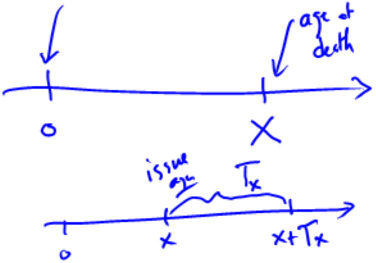
# Survival Models

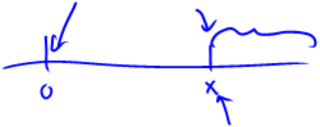
Lecture: Weeks 2-3

# Chapter summary



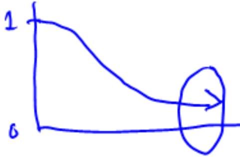

- Survival models
  - Age-at-death random variable
  - Time-until-death random variables
  - Force of mortality (or hazard rate function) ✓
  - Some parametric models
    - De Moivre's (Uniform), Exponential, Weibull, Makeham, Gompertz ✓
    - Generalization of De Moivre's
  - Curtate future lifetime
- Chapter 2 (Dickson, Hardy and Waters = DHW)

# Age-at-death random variable




- $X$ is the age-at-death random variable; continuous, non-negative
- $X$ is interpreted as the lifetime of a newborn (individual from birth)
- Distribution of $X$ is often described by its survival distribution function (SDF):

$$S_0(x) = \mathsf{Pr}[X > x]$$




  - other term used: survival function
- Properties of the survival function:
  - $S_0(0) = 1$: probability a newborn survives 0 years is 1.
  - $S_0(\infty) = \lim_{x \to \infty} S_0(x) = 0$: all lives eventually die.
  - non-increasing function of $x$: not possible to have a higher probability of surviving for a longer period.

# Cumulative distribution and density functions

$S_0(x) := Pr(X > x)$

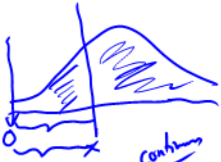
$\Downarrow$

$Pr(X \geq x)$

$1 - S_0(x)$

- Cumulative distribution function (CDF): $F_0(x) = \text{Pr}[X \leq x]$
  - nondecreasing; $F_0(0) = 0$; and $F_0(\infty) = 1$.

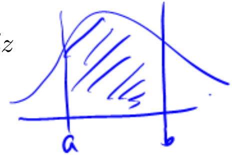
- Clearly we have: $F_0(x) = 1 - S_0(x)$
- Density function: $f_0(x) = \dfrac{dF_0(x)}{dx} = -\dfrac{dS_0(x)}{dx}$
  - non-negative: $f_0(x) \geq 0$ for any $x \geq 0$
  - in terms of CDF: $F_0(x) = \displaystyle\int_0^x f_0(z)dz$
  - in terms of SDF: $S_0(x) = \displaystyle\int_x^\infty f_0(z)dz$



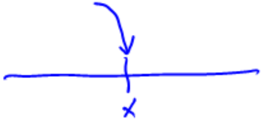

continuous

$X \geq 0$

$\int_c^b$



# Force of mortality




- The force of mortality for a newborn at age $x$:

$$\mu_x = \frac{f_0(x)}{1-F_0(x)} = \frac{f_0(x)}{S_0(x)} = -\frac{1}{S_0(x)}\frac{dS_0(x)}{dx} = -\frac{d\log S_0(x)}{dx}$$

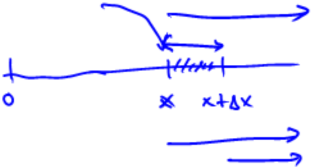
- Interpreted as the conditional instantaneous measure of death at $x$.

- For very small $\Delta x$, $\mu_x \Delta x$ can be interpreted as the probability that a newborn who has attained age $x$ dies between $x$ and $x+\Delta x$:

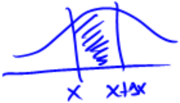
$$\mu_x \Delta x \approx \Pr[x < X \leq x+\Delta x | X > x]$$

- Other term used: hazard rate at age $x$.

$$\Pr(A|B) = \Pr(A,B)/\Pr(B)$$




$$\lim_{\Delta x \to 0} \frac{1}{\Delta x} \Pr\left[ x < X \leq x+\Delta x \,\middle|\, X > x \right] = \mu_x$$




$$\lim_{\Delta x \to 0} \frac{1}{\Delta x} \frac{\Pr[x < X \leq x+\Delta x]}{\Pr[X > x]} = \lim_{\Delta x \to 0} \frac{1}{\Delta x} \frac{S_0(x) - S_0(x+\Delta x)}{S_0(x)}$$

$$= -\frac{1}{S_0(x)} \underbrace{\lim_{\Delta x \to 0} \frac{S_0(x+\Delta x) - S_0(x)}{\Delta x}}_{\frac{dS_0(x)}{dx}}$$

$X$

$S_0(x)$

$F_0(x)$

$f_0(x)$

$\mu_x$

$$\mu_x = -\frac{1}{S_0(x)} \frac{dS_0(x)}{dx} = \frac{f_0(x)}{S_0(x)} = \frac{f_0(x)}{1 - F_0(x)}$$

$$\mu_x = \frac{-\frac{d}{dx} S_0(x)}{S_0(x)} = -\frac{d}{dx} \log S_0(x)$$

natural log ✓

$\log = \log_e$
$= \ln$ ✓

$$\int dF(x) = F(x) + c$$

$$\mu_x \, dx = -d \log S_0(x)$$

$$\int_0^x \mu_z \, dz = \int_0^x -d \log S_0(z)$$

$$e^{-\int_0^x \mu_z dz} = e^{\underbrace{\int_0^x d \log S_0(z)}_{\log S_0(z)\big|_0^x}} = e^{\log S_0(x) - \cancel{\log \underbrace{S_0(0)}_{1}}}$$

$$S_0(x) = e^{-\int_0^x \mu_z \, dz} \longrightarrow$$

Given


$f_0(x)$

$$F_0(x)=\int_0^x f_0(z)\,dz\ ,\quad S_0(x)=1-\int_0^x f_0(z)\,dz=\int_x^\infty f_0(z)\,dz\ ,\quad \mu_x=\frac{f_0(x)}{\int_x^\infty f_0(z)\,dz}$$

$F_0(x)$

$$f_0(x)=\frac{d}{dx}F_0(x)\ ,\quad S_0(x)=1-F_0(x)\ ,\quad \mu_x=\frac{\frac{d}{dx}F_0(x)}{1-F_0(x)}$$

$\checkmark\, S_0(x)$

$$f_0(x)=-\frac{d}{dx}S_0(x)\ ,\quad F_0(x)=1-S_0(x)\ ,\quad \mu_x=\frac{-1}{S_0(x)}\frac{d}{dx}S_0(x)$$

$\checkmark\, \mu_x$

$$S_0(x)=e^{-\int_0^x \mu_z\,dz}\ ,\quad F_0(x)=1-e^{-\int_0^x \mu_z\,dz}\ ,\quad f_0(x)=\mu_x\cdot e^{-\int_0^x \mu_z\,dz}$$

$$\mu_x=\frac{f_0(x)}{S_0(x)} \Rightarrow f_0(x)=\mu_x S_0(x)$$
$$=\mu_x e^{-\int_0^x \mu_z\,dz}$$

<u>Example</u>: $\mu_x = .02$, for $x \geq 0$

$$S_0(x) = e^{-\int_0^x .02\,dz} = e^{-.02x}$$

$$F_0(x) = 1 - e^{-.02x}$$

$$f_0(x) = \mu_x e^{-\int_0^x \mu_z dz} = \underbrace{.02\, e^{-.02x}}_{\text{Exponential distribution}}, \quad x \geq 0$$

$\mu_x = \mu$, independent of age $x$ $\Rightarrow$ Exponential $\underbrace{f_0(x) = \mu e^{-\mu x}}$

constant force

$E(X) = \frac{1}{\mu}$, $\mu = .02$

$= \underline{\underline{50}}$

Example  Uniform distribution  $f_0(x) = \frac{1}{\omega}, \quad 0 \le x < \omega$

$\omega =$ limiting age


$$F_0(x) = \Pr[X \le x] = \frac{x}{\omega}$$

$$\int_0^x \frac{1}{\omega}\,dz = \frac{x}{\omega}$$

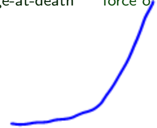
$$S_0(x) = 1 - \frac{x}{\omega}$$

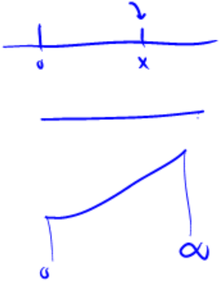
$$\mu_x = \frac{f_0(x)}{\int_x^\infty f_0(z)\,dz} = \frac{1/\omega}{\int_x^\omega \frac{1}{\omega}\,dz} = \frac{1/\omega}{1 - x/\omega} = \frac{1/\omega}{\frac{\omega - x}{\omega}} = \frac{1}{\omega - x}$$

"De Moivre's"  De Moivre's law

e.g. Mortality follows De Moivre's with $\omega = 100$

# Some properties of $\mu_x$






Some important properties of the force of mortality:

- non-negative: $\mu_x \geq 0$ for every $x > 0$ ✓
- divergence: $\int_0^\infty \mu_x dx = \infty$.
- in terms of SDF: $S_0(x) = \exp\left(-\int_0^x \mu_z dz\right)$. ✓ $e^{-\int_0^x \mu_z dz}$
- in terms of PDF: $f_0(x) = \mu_x \exp\left(-\int_0^x \mu_z dz\right)$. ✓ $\mu_x e^{-\int_0^x \mu_z dz}$

# Moments of age-at-death random variable

$\int_0^\infty [1-F_0(x)]\,dx$

- The mean of $X$ is called the complete expectation of life at birth:

average age of a newborn

$$\underline{\underline{\mathring{e}_0}} = \underline{\underline{\mathsf{E}[X]}} = \int_0^\infty x f_0(x)\,dx = \int_0^\infty S_0(x)\,dx.$$

- The RHS of the equation can be derived using integration by parts.
- Variance:

$$\mathsf{Var}[X] = \mathsf{E}\left[X^2\right] - (\mathsf{E}[X])^2 = \mathsf{E}\left[X^2\right] - (\mathring{e}_0)^2.$$


- The median age-at-death $m$ is the solution to

median age at death

$$S_0(m) = F_0(m) = \frac{1}{2}.$$

$\mu_x = .02,\ x \geq 0$ exponential $f_0(x) = .02\, e^{-.02x}$

$E(x) = \frac{1}{.02} = 50$

$Var(x) = \frac{1}{.02^2} = 2500$

$S_0(x) = e^{-.02x}$

$S_0(m) = e^{-.02m} = \frac{1}{2} = 0.5$

$m = -\frac{1}{.02} \log(0.5) = \underline{\underline{34.65736}}$

# Some special parametric laws of mortality

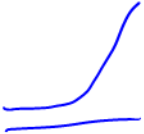
mortality laws

| Law/distribution | $\mu_x$ | $S_0(x)$ | Restrictions |
|---|---|---|---|
| ✓ De Moivre (uniform) | $1/(\omega - x)$ | $1 - (x/\omega)$ | $0 \leq x < \omega$ |
| ✓ Constant force (exponential) | $\mu$ | $\exp(-\mu x)$ | $x \geq 0, \mu > 0$ |
| ✓ Gompertz | $Bc^x$ | $\exp\left[-\frac{B}{\log c}(c^x - 1)\right]$ | $x \geq 0, B > 0, c > 1$ |
| ✓ Makeham | $A + Bc^x$ | $\exp\left[-Ax - \frac{B}{\log c}(c^x - 1)\right]$ | $x \geq 0, B > 0, c > 1,$ $A \geq -B$ |
| Weibull | $kx^n$ | $\exp\left(-\frac{k}{n+1}x^{n+1}\right)$ | $x \geq 0, k > 0, n > 1$ |

$$\mu_x = \underbrace{Bc^x}_{\text{exponentially increasing}}$$

$B, c$ constants

$B>0,\ c>1$



$$S_0(x) = e^{-\int_0^x Bc^z\,dz} = e^{-B\underbrace{\int_0^x c^z\,dz}}$$

$$= e^{-\frac{B}{\log c}\left[c^z\big|_0^x\right]}$$

$c^x - 1$

$$= e^{-\frac{B}{\log c}(c^x-1)}$$

$c^z = e^{z\log c}$

$\int c^z dz = \int e^{\underline{z\log c}}$

$= \frac{1}{\log c} e^{z\log c}$

$= \frac{1}{\log c} c^z$

$c^\infty \to \infty$

$F_0(x)$

$f_0(x)$

properties

- $S_0(1) = e^{-\frac{B}{\log c}(c^0-1)} = e^0 = 1$
- $S_0(\infty) = e^{-\frac{B}{\log c}(c^\infty-1)} = e^{-\infty} \to 0$ eventually dies
- nonincreasing $\frac{d}{dx}S_0(x) = \underbrace{e^{-\frac{B}{\log c}(c^x-1)}}_{>0} \cdot \left(\frac{-B}{\log c} \cdot \underbrace{\frac{d}{dx}c^x}_{>0}\right) < 0$

Gompertz: $\mu_x = Bc^x$

constant

Makeham: $\mu_x = Bc^x + A$

exponentially increasing

independent of age

causes of death

accident

derive

$$S_0(x) = e^{-\int_0^x (A + Bc^x)\,dx}$$

$$= e^{-Ax - \frac{B}{\log c}(c^x - 1)}$$

$S_0(0) = 1$

$S_0(\infty) \to 0$

$\frac{d}{dx} S_0(x) < 0$

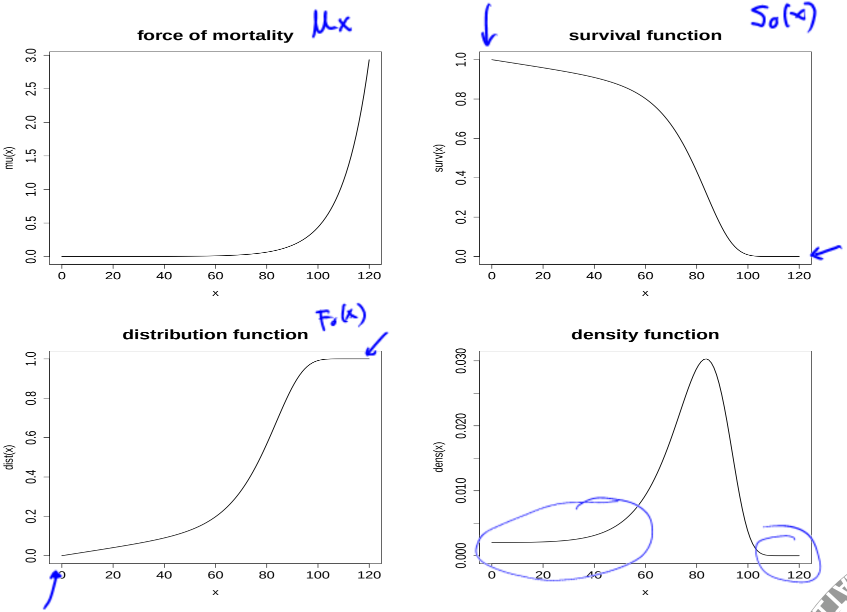



Figure: Makeham's law: $A = 0.002$, $B = 10^{-4.5}$, $c = 1.10$

# Illustrative example 1

Suppose $X$ has survival function defined by

$$S_0(x) = \frac{1}{10}(100 - x)^{1/2}, \quad \text{for} \quad 0 \leq x \leq 100.$$

1. Explain why this is a legitimate survival function.
2. Find the corresponding expression for the density of $X$.
3. Find the corresponding expression for the force of mortality at $x$.
4. Compute the probability that a newborn with survival function defined above will die between the ages 65 and 75.

Solution to be discussed in lecture.

Rewrite $S_0(x) = \frac{1}{10}(100-x)^{1/2} = \left(\frac{100-x}{100}\right)^{1/2} = \left(1-\frac{x}{100}\right)^{1/2}$ form $\left(1-\frac{x}{\omega}\right)^{\alpha}$

$0 \le x \le 100$ $\alpha = 1/2,\ \omega = 100$

① $S_0(0) = 1$

$S_0(\infty) = S_0(100) = 0$

$\frac{d}{dx}S_0(x) = \frac{-1}{200}\left(1-\frac{x}{100}\right)^{-1/2} < 0$ non-increasing

$\rightarrow$ legitimate survival function


② $f_0(x) = -\frac{d}{dx}S_0(x) = \frac{1}{200}\left(1-\frac{x}{100}\right)^{-1/2}$

③ $\mu_x = \frac{f_0(x)}{S_0(x)} = \frac{\frac{1}{200}\left(1-\frac{x}{100}\right)^{-1/2}}{\left(1-\frac{x}{100}\right)^{1/2}} = \frac{\frac{1}{200}}{\left(1-\frac{x}{100}\right)} = \frac{1}{2(100-x)}$

④ $Pr(65 < X \le 75) = S_0(65) - S_0(75) = \frac{1}{10}\sqrt{35} - \frac{1}{10}\sqrt{25} \approx \underline{\underline{.09161}}$

$\rightarrow Pr(X \le 75) - Pr(X \le 65)$

Using same survival function $S_0(x) = \frac{1}{10}(100-x)^{1/2}$

Calculate the probability that a 50-year-old will die between ages 65 and 75.


$$Pr(65 < X \leq 75 \mid X > 50)$$

$$= \frac{Pr(65 < X \leq 75,\ X > 50)}{Pr(X > 50)} = \frac{Pr(65 < X \leq 75)}{Pr(X > 50)}$$

$$= \frac{\cancel{\frac{1}{10}}\sqrt{35} - \cancel{\frac{1}{10}}\sqrt{25}}{\cancel{\frac{1}{10}}\sqrt{50}} = \frac{\sqrt{35} - \sqrt{25}}{\sqrt{50}} = \underline{\underline{0.1295532}}$$

why larger than

.09161 ???

# Practice problem - SOA MLC Spring 2016 Question #2

Use $\mu_x = -\frac{d}{dx} \log S_0(x)$ since $\log S_0(x) = \frac{1}{3} \log\left(1 - \frac{x}{60}\right)$

You are given the survival function:

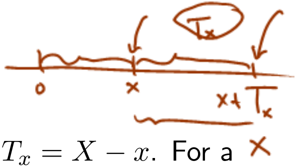
$$S_0(x) = \left(1 - \frac{x}{60}\right)^{1/3}, \quad \text{for} \quad 0 \leq x \leq 60.$$

Calculate $1000\mu_{35}$.

$$\mu_x = -\frac{d}{dx} \log S_0(x) = -\frac{1}{3} \frac{1}{1-\frac{x}{60}} \cdot \left(-\frac{1}{60}\right) = \frac{1}{3(60-x)}$$

$$1000\mu_{35} = 1000 \frac{1}{3\underbrace{(60-35)}_{25}} = 1000 \frac{1}{75} \approx \underline{\underline{13.3333}}$$

## 2.2 Future lifetime random variable




- For a person now age $x$, its future lifetime is $T_x = X - x$. For a newborn, $x = 0$, so that we have $T_0 = X$.
- Life-age-$x$ is denoted by $(x)$.
- SDF: It refers to the probability that $(x)$ will survive for another $t$ years.

$$S_x(t) = \Pr[T_0 > x + t | T_0 > x] = \frac{S_0(x+t)}{S_0(x)} = {}_t p_x = 1 - {}_t q_x$$

- CDF: It refers to the probability that $(x)$ will die within $t$ years.

$$F_x(t) = \Pr[T_0 \leq x + t | T_0 > x] = \frac{S_0(x) - S_0(x+t)}{S_0(x)} = {}_t q_x$$

$f, S, F, \mu$


$$\underbrace{\Pr[T_x > t]}_{S_x(t)} \quad \underbrace{\Pr[X > x+t \mid X > x]}_{= \frac{\Pr[X > x+t,\, X > x]}{\Pr[X > x]}} = \frac{\Pr[X > x+t]}{\Pr[X > x]} = \frac{S_0(x+t)}{S_0(x)}$$

$x = 0 \Rightarrow$ newborn

$X = T_0$

Survival function of $T_x$.

probability that $(x)$ dies before $t$ years

$$\Pr[T_x \le t] = 1 - S_x(t) = 1 - \Pr[T_x > t] = 1 - {}_tp_x = {}_tq_x$$

$$\Pr[T_x > t] = {}_tp_x = \text{prob that } (x) \text{ will live or survive another } t \text{ years}$$

$t$ in years

Remark: $t = 1 \Rightarrow$ drop $t=1$ in symbol

$p_x \qquad {}_1p_x = p_x$

$q_x \qquad {}_1q_x = q_x$

$$\Pr[T_x>t] = {}_tp_x = S_x(t) = \frac{S_0(x+t)}{S_0(x)} \checkmark \qquad S_0 = 1-F_0$$

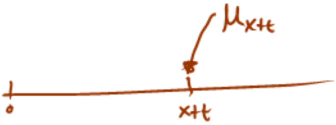
$$\Pr[T_x\le t] = {}_tq_x = 1-{}_tp_x = 1-S_x(t) = \frac{S_0(x)-S_0(x+t)}{S_0(x)} = \underbrace{\frac{F_0(x+t)-F_0(x)}{1-F_0(x)}}_{F_x(t)}$$

$$f_x(t) = \frac{d}{dt}F_x(t) = -\frac{d}{dx}S_x(t) = -\frac{d}{dt}\frac{S_0(x+t)}{S_0(x)} = +\frac{1}{S_0(x)}f_0(x+t) = \frac{f_0(x+t)}{S_0(x)}$$

force of mortality of $T_x$

$$\mu_x(t) = \frac{f_x(t)}{S_x(t)} = \frac{f_0(x+t)/\cancel{S_0(x)}}{S_0(x+t)/\cancel{S_0(x)}} = \frac{f_0(x+t)}{S_0(x+t)} = \mu_{x+t}$$

$$\mu_0(x) \longrightarrow \mu_x$$



$E[X] = \mathring{e}_0$ average lifetime of newborn


$E[T_x] = \mathring{e}_x$ average lifetime of (x)

$$x + \mathring{e}_x \geq \mathring{e}_0$$

$$\int_0^\infty S_0(x)\,dx$$

$$\int_0^\infty \underbrace{S_x(t)}_{{}_tp_x}\,dt = \int_0^\infty {}_tp_x\,dt \implies \boxed{\mathring{e}_x = \int_0^\infty {}_tp_x\,dt}$$

$$Pr[t < T_x \le u+t] =$$

$$Pr[T_x \le u+t] - Pr[T_x \le t]$$

$$\underbrace{{}_{t+u}q_x - {}_tq_x}$$

$$\underbrace{{}_tp_x - {}_{t+u}p_x}$$


$$p = 1 - q$$

probability that (x) will live another t years and then die the following u years

$$= \underbrace{{}_tp_x}\cdot {}_uq_{x+t} = \text{deferred probability} = {}_{t|u}q_x$$

age today

deferred period

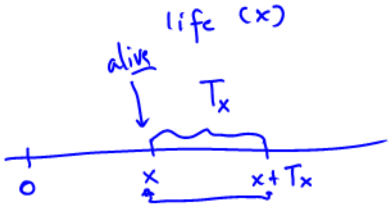
death period

Remark $u=1 \Rightarrow {}_{t|1}q_x = {}_{t|}q_x$

$X$ = age at death (newborn)

$T_x$ = future lifetime of a person age now age $x$

= future lifetime of $(x)$




$$S_x(t) = \Pr[T_x > t] = \text{probability } (x) \text{ will live another } t \text{ years}$$

$$= \Pr[X > x+t \mid X > x] = \frac{\Pr[X > x+t,\ X > x]}{\Pr[X > x]}$$

$$= \frac{\Pr[X > x+t]}{\Pr[X > x]} = \frac{S_0(x+t)}{S_0(x)}$$

$\underline{S_0} \rightarrow \underline{\underline{S_x}}$

$$E(x) = \int_0^{\infty} x \underbrace{f_0(x)}\,dx = \int_0^{\infty} S_0(x)\,dx$$

$$f_0(x) = \frac{d}{dx} F_0(x)$$

$$= -\frac{d}{dx}\left[1 - F_0(x)\right] = -\frac{d}{dx} S_0(x)$$

$$\underbrace{-\int_0^{\infty} x \, dS_0(x)}_{\text{try this}} = \int_0^{\infty} S_0(x)\,dx$$

$$\int u\,dv = uv - \int v\,du$$

# - continued

- Density:

$$f_x(t) = \frac{dF_x(t)}{dt} = -\frac{dS_x(t)}{dt} = \frac{f_0(x+t)}{S_0(x)}.$$

- Remark: If $t = 1$, simply use $p_x$ and $q_x$.
- $p_x$ refers to the probability that $(x)$ survives for another year.
- $q_x = 1 - p_x$, on the other hand, refers to the probability that $(x)$ dies within one year.

# Conditions to be valid

To reiterate, these are the conditions for a survival function to be considered valid:

- $S_x(0) = 1$: probability a person age $x$ survives 0 years is 1.
- $S_x(\infty) = \lim_{t\to\infty} S_x(0) = 0$: all lives, regardless of age, eventually die. $=1$
- The survival function $S_x(t)$ for a life $(x)$ must be a non-increasing function of $t$.

$$\frac{d}{dt} S_x(t) \leq 0$$

## 2.3 Force of mortality of $T_x$

$= \mu_{x+t}$

- In deriving the force of mortality, we can use the basic definition:

${}_tp_x \, \mu_{x+t}$

$$\mu_x(t) = \frac{f_x(t)}{S_x(t)} = \frac{f_0(x+t)}{S_0(x)} \cdot \frac{S_0(x)}{S_0(x+t)}$$
$$= \frac{f_0(x+t)}{S_0(x+t)} = \mu_{x+t}.$$

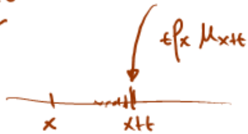
- This is easy to see because the condition of survival to age $x+t$ supercedes the condition of survival to age $x$.
- This results implies the following very useful formula for evaluating the density of $T_x$:

$$f_x(t) = {}_tp_x \times \mu_{x+t} \Rightarrow \quad {}_tp_x \, \mu_{x+t}$$

$$E[T_x] = \int_0^\infty t \cdot f_x(t)\,dt = \int_0^\infty t \cdot \underbrace{{}_tp_x\, \mu_{x+t}}\,dt$$

$$E[g(T_x)] = \int_0^\infty \underbrace{g(t) \cdot \underbrace{{}_tp_x\, \mu_{x+t}}\,dt}$$

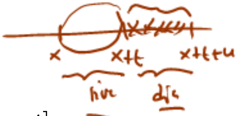
overall possible values of $t$



# Special probability symbol

deferred probability

- The probability that $(x)$ will survive for $t$ years and die within the next $u$ years is denoted by ${}_{t|u}q_x$. This is equivalent to the probability that $(x)$ will die between the ages of $x+t$ and $x+t+u$.
- This can be computed in several ways:




deferred

death

$$
\begin{aligned}
{}_{t|u}q_x &= \Pr[t < T_x \le t+u] \\
&= \Pr[T_x \le t+u] - \Pr[T_x < t] \\
&= {}_{t+u}q_x - {}_tq_x \\
&= {}_tp_x - {}_{t+u}p_x \\
&= \underbrace{{}_tp_x \times {}_uq_{x+t}}.
\end{aligned}
$$

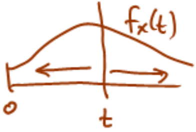
- If $u = 1$, prefix is deleted and simply use ${}_{t|}q_x$.

# Other useful formulas

$$\underbrace{f_x, S_x, F_x, \mu_{x+t}}$$

- It is easy to see that

$$F_x(t) = \int_0^t f_x(s)ds$$




which in actual notation can be written as

$${}_sp_x\,\mu_{x+s}$$

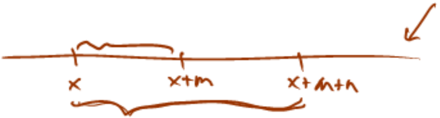
$${}_tq_x = \int_0^t {}_sp_x\ \mu_{x+s}ds$$

$$\int_0^t {}_sp_x\,\mu_{x+s}\,ds$$


- See Figure 2.3 for a very nice interpretation.
- We can generalize this to

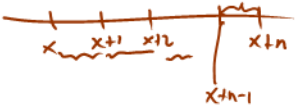
$$\underbrace{{}_{t|u}q_x = \int_t^{t+u} {}_sp_x\ \mu_{x+s}ds}$$

Survival functions p's are <u>multiplicative</u>, but q's <u>not</u>




$${}_{m+n}p_x = {}_m p_x \cdot {}_n p_{x+m}$$

$${}_n p_x = \underbrace{p_x \, p_{x+1} \, p_{x+2} \cdots p_{x+n-1}}_{n \text{ years}}$$




$$\cancel{{}_{m+n}q_x \neq {}_m q_x \cdot {}_n q_{x+m}}$$ not true

$${}_{m+n}q_x = 1 - {}_{m+n}p_x = 1 - {}_m p_x \cdot {}_n p_{x+m} = 1 - {}_n p_x \cdot {}_m p_{x+n}$$

$$= {}_m q_x + {}_m p_x \, {}_n q_{x+m}$$

$X \sim$ exponential $\Rightarrow \mu_x = \mu$

$T_x \sim$ exponential $\Leftarrow \underline{\mu_{x+t}} = \underline{\mu}$

realistic newborn $(x)$ $\Rightarrow$ same lifetime distribution

$$f_x(t) = \underbrace{{}_tp_x}_{e^{-\mu t}} \underbrace{\mu_{x+t}}_{\mu} = \mu e^{-\mu t}$$

$$S_x(t) = \frac{S_0(x+t)}{S_0(x)} = \frac{e^{-\int_0^{x+t} \mu_s \, ds}}{e^{-\int_0^{x} \mu_s \, ds}} = e^{-\int_x^{x+t} \underbrace{\mu_s}_{\mu} \, ds} = e^{-\mu t}$$

$X \sim$ de Moivre's law  Uniform $f_0(x) = \frac{1}{\omega}, \ 0 \le x \le \omega$

$T_x \sim$ de Moivre's law

$$f_x(t) = \frac{f_0(x+t)}{S_0(x)} = \frac{\frac{1}{\omega}}{1 - \frac{x}{\omega}} = \frac{\frac{1}{\omega}}{\frac{\omega - x}{\omega}} = \frac{1}{\omega - x}, \quad 0 \le t \le \omega - x$$

# 2.6 Curtate future lifetime

- Curtate future lifetime of $(x)$ is the number of future years completed by $(x)$ prior to death.
- $K_x = \lfloor T_x \rfloor$, the greatest integer of $T_x$.
- Its probability mass function is

$$\begin{aligned} \mathsf{Pr}[K_x = k] &= \mathsf{Pr}[k \leq T_x < k+1] = \mathsf{Pr}[k < T_x \leq k+1] \\ &= S_x(k) - S_x(k+1) = {}_{k+1}q_x - {}_k q_x = {}_{k|}q_x, \end{aligned}$$

for $k = 0, 1, 2, \ldots$

- Its distribution function is

$$\mathsf{Pr}[K_x \leq k] = \sum_{h=0}^{k} {}_{h|}q_x = {}_{k+1}q_x.$$

(x) future lifetime $T_x$ continuous

curtate future lifetime $K_x$ discrete

→ exact number of years you have lived before death

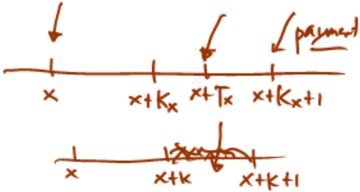
$K_x = \lfloor T_x \rfloor$ = greatest integer function of $T_x$

$\lfloor 4.3 \rfloor = 4$

$\lfloor 51.2 \rfloor = 51$

$\lfloor 69.99999 \rfloor = 69$

$K_x \in \{0, 1, 2, \ldots, \infty\}$




probability mass function

$$\Pr[K_x = k] = {}_{k|1}q_x = {}_{k|}q_x$$

$$= {}_{k+1}q_x - {}_{k}q_x$$

$$= {}_{k}p_x - {}_{k+1}p_x = {}_{k}p_x \cdot q_{x+k}$$

$$\underbrace{\Pr[K_x \le k]}_{\text{distribution of } K_x} = \underbrace{\sum_{j=0}^{k} \underbrace{\Pr[K_x = j]}}$$


$$\sum_{j=0}^{k} {}_{j|}q_x = {}_{k+1}q_x$$

---

de Moivre's law

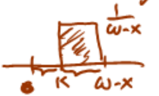
$X \sim$ Uniform on $[0, \omega]$

$T_x \sim$ Uniform on $[0, \omega - x]$

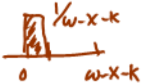
$K_x \sim$ ???


$$\Pr[K_x = k] = {}_{k|}q_x = \underbrace{{}_kp_x} \cdot \underbrace{q_{x+k}}$$

discrete uniform




$$= \underbrace{\frac{\omega - \cancel{x-k}}{\omega - x}} \cdot \underbrace{\frac{1}{\omega - \cancel{x-k}}} = \frac{1}{\omega - x}, \quad k = 0, 1, \ldots, \omega - x - 1$$




$$\sum_{k=0}^{\omega - x - 1} \Pr[K_x = k] = \underbrace{\frac{1}{\omega - x} + \frac{1}{\omega - x} + \cdots \frac{1}{\omega - x}}_{\omega - x \text{ terms}} = 1$$

$X \sim \text{Exponential}(\mu)$

$T_x \sim \text{Exponential}(\mu)$

$K_x \sim ??$

$p = e^{-\mu}$

$q = 1 - e^{-\mu}$

$= 1 - p$

$(1-p)p^k$

$qp^k$

$\Downarrow$ special

geometric

$$\Pr[K_x = k] = {}_{k|}q_x = {}_k p_x \cdot q_{x+k}$$

$${}_k p_x \to e^{-\mu k} \qquad q_{x+k} \to (1 - e^{-\mu})$$

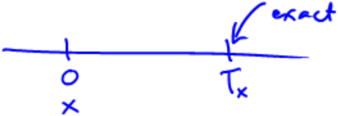
$$= \underbrace{(1-e^{-\mu})e^{-\mu k}}, \quad k = 0, 1, 2, \ldots \infty$$

Show $\sum_{k=0}^{\infty} (1-e^{-\mu})e^{-\mu k} = 1$ show

$(x) \Rightarrow T_x$

$\Downarrow$

$K_x = \lfloor T_x \rfloor$




(discrete) curtate future lifetime

$\lfloor \cdot \rfloor \Rightarrow$ integer part of $T_x$

29.3 $\Downarrow$ 29

38.99 $\Downarrow$ die 38

$\Pr[K_x = k] = {}_{k|}q_x = {}_kp_x \, q_{x+k}$

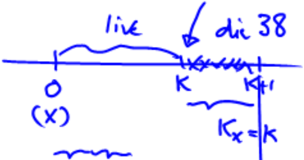
$= {}_{k+1}q_x - {}_kq_x$

$= {}_kp_x - {}_{k+1}p_x$

$k = 0, 1, 2, \ldots$

prob mass function

density




cdf $\Pr[K_x \le k] = {}_{k+1}q_x$

$$= \sum_{h=0}^{k} \underbrace{\Pr[K_x = h]}_{{}_{h|}q_x} = {}_{0|}q_x + {}_{1|}q_x + \cdots + {}_{k|}q_x$$

$$= \cancel{q_x} + (\cancel{{}_2q_x} - \cancel{q_x}) + (\cancel{{}_3q_x} - \cancel{{}_2q_x}) + \cdots + ({}_{k+1}q_x - \cancel{{}_kq_x})$$

exponential ($\mu$) $\Rightarrow$ $K_x$ geometric

$$E[T_x] = \mathring{e}_x = \int_0^\infty {}_tp_x \, dt$$

complete expectation of life

$$E[K_x] = e_x = \sum_{k=1}^\infty {}_kp_x$$

curtate expectation of life

$$E[T_x] = \int_0^\infty t \cdot \underbrace{f_{T_x}(t)}_{{}_tp_x \, \mu_{x+t}} \, dt$$

by parts

$$= \int_0^\infty {}_tp_x \, dt$$

$$\Downarrow$$

$$\sum_{k=0}^\infty k \cdot \underbrace{\underbrace{{}_{k|}q_x}_{Pr[K_x = k]}}_{({}_kp_x - {}_{k+1}p_x)}$$

$$= \cancel{0 \cdot ({}_0p_x - {}_1p_x)} + 1({}_1p_x - {}_2p_x) + 2({}_2p_x - {}_3p_x) + \cdots$$

$$= {}_1p_x + {}_2p_x + {}_3p_x + \cdots = \sum_{k=1}^\infty {}_kp_x$$

# 2.5/2.6 Expectation of life

$Var[T_x] = E[T_x^2] - (E[T_x])^2$

- The expected value of $T_x$ is called the complete expectation of life:

$$\mathring{e}_x = \mathsf{E}[T_x] = \int_0^\infty t f_x(t) dt = \int_0^\infty t \, {}_tp_x \mu_{x+t} dt = \boxed{\int_0^\infty {}_tp_x dt.}$$

- The expected value of $K_x$ is called the curtate expectation of life:

$$e_x = \mathsf{E}[K_x] = \sum_{k=0}^\infty k \cdot \Pr[K_x = k] = \sum_{k=0}^\infty k \cdot {}_{k|}q_x = \boxed{\sum_{k=1}^\infty {}_kp_x.}$$

- Proof can be derived using discrete counterpart of integration by parts (summation by parts). Alternative proof will be provided in class.

- Variances of future lifetime can be similarly defined.

# Illustrative example 2

do this!

Let $X$ be the age-at-death random variable with

$$\mu_x = \frac{1}{2(100-x)}, \quad \text{for } 0 \leq x < 100.$$

force of mortality

1. Give an expression for the survival function of $X$. $\to S_0(x)$
2. Find $f_{36}(t)$, the density function of future lifetime of $(36)$.
3. Compute ${}_{20}p_{36}$, the probability that life $(36)$ will survive to reach age 56.
4. Compute $\mathring{e}_{36}$, the average future lifetime of $(36)$.

① $S_0(x) = e^{-\int_0^x \mu_z \, dz} = e^{+\frac{1}{2}\log(100-z)\Big|_0^x}$ $\qquad \mu_x = \frac{1}{2}\frac{1}{100-x}, \quad 0 \le x \le 100$

$\mu_z \Downarrow \frac{1}{2}\frac{1}{100-z}$

$$= e^{\frac{1}{2}\underbrace{\left[\log(100-x) - \log(100)\right]}_{\log\frac{100-x}{100}}} = e^{\log\left(\frac{100-x}{100}\right)^{1/2}} = \left(\frac{100-x}{100}\right)^{1/2}$$

$0 \le x \le 100$

② $f_x(t) = \dfrac{f_0(x+t)}{S_0(x)} = \dfrac{\frac{1}{200}\left(\frac{100-x-t}{100}\right)^{-1/2}}{\left(\frac{100-x}{100}\right)^{1/2}}$ $\qquad f_0(x) = -\frac{d}{dx}S_0(x) = \frac{1}{200}\left(\frac{100-x}{100}\right)^{-1/2}$

$$= \frac{\cancel{\frac{1}{200}}\ \frac{1}{2}\ 1}{\left(\frac{100-x-t}{\cancel{100}}\right)^{1/2}\left(\frac{100-x}{\cancel{100}}\right)^{1/2}}$$

$$\underbrace{f_{36}(t) = \frac{\frac{1}{2}}{(64-t)^{1/2}\,64^{1/2}} = \frac{1}{16\,(64-t)^{1/2}}, \quad 0 \le t \le 64}$$

③ ${}_{20}p_{36}$ = prob (36) lives another 20 years


$$= \int_0^{20} f_{36}(t)\,dt$$

$$f_{36}(t) = \frac{1}{16\,(64-t)^{1/2}}$$

$$\checkmark\; S_x(t) = \frac{S_0(x+t)}{S_0(x)}$$

$$\frac{S_0(56)}{S_0(36)} = \frac{\left(\frac{100-56}{100}\right)^{1/2}}{\left(\frac{100-36}{100}\right)^{1/2}} = \left(\frac{44}{64}\right)^{1/2} = \left(\frac{11}{16}\right)^{1/2} = .8291562$$

$$\int_0^{20} \frac{1}{16}(64-t)^{-1/2}\,dt$$

$$= -\frac{1}{16}\,\frac{(64-t)^{1/2}}{1/2}\Bigg|_0^{20}$$

④ $$\mathring{e}_{36} = E[T_{36}] = \underbrace{\int_0^{64} t\cdot f_{36}(t)\,dt}$$

$0 \le x \le 100$

$0 \le t \le 64$

$$= \int_0^{64} S_{36}(t)\,dt = \int_0^{64} \frac{S_0(36+t)}{S_0(36)}\,dt = \int_0^{64} \frac{(64-t)^{1/2}}{64^{1/2}}\,dt$$

.....

$= 42.6667$

# Illustrative example 3

$\int S_{15}(t)dt = \int_0^{45} \frac{S_0(15+t)}{S_0(15)} dt$

$E[X] = \frac{\omega}{2} = 30 \Rightarrow \omega = 60$

Suppose you are given that:

- $\mathring{e}_0 = 30$; and average 30 years
- $S_0(x) = 1 - \dfrac{x}{\omega}$, for $\underbrace{0 \leq x \leq \omega}$.

$\Rightarrow$ uniform $f_0(x) = -\frac{d}{dx} S_0(x) = \frac{1}{\omega}$

$\Rightarrow$ De Moivre's at birth $0 \leq x \leq 60$

$T_{15} \sim$ De Moivre's $\underbrace{0 \leq t \leq 45}_{\text{uniform}}$

Evaluate $\mathring{e}_{15}$.

Solution to be discussed in lecture.

$E[T_{15}] \rightarrow \frac{45}{2} = \underline{\underline{22.5}}$

# Illustrative example 4

$_tp_x = e^{-\mu t}$ $\quad q_x = 1 - e^{-\mu}$

$_tq_x = 1 - e^{-\mu t}$

sm 30% | ns 70% | 40

For a group of lives aged 40 consisting of 30% smokers (sm) and the rest, non-smokers (ns), you are given:

- For non-smokers, $\mu_x^{\text{ns}} = 0.05$, for $x \geq 40$
- For smokers, $\mu_x^{\text{sm}} = 0.10$, for $x \geq 40$

Calculate $q_{65}$ for a life randomly selected from those who reach age 65.

prob that (65) will die within one year

$1 - e^{-.10}$

$$.05384395 = q_{65} = q_{65}^{sm} \Pr(sm) + q_{65}^{ns} \Pr(ns)$$

$1 - e^{-.05}$

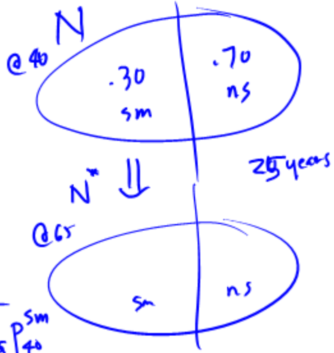
$\cancel{.30}$ $\quad$ 100% $\quad$ $\cancel{.70}$

.1093517 $\quad$ .8906403

$$\Pr(ns@65) = \frac{\#\ ns\ @\ 65}{\#\ total}$$

$$= \frac{\cancel{N}\cdot 0.70\ {}_{25}p_{40}^{ns}}{\cancel{N}\cdot 0.70\ {}_{25}p_{40}^{ns} + \cancel{N}\cdot 0.30\ {}_{25}p_{40}^{sm}}$$

$$= \frac{.70\, e^{-.05(25)}}{.70\, e^{-.05(25)} + .30\, e^{-.10(25)}} = .8906403$$




$$\Pr(sm@65) = \frac{.30\ {}_{25}p_{40}^{sm}}{.30\ {}_{25}p_{40}^{sm} + .70\ {}_{25}p_{40}^{ns}} = 1 - .8906403 = .1093597$$

# Temporary (partial) expectation of life

$\mathring{e}_x$ -

$0$

$(x)$

We can also define temporary (or partial) expectation of life:

$\min(T_x, n)$

$$\mathring{e}_{x:\overline{n}|} = \underbrace{\mathsf{E}\big[\min(T_x, n)\big]}_{\int_0^\infty \min(t,n)\cdot {}_tp_x\,\mu_{x+t}\,dt} = \mathring{e}_{x:\overline{n}|} = \boxed{\int_0^n {}_tp_x\,dt}$$

$0 \qquad n$

This can be interpreted as the average future lifetime of $(x)$ within the next $n$ years.

$\mathring{e}_{x:\overline{\infty}|} = \mathring{e}_x$

$\int_0^\infty {}_tp_x\,dt$

$\int_0^n {}_tp_x\,dt$

Suppose you are given:

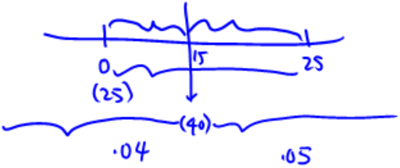
$$\mu_x = \begin{cases} 0.04, & 0 < x < 40 \\ 0.05, & x \geq 40 \end{cases}$$

Calculate $\mathring{e}_{25:\overline{25}|}$

$$\int_0^\infty \min(t,n)\cdot {}_tp_x\,\mu_{x+t}\,dt$$

$\mathring{e}_{25:\overline{25}|}$

$$\int_0^n {}_tp_x\,dt = \int_0^n e^{-\mu t}\,dt$$




$$\mathring{e}_{x:\overline{n+m}|} = \underbrace{\int_0^{n+m} {}_tp_x\,dt} = \int_0^n {}_tp_x\,dt + \int_n^{n+m} {}_tp_x\,dt$$

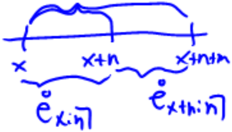
$s = t - n$

$ds = dt$

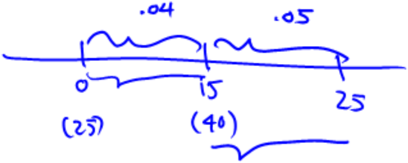
$$= \mathring{e}_{x:\overline{n}|} + \int_0^m {}_{n+s}p_x\,ds$$

$${}_np_x \cdot {}_sp_{x+n}$$

$$= \mathring{e}_{x:\overline{n}|} + {}_np_x \underbrace{\int_0^m {}_tp_{x+n}\,dt}_{\mathring{e}_{x+n:\overline{m}|}}$$




$$\mathring{e}_{x:\overline{n+m}|} = \mathring{e}_{x:\overline{n}|} + {}_np_x\,\mathring{e}_{x+n:\overline{m}|}$$




$$\mathring{e}_{25:\overline{25}|}$$

$${}_tp_x = e^{-\mu t}$$

indep of $x$

$$= \mathring{e}_{25:\overline{15}|} + {}_{15}p_{25}\,\mathring{e}_{40:\overline{10}|}$$

$$= \int_0^{15} e^{-.04t}\,dt + e^{-.04(15)} \int_0^{10} e^{-.05t}\,dt$$

$$\frac{1}{.04}\left(1-e^{-.04(15)}\right) + e^{-.04(15)}\,\frac{1}{.05}\left(1-e^{-.05(10)}\right)$$

$$= 15.59852$$

Special

1. De Moivre's
2. Exponential
3. Gompertz
4. Makeham
5. Generalized deMoivres

$$S_0(x) = \left(1 - \frac{x}{\omega}\right)^{\alpha}, \quad 0 \le x \le \omega$$

if $\alpha = 1 \Rightarrow$ de Moivre's

force of mortality

$$\mu_x = -\frac{d}{dx}\log S_0(x) = \underbrace{-\frac{d}{dx}\,\alpha\cdot\log\left(1-\frac{x}{\omega}\right)}$$

$$= -\alpha \cdot \frac{1}{1-\frac{x}{\omega}} \cdot \frac{-1}{\omega} = \frac{\alpha}{\omega - x} \checkmark$$

$$E[X] = \int_0^{\omega} S_0(x)\,dx = \int_0^{\omega}\left(\frac{\omega - x}{\omega}\right)^{\alpha} dx$$

$$= \frac{1}{\omega^{\alpha}}\int_0^{\omega}(\omega - x)^{\alpha}\,dx = \frac{1}{\omega^{\alpha}}\left[\left.\frac{-(\omega-x)^{\alpha+1}}{\alpha+1}\right|_0^{\omega}\right]$$

$$= \frac{1}{\cancel{\omega^{\alpha}}}\,\frac{1}{\alpha+1}\,\omega^{\cancel{\alpha}+1} = \frac{\omega}{\alpha+1} \checkmark$$

e.g. $S_0(x) = \left(1 - \frac{x}{110}\right)^{1/4}$, $0 \le x \le 110$

$$E[T_{25}] = \frac{85}{5/4} = \frac{85}{\cancel{5}} 4 = 68$$

$X \sim GDM(\omega, \alpha)$

$S_0(x) = \left(1 - \frac{x}{\omega}\right)^{\alpha}$

$T_x \sim GDM(\omega - x, \alpha)$

$$S_x(t) = \frac{S_0(x+t)}{S_0(x)} = \frac{\left(1 - \frac{x+t}{\omega}\right)^{\alpha}}{\left(1 - \frac{x}{\omega}\right)^{\alpha}} = \frac{\left(\frac{\omega - x - t}{\cancel{\omega}}\right)^{\alpha}}{\left(\frac{\omega - x}{\cancel{\omega}}\right)^{\alpha}}$$

$$= \left(\frac{\omega - x - t}{\omega - x}\right)^{\alpha}$$

$$= \left(1 - \frac{t}{\omega - x}\right)^{\alpha}$$

$\mu_x = \frac{\alpha}{\omega - x}$

$\Downarrow$

$\mu_{x+t} = \frac{\alpha}{\omega - x - t}$

$E[X] = \frac{\omega}{\alpha + 1}$

$\Downarrow$

$E[T_x] = \frac{\omega - x}{\alpha + 1}$

# Generalized De Moivre's law

De Moivre's $\Rightarrow$ uniform $(0,\omega)$

$S_0(x) = 1-\frac{x}{\omega} = \frac{\omega - x}{\omega}$

GDM $\Rightarrow$

The SDF of the so-called Generalized De Moivre's Law is expressed as

$$S_0(x) = \left(1 - \frac{x}{\omega}\right)^{\alpha} \text{ for } 0 \leq x \leq \omega.$$

limiting age. $\Rightarrow$

Derive the following for this special type of law of mortality:

1. force of mortality
2. survival function associated with $T_x$
3. expectation of future lifetime of $x$
4. can you find explicit expression for the variance of $T_x$?

$GDM(\omega, \alpha) \rightarrow X$

$$S_0(x) = \left(1 - \frac{x}{\omega}\right)^{\alpha}$$

$$\mu_x = \frac{\alpha}{\omega - x}$$

$$f_0(x) = \underbrace{\frac{\alpha}{\omega}\left(1 - \frac{x}{\omega}\right)^{\alpha-1}} \qquad F_0(x) = 1 - \left(1 - \frac{x}{\omega}\right)^{\alpha}$$


$$T_x \Rightarrow GDM(\omega - x, \alpha)$$

$$S_x(t) = \left(1 - \frac{t}{\omega - x}\right)^{\alpha}$$

$$\mu_{x+t} = \frac{\alpha}{\omega - x - t}$$

$$f_x(t) = \frac{\alpha}{\omega - x}\left(1 - \frac{t}{\omega - x}\right)^{\alpha}$$

$$F_x(t) = 1 - \left(1 - \frac{t}{\omega - x}\right)^{\alpha}$$

Ex 2.6

$\Downarrow$

read this!

# Illustrative example

- We will do **Example 2.6** in class.

# Example 2.3

$T_0 = X$

$\int c^s = \int e^{s \log c} ds = \frac{e^{s \log c}}{\log c} = \frac{c^s}{\log c}$

Gompertz for a newborn, $E[X] = ?$

Let $\mu_x = Bc^x$, for $x > 0$, where $B$ and $c$ are constants such that $0 < B < 1$ and $c > 1$.

Derive an expression for $S_x(t)$. $= \Pr[T_x > t] = {}_tp_x = e^{-\int_0^t \mu_{x+s} ds}$

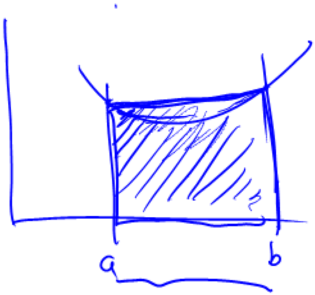
$\mu_{x+s} = Bc^{x+s}$

$= e^{-Bc^x \int_0^t c^s ds} = e^{-\frac{Bc^x}{\log c} c^s \Big|_0^t} = e^{-\frac{Bc^x}{\log c}(c^t - 1)}$

$E[T_x] = \int_0^\infty e^{-\frac{Bc^x}{\log c}(c^t-1)} dt$

implicitly not possible.

$\mathring{e}_{x:\overline{2}|} = \int_0^2$




1

$$\int_a^b f(x)\,dx \approx \underbrace{(b-a)\tfrac{1}{2}\left[f(a)+f(b)\right]}_{\text{trapezoidal rule}}$$


rectangle

trapezoid

2 Simpson's rule

$$\int_a^b f(x)\,dx = \frac{h}{3}\left[f(a) + 4f(a+h) + f(b)\right]$$

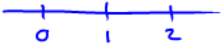
$h = b - a$

Makeham's law $\mu_x = A + Bc^x$ $A = .002$, $B = 10^{-4.5}$

$C = 1.10$

$${}_tp_x = e^{-At - \frac{Bc^x}{\log c}(c^t - 1)}$$

$\mathring{e}_{35:\overline{2}|}$ approximate this with two one-year intervals using Trapezoidal rule



$$\int_0^2 {}_tp_{35}\,dt = \underbrace{\int_0^1 {}_tp_{35}\,dt}_{\frac{1}{2}({}_0p_{35} + {}_1p_{35})} + \underbrace{\int_1^2 {}_tp_{35}\,dt}_{\frac{1}{2}({}_1p_{35} + {}_2p_{35})}$$

${}_0p_{35} = 1$

$\underbrace{\qquad\qquad}_{1.994102}$ ⟷ $\mathring{e}_{35:\overline{2}|} = 1.994116$

verify this

| $t$ | ${}_tp_{35}$ |
|---|---|
| 0 | 1 |
| 1 | .9970719 |
| 2 | .9940579 |

# Typical mortality pattern observed


- High (infant) mortality rate in the first year after birth.
- Average lifetime (nowadays) range between 70-80 - varies from country to country. 77.8 78.5
- Fewer lives/deaths observed after age 110 - supercentenarian is the term used to refer to someone who has reached age 110 or more.
- The highest recorded age at death (I believe) is 122
- Different male/female mortality pattern - females are believed to live longer.

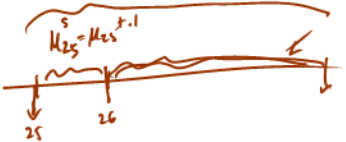
# Substandard mortality

underwriting → selection.

$x$ $x+1$

- A substandard risk is generally referred to someone classified by the insurance company as having a higher chance of dying because of:
  - some physical condition
  - family or personal medical history
  - risky occupation
  - dangerous habits or lifestyle (e.g. skydiving)
- Mortality functions are superscripted with $s$ to denote substandard: $q_x^s$ and $\mu_x^s$.
- For example, substandard mortality may be obtained from a standard table using:
  1. adding a constant to force of mortality: $\mu_x^s = \mu_x + c$
  2. multiplying a fixed constant to probability: $q_x^s = \min(kq_x, 1)$ $k > 1$
- The opposite of a substandard risk is preferred risk where someone is classified to have better chance of survival.

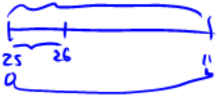
# Practice problem - SOA MLC Fall 2000 Question #4

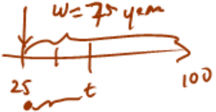



Mortality for Audra, age 25, follows De Moivre's law with $\omega = 100$. If she takes up hot air ballooning for the coming year, her assumed mortality will be adjusted so that for the coming year only she will have a constant force of mortality of 0.1.

Calculate the decrease in the 11-year temporary complete life expectancy for Audra if she takes up hot air ballooning.

$$\mathring{e}^{s}_{25:\overline{11|}} = \int_0^{11} {}_tp^s_{25}\,dt = \underbrace{\int_0^1 {}_tp^s_{25}\,dt} + p^s_{25}\,\mathring{e}_{26:\overline{10|}}$$






$${}_tp_{25} = 1 - {}_tq_{25}$$

$$= 1 - \frac{t}{75}, \quad t \leq 75$$

Misinterpreted solution but still a nice problem to solve

$$\int_0^1 \underbrace{{}_tp_{25}^{s}}\, dt \quad + \quad \frac{74}{75} e^{-.1} \underbrace{\int_0^{10} \left(1 - \frac{t}{74}\right) dt}_{\left(10 - \frac{1}{74}\frac{1}{2}10^2\right)}$$

$$e^{-\int_0^t \mu^s_{x+z}\, dz}$$

$$e^{-\left(\int_0^t \mu_{x+z} + .1\, dz\right)}$$

$$\underbrace{e^{-\int_0^t \mu_{x+z}\, dz}}$$

$$\int_0^1 {}_tp_{25}\ \underbrace{e^{-\int_0^t .1\, dz}}\ dt$$

$$\int_0^1 {}_tp_{25}\ e^{-.1t}\, dt$$

$$\int_0^1 \left(1 - \frac{t}{75}\right) e^{-.1t}\, dt$$

$$\underbrace{\frac{1}{.1}\left(1 - e^{-.1}\right)}_{.9516258} - \frac{1}{75} \underbrace{\int_0^1 t e^{-.1t}\, dt}_{0.467884} +$$

$$\frac{74}{75} e^{-.1} * 9.324324$$

here

$$= 9.269892$$

Sorry, but in the previous solution, I misinterpreted the force of mortality as:

$$\mu^s_{25+t} = \begin{cases} \mu_{x+t} + 0.1, & 0 \le t < 1 \\ \mu_{x+t}, & t \ge 1 \end{cases}$$

But the problem says that the force of mortality for a substandard is

$$\mu^s_{x+t} = \begin{cases} 0.1, & 0 \le t < 1 \\ \mu_{x+t}, & t \ge 1 \end{cases}$$

easier solution but same principle follows!

Next slides details the solution.

$$\mathring{e}^{s}_{25:\overline{11|}} = \int_0^{11} {}_tp^s_{25}\,dt = \int_0^1 {}_tp^s_{25}\,dt + p^s_{25}\,\underbrace{\mathring{e}_{26:\overline{10|}}}_{\text{becomes standard after one year}}$$

$$\mathring{e}_{26:\overline{10|}} = \int_0^{10} {}_tp_{26}\,dt \qquad \Pr[T_{26}>t] = 1-t/74$$

$T_{26} \sim$ De Moivre's from 0 to 74

$$= \int_0^{10} (1 - t/74)\,dt$$

$$= 10 - \frac{1}{2}\frac{1}{74}10^2 = 9.324324$$

$$\int_0^1 {}_tp^s_{25}\,dt = \int_0^1 e^{-.1t}\,dt = \frac{1}{.1}(1-e^{-.1}) = 0.9516258$$

$$p^s_{25} = e^{-\int_0^1 .1\,dt} = e^{-.1}$$

$$= \underbrace{0.9516258 + e^{-.1}(9.324324)}$$

$$= 9.388625$$


$$\mathring{e}_{x:\overline{n+m}|} = \mathring{e}_{x:\overline{n}|} + {}_np_x \, \mathring{e}_{x+n:\overline{m}|}$$

$$e_{x:\overline{n+m}|} = e_{x:\overline{n}|} + {}_np_x \, e_{x+n:\overline{m}|}$$

# Illustrative example 5

You are given:

- Mortality for standard lives follows the Standard Ultimate Life Table (SULT).
- The force of mortality for standard lives age $45+t$ is represented as $\mu_{45+t}^{\text{SULT}}$.
- The force of mortality for substandard lives age $45+t$, $\mu_{45+t}^{sub}$, is defined by

$$\mu_{45+t}^{sub} = \begin{cases} \mu_{45+t}^{\text{SULT}} + 0.05, & \text{for } 0 \le t < 1 \\ \mu_{45+t}^{\text{SULT}}, & \text{for } t \ge 1 \end{cases}$$

Calculate the probability that a substandard 45-year-old will die within the next two years.

sub


$${}_2q_{45}^{sub} = 1 - {}_2p_{45}^{sub}$$

$$= 1 - e^{-\int_0^2 \mu_{45+s}^{sub}\,ds}$$

$$= 1 - p_{45}^{sub}\, p_{46}^{sub}$$

$$= 1 - \underbrace{e^{-\int_0^1 \mu_{45+s}^{sub}\,ds}}_{} \cdot p_{46}^{SULT}$$

$$e^{-\left(\int_0^1 \mu_{45+s}^{SULT} + .05\right)ds} \cdot p_{46}^{SULT}$$

$$\underbrace{e^{-\int_0^1 \mu_{45+s}^{SULT}\,ds}}_{} \; e^{-.05} \quad p_{46}^{SULT}$$

$$= 1 - p_{45}^{SULT}\, p_{46}^{SULT} \cdot e^{-.05} = \underbrace{1-(1-.000771)(1-.000839)\cdot e^{-.05}}_{} \approx .05030144$$

unemployment —

# Final remark - other contexts

social pen    work pen    own

- The notion of a lifetime or survival learned in this chapter can be applied in several other contexts:
  - engineering: lifetime of a machine, lifetime of a lightbulb
  - medical statistics: time-until-death from diagnosis of a disease, survival after surgery
  - finance: time-until-default of credit payment in a bond, time-until-bankruptcy of a company
  - space probe: probability radios installed in space continue to transmit
  - biology: lifetime of an organism
  - other actuarial context: disability, sickness/illness, retirement, unemployment

# Other symbols and notations used

| Expression | Other symbols used |
|---|---|
| probability function | $P(\cdot)$ $\quad$ $\Pr(\cdot)$ |
| survival function of newborn | $S_X(x)$ $\quad$ $S(x)$ $\quad$ $s(x)$ |
| future lifetime of $x$ | $T(x)$ $\quad$ $T$ |
| curtate future lifetime of $x$ | $K(x)$ $\quad$ $K$ |
| survival function of $x$ | $S_{T_x}(t)$ $\quad$ $S_T(t)$ |
| force of mortality of $T_x$ | $\mu_{T_x}(t)$ $\quad$ $\mu_x(t)$ |