9.4 The Reynolds decomposition

So far, we have focussed on homogeneous isotropic turbulence only, without any mean flow. However, while these assumptions have allowed us to make progress deriving exact results, they are overly restrictive in terms of modeling real turbulent flows. For turbulence that is not necessarily homogeneous and isotropic (but still unstratified and non-rotating) many of the formal results derived above are invalid, though the concept of a turbulence energy cascade from large to small scales persists.

Another way of studying this more general type of turbulence is through the Reynolds decomposition. The idea behind Reynolds decomposition draws heavily from the statistical approach to turbulence discussed in previous lectures, but can be extended to account for anisotropy and inhomogeneity. Suppose that it is possible to decompose each of the dynamical quantities ($\boldsymbol{u}$, p, and other scalar fields) as

$$q(\boldsymbol{x},t) = \bar{q}(\boldsymbol{x},t) + q'(\boldsymbol{x},t) \tag{9.1}$$

The overbar denotes some averaging process; this could be for instance the statistical average used in previous sections (i.e. a statistical average over many different realizations of exactly the same experiment), or a global spatial average (e.g. horizontal average), or some local filtering process in time or space or both (i.e. a low pass filter). By definition, $\bar{q}' \equiv 0$. For the steps taken below to work, this averaging process needs to commute with time and space derivatives of q, so that

$$\overline{\nabla q} = \nabla \bar{q} \text{ and } \overline{\partial q/\partial t} = \partial \bar{q}/\partial t \tag{9.2}$$

This condition is difficult to meet in practice, and is only strictly realized in a few idealized cases. If, for instance, the averaging process is a global spatial average (such as a horizontal average in over a fixed domain D), then this is true. It is also true of the statistical average. Finally, this is true if the averaging process is a low-pass filtering, and the flow exhibits a true scale separation in both time and space (so that the large scales are much larger than the small scales, and there is very little energy at intermediate scales).

9.4.1 Mean flow equation

Let's consider the most basic form of the Boussinesq equations, in the absence of stratification, and taking $\rho_m = 1$ as before:

$$\frac{\partial \boldsymbol{u}}{\partial t} + \boldsymbol{u}\cdot\nabla\boldsymbol{u} = -\nabla p + \nu\nabla^2\boldsymbol{u} \tag{9.3}$$

with $\nabla\cdot\boldsymbol{u} = 0$. Using the decomposition above, we have

$$\frac{\partial \bar{\boldsymbol{u}}}{\partial t} + \frac{\partial \boldsymbol{u}'}{\partial t} + (\bar{\boldsymbol{u}} + \boldsymbol{u}')\cdot\nabla(\bar{\boldsymbol{u}} + \boldsymbol{u}') = -\nabla(\bar{p} + p') + \nu\nabla^2(\bar{\boldsymbol{u}} + \boldsymbol{u}') \tag{9.4}$$

We then perform the averaging process on that equation, recalling that it commutes with differential operators, and that the average of a single primed quantity is 0, to get

$$\frac{\partial \bar{\boldsymbol{u}}}{\partial t} + \bar{\boldsymbol{u}} \cdot \nabla \bar{\boldsymbol{u}} + \overline{\boldsymbol{u}' \cdot \nabla \boldsymbol{u}'} = -\nabla \bar{p} + \nu \nabla^2 \bar{\boldsymbol{u}} \tag{9.5}$$

where we have used the fact that $\bar{\bar{q}} = \bar{q}$. This is the mean equation, which describes the evolution of the mean flow $\bar{\boldsymbol{u}}$. We see that it mostly involves mean quantities, *except* for the crucial term in $\overline{\boldsymbol{u}' \cdot \nabla \boldsymbol{u}'}$. Note that the incompressibility condition simply becomes $\nabla \cdot \bar{\boldsymbol{u}} = 0$, so that equation also only involves mean quantities.

Subtracting $\nabla \cdot \bar{\boldsymbol{u}} = 0$ from $\nabla \cdot \boldsymbol{u} = 0$, we obtain

$$\nabla \cdot \boldsymbol{u}' = 0 \tag{9.6}$$

so in this decomposition, the perturbations are also incompressible. Using this, we can write the i-th component of $\overline{\boldsymbol{u}' \cdot \nabla \boldsymbol{u}'}$ as

$$\overline{\boldsymbol{u}' \cdot \nabla u_i'} = \overline{\nabla \cdot (\boldsymbol{u}' u_i')} \equiv \partial_j \left(\overline{u_i' u_j'} \right) \tag{9.7}$$

where we have again used Einstein's convention of implicit summation over repeated indices. The correlation between the perturbations is the *Reynolds stress tensor*[1]

$$R_{ij} = u_i' u_j' \tag{9.8}$$

which is a function of time and space. The tensor $R_{ij}(\boldsymbol{x}, t)$ is equal to $\Phi_{ij}(\boldsymbol{x}, 0, t)$ defined in Section 9.2, since the definition of Φ_{ij} assumed there is no mean flow, so all the velocities in that section were, by definition, perturbations). As was the case for Φ_{ij}, the tensor $R_{ij}(\boldsymbol{x}, t)$ is quadratic in the perturbations, and symmetric. The trace of R is related to the local kinetic energy of the perturbations E^{pert} namely

$$E^{pert} = \frac{1}{2} \left(u_x'^2 + u_y'^2 + u_z'^2 \right) = \frac{1}{2} Tr(R) = \frac{1}{2} R_{ii} \tag{9.9}$$

Substituting (8.8) into (8.5) we therefore have for the $i-$component of the averaged momentum equation, that

$$\frac{\partial \bar{u}_i}{\partial t} + \bar{u}_j \partial_j \bar{u}_i + \partial_j \bar{R}_{ij} = -\partial_i \bar{p} + \nu \partial_{jj} \bar{u}_i \tag{9.10}$$

This confirms and generalizes a result we already saw in Chapter 4 for wave - mean flow interaction, namely that the divergence of the Reynolds stress tensor can affect the mean flow.

[1] Note that there are several possible ways of defining the Reynolds stress; some definitions include the density, and this should be preferred for compressible flows. Other definitions use a negative sign instead.

9.4.2 Reynolds stress equation

Unfortunately, the mean flow equation cannot be solved without knowing what $\bar{R}_{ij}$ is, and that requires knowing how the perturbations evolve. Let us therefore construct an evolution equation for the perturbations, and then for $\bar{R}_{ij}$. The perturbation equation is obtained by subtracting the average equation from the total equation (8.4):

$$\frac{\partial \boldsymbol{u}'}{\partial t} + \boldsymbol{u}' \cdot \nabla \boldsymbol{u}' + \boldsymbol{u}' \cdot \nabla \bar{\boldsymbol{u}} + \bar{\boldsymbol{u}} \cdot \nabla \boldsymbol{u}' - \overline{\boldsymbol{u}' \cdot \nabla \boldsymbol{u}'} = -\nabla p' + \nu \nabla^2 \boldsymbol{u}' \tag{9.11}$$

which, in index notation, is

$$\frac{\partial u_i'}{\partial t} + u_k' \partial_k u_i' + u_k' \partial_k \bar{u}_i + \bar{u}_k \partial_k u_i' - \partial_k \bar{R}_{ik} = -\partial_i p' + \nu \partial_{kk} u_i' \tag{9.12}$$

A similar equation can be constructed for u_j', and using the product rule, we can then construct an equation for R_{ij}:

$$\begin{aligned} \frac{\partial R_{ij}}{\partial t} + u_k' \partial_k R_{ij} + \bar{u}_k \partial_k R_{ij} + R_{jk} \partial_k \bar{u}_i + R_{ik} \partial_k \bar{u}_j - u_j' \partial_k \bar{R}_{ik} - u_i' \partial_k \bar{R}_{jk} \\ = -u_j' \partial_i p' - u_i' \rho_m^{-1} \partial_j p' + \nu \left(u_j' \partial_{kk} u_i' + u_i' \partial_{kk} u_j' \right) \end{aligned} \tag{9.13}$$

Note that

$$u_j' \partial_{kk} u_i' + u_i' \partial_{kk} u_j' = \partial_k (u_j' \partial_k u_i' + u_i' \partial_k u_j') - 2 \partial_k u_j' \partial_k u_i' = \partial_{kk} R_{ij} - 2 \partial_k u_j' \partial_k u_i' \tag{9.14}$$

Finally, we average this equation, to obtain an evolution equation for $\bar{R}_{ij}$:

$$\begin{aligned} \frac{\partial \bar{R}_{ij}}{\partial t} + \overline{u_k' \partial_k R_{ij}} + \bar{u}_k \partial_k \bar{R}_{ij} + \bar{R}_{jk} \partial_k \bar{u}_i + \bar{R}_{ik} \partial_k \bar{u}_j \\ = -\rho_m^{-1} \overline{u_j' \partial_i p' + u_i' \partial_j p'} + \nu \partial_{kk} \bar{R}_{ij} - 2\nu \overline{\partial_k u_j' \partial_k u_i'} \end{aligned} \tag{9.15}$$

This equation contains a number of terms that are known, or can be solved for, namely all the terms that only contain $\bar{\boldsymbol{u}}$ and $\bar{R}$. However, three other kinds of terms emerge, that are problematic:

- the $\overline{u_k' \partial_k R_{ij}}$ term
- the $\overline{u_j' \partial_i p' + u_i' \partial_j p'}$ term
- the $\nu \overline{\partial_k u_j' \partial_k u_i'}$ term

Let's discuss them in turn. The term in $\overline{u_k' \partial_k R_{ij}}$ is called a triple-correlation term, because it effectively involves 3 perturbation terms (one in u_k', and two in R_{ij}). As a result, it is not possible to write this term *exactly* in terms of quadratic correlations or other mean terms, such as $\bar{R}_{ij}$ or $\bar{\boldsymbol{u}}$. The appearance of third-order correlations in the evolution equation for quadratic correlations should be familiar, since we already encountered it twice in this Chapter: once again we are hitting the problem of closure, i.e. it is not possible to create a system of so-called closed equations at this order (in fact, it is not possible to do it at any order). We will discuss closure ideas below.

Insight into the pressure term can be obtained by computing the divergence of the perturbation equation: using the fact that $\nabla \cdot \boldsymbol{u}' = 0$, we get

$$\partial_i(u'_k\partial_k u'_i + u'_k\partial_k\bar{u}_i + \bar{u}_k\partial_k u'_i) - \partial_i\partial_k\bar{R}_{ik} = -\nabla^2 p' \tag{9.16}$$

which can be (formally) solved as

$$p' = -\nabla^{-2}\left[\partial_i(u'_k\partial_k u'_i + u'_k\partial_k\bar{u}_i + \bar{u}_k\partial_k u'_i) - \partial_i\partial_k\bar{R}_{ik}\right] \tag{9.17}$$

Substituting this back into (8.15), we see that there will be terms containing further triple-correlations (i.e. terms containing 3 separate u's), terms containing quadratic correlations and one $\bar{u}$, and terms containing 1 u' and one $\bar{R}$. This last one will vanish under the averaging process, but the other two kinds remain. Again, we see that it is not possible to write this pressure term in closed form in terms of $\bar{R}$ and/or $\bar{\boldsymbol{u}}$ only.

The $\nu\overline{\partial_k u'_j\partial_k u'_i}$ term, finally, is clearly associated with dissipation. In fact, that trace of that term is

$$Tr(\nu\overline{\partial_k u'_j\partial_k u'_i}) = \nu\overline{\partial_k u'_i\partial_k u'_i} = \nu\overline{|\nabla\boldsymbol{u}'|^2} \tag{9.18}$$

which is the kinetic energy dissipation rate associated with the perturbations. But as before, it is not usually possible to write it exactly in terms of $\bar{R}$ and/or $\bar{\boldsymbol{u}}$ only.

9.4.3 Energetics

We now consider the energetics of the flow. The total kinetic energy equation derived from (8.3) in the usual way (i.e. dotting the momentum equation with $\boldsymbol{u}$) is

$$\frac{\partial E}{\partial t} + \partial_j(u_j E) = -\partial_j(u_j p) + \partial_j(\nu\partial_j E) - \nu(\partial_j u_i)(\partial_j u_i) \tag{9.19}$$

using index notation, where $E = \sum_i u_i^2/2$ as usual. As we showed earlier in this chapter, this can be written in conservative form as

$$\frac{\partial E}{\partial t} + \partial_j(u_j E + u_j p - \nu\partial_j E) = -\epsilon \tag{9.20}$$

where the only non-conservative term is the kinetic energy dissipation rate

$$\epsilon = \nu(\partial_j u_i)(\partial_j u_i) = \nu|\nabla\boldsymbol{u}|^2 \tag{9.21}$$

Let us now look at energetics in the mean and perturbation equations separately. We are going to define $E^{mean} = \sum_i \bar{u}_i^2/2$ as the kinetic energy in the mean flow, and $E^{pert} = \sum_i u_i'^2/2$ as the kinetic energy in the perturbations. Since

$$\boldsymbol{u}\cdot\boldsymbol{u} = (\bar{\boldsymbol{u}} + \boldsymbol{u}')\cdot(\bar{\boldsymbol{u}} + \boldsymbol{u}') = \bar{\boldsymbol{u}}\cdot\bar{\boldsymbol{u}} + 2\bar{\boldsymbol{u}}\cdot\boldsymbol{u}' + \boldsymbol{u}'\cdot\boldsymbol{u}' \tag{9.22}$$

we also have, averaging the total kinetic energy,

$$\bar{E} = E^{mean} + \bar{E}^{pert} \tag{9.23}$$

which has implications for their respective evolution equations.

By dotting the mean equation (8.5) with $\bar{\boldsymbol{u}}$, we get

$$\frac{\partial E^{mean}}{\partial t} + \partial_j(\bar{u}_j E^{mean} + \bar{u}_j \bar{p} - \nu \partial_j E^{mean}) + \bar{u}_i \partial_j \bar{R}_{ij} = -\epsilon^{mean} \tag{9.24}$$

where $\epsilon^{mean} = \nu(\partial_j \bar{u}_i)(\partial_j \bar{u}_i)$. This equation has almost exactly the same structure as the total kinetic energy equation, with E^{tot} replaced by E^{mean}, and the local energy dissipation rate of the total flow ϵ^{tot} replaced by the local energy dissipation rate of the mean flow ϵ^{mean}. A new term however has appeared, namely $\bar{u}_i \partial_j \bar{R}_{ij}$. We can rewrite it as

$$\bar{u}_i \partial_j \bar{R}_{ij} = \partial_j(\bar{u}_i \bar{R}_{ij}) - \bar{R}_{ij} \partial_j \bar{u}_i \tag{9.25}$$

which has two parts: another conservative term, as well as a new local term $-\bar{R}_{ij}\partial_j \bar{u}_i$. This term will cause a *local* increase or decrease of the energy in the mean flow, and since it was not there in the total energy equation, can only be interpreted as a *transfer* of energy from the mean flow to the perturbations (or vice versa, depending on its sign).

Finally, let's dot the perturbation equation (8.11) with $\boldsymbol{u}'$, to get

$$\frac{\partial E^{pert}}{\partial t} + \partial_j \left[(u'_j + \bar{u}_j) E^{pert}\right] + R_{ij}\partial_j \bar{u}_i - u'_i \partial_j \bar{R}_{ij} = -\partial_j(u'_j p') + \partial_j(\nu \partial_j E^{pert}) - \epsilon^{pert} \tag{9.26}$$

where $\epsilon^{pert} = \nu(\partial_j u'_i)(\partial_j u'_i)$. We can then take the average of this equation to get

$$\frac{\partial \bar{E}^{pert}}{\partial t} + \partial_j \left[\overline{u'_j E^{pert}} + \bar{u}_j \bar{E}^{pert} + \overline{u'_j p'} - \nu \partial_j \bar{E}^{pert}\right] + \bar{R}_{ij}\partial_j \bar{u}_i = -\bar{\epsilon}^{pert} \tag{9.27}$$

Again, we see that all the terms in this equation are in conservative form, except the averaged kinetic energy decay rate of the perturbations $\bar{\epsilon}^{pert}$, and the exchange term $+\bar{R}_{ij}\partial_j \bar{u}_i$ which, as suspected, has the opposite sign from what it was in the mean equation. In order words, energy lost by the mean flow is gained by the perturbations, and vice versa. It is key, however, to see that this term is proportional to the local shearing rate $\partial_j \bar{u}_i$. In other words, it is the local shear (which is inherently anisotropic) that causes this exchange of energy between mean flow and perturbations.

Finally, we also see another new term appear, namely $\partial_j \overline{u'_j E_K^{pert}}$. This term is conservative, and is not associated with a mean flow, so can only really be interpreted as a diffusive turbulent kinetic energy flux. More on this later.

9.5 Closure models

The idea of a *closure model* is to *approximate* the terms that cannot be written in closed form, with formulae that *are* in closed form at the order considered. Closure models can be used in any of the approaches discussed so far. In the statistical approach discussed in Section 9.2, models for the function K in terms of f can be constructed. In the Fourier approach discussed in Section 9.3, models for the transfer function $T(\boldsymbol{k}, \boldsymbol{k}')$ exist. See the textbooks for examples of such models.

In this section, we will focus on closure models in the Reynolds decomposition approach, as they can more generally be applied to non-homogeneous, non-isotropic turbulence (notably in the presence of shear). Different levels of closure exist. For instance, a *first-order* closure model uses the mean flow equation, and attempts to model the divergence of the Reynolds stress using only information from the mean flow. By contrast, a *second-order* closure model uses the mean flow equation and the Reynolds stress equation, and attempts to model the triple-correlation terms, pressure terms, and dissipation terms discussed in the previous section only using information from the mean flow and the Reynolds stress. There are very many different ways of creating first and second-order closure models, and these models usually contain unknown constants that must be fitted to the data. As such, scientists generally adopt the model that best fits the experimental data for a given problem. It is important to realize, however, that a model that performs well in some applications may not perform well in others. As such, the predictive power of a model must always be questioned until data becomes available to test it.

9.5.1 First order closure models

First order closure models focus on modeling the Reynolds stress divergence in the mean flow equation. The most commonly used first order closure is the *turbulent diffusivity* model, in which the off-diagonal components of the Reynolds stress is modeled as

$$R_{ij} = -\nu_t \left(\partial_i \bar{u}_j + \partial_j \bar{u}_i\right) \tag{9.28}$$

where ν_t depends on the properties of the turbulence and is often called the *eddy viscosity*. The diagonal components are usually ignored as they can be folded into a turbulent pressure term. The mean flow equation becomes

$$\frac{\partial \bar{\boldsymbol{u}}}{\partial t} + \bar{\boldsymbol{u}} \cdot \nabla \bar{\boldsymbol{u}} = -\nabla \bar{p} + \nu \nabla^2 \bar{\boldsymbol{u}} + \partial_j \left(\nu_t (\partial_i \bar{u}_j + \partial_j \bar{u}_i)\right) = -\nabla \bar{p} + \nabla^2 \bar{\boldsymbol{u}} + \partial_j \left(\nu_t \partial_j \bar{u}_i\right) \tag{9.29}$$

This expression is analogous to the one associated with the natural (microscopic) viscous stress, hence the name of the model.

The remaining question is *how to choose* ν_t*?*. Again, the choice should only depend on known mean quantities of the flow, and in this case the flow field $\bar{\boldsymbol{u}}$ is the only possibility. From a dimensional perspective, ν_t has the dimensions of a diffusivity, i.e. a length squared over a time, or a velocity times a length.

The α-viscosity model for ν_t used in accretion disks

A possible expression for ν_t may therefore be

$$\nu_t = aUL \tag{9.30}$$

where a is a constant of order unity, L is a characteristic lengthscale of the flow field $\bar{\boldsymbol{u}}$, and U is a characteristic velocity scale of $\bar{\boldsymbol{u}}$. This choice is very extensively used in the study of astrophysical accretion disks for example, and is called the α-viscosity model (because the constant a is called α in that community). Since the flow velocities are usually close to the sound speed in accretion disks, U is chosen to be equal to it, and the scale L is chosen to be the scaleheight (i.e. thickness) of the disk. The model is also referred to as the Shakura-Sunyaev model in that context.

Prandtl's mixing-length model

The eddy diffusivity model is also at the heart of Prandtl's mixing length model, which is most appropriate for shear flows. In that case, ν_t is chosen to be

$$\nu_t = aL^2 S \tag{9.31}$$

where L is a characteristic lengthscale of the system, and S is the local shearing rate (which has units of one over time).

In fact, this model can be used to solve for the mean flow profile of a turbulent flow past a wall (or between two walls). In that case, L is taken to be the distance to the nearest wall. Consider for example a wall that is located at $y = 0$, and a flow past it that has mean velocity in the x-direction (along the wall). In this example, the averaging process is taken to be a horizontal average, so that the mean quantities are horizontally invariant and only depend on the distance to the wall, y. The wall is assumed to have no-slip boundary conditions. The steady-state mean flow equation with Prandtl's model is

$$\frac{\partial}{\partial y}\left[(\nu_t + \nu)\frac{\partial \bar{u}_x}{\partial y}\right] = 0 \tag{9.32}$$

which implies

$$(\nu_t + \nu)\frac{\partial \bar{u}_x}{\partial y} = C \text{ with } \nu_t = ay^2\left|\frac{\partial \bar{u}_x}{\partial y}\right| \tag{9.33}$$

where C is an integration constant. Assuming that the mean flow is in the positive x direction, the shear is positive and we therefore have the equation

$$\left(ay^2\frac{\partial \bar{u}_x}{\partial y} + \nu\right)\frac{\partial \bar{u}_x}{\partial y} = C \tag{9.34}$$

Far from the wall, ν is negligible compared with ν_t, and this takes the approximate form

$$ay^2\left(\frac{\partial \bar{u}_x}{\partial y}\right)^2 = C \tag{9.35}$$

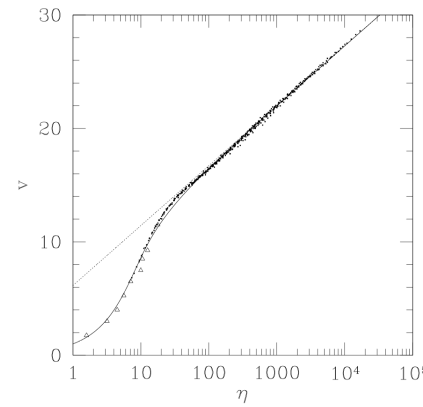

Figure 9.1: Experimental data from Zagarola and Smits 1998, compared with the dimensionless law $v = \ln(\eta)/0.436$ (dotted line), where η is a dimensionless distance to the wall. The solid line corresponds to a second-order closure model of Ogilvie discussed in the following sections.

which has the solution

$$\bar{u}_x = \sqrt{\frac{C}{a}}\ln y + A \tag{9.36}$$

This is called the *law of the wall*, and was first discussed by von Kárman (1930). It fits experimental data on the velocity of turbulent flows past a wall remarkably well (see Figure 8.1), except in the viscous layer where the microscopic viscosity becomes important. The latter can be taken into account by solving the full equation (8.32) numerically.

First order closure models have the advantage of being very easy to use and implement numerically. However, the form of ν_t is only justified from a dimensional perspective, which is not particularly satisfactory. It can also fail dramatically when, e.g. the typical velocity of the perturbations is very different from that of the mean flow, or when the turbulence is highly anisotropic.

The $K - \epsilon$ model

The $K-\epsilon$ model, and its many variants, is the most commonly used turbulence closure model in a variety of applications (engineering, geophysics, etc). The K in the name refers to the kinetic energy and ϵ refers to the kinetic energy dissipation rate. It can be viewed as a 1.5 order closure model, in as much as it doesn't solve the Reynolds equations themselves, but solves an equation to model the energy in the perturbations instead.

We begin, as before, by assuming a form for the Reynolds stress tensor, this time as

$$\bar{R}_{ij} = -\nu_t\left(\partial_i \bar{u}_j + \partial_j \bar{u}_i\right) + \frac{1}{3}\bar{R}\delta_{ij} \tag{9.37}$$

where $\bar{R} = \sum_i \bar{R}_{ii}$ is the trace of the Reynolds stress tensor. This effectively assumes that the diagonal components are isotropic and each share 1/3 of the total. Recalling that $\bar{R}$ is related to the total energy in the flow perturbations, since

$$\frac{1}{2}\bar{R} = \frac{\overline{u_i' u_i'}}{2} = \bar{E}^{pert} \tag{9.38}$$

That energy is the K in the model, and so for consistency with the literature, we will call it K in this section, and then return to calling it $\bar{E}^{pert}$ later. So

$$\bar{R}_{ij} = -\nu_t \left(\partial_i \bar{u}_j + \partial_j \bar{u}_i\right) + \frac{2}{3} K \delta_{ij} \tag{9.39}$$

As in first order closure models, we need to come up with a formula for ν_t. The idea of the $K - \epsilon$ model is based on the same standard mixing-length argument $\nu_t \propto UL$ where U is a characteristic velocity, and L is a characteristic length of the flow. However, U and L are this time related to K and to the dissipation in the perturbations ϵ as follows:

$$U \propto K^{1/2} \tag{9.40}$$

and, since ϵ is, from a dimensional perspective, a velocity squared over time, and therefore also a velocity cubed over a length, namely $\epsilon \propto U^3/L$, then the lengthscale is assumed to be

$$L \propto U^3 \epsilon^{-1} = K^{3/2} \epsilon^{-1} \tag{9.41}$$

The turbulent diffusivity is therefore defined as

$$\nu_t = c_\mu K^2 \epsilon^{-1} \tag{9.42}$$

where c_μ is one of the model constants.

The model then proceeds by constructing evolution equations for both K and ϵ. The evolution of K is modeled based on the exact evolution equation for the perturbations (8.27), namely

$$\frac{\partial K}{\partial t} + \bar{\boldsymbol{u}} \cdot \nabla K + \bar{R}_{ij} \partial_j \bar{u}_i = -\partial_j (\overline{u_j' p'}) + \partial_j (\nu \partial_j K - \overline{u_j' E_K^{pert}}) - \epsilon \tag{9.43}$$

where $\bar{\epsilon}^{pert} \equiv \epsilon$ in this section only, again for consistency with the literature. Almost all terms in this equation are known, except the pressure terms and the triple term, which are then modeled as a diffusion of turbulent kinetic energy, with a turbulent diffusivity that is ν_t defined above. This leads to the evolution equation for K,

$$\frac{\partial K}{\partial t} + \bar{\boldsymbol{u}} \cdot \nabla K + \bar{R}_{ij} \partial_j \bar{u}_i = \partial_j ((\nu + \nu_t) \partial_j K) - \epsilon \tag{9.44}$$

The evolution equation for ϵ on the other hand, in the words of Davidson, is *pure invention*. It is usually written as

$$\frac{\partial \epsilon}{\partial t} + \bar{\boldsymbol{u}} \cdot \nabla \epsilon = \partial_j ((\nu + \frac{\nu_t}{\sigma_\epsilon}) \partial_j \epsilon) + c_1 \frac{\epsilon}{K} \bar{R}_{ij} \partial_j \bar{u}_i - c_2 \frac{\epsilon^2}{K} \tag{9.45}$$

where the constants c_1, c_2, and σ_ϵ are three additional model constants.

One of the known problems of the $K-\epsilon$ model as it is written here is that K and ϵ by definition should always be positive, but the evolution equations proposed do not necessarily guarantee that. This is the *realizability* problem – a good closure model should be *realizable* (ie. should be guaranteed to provide physically meaningful predictions) but many models are not. Realizable variants of the $K-\epsilon$ models have therefore been proposed to address this problem.

9.5.2 Second order closure models

Second order closure models typically attempt to solve for the Reynolds stresses by evolving an approximate representation of their governing equation in which the triple-correlation terms and the pressure-strain terms are modeled. From a numerical perspective, this is substantially more expensive than first order closure models. Indeed, since there are 6 independent components to the Reynolds stress tensor (bearing in mind its symmetry), this involves evolving 6 additional equations.

Note that second-order closure models also usually face the problem of realizability. Indeed, quantities such as the diagonal components of $\bar{R}_{ij}$ *must* be positive. While this is guaranteed in the exact Reynolds equations, it is not unusual to discover that a particular choice of closure breaks the realizability condition, and leads to negative values for these diagonal components, which is unphysical.

Let us visit a simple example of a second-order closure model. This example is not representative of most second-order closure models used in the engineering literature (to my knowledge, there are very few examples of second-order closure models used anywhere other than the engineering literature!). Rather, it will serve as a good excuse to discuss some of the thought process that goes into the construction of a closure.

The model proposed by Prof. Gordon Ogilvie (U. Cambridge).

The following model was proposed by my colleague Gordon Ogilvie (DAMTP, Cambridge), and I personally enjoy working with it for two reasons: (1) each term in the closure has a very well-defined, simple-to-understand role and (2) the model in its original form satisfies the realizability condition (see Ogilvie 2003). See, e.g. Garaud and Ogilvie 2005 for a description of the model applied to wall-bounded flows and comparisons with experiments. The idea behind the model is based on the notion that turbulence has a well-defined integral scale (i.e. usually the scale of the large, most energetic eddies), say L, and that the turbulent dynamics on scales much smaller than L have two effects. The first is to transfer energy from the large scales to the small scales at a certain rate (this energy will ultimately be dissipated viscously on the smallest scales), and the second is to isotropize the turbulence on a certain timescale. It is also possible that a direct energy dissipation may occur on the scale L when L itself is quite small. With these physical processes in mind, the proposed model is :

$$\frac{\partial \bar{R}_{ij}}{\partial t} + \bar{u}_k \partial_k \bar{R}_{ij} + \bar{R}_{jk}\partial_k \bar{u}_i + \bar{R}_{ik}\partial_k \bar{u}_j - \nu \partial_{kk}\bar{R}_{ij}$$
$$= -C_1 \frac{\bar{R}_{ij}}{\tau} - C_2 \frac{\bar{R}_{ij} - \frac{\delta_{ij}}{3}\bar{R}}{\tau} - \nu \frac{C_\nu}{L^2}\bar{R}_{ij} \tag{9.46}$$

where L is the aforementioned scale of energy-bearing eddies. The quantity $\tau = L/\bar{R}^{1/2}$ can be interpreted as an eddy turnover timescale, since $\bar{R}^{1/2}$ is a typical velocity scale. All the terms on the left-hand-side of this equation are exact, and were present in the original Reynolds equations (8.15). All the terms on the right-hand-side are closure terms, that are part of the model. We can interpret these terms as:

- The C_1 term causes a local (in time and space) decay of each component of the Reynolds stress tensor at the same rate, which can be attributed to a transfer of energy from the eddies on scale L to smaller-scale eddies, which eventually dissipate that energy viscously. The energy transfer rate is, crucially, independent of ν, consistent with the notion that this occurs through a turbulent energy cascade.

- The C_2 term causes a redistribution of energy from the off-diagonal components of the Reynolds stress tensor to the diagonal components, and a homogenization across the diagonal ones. As such, this term serves to isotropize the turbulence.

- The C_ν term can be interpreted as a local viscous decay of energy of the perturbations, that takes place when a turbulent cascade does not fully form (this term is usually negligible except in viscous boundary layers).

The exact energy equation for $\bar{E}^{pert}$ must therefore be well-represented by the closure model equation for $\bar{R}$. The latter is obtained by summing the diagonal components of (8.47) and dividing by 2,

$$\frac{\partial}{\partial t}\left(\frac{\bar{R}}{2}\right) + \bar{u}_j \partial_j \left(\frac{\bar{R}}{2}\right) + \bar{R}_{ij}\partial_j \bar{u}_i - \nu \partial_{jj}\left(\frac{\bar{R}}{2}\right) = -C_1 \frac{\frac{1}{2}\bar{R}}{\tau} - \nu \frac{C_\nu}{L^2}\frac{1}{2}\bar{R} \tag{9.47}$$

to be compared with (8.27)

$$\frac{\partial \bar{E}^{pert}}{\partial t} + \partial_j\left(\bar{u}_j \bar{E}^{pert}\right) + \bar{R}_{ij}\partial_j \bar{u}_i - \nu \partial_{jj}\bar{E}^{pert} = -\partial_j(\overline{u_j' p'}) - \partial_j\left(\overline{u_j' E^{pert}}\right) - \bar{\epsilon}^{pert} \tag{9.48}$$

As expected from the model construction, all the exact terms on the left-hand-side match exactly, while the right-hand-side shows how the model attempts to capture the effects of the turbulence. We can see immediately that the closure model only contain local dissipation terms, by contrast with the full equations that also contain transport terms (i.e. turbulent flux of energy). As such, this closure model is purely local, and will not be able to account for dynamics in which turbulent energy is created in one part of the domain, and dissipated in another (which happens for instance in the case of internal gravity waves, as we saw in Chapter 4). The model does however have some very nice properties, one of the most important ones being realizability (Ogilvie 2003). In addition, it is often possible to derive from the model some insight into the properties of fully turbulent flows, that would otherwise be difficult to obtain.

Consider for instance the case where the mean flow $\bar{\boldsymbol{u}}$ is just a linear shear flow with shearing rate S, e.g. $\bar{\boldsymbol{u}} = Sy\boldsymbol{e}_x$. We have seen in Chapter 7 by analyzing the linear stability of this problem that this flow is always linearly stable, but can sustain finite amplitude instabilities. Let us study what the model predicts for the turbulence induced by this finite-amplitude instability. We begin by assuming that the turbulence reaches a steady-state, and that the microscopic diffusion term is negligible. We also assume that the turbulence is completely homogeneous, so spatial derivatives of $\bar{R}_{ij}$ are null. Then, the closure equations are

$$\bar{R}_{jk}\partial_k\bar{u}_i + \bar{R}_{ik}\partial_k\bar{u}_j = -C_1\frac{\bar{R}_{ij}}{\tau} - C_2\frac{\bar{R}_{ij} - \frac{\delta_{ij}}{3}\bar{R}}{\tau} - \nu\frac{C_\nu}{L^2}\bar{R}_{ij} \tag{9.49}$$

Expanding this in a Cartesian coordinate system, we then get

$$2S\bar{R}_{xy} = -\frac{C_1+C_2}{L}\bar{R}^{1/2}\bar{R}_{xx} + \frac{C_2}{3L}\bar{R}^{3/2} - \nu\frac{C_\nu}{L^2}\bar{R}_{xx} \tag{9.50}$$

$$S\bar{R}_{yy} = -\frac{C_1+C_2}{L}\bar{R}^{1/2}\bar{R}_{xy} - \nu\frac{C_\nu}{L^2}\bar{R}_{xy} \tag{9.51}$$

$$S\bar{R}_{yz} = -\frac{C_1+C_2}{L}\bar{R}^{1/2}\bar{R}_{xz} - \nu\frac{C_\nu}{L^2}\bar{R}_{xz} \tag{9.52}$$

$$0 = -\frac{C_1+C_2}{L}\bar{R}^{1/2}\bar{R}_{yz} - \nu\frac{C_\nu}{L^2}\bar{R}_{yz} \tag{9.53}$$

$$0 = -\frac{C_1+C_2}{L}\bar{R}^{1/2}\bar{R}_{yy} + \frac{C_2}{3L}\bar{R}^{3/2} - \nu\frac{C_\nu}{L^2}\bar{R}_{yy} \tag{9.54}$$

$$0 = -\frac{C_1+C_2}{L}\bar{R}^{1/2}\bar{R}_{zz} + \frac{C_2}{3L}\bar{R}^{3/2} - \nu\frac{C_\nu}{L^2}\bar{R}_{zz} \tag{9.55}$$

The 3rd and 4th equations immediately show that $\bar{R}_{xz} = \bar{R}_{yz} = 0$, which makes sense from the perspective of the symmetries of the system: since nothing forces the flow to know the difference between positive and negative z, one would expect these correlations to be 0.

We can then solve for $\bar{R}_{yy}$ and $\bar{R}_{zz}$ in terms of $\bar{R}$ using the last two equations, and adding the first and last 2 equations, we get

$$2S\bar{R}_{xy} = -\frac{C_1}{L}\bar{R}^{3/2} - \nu\frac{C_\nu}{L^2}\bar{R} \tag{9.56}$$

Using this, we can get $\bar{R}_{xx}$ in terms of $\bar{R}$. Finally, using $\bar{R}_{xx} + \bar{R}_{yy} + \bar{R}_{zz} = \bar{R}$, we can get an equation for $\bar{R}$ only:

$$\frac{2C_2}{3L}S^2\bar{R}^{3/2} = \left[\frac{(C_1+C_2)\bar{R}^{1/2}}{L} + \frac{C_\nu\nu}{L^2}\right]^2\left[\frac{C_1}{L}\bar{R}^{3/2} + \frac{C_\nu\nu}{L^2}\bar{R}\right] \tag{9.57}$$

This has 2 solutions: $\bar{R} = 0$ (the laminar solution), as well as (possibly) two other solutions.

Let's first look at the limit in which $\nu \to 0$. Then, we have

$$\bar{R} = \frac{C_2}{C_1(C_1 + C_2)^2} L^2 S^2 \tag{9.58}$$

(note how this is indeed always positive) so

$$\bar{R}_{xy} = -\frac{C_1}{2}\left[\frac{C_2}{C_1(C_1 + C_2)^2}\right]^{3/2} S|S|L^2 \tag{9.59}$$

This effectively recovers the standard turbulent eddy diffusivity formula in which (for this system)

$$\bar{R}_{xy} = -\nu_t S \tag{9.60}$$

which identifies $\nu_t = \frac{C_1}{2}\left[\frac{C_2}{C_1(C_1+C_2)^2}\right]^{3/2} |S|L^2$, as in Prandtl's model.

More interestingly, however, if we *keep* the viscous terms, the problem yields following equation for $\bar{R}$:

$$\frac{2C_2}{3L} S^2 \bar{R}^{1/2} = \left[\frac{(C_1 + C_2)\bar{R}^{1/2}}{L} + \frac{C_\nu \nu}{L^2}\right]^2 \left[\frac{C_1}{L}\bar{R}^{1/2} + \frac{C_\nu \nu}{L^2}\right] \tag{9.61}$$

If we define $U = \bar{R}^{1/2}/L|S|$ as a nondimensional velocity, and $Re = L^2|S|/\nu$ as a Reynolds number, this becomes

$$\frac{2C_2}{3} U = \left[(C_1 + C_2)U + \frac{C_\nu}{Re}\right]^2 \left[C_1 U + \frac{C_\nu}{Re}\right] \tag{9.62}$$

so for a given Reynolds number, there can be 1, 2 or 3 solutions. Figure 8.2 from Garaud and Ogilvie (2005) show the positive solutions for U as a function of Re. We see that there exists a critical threshold in Re below which turbulent solutions (i.e solutions with $U > 0$) exist. Two solutions then exist, and it can be shown using a linear stability analysis that the lower one is always unstable, while the upper one is stable. When $Re \to \infty$, we recover the inviscid solution discussed above.

According to this model, homogeneous shear flows are subcritically unstable, and bifurcate from the stable laminar state at $Re = \infty$ (rather than at a finite Re). The subcritical nature of turbulence in this model setup is well known experimentally (in numerical experiments) – and is correctly captured by the model.

This model has also been tested successfully on various wall-bounded flows, such as pipe flows (see Figure 8.1), Taylor-Couette flows (after adding rotation), Rayleigh-Bénard convection (after adding a scalar field), stratified shear instabilities, and magnetohydrodynamic problems (after adding magnetic fields).

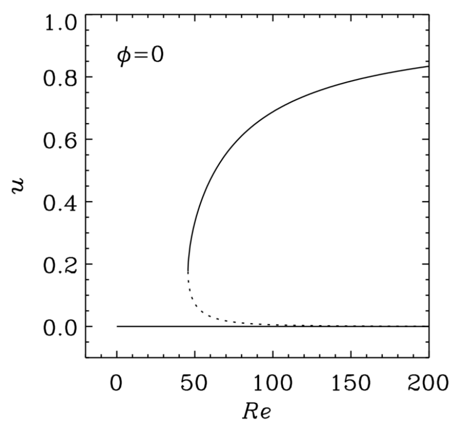

Figure 9.2: Solutions of (8.62) as a function of Re, for model parameters $C_1 = 0.42$, $C_2 = 0.6$ and $C_\nu = 12$.