Total 55

AM 213A: Homework 4 (Theory: Part 2)

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1. Show $\lim_{n\to\infty} ||A^n|| \iff \rho(A) \le 1$ Consider the Schur Decomposition

$$A = QUQ^*.$$

Then taking powers of A

$$A^n = QU^nQ^*$$
.

and since U is upper triangular, successive powers yield successive powers on the diagonal. So after each power of U we can write $U^n = D^n + T_n$ where D is a diagonal matrix containing the eigenvalues of A and T_n is strictly upper triangular. Then taking the limit

"only it'?

2. Show
$$\begin{pmatrix} AB & 0 \\ B & 0 \end{pmatrix}$$
 and $\begin{pmatrix} 0 & 0 \\ B & BA \end{pmatrix}$ have the same eigenvalues.

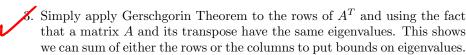
Consider an eigenvector $\begin{pmatrix} u \\ v \end{pmatrix}$ where $\begin{pmatrix} aB & 0 \\ B & 0 \end{pmatrix}\begin{pmatrix} u \\ v \end{pmatrix} = \lambda \begin{pmatrix} u \\ v \end{pmatrix}$.

And considering an eigenvector $\begin{pmatrix} \hat{u} \\ \hat{v} \end{pmatrix}$ of $\begin{pmatrix} 0 & 0 \\ B & BA \end{pmatrix}^T = \begin{pmatrix} 0 & B^T \\ 0 & A^T B^T \end{pmatrix}$, we get

$$\begin{pmatrix} 0 & B^T \\ 0 & A^T B^T \end{pmatrix} \begin{pmatrix} \hat{u} \\ \hat{v} \end{pmatrix} = \lambda \begin{pmatrix} \hat{u} \\ \hat{v} \end{pmatrix}.$$

Since B^T and A^TB^T have the same eigenvalues of B and AB respectively, and $\begin{pmatrix} 0 & B^T \\ 0 & A^TB^T \end{pmatrix}$ has the same eigenvalues as $\begin{pmatrix} 0 & 0 \\ B & BA \end{pmatrix}$. This proves

These we like our sque (They are diff. Sizes.)



4. Use Gerschgorin to bound eigenvalues

Applying bounnds using partial sum on rows

$$|\lambda - 1| \le 0.3 + 0.1 + 0.4 = 0.8$$

 $|\lambda - 2| \le 0.1$
 $|\lambda - 3| \le 0.4$
 $|\lambda - 4| \le 0.1$

Now applying bounnds using partial sum over columns

$$\begin{aligned} |\lambda - 1| &\leq 0.1 \\ |\lambda - 2| &\leq \dots \\ |\lambda - 3| &\leq 0.1 \\ |\lambda - 4| &\leq \dots \end{aligned}$$

Combining, for a tighter bound, we get

$$|\lambda - k| \le 0.1, \ k = 1, 2, 3, 4.$$

5. Show $\lim_{k\to\infty} \frac{y^T A^{k+1} y}{y^T A^k y}$ converges.

Assuming A is real positive definite, then A can be diagonalized by a orthogonal matrix Q

 $A = QDQ^T$.

let
$$x = Q^T y$$
, then

$$\frac{y^T A^{k+1} y}{y^T A^k y} = \frac{x^T D^{k+1} x}{x^T D^k x}.$$

moreover, since each eigenvalue is positive (A positive definite), then we can order the eigenvalues

$$\lambda_1 > \cdots > \lambda_n$$
.

then

$$\frac{x^T D^{k+1} x}{x^T D^k x} = \frac{\lambda_1^{k+1} x_1^2 + \dots \lambda_n^{k+1} x_n^2}{\lambda_1^k x_1^2 + \dots \lambda_n^k x_n^2}.$$

taking the limit, $\lim_{k\to\infty}$, λ_1 dominates the numerator and denominator,

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$$\frac{\lambda_1^k \lambda_1 x_1^2}{\lambda_1^k x_1^2} = \lambda_1.$$

so the limit converges to the largest eigenvalue

6. A real with nonnegative entries such that, for all i

$$\sum_{j=1}^{m} a_{ij} = 1.$$

By Gerschgorin theorem

$$|\lambda - a_{ii}| \le \sum_{j \ne i} |a_{ij}|$$

$$\le \sum_{j \ne i} a_{ij} \qquad a_{ij} \ge 0$$

then

$$-\sum_{j\neq i} a_{ij} \le \lambda - a_{ii} \le \sum_{j\neq i} a_{ij}$$
$$|\lambda| \le \sum_{j\neq i} a_{ij} + a_{ii} = 1$$

since $|\lambda| \leq 1$, no eigenvalue (absolute value) can be greater than 1

- 7. A a normal matrix
 - a) Since A is normal, by the Spectral Theorem, it is orthoganly diago-Duly unitaily nalizable. Namely,

Where V is an orthogonal matrix and D is diagonal. The matrix V also consists of the right singular vectors.

$$A^T A \neq V D^T D V^T = V \hat{D} V^T.$$

Where the singular values are defined as the positive roots of the eigenvalues of $A^T A$

$$A^T A = V \Sigma^T \Sigma V^T.$$

which in this case happen to be the eigenvalues of A squared. We have

be the eigenvalues of
$$A$$
 squared. We
$$\sigma_{i}^{2} = |\lambda_{i}^{2}| \qquad \text{lead} \qquad |\lambda_{i}|^{2} = |\lambda_{i}| + |\lambda_{i}|^{2}$$

$$\sigma_{i} = |\lambda_{i}| \qquad \text{since } \lambda_{i} \in \mathbb{C}$$

V my be &

b) By problem 7 in homework hw1

$$\rho(A) \le ||A||_2.$$

Then by definition

$$\rho(A) = \max |\lambda_i| \le \sigma_{\max} = ||A||_2$$

$$\implies |\lambda_1| \le \sigma_1$$

Then by part (a), $\rho(A) = |\lambda_1| = \sigma_1 = ||A||_2$