EECS 127/227AT Optimization Models in Engineering
Fall 2019 Homework 9

Release date: 10/31/19.
Due date: 11/06/19, 23:00 (11 pm). Please LaTeX or handwrite your homework solution and submit an electronic version.

Submission Format
Your homework submission should consist of a single PDF file that contains all of your answers (any handwritten answers should be scanned) as well as your IPython notebook saved as a PDF.

If you do not attach a PDF "printout" of your IPython notebook, you will not receive credit for problems that involve coding. Make sure that your results and your plots are visible. Assign the IPython printout to the correct problem(s) on Gradescope.

1. Formulating problems as LP/QP/QCQP/SOCP

Formulate the problem in the parts given below as LP/QP/QCQP/SOCP, or, if you cannot, explain why. In our formulations, unless explicitly stated, $x \in \mathbb{R}^n$, $A \in \mathbb{R}^{m,n}$, $y \in \mathbb{R}^m$. If you obtain an LP/QP/QCQP/SOCP formulation, make sure to put the problem in standard form, stating precisely what the variables, objective, and constraints are.

[Optimization Problem Formulation or Few Lines of Justification]

(a) $p_1^* \doteq \min_x \ f(x)$, where $f(x) = \|Ax - y\|_\infty + \|x\|_1$

(b) $p_2^* \doteq \min_x \ f(x)$, where $f(x) = \|Ax - y\|_2^2 + \|x\|_1$

(c) $p_3^* \doteq \min_x \ f(x)$, where $f(x) = \|Ax - y\|_2^2 - \|x\|_1$

(d) $p_4^* \doteq \min_x \ f(x)$, where $f(x) = \|Ax - y\|_2^2 + \|x\|_1^2$

(e) $p_5^* \doteq \min_x \ f(x)$, where $f(x) = \frac{x_1^2+x_2^2+x_3^2}{4(1-x_1)}$, where $x \in \mathrm{dom}(f) = \{x \in \mathbb{R}^3, x_1 < 1\}$
Hint: $4ab = (a+b)^2 - (a-b)^2$

(f) $p_6^* \doteq \max_x \ f(x)$, where $f(x) = \frac{b^\top x}{1-a^\top x}$, and $a, b \in \mathbb{R}^n$ are given, with $a, b \neq 0$ and $\|a\|_2 < 1$

2. Robust linear programming [Extra points]

In this problem we will consider a version of linear programming under uncertainty.

(a) [Short justification, 1-2 lines] Let $x \in \mathbb{R}^n$ be a given vector. Prove that $x^T y \leq \|x\|_1$ for all y such that $||y||_\infty \leq 1$. Is this inequality tight? (Is there always a y such that the equality holds ?)

Let us focus now on a LP in standard form:

$$\begin{aligned} &\min_x c^T x \\ \text{s.t.} \;\; & a_i^T x \leq b_i, \;\; i = 1, ..., m \end{aligned} \tag{1}$$

Consider the set of linear inequalities in (1). Suppose you don't know the coefficients a_i exactly. Instead you are given nominal values $\overline{a}_i$, and you know that the actual coefficient vectors satisfy $\|a_i - \overline{a}_i\|_\infty \leq \rho$ for a given $\rho > 0$. In other words, the actual coefficients a_{ij} can be anywhere in the intervals $[\overline{a}_{ij} - \rho, \overline{a}_{ij} + \rho]$. or equivalently, each vector a_i can lie anywhere in a rectangle with corners $\overline{a}_i + v$ where $v \in \{-\rho, \rho\}^n$. The set of inequalities that constrain problem 1 must be satisfied for all possible values of a_i; i.e., we replace these with the constraints

$$a_i^T x \leq b_i \;\; \forall a_i \in \{\overline{a}_i + v \;\mid \|v\|_\infty \leq \rho\} \;\; i = 1, ..., m. \tag{2}$$

A straightforward but very inefficient way to express this constraint is to enumerate the 2^n corners of the rectangle of possible values a_i and to require that

$$\overline{a}_i^T x + v^T x \leq b_i \;\; \forall v \in \{-\rho, \rho\}^n \;\; i = 1, ..., m.$$

(b) [Few lines of justification] Use the previous result to show that (2) is in fact equivalent to the much more compact set of nonlinear inequalities

$$\overline{a}_i^T x + \rho \|x\|_1 \leq b_i, \;\; i = 1, ..., m.$$

We now would like to formulate the uncertainty in the LP we introduced. We are therefore interested in situations where the coefficient vectors a_i are uncertain, but satisfy bounds $\|a_i - \overline{a}_i\|_\infty \leq \rho$ for given $\overline{a}_i$ and ρ. We want to minimize $c^T x$ subject to the constraint that the inequalities $a_i^T x \leq b_i$ are satisfied for *all* possible values of a_i.

We call this a *robust LP*:

$$\begin{aligned} &\min_x c^T x \\ \text{s.t.} \;\; & a_i^T x \leq b_i, \;\; \forall a_i \in \{\overline{a}_i + v \mid \|v\|_\infty \leq \rho\} \;\;\; i = 1, ..., m. \end{aligned} \tag{3}$$

(c) [Few lines of justification] Using the result from part (b), express the above optimization problem as an LP.

3. A Portfolio Design Problem The returns on $n = 4$ assets are described by a Gaussian (normal) random vector $r \in \mathbb{R}^n$, having the following expected value $\hat{r}$ and covariance matrix Σ:

$$\hat{r} = \begin{bmatrix} 0.12 \\ 0.10 \\ 0.07 \\ 0.03 \end{bmatrix}, \ \Sigma = \begin{bmatrix} 0.0064 & 0.0008 & -0.0011 & 0 \\ 0.0008 & 0.0025 & 0 & 0 \\ -0.0011 & 0 & 0.0004 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

The last (fourth) asset corresponds to a risk-free investment. An investor wants to design a portfolio mix with weights $x \in \mathbb{R}^n$ (each weight x_i is non-negative, and the sum of the weights is one) so as to obtain the best possible expected return $\hat{r}^T x$, while guaranteeing that:

(i) No single asset weighs more than 40%.
(ii) The risk-free assets should not weigh more than 20%.
(iii) No asset should weigh less than 5%.
(iv) The probability of experiencing a return lower than $q = -1\%$ should be no larger than $\epsilon = 10^{-4}$.

(a) [Short justification, 1-2 lines] For a scalar standard normal $y \sim \mathcal{N}(0, 1)$, show that the constraint

$$\mathbf{Pr}(y \leq q) \leq \epsilon$$

can be written as

$$q \leq \Phi^{-1}(\epsilon),$$

where $\Phi(y)$ is the CDF of y, and Φ^{-1} is the inverse CDF. In particular, justify taking any inverse across inequality.

(b) [Short justification, 1-2 lines] Given a multivariable Gaussian $r \sim \mathcal{N}(\hat{r}, \Sigma), r \in \mathcal{R}^n$, its projection onto any $x \in \mathcal{R}^n$ is also a scalar Gaussian. That is,

$$r^T x \sim \mathcal{N}(\hat{r}^T x, x^T \Sigma x).$$

Show that the fourth constraint, also known as the chance constraint, can be written as

$$\Phi^{-1}(10^{-4}) ||\Sigma^{1/2} x||_2 \geq -\hat{r}^T x - 0.01.$$

Note that $\Phi^{-1}(10^{-4})$ is negative so this constraint can be converted to one in standard SOCP format as

$$||\Sigma^{1/2} x||_2 \leq \frac{1}{\Phi^{-1}(10^{-4})}(-\hat{r}^T x - 0.01).$$

Hint: Write $r^\top x$ as a standard Normal.

(c) [Optimization Problem Formulation] Formulate the portfolio optimization problem as a SOCP. Solve the problem using CVXPY in the Jupyter notebook. What is the maximal achievable expected return, under the above constraints?

(d) Solve the problem for a large number of values of ϵ between 10^{-6} and 10^{-1} and plot the optimal values of $\hat{r}^T x$ versus ϵ. Also make an area plot of the optimal portfolios x versus ϵ. What do you observe as the risk tolerance ϵ decreases? Read here for what an area plot is.

(e) *Monte Carlo simulation.* Let x be the optimal portfolio found in part 1, with $\epsilon = 10^{-4}$. This portfolio maximizes the expected return, subject to the probability of a loss being no more than 1 %. Generate 10000 samples of r, and plot a histogram of the returns. Find the empirical mean of the return, and calculate the percentage of samples for which a loss occurs. For more on Monte Carlo, see here.

4. Variation of step-size with Gradient Descent vs Newton Raphson Method

Run and answer questions in the jupyter-notebook "Gradient_vs_Newton.ipynb" which demonstrates the convergence of Gradient Descent and Newton-Raphson method with variation in step-size.