

# Moments and Characteristic Functions

## Instructor's Notes

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# 1 Moments

## 1.1 Definition of Moments

Generally, in math, the  $n$ -th moment of a real-valued continuous function about center  $c$  is: [1]

$$\mu_n = \int_{-\infty}^{\infty} (x - c)^n f(x) dx$$

In particular, for probability density functions  $f$  (or cumulative density function  $F$ ), the moments are given by

$$\mu'_n = E[X^n] = \int_{-\infty}^{\infty} x^n dF(x) = \int_{-\infty}^{\infty} x^n f(x) dx$$

Also we have the definition of the central moment [2]:

$$\mu_n = E[(X - E[X])^n] = \int_{-\infty}^{\infty} (x - \mu)^n f(x) dx$$

Generally central moments are more useful. Not to be confused with mean  $\mu$ .

## 1.2 Description of Moments

The first moment is the mean of a random variable, i.e.

$$\mu = E[X]$$

The second moment is related to the variance of a random variable:

$$\text{Var}[X] = E[X^2] - E[X]^2$$

In fact the variance is just the second central moment:

$$\text{Var}[X] = \mu_2 = E[(X - E[X])^2]$$

As for the third central moment, a related concept is skewness. Below shows two random variables with the same mean variance however different in skewness[3]:

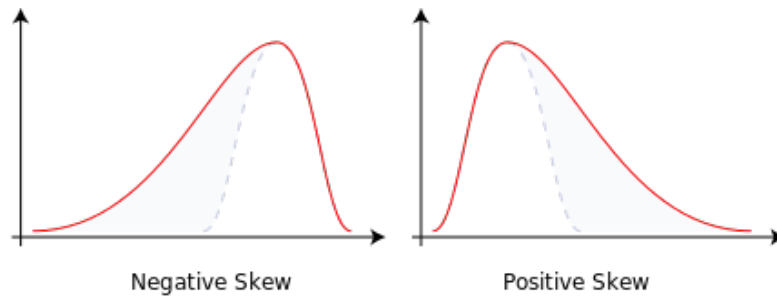


Figure 1: Negative and Positive Skew Diagrams

With all moments up to the order of infinity we can describe the **characteristics** of a probability distribution.

## 2 Characteristic Functions

### 2.1 Moment Generating Functions

**Definition 2.1** Let  $X$  be a random variable with probability density function  $f(x)$ . If there is a positive number  $h$  such that

$$\int_{-\infty}^{\infty} e^{tx} f(x) dx$$

exists and is finite for  $h < t < h$ , then the function defined by

$$M(t) = E[e^{tX}]$$

is called the moment-generating function of  $X$  (or of the distribution of  $X$ ). [4]

The  $r$ -th moment about the origin can be achieved from the moment generating function by evaluating the  $r$ -th derivative[5]:

$$M^{(r)}(0) = E[X^r]$$

Also notice the relation between the Taylor Expansion and the moments.

### 2.2 Characteristic Function

Notice that  $e^{tx}$  is not a "good" function in the sense that it is not bounded and may not converge under some circumstances. Before going to characteristic functions, we first get acquainted with knowledge of complex numbers:

#### 2.2.1 Basic information about complex numbers

Let  $z = a + bi$ , where  $a, b \in \mathbb{R}$ , and  $i = \sqrt{-1}$  is the imaginary unit.  $z$  is then called a complex number and  $a, b$  are called the real and imaginary parts of  $z$ , denoted by  $a = \text{Re}(z), b = \text{Im}(z)$  respectively. (Consider  $i$  as rotation by  $\frac{\pi}{2}$  counterclockwise in the complex plane)

The conjugate of a complex number  $z = a + bi, a, b \in \mathbb{R}$  is  $\hat{z} = a - bi$ , we also define the modulus (or length) of  $z$  to be  $|z| = z\hat{z}$ . Notice that  $|z|$  is a non-negative real number. Euler's formula:

$$e^{i\theta} = \cos \theta + i \sin \theta$$

The formula comes from Taylor's Series. It also gives rise to the polar representation of a complex number, i.e.  $z = re^{i\theta}$ , where  $r$  is the modulus and  $\theta$  is the phase.

From this we also have that  $|e^{i\theta}| = 1$  for any  $\theta$ .

#### 2.2.2 Definition of Characteristic Functions

**Definition 2.2** Let  $X$  be a random variable and denote by  $F$  the cumulative distribution function of  $X$ . The characteristic function  $\varphi = \varphi_X$  of  $X$  (or of  $F$ , in which case we also write  $\varphi_F$ ) is defined by [6]

$$\varphi_X(t) := E[e^{itX}] = \int_{-\infty}^{\infty} e^{itx} dF(x), t \in \mathbb{R}$$

### 2.2.3 Basic Properties

**Theorem 2.1 (Uniqueness Theorem [7])** *Let  $X$  be a real random variable with distribution function  $F$  and characteristic function  $\varphi$ . Similarly, let  $Y$  have distribution function  $G$  and characteristic function  $\psi$ . If  $\varphi(t) = \psi(t)$  for all  $t \in \mathbb{R}$  then  $F(x) = G(x)$  for all  $x \in \mathbb{R}$ .*

Properties from here on come from Bisgaard and Zoltan's book [6].

**Theorem 2.2** *If  $X$  and  $Y$  are independent random variables then the characteristic function of their sum is*

$$\varphi_{X+Y}(t) = \varphi_X \cdot \varphi_Y.$$

**Corollary 2.2.1** *The product of two characteristic functions is a characteristic function.*

**Remark** If  $X$  and  $Y$  are random variables such that  $\varphi_{X+Y} = \varphi_X \cdot \varphi_Y$ , then in general we do not conclude  $X$  and  $Y$  are independent. (See page 13 in [6])

**Theorem 2.3** *For any  $a, b \in \mathbb{R}$ ,*

$$\varphi_{aX+b}(t) = e^{ibt} \varphi_X(at).$$

**Theorem 2.4** *Every characteristic function  $\varphi$  has the following properties:*

- (i)  $\varphi(0) = 1$ ,
- (ii)  $|\varphi(t)| \leq 1$ ,
- (iii)  $\varphi(-t) = \overline{\varphi(t)}$
- (iv)  $\varphi$  is continuous on  $\mathbb{R}$

## 2.3 Common Distributions and Their Characteristic Functions

Table 1: Characteristic Functions for Common Distributions[8]

| Distribution                                     | PMF/PDF   | Characteristic Function                             |
|--|---|---|
| Constant $X \equiv a$                            | -   | $\varphi_X(t) = e^{iat}$ .                          |
| Binomial $X \sim \text{Binomial}(m, p)$          | $p_X(n) = \binom{m}{n} p^n (1-p)^{m-n}$                                 | $\varphi_X(t) = (pe^{it} + (1-p))^m$                |
| Poisson $X \sim \text{Poisson}(\lambda)$         | $p_X(n) = \frac{\lambda^n}{n!} e^{-\lambda}$                            | $\varphi_X(t) = e^{\lambda(e^{it}-1)}$              |
| Exponential $X \sim \text{Exponential}(\lambda)$ | $p_X(n) = \lambda e^{-\lambda x}$                                       | $\varphi_X(t) = \frac{\lambda}{\lambda - it}$       |
| Normal $X \sim N(0, 1)$                          | $f_X(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}$                     | $\varphi_X(t) = e^{-\frac{t^2}{2}}$                 |
| Normal $Y \sim N(\mu, \sigma^2)$                 | $f_Y(y) = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{(y-\mu)^2}{2\sigma^2}}$ | $\varphi_Y(t) = e^{it\mu - \frac{\sigma^2 t^2}{2}}$ |

## 3 Examples and Applications of Characteristic Functions

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