Advanced Fluid Dynamics

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1 Sound waves with gravity

Question 1: Find the equation for isothermal sound waves in an isothermal atmosphere without neglecting gravity.

Conservation of momentum including gravity (1), and conservation of mass (2)

$$\rho \frac{D\mathbf{u}}{Dt} = -\nabla p + \rho \mathbf{g} \tag{1}$$

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{u}) = 0 \tag{2}$$

Then considering small perturbations in a motionless and homogeneous background state

$$\begin{cases} p = p_0 + \tilde{p} \\ \rho = \rho_0 + \tilde{\rho} \\ \mathbf{u} = \underbrace{\mathbf{u_0}}_{=\mathbf{0}} + \tilde{\mathbf{u}} = \tilde{\mathbf{u}} \end{cases} .$$

Plugging these perturbations into the governing equations, expanding the substantial derivative

$$(\rho_0+\tilde{\rho})\frac{\partial \mathbf{u}}{\partial t}+(\rho_0+\tilde{\rho})\mathbf{u}\cdot\nabla\mathbf{u}=-\nabla(\mathbf{p_0}+\tilde{\mathbf{p}})+(\rho_{\mathbf{0}}+\tilde{\rho})\mathbf{g}$$
$$\frac{\partial(\rho_0+\tilde{\rho})}{\partial t}+\nabla\cdot((\rho_0+\tilde{\rho})\mathbf{u})=0$$

Ignoring quadratic terms in small perturbations, and using $\nabla p_0 = 0$ and $\nabla \rho_0 = 0$ for change in homogeneous steady state

$$\rho_0\frac{\partial \tilde{\mathbf{u}}}{\partial t}=-\nabla\tilde{p}+(\rho_0+\tilde{\rho})\mathbf{g} \tag{3}$$
$$\frac{\partial\tilde{\rho}}{\partial t}+\nabla\cdot(\rho_0\mathbf{u})=0 \tag{4}$$

Taking the time derivative of (4)

$$\frac{\partial^2\tilde{\rho}}{\partial t^2}+\nabla\cdot\left(\rho_0\frac{\partial\tilde{\mathbf{u}}}{\partial t}\right)=0$$

Plugging in RHS of (3) for $\rho_0\frac{\partial\tilde{\mathbf{u}}}{\partial t}$

$$\frac{\partial^2\tilde{\rho}}{\partial t^2}+\nabla\cdot(-\nabla\tilde{p}+(\rho_0+\tilde{\rho})\mathbf{g})=0$$
$$\frac{\partial^2\tilde{\rho}}{\partial t^2}=\nabla^2\tilde{p}-(\nabla\cdot\tilde{\rho}\mathbf{g})$$

Considering $\mathbf{g}=\begin{pmatrix}0\\0\\-g\end{pmatrix}$

$$\frac{\partial^2\tilde{\rho}}{\partial t^2}=\nabla^2\tilde{p}+g\frac{\partial\tilde{\rho}}{\partial z} \tag{5}$$

Linearizing around the EOS for an isothermal, $T = T_0$, perfect gas $p = R\rho T_0$, and assuming temperature pertubations diffuse rapidly, $\tilde{T} = 0$, we get the following relationship between $\tilde{p}$ and $\tilde{\rho}$

$$\tilde{p}=RT_0\tilde{\rho} \tag{6}$$

Letting $c^2 = RT_0$, and plugging this, (6), into (5)

$$\boxed{\frac{\partial^2\tilde{\rho}}{\partial t^2}=c^2\nabla^2\tilde{\rho}+g\frac{\partial\tilde{\rho}}{\partial z}} \tag{7}$$

Question 2: Ignoring x and y components, and plugging in the monochramatic wave solution $\rho(z,t) = \hat{\rho}e^{ikz-i\omega t}$ in to (7) with

$$\frac{\partial \rho}{\partial z} = ik\rho, \quad \frac{\partial^2 \rho}{\partial z^2} = -k^2\rho$$
$$\frac{\partial^2 \rho}{\partial t^2} = -\omega^2\rho$$

and simplifying, we get the dispersion relation

$$-\omega^2\rho = -c^2k^2\rho + ik\rho g$$
$$\boxed{\omega^2 = c^2k^2\left(1 - i\frac{g}{kc^2}\right)} \tag{8}$$

The term that includes gravity is negligible when

$$kc^2 \gg g.$$

using $g = 9.8\frac{m}{s^2}$, $c = 290\frac{m}{s}$, and frequency $\omega = 3000$ Hz, we can see that $\frac{g}{c^2} = 0.000116527942925089$. If we assume that this is already negligible then $k = \frac{\omega}{c} = 10.3448275862069$. This is of course already assuming that gravity is negligible, but seems unlikely to be off by a factor of 10. A similar analysis suggests that gravity may not be negligible for very low frequency.

2 Superposition of monochromatic waves vs. d'Alembert's solution

Solve the 1D Cartesian wave equation on domain $t > 0$, and $-\infty < x < \infty$

$$p_{tt} = c^2 p_{xx}$$
$$p(x,0) = f(x)$$
$$p_t(x,0) = g(x)$$

Applying the Fourier transform in x, then Fourier coefficient $\hat{p}_{xx} = (ik)^2\hat{p}$, and

$$\hat{p}_{tt}(k,t) = -(ck)^2\hat{p}(k,t).$$

Which is an ODE in time with solution

$$\hat{p}(k,t) = Ae^{ickt} + Be^{-ickt} \tag{9}$$

with initial conditions

$$\hat{p}(k,0) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \underbrace{p(x,0)}_{=f(x)} e^{-ikx}\,dx$$
$$= \hat{f}(k) \qquad \text{and,}$$
$$\hat{p}_t(k,0) = \hat{g}(k)$$

Applying these to the general solution (9), we get

$$A = \frac{1}{2}\left(\hat{f} + \frac{\hat{g}}{ick}\right)$$
$$B = \frac{1}{2}\left(\hat{f} - \frac{\hat{g}}{ick}\right)$$

then the solution is

$$\hat{p}(k,t) = \frac{1}{2}\left(\hat{f} + \frac{\hat{g}}{ick}\right)e^{ickt} + \frac{1}{2}\left(\hat{f} - \frac{\hat{g}}{ick}\right)e^{-ickt}.$$

Applying the inverse Fourier transform

$$\begin{aligned} u(x,t) &= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \frac{1}{2}\hat{f}(k)e^{ik(x+ct)} + \frac{1}{2}\hat{f}(k)e^{ik(x-ct)}\,dk \\ &+ \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \frac{1}{2ick}\hat{g}(k)e^{ik(x+ct)} - \frac{1}{2ick}\hat{g}(k)e^{ik(x-ct)}\,dk \end{aligned}$$

we recover d Alembert's solution first noting that

$$f(x \pm ct) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \hat{f}(k)e^{ik(x\pm ct)}\,dk.$$

and

$$\begin{aligned} &\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \frac{1}{2ick}\hat{g}(k)e^{ik(x+ct)} - \frac{1}{2ick}\hat{g}(k)e^{ik(x-ct)}\,dk \\ &= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \frac{1}{2cik}\hat{g}(k)e^{ik\xi}\Big|_{x-ct}^{x+ct}\,dk \\ &= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \frac{1}{2c}\hat{g}(k)\int_{x-ct}^{x+ct} e^{ik\xi}\,d\xi\,dk \\ &= \frac{1}{2c}\int_{x-ct}^{x+ct} g(\xi)\,d\xi \end{aligned}$$

putting everything together

$$p(x,t) = \frac{1}{2}\left(f(x-ct) + f(x+ct)\right) + \frac{1}{2c}\int_{x-ct}^{x+ct} g(\xi)\,d\xi.$$

if $p_t(x,0) = g(x) = 0$, and $f(x) = p_0 e^{-\frac{x^2}{2}}$ then

$$\boxed{p(x,t) = \frac{1}{2}p_0\left(e^{-\frac{(x-ct)^2}{2}} + e^{-\frac{(x+ct)^2}{2}}\right)}.$$

3 Global modes in a square

Find the 2D eigenmodes and eigenvalues of the wave equation in a square with side length 1

$$\frac{\partial^2 p}{\partial t^2} = c^2 \nabla^2 p.$$

subject to $p = 0$ on the boundary

$$\begin{aligned} p(0, y, t) = p(1, y, t) = 0 \qquad & y \in [0, 1] \\ p(x, 0, t) = p(x, 1, t) = 0 \qquad & x \in [0, 1] \end{aligned}$$

Using separation of variables for a solution of the form

$$p(x, y, t) = X(x)Y(y)T(t).$$

and plugging this into the wave equations

$$\begin{aligned} XYT'' &= c^2 \left(X''YT + XY''T\right) \\ \frac{T''}{c^2 T} &= \frac{X''}{X} + \frac{Y''}{Y} \end{aligned}$$

Since both sides are functions of different variables, in order for them to equate, they must be equal to a constant

$$\frac{T''}{c^2 T} = \frac{X''}{X} + \frac{Y''}{Y} = A.$$

So we have

$$\begin{aligned} T'' &= c^2 AT \\ \frac{X''}{X} &= A - \frac{Y''}{Y} = B \end{aligned} \tag{10}$$

where (10) must equal another constant. Defining $C = A - B$, we get

$$\begin{aligned} X'' &= BX \\ Y'' &= CY \end{aligned}$$

Considering the different cases for sign of B and C, only $B < 0$ and $C < 0$ lead to non trivial solutions. Applying boundary conditions and ignoring the constant in front of solutions

$$\begin{cases} X_\omega(x) = \sin(\omega\pi x), & \text{for } \omega = 1, 2, 3, \ldots \\ Y_k(y) = \sin(k\pi y), & \text{for } k = 1, 2, 3, \ldots \end{cases}.$$

where $B = -(\omega\pi)^2$ and $C = -(k\pi)^2$ implies $A = -((\omega\pi)^2 + (k\pi)^2)$. Defining $\lambda_{\omega,k} = c^2 A$, then

$$\lambda^2_{(\omega,k)} = -((c\omega\pi)^2 + (ck\pi)^2) < 0.$$

and $T(t)$ has solution

$$T_{(\omega,k)}(t) = a\cos\left(\lambda_{(\omega,k)}t\right) + b\sin\left(\lambda_{(\omega,k)}t\right).$$

So the eigenmodes (11)

$$p_{\omega,k}(x,y,t) = X_\omega(x)Y_k(y)T_{\omega,k}(t)$$

$$\boxed{p_{\omega,k}(x,y,t) = \sin(\omega\pi x)\sin(k\pi y)\left(a\cos\left(\lambda_{(\omega,k)}t\right) + b\sin\left(\lambda_{(\omega,k)}t\right)\right)} \tag{11}$$

and eigenvalues (12)

$$\boxed{\lambda_{(\omega,k)} = c\pi\sqrt{\omega^2 + k^2}} \tag{12}$$

where coefficients a and b are found using initial conditions. Letting $a = b = 1$, and plotting different modes

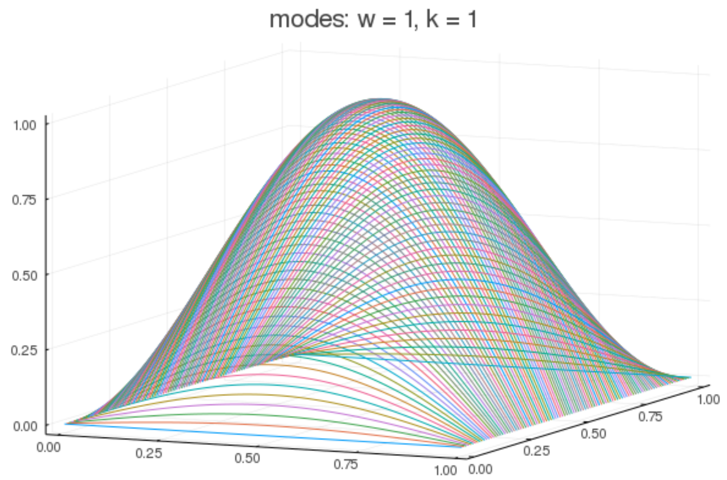

Figure 1: modesw1k1

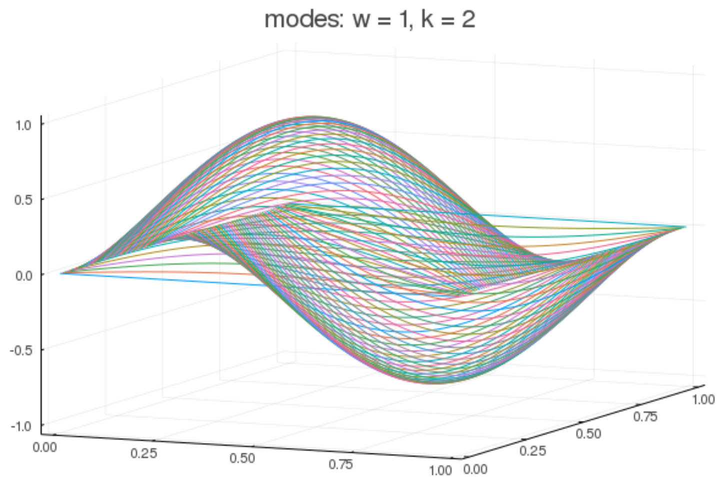

Figure 2: modesw1k2

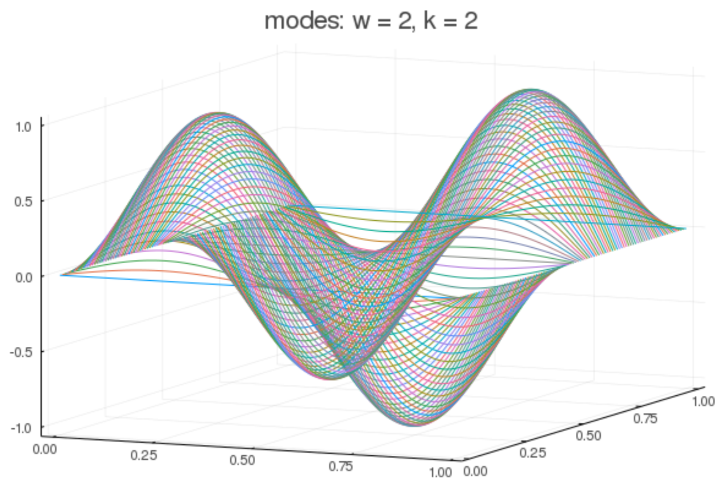

Figure 3: modesw2k2

4 Multi scale expansion for the damped oscillator

Actual Question: Consider the ODE

$$\frac{\mathrm{d}^2 f}{\mathrm{d}t^2} + f = -\varepsilon \frac{\mathrm{d}f}{\mathrm{d}t}$$
$$f(0) = 1$$
$$\frac{\mathrm{d}f(0)}{\mathrm{d}t} = 0$$

Write $f(t) = f(T_s(t), T_f(t))$ where $T_s = \varepsilon t$ is the slow time and $T_f = t$ is the fast time. Using the chain rule

$$
\begin{aligned}
\frac{\mathrm{d}f}{\mathrm{d}t} &= \frac{\partial f}{\partial T_s}\frac{\mathrm{d}T_s}{\mathrm{d}t} + \frac{\partial f}{\partial T_f}\frac{\mathrm{d}T_f}{\mathrm{d}t} \\
&= \varepsilon\frac{\partial f}{\partial T_s} + \frac{\partial f}{\partial T_f} \\
&= \varepsilon\frac{\partial f}{\partial T_s} + \frac{\partial f}{\partial t}
\end{aligned}
$$

Letting $T = T_s$, and assuming solution is of the form $f(T,t) = A(T)e^{i\theta(t)}$ where amplitude is a slowly varying function of time and frequency is a fast varying function of time. Then, using $\omega = -\frac{\partial \theta}{\partial t}$

$$
\begin{aligned}
\frac{\mathrm{d}f}{\mathrm{d}t} &= \varepsilon\frac{\mathrm{d}A}{\mathrm{d}T}e^{i\theta} + i\frac{\mathrm{d}\theta}{\mathrm{d}t}Ae^{i\theta} \\
&= \varepsilon\frac{\mathrm{d}A}{\mathrm{d}T} - i\omega f
\end{aligned}
$$

and

$$
\begin{aligned}
\frac{\partial^2 f}{\partial t^2} &= \left(\varepsilon\frac{\partial}{\partial T} + \frac{\partial}{\partial t}\right)\left(\varepsilon\frac{\partial f}{\partial T} + \frac{\partial f}{\partial t}\right) \\
&= \left(\varepsilon\frac{\partial}{\partial T} + \frac{\partial}{\partial t}\right)\left(\varepsilon\frac{\mathrm{d}A}{\mathrm{d}t}e^{i\theta} + Ai\frac{\mathrm{d}\theta}{\mathrm{d}t}e^{i\theta}\right) \\
&= -\omega^2 Ae^{i\theta} - 2i\varepsilon\omega\frac{\mathrm{d}A}{\mathrm{d}T}e^{i\theta} - i\varepsilon\frac{\mathrm{d}\omega}{\mathrm{d}T}Ae^{i\theta} + O(\varepsilon^2)
\end{aligned}
$$

equating constant terms in resulting ODE

$$
\begin{aligned}
-\omega^2 f + f &= 0 \\
\Rightarrow (-\omega^2 + 1)f &= 0 \\
\Rightarrow \omega &= \pm 1 \\
\Rightarrow \frac{\mathrm{d}\theta}{\mathrm{d}t} &= \pm 1
\end{aligned}
$$

$$
\boxed{\Rightarrow \theta(t) = \pm t + c_{1,2}}.
$$

equating $O(\varepsilon)$ terms

$$
\begin{aligned}
-2i\varepsilon\omega\frac{\mathrm{d}A}{\mathrm{d}T}e^{i\theta} - \underbrace{i\varepsilon\frac{\mathrm{d}\omega}{\mathrm{d}T}}_{=0}f &= i\omega\varepsilon f \\
\frac{\mathrm{d}A}{\mathrm{d}T} &= -\frac{1}{2}A \\
&\boxed{\Rightarrow A(T) = A_0 e^{-\frac{T}{2}}}
\end{aligned}
$$

Finally the general solution is

$$\boxed{f(t) = e^{-\frac{\varepsilon}{2}t}\left(c_1 e^{it} + c_2 e^{-it}\right)}.$$

which is exactly what you'd get solving the equation $f_{tt} + \varepsilon f_t + f = 0$ directly using the anzatz $f(t) = e^{rt}$ or any other second order method.