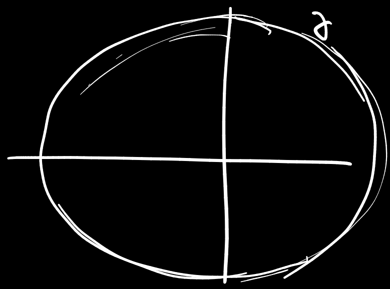
15.3 HW#9

4. $\underbrace{x^2+2y^2}_{}$, $\underbrace{x^2+y^2 \le 4}_{}$ ← Constraint

$f(x,y)$



$$\mathcal{L}(x,y,\lambda) = x^2+2y^2 - \lambda(x^2+y^2-4)$$

$$\nabla \mathcal{L}(x,y,\lambda) = 0$$

$$\begin{pmatrix} 2x - 2\lambda x \\ 4y - 2\lambda y \\ -(x^2+y^2)+4 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

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check for min/max in interior of constraint.

$$f(x,y) = x^2+2y^2$$

$$\nabla f(x,y) = \begin{pmatrix} 2x \\ 4y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} x \\ y \end{pmatrix} = 0$$ which is a minimum

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Hessian

$$Hf = \begin{pmatrix} f_{xx} & f_{xy} \\ f_{xy} & f_{yy} \end{pmatrix}$$

analogous "second derivative"

How do we know if this is positive? at some point

$$\vec{x}^T \underbrace{\begin{pmatrix} f_{xx} & f_{xy} \\ f_{xy} & f_{yy} \end{pmatrix}}_{H} \vec{x} \in \mathbb{R}$$ → we check the sign of this

Hessian is symmetric (granted mixed partials of the function are equal)

∴ orthogonally Diagonalizable

$H = Q^T D Q \Rightarrow$ let $y = Q\vec{x}$

$\Rightarrow \vec{x}^T H \vec{x} = y^T D y = \sum \lambda_i y_i^2$

$\sum \lambda_i y_i^2 > 0$ when $\lambda_i > 0$ for all i

< 0 when $\lambda_i < 0$ for all i

$\det(H) = \prod \lambda_i$

so if $\det(H) = f_{xx}(a,b) f_{yy}(a,b) - f_{xy}(a,b)^2 = D > 0$

and $f_{xx}(a,b) > 0$ or $f_{yy}(a,b) > 0$ then

there is a <u>relative min</u> at (a,b)

if $D > 0$

... and $f_{xx} < 0$ relative <u>max</u>

$D < 0$ saddle point (ie $f_{xx} \overset{1}{<} 0$ and $f_{yy} \overset{2}{>} 0$; opp signs)

$$\begin{pmatrix} 2x - 2\lambda x \\ 4y - 2\lambda y \\ -(x^2+y^2)+4 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

...

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meaning of λ in lagrange multipliers

(besides being a proportionality constant)

say

max $f(x_1, \ldots, x_n)$

s.t. $g(x_1, \ldots, x_n) = c$

$$\Rightarrow \mathcal{L}(x_1, \ldots, x_n, \lambda) = f(x_1, \ldots, x_n) - \lambda\left(g(x_1, \ldots, x_n) - c\right)$$

$\nabla \mathcal{L} = 0$ encompasses the constrained optimization problem

so $\frac{\partial L}{\partial c} = \lambda$ $\therefore$ λ is the rate of change of the quantity being optimized with respect to c

$\max f(x,y)$ s.t. $g(x,y)=c$ } if we vary c; the optimal point changes
let $P(c)$ be that point

$\Rightarrow \max M = f(P(c))$

$$\frac{df(P(c))}{dc} = \nabla f(P(c)) \cdot P'(c) \quad \text{by chain rule}$$

$$\nabla f(P(c)) = \lambda(c) \nabla g(P(c)) \qquad \nabla f(x,y) = \lambda \nabla g(x,y)$$

$$= \lambda(c) \nabla g(P(c)) \cdot P'(c)$$

$$= \lambda(c) \frac{dg(P(c))}{dc} \quad \text{by "reverse" chain rule}$$

$$= \lambda(c) \frac{dc}{dc} \qquad g(x,y)=c$$

$$= \lambda(c)$$

$$\Rightarrow \frac{df(P(c))}{dc} = \frac{dM}{dc} = \lambda(c)$$

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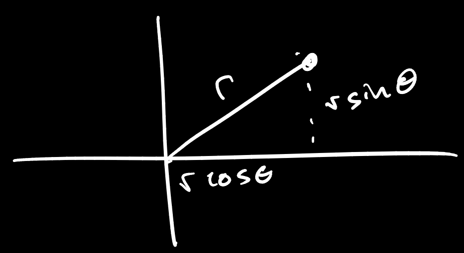
cylindrical and spherical coordinates

Polar coordinates

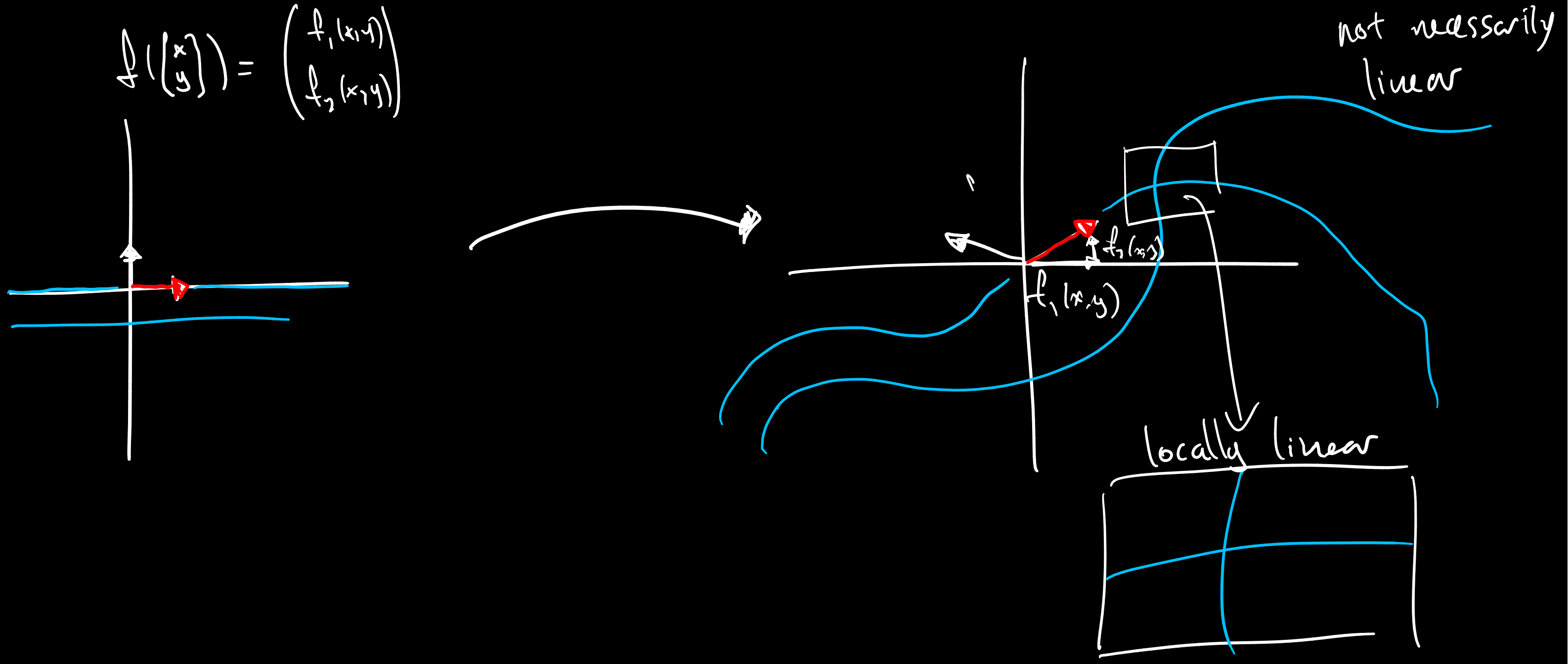


$x = r\cos\theta \qquad y = r\sin\theta$

consider a function

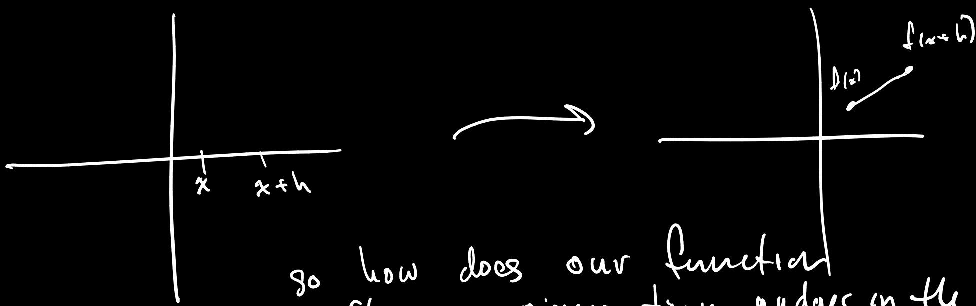
$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} r\cos\theta \\ r\sin\theta \end{pmatrix}$$

$$f\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{pmatrix} f_1(x,y) \\ f_2(x,y) \end{pmatrix}$$



our zoomed in transformation

So we should be able to find a matrix that represents the zoomed in version of the transformation

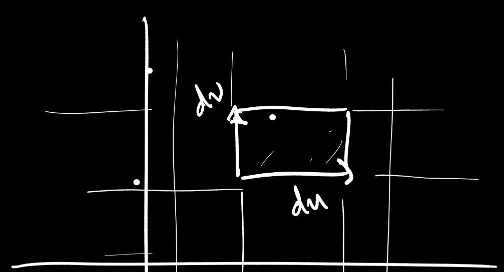


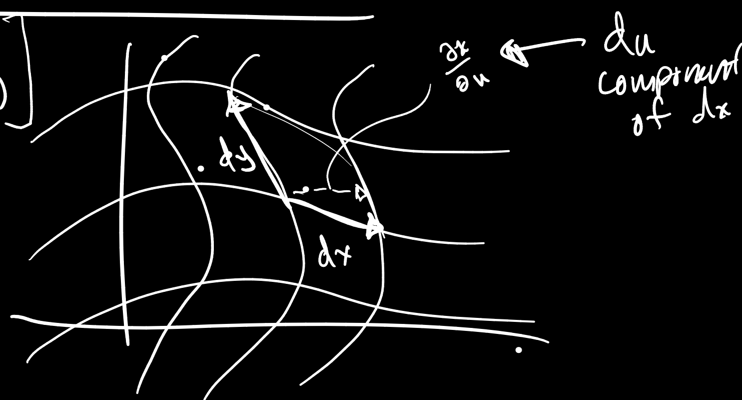
so how does our function change given tiny nudges in the x-direction

this is a derivative!

$$\begin{pmatrix} \partial f_1 / \partial x \\ \partial f_2 / \partial x \end{pmatrix} \quad \cdots$$

$$\boxed{\begin{aligned} x &= f_1(u,v) \\ y &= f_2(u,v) \end{aligned}}$$





(This tells us how changes in u, v effect changes in our new coordinates x, y)

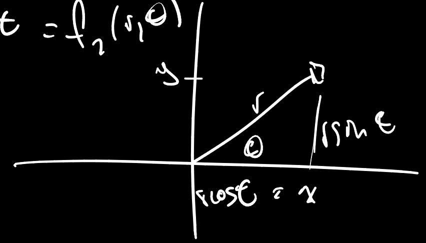
$\hookrightarrow$ differentials

$$\left.\begin{aligned} dx &= \frac{\partial x}{\partial u} du + \frac{\partial x}{\partial v} dv \\ dy &= \frac{\partial y}{\partial u} du + \frac{\partial y}{\partial v} dv \end{aligned}\right\} . \qquad \begin{pmatrix} dx \\ dy \end{pmatrix} = \underbrace{\overbrace{\begin{pmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{pmatrix}}}_{\text{Jacobian of Transformation} \approx \text{zoomed in}} \begin{pmatrix} du \\ dv \end{pmatrix}$$

almost like a directional derivative?

$$\iint f(x,y)\,dx\,dy$$

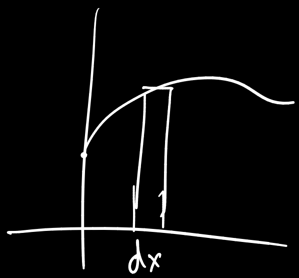
let $x = r\cos\theta = f_1(r,\theta)$
$y = r\sin\theta = f_2(r,\theta)$

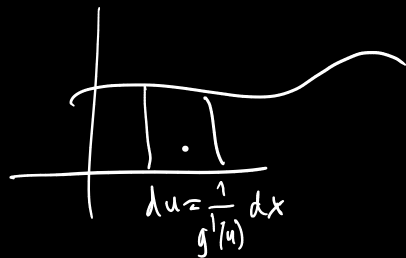


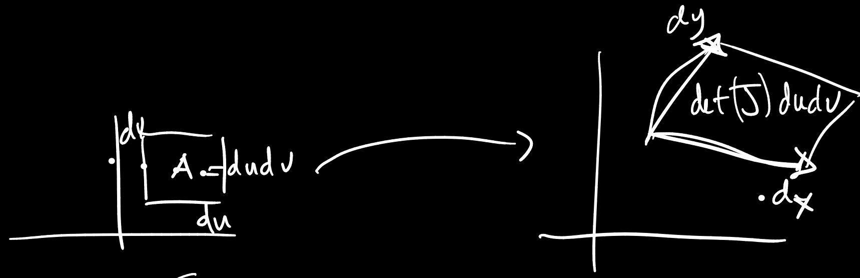
In 1 dimension

$$\int_a^b f(x)\,dx = \int f(g(u))\,g'(u)\,du$$

let $x = g(u)$
$dx = g'(u)\,du$ ~ differential
$= \frac{dx}{du}$







$$\iint f(x,y)\,dx\,dy \qquad du\,dv = \frac{1}{\det J}\,dx\,dy$$

$$= \iint f(r\cos\theta, r\sin\theta)\,\det(J)\,du\,dv$$

$x = r\cos\theta$
$y = r\sin\theta$

$dx = \frac{\partial x}{\partial r}dr + \frac{\partial x}{\partial \theta}d\theta$
$=$
$dy = \frac{\partial y}{\partial r}dr + \frac{\partial y}{\partial \theta}d\theta$
$= \sin\theta\,dr$

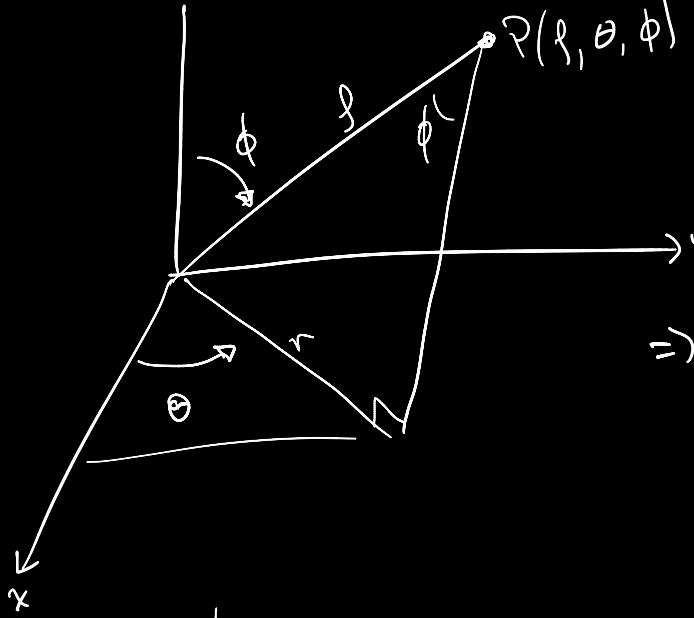
$$\begin{pmatrix} dx \\ dy \end{pmatrix} = \begin{pmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{pmatrix} \begin{pmatrix} dr \\ d\theta \end{pmatrix} = J \begin{pmatrix} dr \\ d\theta \end{pmatrix}$$

$$= \underbrace{\begin{pmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{pmatrix}}_{J} \begin{pmatrix} dr \\ d\theta \end{pmatrix} \Rightarrow \det J = r$$

$$\boxed{\iint f(r\cos\theta, r\sin\theta)\,r\,dr\,d\theta = \iint f(x,y)\,dx\,dy}$$

Polar coordinates

Spherical coordinates

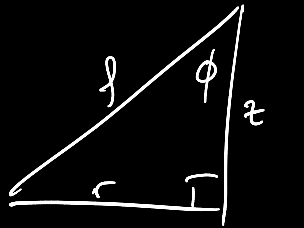


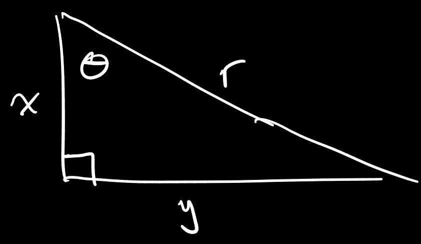
$z = \rho \cos\phi$

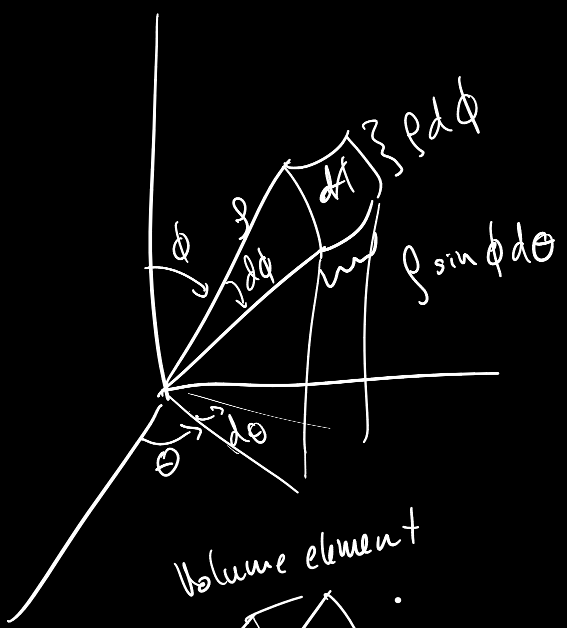
$r = \rho \sin\phi$

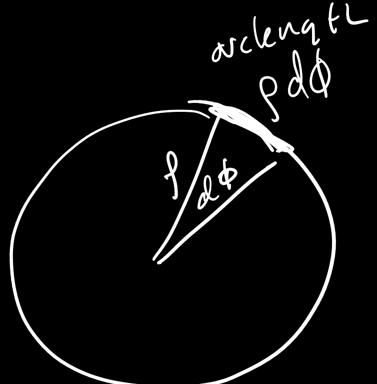
$\Rightarrow x = r \cos\theta$

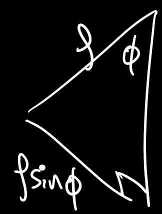
$y = r \sin\theta$

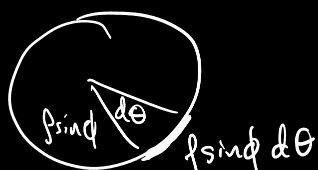












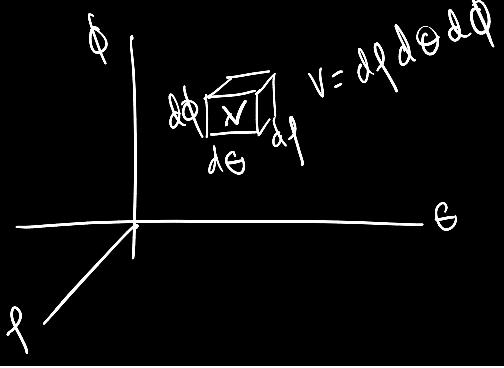
Volume element

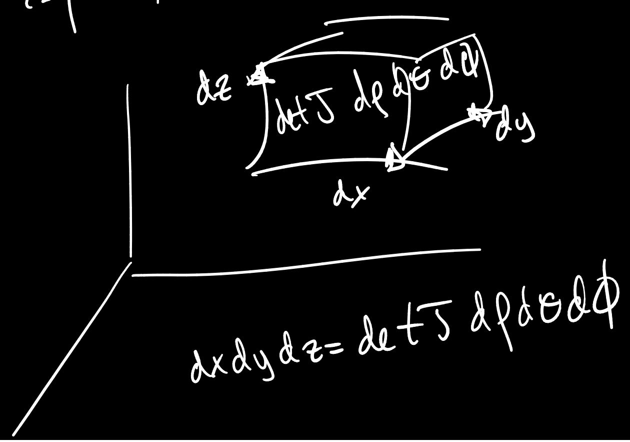
dV

$$dV = dA\, d\rho = \cancel{\rho\, \rho \sin\phi\, d\phi\, d\theta\, d\rho}$$
$$= \rho^2 \sin\phi\, d\rho\, d\theta\, d\phi$$

$$\iiint f(x,y,z)\, dx\, dy\, dz = \iiint f(\rho \sin\phi \cos\theta, \rho \sin\phi \sin\theta, \rho \cos\phi)\, \rho^2 \sin\phi\, d\rho\, d\theta\, d\phi$$

$x = \rho \sin\phi \cos\theta \qquad y = \rho \sin\phi \sin\theta \qquad z = \rho \cos\phi$





$dx\,dy\,dz = \det J\, d\rho\, d\theta\, d\phi$