1. Convexity of Sets and Functions

(a) Determine if the set is convex or not

(i) The empty set $\emptyset$

This set is convex since any line segment between any two elements in the empty set is also in the empty set.

(ii) The singleton set $\{\mathbf{x_0}\}$

This set is convex since any line segment between any two elements in the singleton set is also in the singlton set.

(iii) $\mathbb{R}^\mathbf{n}$

This set is convex since any line segment between any two elements in $\mathbb{R}^n$ is in $\mathbb{R}^n$. $\mathbb{R}$ is a vector space and hence closed under any linear combination, namely the line segment, $\lambda x + (1-\lambda)y$

(iv) $\mathcal{P} = \{(x_1, x_2)$, where x_1 is the date (1-31) and x_2 is the month (1-12) corresponding to the date of birth of EECS 127/227A students$\}$

if $(x_1, y_1), (x_2, y_2) \in \mathcal{P}$ then $\lambda(x_1, y_1) + (1-\lambda)(x_2, y_x) \notin \mathcal{P}$
Since $\lambda \in (0,1)$ $\lambda(x_1, y_1) = (\lambda x_1, \lambda x_2)$ are no longer integers between $1-31$

(v) $\mathcal{P} = \{z : z = (1-t)\vec{a} + t\vec{b}$, where a and b are two vectors with same dimension as z and $t \in [0,1]$}

This set is convex since the set is made up of all line segments, and a line segment between two line segments is still a line segment

(vi) $\mathcal{P} = \{z : \vec{a}^\top z = b, a \neq 0\}$

This set is convex since a hyperplane is convex.

(vii) $\mathcal{P} = \{z : \vec{a}^\top z \leq b, a \neq 0\}$

This set is convex since a half space is convex.

(viii) $\mathcal{P} = \{z : \|z - \overrightarrow{z_0}\|_2 = \epsilon$}

This set is **not** convex since it is a hypercirle of radius ϵ, which does not contain its interior. So any line segment between points in the set, on the surface of the circle, would not be in the set.

(xi) $\mathcal{P} = \{z : \|z - \overrightarrow{z_0}\|_2 \leq \epsilon$}

This set is convex since it is a circle that *does* contain its interior, and circle is convex.

(x) $\mathcal{P}$ = epigraph(f) = $\{(z,t) : z \in domain(f)$ and $t \geq f(z)\}$, where f is a convex function

Since f is convex, $domain(f)$ is convex and

$$f(\lambda x + (1-\lambda)y) \leq \lambda f(x) + (1-\lambda)f(y) \in epigraph(f)$$

Which can be seen by letting $t = \lambda f(x) + (1-\lambda)f(y)$ and $z = \lambda x + (1-\lambda)y$, then

$$(z, t) \in epigraph(f)$$

Since $t \geq f(z)$, and $z \in domain(f)$

So this set is convex

(xi) $\mathcal{P} = Q \bigcap R$, where Q and R are convex sets

If $x, y \in Q \bigcap R, \text{ then } x, y \in Q, \text{ and } x, y \in R \text{ By definition}$

Since both Q and R are convex then,

$$\lambda x + (1-\lambda)y \in Q \text{ and } \lambda x + (1-\lambda)y \in R$$
$$\implies \lambda x + (1-\lambda)y \in Q \bigcap R$$

Therefore this set is convex

(xii) $\mathcal{P}$ = Minkowski sum of sets Q and R, where Q and R are convex sets, where Minkowski sum of two sets Q and R is defined as $Q + R = \{q + r \,|\, q \in Q,\ r \in R\}$.

Let $q_1 + r_1,\ q_2 + r_2 \in Q + R$, then

$$\lambda(q_1 + r_1) + (1-\lambda)(q_2 + r_2)$$
$$= \lambda q_1 + (1-\lambda)q_2 + \lambda r_1 + (1-\lambda)r_2 \in Q + R$$

Since Q and R are convex

Therefore this set is convex

(xiii) $\mathcal{P} = \{(x_1, x_2) : (x_1 \geq x_2 - 1 \ \text{ and } \ x_2 \geq 0) \ \text{ OR } \ (x_1 \leq x_2 - 1 \ \text{ and } \ x_2 \leq 0)\}$

let $x_2 = x_1 + 1 \leq 0$

(b) Determine if the function is convex or not

(i) $f(x) = a^\top \vec{x} + b$

if $a,\ x \in \mathbb{R}^n$, then $domain(f) = \mathbb{R}^n$, which is a convex set, and

$$f(\lambda x + (1-\lambda)y) = a^\top(\lambda x + (1-\lambda)y) + b$$
$$\lambda f(x) + (1-\lambda)f(y) = \lambda(a^\top x + b) + (1-\lambda)(a^\top y + b)$$
$$= a^\top(\lambda x + (1-\lambda)y) + b$$

so $f(\lambda x + (1-\lambda)y) = \lambda f(x) + (1-\lambda)f(y)$, and so $f(x)$ is a convex function

(ii) $f(x) = \cos(x), x \in [\frac{\pi}{2}, \frac{3\pi}{2}]$

This function is convex by observation

(iii) $f(\vec{x}) = \vec{x}^\top Q\vec{x} + a^\top \vec{x} + b$**, where** Q **is Positive Semi-definite**

From part (a) we know $g(x) := a^\top \vec{x} + b$ is convex, and $g(\lambda x + (1-\lambda)y) = \lambda g(x) + (1-\lambda)g(y)$ so we only need to check the conditions on $f(x) := \vec{x}^\top Q\vec{x}$

$$\begin{aligned} f(\lambda x + (1-\lambda)y) &= (\lambda x^\top + (1-\lambda)y^\top)Q(\lambda x + (1-\lambda)y) \\ &= \lambda^2 x^\top Qx + (1-\lambda)^2 y^\top Qy^\top \\ &\leq \lambda f(x) + (1-\lambda)f(y) \\ &= \lambda x^\top Qx + (1-\lambda)y^\top Qy^\top \end{aligned}$$

Since $x^\top Qx \geq 0 \; \forall x$, $\lambda < \lambda^2$, and $(1-\lambda)^2 < 1-\lambda$

Therefore $f(\vec{x}) = \vec{x}^\top Q\vec{x} + a^\top \vec{x} + b$ is a convex function

(iv) $\sum_i w_i f_i(x), w_i \in \mathbb{R}$ **and each** f_i **convex**

For each f_i we know

$$\begin{aligned} f_i(\lambda x + (1-\lambda)y) \leq \lambda f_i(x) + (1-\lambda)f_i(y) \\ \implies -w_i\left(f_i(\lambda x + (1-\lambda)y)\right) \geq -w_i\left(\lambda f_i(x) + (1-\lambda)f_i(y)\right) \end{aligned}$$

So if all $w_i < 0$, then

$$\sum_i w_i f_i(\lambda x + (1-\lambda)y) \geq \sum_i w_i\left(\lambda f_i(x) + (1-\lambda)f_i(y)\right)$$

Therefore this function is **not** convex

(v) $f(A\vec{x} + b)$**, where** f **is a convex function**

Since f is convex it is sufficient to show that the set $\{x : g(x) = Ax + b \in domain(f)\}$ is a convex set.

$$\begin{aligned} g(\lambda x + (1-\lambda)y) &= A(\lambda x + (1-\lambda)y) + b \\ &= \lambda Ax + (1-\lambda)Ay + b \\ \lambda g(x) + (1-\lambda)g(y) &= \lambda Ax + (1-\lambda)Ay + b \end{aligned}$$

Since this set is convex, then the function $f(A\vec{x} + b)$ is convex

(vi) $f(x) = \max_i f_i(x)$, where f_i are convex functions

let f_k be the max of $\lambda x + (1-\lambda)y$, then

$$\begin{aligned} f(\lambda x + (1-\lambda)y) &= f_k(\lambda x + (1-\lambda)y) \\ &\leq \lambda f_k(x) + (1-\lambda) f_k(y) \end{aligned}$$

(vi) $g(f(x))$, where f(x) is convex and g(x) is convex and non-decreasing

Since g is non-decreasing, and

$$f(\lambda + (1-\lambda y)) \leq \lambda f(x) + (1-\lambda) f(y)$$

then,

$$g\left(f(\lambda + (1-\lambda y))\right) \leq g\left(\lambda f(x) + (1-\lambda) f(y)\right)$$

and since g is convex,

$$g\left(\lambda f(x) + (1-\lambda) f(y)\right) \leq \lambda g(f(x)) + (1-\lambda) g(f(y))$$

Therefore,

$$g\left(f(\lambda + (1-\lambda y))\right) \leq g(f(x)) + (1-\lambda) g(f(y))$$

Implying $g(f(x))$ is convex

(vii) $f(x)$ a function whose epigraph is a convex set

Let $(z_1, t_1),\ (z_2, t_t) \in epi(f)$, then $\lambda(z_1, t_1) + (1-\lambda)(z_2, t_2) \in epi(f)$, and

$$\lambda(z_1, f(z_1)) + (1-\lambda)(z_2, f(z_2)) \in epi(f)$$

Then

$$f(\lambda z_1 + (1-\lambda) z_2) \leq \lambda f(z_1) + (1-\lambda) f(z_2)$$

since $\lambda z_1 + (1-\lambda) z_2 \in dom(f)$, and $\lambda f(z_1) + (1-\lambda) f(z_2)$ by definition of being in the $epi(f)$

(c) For all the sets given in the following question, find the convex hull of the set.

(i)

Draw a line between each point and fill it in

(ii)

Draw a tangent line to the max and min of $sin(x)$ and fill it in

(iii)

This is a circle of radius ϵ centered at z_0, so filling it in results in the convex hull.

(d) Laptop wires

Bayen's, and Theo's laptop charger wires are convex since they form a catenary.

Oladapo's is convex, since the epigraph is a convex set.

Sukrit's, and Hari's are not since you can draw a line between two points that are not contained in the epigraph.

2. PSD equivalence: the Schur complement

(a)

Expanding $f(x, y)$

$$\begin{aligned} f(x,y) &= \begin{pmatrix} x \\ y \end{pmatrix}^\top \begin{pmatrix} A & B \\ B^\top & C \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} \\ &= x^\top A x + x^\top B y + y^\top B^\top x + y^\top C y \end{aligned}$$

Taking the gradient $\nabla_x f(x, y)$ and set it equal to zero to find x that minimizes $\min_x f(x, y)$

$$\begin{aligned} \nabla_x f(x,y) &= (A + A^\top)x + By + By = 0 \\ \implies 2Ax &= -2By \\ \implies x &= -A^{-1}By \end{aligned}$$

Pluggin x back into $f(x, y)$

$$\begin{aligned} \min_x f(x,y) = f(-A^{-1}By, y) &= \begin{pmatrix} -y^\top B^\top A^{-1} \\ y^\top \end{pmatrix} \begin{pmatrix} A & B \\ B^\top & C \end{pmatrix} \begin{pmatrix} -A^{-1}By \\ y \end{pmatrix} \\ &= \begin{pmatrix} -y^\top B^\top A^{-1} \\ y^\top \end{pmatrix} \begin{pmatrix} 0 \\ -B^\top A^{-1}By + Cy \end{pmatrix} \\ &= y^\top \left(C - B^\top A^{-1} B\right) y \end{aligned}$$

Therefore $\min_x f(x, y) = y^\top \left(C - B^\top A^{-1} B\right) y$

(b)

Using the hint to minimize the result from part (a) over y
(i.e. $\min\limits_{x,\,y} f(x, y)$)

Let $g(y) = y^\top \left(C - B^\top A^{-1} B\right) y$,
then taking the gradient with respect to y, and setting it equal to zero

$$\begin{aligned} \nabla_y g(y) &= (C + C^\top)y - 2B^\top A^{-1} y = 0 \\ &= \left(C + C^\top - 2B^\top A^{-1}\right) y = 0 \\ \implies y = 0 \end{aligned}$$

Therefore, $\min\limits_{x,\,y} f(x, y) = g(0) = 0$, and since this is the minimum, it implies that that $f(x, y) \geq 0$.

Proving that $M \iff C - B^\top A^{-1} B$ is PSD

(c)

noting the symmetry, $A - BC^{-1}B^\top$, follows from minimizing the following,

$$\begin{aligned} \min_x f^\top(x, y) &= \min_x \begin{pmatrix} x \\ y \end{pmatrix}^\top \begin{pmatrix} C & B^\top \\ B & A \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} \dots \\ &= y^\top (A - BC^{-1}B^\top) y \end{aligned}$$

Then, a similar analysis as in part (b) proves the result.

3. Catenary

(a)

Since the string is static the tension at x and $x + \Delta x$ should be equal and opposite, so the horizontal component is given by:

$$\begin{aligned} -T(x)cos(\theta(x)) + T(x + \Delta x) cos\left(\theta(x + \Delta(x)\right) = 0 \\ \implies T(x)cos(\theta(x)) = T(x + \Delta x) cos\left(\theta(x + \Delta(x)\right) \end{aligned}$$

Since the tension doesn't change from one segment to the next, then there must exist some T_0 such that

$$T(x)cos(\theta(x)) = T_0$$

(b)

The vertical force is given by:

$$-T(x)sin(\theta(x)) + T(x+\Delta x)sin\left(\theta(x+\Delta(x)\right) + P(x,\Delta x) = 0$$

$P(x, \Delta x)$ is given by $\mu g \Delta s(x, \Delta x)$, where μ is the linear density of the string (i.e. mass per unit length), and $\Delta s(x, \Delta x)$ is the length of the string between x and Δx.

Then $\mu g \Delta s(x, \Delta x)$ gives the familiar formula for the force from gravity on a object.

Therefore we can write the vertical force of a line segment by:

$$-T(x)sin(\theta(x)) + T(x+\Delta x)sin\left(\theta(x+\Delta(x)\right) = \mu g \Delta s(x,\Delta x)$$

Using the convention that the force of gravity is negative (for down).

(c)

$$\frac{-T(x)sin(\theta(x)) + T(x+\Delta x)sin\left(\theta(x+\Delta(x)\right) = \mu g \Delta s(x,\Delta x)}{T(x)cos(\theta(x)) = T(x+\Delta x)cos\left(\theta(x+\Delta(x)\right) = T_0}$$

$$= -tan(\theta(x)) + tan(\theta(x+\Delta x)) = \frac{\mu g \Delta s(x,\Delta x)}{T_0}$$

(d)

From definition of $tan(\theta)$ it follows that:

$$tan(\theta(x)) = \frac{\Delta z(x,\Delta x)}{\Delta x}$$

$$tan(\theta(x+\Delta x)) = \frac{\Delta z(x+\Delta x,\Delta x)}{\Delta x}$$

Therefore, from part (c), it follows that

$$-\frac{\Delta z(x,\Delta x)}{\Delta x} + \frac{\Delta z(x+\Delta x,\Delta x)}{\Delta x} = \frac{\mu g}{T_0}\Delta s(x,\Delta x)$$

(e)

By using the standard limit definition of the the derivative, then the second derivative can be expressed as:

$$f''(x) = \lim_{h\to\infty}\frac{f(x+2h-2f(x+h)+f(x)}{h^2}$$

Then expanding part (d), using the definition of $\Delta z(x, \Delta x)$, we get

$$\frac{z(x+2\Delta)-2z(x+\Delta x)+z(x)}{\Delta x} = \frac{\mu g}{T_0}\Delta s(x,\Delta x)$$

Then multiplying both sides by $\frac{1}{\Delta x}$ and taking the limit as Δx tends to 0, we get the result

$$\frac{\partial^2 z}{\partial x^2}(x) = \lim_{\Delta x\to 0}\frac{\mu g}{T_0}\frac{\Delta s(x,\Delta x)}{\Delta x}$$

(f)

Since μ, g, T_0, and $\Delta s(x, \Delta)$ are all greater than 0, then the function $z(x)$ is convex by the second derivative test

$$\frac{\partial^2 z}{\partial x^2}(x) = \lim_{\Delta x \to 0} \frac{\mu g}{T_0} \frac{\Delta s(x, \Delta x)}{\Delta x} \geq 0$$

Noting this is true even as Δx tends to infinity

(g)

From the definition of $\Delta s(x, \Delta x)$

$$\begin{aligned}\frac{\Delta s(x, \Delta x)}{\Delta x} &= \frac{\sqrt{(\Delta x)^2 + (\Delta z(x, \Delta x))^2}}{\Delta x} \\ &= \sqrt{\frac{(\Delta x)^2}{(\Delta x)^2} + \frac{(\Delta z(x, \Delta x))^2}{(\Delta x)^2}} \\ &= \sqrt{1 + \left(\frac{\Delta z(x, \Delta x)}{\Delta x}\right)^2}\end{aligned}$$

(h)

Using the result from parts (e) and (g), we have

$$\frac{\partial^2 z}{\partial x^2}(x) = \lim_{\Delta x \to 0} \frac{\mu g}{T_0} \sqrt{1 + \left(\frac{\Delta z(x, \Delta x)}{\Delta x}\right)^2}$$

Then applying $\frac{\Delta z(x, \Delta x)}{\Delta x} \sim 0$,

$$\frac{\partial^2 z}{\partial x^2}(x) = \frac{\mu g}{T_0}$$

Integrating once we have (where b is the constant of integration)

$$\frac{\partial z}{\partial x}(x) = \frac{\mu g}{T_0} x + b$$

Integrating again (c is the constant of integration here)

$$z(x) = \frac{\mu g}{2T_0} x^2 + bx + c$$

and $a = \frac{\mu g}{2T_0}$

(i)

First noting,

$$\frac{\Delta z(x, \Delta x)}{\Delta x} = \frac{z(x + \Delta x) - z(x)}{\Delta x}$$
$$\implies \lim_{h \to \infty} \frac{\Delta z(x, \Delta x)}{\Delta x} = \frac{\partial z}{\partial x}(x)$$

Then the result follows from (e) and (g),

$$\frac{\partial^2 z}{\partial x^2}(x) = \lim_{\Delta x \to 0} \frac{\mu g}{T_0} \sqrt{1 + \left(\frac{\Delta z(x, \Delta x)}{\Delta x}\right)^2}$$
$$= \frac{\mu g}{T_0} \sqrt{1 + \left(\frac{\partial z}{\partial x}(x)\right)^2}$$

(j)

$z = d \cosh\left(\frac{x}{d}\right)$

$$\frac{\partial z}{\partial x}(x) = \sinh\left(\frac{x}{d}\right)$$
$$\frac{\partial^2 z}{\partial x^2}(x) = \frac{1}{d} \cosh\left(\frac{x}{d}\right)$$

Then using part (i) and the hyperbolic trig identity $\cosh^2 x - \sinh^2 x = 1$

$$\frac{\partial^2 z}{\partial x^2}(x) = \frac{\mu g}{T_0} \sqrt{1 + \left(\frac{\partial z}{\partial x}(x)\right)^2}$$
$$\implies \frac{1}{d} \cosh\left(\frac{x}{d}\right) = \frac{\mu g}{T_0} \cosh\left(\frac{x}{d}\right)$$

The equation is satisfied if

$$d = \frac{T_0}{\mu g} = \frac{1}{2a}$$

4. Solving a simple optimization problem using cvxpy

In the notebook