Two principles :

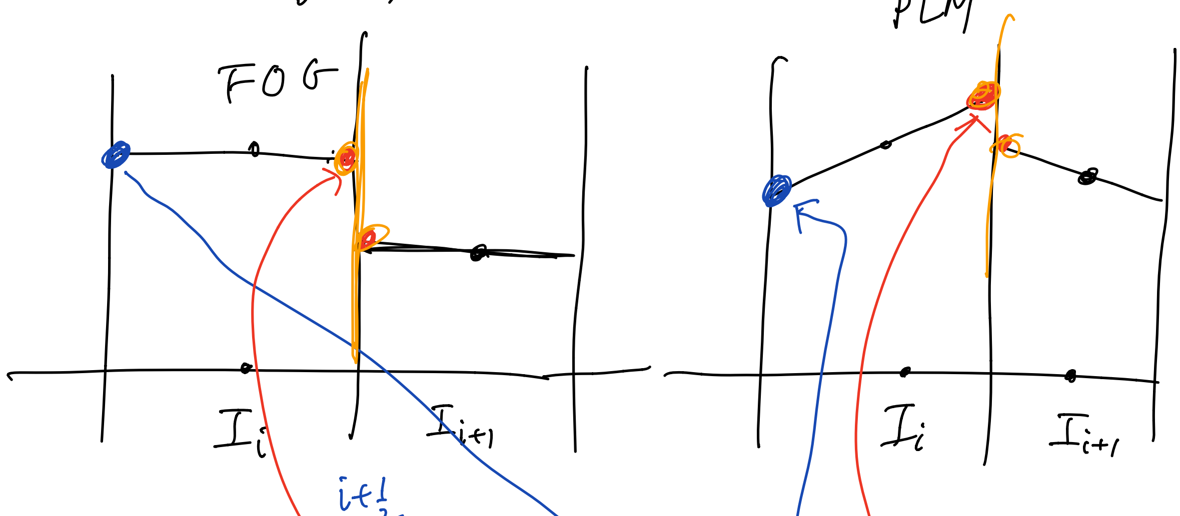
FOG, PLM, PPM, WENo

$\begin{cases} \text{① Monotonic reconstruction (spatial part)} \\ \text{② Half-time step evolution (temporal part)} \end{cases}$

mth degree piecewise poly on I_i

$P_i(x)$ on $I_i = [x_{i-\frac{1}{2}}, x_{i+\frac{1}{2}}]$

$$= \sum_{k=0}^{m} C_k (x - x_i)^k$$

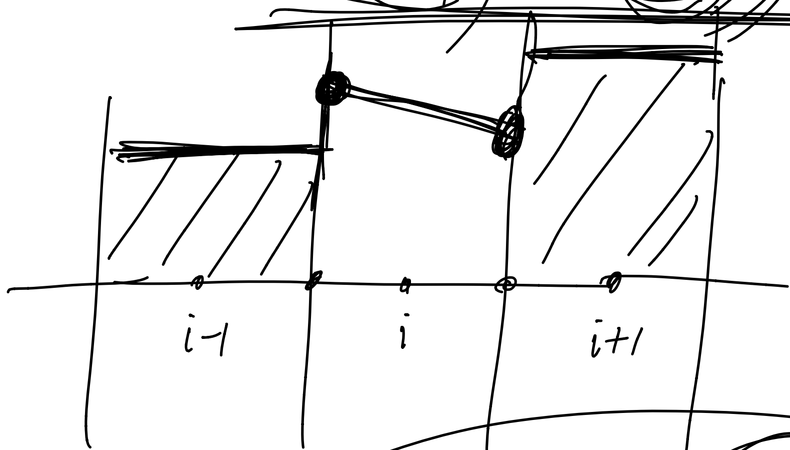


$$\begin{cases} P_i(x_{i-\frac{1}{2}}) = q^n_{L,i} \\ P_i(x_{i+\frac{1}{2}}) = q^n_{R,i} \end{cases}$$

- **Cond 1:** the profile of $P_i(x)$ must be monotonic over I_i

- **Cond 2:** $q^n_{L,i}$, $q^n_{R,i}$

$$\bar{q}^n_{i-1} \le q^n_{L,i} \le \bar{q}^n_{i+1}$$



$$\bar{q}^n_{i-1} \le q^n_{R,i} \le \bar{q}^n_{i+1}$$

$$\frac{1}{\Delta x}\int_{I_i} P_i(x,t^n)\,dx = \bar{q}^n_i$$

② Half time-step evolution

$$q^{(n)}_{L,i} \longrightarrow q^{n+\frac{1}{2}}_{L,i}$$

$$q^{(n)}_{R,i} \longrightarrow q^{n+\frac{1}{2}}_{R,i}$$

MUSCL — Hancock

$$\bar{A} = R \Lambda L$$

$$\frac{\partial w_i^{(k)}}{\partial t} + \lambda_i \frac{\partial w_i^{(k)}}{\partial x} = 0 \quad \text{for each } k.$$

$$W_i = (w_i^{(1)}, w_i^{(2)}, w_i^{(3)})^T$$

$$W = L U$$

$$= L \cdot V$$

L^c

L^p

$$w_i^{(k)} = l_i^{(k,p)} \cdot V_i$$

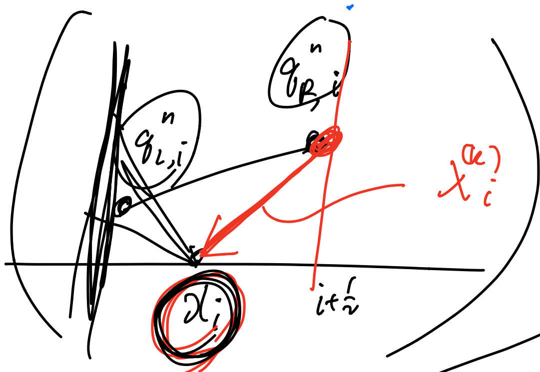
$$V_i = \sum_{k=1}^{3} r_i^{(k,p)} w_i^{(k)}$$

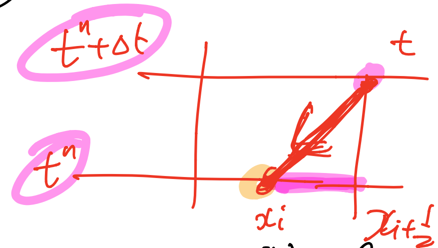
$$V = R^p W$$

$$(q_{i:1}, q_{i:2}, q_{i:3})^T$$

$$= (\rho_i, u_i, p_i)^T \text{ — } (p_{i:1}, p_{i:2}, p_{i:3})^T$$

$$\rightarrow \quad q^{n+\frac{1}{2}}_{R,i:m} = \frac{1}{\Delta t}\int_{t^n}^{t^{n+1}} P_{i:m}(x_{i+\frac{1}{2}})\,dt$$





$$r_{i:m}^{(k)} = r_i^{(k)} \cdot \ell_m$$

$$= \sum_k \frac{1}{\Delta t}\int_{t^n}^{t^{n+1}} r_{i:m}^{(k)}\, w_i^{(k)}(x_{i+\frac{1}{2}}, t)\,dt$$

$\rightarrow$ Consider $\quad w_i^{(k)}(x_{i+\frac{1}{2}}, t^n+\Delta t)$

$$\parallel$$

$\lambda_i^{(k)} > 0$

$$w_i^{(k)}(x_{i+\frac{1}{2}} - \lambda_i^{(k)}\Delta t, t^n) \leftarrow$$

$$= l_i^{(k,p)} \cdot V_i^n(x_{i+\frac{1}{2}} - \lambda_i^{(k)}\Delta t)$$

$$= l_i^{(k,p)} \cdot P_i(x_{i+\frac{1}{2}} - \lambda_i^{(k)}\Delta t)$$

$$l_i^{(k,p)} = \left(l_{i:1}^{(k)},\ l_{i:2}^{(k)},\ l_{i:3}^{(k)} \right)$$

$$q_{R,i:m}^{n+\frac{1}{2}} = \sum_{k=1}^{3} \frac{1}{\Delta t} \int_{t^n}^{t^{n+1}} r_{i:m}^{(k,p)}\, l_i^{(k,p)} \cdot P_i\Big(\underbrace{x_{i+\frac{1}{2}} - \lambda_i^{(k)}(t-t^n)}_{= x} \Big)\, dt$$

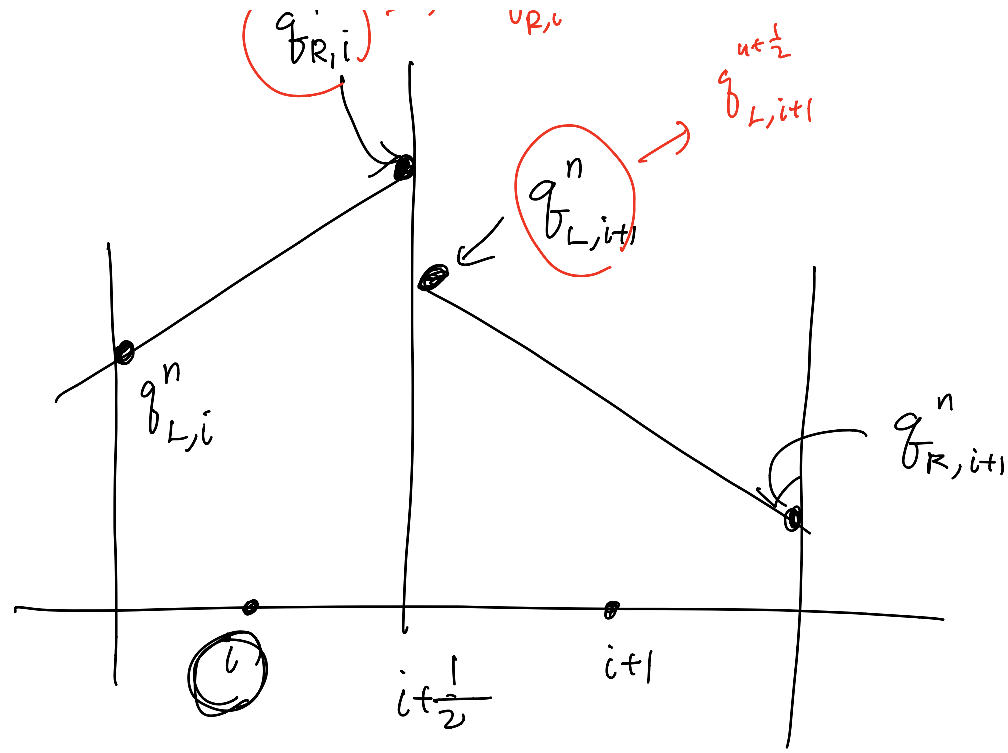
$$= \sum_{\substack{k \\ \lambda_i^{(k)} > 0}} \frac{1}{\lambda_i^{(k)} \Delta t} \int_{x_{i+\frac{1}{2}} - \lambda_i^{(k)} \Delta t}^{x_{i+\frac{1}{2}}} r_{i:m}^{(k,p)}\, l_i^{(k,p)} \cdot P_i(x)\, dx$$

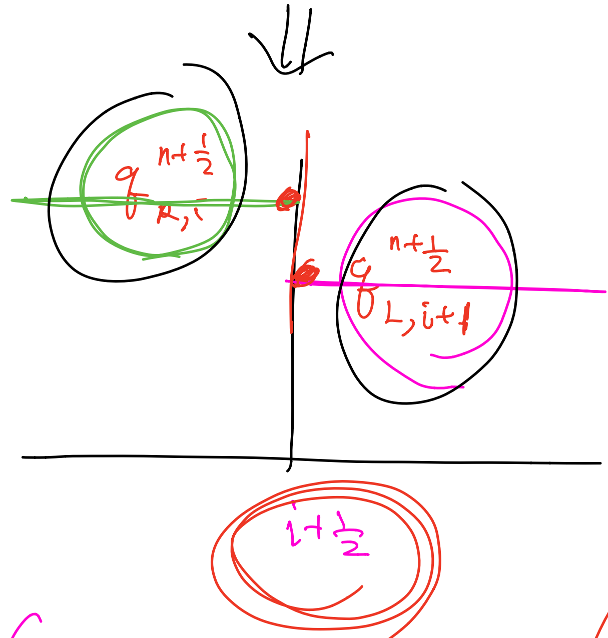
$$\left[x_{i+\frac{1}{2}} - \lambda_i^{(k)} \Delta t,\ x_{i+\frac{1}{2}} \right]$$

$$q_{L,i:m}^{n+\frac{1}{2}} = \sum_{\substack{k \\ \lambda_i^{(k)} < 0}} \frac{1}{\lambda_i^{(k)} \Delta t} \left(* \right)$$

$$(*) = \int_{x_{i-\frac{1}{2}} - \lambda_i^{(k)} \Delta t}^{x_{i-\frac{1}{2}}} r_{i:m}^{(k,p)}\, l_i^{(k,p)} \cdot P_i(x)\, dx$$

$n \rightarrow q_{?}^{n+\frac{1}{2}}$





@ $i+\frac{1}{2}$: $(q_L, q_R) = (q^{n+\frac{1}{2}}_{R,i}, q^{n+\frac{1}{2}}_{L,i+1})$

$\overset{FOG}{=} (\bar{q}^n_i, \bar{q}^n_{i+1})$

Solve $\begin{cases} U_t + \bar{A} U_x = 0 \\ \text{IC} \begin{cases} q_{R,i}^{n+\frac{1}{2}} \; (= q_L), & x < x_{i+\frac{1}{2}} \\ q_{L,i+1}^{n+\frac{1}{2}} \; (= q_R), & x > x_{i+\frac{1}{2}}. \end{cases} \end{cases}$

$\rightarrow \quad RP = RP\left(q_{R,i}^{n+\frac{1}{2}}, \; q_{L,i+1}^{n+\frac{1}{2}}\right)$

to compute numerical Godunov fluxes at $i+\frac{1}{2}$ & $\underline{n+\frac{1}{2}}$

i.e $F_{i+\frac{1}{2}}^{n+\frac{1}{2}}$.

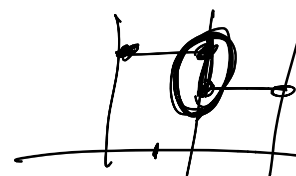
$$U_i^{n+1} = U_i^n - \frac{\Delta t}{\Delta x}\left(F_{i+\frac{1}{2}}^{n+\frac{1}{2}} - F_{i-\frac{1}{2}}^{n+\frac{1}{2}}\right)$$

FOG. $P_i(x) = C_0 = \bar{q}_i^n$

↳ Godunov, 1959

$$\frac{1}{\Delta x}\int_{I_i} C_0 \, dx = \bar{q}_i^n$$

$$= C_0$$



I_i

$$q^{n+\frac{1}{2}}_{L;R,i} = \bar{q}_i^n$$

PLM $P_i(x) = C_0 + C_1(x - x_i)$ $x \in I_i$.

Step1 $\frac{1}{\Delta x}\int_{x_{i-\frac{1}{2}}}^{x_{i+\frac{1}{2}}} p_i(x)\,dx = \bar{q}_i^n$

$$= \frac{1}{\Delta x}\left[C_0 \Delta x + \cancel{\frac{c_1}{2}\left[\frac{\Delta x^2}{4} - \frac{\Delta x^2}{4}\right]} \right]$$

: i $\quad c_0 = \bar{q}_i^n$

$\rightarrow \quad p_i'(x) = c_1 \approx q_i'(x) \cong \frac{\Delta q_i^n}{\Delta x}$

$\rightarrow \quad p_{i:m}(x) = \bar{q}^n_{i:m} + \frac{\Delta \bar{q}^n_{i:m}}{\Delta x}(x - x_i)$

for each q: $\rho \quad u \quad p$

Eval @ $x_{i \pm \frac{1}{2}}$;

$$\rightarrow \quad \begin{cases} q^n_{L,i:m} = \bar{q}^n_{i:m} - \frac{\Delta q^n_{i:m}}{\Delta x} \\ q^n_{R,i:m} = \bar{q}^n_{i:m} + \frac{\Delta q^n_{i:m}}{\Delta x} \end{cases}$$

Step 2 Char. tracing (m)

$$q^{n+\frac{1}{2}}_{R,i;m} = \sum_{\substack{k \\ \lambda_i^{(k)}>0}} \frac{1}{\lambda_i^{(k)}\Delta t} \int_{x_{i+\frac{1}{2}}-\lambda_i^{(k)}\Delta t}^{x_{i+\frac{1}{2}}} r_{i:m}^{(k)}\, l_i^{(k)} \cdot P_i(x)\, dx$$

For each m, m=1, 2, 3

$$r_{i:m}^{(k)} \sum_{s=1}^{3} l_{i:s}^{(k)} P_{i:s}(x)$$

s^{th} component

$$= r_{i:m}^{(k)} \sum_{s=1}^{3} l_{i:s}^{(k)} \left(\bar{q}^{n}_{i:s} + (x-x_i) \frac{\Delta \bar{q}^{n}_{i:s}}{\Delta x} \right)$$

$$= r_{i:m}^{(k)} \sum_{s=1}^{3} l_{i:s}^{(k)} \bar{q}^{n}_{i:s} + \frac{(x-x_i)}{\Delta x} r_{i:m}^{(k)} \sum_{s=1}^{3} l_{i:s}^{(k)} \Delta \bar{q}^{n}_{i:s}$$

(1) = Constant (2)

→ The integration of (1) ;

$$\sum_{k;\lambda_i^{(k)}>0} r_{i:m}^{(k)}\, l_i^{(k)} \cdot \bar{V}_i^{n} \quad \text{---} \quad (*)$$

ROE $\Rightarrow$ ×

$\frac{\text{HLL}}{\text{HLLC}} \Rightarrow \sum_{k=1}^{3}$

$$= l_i^{(k)} \cdot \Delta V_i^n$$

Integrate ② :

$$② = \frac{(x-x_i)}{\Delta x}\, r_{i:m}^{(k)} \sum_{s=1}^{3} l_{i:s}^{(k)}\, \Delta q_{i:s}^{n}$$

$\Delta\omega_i^{(k)}$

char. limiting

$W = LV \qquad LU = W$

k^{th} char. slope

$$= \text{TVD\_limiter}\left[\, l_i^{(k)} \cdot (V_i^n - V_{i-1}^n),\; l_i^{(k)} \cdot (V_{i+1}^n - V_i^n) \,\right]$$

MC   minmod   van Leer

The integration :

$x_{i+\frac{1}{2}}$

$$\sum_{k,\lambda_i^{(k)}>0} \frac{1}{\lambda_i^{(k)}\Delta t} r_{i:m}^{(k)} \Delta w_i^{(k)} \frac{1}{\Delta x} \int_{x_{i+\frac{1}{2}}-\lambda_i^{(k)}\Delta t} (x-x_i)\, dx$$

$$= \boxed{\frac{1}{2} \sum_{\substack{k \\ \lambda_i^{(k)}>0}} \left(1-\frac{\lambda_i^{(k)}\Delta t}{\Delta x}\right) r_{i:m}^{(k)} \Delta w_i^{(k)}}$$

prim. limiting :

$$\Delta w_i^{(k)} = l_i^{(k)} \cdot \text{TVD\_Limiter}\left(\cancel{V_{i+1}^n - V_i^n},\ \cancel{V_i^n - V_{i-1}^n}\right)$$

$$\begin{pmatrix} \rho \\ u \\ p \end{pmatrix}_{R,i}^{n+\frac{1}{2}} \quad l \cdot \Delta V$$

$$V_{R,i}^{n+\frac{1}{2}} = \bar{V}_i^n + \frac{1}{2} \sum_{k;\lambda_i^{(k)}>0} \left(1-\frac{\lambda_i^{(k)}\Delta t}{\Delta x}\right) r_i^{(k)} \Delta w_i^{(k)}$$

$$V_{L,i}^{n+\frac{1}{2}} = \bar{V}_i^n + \frac{1}{2} \sum_{k;\lambda_i^{(k)}<0} \left(-1-\frac{\lambda_i^{(k)}\Delta t}{\Delta x}\right) r_i^{(k)} \Delta w_i^{(k)}.$$

RoE.

$$\sum_{k=1}^{3}$$ for HLL and HLLC