Numerical Linear Algebra

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Contents

2: Assignment 2

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1 Part 1

1.1 Output for Trace, Gaussian Elimination, LU decomposition and Two norm of Error in solution to AX = B

```
A


  4    x    4
   2.0000      1.0000      1.0000      0.0000
   4.0000      3.0000      3.0000      1.0000
   8.0000      7.0000      9.0000      5.0000
   6.0000      7.0000      9.0000      8.0000


Trace of A
```

```
  22.000000000000000

Norm column:           1
  10.954451150103322
Norm column:           2
  10.392304845413264
Norm column:           3
  13.114877048604001
Norm column:           4
  9.4868329805051381

B

   4    x    6
    3.0000     0.0000     1.0000     0.9000     2.1000     3.1416
    6.0000    -2.0000     1.0000    10.4000  -491.2000    -4.7124
   10.0000     2.0000     0.0000   -20.2000     0.1200     2.2440
    1.0000    10.0000    -5.0000    -5.1200   -51.3000     2.3562



matrix A after gauss

   4    x    4
    8.0000     7.0000     9.0000     5.0000
    0.0000     1.7500     2.2500     4.2500
    0.0000     0.0000    -0.8571    -0.2857
    0.0000     0.0000     0.0000     0.6667



matrix B after gauss

   4    x    6
   10.0000     2.0000     0.0000   -20.2000     0.1200     2.2440
   -6.5000     8.5000    -5.0000    10.0300   -51.3900     0.6732
   -0.8571    -0.5714    -0.4286    23.3657  -505.9429    -5.6420
   -2.0000     3.3333    -1.0000     2.4600   148.6933     4.7498

F


solution X to AX = B

   4    x    6
   -0.0000     3.5000     0.2500    -2.0050   360.2700    10.6309
    1.0000    -6.0000    -0.5000    33.4000 -1234.3600   -22.3277
    2.0000    -1.0000     1.0000   -28.4900   515.9200     4.2075
   -3.0000     5.0000    -1.5000     3.6900   223.0400     7.1247
```

```
Error Matrix

   4    x    6
   -0.0000     0.0000    -0.0000    -0.0000    -0.0000     0.0000
    0.0000     0.0000     0.0000    -0.0000    -0.0000     0.0000
    0.0000     0.0000     0.0000    -0.0000    -0.0000    -0.0000
    0.0000     0.0000     0.0000    -0.0000    -0.0000     0.0000



Two norm of columns of error matrix

Norm column:           1
  2.6645352591003757E-015
Norm column:           2
  2.5727487310015434E-015
Norm column:           3
  9.0153937873535153E-015
Norm column:           4
  2.9913803108461275E-013
Norm column:           5
  3.6757338276243205E-013
Norm column:           6
  7.3777640556099254E-015



A before LU

   4    x    4
    2.0000     1.0000     1.0000     0.0000
    4.0000     3.0000     3.0000     1.0000
    8.0000     7.0000     9.0000     5.0000
    6.0000     7.0000     9.0000     8.0000

A after LU

   4    x    4
    8.0000     7.0000     9.0000     5.0000
    0.7500     1.7500     2.2500     4.2500
    0.5000    -0.2857    -0.8571    -0.2857
    0.2500    -0.4286     0.3333     0.6667

Permutation vector

   4
  3   4   2   1
```

```
Solution AX = B using LU decomposition:

   4     x     6
   -0.0000      3.5000      0.2500     -2.0050    360.2700     10.6309
    1.0000     -6.0000     -0.5000     33.4000 -1234.3600    -22.3277
    2.0000     -1.0000      1.0000    -28.4900    515.9200      4.2075
   -3.0000      5.0000     -1.5000      3.6900    223.0400      7.1247


Two norm of error matrix using LU decomposition

Norm column:           1
  8.8817841970012523E-016
Norm column:           2
  3.6620534388177900E-015
Norm column:           3
  1.8322380275993532E-014
Norm column:           4
  2.1609931492617831E-013
Norm column:           5
  3.8138914537411967E-013
Norm column:           6
  7.3777640556099254E-015
```

1.2 Plane Equation

The equation of a plane can be solved for a, b, c, d using the points $A(1, 2, 3)$, $B(-3, 2, 5)$, and $C(\pi, e, -\sqrt{2})$, by solving the following equation for

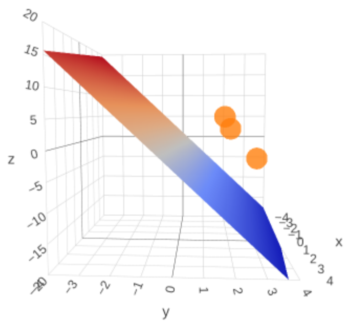
$$\begin{aligned} a + 2b + 3c + d &= 0 \\ -3a + 2b + 5c + d &= 0 \\ \pi a + eb - \sqrt{2}c + d &= 0 \end{aligned}$$

```
   3     x     4
    1.0000      2.0000      3.0000      1.0000
   -3.0000      2.0000      5.0000      1.0000
    3.1416      2.7183     -1.4142      1.0000


   4     x     1
    0.0000
    0.0000
    0.0000
    0.0000
```

```
matrix after gauss

   3    x    4
    3.1416      2.7183     -1.4142      1.0000
    0.0000      4.5958      3.6495      1.9549
    0.0000      0.0000      2.5491      0.1990
```



2 Part 2

2.1 Schur decomposition of a symmetric matrix

If $A \in \mathbb{C}^{m \times m}$, then there exists a unitary matrix Q and an upper triangular matrix U such that $A = QUQ^*$.

If A is (real) symmetric then

$$A = QUQ^T = QU^TQ^T = A^T.$$

This implies

$$U = U^T.$$

and the only way this can happen is if $U = D$ is diagonal, so the Schur decomposition implies that A is orthogonally diagonalizable

$$A = QDQ^T.$$

where the columns of Q form an eigenbasis for A, and Q^T is the change of basis matrix that transforms A into a diagonal matrix D of eigenvalues of A.

2.2 Stability of Gaussian elimination

Consider

$$\begin{pmatrix} 1 & 1 \\ \varepsilon & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}.$$

Multiply the last row by c so that $c\varepsilon \gg 1$, then swapping row one with row two. The augmented system becomes

$$\begin{pmatrix} c\varepsilon & c & c \\ 1 & 1 & 2 \end{pmatrix}.$$

the elimination step

$$\begin{pmatrix} c\varepsilon & c & c \\ 0 & 1 - \frac{1}{\varepsilon} & 2 - \frac{1}{\varepsilon} \end{pmatrix}.$$

which is numerically equivalent to

$$\begin{pmatrix} c\varepsilon & c & c \\ 0 & -\varepsilon^{-1} & \varepsilon^{-1} \end{pmatrix}.$$

and leads to the incorrect solution

$$y = \frac{-\varepsilon^{-1}}{-\varepsilon^{-1}} = 1 \qquad x = \frac{1-1}{\varepsilon} = 0.$$

The issue partial pivoting hoped to correct was reintroduced by scaling the row with a smaller pivot.

2.3 Diagonal entries of a symmetric positive definite matrix

If A is symmetric positive definite then for all v

$$v^T A v > 0.$$

Considering the standard basis vectors e_j

$$e_j^T A e_j = a_{jj}.$$

Since any vector v can be written as a linear combination of these basis vectors

$$v = v_1 e_1 + \cdots + v_n e_n.$$

then

$$v^T A v = \sum_{j=1}^{n} a_{jj} v_j^2 > 0.$$

implies $a_{jj} > 0$

2.4 LU decomposition of a block matrix

2.4.1 Verify the formula

$$\begin{aligned}
\begin{pmatrix} I & 0 \\ -A_{21}A_{11}^{-1} & I \end{pmatrix}\begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} &= \begin{pmatrix} A_{11} & A_{12} \\ -A_{21}A_{11}^{-1}A_{11} + A_{21} & -A_{21}A_{11}^{-1}A_{12} + A_{22} \end{pmatrix} \\
&= \begin{pmatrix} A_{11} & A_{12} \\ 0 & A_{22} - A_{21}A_{11}^{-1}A_{12} \end{pmatrix}
\end{aligned}$$

2.4.2 Show $\mathbf{D = A_{22} - A_{21}A_{11}^{-1}A_{12}}$ after n steps of Gaussian elimination

$$\begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} \Rightarrow \begin{pmatrix} A_{11} & C \\ 0 & D \end{pmatrix}.$$

First defining the matrix R_i as the matrix that contains the negation of the i^{th} row of A_{21} and zeros everywhere else. Then performing n steps of Gaussian elimination on A using block elementary matrices

$$E_i = \begin{pmatrix} I & 0 \\ R_iA_{11}^{-1} & I \end{pmatrix}.$$

eliminates the block matrix A_{21}

$$E_n \dots E_1 A = \begin{pmatrix} I & 0 \\ R_nA_{11}^{-1} & I \end{pmatrix} \cdots \begin{pmatrix} I & 0 \\ R_1A_{11}^{-1} & I \end{pmatrix}\begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} = \begin{pmatrix} A_{11} & C \\ 0 & D \end{pmatrix}$$

where

$$\begin{aligned}
E_n \dots E_1 &= \begin{pmatrix} I & 0 \\ R_nA_{11}^{-1} & I \end{pmatrix} \cdots \begin{pmatrix} I & 0 \\ R_1A_{11}^{-1} & I \end{pmatrix} \\
&= \begin{pmatrix} I & 0 \\ R_nA_{11}^{-1} + \cdots + R_1A_{11}^{-1} & I \end{pmatrix} \\
&= \begin{pmatrix} I & 0 \\ (R_n + \cdots + R_1)A_{11}^{-1} & I \end{pmatrix} \\
&= \begin{pmatrix} I & 0 \\ -A_{21}A_{11}^{-1} & I \end{pmatrix}
\end{aligned}$$

which is exactly what we used to verify the formula in (2.4.1)

2.5 $\mathbf{Ax = b}$ complex valued

2.5.1 Modify problem

Decompose $A = A_1 + iA_2$ and $b = b_1 + ib_2$ then

$$\begin{aligned}
Ax &= b \\
(A_1 + iA_2)x &= b_1 + ib_2
\end{aligned}$$

equating the real and imaginary parts this is equivalent to solving

$$A_1x_1 = b_1 \qquad A_2x_2 = b_2.$$

where $x_1 = \mathrm{Re}(x)$, $x_2 = \mathrm{Im}(x)$, and both systems are real

2.5.2 Compare storage and number of floating point operations

Consider complex numbers a_{ij}/a_{jj} used in Gaussian elimination

$$\frac{a_{ij}}{a_{jj}} = \frac{(a,b)}{(c,d)} = \frac{(ac+bd, -ad+bc)}{c^2+d^2}.$$

this introduces 6 multiplications and 3 additions, compared to just 1 operation if they were real. The storage requirements for the complex case $Ax = b$ is the same as for the two systems $A_1x_1 = b_1$ and $A_2x_2 = b_2$.