Discretization

$\rightarrow$ $\boxed{u_t + \big(f(u)\big)_x = 0}$

$\uparrow$ model PDE

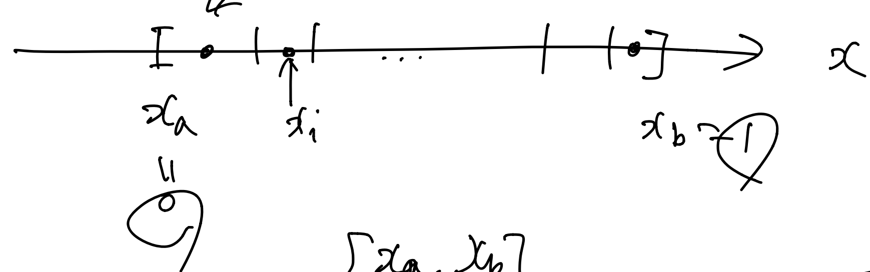
e.g. $f(u) = au$

(i) time : t

(ii) space : x, y, z.

$$\Rightarrow \begin{cases} \text{(a)} \quad t^n = \underset{(=0)}{t_0} + n\Delta t \qquad \text{(CFL)} \\ \text{(b)} \quad x_i = \left(i - \tfrac{1}{2}\right)\Delta x, \quad i = 1, \dots, N \end{cases}$$

$[x_{i-\frac{1}{2}}, x_{i+\frac{1}{2}}] = I_i \Rightarrow x_i \leftarrow$ cell-centers



$[x_a, x_b]$

$x_{i\pm\frac{1}{2}} = x_i \pm \frac{\Delta x}{2}$, $\quad \Delta x = \frac{x_b - x_a}{N}$

$\rightarrow$ cell-interfaces

$\Delta x = \frac{1}{N}$

Def. Let $u_i^n = u(x_i, t^n)$

$\leftarrow$ ptwise values of the exact soln u at (x_i, t^n).

Def. Let U_i^n be the num. appn. to the exact soln u_i^n

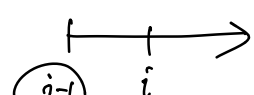
differentiable PDE

Rmk.
- FDM : $U_i^n \approx u_i^n$ ← pt wise
- FVM : $U_i^n \approx \frac{1}{\Delta x}\int_{I_i} u(x,t^n)\,dx$ ← cell-averaged value

integral PDE.

Def. D_i^n : exact soln to the associated "difference" eqn of the original PDE

(e.g) $\frac{\partial u}{\partial t} + a\frac{\partial u}{\partial x} = 0 \quad (a>0)$



FTBS :
$$\boxed{\frac{U_i^{n+1} - U_i^n}{\delta x} + a\,\frac{U_i^n - U_{i-1}^n}{\Delta x} = 0} \quad (*)$$

$$\boxed{\frac{D_i^{n+1} - D_i^n}{\Delta t} + a\,\frac{D_i^n - D_{i-1}^n}{\Delta x} = 0}$$

i.e, D_i^n satisfies $(*)$ exactly.

Def. Discretization error at (x_i, t^n)

$$E_{d,i}^n \equiv u_i^n - D_i^n \quad \left(\Leftarrow \text{truncation errors} + \text{BC errors}\right)$$

Def. Round-off error at (x_i, t^n) :

$$E_{r,i}^n = D_i^n - U_i^n$$

Def. Global errors at (x_i, t^n)

$$E_{g,i}^n = u_i^n - U_i^n$$

$$\left(= E_{d,i}^n + E_{r,i}^n \right)$$

Def. A num. method is "convergent" at t^n in $\|\cdot\|$ if

$$\lim_{\Delta t, \Delta x \to 0} \|E_g^n\| = 0, \quad \text{where}$$

given $E_g^n = (E_{g,1}^n, \cdots, E_{g,N}^n)^T$.

(ex) $\|E_g^n\|_1 = \frac{1}{N} \sum_{i=1}^{N} |E_{g,i}^n|$

$$= \Delta x \sum_{i=1}^{N} |E_{g,i}^n|$$

$$= \Delta x \sum_{i=1}^{N} |u_i^n - U_i^n|$$

?

(ex) $\|E_g^n\|_p = \left(\Delta x \sum_{i=1}^{N} |E_{g_i}^n|^p \right)^{\frac{1}{p}}$

<u>Thm</u>. The <u>Lax-equivalence</u> Thm for <u>linear</u> PDE

Consistency + absolute Stability $\Longleftrightarrow$ "Convergence"

<u>Def</u>. Let N be <u>a linear num. operator</u> mapping $U_i^n \longrightarrow U_i^{n+1}$ i.e.

$$U_i^{n+1} = N(U_i^n), \quad \forall i.$$

or $\quad U^{n+1} = N(U^n),$

$$U^n = (U_1^n, \cdots, U_N^n)^T.$$

one-step explicit time-marching operator

Def. <u>One-step error</u>

$E^{n+1}_{1step,i}$ at (x_i, t^{n+1})

$$\overset{def}{=} u_i^{n+1} - \Gamma(u_i^n), \quad i = 1, \dots, N$$

Def. <u>Local truncation error</u>

$E^{n+1}_{LT,i}$ at (x_i, t^{n+1})

$$\overset{def}{=} \frac{1}{\Delta t} E^{n+1}_{1step,i}, \quad i = 1, \dots, N$$

Def. A numerical method is of <u>order p</u> in t & of q <u>in x</u> for suff. smooth data (with "compact support")

$$\Longleftrightarrow E^{n+1}_{LT,i} = O(\Delta x^q + \Delta t^p) \quad 1D$$

$$\approx O(\Delta t^1)$$

Rmk. $E_{LT,i}^{n+1} = O\left((\Delta x)^{q_1} + (\Delta y)^{q_2} + (\Delta z)^{q_3} + (\Delta t)^{p}\right)$

$p = 1$, $\quad q_1 = 5$, $\quad q_2 = 2 = q_3$

$q_1 = q_2 = q_3 = p = 4$

$\rightarrow \boxed{q_1 = q_2 = q_3 = 5 \quad p = 2}$

→ difference Eq

Def A num. method is consistent with the original PDE in $\|\cdot\|$ if

$$\lim_{\Delta t, \Delta x \to 0} \| E_{LT}^{n} \| = 0$$

for all "smooth ftn" $u(x,t)$

Def. A linear num. method defined by

1 $\mathcal{N}$ is called **"stable"** in $\|\cdot\|$

if $\exists C_T$ s.t. $\|\mathcal{N}^n\| \le C_T$,

"$\forall\, n\Delta t \le T$" for each T.

Rmks. A num. method is stable if

$\|\mathcal{N}\| \le 1$.

$\because$ Claim: $\|\mathcal{N}^n\| \le C$ ✓

pf $\|\mathcal{N}^n\| \le \|\mathcal{N}\|^n \le 1$.

Rmk. A num. method is stable if

$\|U^{n+1}\| \le \|U^n\|$ (for $U^n \ne 0$)

$\because$ Assume $\|U^{n+1}\| \le \|U^n\|$

$$\boxed{1 \ge \frac{\|U^{n+1}\|}{\|U^n\|} = \frac{\|\mathcal{N}(U^n)\|}{\|U^n\|}} \leftarrow (*)$$

$\Rightarrow$ Since ⊛ is true for "all"

$U^n \neq 0$,

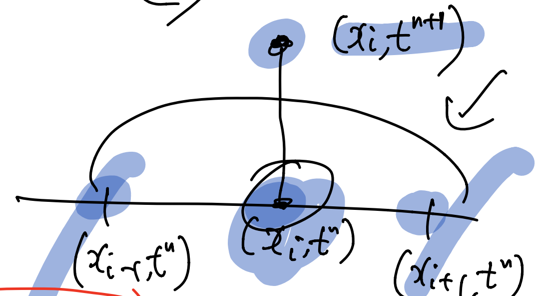
$$1 \geq \sup_{U^n \neq 0} \frac{\|N(U^n)\|}{\|U^n\|} = \|N\|$$

$u_t + a u_x = 0$. ← $\boxed{a>0}$, $\boxed{a<0}$

→ ←

$t>0$

exact soln : $u(x,t) = u_0(x - at)$



Ex FTBS ✓

Ex FTCS → NOT stable

$$\boxed{\frac{U_i^{n+1} - U_i^n}{\Delta t} = -\frac{a}{2\Delta x}\left[U_{i+1}^n - U_{i-1}^n\right]}$$

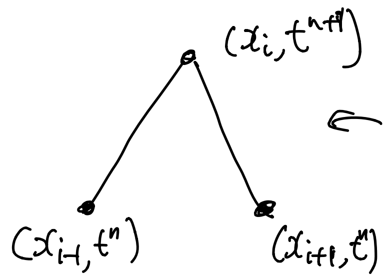
$\sim O(\Delta t^1 + \Delta x^2)$ ⊛⊛

$$\boxed{U_i^{n+1} = U_i^n - \frac{a\Delta t}{2\Delta x}\left[U_{i+1}^n - U_{i-1}^n\right]}$$

Ex Lax-Friedrichs (LF)

$$U_i^{n+1} = \frac{U_{i+1}^n + U_{i-1}^n}{2} - \frac{a\Delta t}{2\Delta x}\left[U_{i+1}^n - U_{i-1}^n\right]$$

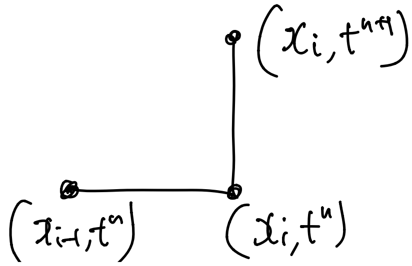
$$\sim O(\Delta t^1 + \Delta x^1)$$



LF is Stable if $\left|\frac{a\Delta t}{\Delta x}\right| \le 1$

Ex FTBS $\sim O(\Delta t + \Delta x)$ ⇝ stable when $a > 0$

$$U_i^{n+1} = U_i^n - \frac{a\Delta t}{\Delta x}\left[U_i^n - U_{i-1}^n\right]$$

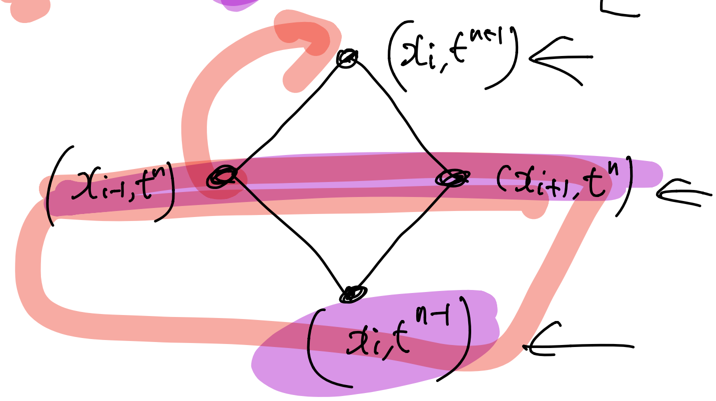


Ex FTFS $\sim O(\Delta t + \Delta x)$ stable when $a<0$

$$U_i^{n+1} = U_i^n - \frac{a\Delta t}{\Delta x}\left[U_{i+1}^n - U_i^n\right]$$

Ex Leapfrog (CTCS) $\sim O(\Delta t^2 + \Delta x^2)$

$$U_i^{n+1} = U_i^{n-1} - \frac{a\Delta t}{\Delta x}\left[U_{i+1}^n - U_{i-1}^n\right]$$



(three-level method)

(two-step method)

Ex Lax-Wendroff (LW) $\sim O(\Delta t^2 + \Delta x^2)$

$\rightarrow$ "Cauchy-Kowalewski procedure"

LW "

$$u_t + a u_x = 0 \quad \rightarrow \quad u_t = -a u_x$$

$$\rightarrow u(x, t+\Delta t) = u(x,t) + \Delta t\, u_t + \frac{\Delta t^2}{2} u_{tt} + O(\Delta t^3)$$

$$u_{tt} = (-a u_x)_t$$

$$= -a (u_t)_x$$

$$= -a(-a u_x)_x$$

$$= a^2 u_{xx}$$

$$\rightarrow \boxed{u(x, t+\Delta t) = u - a\Delta t\, u_x - \frac{a^2 \Delta t^2}{2} u_{xx} + \cancel{O(\Delta t^3)}}$$

$$u_x \approx \frac{1}{2\Delta x}\left[u(x+\Delta x, t) - u(x-\Delta t, t)\right]$$

$$u_{xx} \approx \frac{1}{\Delta x^2}\left[u(x+\Delta x, t) - 2u(x,t) + u(x-\Delta x, t)\right]$$