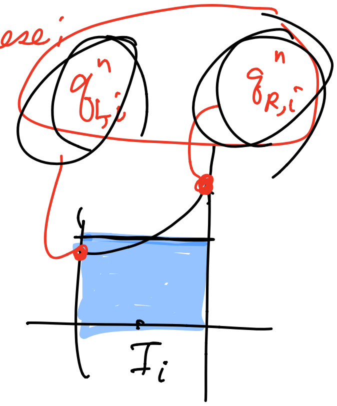
$$\bar{P}_i = \frac{1}{\Delta x}\int_{I_i} P_i(x)\,dx = \frac{1}{\Delta x}\int_{I_i} q_i^n(x)\,dx = \underline{\bar{q}_i^n}$$

$$\| $$

$$\frac{1}{\Delta x}\left(C_0\,\Delta x + \frac{2}{3}C_2\left(\frac{\Delta x}{2}\right)^3\right)$$

$$= C_0 + \frac{1}{12}C_2\,\Delta x^2$$

Assume we know these:



$$\Rightarrow \quad (i) \quad C_0 + \frac{1}{12}C_2\,\Delta x^2 = \bar{q}_i^n$$

$$(ii)\ \&\ (iii) \quad v_{L,R,i}^n = \underline{P_i(x_{i\pm\frac{1}{2}})} = C_0 \pm \frac{C_1}{2}\Delta x + \frac{C_2}{4}\Delta x^2$$

$$\Rightarrow \begin{cases} c_0 = \bar{q}_i^n - \dfrac{c_2}{12}\Delta x^2 \\ c_1 = \dfrac{1}{\Delta x}\left(q_{R,i}^n - q_{L,i}^n\right) \\ c_2 = \dfrac{6}{\Delta x^2}\left(\dfrac{q_{L,i}^n + q_{R,i}^n}{2} - \bar{q}_i^n\right) \end{cases}$$

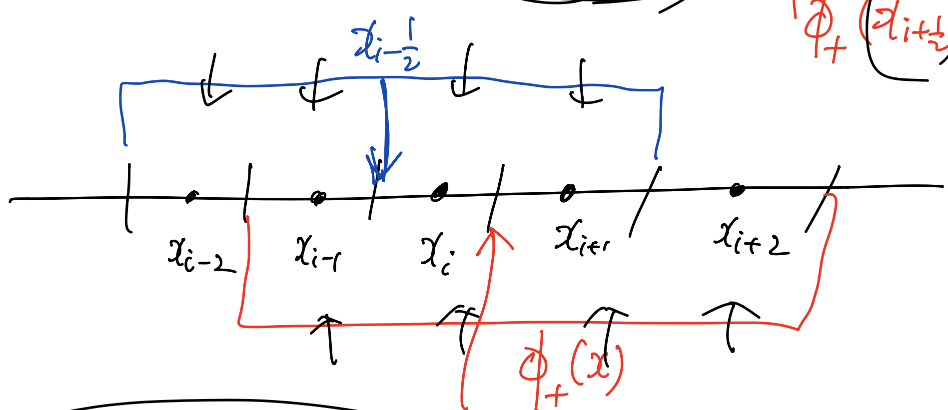
Consider two additional 4th degree poly

$$a_0^{\pm}$$

$$\phi_{\pm}(x) = \sum_{k=0}^{3} a_k^{\pm}\,(x - x_{i\pm\frac{1}{2}})^k$$

to determine $q_{L,R,i}^n$

$$\begin{cases} \phi_-(x_{i-\frac{1}{2}}) = q_{L,i}^n \\ \phi_+(x_{i+\frac{1}{2}}) = q_{R,i}^n \end{cases}$$



$$\frac{1}{\Delta x}\int^{x_{i+\frac{1}{2}}} \phi_-(x)\,dx = \bar{q}_k^n, \quad i-2 \le k \le i-1$$

I_k

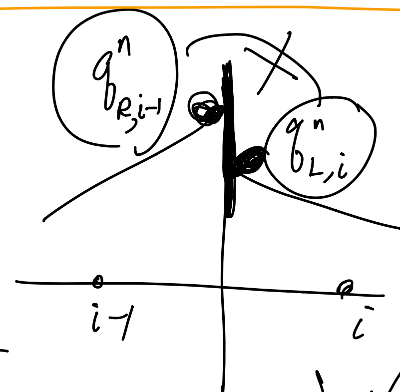
$$\frac{1}{\Delta x}\int \phi_+(x)\,dx = \bar{q}_k^n, \qquad i-1 \le k \le i+2$$

(*)

$$\boxed{a_0^{\pm} = \frac{1}{12}\left(-\bar{q}^n_{i-2+s} + 7\bar{q}^n_{i-1+s} + 7\bar{q}^n_{i+s} - \bar{q}^n_{i+1+s}\right)}$$

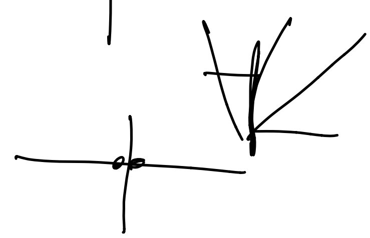
$s=0$ for a_k^-

$s=1$ for a_k^+

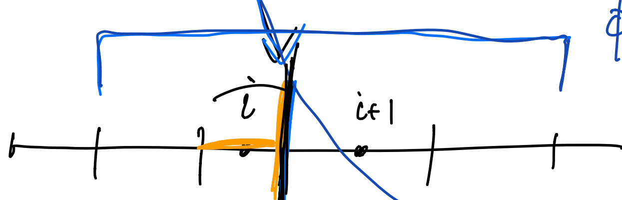


$$q^n_{L,i} = \phi_-(x_{i-\frac{1}{2}}) = a_0^-$$

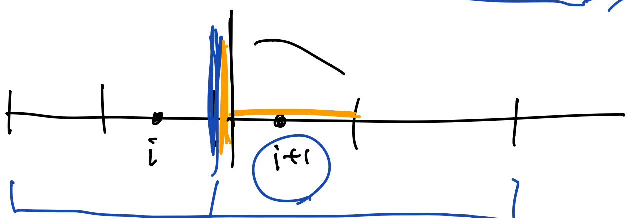
$$q^n_{R,i} = \phi_+(x_{i+\frac{1}{2}}) = a_0^+$$



$$\phi_+^i(x) = \sum a_k^+ (x - x_{i+\frac{1}{2}})^k$$



$\phi_+^i(x_{i+\frac{1}{2}})$



$q^{n}_{L,i+1}$

$\phi_{-}^{i+1}(x) = \sum a_k^{-} (x - x_{i+\frac{1}{2}})^k$

$\phi_{-}^{i+1}(x_{i+\frac{1}{2}})$

n

$q^{n+\frac{1}{2}}_{R,i}$ L

$n+\frac{1}{2}$

$q^{n+\frac{1}{2}}_{L,i+1}$ R

$i+\frac{1}{2}$

$s = 0, 1$

$$a_0^{\pm} = \frac{1}{2}\left(\bar{q}^n_{i-1+s} + \bar{q}^n_{i+s}\right) - \frac{1}{6}\left(\Delta\bar{q}^n_{i+s} - \Delta\bar{q}^n_{i-1+s}\right)$$

where $$\Delta\bar{q}^n_i = \frac{1}{2}\left(\bar{q}^n_{i+1} - \bar{q}^n_{i-1}\right)$$

prim. limiting

$$\Delta^{TVD}\bar{q}^n_i = \text{TVD\_limiter}\left[\bar{q}^n_{i+1} - \bar{q}^n_i,\ \bar{q}^n_i - \bar{q}_{i-1}\right]$$

$$\Rightarrow a_0^{\pm} = \frac{1}{2}\left(\bar{q}^n_{i-1+s} + \bar{q}^n_{i+s}\right) - \frac{1}{6}\left(\Delta^{TVD}\bar{q}^n_{i+s} - \Delta^{TVD}\bar{q}^n_{i-1+s}\right)$$

$W = L^c U = L^p V$ / $s = 0$; $-$

or $V = R^p W$ ($s=1 : t$)

Char. limiting :

$$\Delta^{TVD} q^n_{i:m} = \sum_{k=1}^{3} r^{(k)}_{i:m} \, \Delta^{TVD} \omega^{(k)}_i$$

$$\Delta^{TVD} \omega^{(k)}_i = \text{TVD\_limiter}\left[\underset{\sim}{l}^{(k)}_i \cdot (\bar{V}^n_{i+1} - \bar{V}^n_i), \; \underset{\sim}{l}^{(k)}_i \cdot (\bar{V}^n_i - \bar{V}^n_{i-1}) \right]$$

$\cancel{l^{(k)}_{i+1}}$

$\cancel{l^{(k)}_{i-1}}$

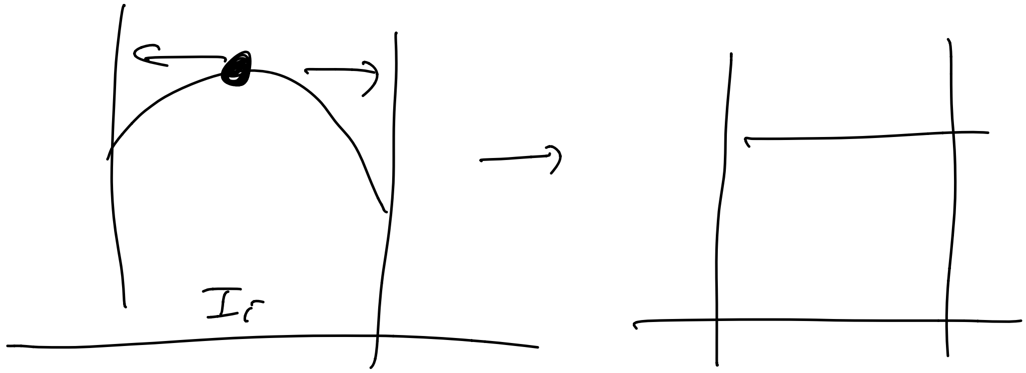
i

Cond 1 : $P_i(x)$ is monotonic over I_i

(i) PPM reduces to FOG, i.e,

$$q^n_{L,R,i} = \bar{q}^n_i \quad \text{when}$$

$$\boxed{(q^n_{R,i} - \bar{q}^n_i)(\bar{q}^n_i - q^n_{L,i}) \le 0}$$



(ii)

Cnd 2 :

$$\bar{q}^n_{i-1} \le q^n_{L,i} \le \bar{q}^n_{i} \quad \&$$

$$\bar{q}^n_{i} \le q^n_{R,i} \le \bar{q}^n_{i+1}$$

Step 2

Char tracing:

$$q^{n}_{R,L,i} \longrightarrow q^{n+\frac{1}{2}}_{R,L,i}$$

$$q^{n+\frac{1}{2}}_{R,i:m} = \sum_{k,\ \lambda^{(k)}_i > 0} \frac{1}{\lambda^{(k)}_i \Delta t} \int_{x_{i+\frac{1}{2}} - \lambda^{(k)}_i \Delta t}^{x_{i+\frac{1}{2}}} \underbrace{\overline{r^{(k)}_{i:m}}\ \overline{l^{(k)}_i} \cdot \overline{P_i(x)}}_{*}\, dx$$

For each m,

$$* \quad \underbrace{r^{(k)}_{i:m} \sum_{s=1}^{3} l^{(k)}_{i:s} \left(\underline{C_{0:s}}\right.}_{} + \underline{C_{1:s}}(x-x_i) + \left.\underline{C_{2:s}}(x-x_i)^2\right)$$

$$\overline{C_\ell} = (C_{\ell:1}, C_{\ell:2}, C_{\ell:3})^T, \quad \ell = 0, 1, 2$$

$\Rightarrow$ the integral of the first term:

$$\sum_k r^{(k)}_{i:m}\, l^{(k)}_i \cdot \overline{C_0} \quad \checkmark$$

$\Rightarrow$ The int. of the 2nd term:

$$\frac{1}{2}\sum_{\substack{k,\\ \lambda_i^{(k)}>0}}\left(1-\frac{\lambda_i^{(k)}\Delta t}{\Delta x}\right) r_{i:m}^{(k)}\,\Delta\overline{C}_1^{(k)}$$

where $\Delta\overline{C}_1^{(k)} = \sum_{s=1}^{3} l_{i:s}^{(k)}\, C_{1:s}\,\Delta x$

$$= l_i^{(k)}\cdot\overline{C}_1\,\Delta x$$

$\Rightarrow$ The int. of the 3^{rd} term;

$$\frac{1}{4}\sum_{\substack{k\\ \lambda_i^{(k)}>0}}\left(1-\frac{2\lambda_i^{(k)}\Delta t}{\Delta x}+\frac{4}{3}\left(\frac{\lambda_i^{(k)}\Delta t}{\Delta x}\right)^2\right) r_{i:m}^{(k)}\,\Delta\overline{C}_2^{(k)}$$

where

$$\Delta\overline{C}_2^{(k)} = \sum_{s=1}^{3} l_{i:s}^{(k)}\, C_{2:s}\,\Delta x^2 = l_i^{(k)}\cdot\Delta\overline{C}_2\,\Delta x^2$$

$\Rightarrow$ The final form; $\approx$ PLM

$$V_{R,i}^{n+\frac{1}{2}} = \overline{C}_0 + \frac{1}{2}\sum_{\substack{k\\ \lambda_i^{(k)}>0}}\left(1-\frac{\lambda_i^{(k)}\Delta t}{\Delta x}\right) r_i^{(k)}\,\Delta\overline{C}_1^{(k)}$$

$$+\frac{1}{4}\sum_{\substack{k\\ \lambda_i^{(k)}>0}}\left(1-\frac{2\lambda_i^{(k)}\Delta t}{\Delta x}+\frac{4}{3}\left(\frac{\lambda_i^{(k)}\Delta t}{\Delta x}\right)^2\right) r_i^{(k)}\,\Delta\overline{C}_2^{(k)}$$

$$V_{L,i}^{n+\frac{1}{2}} = \bar{C}_0 + \frac{1}{2} \sum_{\substack{k \\ \lambda_i^{(k)}<0}} \left(-1 - \frac{\lambda_i^{(k)}\Delta t}{\Delta x}\right) r_i^{(k)} \Delta \bar{C}_1^{(k)}$$

$$+ \frac{1}{4} \sum_{\substack{k \\ \lambda_i^{(k)}<0}} \left(1 + \frac{2\lambda_i^{(k)}\Delta t}{\Delta x} + \frac{4}{3}\left(\frac{\lambda_i^{(k)}\Delta t}{\Delta x}\right)^2\right) r_i^{(k)} \Delta \bar{C}_2^{(k)}$$