Advanced Fluid Dynamics

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1 Mass conservation and other conservation laws

1.1 Mass conservation in polar coordinates

Question 1: Express the mass conservation equation in 2D polar coordinates

mass conservation

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \vec{u}) = 0.$$

expressed in 2D polar coordinates

$$\boxed{\frac{\partial \rho}{\partial t} + \frac{1}{r}\frac{\partial (r\rho \vec{u}_r)}{\partial r} + \frac{1}{r}\frac{\partial (\rho \vec{u}_\theta)}{\partial \theta} = 0}.$$

where $\vec{u} = \begin{pmatrix} u_r \\ u_\theta \end{pmatrix}$ is the velocity vector field expressed in polar coordinates.

Velocity field in polar coordinates

Let $\beta = \{\vec{v}_r, \vec{v}_\theta\}$ be an orthonormal basis where

$$\vec{v}_r = \begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix}, \quad \vec{v}_\theta = \begin{pmatrix} -\sin\theta \\ \cos\theta \end{pmatrix}..$$

Then at each point (r, θ) in space the position of a particle can be expressed as

$$\vec{r}(r, \theta) = r\vec{v}_r.$$

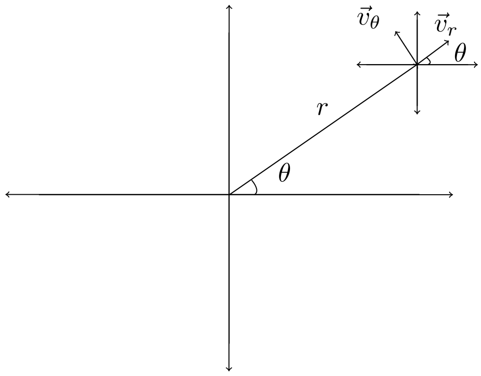

Figure 1: Tangent space? coordinate transform

so a fluid particles velocity can be expressed as

$$\begin{aligned}\vec{u}(r, \theta) = \frac{\mathrm{d}[r\vec{v}_r]}{\mathrm{d}t} &= \frac{\mathrm{d}r}{\mathrm{d}t}\vec{v}_r + r\frac{\partial \vec{v}_r}{\partial \theta}\frac{\mathrm{d}\theta}{\mathrm{d}t} \\ &= \frac{\mathrm{d}r}{\mathrm{d}t}\vec{v}_r + r\frac{\mathrm{d}\theta}{\mathrm{d}t}\vec{v}_\theta \\ &= \dot{r}\vec{v}_r + r\dot{\theta}\vec{v}_\theta\end{aligned}$$

then in β (polar) coordinates

$$[\vec{u}(r, \theta)]_\beta = \begin{pmatrix} \dot{r} \\ r\dot{\theta} \end{pmatrix}.$$

Question 2: If $\vec{u} = \begin{pmatrix} u_r \\ u_\theta \end{pmatrix} = \begin{pmatrix} u_0 \\ 0 \end{pmatrix}$, then a fluid particle has speed u_0 pointing in the direction of basis vector $\vec{v}_r$.

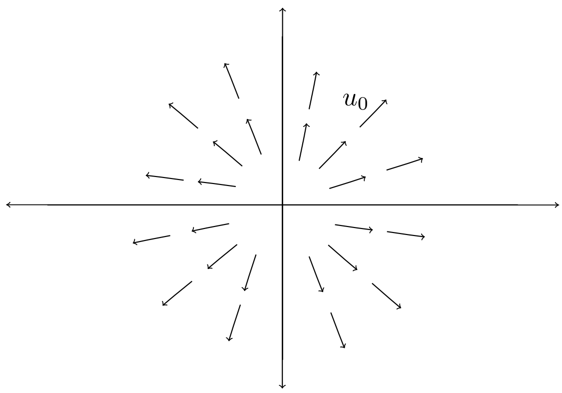

Figure 2: velocity field

Question 3: Solve for $\rho(r,t)$ with initial condition $\rho_0(r,t=0) = re^{-\frac{(r-1)^2}{2}}$ and $\vec{u}_\beta = \begin{pmatrix} u_0 \\ 0 \end{pmatrix}$.

Mass conservation expressed in polar coordinates, and assuming ρ doesn't depend on θ

$$\frac{\partial \rho}{\partial t} + \frac{1}{r}\frac{\partial (r\rho\vec{u}_r)}{\partial r} + \frac{1}{r}\frac{\partial (\rho\vec{u}_\theta)}{\partial \theta} = 0.$$

becomes

$$\frac{\partial \rho}{\partial t} + \frac{u_0}{r}\frac{\partial (r\rho)}{\partial r} = 0.$$

if we let $f(r,t) = r\rho(r,t)$, then we get

$$\begin{aligned} \frac{1}{r}\frac{\partial f}{\partial t} + \frac{u_0}{r}\frac{\partial f}{\partial r} &= 0 \\ \frac{\partial f}{\partial t} + u_0\frac{\partial f}{\partial r} &= 0 \end{aligned}$$

considering the parameterization $f = f(r(s), t(s)) \Rightarrow \frac{\mathrm{d}f}{\mathrm{d}s} = \frac{\mathrm{d}t}{\mathrm{d}s}f_t + \frac{\mathrm{d}r}{\mathrm{d}s}f_r = 0$, we get the following characteristic equations

$$\begin{cases} \dfrac{\mathrm{d}t}{\mathrm{d}s} = 1 & \Rightarrow t = s + t_0 \underbrace{\Rightarrow}_{t_0=0} t = s \\ \dfrac{\mathrm{d}r}{\mathrm{d}s} = u_0 & \Rightarrow r = u_0 s + r_0 \\ \dfrac{\mathrm{d}f}{\mathrm{d}s} = 0 & \Rightarrow f = f_0(r_0, t_0) \end{cases} .$$

putting everything together

$$f(r,t) = f(r_0, t_0 = 0) = (r - u_0 t)\rho_0(r - ru_0 t, 0).$$

$$\Rightarrow \boxed{\rho(r,t) = (r - u_0 t)e^{-\frac{1}{2}(r-u_0 t-1)^2}}.$$

Plotting $\rho(r,t)$ at various snapshots in time, with $u_0 = 1$, we see an agreement with Figure 2. The density gets pushed away from the origin, $r = 0$, at a constant speed u_0.

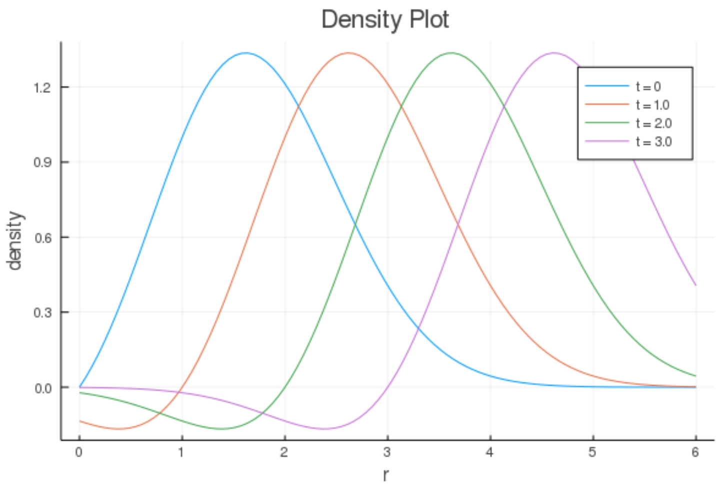

Figure 3

1.2 Other conservation laws

Assuming S is the salt concentration (mass of salt per unit mass of water) in salty water. If salt is conserved, then the equation for the evolution of S is the change in salt concentration w.r.t time equal to the (negative) divergence of salt flux at a point

$$\boxed{\frac{\partial S}{\partial t} + \nabla \cdot (S\vec{u}) = 0}.$$

2 Momentum equation and vorticity

Question 1: Assuming incompressible flow, and constant kinematic viscosity μ is constant, show $\nabla \cdot \Pi = \mu \nabla^2 \vec{u}$

Since the flow is incompressible $\Rightarrow \nabla \cdot \vec{u} = 0$, then the viscous stress tensor becomes

$$
\begin{aligned}
\Pi &= (k - \frac{2}{3}\mu)\underbrace{\nabla \cdot \vec{u}}_{=0} I + \mu(\nabla \vec{u} + (\nabla \vec{u})^T) \\
&= \mu(\nabla \vec{u} + (\nabla \vec{u})^T) \\
&= \mu \left[\begin{pmatrix} \frac{\partial u_1}{\partial x_1} & \frac{\partial u_1}{\partial x_2} & \frac{\partial u_1}{\partial x_3} \\ \frac{\partial u_2}{\partial x_1} & \frac{\partial u_2}{\partial x_2} & \frac{\partial u_2}{\partial x_3} \\ \frac{\partial u_3}{\partial x_1} & \frac{\partial u_3}{\partial x_2} & \frac{\partial u_3}{\partial x_3} \end{pmatrix} + \begin{pmatrix} \frac{\partial u_1}{\partial x_1} & \frac{\partial u_2}{\partial x_1} & \frac{\partial u_3}{\partial x_1} \\ \frac{\partial u_1}{\partial x_2} & \frac{\partial u_2}{\partial x_2} & \frac{\partial u_3}{\partial x_2} \\ \frac{\partial u_1}{\partial x_3} & \frac{\partial u_2}{\partial x_3} & \frac{\partial u_3}{\partial x_3} \end{pmatrix} \right] \\
\Rightarrow \Pi_{ij} &= \mu \left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right)
\end{aligned}
$$

Then the, i^{th} term of the, divergence of Π, for constant kinematic viscosity μ

$$
\begin{aligned}
(\nabla \cdot \Pi)_i &= \sum_j = \frac{\partial \Pi_{ji}}{\partial x_j} \\
&= \mu \sum_j \frac{\partial}{\partial x_j} \left(\frac{\partial u_j}{\partial x_i} + \frac{\partial u_i}{\partial x_j} \right) \\
&= \mu \sum_j \frac{\partial}{\partial x_j} \left(\frac{\partial u_j}{\partial x_i} \right) + \frac{\partial^2 u_i}{\partial x_j^2} \\
&= \mu \sum_j \frac{\partial}{\partial x_i} \underbrace{\left(\frac{\partial u_j}{\partial x_j} \right)}_{=\nabla \cdot \vec{u} = 0} + \frac{\partial^2 u_i}{\partial x_j^2} \qquad \textbf{equality of mixed partials} \\
&= \mu \sum_j \frac{\partial^2 u_i}{\partial x_j^2}
\end{aligned}
$$

Therefore,

$$\boxed{\nabla \cdot \Pi = \mu \nabla^2 \vec{u}}.$$

Question 2: Derive the equation for the evolution of vorticity $\omega = \nabla \times \vec{u}$ from the momentum equation

Assuming density is constant and there is no viscous dissipation ($\nabla \cdot \Pi = 0$), then the momentum equation becomes

$$\rho_0 \frac{D\vec{u}}{Dt} = -\nabla p + \underbrace{\nabla \cdot \Pi}_{=0}.$$

expanding the substantial derivative and multiplying the momentum equation by $(\nabla \times)$

$$\rho_0 \left(\nabla \times \left(\frac{\partial \vec{u}}{\partial t} + \vec{u} \cdot \nabla \vec{u} \right) \right) = \nabla \times (-\nabla p).$$

by continuity

$$\nabla \times \frac{\partial \vec{u}}{\partial t} = \frac{\partial (\nabla \times \vec{u})}{\partial t} = \frac{\partial \omega}{\partial t}.$$

and using the identity

$$(\vec{u} \cdot \nabla)\vec{u} = (\nabla \vec{u}) \cdot \vec{u} - \vec{u} \times \underbrace{(\nabla \times \vec{u})}_{=\omega}.$$

we have

$$\rho_0 \left(\frac{\partial \omega}{\partial t} + \underbrace{\nabla \times [(\nabla \vec{u})}_{=0} \cdot \vec{u} - \vec{u} \times \omega] \right) = -\underbrace{\nabla \times \nabla p}_{=0}.$$

where the curl of grad is zero $\nabla \times \nabla(\cdot) = 0$. Dividing by ρ_0

$$\frac{\partial \vec{\omega}}{\partial t} - \nabla \times \vec{u} \times \vec{\omega} = 0.$$

using the identity $\nabla \times (\vec{u} \times \vec{\omega}) = \vec{u}(\underbrace{\nabla \cdot \vec{\omega}}_{=0}) - \vec{\omega}(\underbrace{\nabla \cdot \vec{u}}_{=0}) + (\vec{\omega} \cdot \nabla)\vec{u} - (\vec{u} \cdot \nabla)\vec{\omega}$. Where $\nabla \cdot \vec{\omega} = 0$ because it is the divergence of curl, and $\nabla \cdot \vec{u} = 0$ due to the flow being incompressible. Therefore, the equation for the evolution of vorticity is

$$\boxed{\frac{\partial \vec{\omega}}{\partial t} + (\vec{u} \cdot \nabla)\vec{\omega} = (\vec{\omega} \cdot \nabla)\vec{u}}.$$

or, using the substantial derivative

$$\boxed{\frac{D\vec{\omega}}{Dt} = (\vec{\omega} \cdot \nabla)\vec{u}}.$$

3 Thermal energy equation and hydrostatic equilibrium

3.1 Energy equation

Using $\nabla \cdot \Pi = \mu \nabla^2 \vec{u}$ for incompressible Newtonian fluid, find the expression for viscous heating ϕ

3.2 Hydrostatics

Hydrostatic equilibrium is derived from the momentum equation assuming there is no fluid motion

$$-\nabla p + \rho \vec{g} = 0.$$

Question 1: In Cartesian coordinates with $\vec{g} = -g\vec{e}_z$, we have

$$-\begin{pmatrix} p_x \\ p_y \\ p_z \end{pmatrix} = \rho \begin{pmatrix} 0 \\ 0 \\ g \end{pmatrix}.$$

Question 2: The liquid equation of state implies density is simply a function of temperature

$$\rho = \rho(T).$$

and since the liquid is isothermal, temperature is constant $T = T_0$. This implies density is constant

$$\rho = \rho_0.$$

$p_x = 0$, and $p_y = 0$ implies pressure is independent of x and y, so we have

$$p_z = -\rho_0 g \quad \text{is constant.}$$

integrating with respect to z

$$p(z) = -g\rho_0 z + p_0.$$

since pressure is zero at the surface $p(H) = 0 \Rightarrow p_0 = g\rho_0 H$, then pressure increases linearly from $z = H$ to $z = 0$

$$\boxed{p(z) = -g\rho_0 z + g\rho_0 H}.$$

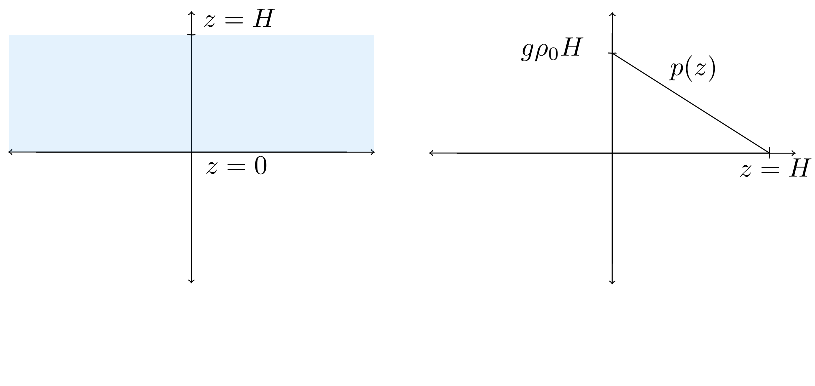

Figure 4: liquidEOS

Question 3: For an isothermal perfect gas equation of state, pressure is a function of density only $p(\rho) = R\rho T_0$. So the equation for hydrostatic equilibrium in terms of density

$$\rho_z = -\rho \frac{g}{RT_0}.$$

Integrating with respect to z

$$\ln \rho = -\frac{g}{RT_0} z + c.$$

$$\Rightarrow \boxed{\rho(z) = \rho_0 e^{-\frac{g}{RT_0} z}}.$$

Here density decreases exponentially from $z = 0$

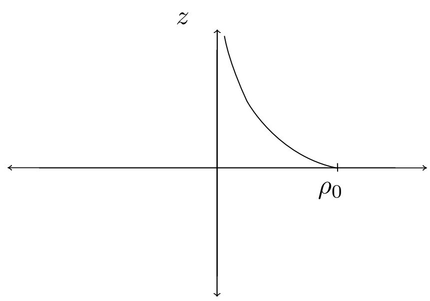

Figure 5: perfectgasEOS

The scaleheight, H, comparing the perfect gas equation of state and Boyles law

$$\frac{p}{\rho} = gmH = RT_0.$$