Ex $\underline{u_t + (au)_x = 0}$; linear scalar advection

$$\begin{cases} a = \text{constant} \\ u \text{ : any quantity} \\ f(u) = au \end{cases}$$

Ex $\underline{u_t + \left(\frac{u^2}{2}\right)_x = 0}$; Burgers nonlinear scalar adv

$f(u) = \frac{u^2}{2}$ (nonlinear)

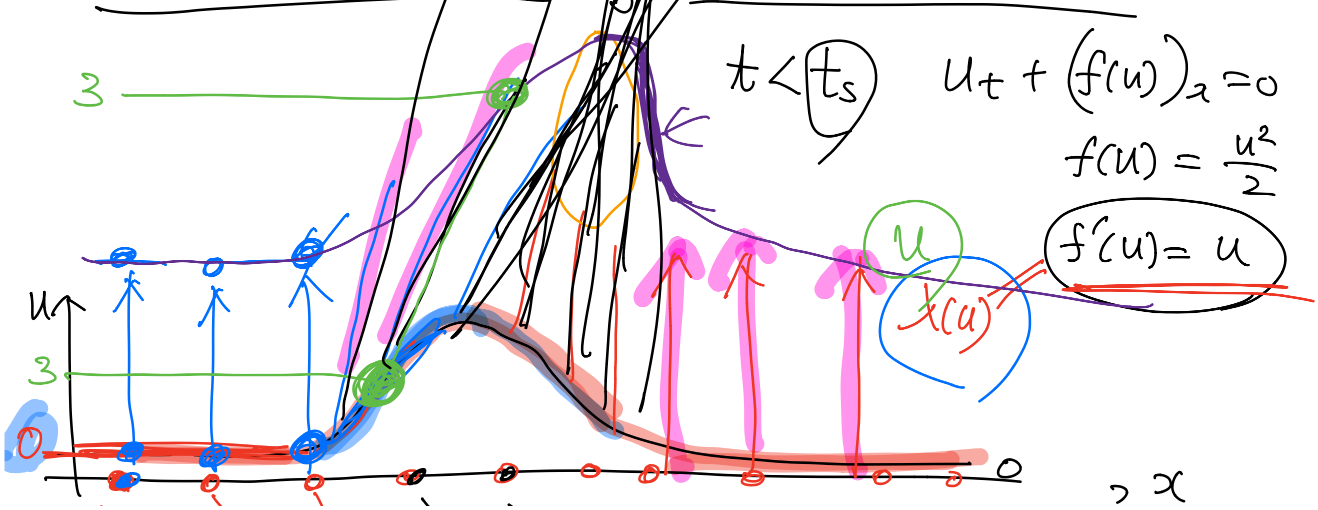
Rmk

$\underline{\text{numerical accuracy}}$

vs.

$\underline{\text{numerical stability}}$

$t \gg t_s$



$t < t_s$ $\quad u_t + (f(u))_x = 0$

$f(u) = \frac{u^2}{2}$

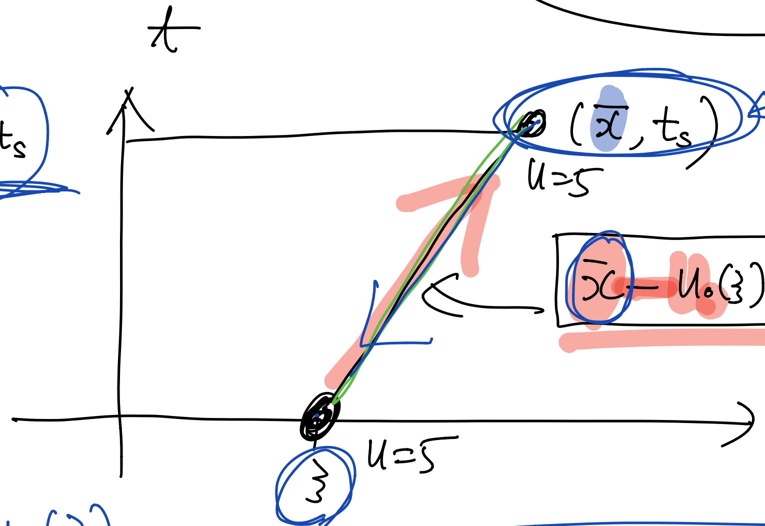
$f'(u) = u$

$\lambda(u) > 0 \quad \lambda(u) > 0 \quad \lambda(u) = 0$

$\xi_1 \quad \xi_2 \quad \cdots$

$$\begin{cases} x'(t) = u(\xi, 0) \\ x(0) = \xi \end{cases}$$

$t < t_s$



$x, \leftarrow t = 0$

$$\boxed{\bar{x} - u_0(\xi)t = \xi}$$

$u_0(\xi)$ known

$$\boxed{\xi + u_0(\xi)t = \bar{x}}$$

$$\boxed{u(\bar{x}, t_s)} = u_0(x - u_0(\xi)t)$$

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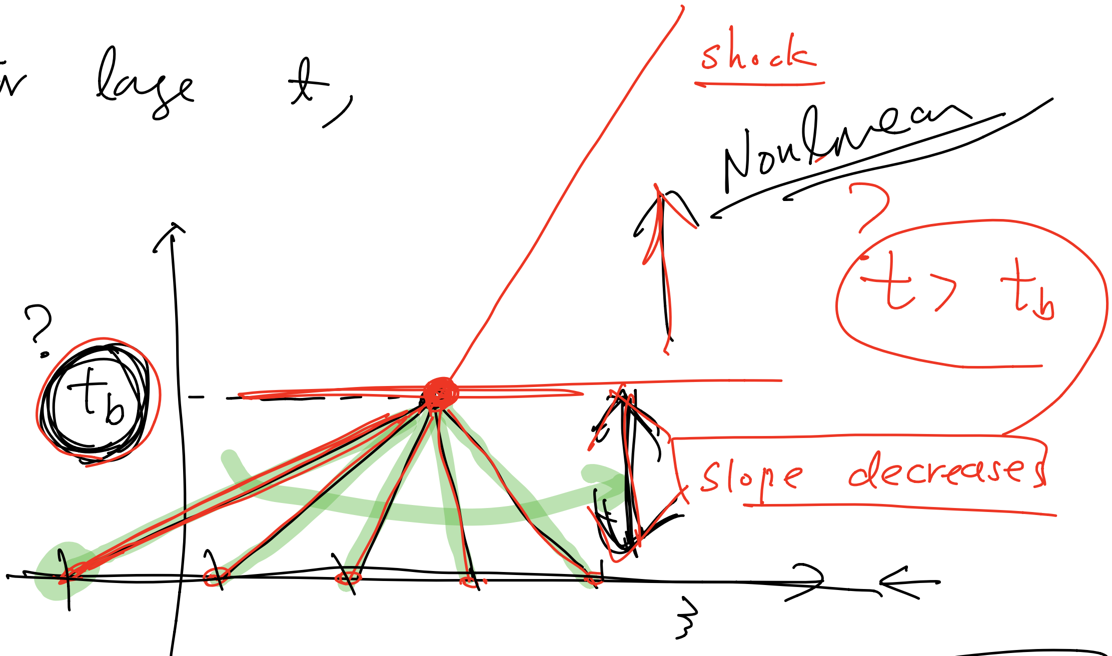
② For t $(= t_b) \gg t_s$

large

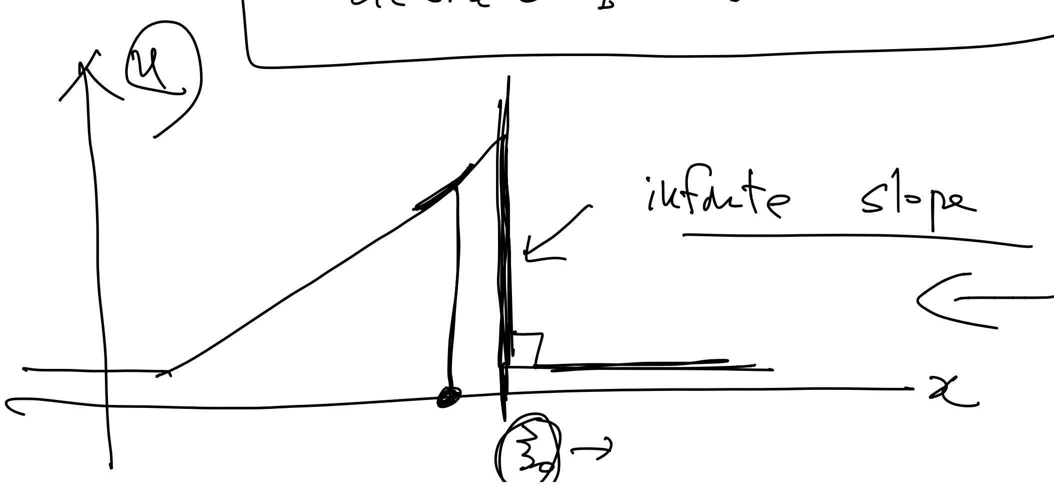
For lage t,



→ Collison will happen if $u_0'(\xi) < 0$

$f(u) = u^2/2$, $\Rightarrow$ $f'(u) = \lambda(u) = u$

your advection u is a decreasing ftn.



t_b is the time when $u'(x) \to \infty$

Thm. $\exists\, t_b$ s.t chars cross each others

at $t_b = -\dfrac{1}{u_0'(\xi_0)}$ where

ξ_0 is a pt. s.t $u_0'(\xi_0) < 0$

pf. Claim: at $t = t_b$, $u_x \to \infty$.

Given $u(x,t) = \underbrace{u_0(x - u_0(\xi)t)}_{u_0(\xi)}$ ... ①

$x - u_0(\xi)t = \xi$ ... ②

From ①: $u_x = \dfrac{du_0}{d\xi}\dfrac{\partial \xi}{\partial x}$ ∞

From ②: $1 - \dfrac{du_0}{d\xi}\dfrac{\partial \xi}{\partial x}t = \dfrac{\partial \xi}{\partial x}$

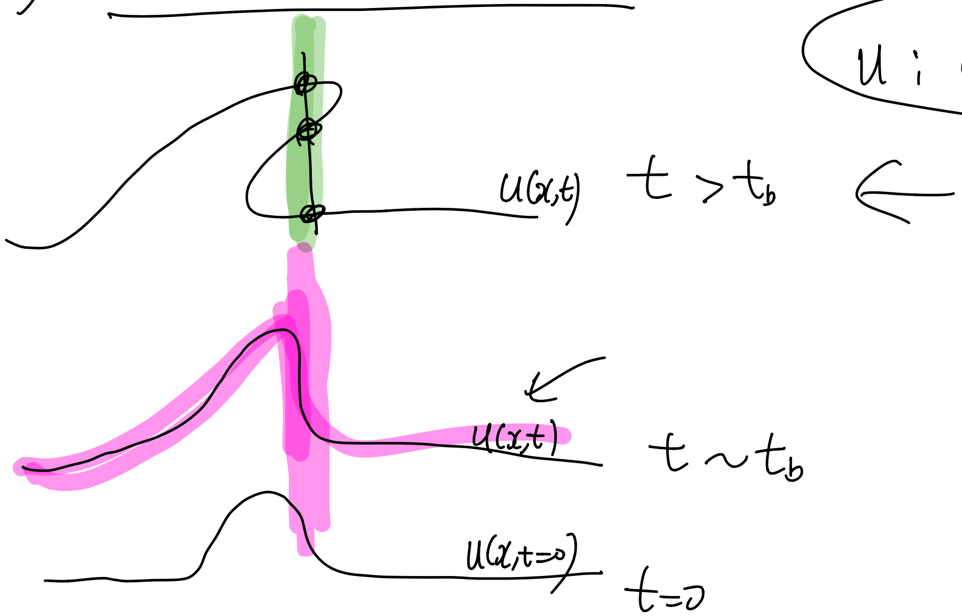
$\to \dfrac{\partial \xi}{\partial x}\left(1 + \dfrac{du_0}{d\xi}t\right) = 1$

$$\longrightarrow \frac{\partial \xi}{\partial x} = \frac{1}{1 + \frac{du_0}{d\xi}\, t} \longrightarrow 0$$

$$\longrightarrow u_x \longrightarrow \infty \quad \text{as} \quad \underset{\searrow\, t_b}{t} \longrightarrow -\frac{1}{\frac{du_0}{d\xi}}$$

(3) For $t > t_b$:

u : density



$\blacksquare$

"Weak" Solution

$$\iint \phi \left(u_t + (f(u))_x \right) = 0$$

→ spatial

$\phi \in C_0^1(\mathbb{R} \times \mathbb{R}^+)$ → $t \geq 0$

at least one-time differentiable

Compact support

$\begin{cases} \phi = 0 \text{ outside some bdd set} \\ \phi \neq 0 \text{ on the bdd set} \end{cases}$

compact support

diff.'ble form

Def. $u(x,t)$: a weak soln of $\boxed{u_t + f(u)_x = 0}$ if u satisfies the following condition for $\forall \phi$:

$$\boxed{\int_{\mathbb{R}^+} \int_{\mathbb{R}} \left[\phi_t u + \phi_x f(u) \right] dx\, dt = - \int_{\mathbb{R}} \phi(x,0)\, u(x,0)\, dx}$$

+ integration form.

pf $0 = \int_{\mathbb{R}^+}\int_{\mathbb{R}} (\phi u_t + \phi f_x)\, dx\, dt$

(i) $\int_{\mathbb{R}}\int_{\mathbb{R}^+} \phi u_t \, dx\, dt$

$= \int_{\mathbb{R}} \left[\phi u \Big|_0^\infty - \int_0^\infty u \phi_t \, dt \right] dx$

$= \int_{\mathbb{R}^+} \left(-\phi(x,0)u(x,0) \right) dx$

$- \int_{\mathbb{R}} \int_{\mathbb{R}^+} u \phi_t \, dt\, dx$

(ii) $\int_{\mathbb{R}^+}\int_{\mathbb{R}} \phi f_x \, dx\, dt$

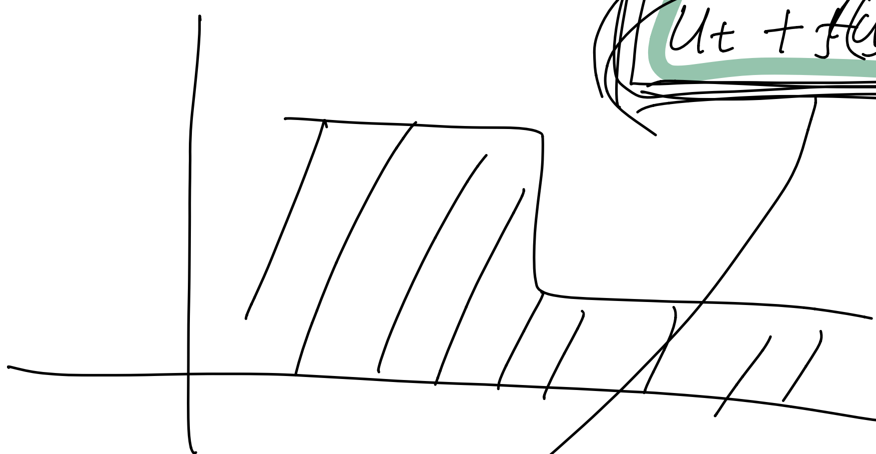
$= \int_{\mathbb{R}^+} \left[\cancel{\phi f \Big|_{-\infty}^{\infty}} - \int_{\mathbb{R}} \phi_x f \, dx \right] dt$ $\xrightarrow{} 0$

$= -\int_{\mathbb{R}^+} \int_{\mathbb{R}} \phi_x f \, dx\, dt$

Rmk. If $u(x,t)$ is a weak sol'n then $u(x,t)$ also satisfies the original integral version of eqn.

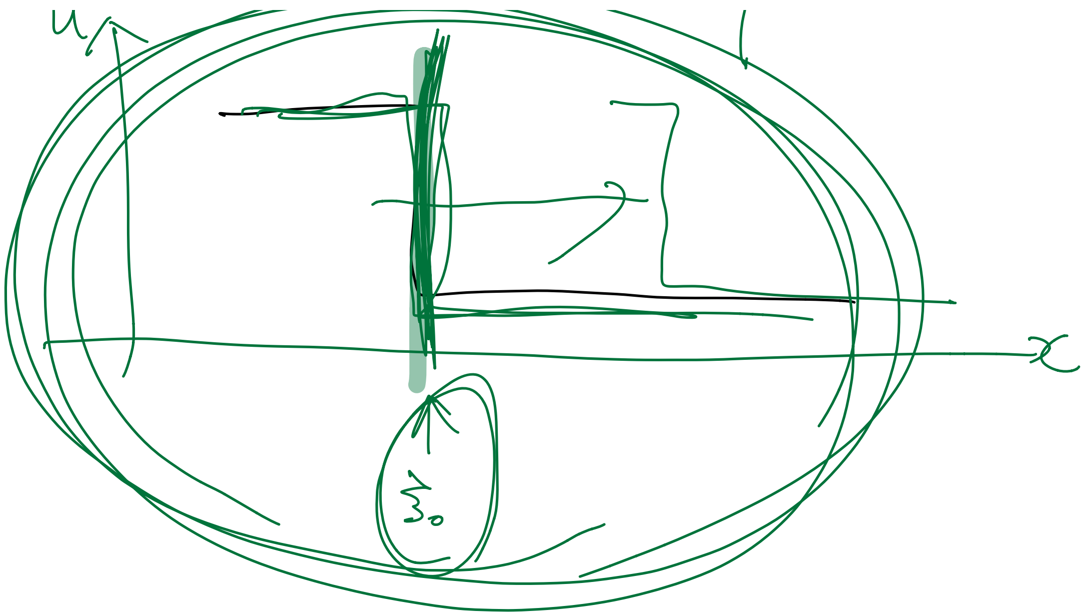
$$\iint u_t + (f(u))_x \, dx\, dt = 0$$

$$\boxed{u_t + f(u)_x = 0}$$



incompressible flow

$$\boxed{u_t + a u_x = 0}$$


u
x
ξ_0