Numerical Linear Algebra

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1: Assignment 1

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1 Problem 1

1.1 If $A \in \mathbb{C}$ is unitary and upper triangular, then A is diagonal

$A \in \mathbb{C}^{m\times m}$ unitary implies, by definition,

$$A^*A = AA^* = I.$$

If A is upper triangular, then

$$A^*A = \begin{pmatrix} \overline{a_{11}} & 0 & \dots & 0 \\ \overline{a_{12}} & \overline{a_{22}} & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ \overline{a_{1m}} & \overline{a_{2m}} & \dots & \overline{a_{mm}} \end{pmatrix} \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1m} \\ 0 & a_{22} & \dots & a_{2m} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & a_{mm} \end{pmatrix}$$

where $\overline{a_{ij}}$ is the conjugate of a_{ij}. Then the first column of $A^*A = \boldsymbol{e_1}$, where $\boldsymbol{e_1}$ is the standard basis vector,

$$\begin{pmatrix} \overline{a_{11}} & 0 & \dots & 0 \\ \overline{a_{12}} & \overline{a_{22}} & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ \overline{a_{1m}} & \overline{a_{2m}} & \dots & \overline{a_{mm}} \end{pmatrix} \begin{pmatrix} a_{11} \\ 0 \\ \vdots \\ 0 \end{pmatrix} = \boldsymbol{e_1}.$$

implies $|a_{11}|^2 = 1$, and

$$\overline{a_{1j}} = a_{1j} = 0 \qquad \text{for } j > i = 1.$$

Continuing for the remaining columns, we find that $|a_{ii}|^2 = 1$ and

$$\overline{a_{ij}} = a_{ij} = 0 \qquad \text{for } j > i = 2, 3, \dots, m.$$

This implies that A is diagonal. This would also hold if A is lower triangular and unitary since, $A^*A = AA^*$. $\square$.

2 Problem 2

2.1 If A is invertible and $\lambda \neq 0$ is an eigenvalue of A, then $\frac{1}{\lambda}$ is an eigenvalue of A^{-1}

Let (λ, v) be an eigenpair for $A \in \mathbb{C}$, then

$$\begin{aligned} Av &= \lambda v \\ A^{-1}Av &= \lambda A^{-1}v && \text{multiplying by } A^{-1} \\ \frac{1}{\lambda}v &= A^{-1}v \end{aligned}$$

$\Rightarrow \frac{1}{\lambda}$ is an eigenvalue of A^{-1}, and A and A^{-1} have the same eigenvectors. $\square$

2.2 Show $AB = BA$ have the same eigenvalues.

Let $Bv = w$ and (λ, v) be an eigenpair of AB, then

$$(AB)\,v = \lambda v$$

implies

$$Aw = \lambda v.$$

Multiplying both sides by B

$$\begin{aligned} (BA)\,w &= \lambda Bv \\ &= \lambda w \end{aligned}$$

$\Rightarrow \lambda$ is an eigenvalue of both AB and BA with different eigenvectors. $\square$

2.3 Show $A \in \mathbb{R}$ and A^* have the same eigenvalues

Let $\langle \cdot, \cdot \rangle$ be an inner product, and (λ, v) be an eigenpair of A, then

$$\begin{aligned} \langle Av, Av \rangle &= \lambda \langle v, Av \rangle \\ &= \lambda\bar{\lambda}\langle v, v \rangle \quad \text{conjugate linearity} \\ &= \lambda \langle A^*v, v \rangle \quad \text{definition of adjoint} \end{aligned}$$

$\Rightarrow A^*v = \bar{\lambda}v$. So $(\bar{\lambda}, v)$ is an eigenpair of A^*, and since complex eigenvalues of a real valued matrix come in conjugate pairs then both λ and $\bar{\lambda}$ are eigenvalues of A and A^*. Otherwise if $\lambda \in \mathbb{R}$, then $\lambda = \bar{\lambda}$, and hence they have the same eigenvalues. $\square$

3 Problem 3

Let $A \in \mathbb{C}$ be hermitian

3.1 Prove all eigenvalues of A are real

$\langle Av, v\rangle = \lambda\langle v, v\rangle \underbrace{=}_{\text{definition of Adjoint}} \langle v, A^*v\rangle \underbrace{=}_{A \text{ hermitian}} \langle v, Av\rangle \underbrace{=}_{\text{conjugate symmetry}} \bar{\lambda}\langle v, v\rangle.$

$\Rightarrow \lambda = \bar{\lambda} \Rightarrow \lambda \in \mathbb{R}$ Since the only way λ equals its conjugate is if it is real.

3.2 If x and y are eigenvectors corresponding to distinct eigenvalues, show they are orthogonal

$$\langle Ax, y\rangle = \lambda_x\langle x, y\rangle \underbrace{=}_{\text{Hermitian}} \langle x, Ay\rangle \underbrace{=}_{\lambda_y \in \mathbb{R}} \lambda_y\langle x, y\rangle.$$

Since $\lambda_x \neq \lambda_y$ (distinct), then $\langle x, y\rangle = 0$

4 Problem 4

Show that a hermitian matrix A is positive definite iff $\lambda_i > 0$ for all λ_i in the spectrum of A

Since A is hermitian, then A is unitarily diagonalizable with real eigenvalues, i.e. $A = UDU^*$ where D is diagonal and U is unitary, ($U^*U = I$). Considering change of basis $y = U^*x$ into the basis where A is diagonal, then $y^* = x^*U$ and

$$\begin{aligned}
\langle Ax, x\rangle &= \langle UDU^*x, x\rangle \\
&= y^*Dy \\
&= \sum_{i=1}^{m} \lambda_i y^*y \\
&= \sum_{i=1}^{m} \lambda_i \|y\|^2
\end{aligned}$$

Since $\|y\|^2 > 0$ for $x \neq 0$, then $\langle Ax, x\rangle > 0$ if and only if $\lambda_i > 0$

5 Problem 5

Let $A \in \mathbb{C}$ be unitary

5.1 Let (λ, x) be an eigenpair of A, show $|\lambda| = 1$

A unitary implies $A^*A = I$

$$\langle Av, Av\rangle = \lambda\bar{\lambda}\langle v, v\rangle = |\lambda|^2\langle v, v\rangle = \langle v, A^*Av\rangle = \langle v, v\rangle.$$

$\Rightarrow |\lambda| = 1$

5.2 Prove or disprove $\|A\|_F = 1$

$$\|A\|_F^2 = Tr(A^*A) = Tr(I) = m \neq 1.$$

6 Problem 6

Let $A \in \mathbb{C}$ be skew-hermition

6.1 Show eigenvalues of A are pure imaginary

$$\langle Av, v\rangle = \underbrace{\lambda\langle v, v\rangle} = \langle v, A^*\rangle = \langle v, -Av\rangle = \underbrace{-\bar{\lambda}\langle v, v\rangle}.$$

$\Rightarrow \lambda = -\bar{\lambda} \Rightarrow \lambda \in \mathbb{C}$ is pure imaginary

6.2 Show $I - A$ is nonsingular

Consider $v \in \ker(I - A)$

$$\begin{aligned}(I - A)v &= 0\\ Iv = v &= Av\end{aligned}$$

Then,

$$\langle v, v\rangle = \langle Av, v\rangle = \langle v, A^*v\rangle = -\langle v, v\rangle.$$

Which is only possible if $v = 0$. This shows the kernal is trivial, and hence $I - A$ is nonsingular.

7 Problem 7

Show $\rho(A) \leq \|A\|$, where $\rho(A)$ is the spectral radius of A

Spectral radius $\rho(A) = \max\{|\lambda_1|, |\lambda_2|, \ldots, |\lambda_n|\}$.

$$\begin{aligned}\|Ax\|^2 = \langle Ax, Ax\rangle &= \lambda\bar{\lambda}\langle x, x\rangle\\ &= |\lambda|^2\cancel{\|x\|^2} \quad \underbrace{\leq}_{\text{property of induced norm}} \quad = \|A\|^2\cancel{\|x\|^2}\end{aligned}$$

$\Rightarrow |\lambda| \leq \|A\| \ \forall\lambda \Rightarrow \rho(A) \leq \|A\|$

8 Problem 8

Let A be defined by the inner product $A = uv^*$

8.1 Prove of disprove $\|A\|_2 = \|u\|_2\|v\|_2$

Consider A^*Av

$$\begin{aligned} A^*Av &= vu^*uv^*v \\ &= \|u\|_2^2 v(v^*v) \\ &= \|u\|_2^2\|v\|_2^2 v \end{aligned}$$

This shows v is the only eigenvector of A^*A, and since A is rank 1, this is the only singular value (squared). So

$$\Rightarrow \|A\|_2 = \sigma_{\max} = \sigma = \|u\|_2\|v\|_2.$$

8.2 Prove or disprove $\|A\|_F = \|u\|_F\|v\|_F$

Since A is rank 1 the equality holds. Where the only singular value of A is $\sigma = \|u\|_2\|v\|_2$. Since, the frobenius norm is the same as the two norm for vectors, then

$$\|A\|_F = \sqrt{Tr(A^*A)} = \sum_{i=1}^{m}\sqrt{\sigma_i^2} = \sigma = \|u\|_2\|v\|_2 = \|u\|_F\|v\|_F = \|A\|_2.$$

9 Problem 9

$A, Q \in \mathbb{C}$ where A is arbitrary and Q is unitary

9.1 Show $\|AQ\|_2 = \|A\|_2$

Definition of 2-norm, and Q unitary, its easy to see that $\|QA\|_2$

$$\|QA\|_2 = \sqrt{\lambda_{\max}(A^*Q^*QA)} = \sqrt{\lambda_{\max}(A^*A)} = \sigma_{\max}(A) = \|A\|_2.$$

If we let $B = QA$, then noting that $B * B$, and BB^* are positive definite

$$\begin{aligned} \langle B^*Bx, x\rangle &= \langle Bx, Bx\rangle > 0 \\ \langle BB^*x, x\rangle &= \langle B^*x, B^*x\rangle > 0 \end{aligned}$$

for $x \neq 0$, then referencing problem (10.1)

$$\begin{aligned} \|AQ\|_2 = \sqrt{\lambda_{\max}(Q^*A^*AQ)} = \sqrt{\lambda_{\max}(B^*B)} &= \sigma_{\max}(B^*B) \\ &= \sigma_{\max}(BB^*) \\ &= \|AQ\|_2 \\ &= \|A\|_2 \end{aligned}$$

9.2 Show $\|AQ\|_F = \|QA\|_F = \|A\|_F$

First it's easy to show $\|QA\|_F = \|A\|_F$

$$\|QA\|_F = \sqrt{\text{trace}(A^*Q^*QA)} = \sqrt{\text{trace}(A^*A)} = \|A\|_F.$$

then using the cyclic nature of the trace

$$\|AQ\|_F = \sqrt{\text{trace}(Q^*A^*AQ)} = \sqrt{\text{trace}(QQ^*A^*A)}\sqrt{\text{trace}(A^*A)} = \|A\|_F.$$

10 Problem 10

10.1 Show that if A and B are unitarily equivalent, then they have the same singular values.

Unitarily equivalent means $A = QBQ^*$ for some unitary $Q \in \mathbb{C}$.

Since A is square and has SVD, write $A = U\Sigma V^*$. Then.

$$\begin{aligned} A = U\Sigma V^* &= QBQ^* \\ Q^*U\Sigma V^*Q &= B \\ \hat{U}\Sigma\hat{V}^* &= B \end{aligned}$$

Which forms the SVD of B. Hence A and B have the same singular values. This can be seen by noting $\hat{U}$ and $\hat{V}$ form the unitary eigendecomposition of BB^* and B^*B respectively. i.e.

$$\begin{aligned} BB^* &= Q^*U\Sigma^2U^*Q \\ &= \hat{U}\Sigma^2\hat{U}^* && \text{and,} \\ B^*B &= Q^*V\Sigma^2V^*Q \\ &= \hat{V}\Sigma^2\hat{V}^* \end{aligned}$$

10.2 Show the converse is not necessarily true

11 Problem 11

Find the relative condition number of the following functions and discuss if there is any concern of being ill-conditioned

11.1 $f(x_1, x_2) = x_1 + x_2$

The Jacobian is $Jf = \begin{pmatrix} 1 & 1 \end{pmatrix}$. Using the infinity norm the relative condition number is

$$\kappa = \frac{\|Jf(x)\|_\infty \|x\|_\infty}{\|f(x)\|_\infty} = \frac{2\max\{|x_1|, |x_2|\}}{|x_1 + x_2|}.$$

This is ill-conditioned for $x_1 \longrightarrow -x_2$

11.2 $f(x_1, x_2) = x_1 x_2$

The Jacobian is $Jf = \begin{pmatrix} x_2 & x_1 \end{pmatrix}$. Using the infinity norm the relative condition number is.

$$\kappa = \frac{\|Jf(x)\|_\infty \|x\|_\infty}{\|f(x)\|_\infty} = \frac{(|x_2| + |x_1|) \max\{|x_1|, |x_2|\}}{|x_1 x_2|}.$$

This is ill-conditioned for x_1 or $x_2 \longrightarrow 0$

11.3 $f(x) = (x-2)^9$

The Jacobian is $Jf = 9(x-2)^8$. Using the infinity norm, the relative condition number is.

$$\begin{aligned} \kappa &= \frac{|9(x-2)^8||x|}{|(x-2)^9|} \\ &= \frac{|9x|}{|x-2|} \frac{|x+2|}{|x+2|} \\ &= \frac{|9x^2 + 18x|}{x^2 + 4} \end{aligned}$$

Which, after simplifying, we see is not ill-conditioned.

12 Problem 12

12.1 Plot $f(x)$

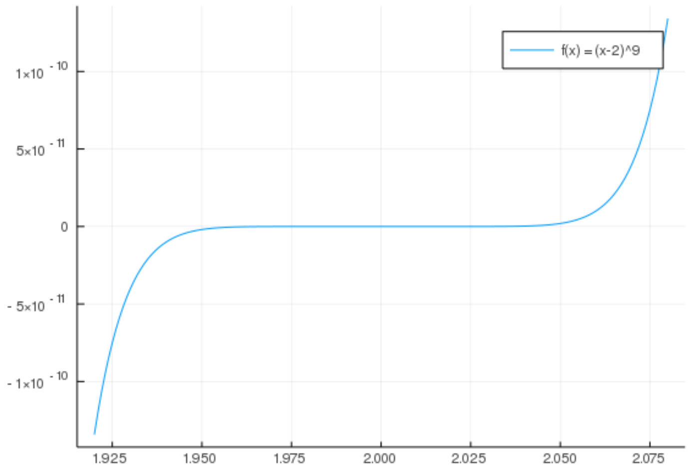

Figure 1: Plot f

12.2 Plot $g(x)$

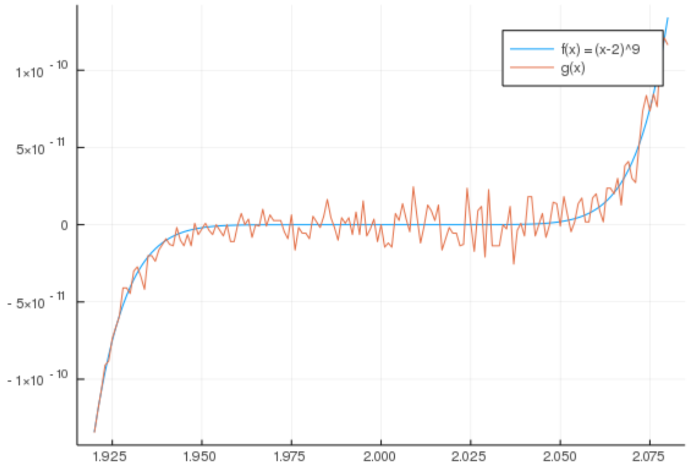

Figure 2: Plot f and g

12.3 Conclusion

It appears the expanded form of $g(x)$ is unable to remove the discontinuity at $x = 2$