Waves, Instabilities, and Turbulence in Fluids

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1 Wave packet approximation

Taking the dot product of the momentum equation with the velocity vector $\mathbf{u}$, and making the appropriate assumptions on the background state, we get an energy equation in the form

$$\frac{\partial E_T}{\partial t} + \nabla \cdot (\text{Flux}) = \frac{\tilde{p}^2}{2\bar{\rho}} \frac{\partial}{\partial t} \left(\frac{1}{c^2} \right) \tag{1}$$

where total energy E_T is the sum of kinetic energy, $E_K = \bar{\rho}\frac{\tilde{\mathbf{u}}^2}{2}$, and compressional energy, $E_P = \frac{\tilde{p}^2}{2\bar{\rho}c^2}$. Using a similar wavepacket approximation for velocity, we relate $U(\mathbf{X}, T)$ to the amplitude $A(\mathbf{X}, T)$ using the momentum equation, and we find there is an equipartition between kinetic and accoustic energies in the wave packet amplitude solution

$$E_T = E_K + E_P = \frac{|A|^2}{2\bar{\rho}c^2} + \frac{|A|^2}{2\bar{\rho}c^2}.$$

So, we seek a similar equation to (1). Multiplying the evolution equation for amplitude with $A^*/\bar{\rho}c^2$, we get

$$\frac{1}{\bar{\rho}c^2}\frac{\partial |A|^2}{\partial T} + \frac{1}{\bar{\rho}c^2}\mathbf{c_g}\cdot\nabla_\varepsilon |A|^2 = -\frac{|A|^2}{\bar{\rho}kc}\left[\frac{\partial}{\partial T}\left(\frac{\omega}{c^2}\right) + \nabla_\varepsilon\cdot\mathbf{k}\right] \tag{2}$$

where $k = |\mathbf{k}|$ is the norm of the wavevector, and $\mathbf{c_g} = \frac{c^2}{\omega}\mathbf{k}$, is the group velocity vector. We want

$$\frac{\partial}{\partial T}\left(\frac{|A|^2}{\bar{\rho}c^2}\right) + \nabla_\varepsilon\cdot\left(\mathbf{c_g}\frac{|A|^2}{\bar{\rho}c^2}\right) \tag{3}$$

using the chain rule we recover (2) with additional terms so we add this in and we get

$$\begin{aligned}\frac{\partial}{\partial T}\left(\frac{|A|^2}{\bar{\rho}c^2}\right) + \nabla_\varepsilon\cdot\left(\mathbf{c_g}\frac{|A|^2}{\bar{\rho}c^2}\right) &= -\frac{|A|^2}{\bar{\rho}kc}\left[\frac{\partial}{\partial T}\left(\frac{\omega}{c^2}\right) + \nabla_\varepsilon\cdot\mathbf{k}\right]\\ &\quad + |A|^2\frac{\partial}{\partial T}\left(\frac{1}{\bar{\rho}c^2}\right) + |A|^2\nabla_\varepsilon\cdot\left(\frac{\mathbf{c_g}}{\bar{\rho}c^2}\right)\end{aligned}$$

using the chain rule on the RHS for $\dfrac{\partial}{\partial T}$ terms and pulling out $\frac{|A|^2}{\bar{\rho}}$, and rearranging and using the dispersion relation $w = ck$

$$\frac{|A|^2}{\bar{\rho}}\left[\frac{1}{c^2}\cancelto{0}{\frac{\partial}{\partial T}\left(\frac{1}{\bar{\rho}}\right)} + \cancel{\frac{\partial}{\partial T}\left(\frac{1}{c^2}\right)} + \frac{1}{\bar{c}^2}\nabla_\varepsilon\cdot\mathbf{c_g} + \frac{\bar{\rho}}{\bar{\rho}}\mathbf{c_g}\cdot\nabla_\varepsilon\frac{1}{c^2}\right]$$
$$\frac{|A|^2}{\bar{\rho}}\left[-\frac{1}{\omega c^2}\frac{\partial\omega}{\partial T} - \cancel{\frac{\omega}{\omega}\frac{\partial}{\partial T}\left(\frac{1}{c^2}\right)} - \frac{1}{\omega}\nabla_\varepsilon\cdot\mathbf{k}\right]$$

$\dfrac{\partial}{\partial T}\dfrac{1}{\bar{\rho}}$ goes to zero and $\nabla_\varepsilon\frac{1}{\bar{\rho}c^2} = \frac{1}{\bar{\rho}}\nabla_\varepsilon\frac{1}{c^2}$ if we make the assumption that $\bar{\rho}$ is constant. Using

$$\frac{1}{c^2}\nabla_\varepsilon\cdot\mathbf{c_g} + \mathbf{c_g}\cdot\nabla_\varepsilon\frac{1}{c^2} = \nabla_\varepsilon\cdot\left(\frac{\mathbf{c_g}}{c^2}\right).$$

we have

$$\frac{|A|^2}{\bar{\rho}}\left[\nabla_\varepsilon\cdot\left(\frac{\mathbf{c_g}}{c^2}\right) - \frac{1}{\omega c^2}\frac{\partial\omega}{\partial T} - \frac{1}{\omega}\nabla_\varepsilon\cdot\mathbf{k}\right].$$

using $\mathbf{c_g} = \frac{c^2}{\omega}\mathbf{k}$, then

$$
\begin{aligned}
\nabla_\varepsilon \cdot \left(\frac{\mathbf{c_g}}{c^2}\right) &= \nabla_\varepsilon \cdot \left(\frac{\mathbf{k}}{\omega}\right) \\
&= \frac{1}{\omega}\nabla_\varepsilon \cdot \mathbf{k} + \mathbf{k} \cdot \nabla_\varepsilon \frac{1}{\omega}
\end{aligned}
$$

putting this together

$$
\begin{aligned}
&\frac{|A|^2}{\bar{\rho}}\left[\cancel{\frac{1}{\omega}\nabla_\varepsilon \cdot \mathbf{k}} + \mathbf{k} \cdot \nabla_\varepsilon \frac{1}{\omega} - \frac{1}{\omega c^2}\frac{\partial \omega}{\partial T} - \cancel{\frac{1}{\omega}\nabla_\varepsilon \cdot \mathbf{k}}\right] \\
&= \frac{|A|^2}{\bar{\rho}}\left[\mathbf{k} \cdot \nabla_\varepsilon \frac{1}{\omega} - \frac{1}{\omega c^2}\frac{\partial \omega}{\partial T}\right] \\
&= \frac{|A|^2}{\bar{\rho}\omega c^2}\left[\omega c^2 \mathbf{k} \cdot \nabla_\varepsilon \frac{1}{\omega} - \frac{\partial \omega}{\partial T}\right]
\end{aligned}
$$

Subbing $\mathbf{k} = \frac{\omega}{c^2}\mathbf{c_g}$, then

$$
\omega c^2 \mathbf{k} \cdot \nabla_\varepsilon \frac{1}{\omega} = \omega^2 \mathbf{c_g} \cdot \nabla_\varepsilon \frac{1}{\omega}.
$$

using $\nabla_\varepsilon \omega^{-1} = -\frac{1}{\omega^2}\nabla_\varepsilon \omega$, we get

$$
-\frac{|A|^2}{\bar{\rho}\omega c^2}\left[\mathbf{c_g} \cdot \nabla_\varepsilon \omega + \frac{\partial \omega}{\partial T}\right].
$$

then using the dispersion relation $\omega = ck$, and the evolution equation for ω

$$
\begin{aligned}
\frac{\partial \omega}{\partial T} + \mathbf{c_g} \cdot \nabla_\varepsilon \omega &= \left(\frac{\partial \Omega}{\partial T}\right)_{\mathbf{k}} \\
&= k\frac{\partial c}{\partial T} \qquad \text{for constant } \mathbf{k}
\end{aligned}
$$

we get

$$
\begin{aligned}
&-\frac{|A|^2}{\bar{\rho}\omega c^2}\left[k\frac{\partial c}{\partial T}\right] \\
&= -\frac{|A|^2}{\bar{\rho}c^3}\left[\frac{\partial c}{\partial T}\right] \qquad \omega = ck
\end{aligned}
$$

using $\dfrac{\partial c^{-2}}{\partial T} = -\dfrac{2}{c^3}\dfrac{\partial c}{\partial T}$, and putting everything together, we finally get

$$
\boxed{\frac{\partial}{\partial T}\left(\frac{|A|^2}{\bar{\rho}c^2}\right) + \nabla_\varepsilon \cdot \left(\mathbf{c_g}\frac{|A|^2}{\bar{\rho}c^2}\right) = \frac{|A|^2}{2\bar{\rho}}\frac{\partial}{\partial T}\left(\frac{1}{c^2}\right)} \tag{4}
$$

2 Application: A sound mirror

Considering the wave packet evolution equations for constant sound speed

$$\frac{\partial \mathbf{k}}{\partial T} + \mathbf{c_g} \cdot \nabla_\varepsilon \mathbf{k} = 0 \tag{5}$$

$$\frac{\partial \omega}{\partial T} + \mathbf{c_g} \cdot \nabla_\varepsilon \omega = 0 \tag{6}$$

$$\frac{\partial A}{\partial T} + \mathbf{c_g} \cdot \nabla_\varepsilon A = -\frac{Ac}{2} \nabla_\varepsilon \cdot \left(\frac{\mathbf{k}}{k}\right) \tag{7}$$

then inside the first half sphere the divergence of the wavevector is positive

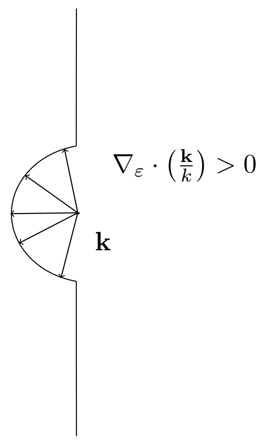

Figure 1: region1

in this region, before the sound hits the wall, the amplitude decreases while being advected. We can see this from the RHS of (7). Then using the property (approximating the half sphere as a parabola), that rays emitted from the focus go to infinity on parallel lines, then

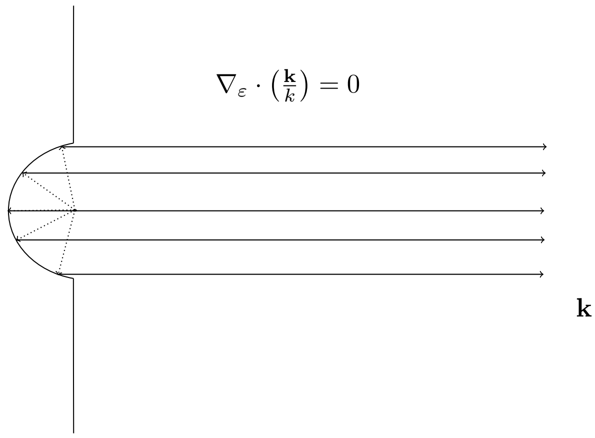

Figure 2: region2

as the sound travels across the room, the amplitude is conserved as it is advected. Finally, when the sound reaches the other half sphere, these parallel lines are then focused as the amplitude grows and converges at a point where the other person stands.

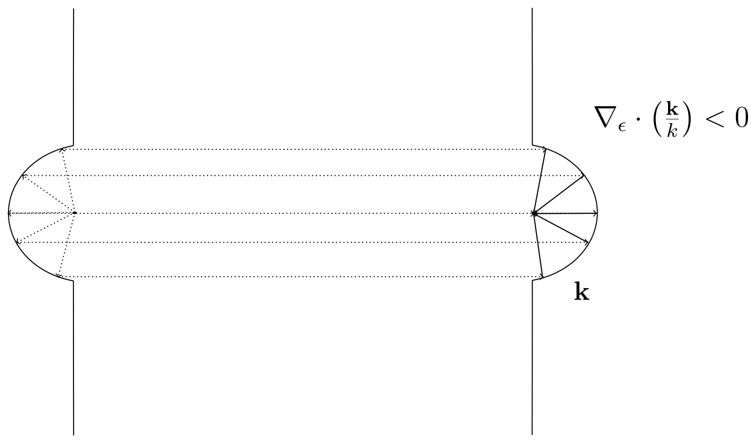

Figure 3: region3

If the speaker were to turn around and speak towards the person across the room, the amplitude would likely dissipate before reaching it's destination.

3 Diffraction of sound waves

Initially we have (5), (6), and (7) for the evolution of the wave packet with constant sound speed, where $\omega = \omega_0$ is advected without change, and dispersion relation $\omega_0^2 = c_0^2|\mathbf{k}|^2$. The evolution equations for non-constant sound speed

$$\frac{\partial \omega}{\partial T} + \mathbf{c_g} \cdot \nabla_\varepsilon = \left(\frac{\partial \Omega}{\partial T}\right)_{\mathbf{k}} = k\frac{\partial c_s}{\partial T} = 0 \tag{8}$$

$$\frac{\partial \mathbf{k}}{\partial T} + \mathbf{c_g} \cdot \nabla_\varepsilon = -(\nabla_\varepsilon \Omega)_{\mathbf{k}} = -k\nabla_\varepsilon c_s(X, Y) \tag{9}$$

$$\frac{\partial A}{\partial T} + \mathbf{c_g} \cdot \nabla_\varepsilon A = -\frac{Ac_s^2}{2\omega}\left[\frac{\partial}{\partial T}\left(\frac{\omega}{c_s^2}\right) + \nabla_\varepsilon \cdot \mathbf{k}\right] \tag{10}$$

Since c_s doesn't depend on time, $\omega = \omega_0$ is still conserved. Then from the dispersion relation $\omega^2 = c_s^2(X,Y)|\mathbf{k}|^2$ for waves coming from $Y \to +\infty$ in the region of the sound speed anomoly, we have

$$\mathbf{k} = \begin{pmatrix} k_x \\ k_y \end{pmatrix} = \begin{pmatrix} 0 \\ k_y \end{pmatrix}.$$

so

$$\omega_0^2 = c^2(X,Y)k_y^2$$
$$\Rightarrow k_y^2 = \frac{\omega_0^2}{c_s^2(X,Y)}$$

for $c_s(X,Y) = c_0 + (\Delta c)e^{-\frac{X^2+Y^2}{2}}$

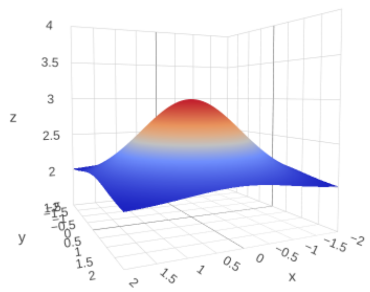

Figure 4: Sound anomaly

we have the following figure for how wave paths change

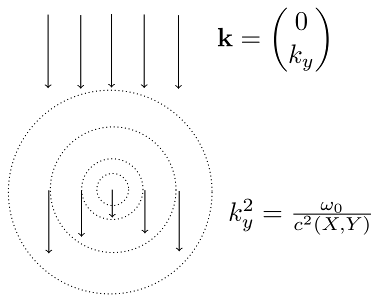

Figure 5: paths

Hence The y component of $\mathbf{k}$ slows down as it approaches the area where the sound speed increases, and we can see roughly how much it slows down by drawing contour lines of the graph of c_s. On the other hand, if $\Delta c < 0$, the vectors $\mathbf{k}$ speed up

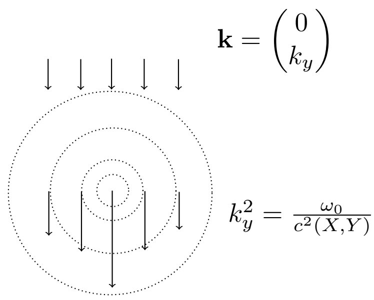

Figure 6: paths2

4 Gravity waves wave packet equations

Governing equation for internal gravity waves

$$\frac{\partial^2}{\partial t^2}\left(\nabla^2 \phi\right) = -\frac{\partial}{\partial x}\left(N^2\left(\frac{\partial \phi}{\partial x}\right)\right).$$

with $\mathbf{k} = \nabla\theta$ and $\omega = -\dfrac{\partial\theta}{\partial t}$. The wave packet approximation

$$\phi = A(X, Z, T)e^{i\theta(x,z,t)}.$$

then

$$\nabla\phi = \varepsilon\nabla A e^{i\theta} + i\nabla\theta A e^{i\theta}.$$

$$\nabla^2\phi = \varepsilon i\nabla A\nabla\theta e^{i\theta} + \varepsilon i\nabla^2\theta A e^{i\theta} + \varepsilon i\nabla\theta\nabla A e^{i\theta} - (\nabla\theta)^2 A e^{i\theta}.$$

...

5 Gravity waves in a linearly stratified medium

$N = aZ$, so evolution equation for ω is conserved, where dispersion relation $\omega = \Omega = \pm\frac{k_x}{k}N(Z)$

$$\frac{\partial\omega}{\partial T} + \mathbf{c_g}\cdot\nabla_\varepsilon\omega = \frac{\partial\Omega}{\partial T} = 0.$$

so set $\omega = \omega_0$, then ray paths can be found from

$$\frac{\mathrm{d}Z}{\mathrm{d}X} = \frac{\mathbf{c_g}\cdot\mathbf{e_z}}{\mathbf{c_g}\cdot\mathbf{e_x}} = \left(\frac{N(Z)^2}{\omega_0^2} - 1\right)^{-\frac{1}{2}}.$$

where group speed $\mathbf{c_g} = \frac{N}{k}\left(\mathbf{e_x} - k_x\frac{\mathbf{k}}{k^2}\right)$, then we want to integrate

$$\left(\frac{a^2Z^2}{\omega_0} - 1\right)^{\frac{1}{2}} dZ = dX.$$

using the change of variable $\sec^2(\theta) = \frac{a^2Z^2}{\omega_0}$, we get

$$X(\theta) = \frac{\sqrt{\omega_0}}{a}\int \sec\theta\tan^2\theta\, d\theta.$$

with solution

$$X(\theta) = \frac{\sqrt{\omega_0}}{2a}\left(\tan\theta\sec\theta - \tanh^{-1}(\sin\theta)\right).$$

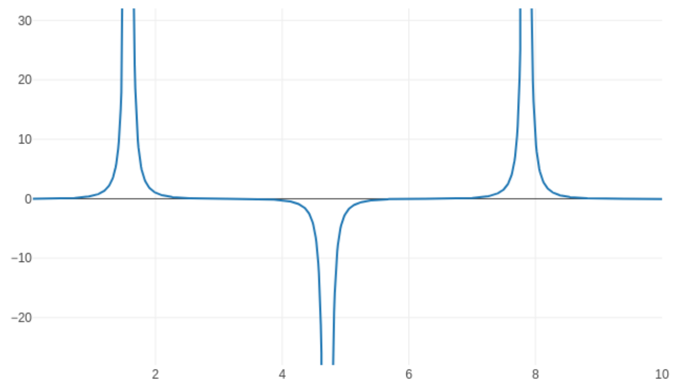

Figure 7: Plot