Moments and Characteristic Functions
Instructor's Notes

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April 4, 2019

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1 Moments

1.1 Definition of Moments

Generally, in math, the n-th moment of a real-valued continuous function about center c is: [1]

$$\mu_n = \int_{-\infty}^{\infty} (x - c)^n f(x) dx$$

In particular, for probability density functions f (or cumulative density function F), the moments are given by

$$\mu'_n = E[X^n] = \int_{-\infty}^{\infty} x^n dF(x) = \int_{-\infty}^{\infty} x^n f(x) dx$$

Also we have the definition of the central moment [2]:

$$\mu_n = E[(X - E[X])^n] = \int_{-\infty}^{\infty} (x - \mu)^n f(x) dx$$

Generally central moments are more useful. Not to be confused with mean μ.

1.2 Description of Moments

The first moment is the mean of a random variable, i.e.

$$\mu = E[X]$$

The second moment is related to the variance of a random variable:

$$\mathrm{Var}[X] = E[X^2] - E[X]^2$$

In fact the variance is just the second central moment:

$$\mathrm{Var}[X] = \mu_2 = E[(X - E[X])^2]$$

As for the third central moment, a related concept is skewness. Below shows two random variables with the same mean variance however different in skewness[3]:

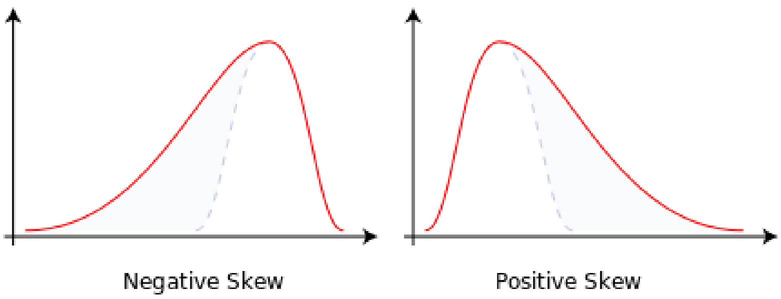

Figure 1: Negative and Positive Skew Diagrams

With all moments up to the order of infinity we can describe the **characteristics** of a probability distribution.

2 Characteristic Functions

2.1 Moment Generating Functions

Definition 2.1 *Let X be a random variable with probability density function $f(x)$. If there is a positive number h such that*

$$\int_{-\infty}^{\infty} e^{tx} f(x) dx$$

exists and is finite for $h < t < h$, then the function defined by

$$M(t) = E[e^{tX}]$$

is called the moment-generating function of X (or of the distribution of X). [4]

The r-th moment about the origin can be achieved from the moment generating function by evaluating the r-th derivative[5]:

$$M^{(r)}(0) = E[X^r]$$

Also notice the relation between the Taylor Expansion and the moments.

2.2 Characteristic Function

Notice that e^{tx} is not a "good" function in the sense that it is not bounded and may not converge under some circumstances. Before going to characteristic functions, we first get acquainted with knowledge of complex numbers:

2.2.1 Basic information about complex numbers

Let $z = a + bi$, where $a, b \in \mathbb{R}$, and $i = \sqrt{-1}$ is the imaginary unit. z is then called a complex number and a, b are called the real and imaginary parts of z, denoted by $a = \mathrm{Re}(z), b = \mathrm{Im}(z)$ respectively. (Consider i as rotation by $\frac{\pi}{2}$ counterclockwise in the complex plane)
The conjugate of a complex number $z = a + bi, a, b \in \mathbb{R}$ is $\hat{z} = a - bi$, we also define the modulus (or length) of z to be $|z| = z\hat{z}$. Notice that $|z|$ is a non-negative real number.
Euler's formula:

$$e^{i\theta} = \cos\theta + i\sin\theta$$

The formula comes from Taylor's Series. It also gives rise to the polar representation of a complex number, i.e. $z = re^{i\theta}$, where r is the modulus and θ is the phase.
From this we also have that $|e^{i\theta}| = 1$ for any θ.

2.2.2 Definition of Characteristic Functions

Definition 2.2 *Let X be a random variable and denote by F the cumulative distribution function of X. The characteristic function $\varphi = \varphi_X$ of X (or of F, in which case we also write φ_F) is defined by* [6]

$$\varphi_X(t) := E[e^{itX}] = \int_{-\infty}^{\infty} e^{itx} dF(x), t \in \mathbb{R}$$

2.2.3 Basic Properties

Theorem 2.1 (Uniqueness Theorem [7]) *Let X be a real random variable with distribution function F and characteristic function φ. Similarly, let Y have distribution function G and characteristic function ψ. If $\varphi(t) = \psi(t)$ for all $t \in \mathbb{R}$ then $F(x) = G(x)$ for all $x \in \mathbb{R}$.*

Properties from here on come from Bisgaard and Zoltan's book [6].

Theorem 2.2 *If X and Y are independent random variables then the characteristic function of their sum is*

$$\varphi_{X+Y}(t) = \varphi_X \cdot \varphi_Y.$$

Corollary 2.2.1 *The product of two characteristic functions is a characteristic function.*

Remark If X and Y are random variables such that $\varphi_{X+Y} = \varphi_X \cdot \varphi_Y$, then in general we do not conclude X and Y are independent. (See page 13 in [6])

Theorem 2.3 *For any $a, b \in \mathbb{R}$,*

$$\varphi_{aX+b}(t) = e^{ibt}\varphi_X(at).$$

Theorem 2.4 *Every characteristic function φ has the following properties:*

(i) $f(0) = 1$,

(ii) $|f(t)| <= 1$,

(iii) $f(-t) = \overline{f(t)}$

(iv) f *is continuous on* $\mathbb{R}$

2.3 Common Distributions and Their Characteristic Functions

Table 1: Characteristic Functions for Common Distributions[8]

Distribution	PMF/PDF	Characteristic Function
Constant $X \equiv a$	-	$\phi_X(t) = e^{iat}$.
Binomial $X \sim Binomial(m, p)$	$p_X(n) = \binom{n}{m} p^n (1-p)^{m-n}$	$\phi_X(t) = (pe^{it} + (1-p))^m$
Poisson $X \sim Poisson(\lambda)$	$p_X(n) = \frac{\lambda^n}{n!} e^{-\lambda}$	$\phi_X(t) = e^{\lambda(e^{it}-1)}$
Normal $X \sim N(0,1)$	$f_X(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}$	$\phi_X(t) = e^{-\frac{t^2}{2}}$
Normal $Y \sim N(\mu, \sigma^2)$	$f_Y(y) = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{(y-\mu)^2}{2\sigma^2}}$	$\phi_Y(t) = e^{it\mu - \frac{\sigma^2 t^2}{2}}$

3 Examples and Applications of Characteristic Functions

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