hw_5

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1. City migrations as matrix iterations: the transition matrix

(a)

The j^{th} column of T corresponds to the annual percentage of migration to that city from each city i. So if we wanted to know how many people from city i moved to city j after one year, we multiply the i^{th} column of $\vec{m}_t$ by the i^{th} row of the j^{th} column, then the total population in the j^{th} city is this summed over all i.

let T_j be the j^{th} column of T. Then

$$T_j \cdot \vec{m}_t = T_j^\top \vec{m}_t = m_{j,t+1}$$

Therefore the new population vector $\vec{m}_{t+1}$ is

$$\vec{m}_{t+1} = T^\top \vec{m}_t$$

(b)

let $\vec{v}$ be an eigenvector with eigen value λ

$$T\vec{v} = \lambda\vec{v}$$

lets consider one row of $\vec{v}$

$$p_{i1}v_1 + \cdots + p_{in}v_n = \lambda v_i$$

taking the $\|\cdot\|_\infty$ norm, and letting $|v_k| = \max_i |\vec{v}_i|$

$$\|p_{i1}v_1 + \cdots + p_{in}v_n\|_\infty = \|\lambda v_i\|_\infty p_{i1}|v_k| + \cdots + p_{in}|v_k| \geq |\lambda||v_k| \left(p_{i1} + \cdots + p_{in}\right)|v_k| \geq |\lambda||v_k|$$

since $\sum_j p_{i,j} = 1$

$$|\lambda| \leq 1$$

(c)

Let $\mathbf{1}$ be a vector with all ones, and since $\sum_j p_{i,j} = 1$

$$T\mathbf{1} = \mathbf{1}$$

Then normalizing $\vec{v} = \frac{1}{\sqrt{n}}\mathbf{1}$ yields an eigenvector with eigenvalue 1 such that $\|\vec{v}\|_2 = 1$

(d)

One component of $T^\top \vec{m}$

$$p_{1j}m_{1,t} + \cdots + p_{nj}m_{n,t} = m_{j,t+1}$$

Then taking $\| \cdot \|_1$ is the sum over j of the absolute values of the components of $m_{j,t+1}$

$$\sum_j (p_{1j} m_{1,t} + \cdots + p_{nj} m_{n,t}) = \sum_j m_{j,t+1}$$

LHS

$$p_{11}|m_{1,t}| + \cdots + p_{n1}|m_{n,t}| + p_{12}|m_{1,t}| + \cdots + p_{n2}|m_{n,t}| + \cdots + + p_{1n}|m_{1,t}| + \cdots + p_{nn}|m_{n,t}|$$

Grouping components

$$(p_{11} + \cdots + p_{1n})|m_{1,t}| + \cdots + (p_{nn} + \cdots + p_{nn})|m_{n,t}| = \sum_j |m_{j,t}|$$

Finally

$$\sum_j |m_{j,t}| = \sum_j |m_{j,t+1}|$$

proving that the total population remains constant

(e)

assume $\vec{v}$ be an eigenvector with eigenvalue 1, then

$$(I - T)\vec{v} = v - Tv = v - v = 0 \implies v \in N(I - T)$$

Now assume $v \in N(I - T)$, then

$$(I - T)v = 0v - Tv = 0Tv = v$$

Implying v is an eigenvector with eigenvalue 1

Proving finding all eigenvectors with eigenvalue 1 is equivalent to finding the nullspace of $I - T$

(f)

Show $Tu = u$**, where** $u = v - v_i 1$

$$Tu = T(v - v_i 1) = Tv - v_i T1 = v - v_i 1$$

Therefore u is an eigenvector with eigevalue 1

Show $u \geq 0$

All components u_k of u

$$u_k = v_k - v_i \geq 0$$

Since $i = \arg\min_j v_j$ Therefore $u \geq 0$

Show $u_i = 0$

i^{th} component of u

$$u_i = v_i - (v_i 1)_i = v_i - v_i = 0$$

Therefore $u_i = 0$

(g)

Follows from part (f)