Ex $u_t + u u_x = 0$ $\rightarrow$ $u_t + a u_x = 0$

$a \geq 0$

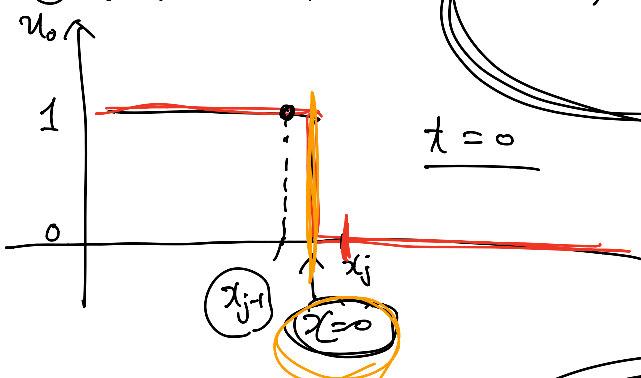
$\rightarrow$ $\boxed{U_i^{n+1} = U_i^n - \frac{U_i^n \Delta t}{\Delta x}\left(U_i^n - U_{i-1}^n\right)}$ $\leftarrow$ $U_i^{n+1} = U_i^n - \frac{a \Delta t}{\Delta x}\left(U_i^n - U_{i-1}^n\right)$

$(U_i^n \geq 0)$, $\forall n, i$ $(*)$

@ (x_i, t^n)

$\rightarrow$ Consider IC

$$u_0(x) = \begin{cases} 1 & x < 0 \\ 0 & x \geq 0 \end{cases}$$



$t = 0$

is NOT a weak soln

$$U_i^0 = \begin{cases} 1 & x_i < 0 \\ 0 & x_i \geq 0 \end{cases}$$

$\rightarrow$ Choose j s.t

$$\begin{cases} U_{j-1}^0 = 1 \\ U_j^0 = 0 \end{cases}$$

$(n = 0) \rightarrow (n = 1)$

$$\to \boxed{U_j^1 = \underset{0}{U_j^0} - \frac{U_j^0 \Delta t}{\Delta x}\left(U_j^n - U_{j-1}^n\right) = 0}$$

$$\to \begin{cases} U_j^n = U_j^0 = 0, \ \forall n, \\ U_i^n = U_i^0, \quad \forall n, \ \forall i. \end{cases}$$

(**)

→ Soln = standing shock.

→ Although, from theory, $f(u) = \frac{u^2}{2}$

$$s = \frac{[f]}{[u]} = \frac{f(1) - f(0)}{1 - 0} = \frac{1}{2}$$

$$\boxed{u_t + (f(u))_x = 0}$$

Ex

$$u_0 = \begin{cases} 2 & x < 0.3 \\ 1 & x \geq 0.3 \end{cases}$$

on $[0, 1]$

Ex $U_i^{n+1} = U_i^n - \frac{\Delta t}{\Delta x}\left[\frac{1}{2}(U_i^n)^2 - \frac{1}{2}(U_{i-1}^n)^2\right]$ ...***

$$- \frac{\Delta t \Delta x}{2}\left(\frac{U_i^n - U_{i-1}^n}{\Delta x}\right)^2 \quad ①$$

$$\sim -\int_{t^n}^{t^{n+1}} \frac{1}{2}\Delta x\, u_x^2 \, dt$$

① → 0 as $\Delta x \to 0$ ⇐

But ① will behave differently for different Δx, U_i^n

$f = \frac{1}{2}u^2$

Rmk Conservative discrete form:

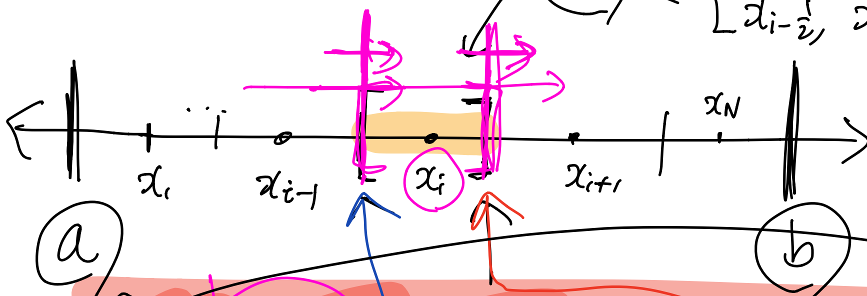
$$U_i^{n+1} = U_i^n - \frac{\Delta t}{\Delta x}\left[F(U_i^n) - F(U_{i-1}^n)\right]$$

(e.g. upwind form)

Rmk. A general conservative

discrete form :

$$I_i = [x_{i-\frac{1}{2}}, x_{i+\frac{1}{2}}] \qquad [a,b] = \bigcup_{i=1}^{N} I_i$$



$$\Rightarrow \Delta x \left[U_i^{n+1} = U_i^n - \frac{\Delta t}{\Delta x}\left[F^n_{i+\frac{1}{2}} - F^n_{i-\frac{1}{2}}\right] \right]$$

$\Rightarrow$ FVM;

$$\begin{cases} U_i^n = \dfrac{1}{\Delta x}\displaystyle\int_{x_{i-\frac{1}{2}}}^{x_{i+\frac{1}{2}}} u(x, t^n)\, dx \\[2ex] F^n_{i+\frac{1}{2}} = \dfrac{1}{\Delta t}\displaystyle\int_{t^n}^{t^{n+1}} f(u(x_{i+\frac{1}{2}}, t))\, dt \end{cases}$$

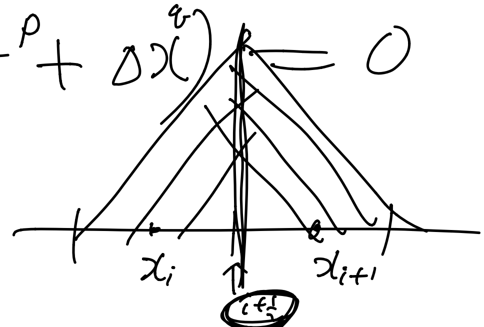
$\star$

$$\Rightarrow \Delta x \sum_{i=1}^{N} U_i^{n+1} = \Delta x \sum_{i=1}^{N} U_i^n - \Delta t \left(F^n_{N+\frac{1}{2}} - F^n_{\frac{1}{2}} \right)$$

$$= \Delta x \sum_{i=1}^{N} U_i^n - \Delta t \left(F^n(b) - F^n(a) \right)$$

Consistency for discont. soln.

For smooth $\lim_{\Delta t, \Delta x \to 0} O(\Delta t^p + \Delta x^q) \to 0$



$$F^n_{i+\frac{1}{2}} = \mathcal{F}(\underline{U_i^n}, \underline{U_{i+1}^n})$$

numerical flux

a final relation of flux using U_i^n & U_{i+1}^n

Def The method

$$U_i^{n+1} = U_i^n - \frac{\Delta t}{\Delta x}\left[\mathcal{F}(U_{i+1}^n, U_i^n) - \mathcal{F}(U_i^n, U_{i-1}^n)\right]$$

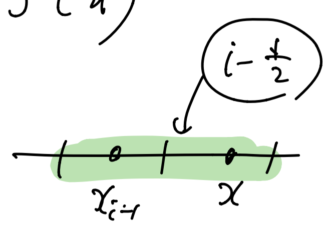
is consistent with the original conservation law if $\mathcal{F}_c$ reduces to the true flux ftn f for the case of constant flows;

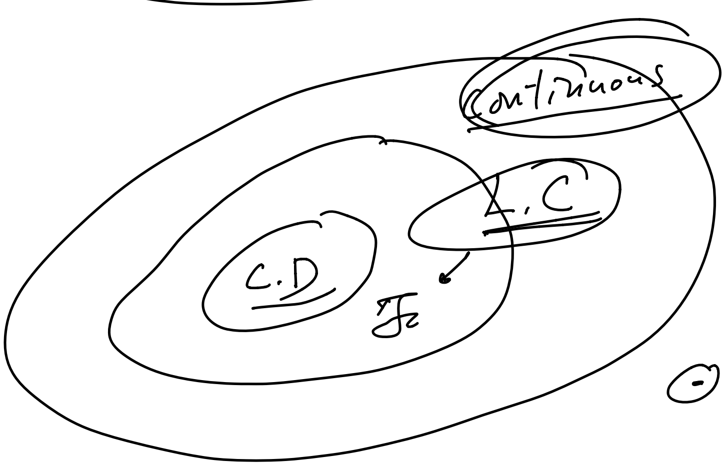
i.e., if $u(x,t) = \bar{u}$ const. then

$$\mathcal{F}_c(\bar{u},\bar{u}) = f(\bar{u}), \ \forall \bar{u} \in \mathbb{R}.$$

Rmk We at least expect $\mathcal{F}_c$ is continuous. so that

$$\lim_{V_{i-1}^n, V_i^n \to \bar{u}} \mathcal{F}_c(V_{i-1}^n, V_i^n) = f(\bar{u})$$





Lipschitz cont.

$\odot$ $f(x) = |x|$

$\odot \; f(x) = \sqrt{x}$

Def. $\mathcal{F}_i$ is Lipschitz at $\bar{u}$ if $\exists\, L$ (may depend on $\bar{u}$) s.t

$$\left| \mathcal{F}_i(U_{i-1}^n, U_i^n) - f(\bar{u}) \right|$$

$$\leq L \max\left(|U_{i-1}^n - \bar{u}|, |U_i^n - \bar{u}| \right)$$

for all U_{i-1}^n, U_i^n with

$|U_i^n - \bar{u}|$ & $|U_{i-1}^n - \bar{u}|$ suff. small.

Rmk. $\mathcal{F}_i$ is L. Cont. if it is Lipschitz at every pt.

Note. If $\mathcal{F}_i$ is L.C. then the method is consistent.

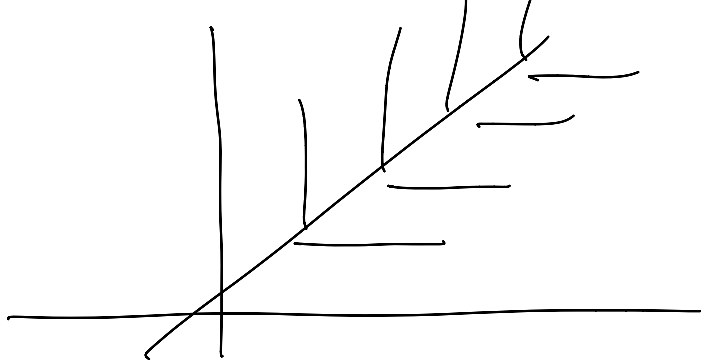
The Lax – Wendroff Thm

Let U_i^n be a num. sol. computed using a conservative & consistent method.

If $U_i^n \longrightarrow u(x,t)$ as $\Delta t, \Delta x \to 0$, then $u(x,t)$ is a weak soln of the conservation law.

Warning ① does NOT guarantee for convergence.

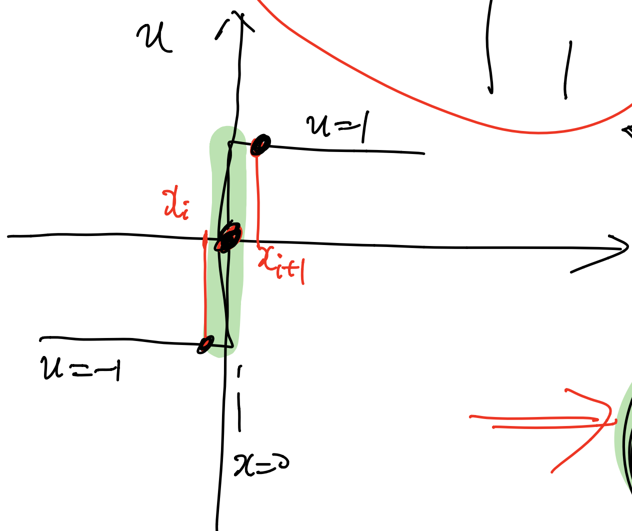
② No guarantee that the convergent weak soln satisfies the entropy condition



Ex Burgers' Eqn

$u_0 = \begin{cases} 1 \\ 0 \end{cases}$

IC $u_0(x) = \begin{cases} -1 & x<0 \\ 1 & x>0 \end{cases}$



weak soln of the Burgers' eqn

$U_i^n = \begin{cases} -1, & x_i < 0 \\ 1, & x_i \geq 0 \end{cases}$

$f(u) = \frac{u^2}{2}$

$$s = \frac{[f]}{[u]} = \frac{f(-1) - f(1)}{-1-1} = \frac{0}{-2} = 0$$

$\longrightarrow$ standing shock.

Ex

$U_i^0 = \begin{cases} -1 & x_i < 0 \\ 0 & x_i = 0 \\ 1 & x_i > 0 \end{cases}$

$$F^n_{i+\frac{1}{2}} = \mathcal{F}(U^n_i, U^n_{i+1}) \leftarrow \text{upwind flux}$$

$$= \begin{cases} f(U^n_i) & S_{i+\frac{1}{2}} \geq 0 \\ f(U^n_{i+1}) & S_{i+\frac{1}{2}} < 0 \end{cases}$$

where

$$S_{i+\frac{1}{2}} = \frac{f(U^n_{i+1}) - f(U^n_i)}{U^n_{i+1} - U^n_i} = \frac{1}{2}$$