Def. **Domain of Dependence**

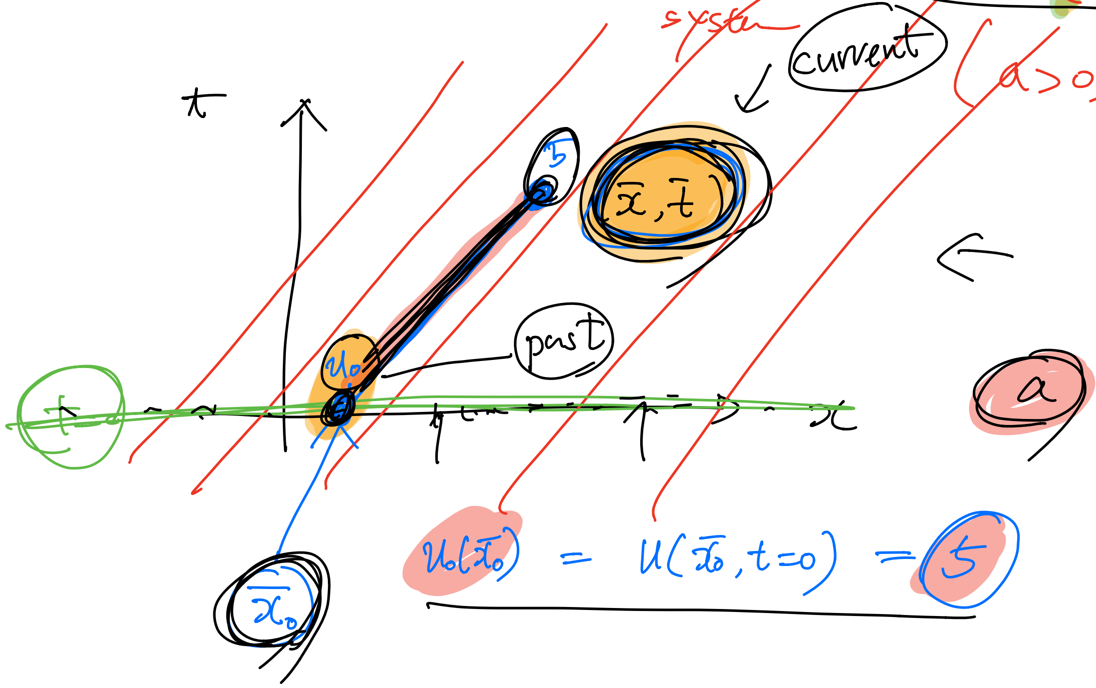
fixed pt.

The soln $u(x,t)$ at any pt $(\bar{x},\bar{t})$ depends only on the initial data at a single pt, namely, $\bar{x}_0$, s.t

char. vel.

linear system $\rightarrow$ $u_t + (au)_x = 0$

$(a>0, \quad a<0)$



$u_0(\bar{x}_0) = u(\bar{x}_0, t=0) = 5$

$(\bar{x},\bar{t})$ lies on the char. line through $\bar{x}_0$

$$u(\bar{x},\bar{t}) = 5$$

Def. DoD $= D(\bar{x},\bar{t})$ $\rightarrow$ char. lines

$$= \{(\bar{x} - \lambda_m \bar{t}) \mid m = 1, \dots, p\}$$

p : # of char. lines

(# of eqns in the given PDE system)

D.D of the current pt $(\bar{x}, \bar{t})$

Ex ① $p = 1$

② Euler Eqns (1D) ($p=3$)

cont. eqn

momentum eqn in x, ~~y~~, z

energy eqn

Rmk. For a given hyperbolic PDEs with p eqns, we have

$$\lambda_1 \le \lambda_2 \le \cdots \le \lambda_p$$

Rmk. If $\lambda_1 \lneq \lambda_2 \lneq \cdots \lneq \lambda_p$

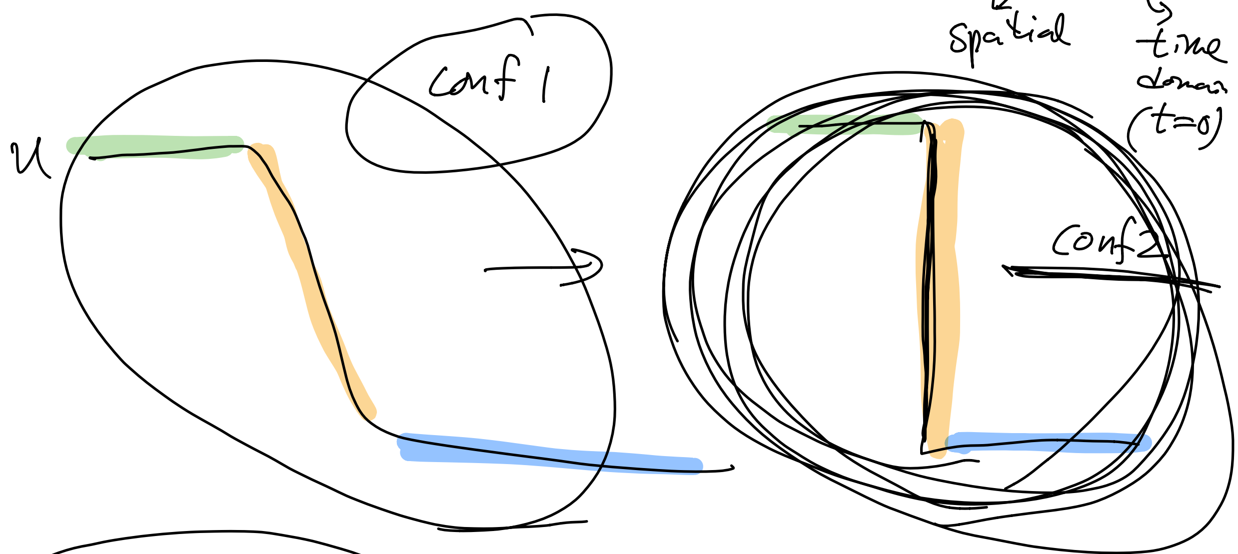
$\Longrightarrow$ "strictly hyperbolic"

Def. R.I $\mathcal{R} = \{ x \mid \lambda_1 t + \bar{x}_0 \le x \le \lambda_p t + \bar{x}_0 \}$

of the pt. $\bar{x}_0 \in \mathbb{R} \times \{0\}$

spatial

time domain $(t=0)$



Non-Smooth Data

linear

$u_t + a u_x = 0$

$(f(u) = au)$

$u_x = ?$

OK for $\forall t > 0$

even though $\exists$ discont @ $t=0$

non-linear

$u_t + \left(\frac{u^2}{2}\right)_x = 0$

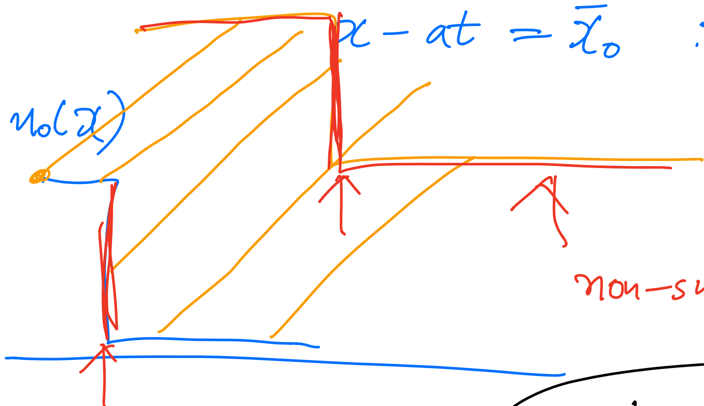
$\left(f(u) = \frac{u^2}{2}\right)$

OK only upto $t < t_b$

$\Rightarrow$ ☺ IF $u_0(\bar{x}_0) = u(\bar{x}_0, t=0)$ is known,

then the later time soln is $u_0(x-at)$, where

$x - at = \bar{x}_0$: char. line.



Two approaches

- Vanishing Viscosity
- Weak soln $\Leftarrow$ ☆

$$\boxed{u_t + a u_x = 0}$$

$$\Downarrow$$

$$\boxed{u_t + a u_x = \varepsilon u_{xx}}$$

$\varepsilon = 0.1$

$\varepsilon = 0.01$

$\varepsilon > 0$

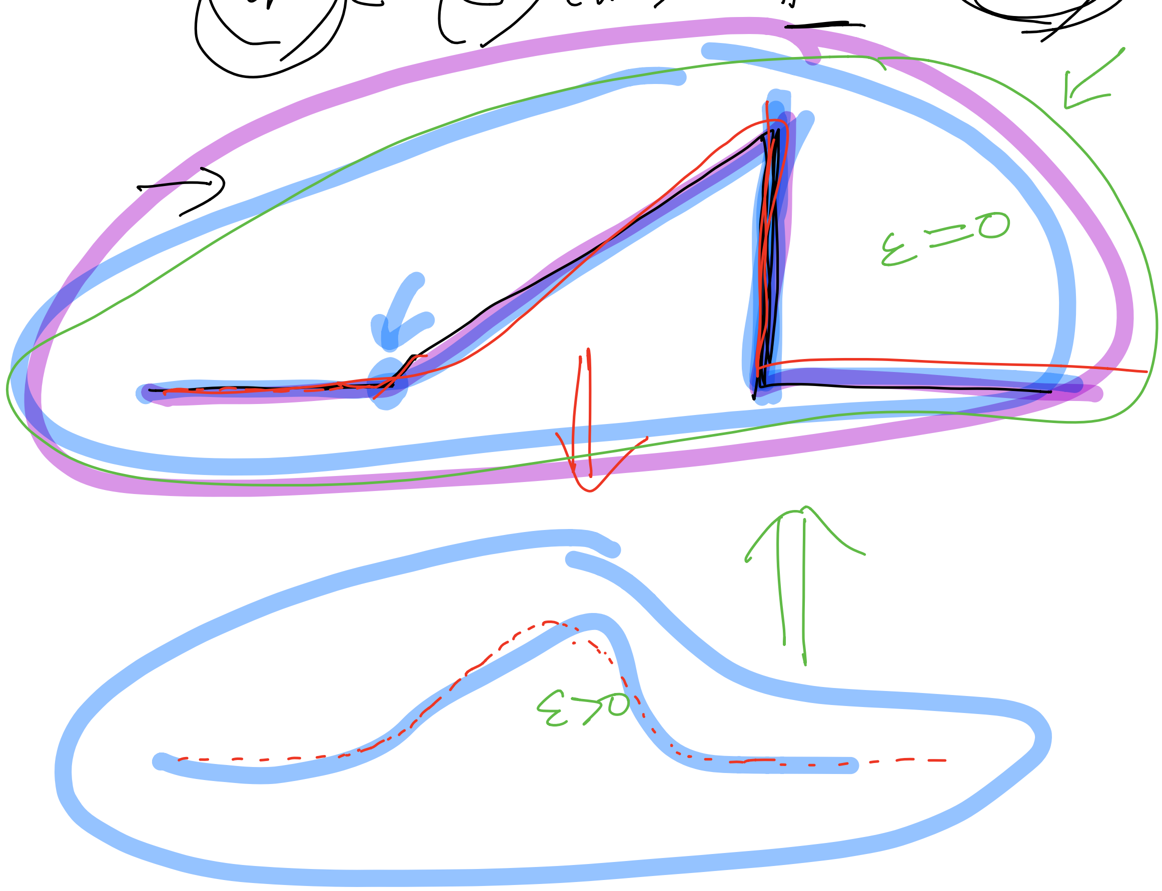
adv-diff. eqn

u_{xx}

$\boxed{\text{parabolic Eq}}$ $\qquad x \to \eta$

$\to$ ... —

$\to$ $u^{\varepsilon} \in C^{\infty}(\mathbb{R}) \times \mathbb{R}^{+}$ $\qquad t>0$

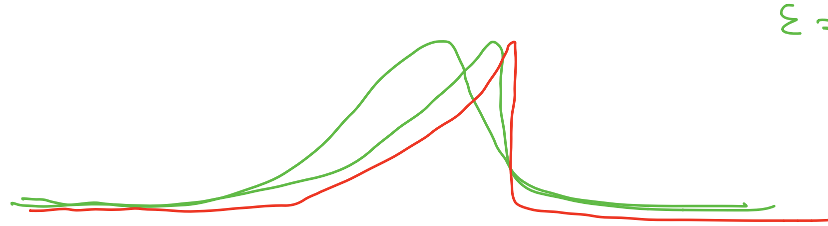


$\varepsilon \to 0$

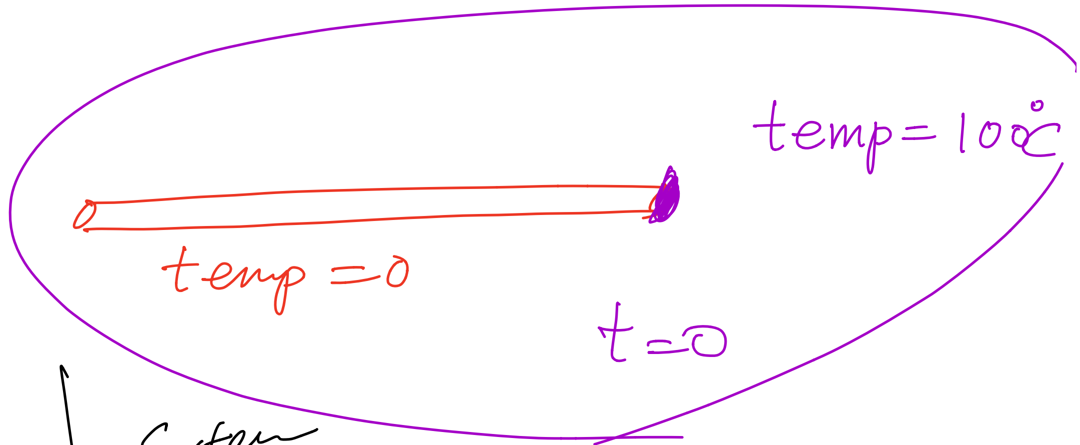
$\varepsilon=0.1$

$\varepsilon=0.01$

$\varepsilon=0,0001$



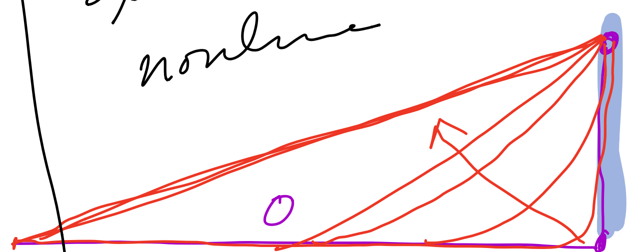
Ex

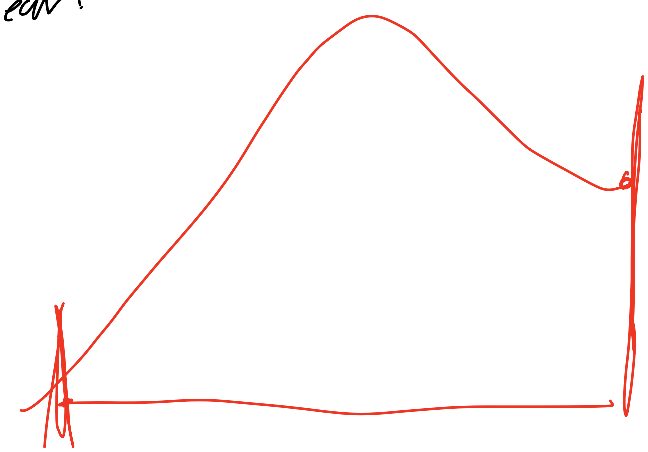


Scalar | System

linear | nonlinear

non-linear





linear.

$u_t + (au)_x = 0$

$f(u) = au$

Nonliear scalar Eqns.

$$u_t + (f(u))_x = 0$$

Flux ftn, nonlinear

(e.g.) $f(u) = u^2/2$

① $f(u)$: convex ($f''(u) > 0, \ \forall u$)

(f is concave if $f''(u) < 0$ $\forall u$)

Ex Burgers

Ex Euler's Equs ← gas dynamics, hydro eqns.

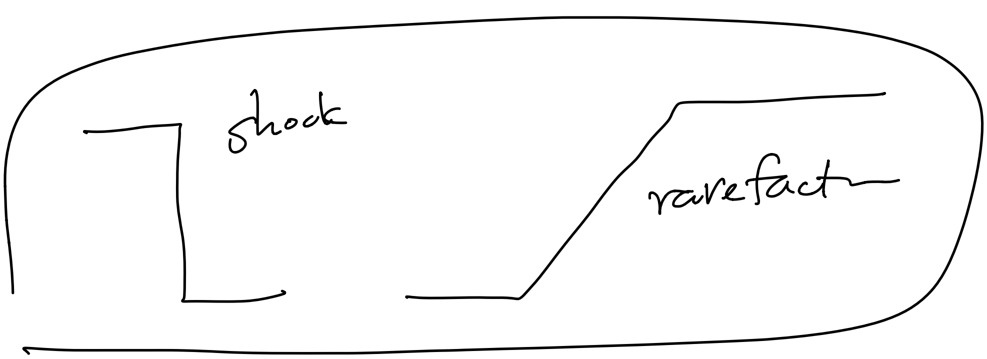
Ex NS Equs

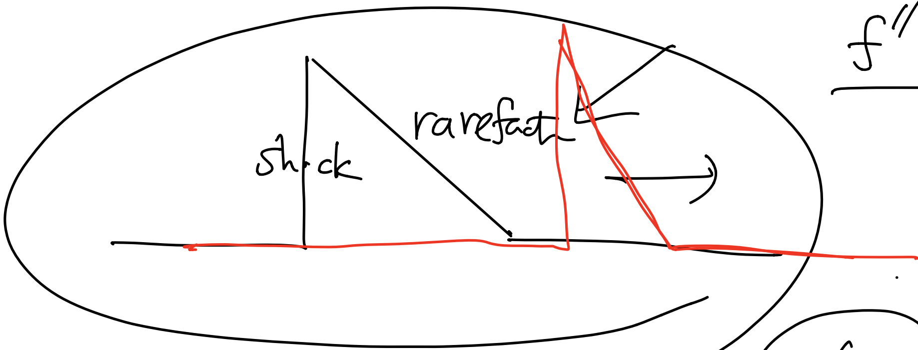
② $f(u)$: non-convex

Ex : Buckley – Leverett equ

$$u_t + \left(\frac{u^2}{u^2 + a^2(1-u)^2} \right)_x = 0$$

Ex MHD equs ← (magnetohydrodynamics)





$f''(u) > 0$

$\Downarrow$

$f''(u) < 0$

$\Rightarrow$ $f'(u)$ can increase & decrease

$\dfrac{df}{du}$ = char. speed

Ex $\quad u_t + \left(\dfrac{u^2}{2}\right)_x = 0, \quad f(u) = u^2/2$

$f'(u) = u$

$\boxed{u_t + u u_x = 0}$

Ex $\quad u_t + (au)_x = 0, \quad f(u) = au$

$f'(u) = a$

Ex $\quad U = \begin{pmatrix} \rho \\ \rho u \\ \rho E \end{pmatrix} \leftarrow$

$\rightarrow \quad U_t + \big(F(U)\big)_x = 0$

$$U_t + \frac{\partial F}{\partial U}\frac{\partial U}{\partial x} = 0$$

3×3

Flux Jacobian matrix

char. velocities are the eigen values of $\frac{\partial F}{\partial U}$.

Burgers Eqn

$\nabla \cdot F(U)$

$$u_t + \left(\frac{u^2}{2}\right)_x = 0 \leftarrow$$ conservative form (or divergence form)

$$u_t + u u_x = 0 \leftarrow$$ non-conservative form

$$\nabla \cdot f(u) = \frac{\partial}{\partial x}\left(f(u)\right)$$

(1) properties of $\boxed{u(x,t)}$ for small t

(i.e, $0 < t \le t_s$) "small"

→ Assume the initial date $u_0(x)$ is smooth,

→ $\boxed{\nexists \text{ singularity for } 0 < t \le t_s}$

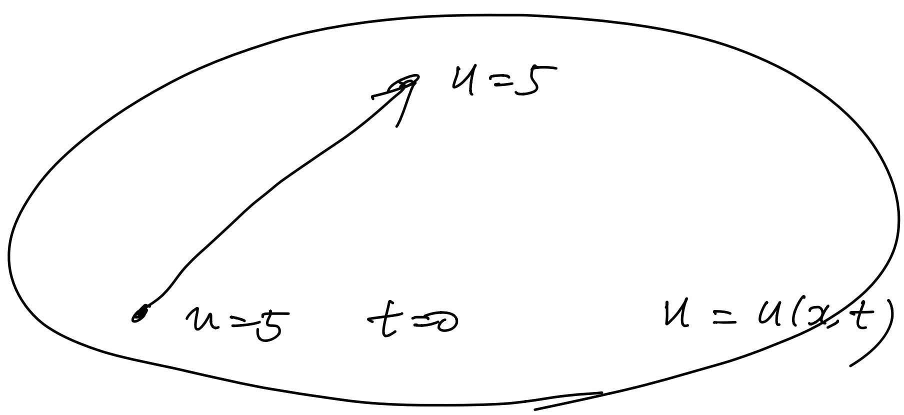
$u_t + a u_x = 0$

$$\begin{cases} x'(t) = a \\ x'(t) = u_0(x) \end{cases}$$

→ char. lines satisfy $= f'(u)$

$$(*) \begin{cases} \cancel{x'(t) = u(x,t)} \\ x(0) = u_0(x). \end{cases}$$

→ $\dfrac{du}{dt} = \ldots = 0$ ✓



pf $\dfrac{du}{dt} = \dfrac{\partial u}{\partial t} + \dfrac{\partial u}{\partial x}\dfrac{dx}{dt}$

$= \dfrac{\partial u}{\partial t} + \dfrac{\partial u}{\partial x} u$

$= 0$

$\therefore$ u remains constant along the char. lines.
