The Riemann Problem (RP)

→ $u_t + \left(f(u)\right)_x = 0$, $\underline{f \text{ convex}}$

→ nonlinear salar

$$f''(u) > 0$$
$$\|$$
$$\lambda'(u) \quad (\because \lambda(u) = f'(u))$$

→ f: convex

$\Rightarrow$ λ is monotone increasing

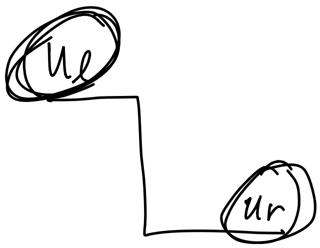
(i.e.g $f(u) = \frac{u^2}{2}$
$\lambda(u) = u$)

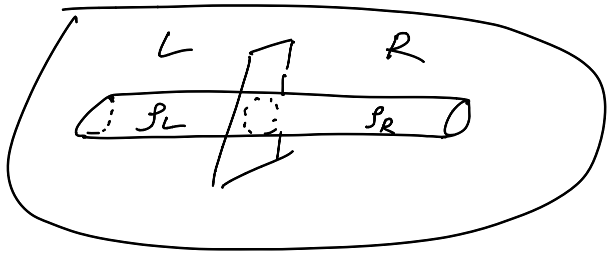
$\Rightarrow$ if $u_l \gtrless u_r$ then
$\lambda(u_l) \gtrless \lambda(u_r)$ ⋯ (*)

Consider a conservation law with piecewise constant discont as IC:

$$u(x,0) = \begin{cases} u_l & x < 0 \\ u_r & x > 0 \end{cases} \qquad t=0$$

Case 1 $u_\ell > u_r \implies \exists!$ weak soln.





$$u(x,t) = \begin{cases} u_\ell & x < st \\ & (t>0) \\ u_r & x > st \end{cases}$$

?

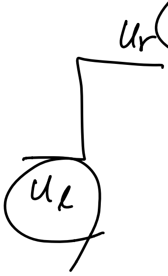
where s : shock speed at which the initial discont. travels.

Case 2 $u_\ell < u_r \implies \exists \infty$ weak solns

(i) Entropy violating soln

→ needs to be rejected

→ wrong one!



(ii) Rarefaction wave soln

→ correct soln!

$$u(x,t) = \begin{cases} u_l & x < u_l t \\ \boxed{x/t} & u_l t \le x \le u_r t \\ u_r & x > u_r t \end{cases} \quad (t>0)$$

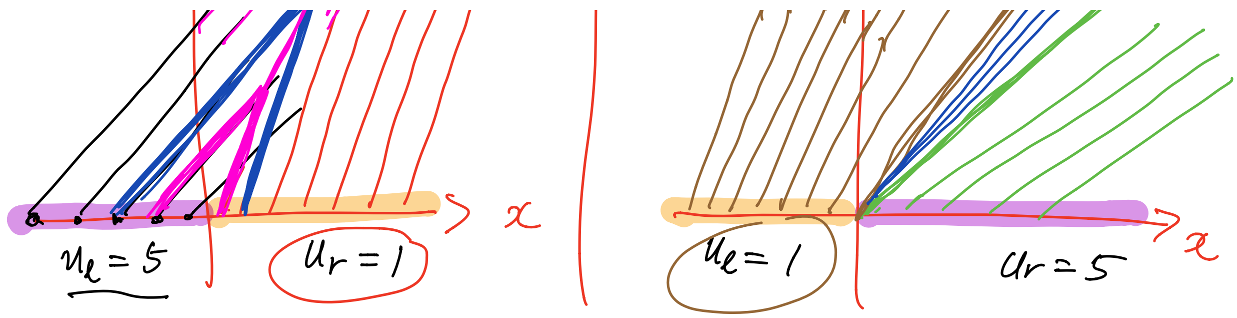
Rmk. Case 1 $u_l > u_r$ (shock soln)

$$\Rightarrow \boxed{\lambda(u_l) > s > \lambda(u_r)}$$

Rmk. Case 2. $u_l < u_r$ (divergent soln or divergent characteristics)

$$\Rightarrow \boxed{\lambda(u_l) < \lambda(u_r)}$$

$\lambda(u) = u$



Rankine–Hugoniot Condition (RH)

Rmk Leibniz Integral rule

$f(x,t)$: a ftn st $\frac{\partial f}{\partial t}$ exists and is conti

$$\Rightarrow \frac{d}{dt}\int_{a(t)}^{b(t)} f(x,t)\,dx$$

$$= \int_{a(t)}^{b(t)} \frac{\partial f}{\partial t}\,dx + b'(t)\,f(b(t),t) - a'(t)\,f(a(t),t)$$

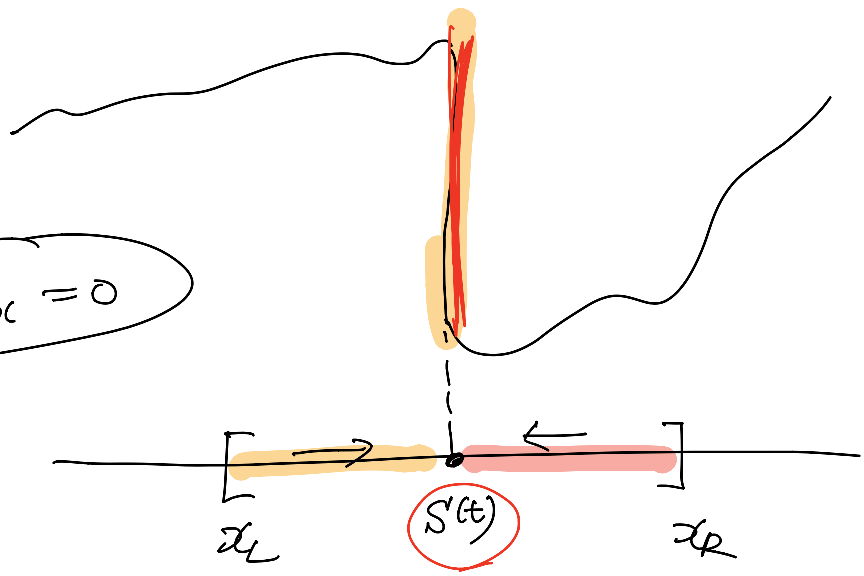
In case 1, the later time soln is given by

$$u(x,t) = \begin{cases} u_l & x < st \\ u_r & x > st \end{cases},$$

$$\boxed{s = \frac{[f]}{[u]}}$$ the shock speed.

Consider

$$u_t + f(u)_x = 0$$



$$\rightarrow \quad 0 = \int_{x_L}^{x_R} \left[u_t + f(u)_x\right] dx$$

$$= \frac{d}{dt}\int_{x_L}^{x_R} u\,dx + \overbrace{f(u(x_R,t))}^{= f_R} - \underbrace{f(u(x_L,t))}_{= f_L}$$

$$\rightarrow \cancel{f_L - f_R} = \frac{d}{dt}\left[\int_{x_L}^{s(t)} u\,dx + \int_{s(t)}^{x_R} u\,dx\right]$$

$$= \underbrace{\lim_{x \uparrow s(t)} u(s(t),t)}_{= u_L} \frac{ds}{dt} + \int_{x_L}^{s(t)} \underbrace{u_t}_{= -f_x}(x,t)\,dx$$

$$- \underbrace{\lim_{x \downarrow s(t)} u(s(t),t)}_{u_R} \frac{ds}{dt} + \int_{s(t)}^{x_R} \underbrace{u_t}_{= -f_x}(x,t)\,dx$$

$$= \frac{ds}{dt}\left[u_L - u_R\right] + \underbrace{\lim_{x \uparrow s(t)} \left(-f(s(t),t)\right)}_{= -f_L} \cancel{+ f_L}$$

$$+ \underbrace{\lim_{x \downarrow s(t)} \left(f(s(t),t)\right)}_{= f_R} \cancel{- f_R}$$

$$\Rightarrow \boxed{\frac{ds}{dt} = \frac{f_L - f_R}{u_L - u_R} = \frac{[f]}{[u]}}$$

$\overset{\text{def}}{|||}$

s

$\boxtimes$

Rmk Entropy Cond. (shock soln) $u_\ell > u_r$

$$\rightarrow \boxed{\lambda(u_\ell) > \sigma > \lambda(u_r)}$$

Rmk Divergent char. cond. (rarefaction soln)

$$\boxed{u_\ell < u_r}$$

$$\rightarrow \boxed{\lambda(u_\ell) < \lambda(u_r)}$$

Ex Solve Burgers Eqn on an ∞ domain with $f(u) = u^2/2$

$$u(x,0) = \begin{cases} \cancel{0} = u_\ell & x<0 \\ 1 \leftarrow u_r & x>0 \end{cases}$$

$\Rightarrow$

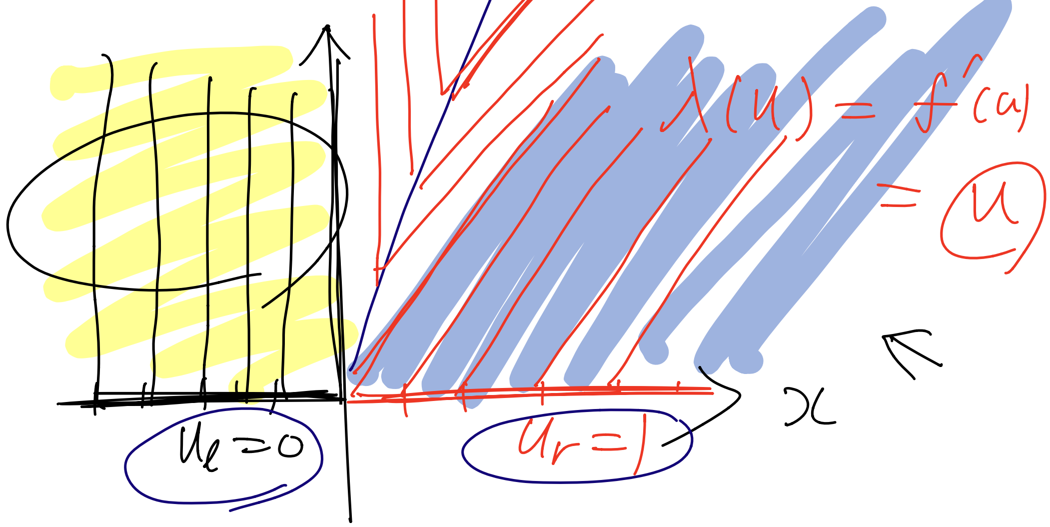
Soln

Case 1 $\quad s = \dfrac{[f]}{[u]} = \dfrac{f_L - f_R}{u_L - u_r} = \dfrac{0 - \frac{1}{2}}{0 - 1}$

$$= \frac{1}{2}$$

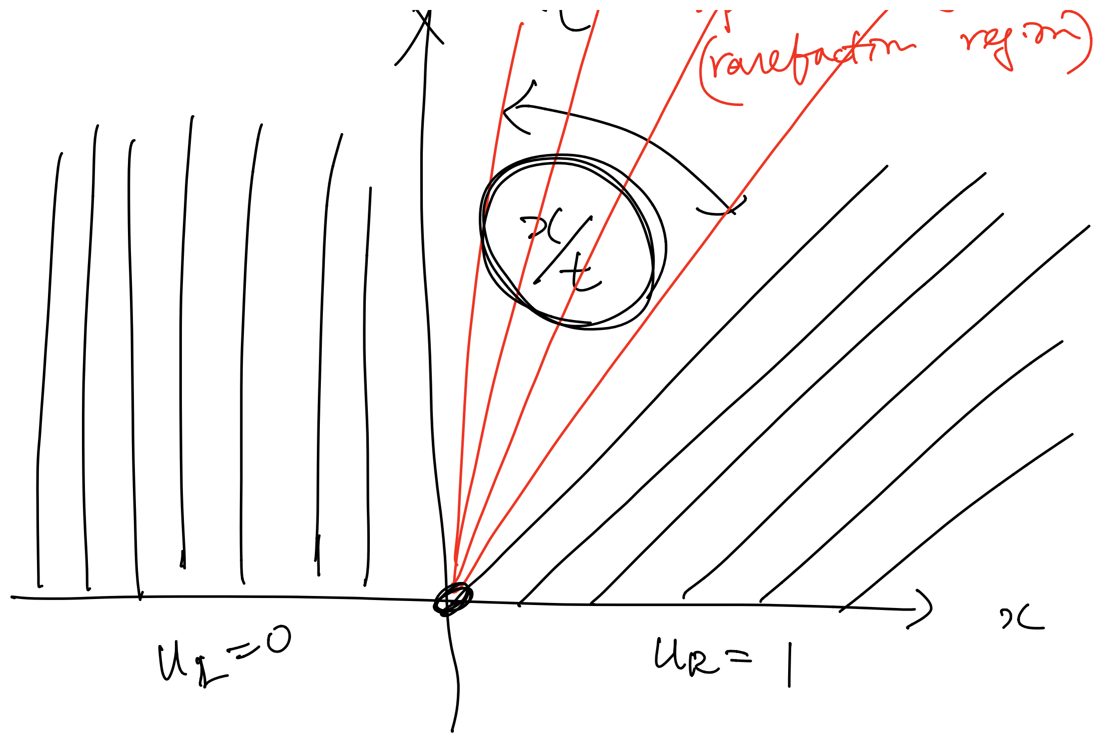
$\Rightarrow$ We may have

$$u(x,t) = \begin{cases} u_l = 0 & x/t < s = \frac{1}{2} \\ u_r = 1 & x/t > s = \frac{1}{2} \end{cases}$$



$$\boxed{\begin{array}{ccccc} \lambda(u_L) & > & \frac{1}{2} & > & \lambda(u_R) \\ \| & & & & \| \\ 0 & & & & 1 \end{array}}$$

expansion region



$$u(x,t) = \begin{cases} u_L & x/t < 0 \\ x/t & 0 \le x/t \le 1 \\ u_R & x/t > 0 \end{cases}$$

Check if $\frac{x}{t}$ ($u(x,t)$) is a proper soln :

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} \stackrel{?}{=} 0$$

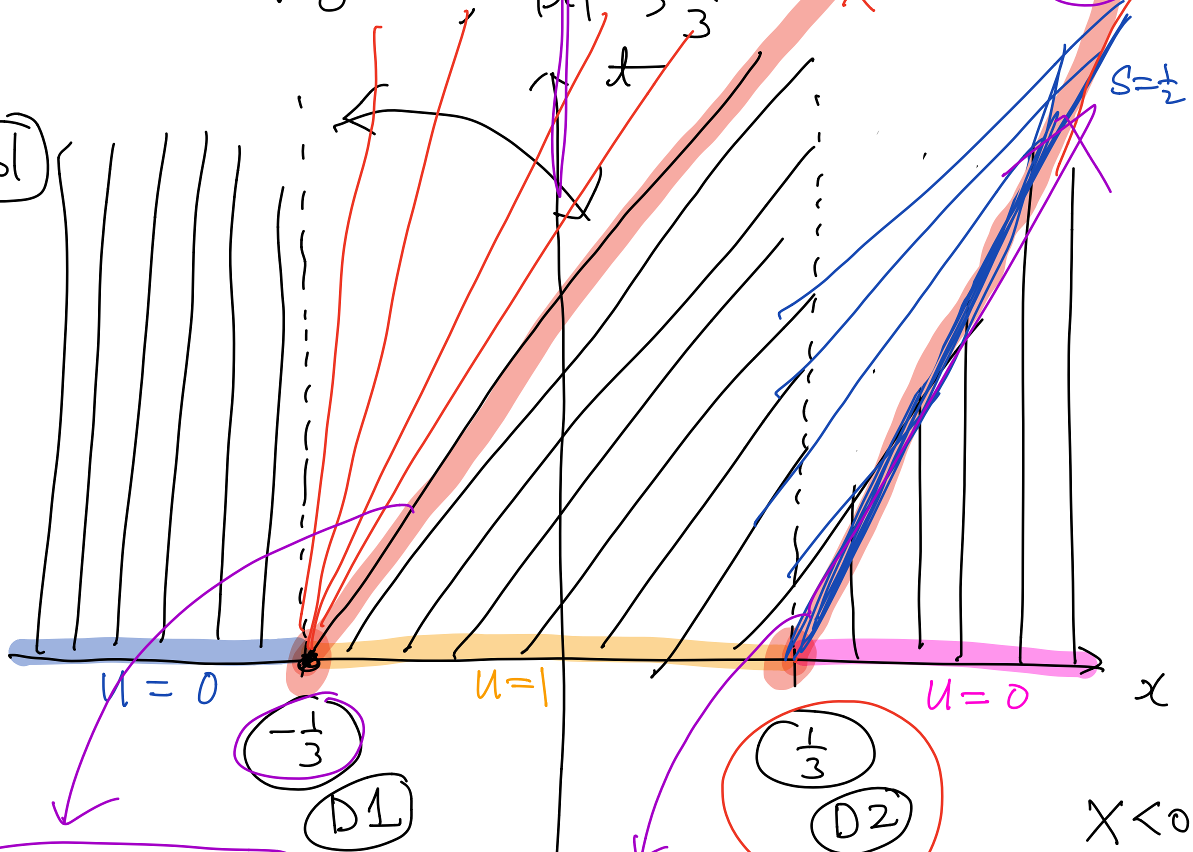
pf $\quad -\frac{x}{t^2} + \left(\frac{x}{t}\right)\frac{1}{t} = 0 \quad \blacksquare$

Ex Solve Burgers Equ on ∞ domain for $t \le \frac{4}{3}$ with IC:

$$u = \begin{cases} 1, & |x| < \frac{1}{3} \\ 0, & |x| > \frac{1}{3} \end{cases}$$

$\lambda(u) = u$

Sol



$x = -\frac{1}{3} + 1t$ $=$ $x = \frac{1}{3} + \frac{1}{2}t$

$t = \frac{4}{3}$

At D1 (i.e, $x = -\frac{1}{3}$) $u = \begin{cases} 0 \\ 1 \end{cases}$

$\rightarrow X = x + \frac{1}{3}$

$X < 0$

$\Updownarrow$

$x < -\frac{1}{3}$

$x > -\frac{1}{3}$

IC $u(X,0) = \begin{cases} 0, & X < 0 \\ 1, & X > 0 \end{cases}$ $X > 0$

The rarefaction soln:

$$\rightarrow \quad u(X,t) = \begin{cases} 0, & X/t < 0 \\ X/t, & 0 \leq X/t \leq 1 \\ 1, & X/t > 1 \end{cases}$$

$$\boxed{\rightarrow \quad u(x+\tfrac{1}{3}, t) = \begin{cases} 0, & x < -\tfrac{1}{3} \\ \dfrac{x+\frac{1}{3}}{t}, & -\tfrac{1}{3} < x < t-\tfrac{1}{3} \\ 1, & x > t-\tfrac{1}{3} \end{cases}}$$

$*$

Now at D2 : $X = x - \frac{1}{3}$

$$u(X,0) = \begin{cases} 1 = u_L & x < 0 \\ 0 = u_R & x > 0 \end{cases}$$

$f(u) = \frac{u^2}{2}$

$$\rightarrow \quad s = \frac{[f]}{[u]} = \frac{f_L - f_R}{u_L - u_R} = \frac{1}{2}$$

$$\rightarrow \quad \overset{(t>0)}{u(x,t)} = \begin{cases} 1, & x < \frac{1}{2}t \\ 0, & x > \frac{1}{2}t \end{cases}$$

(**)

$$\rightarrow \quad \boxed{u\left(x - \frac{1}{3}, t\right) = \begin{cases} 1, & x < \frac{1}{2}t + \frac{1}{3} \\ 0, & x > \frac{1}{2}t + \frac{1}{3} \end{cases}}$$

$$\Rightarrow \quad u(x,t) = \begin{cases} 0, & x < -\frac{1}{3} \\ \frac{x + \frac{1}{3}}{t}, & -\frac{1}{3} < x < t - \frac{1}{3} \\ 1, & t - \frac{1}{3} < x < \frac{1}{2}t + \frac{1}{3} \\ 0 & x > \frac{1}{2}t + \frac{1}{3} \end{cases}$$

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