Homework 2 (Chapter 2 part 1)

January 11, 2022

1 Sound waves with gravity (part 1)

Question 1: Consider an isothermal atmosphere in a Cartesian coordinate system (see Chapter 1). Show that the equation for isothermal sound waves in that isothermal atmosphere, without neglecting gravity, is

$$\frac{\partial^2 \tilde{\rho}}{\partial t^2} = c^2 \nabla^2 \tilde{\rho} + g \frac{\partial \tilde{\rho}}{\partial z} \tag{1}$$

Question 2: Assume 1D monochromatic plane wave solutions to the wave equation derived above, assuming that $\tilde{\rho}$ only varies with z (i.e. ignore x and y dependences to only study $\tilde{\rho}(z,t)$)

- What is the dispersion relation?
- Based on the dispersion relation only, under which circumstances is the term that includes gravity negligible?
- Plug in typical dimensional numbers for g, c, k, etc.. Is gravity typically negligible for isothermal sound waves in air? Are there any circumstances in which it may not be negligible?

2 Superposition of monochromatic waves vs. d'Alembert's solution

Solve the 1D Cartesian wave equation $\partial_{tt} p = c^2 \partial_{xx} p$ subject to initial conditions $p(x,0) = p_0 \exp(-x^2/2)$ and $\partial_t p(x,0) = 0$ using a superposition of monochromatic waves. How does this relate to d'Alembert's solution for the same initial conditions? Hint: I find that it helps to remember that, for any complex number, $\Re(z) = (z + z^*)/2$.

3 Global modes in a square

Find the 2D eigenmodes and eigenvalues of the wave equation $\frac{\partial^2 p}{\partial t^2} = c^2 \nabla^2 p$ in a square whose side length is 1, subject to $p = 0$ on the boundary of the region. Plot a few representative eigenmodes in each case. What are all the possible frequencies achievable?

4 Multiscale expansion for the damped oscillator

Warm up problem: (means you don't have to do it, but if you are stuck on the next one, do this one first). Consider the ODE

- Consider the function $f(x) = e^{-\epsilon x} \sin(x)$. Calculate its derivative $\frac{df}{dx}$.
- Write $f(x)$ as a function of two variables $X_s = \epsilon x$ (the slow space variable), and $X_f = x$ (the fast space ariable) as $f(X_s, X_f) = e^{-X_s} \sin(X_f)$. Compute

$$\frac{df}{dx} = \frac{\partial f}{\partial X_s}\frac{\partial X_s}{\partial x} + \frac{\partial f}{\partial X_f}\frac{\partial X_f}{\partial x} \tag{2}$$

  and verify that you recover the same answer answer as in the first case.

Actual Problem: Consider the ODE

$$\frac{d^2 f}{dt^2} + f = -\epsilon \frac{df}{dt} \tag{3}$$

with initial conditions $f(0) = 1$ and $\frac{df}{dt}(0) = 0$. Write $f(t)$as the function $f(T_s, T_f)$ where $T_s = \epsilon t$ is the slow time and $T_f = t$ is the fast time. Compute $\frac{df}{dt}$, and $\frac{d^2 f}{dt^2}$, and use these to solve the ODE approximately, order-by-order in ϵ. Compare the answer to the exact solution.