Characteristic Functions

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Raindrops Again and Again and ...

Raindrops in two seconds

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But why?

Sum of Independent Normals

The Normal Sum Problem

$X_1 \sim N(\mu_1, \sigma_1^2), X_2 \sim N(\mu_2, \sigma_2^2)$. X_1, X_2 are independent. Distribution of $Y = X_1 + X_2$?

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Observation:

- $E[Y] = E[X_1] + E[X_2]$
- $Var[Y] = Var[X_1] + Var[X_2]$

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But what is the distribution?

To deal with the sums of random variables, **characteristic function** is a powerful weapon for us.

Definition

Let X be a random variable and denote by F the cumulative distribution function of X (or f the probability density function). The characteristic function $\varphi = \varphi_X$ of X (or of F, in which case we also write φ_F) is defined by

$$\varphi_X(t) := E[e^{itX}] = \int_{-\infty}^{\infty} e^{itx} dF(x) = \int_{-\infty}^{\infty} e^{itx} f(x) dx, t \in \mathbb{R}$$

Recall Complex Numbers

If you have forgotten everything about complex numbers, all you need to recall is the following:

- i is the imaginary unit,
- $i^2 = -1$,
- $e^{i\theta} = \cos\theta + i\sin\theta$. (This formula comes from Taylor Series)

Properties

Theorem (Uniqueness Theorem)

Let X be a real random variable with distribution function F and characteristic function φ. Similarly, let Y have distribution function G and characteristic function ψ. If $\varphi(t) = \psi(t)$ for all $t \in \mathbb{R}$ then $F(x) = G(x)$ for all $x \in \mathbb{R}$.

From this we may easily conclude the distribution of a random variable if we can prove that its characteristic function is of the same form as a known distribution.

Properties

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Theorem (Inversion Formula)

If $\int_{\mathbb{R}} |\varphi(t)| dt < \infty$ then X has bounded continuous density

$$f(x) = \frac{1}{2\pi} \int_{\mathbb{R}} e^{-itx} \varphi(t) dt$$

Properties

Theorem

If X and Y are independent random variables then the characteristic function of their sum is

$$\varphi_{X+Y}(t) = \varphi_X \cdot \varphi_Y.$$

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If X and Y are random variables such that $\varphi_{X+Y} = \varphi_X \cdot \varphi_Y$, then in general we do not conclude X and Y are independent. This is called **subindependence**.

Properties

Theorem

For any $a, b \in \mathbb{R}$,

$$\varphi_{aX+b}(t) = e^{ibt}\varphi_X(at).$$

Distributions and Their Characteristic Functions

Example (Characteristic Function for Exponential)

Find the characteristic function for $X \sim \mathit{Exponential}(\lambda)$.

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$$\begin{aligned}
\varphi_X(t) &:= E[e^{itX}] \\
&= \int_{-\infty}^{\infty} e^{itx} f(x)dx \\
&= \int_{0}^{\infty} e^{itx} \cdot \lambda e^{-\lambda x} dx && \text{By distribution of x} \\
&= \frac{\lambda}{it-\lambda} e^{(it-\lambda)x}\Big|_0^{\infty} \\
&= \frac{\lambda}{\lambda - it} && \text{By Squeeze Theorem}
\end{aligned}$$

Distributions and Their Characteristic Functions

The last two steps

$$\frac{\lambda}{it-\lambda}e^{(it-\lambda)x}\Big|_0^{\infty} = \frac{\lambda}{\lambda - it}$$

can be justified by

$$\lim_{x\to\infty} |e^{(it-\lambda)x}| = \lim_{x\to\infty} |e^{-\lambda x}| = 0 \text{ for positive } \lambda$$

Distributions and Their Characteristic Functions

The table below shows some common distributions and their characteristic functions:

Table: Characteristic Functions for Common Distributions

Distribution	Characteristic Function
Constant $X \equiv a$	$\varphi_X(t) = e^{iat}$.
Binomial $X \sim Binomial(m, p)$	$\varphi_X(t) = (pe^{it} + (1-p))^m$
Poisson $X \sim Poisson(\lambda)$	$\varphi_X(t) = e^{\lambda(e^{it}-1)}$
Exponential $X \sim Exponential(\lambda)$	$\varphi_X(t) = \frac{\lambda}{\lambda - it}$
Normal $X \sim N(0, 1)$	$\varphi_X(t) = e^{-\frac{t^2}{2}}$
Normal $Y \sim N(\mu, \sigma^2)$	$\varphi_Y(t) = e^{it\mu - \frac{\sigma^2 t^2}{2}}$

Statistics and Probability

So what "characteristics" are these weird "characteristic functions" talking about?

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To understand this, first we need some knowledge about **moments**.

Definition (Moment)

For probability density functions f (or cumulative density function F), the moments are given by

$$\mu_n' = E[X^n] = \int_{-\infty}^{\infty} x^n dF(x) = \int_{-\infty}^{\infty} x^n f(x)dx$$

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Relation with statistics. Observe that the first moment is simply the expectation. The second moment is related to the variance: $Var[X] = E[X^2] - (E[X])^2$.

What about the third moment?

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A related concept is **skewness**:

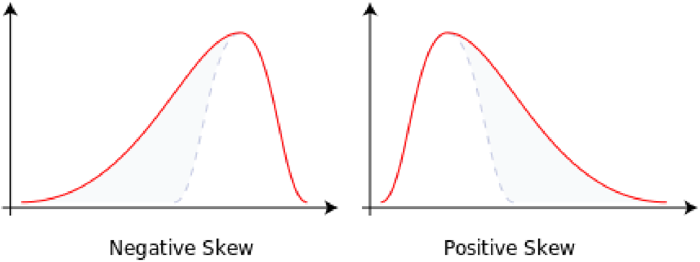

Figure: Same expectation and variance, but different skewness

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Solution

Let X, Y be two independent $Poisson(\lambda)$ random variables. Let $Z = X + Y$. Notice that characteristic functions for X (and respectively Y) is

$$\varphi_X(t) = e^{\lambda(e^{it}-1)}$$

Therefore we have

$$\varphi_Z(t) = \varphi_{X+Y}(t) = (\varphi_X(t))^2 = e^{2\lambda(e^{it}-1)}$$

By uniqueness of characteristic functions we know that $Z \sim Poisson(2\lambda)$.

Raindrops Again

Conclusion:

- This is the same as our intuition.
- By similar ideas, one can show that the sum of two independent poisson random variables has a possion distribution with an expectation of the sum of both expectations

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Solution

Similarly to previous problem

$$\varphi_Y(t) = \varphi_{X_1+X_2}(t) = e^{it\mu_1 - \frac{\sigma_1^2 t^2}{2}} \cdot e^{it\mu_2 - \frac{\sigma_2^2 t^2}{2}}$$

Therefore we have

$$\varphi_Y(t) = e^{it(\mu_1+\mu_2) - \frac{(\sigma_1^2+\sigma_2^2)t^2}{2}}$$

which implies that $Y \sim N(\mu_1 + \mu_2, \sigma_1^2 + \sigma_2^2)$

Sum of Independent Normals

Conclusion:

- Our result is consistent with the statistics.
- Sum of two independent normal random variables is still normal.

From Binomial To Poisson

Example

Show that $Poisson(\lambda)$ is the same as $\lim_{n\to\infty} Binomial(n, \frac{\lambda}{n})$.

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Solution

Take the limit of the characteristic function of the binomial distribution.

$$\begin{aligned}
\lim_{n\to\infty} \varphi_B(t) &= \lim_{n\to\infty} (\frac{\lambda}{n}e^{it} + (1 - \frac{\lambda}{n}))^n \\
&= \lim_{n\to\infty} (1 + (e^{it} - 1)\frac{\lambda}{n})^n \\
&= e^{\lambda(e^{it}-1)}
\end{aligned}$$

First we need to prove a lemma that gives us the common characteristics of all distributions:

Lemma

For any random variable X with $E[X] = 0$, $Var[X] = 1$, we have $\varphi_X(t) = 1 - \frac{1}{2}t^2 + o(t^2)$.

Proof.

Directly from the definition and Taylor Series we have

$$\varphi_X(t) = E[e^{itX}] = E[1 + itX + \frac{1}{2}i^2t^2X^2 + o((tX)^2)]$$

By linearity of expectation,

$$\varphi_X(t) = E[1] + itE[X] - \frac{1}{2}t^2E[X^2] + o(t^2)$$

Also

$$E[X^2] = Var[X] - (E[X])^2 = 1$$

Therefore

$$\varphi_X(t) = 1 - \frac{1}{2}t^2 + o(t^2)$$

$\square$

Central Limit Theorem

Theorem (Central Limit Theorem)

If X_i are independent identically distributed random variables with $E[X_i] = \mu, Var[X_i] = \sigma^2$, then $S_n^* = \frac{1}{\sigma\sqrt{n}}\sum_{i=1}^{n}(X_i - \mu)$ converges weakly to $N(0,1)$.

Proof.

By Lemma 20 we denote the characteristic function of $\frac{X_i - \mu}{\sigma}$ by $\varphi(t)$, then $\varphi(t) = 1 - \frac{1}{2}t^2 + o(t^2)$. Therefore the characteristic function of S_n^* is

$$\varphi^n(t/\sqrt{n}) = [1 - \frac{t^2}{2n} + o(t^2/n)]^n$$

Take $n \to \infty$ and we get the characteristic function of $\lim_{n\to\infty} S_n^*$

$$\Phi(t) = \lim_{n\to\infty} \varphi^n(t/\sqrt{n}) = \lim_{n\to\infty} [1 - \frac{t^2}{2n}]^n = e^{-\frac{t^2}{2}}$$

Therefore $\lim_{n\to\infty} S_n^*$ converges to $N(0,1)$. □

Q&A

Thanks for your attention!