Discrete Time Signal Processing

Alexandre Boé

INTRODUCTION

Difference between analog and digital signals ?

Pros and Cons for using **digital signal processing** ?

- ✓
- ✓
- χ
- χ

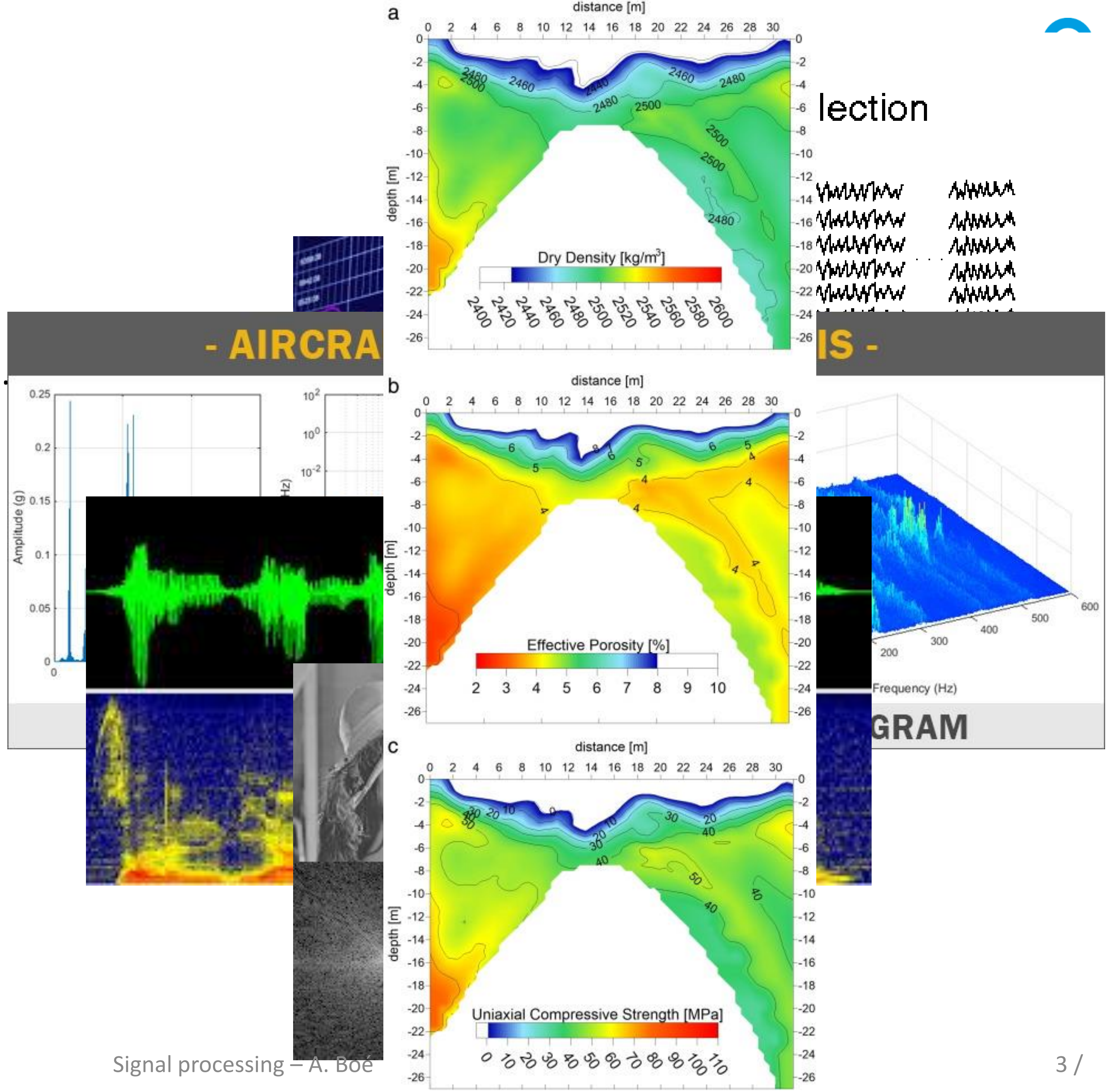
Digital signal => sequence

Scilab software to compute data

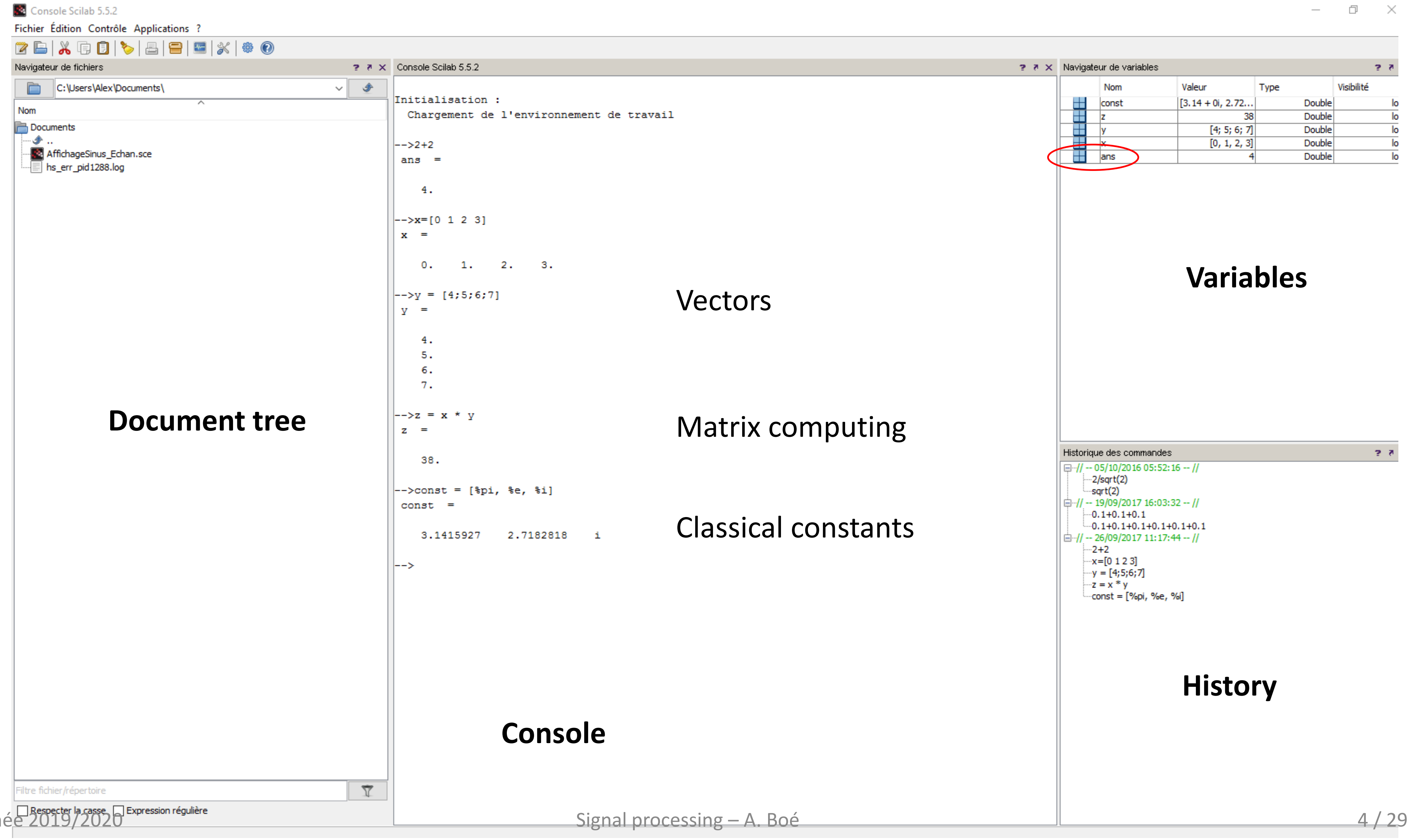
INTRODUCTION

Applications:

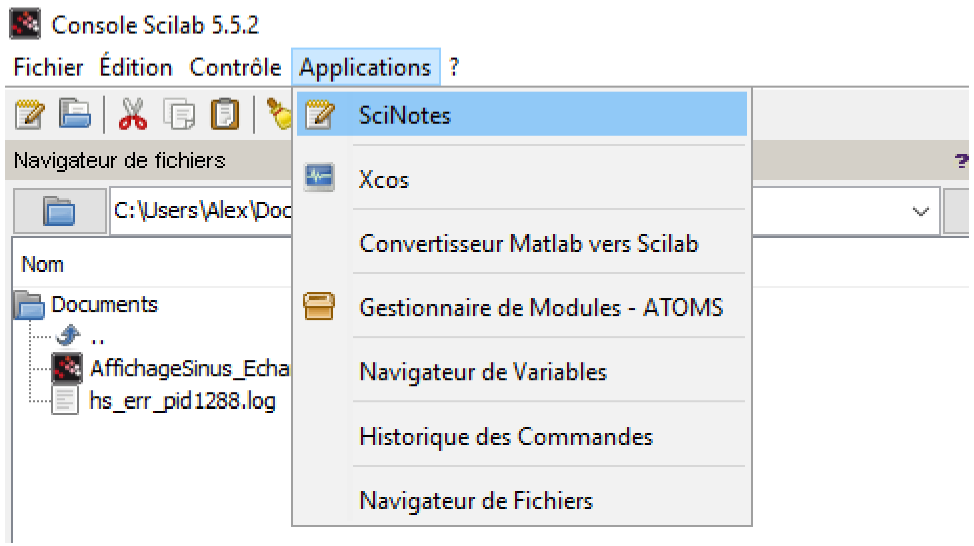
- Medical signals (EEG, MRI, …
- Financial engineering
- Data analysis
- Speech processing
- Image processing
- Seismic survey

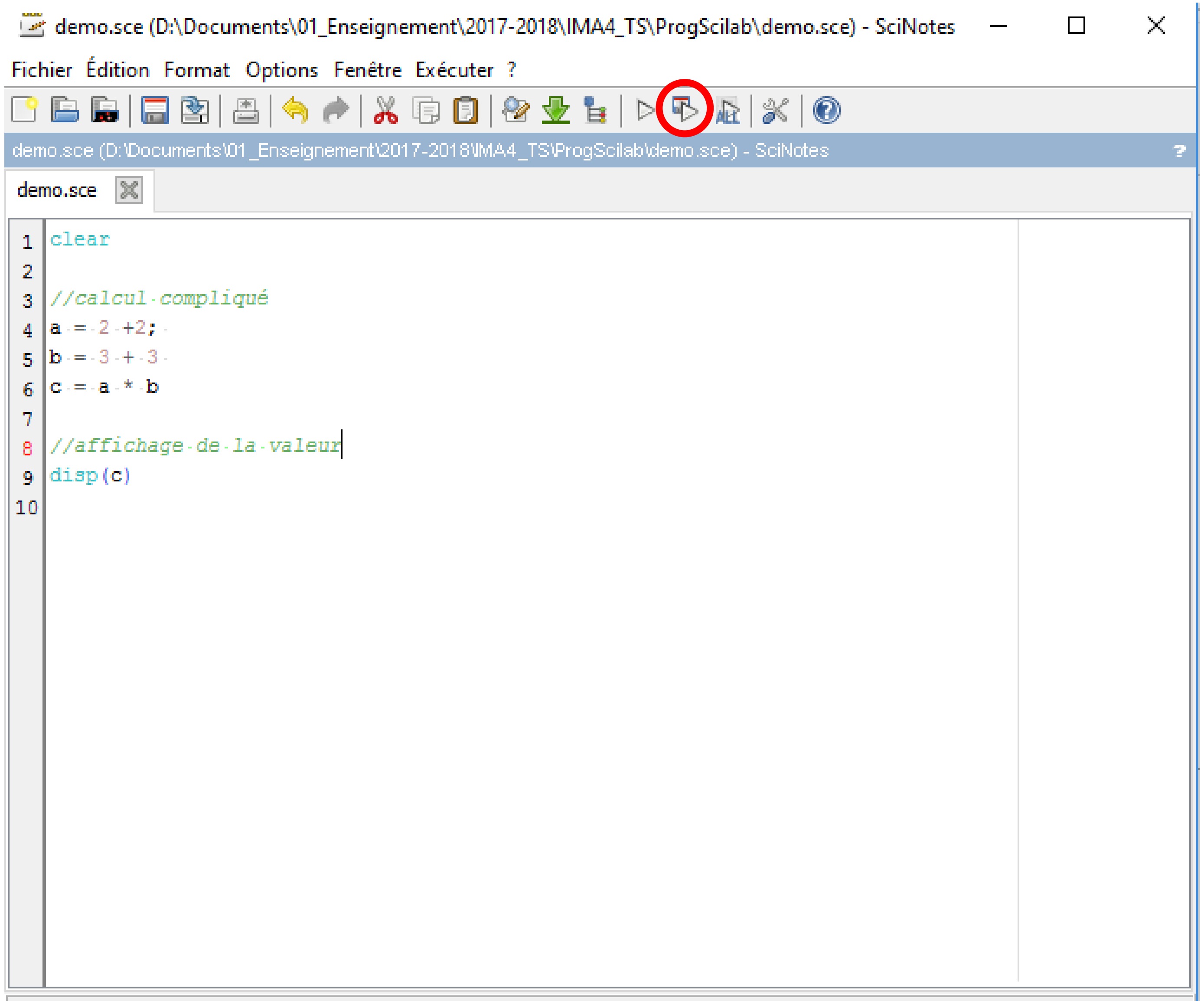


Lesson 1. Scilab



LESSON 1. SCILAB





Numerical computing

Programming language:

- variables
- comments
- structures
- …

Display

Useful fonction : help

Double float by default (64 bit)

Computing on matrices

Display

Programming

Command	Description
abs sign	Module and sign
real imag	Real and imaginary part
exp log log10	
cos sin tan …	
acos asin atan	
cosh acosh …	
sqrt	Square root
floor	Round down
round	Round to nearest integer
ceil	Round up
int	Round towards zero
rand()	Random number generator

LESSON 1. SCILAB

Double float by default (64 bit)
Computing on matrices
Display
Programming

Exercice 1.

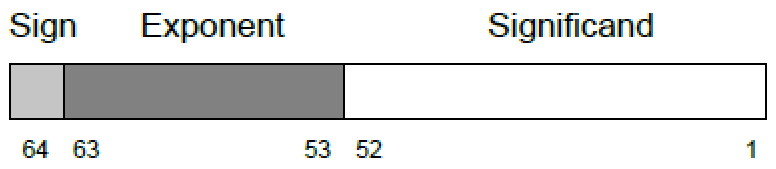
display 2^53, 2^53 +1, … , 2^53+10
conclusion ?

Exercice 2.

compute sinus of 180° and Pi radians
conclusion ?

Voir %eps/2

Exercice 3. IEE-754 64 bit floating point number



compute 1+%eps and 1+%eps/2
conclusion ?

display 0.1 with maximum number of digits
conclusion ?

associativity with respect to addition / multiplication
distributivity of multiplication over addition

Assess derivative of sin(x) at x=1 and conclude on aproximation and convergence

- Double float by default (64 bit)
- Computing on matrices
- Display
- Programmin

Command	Description
[0, 1, 2]	Row vector
[0; 1; 2]	Column vector
[0,1,2;0,1,2]	2*2 matrix
A(i, j)	**A_{ij} element**
A(i1:i2, j1:j2)	Sub matrix
$	Means *last index*
+ - * ^	Operations on matrices
.+ .- .* .^	Elementwise operations
’	Transpose and conjugate
.’	Transpose (no conjugate)
one(x,y) zero(x,y) eye(x,y)	Predefined matrices
min:step:max	**Vector from min to max with step**

POLYTECH LILLE

Double float by default (64 bit)

Computing on matrices

Display

Programming

Command	Description
size()	Size of the matrix
det()	Determinant of the matrix
rank()	Rank of the matrix
inv()	Inverse of the matrix
sum() prod()	Sum and product of vector coefficients
min() max()	Maximum or minimum coefficient
norm()	Norm of a vector
spec()	Eigenvalue / eigenvectors
bdiag	Diagonalized matrix
det()	Determinant of the matrix

Double float by default (64 bit)
Computing on matrices
Display
Programming

Simple graph: two vectors using plot2d
style (negative numbers for different points, positive numbers for colors)
subplot for multiplots
xtitle for titles and axis captions

and many other functions … go to visit help !

Exercice 4. Plot the function $\frac{\sin x}{x}$

Double float by default (64 bit)
Computing on matrices
Display
Programming

Looping and branching

if … then … else … end

select … case … case … else … end

for … end

while … end

break and continue

Exercise 5.

Determine the Fourier's series of a square signal (periodic)
Display on a graph the sum of harmonics for n = 1, 3, 5, 7, 9 …

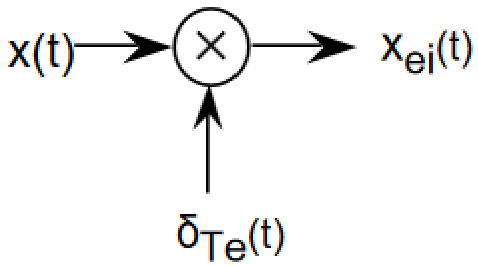
$x[n] = x_c(nT_S)$ avec $-\infty < n < +\infty$

T_S is the sampling period
$F_S = \frac{1}{T_S}$ is the sampling frequency

Experiment the sampling of periodic signals and expose the Shannon criterion

Lesson 2. Sampling

Effect of sampling:



$$x_{ei}(t) = x(t) \cdot \delta_{Te}(t) = \sum_{n=\infty}^{\infty} x(t)\ \delta(t - nT_e)$$

Dirac:

$$e_i(t) = \delta_{Te}(t) = \sum_{n=-\infty}^{\infty} \delta(t - nT_e) = \sum_{n=-\infty}^{\infty} \delta\left(t - \frac{n}{f_e}\right)$$

Fourier’s Transform:

$$\mathcal{F}\{\delta_{Te}(t)\} = \mathcal{F}\left\{\sum_{n=-\infty}^{\infty} \delta\left(t - \frac{n}{f_e}\right)\right\} = f_e\, \delta_{fe}(f) = f_e \sum_{k=-\infty}^{\infty} \delta(f - k\, f_e)$$

$$\boxed{X_e(f) = X(f) * f_e\ \delta_{fe}(f) = f_e \sum_{k=-\infty}^{\infty} X(f - k f_e)}$$

Considering a continuous signal $x(t)$ and its sampled version $x_S(t)$:

$$(t) = \sum_{k=-\infty}^{+\infty} \delta(t - kT_S)$$

Apply the Fourier's transform to both parts:

$$TF[x_S(t)] = TF[x(t)] * F_S \sum_{k=-\infty}^{+\infty} \delta(f - kF_S)$$

We can note that:

$$x(t) * \delta(t - a) = x(t - a)$$

Then:

$$TF[x_S(t)] = F_S \sum_{k=-\infty}^{+\infty} X(f - kF_S)$$

Signal reconstruction:

$$x_S(t) = x(t) \sum_{k=-\infty}^{+\infty} \delta(t - kT_S) = \sum_{k=-\infty}^{+\infty} x(kT_S)\delta(t - kT_S)$$

The sampling repects the Shannon criterion, and applying a low pas filter ($F_{stop} = \frac{F_S}{2}$). The transfer function of the filter is given by: $\frac{1}{F_S}\prod\left(\frac{f}{F_S}\right)$.

At the output of the filter:

$$\left[X(f) * F_S \sum_{k=-\infty}^{+\infty} \delta(f - kF_S)\right] \frac{1}{F_S} \prod\left(\frac{f}{F_S}\right)$$

Apply inverse FT:

$$x(t) = TF^{-1}\left[\left[X(f) * F_S \sum_{k=-\infty}^{+\infty} \delta(f - kF_S)\right] \frac{1}{F_S} \prod\left(\frac{f}{F_S}\right)\right] = x_S(t) * sinc(\pi F_S t)$$

Then:

$$x(t) = \left[\sum_{k=-\infty}^{+\infty} x(kT_S)\delta(t - kT_S)\right] * sinc(\pi F_S t) = \sum_{k=-\infty}^{+\infty} x(kT_S)\, sinc[\pi F_S t(t - KT_S)]$$

DFT

$$X(k) = \frac{1}{N}\sum_{n=0}^{N-1} s(n)e^{-2i\pi n\frac{k}{N}}$$

(Non unique definition …)

Demonstration: …

Exercice: DFT of $x_{k\epsilon[0,15]} = \{1,1,1,0,\ldots,0\}$

DFT inverse

$$x(n) = \frac{1}{N}\sum_{k=0}^{N-1} X(k)e^{2i\pi n\frac{k}{N}}$$

Properties:

- Linearity
- Delay: $x(n-n_0) = X(k)e^{-i2\pi k\frac{n_0}{N}}$
- $(X)_{n\in\mathbb{Z}} = TFD\big((x_k)_{k\epsilon\mathbb{Z}}\big)$
  - $TFD(x_{-k}) = X_{-n}$
  - $TFD(\overline{x_k}) = \overline{Y_{-n}}$
  - $TFD(\overline{y_{-k}}) = \overline{Y_N}$

- Convolution:

  $(x_k)_{k\epsilon\mathbb{Z}}$ and $(y_k)_{k\epsilon\mathbb{Z}}$

  The circular or cyclic convolution is $z = x * y$ defined by $\forall k \in \mathbb{Z}, z_k = x * y = \sum_{q=0}^{N-1} x_q y_{k-q}$

  $X = TFD(x), Y = TFD(y), Z = TFD(z)\ Z_k = NX_kY_k$

  $TFD(x \cdot y) = X * Y$

- Energy: $\sum_{k=0}^{N-1} |y_k|^2 = N \sum_{n=0}^{N-1} |Y_n|^2$

Link with Fourier's Transform

Exercice 1. DFT

On considère un signal rampe échantillonné avec un pas de 1s.

1. Ecrire la suite $x[n]$ pour N (nombre d'échantillon) = 4.
2. Calculer 'à la main' la transformée de Fourier discrète du signal.
3. À partir de cet exemple, écrire un algorithme simple de calcul de la TFD d'un signal échantillonné.
4. Donner le nombre d'opérations à effectuer.

FFT algorithm

Exercice 2. FFT

On considère maintenant la fonction $x(t) = e^{-j2\pi F_0 t}$

1. Donner le spectre de cette fonction par calcul
2. Vérifier à l'aide de Scilab le calcul, en utilisant la fonction écrite précédemment et en utilisant la fonction *fft*
3. Comparer le temps de calcul des deux méthodes

Exercice 3. Propriétés de la TFD sur un signal périodique

1. Calculer la TFD d'un signal sinusoïdal sur un nombre entier de périodes
2. Faire de même sur un nombre non entier de périodes
3. Conclusion ?
4. Faire le même exercice en considérant une somme de deux sinusoïdes d'amplitude égale et de fréquences proches puis de fréquences « éloignées » avec des amplitudes très différentes
5. Conclusions ?

TFD gives exact results for: **a multiple integer of the signal period**

a window length equal to a multiple of sampling period (always true on pratical)²

Fenêtre rectangulaire, qui conduit à l'approximation sigma :

$$h(t) = \begin{cases} 1 & \text{si } t \in [0, T] \\ 0 & \text{sinon.} \end{cases}$$

Fenêtre triangulaire (de Bartlett) :

$$h(t) = \begin{cases} \frac{2t}{T} & \text{si } t \in [0, \frac{T}{2}[\\ \frac{2(T-t)}{T} & \text{si } t \in [\frac{T}{2}, T] \\ 0 & \text{sinon.} \end{cases}$$

Fenêtre de Hann :

$$h(t) = \begin{cases} \frac{1}{2} - \frac{1}{2}\cos(2\pi\frac{t}{T}) & \text{si } t \in [0, T] \\ 0 & \text{sinon.} \end{cases}$$

Fenêtre de Hamming :

$$h(t) = \begin{cases} 0,54 - 0,46\cos(2\pi\frac{t}{T}) & \text{si } t \in [0, T] \\ 0 & \text{sinon.} \end{cases}$$

Fenêtre de Blackman :

$$h(t) = \begin{cases} 0,42 - 0,5\cos(2\pi\frac{t}{T}) + 0,08\cos(4\pi\frac{t}{T}) & \text{si } t \in [0, T] \\ 0 & \text{sinon.} \end{cases}$$

Et d'autres : fenêtres de Kaiser (en) (de paramètre α), gaussienne, *flat top*, en cosinus relevé…

www.wikipedia.fr

Fenêtre	Lobe 2aire (dB)	Pente (dB/oct)	Bande passante (bins)	Perte au pire des cas (dB)
Rectangulaire	−13	−6	1,21	3,92
Triangulaire	−27	−12	1,78	3,07
Hann	−32	−18	2,00	3,18
Hamming	−43	−6	1,81	3,10
Blackman-Harris 3	−67	−6	1,81	3,45

www.wikipedia.fr

How to choose the pertinent window ?

- For signal with distant frequencies, use low secondary lobe window
- For a highest resolution in close frequencies, use low main lobe window.
- For a highest resolution on amplitude (with lower resolution on frequency), use large main lobe window.
- With a wideband signal, use flat window or no window (= rectangular)

- **Hann window is the most versatile, with a good frequency resolution. In case of unknown signal, it should be a good starting choice**

POLYTECH LILLE

Window choice:

Type de signal	Fenêtre
Transitoires dont la durée est inférieure à la longueur de la fenêtre	Rectangulaire
Transitoires dont la durée est supérieure à la longueur de la fenêtre	Exponentielle, Hann
Applications standards	Hann
Analyse spectrale (mesures de réponses fréquentielles)	Hanning (pour excitation aléatoire), rectangulaire (pour excitation pseudo-aléatoire)
Séparation de deux tons dont les fréquences sont très proches mais dont les amplitudes sont très différentes	Kaiser-Bessel
Séparation de deux tons dont les fréquences sont très proches mais dont les amplitudes sont presque identiques	Rectangulaire
Mesures précises de l'amplitude d'un ton unique	À profil plat
Signal sinusoïdal ou combinaison de signaux sinusoïdaux	Hann
Signal sinusoïdal avec nécessité de précision de l'amplitude	À profil plat
Signal aléatoire à bande étroite (données de vibration)	Hann
Signal aléatoire à large bande (bruit blanc)	Uniforme
Signal sinusoïdal avec courbes rapprochées	Uniforme, Hamming
Signaux d'excitation (coup de marteau)	Force
Signaux de réponse	Exponentielle
Signaux dont le contenu est inconnu	Hann

Exercice 4. Fenêtrage

On considère un signal sinusoïdal observé sur un nombre entier de périodes ou sur un nombre non entier de périodes, puis deux signaux de même amplitude avec des fréquences proches, puis deux signaux avec des amplitudes très différentes :

1. Tracer les fenetres Hann, Hamming, Blackman
2. Appliquer au signal une fenêtre de Hann, de Hamming et de Blackman
3. Calculer dans les trois cas le spectre du signal obtenu
4. Conclusions ?

Exercice 5. Zero padding

On veut artificiellement augmenter la résolution de la FFT pour obtenir un affichage de meilleure qualité :

1. Représenter un signal temporel avec deux sinusoïdes de même amplitude et proches en fréquence et une troisième avec une amplitude faible comparée aux deux autres
2. Calculer le spectre du signal
3. Ajouter des zéros sur le signal et observer le spectre obtenu

Summary:

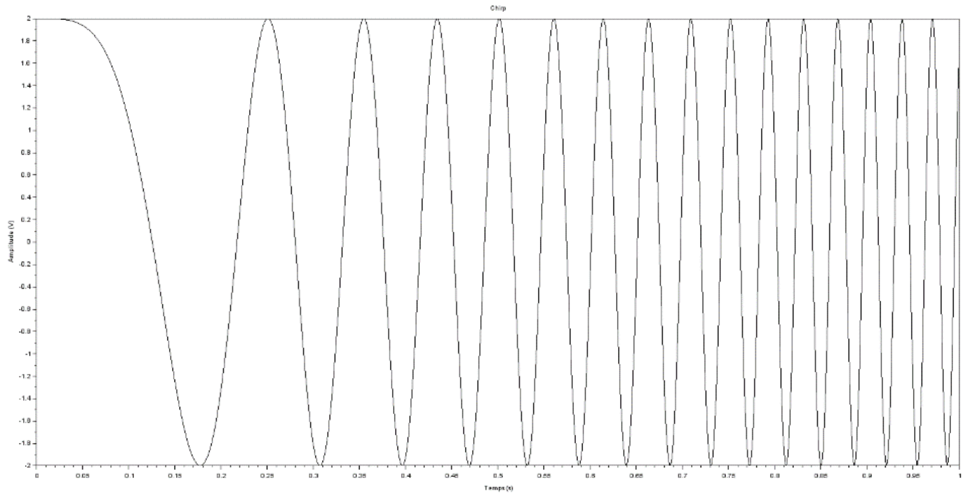
1. Complexity : DFT = N^2, FFT algorithm = N log N when N = 2^P
2. FFT gives exact results for a multiple integer of period
3. k^{th} sample represent the frequency $F_k = k\frac{F_s}{N}$
4. Frequency resolution is roughly proportionnal to NF_s
5. Window

LESSON 4. RANDOM SIGNALS

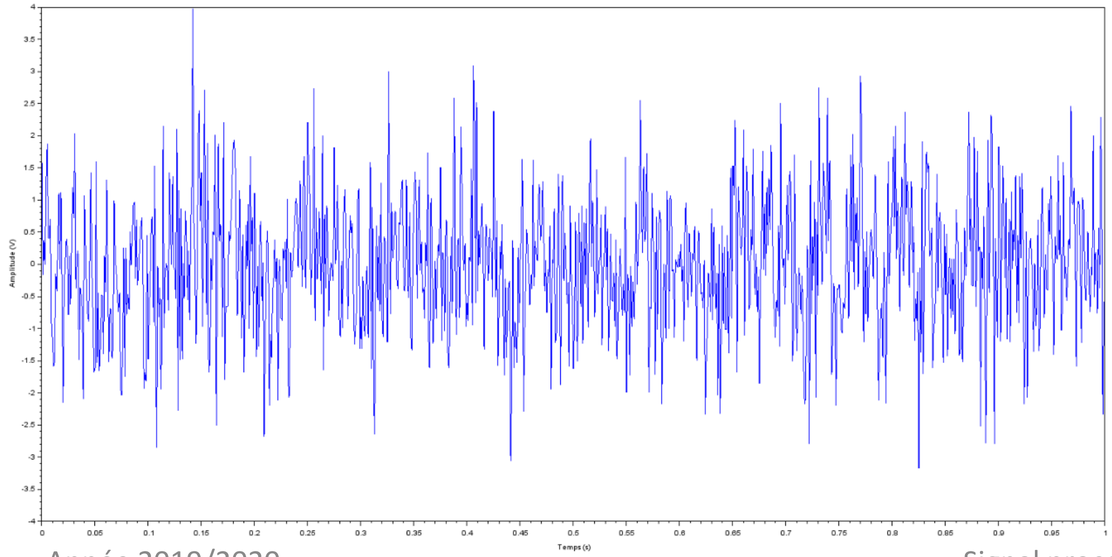
Exercise 4.1 :

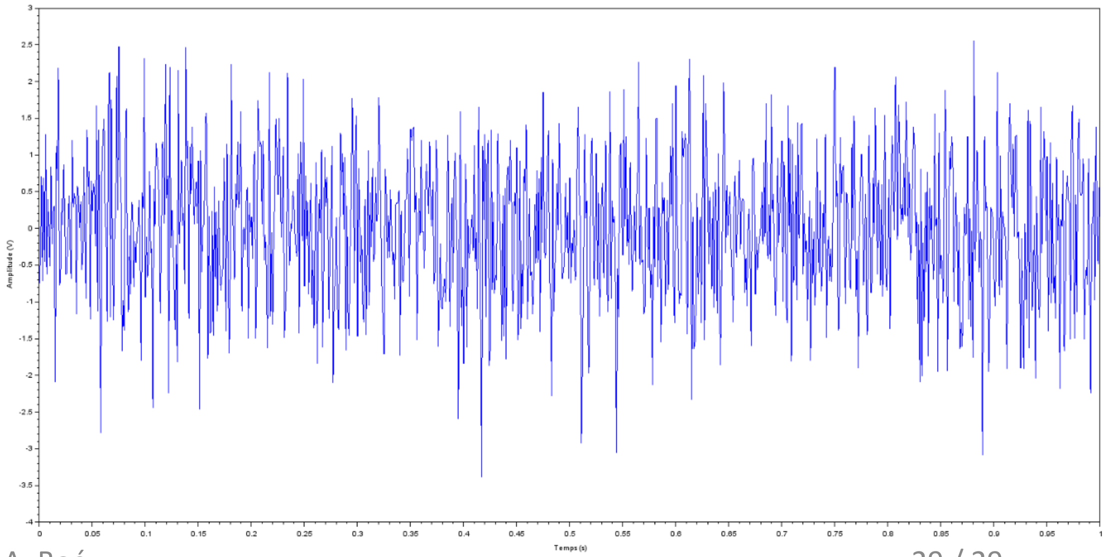
Extraction of a signal: simulation of RADAR signals

- Simulate a chirp emission of a RADAR for plane detection



- Add noise and a return signal if a plane is present
- Have a look at received signals
- Try to find in which signal a plane is present ?





Exercice 4.2: Random signal and Wiener-Khintchine's theorem.

- We consider a Gaussian process (electronics noise)
- Write a random function and draw the probabiliy graph of a Gaussian process
- Assess the power of a signal by summing the samples and by using Wiener-Kintchine theorem

- In real analysis, it is generally not possible to get as many samples or to do mean on different signals. Using a fixed number of samples and working on subset, do the same analysis and conclude

Exercice 4.3: Intercorrelation for leak detection.

- We consider a leak on a pipe, generating noise. By using two tranducers (e.g. microphones) on the pipe, we collect data.
- Give the expression of the signal received at the two transducers
- Write the intercorrelation functions
- Knowing the propagation speed of the signal, deduce the localization of the leak
- Application on data ‘leak’

BIBLIOGRAPHY

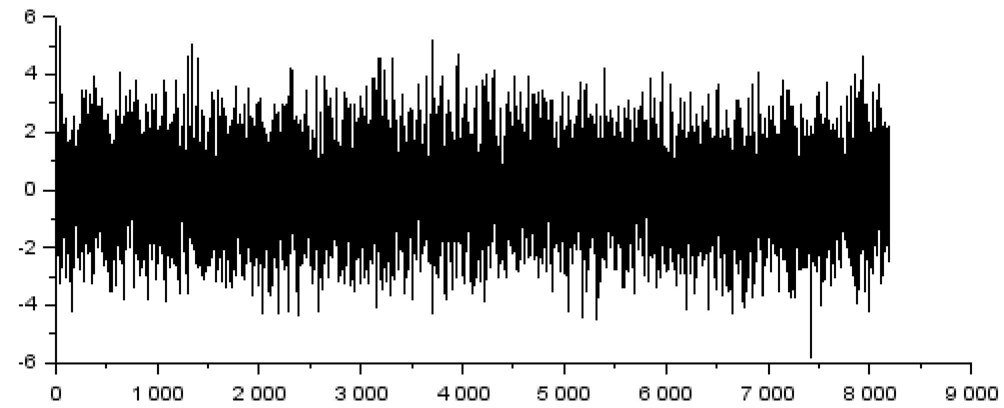
Alan V. Opprenheim and Ronald W. Schafer, **Discrete-Time Signal Processing**, 2nd Edition

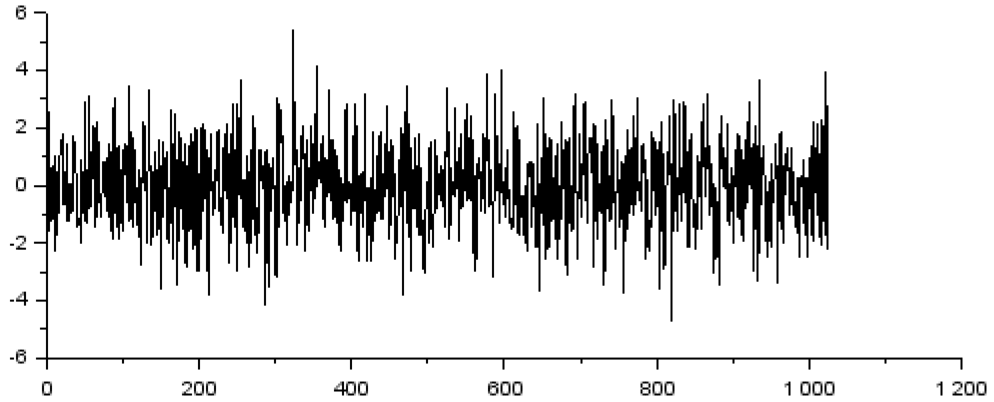
Romain Joly, **Petit Guide de Survie en Scilab**, https://www-fourier.ujf-grenoble.fr/~rjoly/Documents/Pedago/guide-scilab.pdf

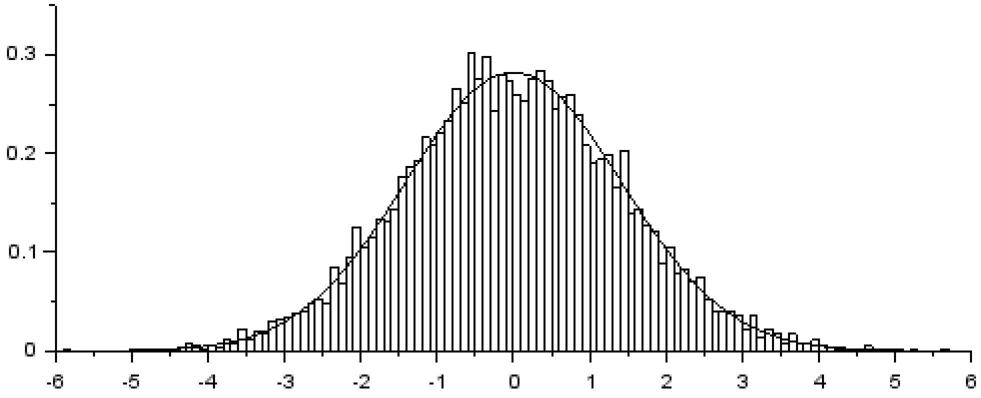
Michaël Baudin, Floating point numbers in Scilab, http://forge.scilab.org/index.php/p/docscifloat/

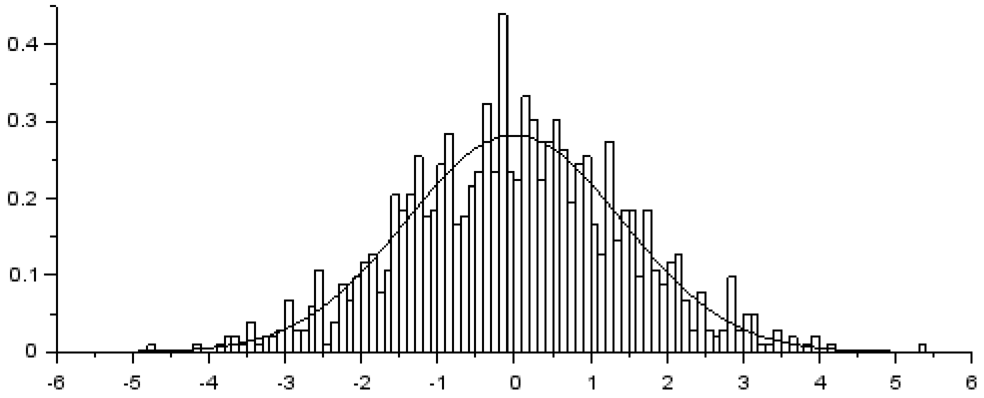
Denis Rabaste, TP sur le traitement du signal en ligne, http://denis.rabaste.free.fr/

p=0,5 moy=0





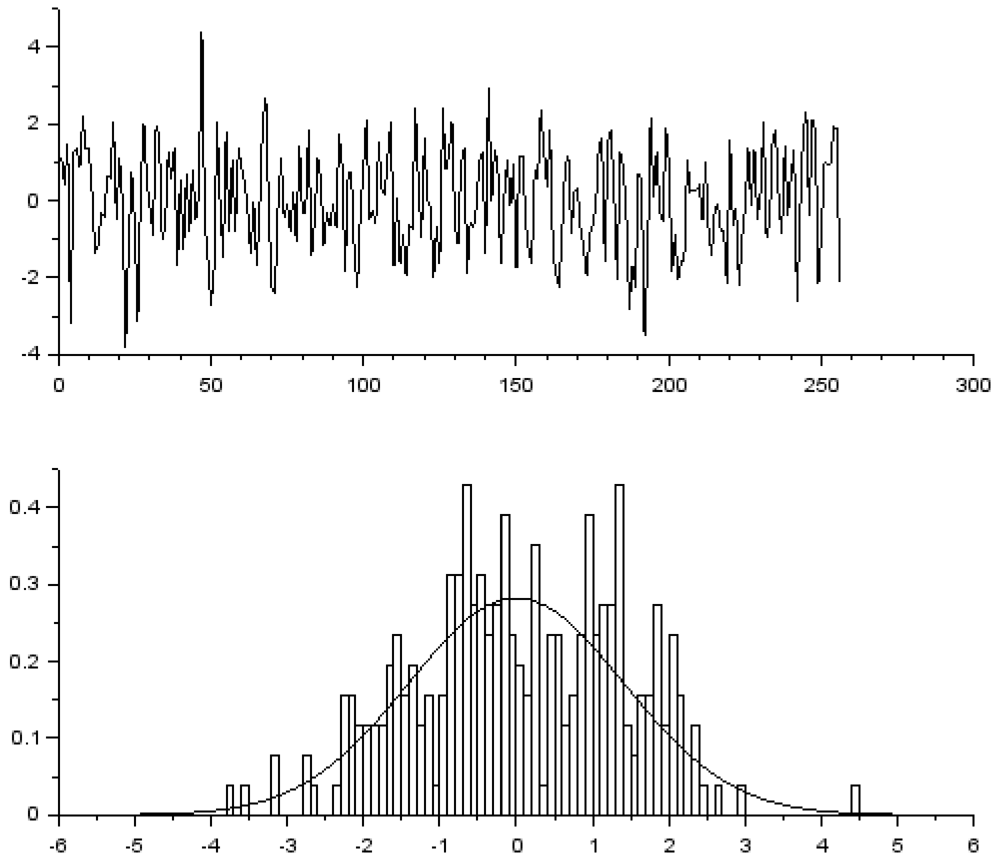




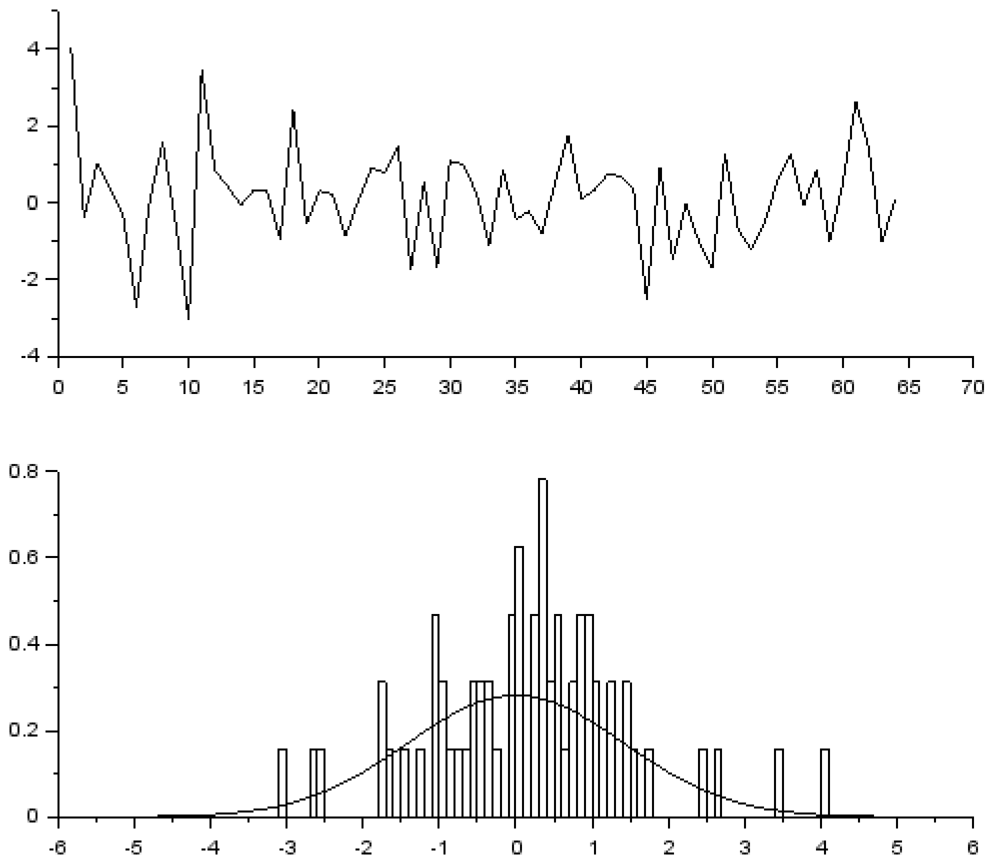
N=8192

N=1024

p=0,5 moy=0

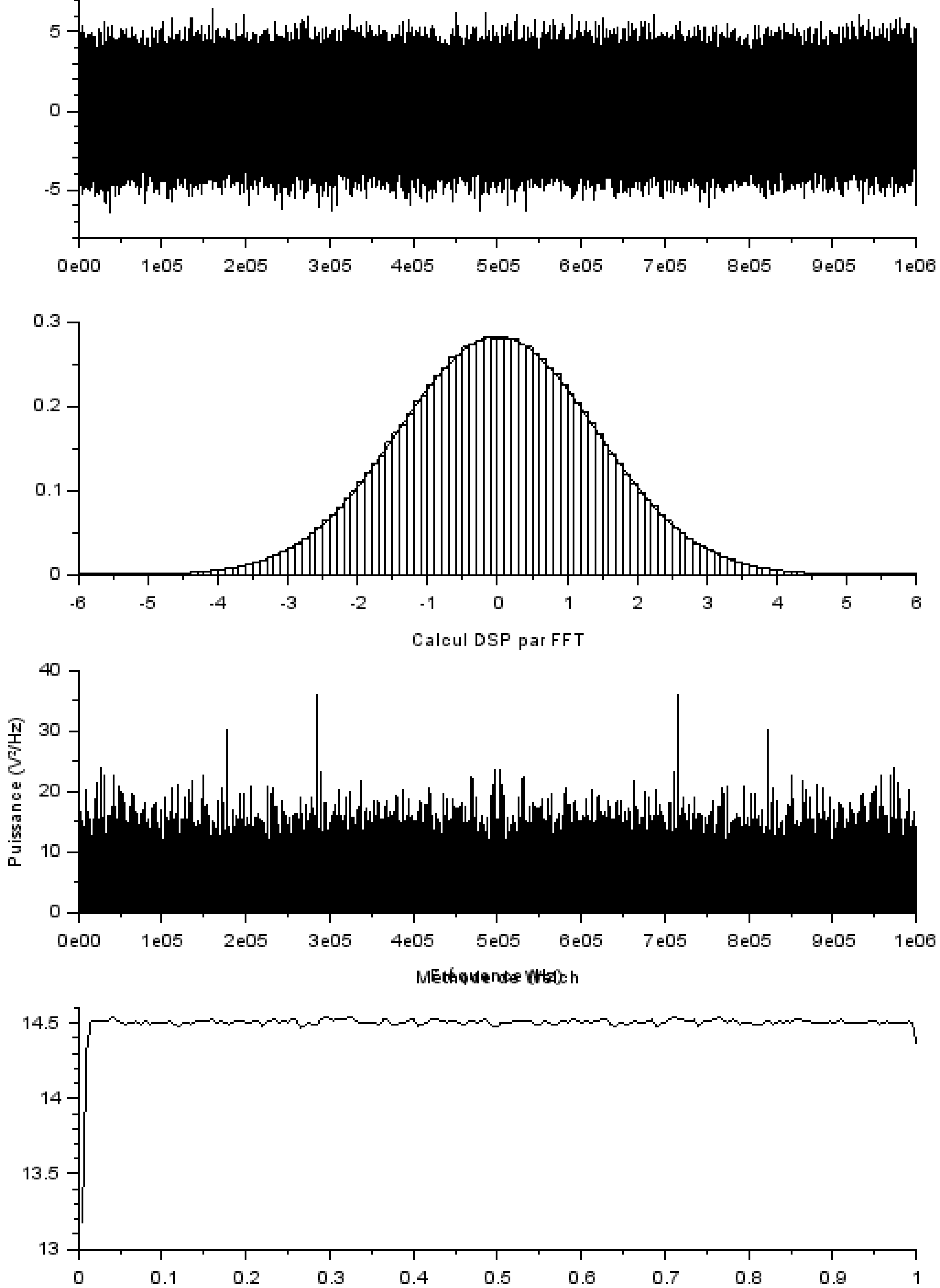


N=256

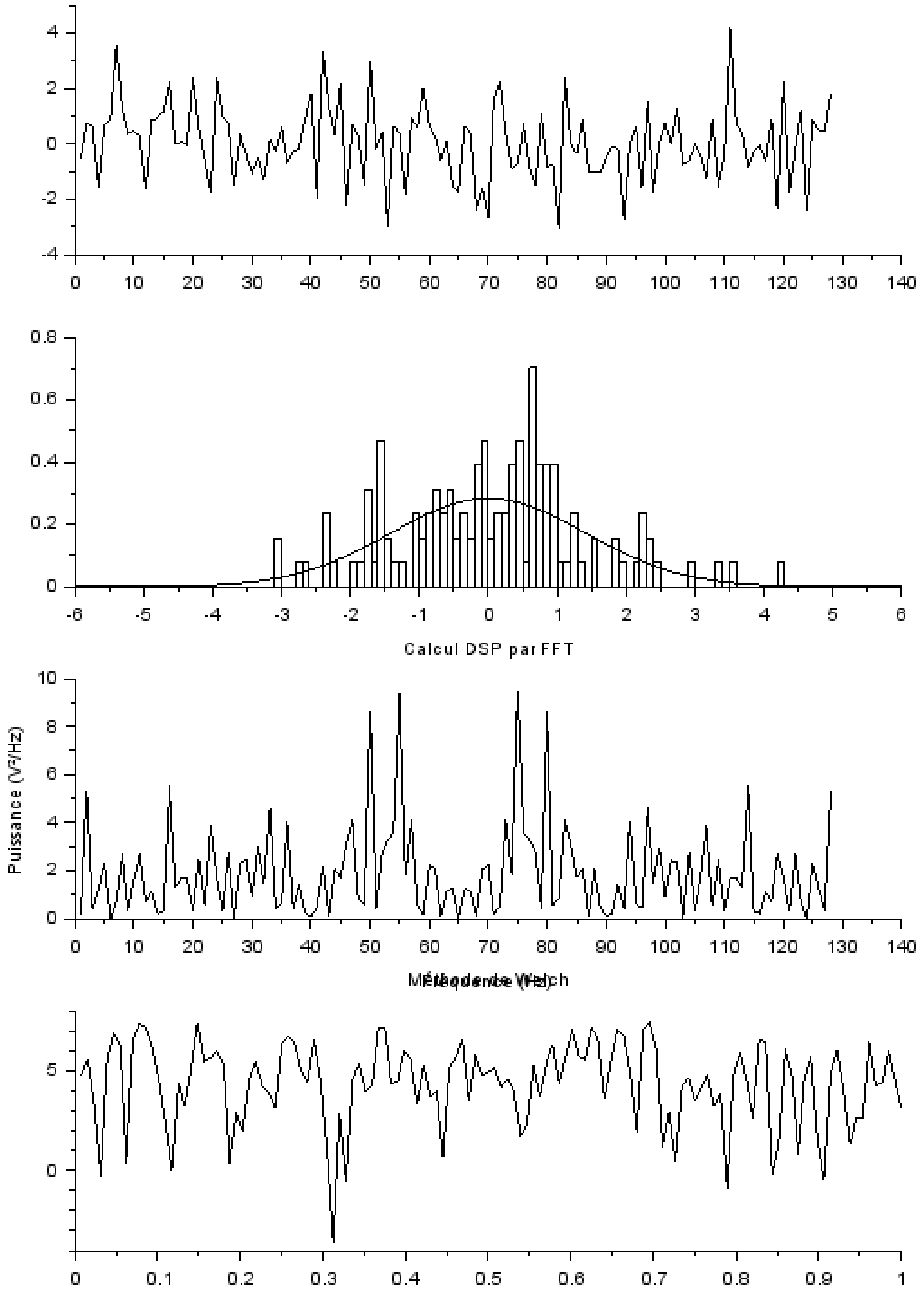


N=64

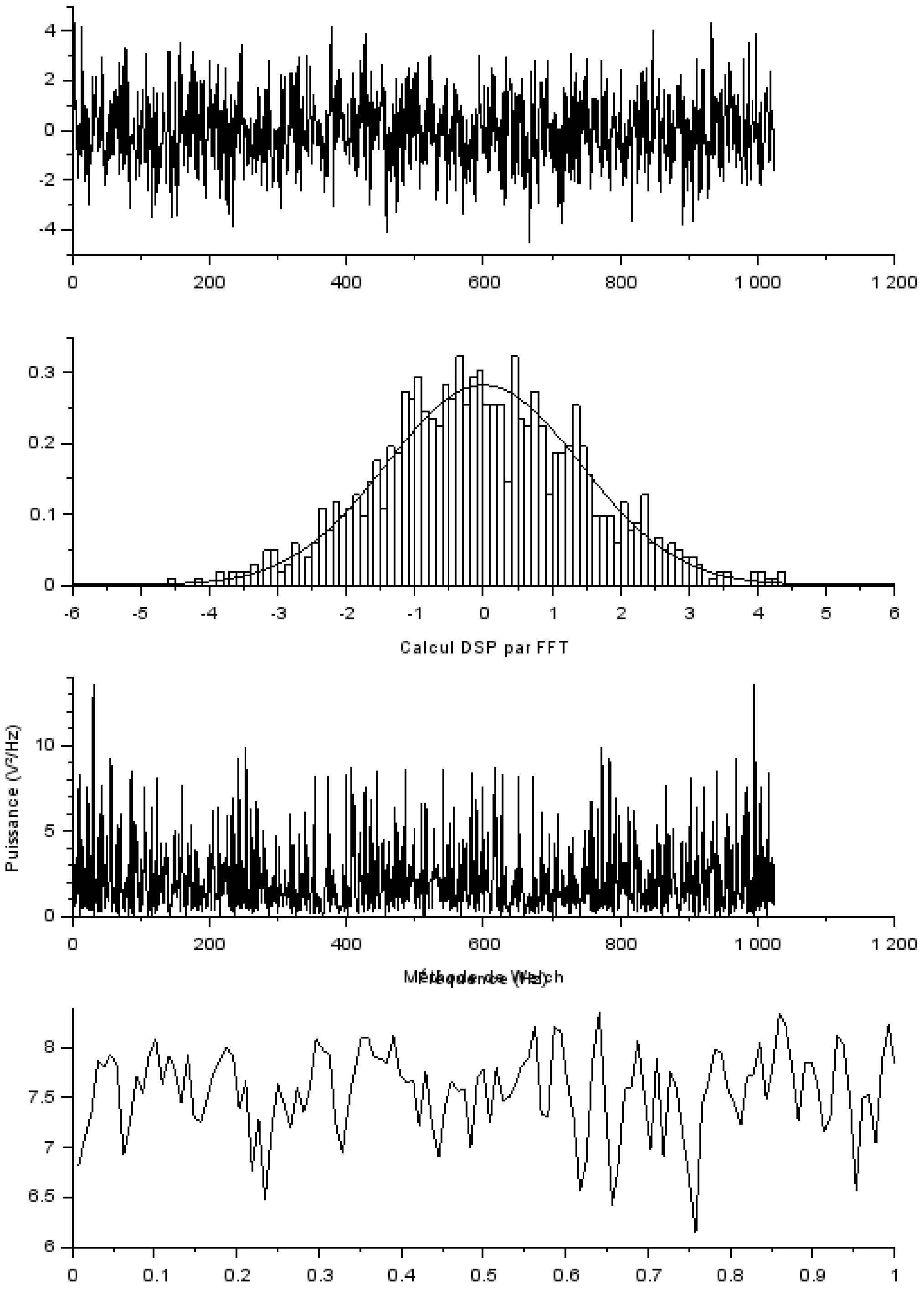
N = 999999



N = 128


Calcul DSP par FFT
Puissance (V²/Hz)
Fréquence (Hz)
Méthode de Welch

N = 1024



Calcul DSP par FFT
Puissance (V²/Hz)
Fréquence (Hz)
Méthode de Welch