Waves, Instabilities, and Turbulence in Fluids

Kevin Corcoran

June 21, 2023

Contents

Lecture 1: The Governing Equations of Fluid Dynamics

Tue 04 Jan 2022 13:52

1 Mass Conservation

Eulerian Description

Lagrangian Description

$$\frac{D\rho}{Dt} = -\rho\nabla \cdot u.$$

Method of characteristics help this make more sense

2 Momentum Conservation

$$\rho\frac{D\vec{u}}{Dt} = \sum \vec{F} \tag{1}$$
$$= \underbrace{\nabla p}_{\text{pressure}} + \underbrace{\nabla \cdot \Pi}_{\text{viscosity}} + F_{\text{external}} \tag{2}$$

where, for Newtonian fluid

$$\Pi = (k - \frac{2}{3}\mu)\nabla \cdot \vec{u}.$$

3 Equation of State (EOS)

Comes from thermodynamics, and is a property of the fluid. Relates various thermodynamics quantities to one another. e.g. p, ρ, T, $\underbrace{s}_{\text{specific entropy / unit mass}}$, $\underbrace{e}_{\text{eternal energy / unit mass}}$. e energy inside gas if on the whole it is not moving?. Equation of state is needed to solve fluid problems.

For a single-component fluid or gas (e.g. water, O2), an equation of state relates one thermodynamic property to two others, eg

$$p = f(\rho, T)e = f(s, p)$$

All of these e.o.s are equivalent. They can be derived from each other.

Examples:

Perfect gas:

$$p = R\rho T.$$

where R = gas constant (specific to gas considered). Or $R = \frac{Ru}{mg}$ where Ru is the universal gas constant, and mg is the molecular weight of gas.

Liquid: incompressible

$$\rho = \rho(T).$$

(no pressure dependence!)

Now we have 3 equations relating 4 variables $(\rho, p, \vec{u}, T\ )$. Still need one more equation.

4 Total Energy Equation (Thermal Energy Equation).

Last equation derives from thermodynamics

$$\frac{\mathrm{d}u}{\mathrm{d}t} = \underbrace{Q}_{\text{heat input / unit time}} + \underbrace{W}_{\text{work done / unit time}}.$$

where u is total internal energy of a parcel of fluid or container. Sometimes written as

$$du = -pdV + TdS.$$

In fluid dynamics, when applied to parcels,

$$\rho\frac{D\rho}{Dt} = \underbrace{-p\nabla\cdot\vec{u} + \underbrace{Q}_{\text{heat}}}_{\text{external}} + \underbrace{\underbrace{\phi}_{\text{viscous heating}} - \nabla\cdot\vec{q}}_{\text{internal}}.$$

ϕ comes from nearby parcels ...

Special cases

for a perfect gas: $e = \underbrace{q}_{\text{specif heat at constant volume}} T$

$$\rho q\frac{DT}{Dt} = -\nabla.$$

hooks law $q = -k\nabla T$

$$\Rightarrow .$$

finally ϕ : is usually negligible, and if not, can be written in terms of $\rho, \vec{u}$

$\Rightarrow$ energy eq relates $\rho, T, \vec{u}, p$

5 Conservation Laws

Only if we ignore dissipation (viscosity?)

5.1 Conservation of Mass

5.2 Conservation of Momentum in Conservative Form

$$\rho\frac{D\vec{u}}{Dt} = -\nabla p\frac{\partial\rho\vec{u}}{\partial t} + \nabla\cdot(\rho\vec{u}\vec{u} + pI)$$

$a = \rho\vec{u}$ (a vector), and $F_a = \rho\vec{u}\vec{u} + pI$ (a tensor)

(Check what this looks like in Cartesian coordinates)

5.3 Conservation of Energy in Conservative Form

(ignore viscosity for now although it can be included)

$$\begin{aligned}\rho\frac{De}{Dt} &= -p\nabla\cdot\vec{u}\\ \rho(\frac{\partial e}{\partial t}+\mu\cdot\nabla e) &= -p\nabla\cdot\vec{u}\\ \frac{\partial\rho e}{\partial t}+\nabla\cdot(\rho\vec{u}e) &= -p\nabla\cdot\vec{u}\end{aligned}$$

Lecture 2: Non-dispersive Waves

Thu 06 Jan 2022 13:40

6 Non-dispersive Waves

Satisfy dispersion relation.

$$\omega = \alpha k.$$

where ω is frequency and k is the wave number. Examples include sound waves, and electromagnetic waves.

7 Sound waves in a homogeneous, invariant medium

homogeneous: u is constant in space invariant: u is constant in time

7.1 The wave equation for small amplitude pertubations

Mass conservation time change in density is equal to divergence of mass (negative)

Momentum (ignoring gravity / viscosity) density times substantial time derivative of velocity equals gradient of pressure (negative)

consider the background has

$$\begin{cases} p = p_0 & \textbf{constant}\\ \rho = \rho_0 & \textbf{constant}\\ \vec{u} = 0 & \textbf{no back flow}\end{cases}.$$

then let

$$\begin{aligned}p(x,t) &= p_0 + \hat{p}(x,t)\\ \rho(x,t) &= \rho_0 + \hat{\rho}(x,t)\\ \vec{u} &= 0 + \hat{\vec{u}}(x,t)\end{aligned}$$

where $\hat{p} << p_0$, $\hat{\rho} << \rho_0$ (i.e. small perturbations) and x is a vector

$\Rightarrow$ applying mass cons. and ignoring quadratic terms due to small perturbations

$$\frac{\partial \hat{\rho}}{\partial t} = -\rho_0 \nabla \vec{\hat{u}}.$$

$\Rightarrow$ applying momentum conservation and ignoring quadratic terms due to small perturbations

$$\rho_0 \frac{\partial \hat{u}}{\partial t} = -\nabla \hat{p}.$$

taking $\dfrac{\partial}{\partial t}$ of mass conservation, then we get

$$\frac{\partial^2 \hat{\rho}}{\partial t^2} = \nabla^2 \hat{p}.$$

which almost looks like the (hyperbolic) wave equation. We need EOS to relate $\hat{\rho}$ and $\hat{p}$

Assume a perfect gas $p = R\rho T$, then liberalizing this (applying small perturbations)

$$p_0 + \hat{p} = R(\rho_0 + \hat{\rho})(T_0 + \hat{T}).$$

in the background $p_0 = R\rho_0 T_0$, removing higher order terms, we get

$$\hat{p} = R(\hat{\rho} T_0 + \rho_0 \hat{T}).$$

Make assumptions to simplify the problem. Assume $\hat{T} = 0$ (at least very close to zero). This would give us the desired relationship between $\hat{\rho}$ and $\hat{p}$. Assuming temperature fluctuations decay very rapidly through radiation or diffusion). This is the isothermal assumption. So we have

$$\hat{p} = R\hat{\rho} T_0.$$

$\Rightarrow$ wave equation is

$$\frac{\partial^2 \hat{\rho}}{\partial t^2} = RT_0 \nabla^2 \hat{\rho}.$$

where the wave speed (isothermal sound speed)

$$c_T = \sqrt{RT_0}.$$

for air $c_T \approx 290 \frac{m}{s}$ (approximately). Although air is not actually isothermal.

Solve this using d'Alembert's technique (only works in 1D). The idea is to use a change of variable $\eta = x - ct$ and $\zeta = x + ct$. Then we get

$$\frac{\partial^2 p}{\partial \eta \partial \zeta} = 0.$$

integrating twice, we get

$$\begin{aligned} p(\eta, \zeta) &= F(\zeta) + G(\eta) \\ p(x,t) &= F(x+ct) + G(x-ct) \end{aligned}$$

True for any function satisfying 1D wave equation over infinite domain $x \in (-\infty, \infty)$. Suppose initial conditions are

$$\begin{cases} p(x,0) = p_0(x) \\ \dfrac{\partial p}{\partial t}(x,0) = q_0(x) \end{cases}.$$

applying these, we get

$$\begin{cases} p_0(x) = F(x) + G(x) \\ q_0(x) = cF'(x) - cG'(x) \end{cases}.$$

integrating q_0

$$\frac{1}{c}\int_0^x q_0(s)\, ds = (F(x) - F(0)) - (G(x) - G(0)).$$

this yields

$$p(x,t) = \frac{1}{2}p_0(x+ct) + \frac{1}{2}p_0(x-ct) + \frac{1}{2c}\int_{x-ct}^{x+ct} q_0(s)\, ds.$$

8 Monochromatic wave solution of the wave equation in an infinite domain

In electromagnetic waves, single frequency of light. For sound, analogously, this is just a single sound frequency. Assume the ansatz

$$p(x,t) = \hat{p}e^{ikx - i\omega t}.$$

where $\hat{p}$ can be complex and is constant. At the end, just take the real part of the solution $Re(p(x,t))$. Plugging this into the wave equation, we get

$$\omega^2 = c^2k^2 \Rightarrow \omega = \pm ck.$$

This is the dispersion relation for monochromatic sound waves, and shows that the waves are non-dispersive. By convention, choose $\omega > 0$, so

$$\omega = c|k|.$$

with this choice, the sign of k tells us the direction of propagation. Taking the real part of $p(x,t)$ as a linear combination of $k > 0$ (propagates to the right) and $k < 0$ (soln propagates to the left) then consider $\hat{p}_+$ and $\hat{p}_-$ are either real, or pure imaginary.

Define Phase Speed

$$\begin{aligned} p(x,t) &= \hat{p}e^{ikx - i\omega t} \\ &= \hat{p}e^{i\theta(x,t)} \end{aligned}$$

where $\theta(x,t) = kx - \omega t$ is the **phase** of the wave. If θ is constant then p is constant, though if θ is constant, then so is $kx - \omega t = \text{const}$.

Drawing lines for $k > 0$, and $k < 0$ in the x, t plane for

$$\begin{aligned} t &= \frac{kx - \text{const}}{\omega} \\ &= \text{sign}(k)\frac{x}{c} + \text{const} \end{aligned}$$

The constant phase propagates at velocity c, which propagates at So .. phase speed and group speed are the same. unique to non-dispersive waves.

Lecture 3: Sound waves in inhomogeneous, time-dependent medium

Tue 18 Jan 2022 13:35

Lecture 4: Chapter 5 - part 2 (nonlinear convection

Thu 03 Feb 2022 13:35

8.1 Weakly nonlinear theory of RBC above onset (above Ra_c)

We know from data that convective rolls are *steady* (settle in steady state with finite amplitude).

- Can we model this?

8.2 Preliminaries: tools needed

- Solvability condition (Fredholm's alternative (here for ODE's, but generalizes))

Consider a linear ODE

$$L[u(x)] = F(x).$$

defined on (a, b) with boundary conditions

$$u(a) = 0$$
$$u(b) = 0$$

Theorem: If L is self-adjoint with respect to the inner-product $\langle ., . \rangle$ and if there is a non-zero solution to

$$L(u_n) = 0.$$

the equation $L(u) = F$ only has solutions if

$$\langle u_n, F \rangle = 0.$$

Definition an operator L is self-adjoint w.r.t the inner product $, . , .,$ if

$$\langle Lv, u \rangle = \langle v, Lu \rangle.$$

where

$$\langle u, v \rangle = \int_a^b g(x)v(x)u(x)\, dx.$$

Note: Self adjoint operator for ODEs are unitarily diagonalizable

Equivalently we know that eigenfunctions of L correspoding to different eigenvalues are orthogonal w.r.t to $\langle ., . \rangle$. Therefore there exists a basis of eigenfunctions for any function on (a, b) can be written as

$$f(x) = \sum_n a_n v_n(x).$$

these are called generalized Fourier Series

Proof. We know that the solution to $L[u(x)] = F(x)$, if it exists can be written as

$$u(x) = \sum a_n v_n.$$

where $L[v_n(x)] = \lambda_n v_n(x)$

We also know

$$F(x) = \sum_n b_n v_n(x).$$

$$\Rightarrow L[\sum_n a_n v_n(s) = \sum b_n v_n(x)].$$

Take dot product with $v_m(x)$

$$\sum_n a_n \lambda_n \langle v_m, v_n \rangle = \sum_n b_n \langle v_m, v_n \rangle.$$

$$\Rightarrow a_m = \frac{b_m}{\lambda_m}.$$

which is fine unless $\lambda_m - 0$ But if there is a solution to the problem

$$Lu_n = 0.$$

then that means u_h is an eigenfunction of L with eigenvalue 0. If thats the case then a_m corresponds to that eigenvalue is undefined unless $b_m = 0$ as well. That happens when

$$\begin{aligned} F(x) &= \sum b_n v_n \\ &= \end{aligned}$$

$\square$

- "Baby step" weakly nonlinear theory on simple PDE

Consider

$$\begin{cases} \dfrac{\partial u}{\partial t} - \sin u = \dfrac{1}{R}\dfrac{\partial^2 u}{\partial z^2} \\ u(0) = 0, \qquad\qquad u(\pi) = 0 \end{cases}.$$

Steady state solution: $u = 0$. Assume u is small and linearize around steady state. Use $\sin(u) = u$, then

$$\frac{\partial u}{\partial t} - u = \frac{1}{R}\frac{\partial^2 u}{\partial z^2}.$$

Seeking solutions of the kind

$$u(z,t) = \hat{u}(z)e^{\lambda t}.$$

then

$$\frac{\mathrm{d}^2 \hat{u}}{\mathrm{d}z^2} = R(\lambda - 1)\hat{u}.$$

Then look for solutions that satisfy boundary conditions $u(0) = 0$, $u(\pi) = 0$, then

$$\hat{u} = \begin{cases} \sin(\sqrt{R(1-\lambda)}) \\ \cos(\sqrt{R(1-\lambda)}) \end{cases}.$$

$u(0) = 0 \Rightarrow$ no cosine. $u(\pi) = 0 implies$

$$\lambda_n = 1 - \frac{n^2}{R}.$$

If $R < 1$, then all eigenvalues < 0 and all perturbations decay

If $1 < R < 4$, then only mode that grows is $n = 1$.

- we expect very simple behavior (single mode excited)
- we want to study the nonlinear saturation of that mode, for $R = 1 + \varepsilon$.

  (R is just a little bit above critical.)

for this R,

$$\lambda_1 = 1 - \frac{1}{R} = 1 - \frac{1}{1+\varepsilon}$$
$$= 1 - (1 - \varepsilon + \varepsilon^2 + \dots) = \varepsilon + \text{H.O.T}$$

- the mode is growing exponentially at rate ε

$$u(z,t) \approx e^{\varepsilon t}.$$

So now the PDE is

$$\varepsilon \frac{\partial u}{\partial T} - \sin u = \frac{1}{1+\varepsilon} \frac{\partial^2 u}{\partial z^2}.$$

and we assume

$$u(z,T) = \varepsilon^{\alpha} u_0 + \varepsilon^{2\alpha} u_1 + \dots.$$

this assumes the nonlinear solutions has small amplitude of ε is small. How small u