

Numerical Linear Algebra

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2: Assignment 2

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1 Part 1

1.1 Output for Trace, Gaussian Elimination, LU decomposition and Two norm of Error in solution to $AX = B$

A

4	x	4		
2.0000		1.0000	1.0000	0.0000
4.0000		3.0000	3.0000	1.0000
8.0000		7.0000	9.0000	5.0000
6.0000		7.0000	9.0000	8.0000

Trace of A

```

22.000000000000000

Norm column:      1
 10.954451150103322
Norm column:      2
 10.392304845413264
Norm column:      3
 13.114877048604001
Norm column:      4
  9.4868329805051381

B

 4      x      6
 3.0000      0.0000      1.0000      0.9000      2.1000      3.1416
 6.0000     -2.0000      1.0000     10.4000    -491.2000    -4.7124
10.0000      2.0000      0.0000    -20.2000      0.1200      2.2440
 1.0000     10.0000     -5.0000     -5.1200    -51.3000      2.3562

matrix A after gauss

 4      x      4
 8.0000      7.0000      9.0000      5.0000
 0.0000      1.7500      2.2500      4.2500
 0.0000      0.0000     -0.8571     -0.2857
 0.0000      0.0000      0.0000      0.6667

matrix B after gauss

 4      x      6
10.0000      2.0000      0.0000    -20.2000      0.1200      2.2440
 -6.5000      8.5000     -5.0000     10.0300    -51.3900      0.6732
 -0.8571     -0.5714     -0.4286     23.3657   -505.9429    -5.6420
 -2.0000      3.3333     -1.0000      2.4600    148.6933      4.7498

F

solution X to AX = B

 4      x      6
 -0.0000      3.5000      0.2500     -2.0050    360.2700     10.6309
  1.0000     -6.0000     -0.5000     33.4000  -1234.3600    -22.3277
  2.0000     -1.0000      1.0000    -28.4900    515.9200      4.2075
 -3.0000      5.0000     -1.5000      3.6900    223.0400      7.1247

```

Error Matrix

4	x	6				
-0.0000		0.0000	-0.0000	-0.0000	-0.0000	0.0000
0.0000		0.0000	0.0000	-0.0000	-0.0000	0.0000
0.0000		0.0000	0.0000	-0.0000	-0.0000	-0.0000
0.0000		0.0000	0.0000	-0.0000	-0.0000	0.0000

Two norm of columns of error matrix

Norm column: 1
2.6645352591003757E-015
Norm column: 2
2.5727487310015434E-015
Norm column: 3
9.0153937873535153E-015
Norm column: 4
2.9913803108461275E-013
Norm column: 5
3.6757338276243205E-013
Norm column: 6
7.3777640556099254E-015

A before LU

4	x	4		
2.0000		1.0000	1.0000	0.0000
4.0000		3.0000	3.0000	1.0000
8.0000		7.0000	9.0000	5.0000
6.0000		7.0000	9.0000	8.0000

A after LU

4	x	4		
8.0000		7.0000	9.0000	5.0000
0.7500		1.7500	2.2500	4.2500
0.5000		-0.2857	-0.8571	-0.2857
0.2500		-0.4286	0.3333	0.6667

Permutation vector

4			
3	4	2	1

Solution $AX = B$ using LU decomposition:

4	x	6				
-0.0000	3.5000	0.2500	-2.0050	360.2700	10.6309	
1.0000	-6.0000	-0.5000	33.4000	-1234.3600	-22.3277	
2.0000	-1.0000	1.0000	-28.4900	515.9200	4.2075	
-3.0000	5.0000	-1.5000	3.6900	223.0400	7.1247	

Two norm of error matrix using LU decomposition

Norm column:	1
	8.8817841970012523E-016
Norm column:	2
	3.6620534388177900E-015
Norm column:	3
	1.8322380275993532E-014
Norm column:	4
	2.1609931492617831E-013
Norm column:	5
	3.8138914537411967E-013
Norm column:	6
	7.3777640556099254E-015

1.2 Plane Equation

The equation of a plane can be solved for a, b, c, d using the points $A(1, 2, 3)$, $B(-3, 2, 5)$, and $C(\pi, e, -\sqrt{2})$, by solving the following equation for

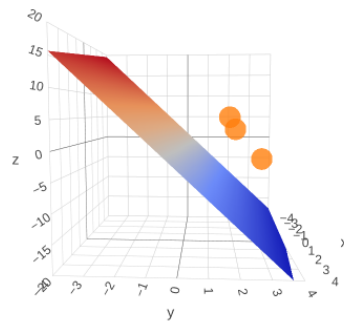
$$\begin{aligned} a + 2b + 3c + d &= 0 \\ -3a + 2b + 5c + d &= 0 \\ \pi a + eb - \sqrt{2}c + d &= 0 \end{aligned}$$

3	x	4		
1.0000	2.0000	3.0000	1.0000	
-3.0000	2.0000	5.0000	1.0000	
3.1416	2.7183	-1.4142	1.0000	

4	x	1
0.0000		
0.0000		
0.0000		
0.0000		

matrix after gauss

3	x	4		
3.1416	2.7183	-1.4142	1.0000	
0.0000	4.5958	3.6495	1.9549	
0.0000	0.0000	2.5491	0.1990	



Theory total
46/50

10/10

2 Part 2

2.1 Schur decomposition of a symmetric matrix

If $A \in \mathbb{C}^{m \times m}$, then there exists a unitary matrix Q and an upper triangular matrix U such that $A = QUQ^*$.

If A is (real) symmetric then

$$A = QUQ^T = QU^TQ^T = A^T.$$

This implies

$$U = U^T.$$

and the only way this can happen is if $U = D$ is diagonal, so the Schur decomposition implies that A is orthogonally diagonalizable

$$A = QDQ^T.$$

where the columns of Q form an eigenbasis for A , and Q^T is the change of basis matrix that transforms A into a diagonal matrix D of eigenvalues of A .

2.2 Stability of Gaussian elimination

Consider

$$\begin{pmatrix} 1 & 1 \\ \varepsilon & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}.$$

Multiply the last row by c so that $c\varepsilon \gg 1$, then swapping row one with row two. The augmented system becomes

$$\begin{pmatrix} c\varepsilon & c & c \\ 1 & 1 & 2 \end{pmatrix}.$$

the elimination step

$$\begin{pmatrix} c\varepsilon & c & c \\ 0 & 1 - \frac{1}{\varepsilon} & 2 - \frac{1}{\varepsilon} \end{pmatrix}.$$

which is numerically equivalent to

$$\begin{pmatrix} c\varepsilon & c & c \\ 0 & -\varepsilon^{-1} & \varepsilon^{-1} \end{pmatrix}.$$

and leads to the incorrect solution

$$y = \frac{-\varepsilon^{-1}}{-\varepsilon^{-1}} = 1 \quad x = \frac{1-1}{\varepsilon} = 0.$$

The issue partial pivoting hoped to correct was reintroduced by scaling the row with a smaller pivot.

How would you improve the method?

2.3 Diagonal entries of a symmetric positive definite matrix

If A is symmetric positive definite then for all v

$$v^T A v > 0.$$

Considering the standard basis vectors e_j

$$e_j^T A e_j = a_{jj}.$$

Since any vector v can be written as a linear combination of these basis vectors

$$v = v_1 e_1 + \cdots + v_n e_n.$$

then

$$v^T A v = \sum_{j=1}^n a_{jj} v_j^2 > 0.$$

implies $a_{jj} > 0$

$\frac{10}{10}$

$\frac{9}{10}$

These 2 lines are all you need

To get this -1

Don't need this.

not true

2.4 LU decomposition of a block matrix

2.4.1 Verify the formula

$$\begin{pmatrix} I & 0 \\ -A_{21}A_{11}^{-1} & I \end{pmatrix} \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} = \begin{pmatrix} A_{11} & A_{12} \\ -A_{21}A_{11}^{-1}A_{11} + A_{21} & -A_{21}A_{11}^{-1}A_{12} + A_{22} \end{pmatrix} \\ = \begin{pmatrix} A_{11} & A_{12} \\ 0 & A_{22} - A_{21}A_{11}^{-1}A_{12} \end{pmatrix} \quad \checkmark$$

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2.4.2 Show $D = A_{22} - A_{21}A_{11}^{-1}A_{12}$ after n steps of Gaussian elimination

$$\begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} \Rightarrow \begin{pmatrix} A_{11} & C \\ 0 & D \end{pmatrix}.$$

Interesting approach. should elaborate on how A_{11}^{-1} flows up via GE algo.

First defining the matrix R_i as the matrix that contains the negation of the i^{th} row of A_{21} and zeros everywhere else. Then performing n steps of Gaussian elimination on A using block elementary matrices

$$E_i = \begin{pmatrix} I & 0 \\ R_i A_{11}^{-1} & I \end{pmatrix}.$$

eliminates the block matrix A_{21}

$$E_n \dots E_1 A = \begin{pmatrix} I & 0 \\ R_n A_{11}^{-1} & I \end{pmatrix} \dots \begin{pmatrix} I & 0 \\ R_1 A_{11}^{-1} & I \end{pmatrix} \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} = \begin{pmatrix} A_{11} & C \\ 0 & D \end{pmatrix}$$

where

$$E_n \dots E_1 = \begin{pmatrix} I & 0 \\ R_n A_{11}^{-1} & I \end{pmatrix} \dots \begin{pmatrix} I & 0 \\ R_1 A_{11}^{-1} & I \end{pmatrix} \\ = \begin{pmatrix} I & 0 \\ R_n A_{11}^{-1} + \dots + R_1 A_{11}^{-1} & I \end{pmatrix} \quad \checkmark \\ = \begin{pmatrix} I & 0 \\ (R_n + \dots + R_1) A_{11}^{-1} & I \end{pmatrix} \\ = \begin{pmatrix} I & 0 \\ -A_{21} A_{11}^{-1} & I \end{pmatrix}$$

which is exactly what we used to verify the formula in (2.4.1)

2.5 $Ax = b$ complex valued

2.5.1 Modify problem

Decompose $A = A_1 + iA_2$ and $b = b_1 + ib_2$ then

$$Ax = b \\ (A_1 + iA_2)x = b_1 + ib_2$$

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This neglects cross terms
e.g. $(iA_2)(ix_2)$ is real.
-2

equating the real and imaginary parts this is equivalent to solving

$$A_1 x_1 = b_1 \quad \cancel{A_2 x_2 = b_2}.$$

where $x_1 = \text{Re}(x)$, $x_2 = \text{Im}(x)$, and both systems are real

Total $2m \times 2m$ system? -1

2.5.2 Compare storage and number of floating point operations

Consider complex numbers a_{ij}/a_{jj} used in Gaussian elimination

$$\frac{a_{ij}}{a_{jj}} = \frac{(a, b)}{(c, d)} = \frac{(ac + bd, -ad + bc)}{c^2 + d^2}.$$

this introduces 6 multiplications and 3 additions, compared to just 1 operation if they were real. The storage requirements for the complex case $Ax = b$ is the same as for the two systems $A_1 x_1 = b_1$ and $A_2 x_2 = b_2$.

Wrong, but consistent w/ part a answer.

Division is a small part of total FPE cost.

What is net cost of the version? -2
& version? -2