

Characteristic Functions

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April 7, 2019

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Sum of Independent Normals

The Normal Sum Problem

$X_1 \sim N(\mu_1, \sigma_1^2), X_2 \sim N(\mu_2, \sigma_2^2)$. X_1, X_2 are independent. Distribution of $Y = X_1 + X_2$?

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But what is the distribution?

Characteristic Functions

To deal with the sums of random variables, **characteristic function** is a powerful weapon for us.

Definition

Let X be a random variable and denote by F the cumulative distribution function of X (or f the probability density function). The characteristic function $\varphi = \varphi_X$ of X (or of F , in which case we also write φ_F) is defined by

$$\varphi_X(t) := E[e^{itX}] = \int_{-\infty}^{\infty} e^{itx} dF(x) = \int_{-\infty}^{\infty} e^{itx} f(x) dx, t \in \mathbb{R}$$

Recall Complex Numbers

If you have forgotten everything about complex numbers, all you need to recall is the following:

- i is the imaginary unit,
- $i^2 = -1$,
- $e^{i\theta} = \cos \theta + i \sin \theta$. (This formula comes from Taylor Series)

Theorem (Uniqueness Theorem)

Let X be a real random variable with distribution function F and characteristic function φ . Similarly, let Y have distribution function G and characteristic function ψ . If $\varphi(t) = \psi(t)$ for all $t \in \mathbb{R}$ then $F(x) = G(x)$ for all $x \in \mathbb{R}$.

From this we may easily conclude the distribution of a random variable if we can prove that its characteristic function is of the same form as a known distribution.

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Theorem (Inversion Formula)

If $\int_{\mathbb{R}} |\varphi(t)| dt < \infty$ then X has bounded continuous density

$$f(x) = \frac{1}{2\pi} \int_{\mathbb{R}} e^{-itx} \varphi(t) dt$$

Theorem

If X and Y are independent random variables then the characteristic function of their sum is

$$\varphi_{X+Y}(t) = \varphi_X \cdot \varphi_Y.$$

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If X and Y are random variables such that $\varphi_{X+Y} = \varphi_X \cdot \varphi_Y$, then in general we do not conclude X and Y are independent. This is called **subindependence**.

Theorem

For any $a, b \in \mathbb{R}$,

$$\varphi_{aX+b}(t) = e^{ibt} \varphi_X(at).$$

Distributions and Their Characteristic Functions

Example (Characteristic Function for Exponential)

Find the characteristic function for $X \sim \text{Exponential}(\lambda)$.

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$$\begin{aligned}\varphi_X(t) &:= E[e^{itX}] \\&= \int_{-\infty}^{\infty} e^{itx} f(x) dx \\&= \int_0^{\infty} e^{itx} \cdot \lambda e^{-\lambda x} dx && \text{By distribution of } x \\&= \frac{\lambda}{it - \lambda} e^{(it - \lambda)x} \Big|_0^{\infty} \\&= \frac{\lambda}{\lambda - it} && \text{By Squeeze Theorem}\end{aligned}$$

Distributions and Their Characteristic Functions

The last two steps

$$\left. \frac{\lambda}{it - \lambda} e^{(it - \lambda)x} \right|_0^\infty = \frac{\lambda}{\lambda - it}$$

can be justified by

$$\lim_{x \rightarrow \infty} |e^{(it - \lambda)x}| = \lim_{x \rightarrow \infty} |e^{-\lambda x}| = 0 \text{ for positive } \lambda$$

Distributions and Their Characteristic Functions

The table below shows some common distributions and their characteristic functions:

Table: Characteristic Functions for Common Distributions

Distribution	Characteristic Function
Constant $X \equiv a$	$\varphi_X(t) = e^{iat}$.
Binomial $X \sim \text{Binomial}(m, p)$	$\varphi_X(t) = (pe^{it} + (1 - p))^m$
Poisson $X \sim \text{Poisson}(\lambda)$	$\varphi_X(t) = e^{\lambda(e^{it}-1)}$
Exponential $X \sim \text{Exponential}(\lambda)$	$\varphi_X(t) = \frac{\lambda}{\lambda - it}$
Normal $X \sim N(0, 1)$	$\varphi_X(t) = e^{-\frac{t^2}{2}}$
Normal $Y \sim N(\mu, \sigma^2)$	$\varphi_Y(t) = e^{it\mu - \frac{\sigma^2 t^2}{2}}$

Statistics and Probability

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To understand this, first we need some knowledge about **moments**.

Definition (Moment)

For probability density functions f (or cumulative density function F), the moments are given by

$$\mu'_n = E[X^n] = \int_{-\infty}^{\infty} x^n dF(x) = \int_{-\infty}^{\infty} x^n f(x) dx$$

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Relation with statistics. Observe that the first moment is simply the expectation. The second moment is related to the variance:

$$\text{Var}[X] = E[X^2] - (E[X])^2.$$

What about the third moment?

Statistics and Probability

What about the third moment?

A related concept is **skewness**:

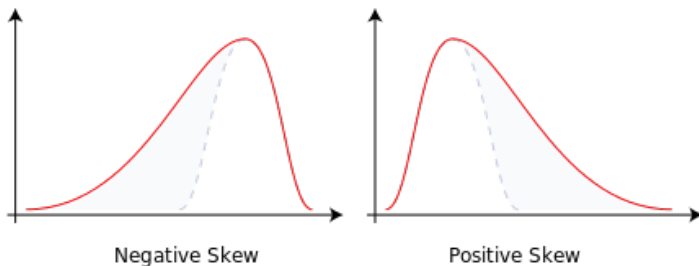


Figure: Same expectation and variance, but different skewness