

Statistics for Data Science

Unit 4 Part 1 Homework: Discrete Random Variables

Kevin Hartman (W203 Wednesday 6:30pm Summer 2019)

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1. Best Game in the Casino†

You flip a fair coin 3 times, and get a different amount of money depending on how many heads you get. For 0 heads, you get \$0. For 1 head, you get \$2. For 2 heads, you get \$4. Your expected winnings from the game are \$6.

Givens:

X is a binomial random variable based on n trials with success probability p .

When $X=x$, let x be the number of heads among the $n = 3$ trials with the probability of a head in each trial being $p = \frac{1}{2}$.

$$b(x; n, p) = \begin{cases} \binom{n}{x} p^x (1-p)^{n-x}, & x \in \{0, 1, 2, 3, \dots\} \\ 0, & \text{otherwise.} \end{cases}$$

$$\text{Probability of 0 Heads: } P(X = 0) = b(0; 3, \frac{1}{2}) = \binom{3}{0} (\frac{1}{2})^3 = \frac{1}{8}$$

$$\text{Probability of 1 Heads: } P(X = 1) = b(1; 3, \frac{1}{2}) = \binom{3}{1} (\frac{1}{2})^3 = \frac{3}{8}$$

$$\text{Probability of 2 Heads: } P(X = 2) = b(2; 3, \frac{1}{2}) = \binom{3}{2} (\frac{1}{2})^3 = \frac{3}{8}$$

$$\text{Probability of 3 Heads: } P(X = 3) = b(3; 3, \frac{1}{2}) = \binom{3}{3} (\frac{1}{2})^3 = \frac{1}{8}$$

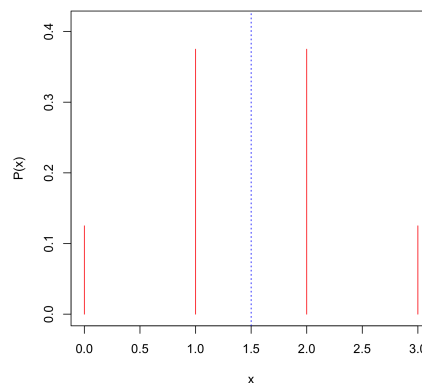


Figure 1: Plot of binomial distribution of $n=3$, $p=\frac{1}{2}$

- (a) How much do you get paid if the coin comes up heads 3 times?

Answer:

Given fair coins, the expectation of X , $E(X) = np = 3 \cdot \frac{1}{2} = \frac{3}{2}$ (also shown in graph above)

The expected winnings from the game is \$6, such that $g(E(X)) = 6$

We also have:

$$G(X = 0) = 0$$

$$G(X = 1) = 2$$

$$G(X = 2) = 4$$

$$G(X = 3) = A$$

Using the expectation that $E(g(X)) = g(E(X))$ we can solve for A :

$$\begin{aligned} E(g(X)) &= \sum_{y=0}^x g(y) \cdot p(y) = 6 \\ (0 \cdot \frac{1}{8}) + (2 \cdot \frac{3}{8}) + (4 \cdot \frac{3}{8}) + (A \cdot \frac{1}{8}) &= 6 \\ (\frac{6}{8}) + (\frac{12}{8}) + (\frac{A}{8}) &= 6 \\ 6 + 12 + A &= 48 \end{aligned}$$

$$A = 30$$

- (b) Write down a complete expression for the cumulative probability function for your winnings from the game.

Answer:

$$\text{With } g(x) = \begin{cases} 0 & ; x < 1 \\ \frac{16}{3} & ; 1 \leq x < 3 \\ 208 & ; x \leq 3 \end{cases}, \quad p = \frac{1}{2}, \text{ and } n = 3$$

$$\text{Winnings} = g(P(X \leq x)) = \sum_{y=0}^x g(y) \cdot \binom{n}{y} p^y (1-p)^{n-y}$$

Proof:

$$G(P(X \leq 0)) = (\frac{1}{8} \cdot 0) = 0$$

$$G(P(X \leq 1)) = (\frac{1}{8} \cdot 0) + (\frac{3}{8} \cdot \frac{16}{3}) = 2$$

$$G(P(X \leq 2)) = (\frac{1}{8} \cdot 0) + (\frac{3}{8} \cdot \frac{16}{3}) + (\frac{3}{8} \cdot \frac{16}{3}) = 4$$

$$G(P(X \leq 3)) = (\frac{1}{8} \cdot 0) + (\frac{3}{8} \cdot \frac{16}{3}) + (\frac{3}{8} \cdot \frac{16}{3}) + (\frac{1}{8} \cdot 208) = 30$$

2. Reciprocal Dice

Let X be a random variable representing the outcome of rolling a 6-sided die. Before the die is rolled, you are given two options:

- (a) You get $1/E(X)$ in dollars right away.

Proof:

Since each die roll has uniform probability we can compute $E(X)$ as:

$$E(X) = \frac{1}{k} \sum_j x_j = \frac{1 + 2 + 3 + 4 + 5 + 6}{6} = 3.5$$

$1/E(X)$ is approx .28571 dollars

- (b) You wait until the die is rolled, then get $1/X$ in dollars.

Proof:

Given $g(X) = 1/X$ and $E(g(X)) = \sum_x g(x) \cdot p(x)$

$$E(g(X)) = (1 \cdot \frac{1}{6}) + (\frac{1}{2} \cdot \frac{1}{6}) + (\frac{1}{3} \cdot \frac{1}{6}) + (\frac{1}{4} \cdot \frac{1}{6}) + (\frac{1}{5} \cdot \frac{1}{6}) + (\frac{1}{6} \cdot \frac{1}{6})$$

$E(g(X))$ is approx .408333 dollars

- (c) Which option is better for you, in expectation?

Answer:

.408333 is greater than .28571. The second option is better in expectation.

3. The Baseline for Measuring Deviations

Given any random variable X and a real number t , we can define another random variable $Y = (X - t)^2$. In other words, for any random variable X , we can choose a real number, t , as a baseline and calculate the squared deviation of X away from t .

You might wonder why we often square deviations (instead of taking an absolute value, or cubing them, etc.). This exercise will shed some light on why this is a natural choice.

- (a) Write down an expression for $E(Y)$ and simplify it as much as you can. Even though we haven't proved this yet, you can use the fact that for any two random variables, A and B , $E(A + B) = E(A) + E(B)$.

Answer:

$$\begin{aligned} E(Y) &= E[(X - t)^2] = E[(X^2 - 2tX + t^2)] \\ &= E[X^2] + E[(-2tX)] + E[t^2] \\ &= E(X^2) - 2tE(X) + t^2 \end{aligned}$$

- (b) Taking a partial derivative with respect to t , compute the value of t that minimizes $E(Y)$. (Hint: Your answer should be a very familiar value)

Answer:

$$E(Y)'_{(x,t)} = \frac{\partial}{\partial t}[E(X^2) - 2tE(X) + t^2] = -2E(X) + 2t$$

$E(Y)$ approaches minimum when

$$t = E(X)$$

- (c) What is the value of $E(Y)$ for this choice of t ? (Hint: this should also be a very familiar value)

Answer:

Replacing t with $E(X)$ in $E(Y) = E(X^2) - 2tE(X) + t^2$:

$$\begin{aligned} E(X^2) - 2tE(X) + t^2 &= E(X^2) - 2E(X)E(X) + [E(X)]^2 \\ &= E(X^2) - 2[E(X)]^2 + [E(X)]^2 \end{aligned}$$

The value of $E(Y)$ at the minimum will be

$$E(Y) = E(X^2) - [E(X)]^2$$

4. Optional Advanced Exercise: Heavy Tails

One reason to study the mathematical foundation of statistics is to recognize situations where common intuition can break down. An unusual class of distributions are those we call *heavy-tailed*. The exact definition varies, but we'll say that a heavy-tailed distribution is one for which not all moments are finite. Consider a random variable M with the following pmf:

$$p_M(x) = \begin{cases} c/x^3, & x \in \{1, 2, 3, \dots\} \\ 0, & \text{otherwise.} \end{cases}$$

where c is a constant (you can calculate its value if you like, but it's not important).

- (a) Is $E(M)$ finite?

Answer:

Yes it is finite because although this is an infinite series, the sum of c/x^3 from 1 to ∞ will eventually converge. It converges because the equation represents a p-series ($1/n^p$) and a p-series converges when $p > 1$ and diverges when $0 < p < 1$.

- (b) Is $V(M)$ finite?

Answer:

Also finite for the same reason as (a). For $V(M)$ to be calculated the infinite series must converge.

Heavy-tailed distributions may seem odd, but they're not as rare as you might suspect. Researchers argue that the distribution of wealth is heavy-tailed; so is the distribution of computer file sizes, insurance payouts, and area burned by forest fires. These random variables are problematic in that a lot of common statistical techniques don't work on them. For this class, we'll assume that all of our variables don't have heavy-tails.