Divergence of velocity fields $\nabla \cdot \vec{V}$

$\dfrac{D\rho}{Dt}$

$\dfrac{\partial \rho}{\partial t} + \nabla \cdot (\rho \vec{V}) = 0$

FCV

mass is conserved.

$\mathcal{V}$, S : vary

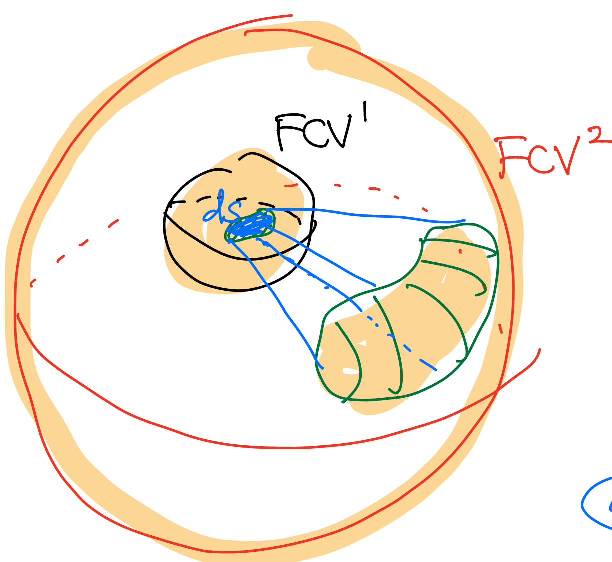
F1 FCV fixed

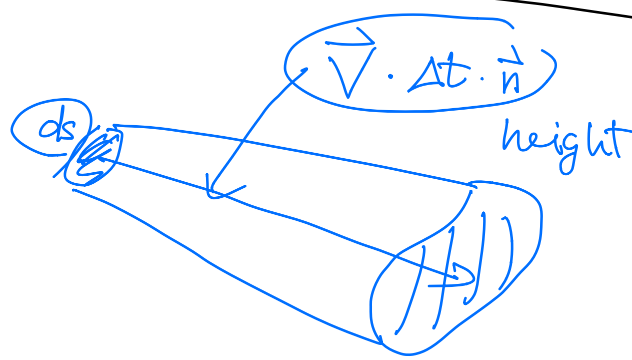
F2 FCV moving

F3 IFE fixed

F4 IFE moving

$\rightarrow d\mathcal{V}$





$\Delta \mathcal{V} = ds\, \vec{V} \cdot \Delta t\, \vec{n}$

$= \vec{V} \cdot \Delta t\, d\vec{s}$

$ds \rightarrow 0$

$$\boxed{\iint_S \Delta \mathcal{V} = \iint_S \vec{V} \cdot \Delta t\, d\vec{s}}$$

$$\rightarrow \frac{1}{\Delta t} \iint_S \Delta V = \iint_S \vec{V} \cdot d\vec{S}$$

$$= \iiint_V \boxed{\nabla \cdot \vec{V}} \, dV$$

$$\frac{DV}{Dt}$$

$$V \rightarrow \delta V$$

$$\rightarrow \frac{D(\delta V)}{Dt} = \iiint_{\delta V} \nabla \cdot \vec{V} \, dV$$

$$= \nabla \cdot \vec{V} \iiint_{\delta V} dV = \delta V$$

$$= (\nabla \cdot \vec{V}) \, \delta V$$

$$\rightarrow \boxed{\nabla \cdot \vec{V} = \frac{1}{\delta V} \frac{D(\delta V)}{Dt}}$$

$\rightarrow$ incompressible flow $\boxed{\nabla \cdot \vec{V} = 0}$

incompressibility condition

The continuity equ. (density equ).

$\Rightarrow$ mass conservation.

Assume F1. (FCV fixed in space)

LHS

$\rightarrow$ The net mass flow "out" of V throug S

$=$ the time rate of "decrease" of mass inside V

RHS

$\rightarrow$ LHS : $\rho \vec{V} \cdot \vec{n} ds = \rho \vec{V} \cdot d\vec{s}$

$$\iint_S \rho \vec{V} \cdot d\vec{s}$$

$\rightarrow$ RHS : $-\frac{\partial}{\partial t} \iiint_V \rho \, dV$

F1

$$\Rightarrow \boxed{\frac{\partial}{\partial t} \iiint_V \rho \, dV + \iint_S \rho \vec{V} \cdot d\vec{s} = 0}$$

⇒ Integral form of the cont. eqn in conservative form.

$$\frac{1}{T}\;\frac{m}{L^3}\,L^3 + \frac{m}{L^3}\,\frac{L}{T}\,L^2$$

$$\underbrace{\frac{1}{T}\,\frac{m}{L^3}\,L^3}_{\frac{m}{T}} + \underbrace{\frac{m}{L^3}\,\frac{L}{T}\,L^2}_{\frac{m}{T}}$$

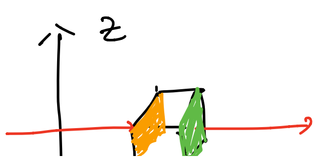
F2 FCV moving with fluids

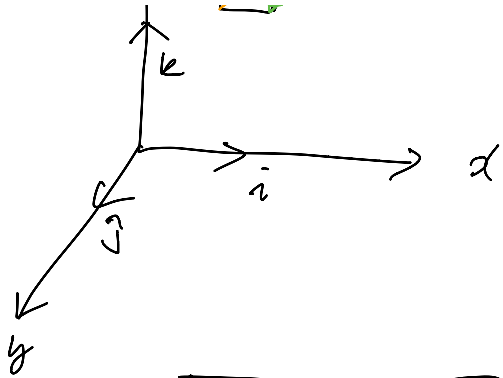
$\frac{D}{Dt}$

$$\boxed{\frac{D}{Dt}\iiint_V \rho\, dV = 0}$$ F2

Integral form in non-conservative form.

F3 IFE fixed in space.





$$\bar{V} = u\hat{i} + v\hat{j} + w\hat{k}$$

(i) the mass flowing in through the left yz face:

$$\rho u \, dy \, dz$$

(ii) the mass flowing out through the right yz face:

$$\left(\rho u + \frac{\partial(\rho u)}{\partial x} dx\right) dy \, dz$$

(iii) The net outflow in x-dir.:

$$\frac{\partial(\rho u)}{\partial x} dx \, dy \, dz.$$

(iv) $\dfrac{\partial(\rho v)}{\partial y} dx \, dy \, dz$ in y-dir

(v) $\dfrac{\partial(\rho w)}{\partial z} dx \, dy \, dz$ in z-dir

$$\Rightarrow \left(\frac{\partial(\rho u)}{\partial x} + \frac{\partial(\rho v)}{\partial y} + \frac{\partial(\rho w)}{\partial z} \right) \cancel{dx\,dy\,dz}$$

= time rate of decrease of the total mass

$$-\frac{\partial}{\partial t}\rho\, \cancel{dx\,dy\,dz}$$

$$\Rightarrow \boxed{\frac{\partial \rho}{\partial t} + \frac{\partial(\rho u)}{\partial x} + \frac{\partial(\rho v)}{\partial y} + \frac{\partial(\rho w)}{\partial z} = 0}$$

$$\Rightarrow \boxed{\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \vec{V}) = 0} \quad F3$$

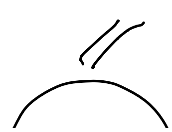
differential form in conservative form

F4 IFE moving with fluids.

substantial derivative

$$0 = \frac{D}{Dt}(\rho\, \delta V)$$

$\delta V\, \nabla \cdot \vec{V}$



$$= \delta V \frac{D\rho}{Dt} + \rho \frac{D(\delta V)}{Dt}$$

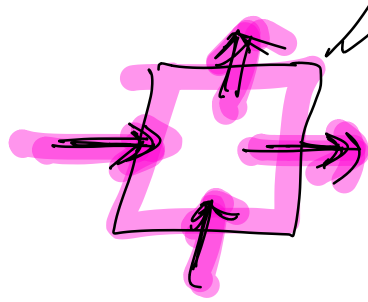
$$= \cancel{\delta V} \frac{D\rho}{Dt} + \rho \cancel{\delta V}\, \nabla \cdot \vec{V}$$

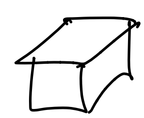
$$\boxed{\frac{D\rho}{Dt} + \rho \nabla \cdot \vec{V} = 0}$$

differential form in non-conservative

$$\boxed{\frac{\partial\rho}{\partial t} + u\frac{\partial\rho}{\partial x} + v\frac{\partial\rho}{\partial y} + w\frac{\partial\rho}{\partial z} + \rho \nabla \cdot \vec{V} = 0}$$ F4

$$\boxed{\frac{\partial\rho}{\partial t} + \nabla \cdot (\rho \vec{V}) = 0}$$ F3





Conservative form

Chapter 2

Scalar Conservation Law (Mathem'l Theory)

→ Two types of PDEs.

(1) Scalar conservation → single equ
- linear conservation (ex) $\frac{\partial u}{\partial t} + \frac{\partial}{\partial x}(au) = 0$
- nonlinear conservation (ex) Burgers Equ $\frac{\partial u}{\partial t} + \frac{\partial}{\partial x}\left(\frac{u^2}{2}\right) = 0$

(2) System of conservation → multiple equs
- nonlinear (ex) Euler Equs, MHD Equs, rad-hydro, etc.

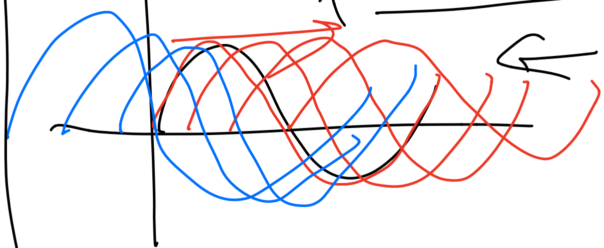
[1] Linear Scalar Equ.

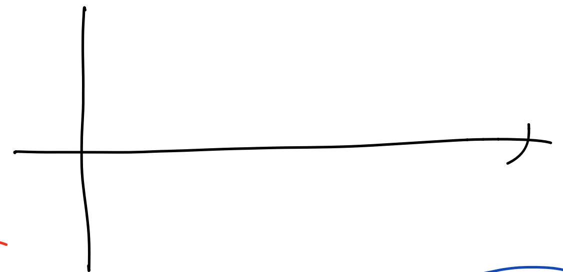
conservative form

$$\boxed{u_t + (au)_x = 0}, \quad t \geq 0 \quad \cdots (1)$$

→ u: quantity gets advected by the advection velocity "a"

IC $u(x,t=0) = u_0(x) = \sin(x)$





left → right

$a>0$

right → left $\quad a<0$

non-conservative form

$$\boxed{u_t + u\frac{\partial a}{\partial x} + a\frac{\partial u}{\partial x} = 0}$$

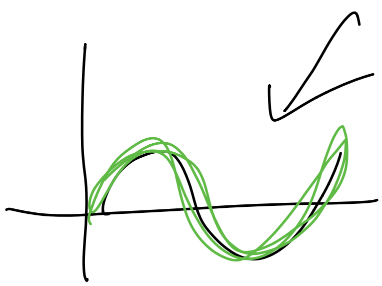
... (2)

$a>0$

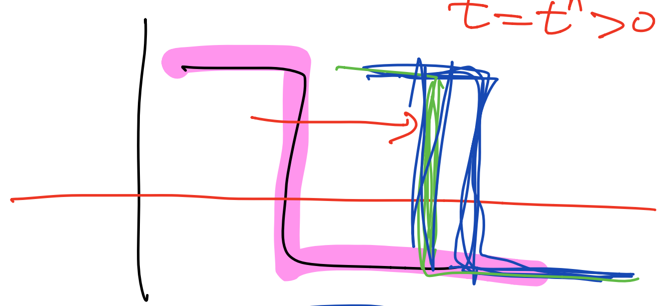
$t=0$

$t=t^n>0$

Case 1



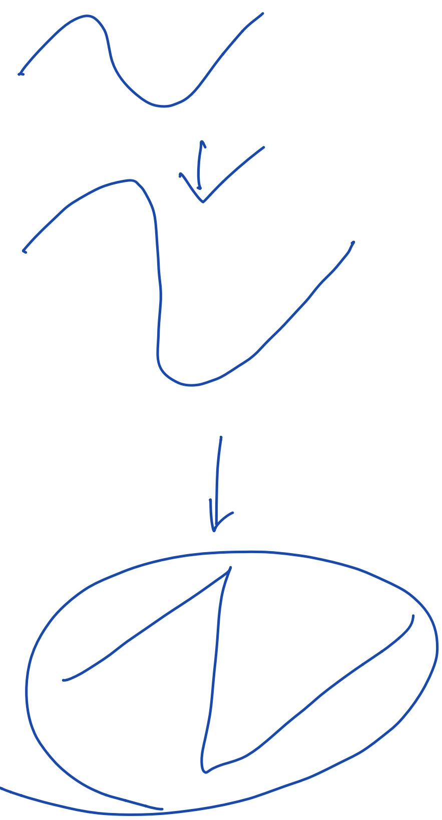
Case 2



incompressible vs. compressible

density doesn't change

density change



$$\boxed{u_t + (au)_x = 0}$$

(i) $\boxed{a = \text{constant}}$

(ii) $\boxed{a = a(x(t))}$

spatial

temporal

IC : $u(x,0) = u_0(x)$ $\quad t>0$

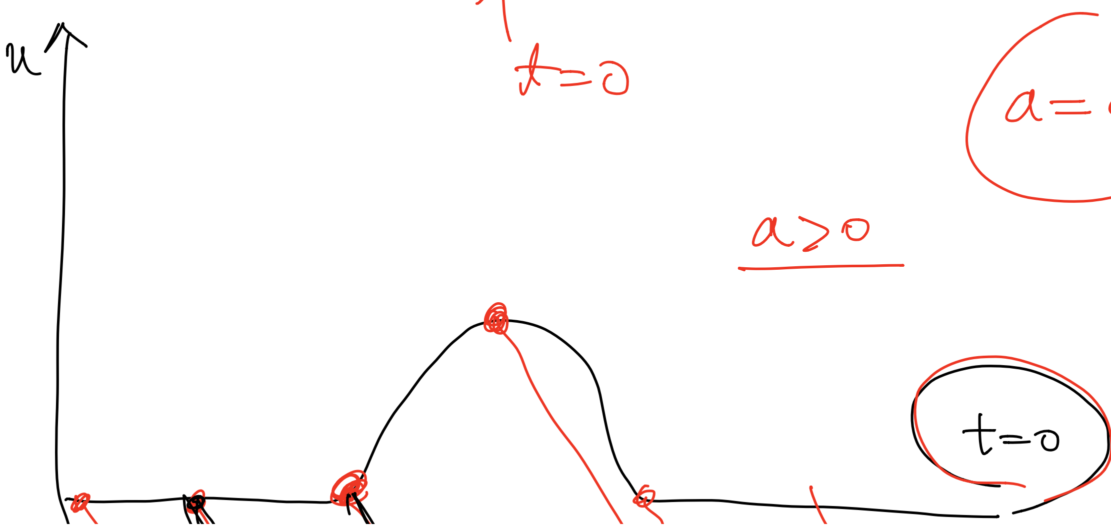
Soln

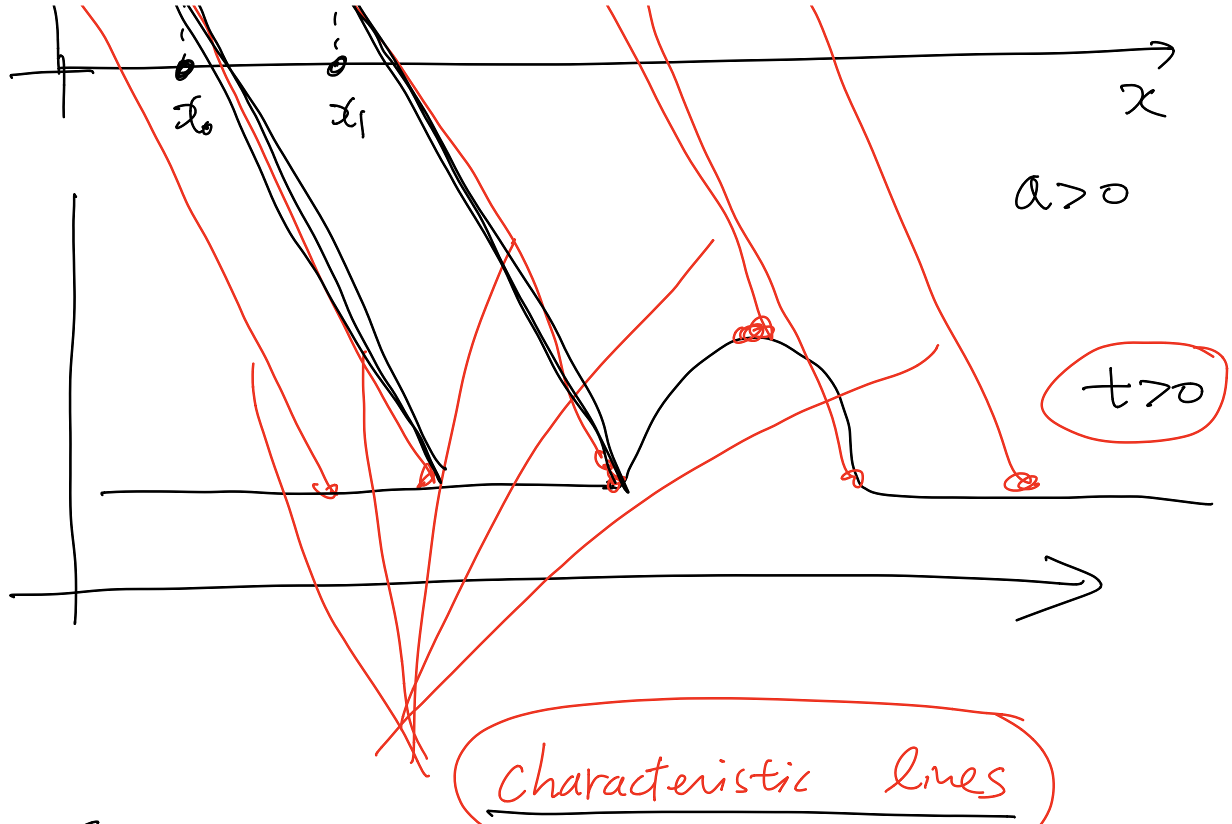
$$u(x,t) = u_0(x - at), \quad t>0$$

$t=0$

$a = a(x)$

$a>0$





ODE

$$\begin{cases} x'(t) = a \\ x(0) = x_0 \end{cases} \qquad \begin{cases} x'(t) = a \\ x(0) = x_1 \end{cases} \quad \cdots$$

Remark

Rmk

The soln $u(x,t)$ is constant along the char. lines when $\boxed{a = \text{const.}}$ $(u_t + a u_x = 0)$

pf $\frac{d}{dt} u(x,t)$

$$= \frac{\partial u}{\partial t} + \frac{\partial u}{\partial x}\left(\frac{dx}{dt}\right) \leftarrow$$

$$= \boxed{\frac{\partial u}{\partial t} + a\frac{\partial u}{\partial x}} = 0 \qquad \square$$

Rmks $u(x,t) \neq \text{const}$ along the char. lines ($u_t + (a(x)u)_x = 0$)

if $a = a(x(t))$.

$u_t + a'(x)u + a(x)u_x = 0$

pf $\frac{d\,u(x,t)}{dt} = \frac{\partial u}{\partial t} + a\frac{\partial u}{\partial x} =$

$= -a'(x)u \neq 0.$

Domain of dependence	Range of Influence.