Riemann Problems for linearized Euler Eqns.

Reconstruction ($\overset{1st}{\text{FOG}}$, $\overset{2nd}{\text{PLM}}$, $\overset{3rd}{\text{PPM}}$, $\overset{5th}{\text{WENO}}$) → "linear"

RP — nonlinear part

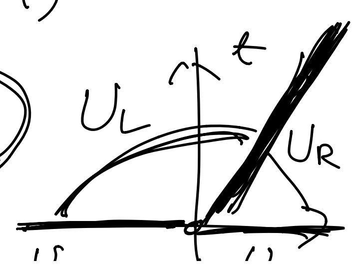
$$\frac{\partial F}{\partial U}$$

$$\begin{cases} \text{(PDE)} & U_t + A U_x = 0 \\ \text{(IC)} & U(x,0) = \begin{cases} U_L, & x<0 \\ U_R, & x>0 \end{cases} \end{cases}$$

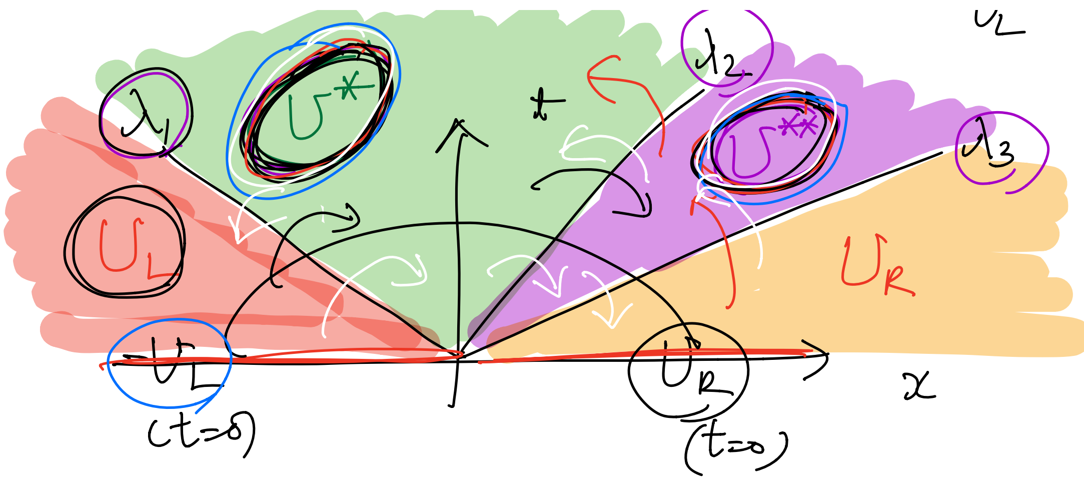
(strictly hyperbolic) $\lambda_1 > \lambda_2 > \lambda_3 \cdots > \lambda_m$

$$U = \begin{pmatrix} \rho \\ \rho u \\ \rho E \end{pmatrix} \qquad F(U) = \begin{pmatrix} \rho u \\ \rho u^2 + p \\ u(\rho E + p) \end{pmatrix}$$

1D Euler $A_{3\times3}$ $m=3$



$U_L \quad \cdot \quad U_R \; x$



Riemann fan

Ex

3D Euler Eqns :

$$U = \begin{pmatrix} \rho \\ \rho u \\ \rho v \\ \rho w \\ \rho E \end{pmatrix} \qquad A_{5\times 5}$$

MHD

$$U = \begin{pmatrix} \rho \\ \rho u \\ \rho v \\ \rho w \\ \rho E \\ B_x \\ B_y \\ B_z \end{pmatrix} \qquad m = 7$$

$A_{7\times 7}$

$\rightarrow U_L$ & U_R : Given $\quad (m=3)$

$\therefore$ $U_L = \sum_{i=1}^{3} \alpha_i r_i$

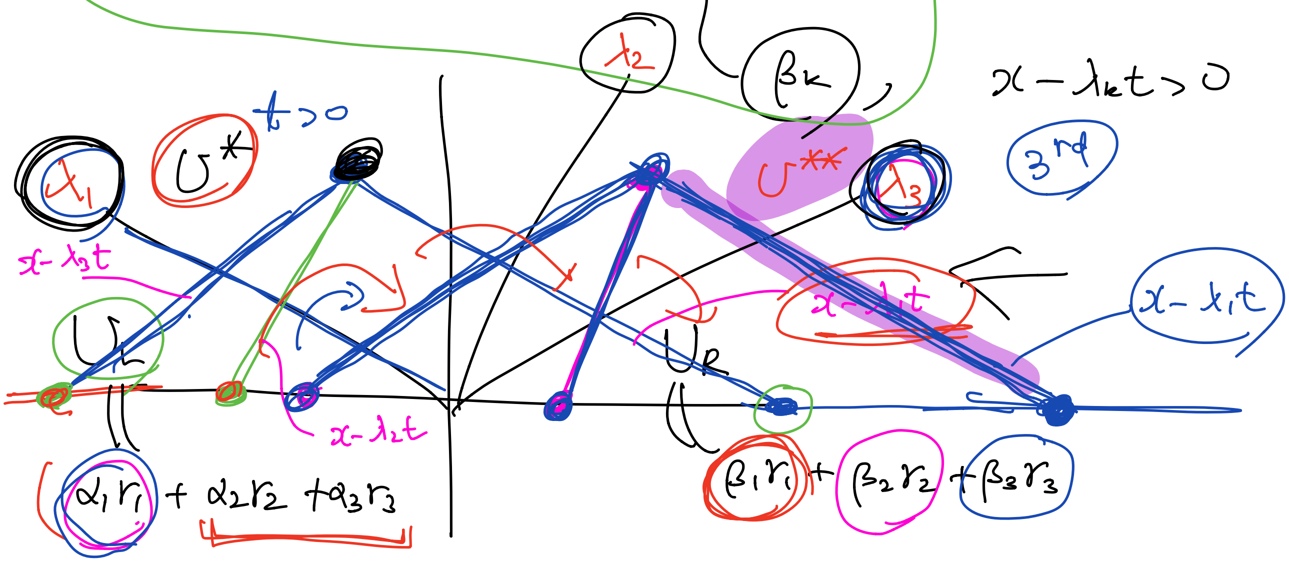
$U_R = \sum_{i=1}^{3} \beta_i r_i$ $\qquad R^{-1} = L$

→ $W = LU$ or $U = RW$

→ $U(x,t) = \sum_{k=1}^{m} r_k w_k(x,t)$ ← $t > 0$

$= \sum_{k=1}^{m} r_k w_k(x - \lambda_k t, 0)$, where

$$w_k(x-\lambda_k t, 0) = \begin{cases} \alpha_k, & x - \lambda_k t < 0 \\ \beta_k, & x - \lambda_k t > 0 \end{cases}$$



→ Let I be the maximum value

s.t. $\quad x - \lambda_k t > 0 \quad , \forall k \le I$

$$\rightarrow U(x,t) = \sum_{k=1}^{I} \beta_k r_k + \sum_{k=I+1}^{m} \alpha_k r_k$$

$$\rightarrow \begin{cases} U^{*} = \underline{\beta_1 r_1} + \alpha_2 r_2 + \alpha_3 r_3 \\ U^{**} = \beta_1 r_1 + \beta_2 r_2 + \alpha_r r_3 \end{cases}$$

$\rightarrow$ Total jump $\Delta U = U_R - U_L$

$$= \sum_{k=1}^{m} (\beta_k - \alpha_k) r_k$$

$$\rightarrow \Delta U_k = (\beta_k - \alpha_k) r_k$$

the strength of the k^{th} wave

Characteristic fields

$, \lambda_k$

→ The char. speed (or eigenvalues) $\lambda_k = \lambda_k(U)$ defines a char. field, the λ_k-field

→ r_k-field

Def. A λ_k-field is said to be $\boxed{\text{linearly degenerate}}$ if

$$\nabla_U \lambda_k(U) \cdot r_k(U) = 0, \quad \forall U$$

contact discont.

Def. A λ_k-field is said to be $\boxed{\text{genuinely nonlinear}}$ if

$$\nabla_U \lambda_k(U) \cdot r_k(U) \neq 0, \quad \forall U.$$

either shock or rarefaction

Rmk $\nabla_U \lambda_k(U) = \left(\frac{\partial \lambda_k}{\partial U_1}, \frac{\partial \lambda_k}{\partial U_2}, \frac{\partial \lambda_k}{\partial U_3} \right)^T$

for each k $\qquad U = \begin{pmatrix} \rho \\ \rho u \\ \rho E \end{pmatrix} = \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix}$

Ex λ_2-field : linearly degenerate

pf $\nabla_v \lambda_2 =$

$$\lambda_2 = u = \frac{\rho u}{\rho} = \frac{u_2}{u_1}$$

$$\nabla_v \lambda_2 = \left(-\frac{u_2}{u_1^2}, \ \frac{1}{u_1}, \ 0 \right)$$

$$= \left(-\frac{u}{\rho}, \ \frac{1}{\rho}, \ 0 \right)$$

$$r_2 = \left(1, \ u, \ \frac{u^2}{2} \right)^T$$

$$\nabla_v \lambda_2 \cdot r_2 = 0 \quad \therefore \text{ lin. deg.}$$

Ex λ_1 & λ_3 : genuinely nonlinear.

(———)

Rmk. λ_k : linearly deg.

$\rightarrow$ λ_k is constant along each integral curve.

Ex. In a const. coeff. $u_t + a u_x = 0$, $\lambda_1 = \lambda_k = a$

$\nabla \lambda_1 = 0$,

Ex. In nonlinear advection.

$u_t + (f(u))_x = 0$ $\quad m=1$

$\lambda_1(u) = f'(u) = A_{1\times 1}$

$$A r_1 = \lambda_1 r_1 = \lambda_1 r_1$$

$$\parallel$$

$$f'(u) r_1 = \lambda_1 \cdot r_1$$

$\therefore$ $\underline{r_1 = 1}$

$$\Rightarrow \nabla \lambda_1 \cdot \underline{r_1} = \underline{f''(u)} = 0$$

$$\neq 0$$

convexity condition

<u>For nonlinear scalar equi</u>

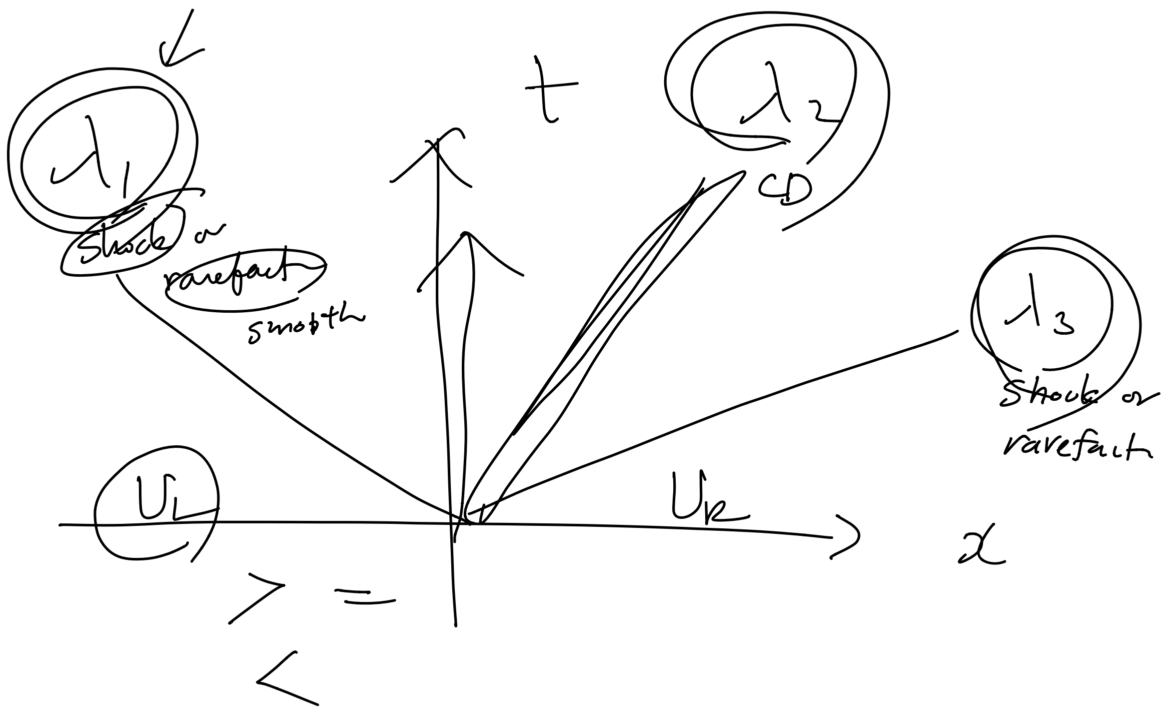
convexity cond $\Leftrightarrow$ gen. nonlinear

<u>Rmk</u>. λ_1 & λ_3 : genuinely nonlinear

$\rightarrow$ either rarefaction or shock

λ_2 : <u>lin. deg.</u>

(contact discont.)



Elementary wave solutions of RP

→ RP consists of a single nontrivial wave

$\boxed{1}$ shock wave ;
U_L & U_R are connected through a single jump discont. in a genuinely nonlinear field k

① RH ; $F(U_R) - F(U_L) = s(U_R - U_L)$

② $\lambda_k(U_L) > s > \lambda_k(U_R)$

2 Contact wave

U_L & U_R are connected through a single jump discont. of speed s in a lin. deg. field k

① RH : $F(U_R) - F(U_L) = s(U_R - U_L)$

② Genneralized Riemann Invariants across the k^{th} wave :

$$\frac{dw_1}{r_k \cdot e_1} = \frac{dw_2}{r_k \cdot e_2} = \frac{dw_3}{r_k \cdot e_3}$$

$(m-1)$ ODE sys.

Constancy relationship across the wave

$W = (w_1, w_2, w_3)^T$; either U or V

3 Rarefaction wave

U_L & U_R : connected through a smooth transition in a gen. nonlinear field k &

① Gen. Riem. Inv.

② $\lambda_k(U_L) < \lambda_k(U_R)$

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ex r_2 – wave

$$r_2 = \begin{bmatrix} 1 \\ u \\ u^2/2 \end{bmatrix}$$

$$\lambda_2 = u$$

$$W = U = (\rho, \rho u, \rho E)^T = (w_1, w_2, w_3)^T$$

$$\frac{dw_1}{r_2 \cdot e_1} = \frac{dw_2}{r_2 \cdot e_2} = \frac{dw_3}{r_2 \cdot e_3}$$

$$\to \frac{d\rho}{1} = \frac{d(\rho u)}{u} = \frac{d(\rho E)}{u^2/2} = c$$

① ② ③

Claim: $\begin{cases} p = \text{constant} \\ u = \text{const.} \end{cases}$

① $d(\rho u) - u\,d\rho = 0$

$\Rightarrow \int d(\rho u) - \int u\,d\rho = 0$

$\Rightarrow \rho u - \rho u = 0$ ✓

② $\int \frac{1}{2} u^2 d\rho - \int d(\rho E) = 0$

$\rightarrow \frac{1}{2}\rho u^2 - \rho E = 0$

$\rho E = \rho\left(\frac{u^2}{2} + e\right)$

$e = \frac{p}{\rho(\gamma-1)}$

$\rightarrow \rho e = c$

$\rightarrow \frac{p}{\gamma - 1} = c$ $\therefore$ $p = \text{const.}$

③ $\int \frac{1}{2} u^2 d(\rho u) - \int d(\rho E) = 0$.

. - - Same as ②

2 ODEs

1' $d\rho = c$

$\rightarrow \int d\rho = \int c \cdot d\rho$

$\rightarrow \rho = c\rho$ $\therefore$ $c = 1$.

2' $\int d(\rho u) = \int u \cdot c \, d(\rho u)$

$\rightarrow \cancel{\rho u} = u \cdot c \cdot \cancel{\rho u}$

$\rightarrow \therefore u = \text{const.}$

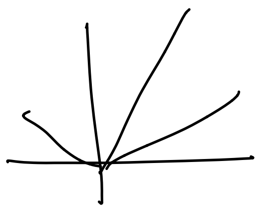
3' $\therefore$ - - $u = \text{constant}$

Riemann Solvers

① HLL

② Roe

HLLC

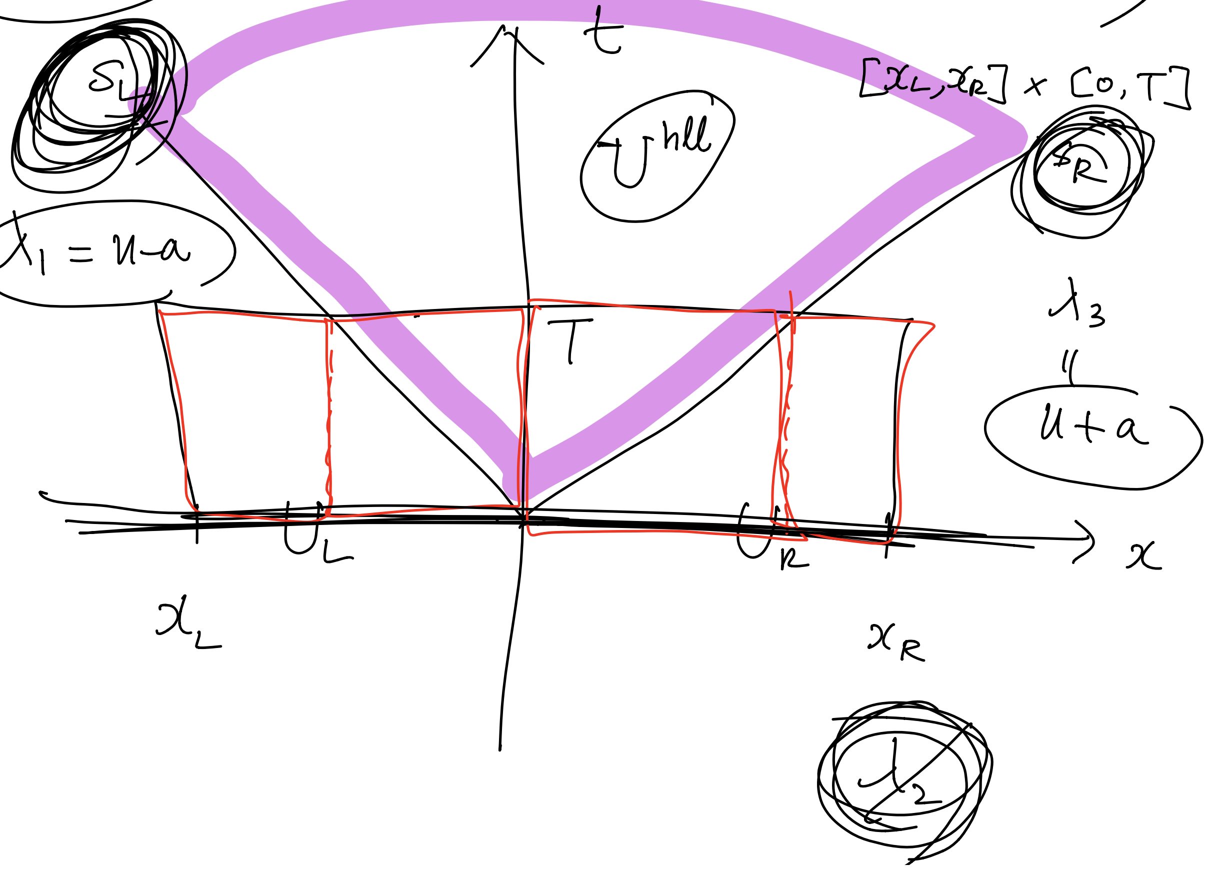


HLLD

LLF

gLF

HLL (Harten – Lax – van Leer)



$$U_t + F(U)_x = 0$$

$t=0$

$$\int_{x_L}^{x_R} U(x,T)\,dx - \int_{x_L}^{x_R} U(x,0)\,dx$$

$$= \int_0^T F(U(x_L,t))\,dt - \int_0^T F(U(x_R,t))\,dt$$