Ex $\boxed{\text{Lax–Wendroff}}$

$$U_i^{n+1} = U_i^n - \frac{1}{2}\left(\frac{a\Delta t}{\Delta x}\right)\left[U_{i+1}^n - U_{i-1}^n\right]$$

$$+ \frac{1}{2}\left(\frac{a\Delta t}{\Delta x}\right)^2\left[U_{i+1}^n - 2U_i^n + U_{i-1}^n\right]$$

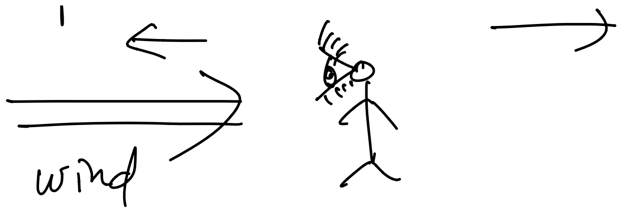
$$\sim O(\Delta t^2 + \Delta x^2)$$

Ex Upwind Method $\sim O(\Delta t + \Delta x)$

Def. The upwind direction is the direction facing the wind. (the direction the wind comes from)

Def. The downwind direction is the direction that the wind goes to.

"upwind" downwind



Ex $u_t + a u_x = 0.$

$a > 0$ Left = upwind

$a < 0$ Right upwind

Case 1 : $a > 0$

upwind

your location

Downwind

x_{i-2} x_{i-1} x_i x_{i+1} . x_{i+2}

informed

uninformed

current

upwind

$$\frac{U_i^{n+1} - U_i^n}{\Delta t} + a \frac{U_i^n - U_{i-1}^n}{\Delta x} = 0 \quad \cdots (*)$$

$(a>0)$

upwind method

$(a>0)$

$$\frac{U_i^{n+1} - U_i^n}{\Delta t} + a\,\frac{U_{i+1}^n - U_i^n}{\Delta x} = 0$$

downwind information

(**) downwind method

Case 2:
$a < 0$

if $a > 0$ then
do (*)
elseif $a \leq 0$ then
do (**)
endif

→ General upwind formulation:

$$U_i^{n+1} = U_i^n - \frac{1}{2}\left(\frac{a\Delta t}{\Delta x}\right)\left[U_{i+1}^n - U_{i-1}^n\right]$$

$$+ \frac{|a|\Delta t}{2\Delta x}\left[U_{i+1}^n - 2U_i^n + U_{i-1}^n\right]$$

$\cdots$ (***)

Ex Let $a>0$, Show the upwind method is $\mathcal{O}(\Delta t + \Delta x)$.

Ca

pf $$U_i^{n+1} = U_i^n - \frac{a\Delta t}{\Delta x}\left(U_i^n - U_{i-1}^n\right)$$

$$= \mathcal{N}(U_i^n)$$

$$E_{LT,i}^{n+1} = \frac{1}{\Delta t}\left[u_i^{n+1} - \mathcal{N}(u_i^n)\right]$$

$$u_i^{n+1} = \mathcal{N}(u_i^n)$$

$$= \frac{1}{\Delta t}\left[u(x_i,t^{n+1}) - u(x_i,t^n)\right] + \frac{a}{\Delta x}\left\{u(x_i,t^n) - u(x_{i-1},t^n)\right\}$$

$$= \frac{1}{\cancel{\Delta t}}\left[\cancel{u(x_i,t^n)} + \cancel{\Delta t}\, u_t + \frac{\Delta t^{2}}{2} u_{tt} + \mathcal{O}(\Delta t^3) - \cancel{u}\right]$$

$$+ \frac{a}{\cancel{\Delta x}}\left[\cancel{u} - \left\{\cancel{u} - \cancel{\Delta x}\, u_x + \frac{\Delta x^2}{2} u_{xx} + \mathcal{O}(\Delta x^3)\right\}\right]$$

$$= u_t + \frac{\Delta t}{2} u_{tt} + \mathcal{O}(\Delta t^2)$$

$$+ a\, u_x - a\,\Delta x\, u_{xx} + \mathcal{O}(\Delta x^2)$$

$$u_t + a u_x = 0$$

$$u_t = -a u_x$$

$$u_{tt} = -a u_{tx}$$

$$= \frac{a^2 \Delta t}{2} - a\Delta x \, u_{xx} + \mathcal{O}(\Delta t^2 + \Delta x^2)$$

$= -a(-au_x)_x$

$= a^2 u_{xx}$

$\therefore \quad \boxed{\mathcal{O}(\Delta t + \Delta x)}$ ✓

$\therefore$ 1st order in both space & time.

To show the upwind method is consistent ✓ ;

$$\lim_{\Delta t, \Delta x \to 0} E_{LT,i}^{n+1} = 0 \qquad \forall i.$$

(Ex) Show LW is $\mathcal{O}(\Delta t^2 + \Delta x^2)$.

(Ex) Show LF is $\mathcal{O}(\Delta t + \Delta x)$

(Ex) Show that the upwind method $(a>0)$ is stable if $0 < C_a \le 1$.

$$\left(Ca = \frac{a\Delta t}{\Delta x} \right)$$

$$U_i^{n+1} = U_i^n - Ca(U_i^n - U_{i-1}^n)$$

Pf

$$\|U^{n+1}\|_1 = \Delta x \sum_{i=1}^{N} |U_i^{n+1}|$$

$$\le \Delta x \sum_{i=1}^{N} |(1-Ca)U_i^n|$$

$$+ \Delta x \sum_{i=0}^{N-1} Ca|U_{i-1}^n|$$

$$\le \Delta x \sum \underline{(1-Ca)} |U_i^n| + \Delta x \sum Ca |U_{i-1}^n|$$

$$= (1-Ca)\, \Delta x \sum |U_i^n| + Ca\, \Delta x \sum |U_{i-1}^n|$$

$$= (1-Ca)\|U^n\| + Ca\|U^n\|$$

$$= \|U^n\|_1 \quad \therefore \text{ stable.}$$

$\|N^n\| < C$

$\|N\| \le 1$

$\|U^{n+1}\| \le \|U^n\|$

Von-Neumann Stability analysis

Ex Show that the <u>downwind method</u> $(a>0)$ is <u>NOT</u> stable for $0 < Ca \le 1$

$$U_i^{n+1} = U_i^n - Ca(U_{i+1}^n - U_i^n)$$

pf $\|U^{n+1}\| = \Delta x \sum |U_i^{n+1}|$

$= \Delta x \sum |(1+C_a)U_i^n - C_a U_{i+1}^n|$

$\rightarrow$ (Note $||x|-|y|| \leq |x-y|$

and "=" holds when $x=y$)

$\geq \Delta x \sum (1+C_a)|U_i^n| - \Delta x \sum C_a |U_{i+1}^n|$

$= (1+C_a)\|U^n\| - C_a\|U^n\|$

$= \|U^n\|$ ▨

Stability using Von Neumann Analysis

$u_t + a u_x = 0 \rightarrow$ (FTFS ⋮)

Real space | Fourier Space

PDE $\xrightarrow{FT}$ ODE

$u_t + a u_x = 0$ $\longrightarrow$ $\hat{u}_t + aIk\hat{u} = 0$

Soln to PDE $\xleftarrow{IFT}$ soln

(1) single Fourier mode rep.

v_j^n

$$v_j^n = A(t)\, e^{Ikj\Delta x} \quad \cdots \quad (*)$$

k ; wave #

$I = \sqrt{-1}$

$A(t)$ = amplification factor

$= e^{\alpha n \Delta t}$ $\qquad \alpha$: const.

(2) $\|U^{n+1}\|_1 \le \|U^n\|_1$

$\longrightarrow$ Looking at j^{th} ;

$$1 \geq \left| \frac{U_j^{n+1}}{U_j^n} \right| = \left| \frac{e^{\alpha(n+1)\Delta t} \cancel{e^{Ikj\Delta x}}}{e^{\alpha n\Delta t} \cancel{e^{Ikj\Delta x}}} \right|$$

$$= \left| e^{\alpha\Delta t} \right|$$

(***)

Ex Consider the upwind method $a > 0$

$$U_j^{n+1} = U_j^n - \underbrace{\frac{a\Delta t}{\Delta x}}_{Ca} \left[U_j^n - U_{j-1}^n \right] \quad \leftarrow (**)$$

$$\Rightarrow e^{\alpha(n+1)\Delta t} e^{Ikj\Delta x} = e^{\alpha\Delta t} e^{Ikj\Delta x}$$

$$- Ca\, e^{\alpha n\Delta t} \left[e^{Ikj\Delta x} - e^{Ik(j-1)\Delta x} \right]$$

$$\Rightarrow e^{\alpha\Delta t} = 1 - Ca \left[1 - e^{-Ik\Delta x} \right]$$

$$= 1 - Ca + Ca \cos(k\Delta x)$$

$$+ I(\sin(k\Delta x))$$

Claim: $\left| e^{\alpha \Delta t} \right|^2 \leq \cancel{1}$?

$$\cancel{1} - 2\,Ca\,(1 - Ca)\,(1 - \cos(k\Delta x))$$

$$\Rightarrow \cancel{2}\,Ca\,(1 - Ca)\,\cancel{(1 - \cos(k(\Delta x))} \geq 0$$

$$\Rightarrow \boxed{Ca\,(1 - Ca) \geq 0}$$

$$\Rightarrow \boxed{0 \leq Ca \leq 1}$$

$Ca = \frac{a\Delta t}{\Delta x}$

Ex Show FTCS is stable for solving $u_t = b\,u_{xx}$ if

$$\frac{b\Delta t}{\Delta x^2} \leq \frac{1}{2}.$$

Soln
$$U_j^{n+1} = U_j^n + \frac{b\Delta t}{\Delta x^2}\left[U_{j+1}^n - 2U_j^n + U_{j-1}^n\right]$$

$$\rightarrow \boxed{U_j^n = e^{\alpha n \Delta t}\, e^{Ikj\Delta x}}$$

$\left| e^{\alpha \Delta t} \right| \leq 1$

$\boxed{\dfrac{b \Delta t}{\Delta x^2} \leq \dfrac{1}{2}}$ ▨

Ex Lax – Fredrichs method is stable for $-1 \leq Ca \leq 1$.

▨

Fredrichs

CFL Condition

Courant

Lewy

consistent

A (consistent) numerical method can be convergent only if its numerical DoD $\supset$ analytical DoD at least as $\Delta t, \Delta x \to 0$

Lax - Equiv. Thm

consisten + stable $\Longleftrightarrow$ Convergent