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Lecture 17 Notes

These notes correspond to Lesson 24 in the text.

The Wave Equation in Two and Three Dimensions

Waves in Three Dimensions

Consider the IVP

$$u_{tt} = c^2\nabla^2 u, \quad (x,y,z) \in \mathbb{R}^3, \quad t > 0,$$
$$u(x,y,z,0) = 0, \quad u_t(x,y,z,0) = \psi(x,y,z),$$

where $\nabla^2 u$ is the *Laplacian*

$$\nabla^2 u = \nabla \cdot \nabla u = u_{xx} + u_{yy} + u_{zz}.$$

Using the 3-D Fourier transform,

$$\hat{u}(\omega_1,\omega_2,\omega_3,t) = \frac{1}{(2\pi)^{3/2}}\int_{\mathbb{R}^3} e^{-i(\omega_1 x+\omega_2 y+\omega_3 z)}u(x,y,z,t)\,dV,$$

we obtain the ODE

$$\hat{u}_{tt} = -c^2(\omega_1^2+\omega_2^2+\omega_3^2)\hat{u},$$

with initial conditions

$$\hat{u}(\omega_1,\omega_2,\omega_3,0) = 0, \quad \hat{u}_t(\omega_1,\omega_2,\omega_3,0) = \hat{\psi}(\omega_1,\omega_2,\omega_3),$$

where $\hat{\psi}$ is the Fourier transform of ψ. This IVP has the solution

$$\hat{u}(\omega_1,\omega_2,\omega_3,t) = \frac{1}{c\|\vec{\omega}\|}\hat{\psi}(\omega_1,\omega_2,\omega_3)\sin(c\|\vec{\omega}\|t),$$

where $\|\vec{\omega}\| = \sqrt{\omega_1^2+\omega_2^2+\omega_3^2}$. To compute the inverse Fourier transform, we first compute

$$\begin{aligned}
\mathscr{F}^{-1}\left[\frac{\sin(c\|\vec{\omega}\|t)}{c\|\vec{\omega}\|}\right] &= \frac{1}{(2\pi)^{3/2}}\int_0^{2\pi}\int_0^{\pi}\int_0^{\infty} e^{ir\rho\cos\phi}\frac{\sin(c\rho t)}{c\rho}\rho^2\sin\phi\,d\rho\,d\phi\,d\theta \\
&= \frac{1}{c\sqrt{2\pi}}\int_0^{\infty}\rho\sin(c\rho t)\int_0^{\pi} e^{ir\rho\cos\phi}\sin\phi\,d\phi\,d\rho \\
&= \frac{1}{c\sqrt{2\pi}}\int_0^{\infty}\rho\sin(c\rho t)\int_{-1}^{1} e^{ir\rho u}\,du\,d\rho, \quad u = \cos\phi \\
&= \frac{1}{c\sqrt{2\pi}}\int_0^{\infty}\sin(c\rho t)\frac{1}{ir}[e^{ir\rho} - e^{-ir\rho}]\,d\rho \\
&= \frac{1}{cr}\sqrt{\frac{2}{\pi}}\int_0^{\infty}\sin(c\rho t)\sin(r\rho)\,d\rho \\
&= \frac{1}{cr\sqrt{2\pi}}\int_0^{\infty}\cos[\rho(r-ct)] - \cos[\rho(r+ct)]\,d\rho \\
&= \frac{\pi}{cr\sqrt{2\pi}}\delta(r-ct)
\end{aligned}$$

where $r = \sqrt{x^2+y^2+z^2}$ and $\delta(x)$ is the Dirac delta function. We have used spherical coordinates, with the "north pole" $\phi = 0$ pointing in the direction of the position vector $(x, y, z) = (0, 0, r)$. We then apply the convolution theorem to obtain

$$\begin{aligned}
u(x,y,z,t) &= \frac{1}{(2\pi)^{3/2}} \int_{\mathbb{R}^3} \psi(x', y', z') \left[\frac{\pi}{cr\sqrt{2\pi}} \delta(r' - ct) \right] dx'\, dy'\, dz' \\
&= \frac{1}{4\pi c^2 t} \int_0^{2\pi} \int_0^{\pi} \psi(x + ct\cos\theta\sin\phi, y + ct\sin\theta\sin\phi, z + ct\cos\phi)(ct)^2 \sin\phi\, d\phi\, d\theta \\
&= t\overline{\psi},
\end{aligned}$$

where $r' = \sqrt{(x-x')^2 + (y-y')^2 + (z-z')^2}$ and

$$\overline{\psi} = \frac{1}{4\pi c^2 t^2} \int_0^{2\pi} \int_0^{\pi} \psi(x + ct\cos\theta\sin\phi, y + ct\sin\theta\sin\phi, z + ct\cos\phi)(ct)^2 \sin\phi\, d\phi\, d\theta$$

is the average of ψ over the sphere of center (x, y, z) and radius ct.

Now, we consider the IVP

$$u_{tt} = c^2 \nabla^2 u, \quad (x, y, z) \in \mathbb{R}^3, \quad t > 0,$$

$$u(x, y, z, 0) = \phi(x, y, z), \quad u_t(x, y, z, 0) = 0.$$

Let v be the solution of

$$v_{tt} = c^2 \nabla^2 v, \quad (x, y, z) \in \mathbb{R}^3, \quad t > 0,$$

$$v(x, y, z, 0) = 0, \quad v_t(x, y, z, 0) = \phi(x, y, z).$$

Then

$$v(x, y, z, t) = t\overline{\phi},$$

where $\overline{\phi}$ is the average of ϕ over the sphere centered at (x, y, z) with radius ct. We then have

$$(v_t)_{tt} = (v_{tt})_t = (c^2 \nabla^2 v)_t = c^2 \nabla^2 (v_t),$$

and

$$v_t(x, y, z, 0) = \phi(x, y, z), \quad (v_t)_t(x, y, z, 0) = 2\overline{\phi}_t\big|_{t=0} = 0,$$

which means that $u = v_t$, as it satisfies the PDE and initial conditions of the IVP for u.

We conclude that the solution of the IVP

$$u_{tt} = c^2 \nabla^2 u, \quad (x, y, z) \in \mathbb{R}^3, \quad t > 0,$$

$$u(x, y, z, 0) = \phi(x, y, z), \quad u_t(x, y, z, 0) = \psi,$$

is

$$u(x, y, z, t) = \frac{\partial}{\partial t}[t\overline{\phi}] + t\overline{\psi}.$$

This formula for the solution is known as *Kirchhoff's formula*, as well as *Poisson's formula*.

Because the solution depends on integrals over a sphere of radius ct, it follows that if the initial data are zero except within a small sphere, then the solution is zero at any point (x_0, y_0, z_0) outside this sphere until ct is large enough so that the sphere centered at (x_0, y_0, z_0) with radius ct overlaps the sphere within which the initial data is nonzero. That is, the solution exhibits a *sharp leading edge*. Then, once ct is so large that the sphere centered at (x_0, y_0, z_0) with radius ct *contains* the sphere within which the initial data is nonzero, the solution at (x_0, y_0, z_0) is zero again, and will remain zero. That is, the solution also exhibits a *sharp trailing edge*. This is known as *Huygen's principle*. It holds in dimensions 3, 5, 7, and so on, but not in another dimension. We have already seen that it does not hold in one dimension, as D'Alembert's solution integrates $u_t(x, 0) = g(x)$ from $x - ct$ to $x + ct$.

Two-Dimensional Wave Equation

The solution of the wave equation in two dimensions can be obtained by solving the three-dimensional wave equation in the case where the initial data depends only on x and y, but not z. In this case, the three-dimensional solution consists of *cylindrical waves*.

We first consider the IVP

$$u_{tt} = c^2\nabla^2 u, \quad (x, y) \in \mathbb{R}^2, \quad t > 0,$$
$$u(x, y, 0) = 0, \quad u_t(x, y) = \psi(x, y).$$

Extending this problem to three dimensions, we obtain the solution $u = t\overline{\psi}$, where

$$\overline{\psi} = \frac{1}{4\pi c^2 t^2}\int_0^{2\pi}\int_0^{\pi} \psi(x + ct\cos\theta\sin\phi, y + ct\sin\theta\sin\phi, z + ct\cos\phi)(ct)^2\sin\phi\,d\phi\,d\theta.$$

To obtain the solution of the original two-dimensional problem, we need to convert this integral over a sphere of radius ct to an integral over a disc of radius ct. This approach to obtaining the solution is called the *method of descent*.

We use a substitution to change the spherical coordinate variable ϕ to the polar coordinate variable r. From the relation between them,

$$\phi = \cos^{-1}\left(\frac{z}{ct}\right) = \cos^{-1}\left(\frac{\sqrt{(ct)^2 - r^2}}{ct}\right)$$

we obtain

$$d\phi = \frac{1}{\sqrt{1 - \frac{(ct)^2 - r^2}{(ct)^2}}}\frac{1}{ct}\frac{1}{2}\frac{1}{\sqrt{(ct)^2 - r^2}}(-2r) = \frac{1}{\sqrt{(ct)^2 - r^2}}\,dr.$$

From

$$\sin\phi = \frac{r}{ct}$$

our solution becomes

$$\begin{aligned}
t\overline{\psi} &= \frac{t}{4\pi c^2 t^2}\int_0^{2\pi}\int_0^{\pi}\psi(x + ct\cos\theta\sin\phi, y + ct\sin\theta\sin\phi)(ct)^2\sin\phi\,d\phi\,d\theta \\
&= \frac{2}{4\pi c^2 t}\int_0^{2\pi}\int_0^{ct}\frac{\psi(x + r\cos\theta, y + r\sin\theta)}{\sqrt{(ct)^2 - r^2}}(ct)^2\frac{r}{ct}\,dr\,d\theta \\
&= \frac{1}{2\pi c}\int_0^{2\pi}\int_0^{ct}\frac{\psi(x + r\cos\theta, y + r\sin\theta)}{\sqrt{(ct)^2 - r^2}}r\,dr\,d\theta.
\end{aligned}$$

It follows from Kirchhoff's formula that the solution of the IVP

$$u_{tt} = c^2\nabla^2 u, \quad (x, y) \in \mathbb{R}^2, \quad t > 0,$$
$$u(x, y, 0) = \phi(x, y), \quad u_t(x, y) = \psi(x, y).$$

is

$$\begin{aligned}
u(x, t) \;=\; &\frac{1}{2\pi c}\int_0^{2\pi}\int_0^{ct}\frac{\psi(x + r\cos\theta, y + r\sin\theta)}{\sqrt{(ct)^2 - r^2}}r\,dr\,d\theta + \\
&\frac{\partial}{\partial t}\left[\frac{1}{2\pi c}\int_0^{2\pi}\int_0^{ct}\frac{\phi(x + r\cos\theta, y + r\sin\theta)}{\sqrt{(ct)^2 - r^2}}r\,dr\,d\theta\right].
\end{aligned}$$

Because these integrals are over the *interior* of the disc, as opposed to the boundary of a sphere in the three-dimensional case, it can be concluded that Huygen's principle does *not* hold in two dimensions.