Total $\frac{55}{70}$

# AM 213A: Homework 4 (Theory: Part 2)

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1. Show $\lim_{n\to\infty} ||A^n|| \iff \rho(A) \leq 1$

   $\frac{2}{10}$

   Consider the Schur Decomposition

$$A = QUQ^*.$$

   Then taking powers of $A$

$$A^n = QU^nQ^*.$$

   and since $U$ is upper triangular, successive powers yield successive powers on the diagonal. So after each power of $U$ we can write $U^n = D^n + T_n$ where $D$ is a diagonal matrix containing the eigenvalues of $A$ and $T_n$ is strictly upper triangular. Then taking the limit

   "only if"? -5

$$U \quad \ldots$$

   -3

2. Show $\begin{pmatrix} AB & 0 \\ B & 0 \end{pmatrix}$ and $\begin{pmatrix} 0 & 0 \\ B & BA \end{pmatrix}$ have the same eigenvalues.

   $\frac{7}{10}$

   Consider an eigenvector $\begin{pmatrix} u \\ v \end{pmatrix}$

   what do these tell you about v? -1

$$\begin{pmatrix} AB & 0 \\ B & 0 \end{pmatrix}\begin{pmatrix} u \\ v \end{pmatrix} = \lambda \begin{pmatrix} u \\ v \end{pmatrix}.$$

   And considering an eigenvector $\begin{pmatrix} \hat{u} \\ \hat{v} \end{pmatrix}$ of $\begin{pmatrix} 0 & 0 \\ B & BA \end{pmatrix}^T = \begin{pmatrix} 0 & B^T \\ 0 & A^TB^T \end{pmatrix}$, we get

   similarly, what do these say about $\hat{u}$?

$$\begin{pmatrix} 0 & B^T \\ 0 & A^TB^T \end{pmatrix}\begin{pmatrix} \hat{u} \\ \hat{v} \end{pmatrix} = \lambda \begin{pmatrix} \hat{u} \\ \hat{v} \end{pmatrix}.$$

   B not square, so doesn't have e-vals -1

   Since $B^T$ and $A^TB^T$ have the same eigenvalues of $B$ and $AB$ respectively, and $\begin{pmatrix} 0 & B^T \\ 0 & A^TB^T \end{pmatrix}$ has the same eigenvalues as $\begin{pmatrix} 0 & 0 \\ B & BA \end{pmatrix}$. This proves the result.

   These have different e-vals since A, B not square (They are diff. sizes.) -1

10/10

3. Simply apply Gerschgorin Theorem to the rows of $A^T$ and using the fact that a matrix $A$ and its transpose have the same eigenvalues. This shows we can sum of either the rows or the columns to put bounds on eigenvalues.

4. Use Gerschgorin to bound eigenvalues

   8/10

   Applying bounnds using partial sum on rows

$$
\begin{aligned}
|\lambda - 1| &\leq 0.3 + 0.1 + 0.4 = 0.8 \\
|\lambda - 2| &\leq 0.1 \\
|\lambda - 3| &\leq 0.4 \\
|\lambda - 4| &\leq 0.1
\end{aligned}
$$

   Now applying bounnds using partial sum over columns

   how do you know exactly one per circle? -2

$$
\begin{aligned}
|\lambda - 1| &\leq 0.1 \\
|\lambda - 2| &\leq \dots \\
|\lambda - 3| &\leq 0.1 \\
|\lambda - 4| &\leq \dots
\end{aligned}
$$

   Combining, for a tighter bound, we get

$$|\lambda - k| \leq 0.1, \; k = 1, 2, 3, 4.$$

5. Show $\lim_{k\to\infty} \frac{y^T A^{k+1} y}{y^T A^k y}$ converges.

   10/10

   Assuming $A$ is real positive definite, then $A$ can be diagonalized by a orthogonal matrix $Q$

$$A = QDQ^T.$$

   let $x = Q^T y$, then

$$\frac{y^T A^{k+1} y}{y^T A^k y} = \frac{x^T D^{k+1} x}{x^T D^k x}.$$

   moreover, since each eigenvalue is positive ($A$ positive definite), then we can order the eigenvalues

$$\lambda_1 > \dots > \lambda_n.$$

   then

$$\frac{x^T D^{k+1} x}{x^T D^k x} = \frac{\lambda_1^{k+1} x_1^2 + \dots \lambda_n^{k+1} x_n^2}{\lambda_1^k x_1^2 + \dots \lambda_n^k x_n^2}.$$

   taking the limit, $\lim_{k\to\infty}$, $\lambda_1$ dominates the numerator and denominator, we get

Minor point: what happens if $y \perp v_1$ with $v_1$ assoc. to $\lambda_1$?

$$\frac{\lambda_1^k \lambda_1 x_1^2}{\lambda_1^k x_1^2} = \lambda_1.$$

so the limit converges to the largest eigenvalue

$\frac{10}{10}$

6. $A$ real with nonnegative entries such that, for all $i$

$$\sum_{j=1}^{m} a_{ij} = 1.$$

By Gerschgorin theorem

$$\begin{aligned}|\lambda - a_{ii}| &\le \sum_{j\neq i} |a_{ij}| \\ &\le \sum_{j\neq i} a_{ij} \qquad a_{ij} \ge 0\end{aligned}$$

then

$$\begin{aligned}-\sum_{j\neq i} a_{ij} \le \lambda - a_{ii} &\le \sum_{j\neq i} a_{ij} \\ |\lambda| &\le \sum_{j\neq i} a_{ij} + a_{ii} = 1\end{aligned}$$

since $|\lambda| \le 1$, no eigenvalue (absolute value) can be greater than 1

7. $A$ a normal matrix

$\frac{8}{10}$

**a)** Since $A$ is normal, by the Spectral Theorem, it is ~~orthoganlly~~ diagonalizable. Namely, only unitarily

$V$ may be $\mathbb{C}$

$$A = VDV^T.$$

Where $V$ is an orthogonal matrix and $D$ is diagonal. The matrix $V$ also consists of the right ~~singular~~ vectors. It does not. $-1$

but then must be [illegible]

$$A^T A = VD^T DV^T = V\hat{D}V^T.$$

Where the singular values are defined as the positive roots of the eigenvalues of $A^T A$

$$A^T A = V\Sigma^T \Sigma V^T.$$

which in this case happen to be the eigenvalues of $A$ squared. We have

$$\begin{aligned}\sigma_i^2 &= \lambda_i^2 \quad |\lambda_i| \\ \sigma_i &= |\lambda_i|\end{aligned}$$

need $|\lambda_i|^2 = \bar{\lambda}_i \lambda_i \neq \lambda_i^2$ $-1$

since $\lambda_i \in \mathbb{C}$

**b)** By problem 7 in homework hw1

$$\rho(A) \leq ||A||_2.$$

Then by definition

$$\rho(A) = \max|\lambda_i| \leq \sigma_{\max} = \|A\|_2$$
$$\implies |\lambda_1| \leq \sigma_1$$

.

Then by part $(a)$, $\rho(A) = |\lambda_1| = \sigma_1 = ||A||_2$