

# Homework 4

February 3, 2022

Do problems 1 and 2, and then *either* problem 3 *or* problem 4.

## 1 Problem 1: Local analysis for convection

Redo the local stability analysis for homogeneous convection we did in class, but this time add the dissipation terms we neglected: the viscous dissipation term in the momentum equation, and the thermal dissipation term in the temperature equation. What changes and what stays the same? Is  $\lambda$  still independent of  $|k|$ ? What do the fastest-growing modes look like? You may assume that the system is 2D. It is not necessary, but will make your calculation easier.

## 2 Problem 2: Threshold for instability of various modes of convection

Starting from the quadratic equation describing the growth rate of linearly unstable modes in Rayleigh-Bénard convection, (equation 5.24 of the Chapter 5 part 1 PDF document), determine the critical Rayleigh number and associated value of  $k_x$  for the onset of the  $n = 2$  and  $n = 3$  modes. Then, plot the corresponding curves  $Ra_c(k_x)$  (similar to figure 5.4) for the  $n = 2$  and  $n = 3$  modes.

## 3 Rayleigh Bénard convection with rotation (option 1)

*This is a hard problem. See, e.g. Hydrodynamic Stability by Chandrasekhar if you need help.*

Consider the Rayleigh-Bénard problem we studied in class (i.e. convection between bounded plates), with exactly the same setup. This question guides you towards finding a criterion for linear instability in the presence of rotation, when the rotation axis is in the same direction as gravity.

In the presence of rotation, oriented as in  $\mathbf{\Omega} = (0, 0, \Omega)$ , the governing equations become

$$\begin{aligned}\nabla \cdot \mathbf{u} &= 0 \\ \rho_m \left( \frac{\partial \mathbf{u}}{\partial t} + \mathbf{u} \cdot \nabla \mathbf{u} + 2\mathbf{\Omega} \times \mathbf{u} \right) &= -\nabla p + \rho \mathbf{g} + \rho_m \nu \nabla^2 \mathbf{u} \\ \frac{\partial T}{\partial t} + \mathbf{u} \cdot \nabla T &= \kappa_T \nabla^2 T \\ \frac{\rho}{\rho_m} &= -\alpha T\end{aligned}\tag{1}$$

Question 1: Show that there exists a steady state with no fluid motions in which  $\bar{T}(z) = T_m - z\Delta T/H$ .

Question 2: Non-dimensionalize the equations exactly as in Section 5.2.6 of the notes. Show that an additional parameter  $E = \frac{2\Omega H^2}{\kappa_T}$  appears in the presence of rotation.

Question 3: Assume 3D perturbations and write  $T(x, y, z, t) = \bar{T}(z) + \tilde{T}(x, y, z, t)$ . What system of equations do the perturbations satisfy? Linearize this system.

Question 4: Assume an ansatz of the form  $q(x, y, z, t) = \hat{q}(z)e^{ilx + iky + \lambda t}$  for the modes, and substitute it into the equations. Show that the resulting equation for, say,  $\hat{T}(z)$ , is

$$(\lambda - \text{Pr}D)^2(\lambda - D)D\hat{T} + (\lambda - \text{Pr}D)(l^2 + k^2)\text{RaPr}\hat{T} + E^2(\lambda - D)\frac{d^2\hat{T}}{dz^2} = 0\tag{2}$$

where  $D$  is an operator defined as  $D \equiv \frac{d^2}{dz^2} - (k^2 + l^2)$ .

Question 5: Check that when  $E = 0$  you recover the same equation as we had in the non-rotating case. Also check that modes with  $\hat{T}(z) = \sin(n\pi z)$  are solutions of the equation. What equation does this imply for  $\lambda$ ?

Question 6: Set  $\lambda = 0$ , and thus determine the critical value of needed for the onset of convection, as a function of  $n$ ,  $k$ ,  $l$ , and  $E$ . How does it differ from the non-rotating case? Does rotation inhibit or promote convection?

## 4 Double-diffusive convection (option 2)

*This is also a hard problem, though you can find the answer or help with the answer in, e.g., Radko 2013, Double-diffusive convection, or some of my early papers on the topic on my website (e.g. Traxler et al. 2011)*

Consider the following set of equations, which model homogeneous double-diffusive convection (ignoring the effects of boundaries). This is essentially the same as normal convection, except that there is a

second scalar field  $C$  (for concentration of some solute) that contributes to the density.

$$\begin{aligned}
\nabla \cdot \mathbf{u} &= 0 \\
\rho_m \left( \frac{\partial \mathbf{u}}{\partial t} + \mathbf{u} \cdot \nabla \mathbf{u} \right) &= -\nabla p + \rho \mathbf{g} + \rho_m \nu \nabla^2 \mathbf{u} \\
\frac{\partial T}{\partial t} + \mathbf{u} \cdot \nabla T &= \kappa_T \nabla^2 T \\
\frac{\partial C}{\partial t} + \mathbf{u} \cdot \nabla C &= \kappa_C \nabla^2 C \\
\frac{\rho}{\rho_m} &= -\alpha T + \beta C
\end{aligned} \tag{3}$$

Question 1: Show that there exists a steady state with no fluid motions in which  $\bar{T}(z) = T_m + zT_{0z}$ , and  $\bar{C}(z) = C_m + zC_{0z}$ .

Question 2: In what follows, assume for simplicity that  $T_{0z} > 0$  and  $C_{0z} > 0$  (this selects the so-called fingering regime). Non-dimensionalize the equations using the unit length  $[l] = d = \left( \frac{\kappa_T \nu}{\alpha g T_{0z}} \right)^{1/4}$ , the unit velocity  $[u] = \kappa_T / d$ , the unit time  $t = d^2 / \kappa_T$ , the unit temperature  $[T] = dT_{0z}$  and the unit concentration  $[C] = (\alpha/\beta)dT_{0z}$ . (Ask me why this is a good choice in office hours if you are interested). Show that three interesting parameters pop up:

$$\text{Pr} = \frac{\nu}{\kappa_T}, \quad \tau = \frac{\kappa_C}{\kappa_T}, \quad \text{and} \quad R_0 = \frac{\alpha T_{0z}}{\beta C_{0z}} \tag{4}$$

Question 3: Assume 2D perturbations and write  $T(x, z, t) = \bar{T}(z) + \tilde{T}(x, z, t)$  (and similarly for  $C$ ). What system of equations do the perturbations satisfy? Linearize this system.

Question 4: Assume normal modes of the form  $q(x, z, t) = \hat{q} e^{ilx + ikz + \lambda t}$ , and substitute this ansatz into the equations. What equation does  $\lambda$  satisfy? It can be shown (though you do not have to do that!) that modes with  $k = 0$  grow the fastest. What equation does  $\lambda$  satisfy for these modes?

Question 5: Set  $\lambda = 0$  (in the equation for  $k = 0$  modes), and thus determine the critical value of  $R_0$  needed for the onset of fingering convection, as a function of  $l$ , and  $\tau$ . Interpret your findings.