ECS 315: Probability and Random Processes **2016/1**

HW Solution 7 — Due: Oct 25, 5 PM

Lecturer: Prapun Suksompong, Ph.D.

Problem 1. For each description of a random variable X below, indicate whether X is a discrete random variable.

(a) X is the number of websites visited by a randomly chosen software engineer in a day.

(b) X is the number of classes a randomly chosen student is taking.

(c) X is the average height of the passengers on a randomly chosen bus.

(d) A game involves a circular spinner with eight sections labeled with numbers. X is the amount of time the spinner spins before coming to a rest.

(e) X is the thickness of the longest book in a randomly chosen library.

(f) X is the number of keys on a randomly chosen keyboard.

(g) X is the length of a randomly chosen person's arm.

Solution:We consider the number of possibilities for the values of X in each part. The X defined in parts (a), (b), and (f) are discrete. The X defined in other parts are not discrete.

Problem 2. An optical inspection system is to distinguish among different part types. The probability of a correct classification of any part is 0.98. Suppose that three parts are inspected and that the classifications are independent.

(a) Let the random variable X denote the number of parts that are correctly classified. Determine the probability mass function of X. [Montgomery and Runger, 2010, Q3-20]

(b) Let the random variable Y denote the number of parts that are incorrectly classified. Determine the probability mass function of Y.

Solution[1]:

We will reexpress the problem in terms of Bernoulli trials so that we can use the results discussed in class. In this problem, we have three Bernoulli trials. Each trial deals with classification.

[1] The solution provided here assumes that we still haven't reached the part of the course where binomial random variable is discussed. Therefore, the pmf is derived by relying on the concept of Bernoulli trials and the formula discussed back when we studied that topic.

(a) To find $p_X(x)$, first we find its support. Three parts are inspected here. Therefore, X can be 0, 1, 2, or 3. So, we need to find $p_X(x) = P[X = x]$ when $x = 0, 1, 2$ or 3. The pmf $p_X(x)$ for other x values are all 0 because X cannot take the value of those x.

For each $x \in \{0, 1, 2, 3\}$, $p_X(x) = P[X = x]$ is simply the probability that exactly x parts are correctly classified. Note that, because we are interested in the correctly classified part, we define the "success" event for a trial to be the event that the part is classified correctly. We are given that the probability of a correct classification of any part is 0.98. Therefore, for each of our Bernoulli trials, the probability of success is $p = 0.98$. Under such interpretation (of "success"), $p_X(x)$ is then the same as finding the probability of having exactly x successes in $n = 3$ Bernoulli trials. We have seen in class that the probability of this is $\binom{3}{x}p^x(1-p)^{3-x}$. Plugging in $p = 0.98$, we have $p_X(x) = \binom{3}{x}0.98^x(0.02)^{3-x}$ for $x \in \{0, 1, 2, 3\}$.

Combining the expression above with the cases for other x values, we then have

$$p_X(x) = \begin{cases} \binom{3}{x}0.98^x(0.02)^{3-x}, & x \in \{0,1,2,3\}, \\ 0, & \text{otherwise} \end{cases} \tag{7.1}$$

In particular, $p_X(0) = 8 \times 10^{-6}$, $p_X(1) = 0.001176$, $p_X(2) = 0.057624$, and $p_X(3) = 0.941192$.

Remark: In fact, this X is a binomial random variable with $n = 3$ and $p = 0.98$. In `MATLAB`, the probabilities above can be calculated via the command
`binopdf(0:3,3,0.98)`.

(b) ***Method 1***: Similar analysis is performed on the random variable Y. The only difference here is that, now, we are interested in the number of parts that are *incorrectly* classified. Therefore, we will define the "success" event for a trial to be the event that the part is classified *incorrectly*. We are given that the probability of a correct classification of any part is 0.98. Therefore, for each of our Bernoulli trials, the probability of success is $1 - p = 1 - 0.98 = 0.02$. With this new probability of success, we have

$$p_Y(y) = \begin{cases} \binom{3}{y}0.02^y(0.98)^{3-y}, & y \in \{0,1,2,3\}, \\ 0, & \text{otherwise} \end{cases} \tag{7.2}$$

In particular, $p_Y(0) = 0.941192$, $p_Y(1) = 0.057624$, $p_Y(2) = 0.001176$, and $p_Y(3) = 8 \times 10^{-6}$.

Remark: In fact, this Y is a binomial random variable with $n = 3$ and $p = 0.02$. In `MATLAB`, the probability values above can be calculated via the command
`binopdf(0:3,3,0.02)`.

Method 2: Alternatively, note that there are three parts. If X of them are classified correctly, then the number of incorrectly classified parts is $n - X$, which is what we defined as Y. Therefore, $Y = 3 - X$. Hence, $p_Y(y) = P[Y = y] = P[3 - X = y] = P[X = 3 - y] = p_X(3 - y)$.

Problem 3. Consider the sample space $\Omega = \{-2, -1, 0, 1, 2, 3, 4\}$. Suppose that $P(A) = |A|/|\Omega|$ for any event $A \subset \Omega$. Define the random variable $X(\omega) = \omega^2$. Find the probability mass function of X.

Solution: The random variable maps the outcomes $\omega = -2, -1, 0, 1, 2, 3, 4$ to numbers $x = 4, 1, 0, 1, 4, 9, 16$, respectively. Therefore,

$$
\begin{aligned}
p_X(0) &= P(\{\omega : X(\omega) = 0\}) = P(\{0\}) = \frac{1}{7},\\
p_X(1) &= P(\{\omega : X(\omega) = 1\}) = P(\{-1, 1\}) = \frac{2}{7},\\
p_X(4) &= P(\{\omega : X(\omega) = 4\}) = P(\{-2, 2\}) = \frac{2}{7},\\
p_X(9) &= P(\{\omega : X(\omega) = 9\}) = P(\{3\}) = \frac{1}{7}, \text{ and}\\
p_X(16) &= P(\{\omega : X(\omega) = 16\}) = P(\{4\}) = \frac{1}{7}.
\end{aligned}
$$

Combining the results above, we get the complete pmf:

$$
p_X(x) = \boxed{\begin{cases} \frac{1}{7}, & x = 0, 9, 16, \\ \frac{2}{7}, & x = 1, 4, \\ 0, & \text{otherwise.} \end{cases}}
$$

ECS 315: Probability and Random Processes **2016/1**

HW Solution 8 — Due: November 1, 5 PM

Lecturer: Prapun Suksompong, Ph.D.

Problem 1. Suppose X is a random variable whose pmf at $x = 0, 1, 2, 3, 4$ is given by $p_X(x) = \frac{2x+1}{25}$.

Remark: Note that the statement above does not specify the value of the $p_X(x)$ at the value of x that is not 0,1,2,3, or 4.

(a) What is $p_X(5)$?

(b) Determine the following probabilities:

(i) $P[X = 4]$

(ii) $P[X \le 1]$

(iii) $P[2 \le X < 4]$

(iv) $P[X > -10]$

Solution:

(a) First, we calculate

$$\sum_{x=0}^{4} p_X(x) = \sum_{x=0}^{4} \frac{2x+1}{25} = \frac{1+3+5+7+9}{25} = \frac{25}{25} = 1.$$

Therefore, there can't be any other x with $p_X(x) > 0$. At $x = 5$, we then conclude that $p_X(5) = \boxed{0.}$ The same reasoning also implies that $p_X(x) = 0$ at any x that is not 0,1,2,3, or 4.

(b) Recall that, for discrete random variable X, the probability

$$P[\text{some condition(s) on } X]$$

can be calculated by adding $p_X(x)$ for all x in the support of X that satisfies the given condition(s).

(i) $P[X = 4] = p_X(4) = \frac{2\times4+1}{25} = \boxed{\frac{9}{25}.}$

(ii) $P[X \le 1] = p_X(0) + p_X(1) = \frac{2\times0+1}{25} + \frac{2\times1+1}{25} = \frac{1}{25} + \frac{3}{25} = \boxed{\frac{4}{25}.}$

(iii) $P[2 \le X < 4] = p_X(2) + p_X(3) = \frac{2\times 2+1}{25} + \frac{2\times 3+1}{25} = \frac{5}{25} + \frac{7}{25} = \boxed{\frac{12}{25}}.$

(iv) $P[X > -10] = \boxed{1}$ because all the x in the support of X satisfies $x > -10$.

Problem 2. The random variable V has pmf

$$p_V(v) = \begin{cases} cv^2, & v = 1, 2, 3, 4, \\ 0, & \text{otherwise.} \end{cases}$$

(a) Find the value of the constant c.

(b) Find $P[V \in \{u^2 : u = 1, 2, 3, \ldots\}]$.

(c) Find the probability that V is an even number.

(d) Find $P[V > 2]$.

(e) Sketch $p_V(v)$.

(f) Sketch $F_V(v)$. (Note that $F_V(v) = P[V \le v]$.)

Solution: [Y&G, Q2.2.3]

(a) We choose c so that the pmf sums to one:

$$\sum_v p_V(v) = c(1^2 + 2^2 + 3^2 + 4^2) = 30c = 1.$$

Hence, $c = \boxed{1/30}$.

(b) $P[V \in \{u^2 : u = 1, 2, 3, \ldots\}] = P[V \in \{1, 4, 9, 16, 25\}] = p_V(1) + p_V(4) = c(1^2 + 4^2) = \boxed{17/30}$.

(c) $P[V \text{ even}] = p_V(2) + p_V(4) = c(2^2 + 4^2) = 20/30 = \boxed{2/3}$.

(d) $P[V > 2] = p_V(3) + p_V(4) = c(3^2 + 4^2) = 25/30 = \boxed{5/6}$.

(e) See Figure 8.1 for the sketch of $p_V(v)$:

(f) See Figure 8.2 for the sketch of $F_V(v)$:

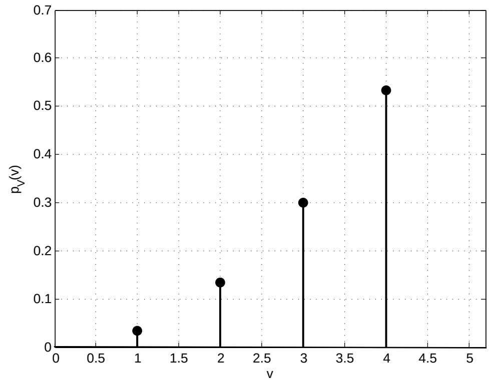


Figure 8.1: Sketch of $p_V(v)$ for Question 2

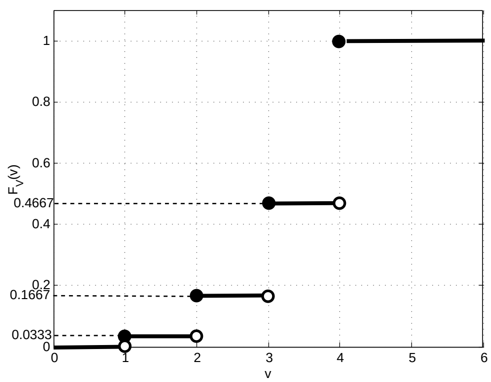


Figure 8.2: Sketch of $F_V(v)$ for Question 2

Problem 3. The thickness of the wood paneling (in inches) that a customer orders is a random variable with the following cdf:

$$F_X(x) = \begin{cases} 0, & x < \frac{1}{8} \\ 0.2, & \frac{1}{8} \le x < \frac{1}{4} \\ 0.9, & \frac{1}{4} \le x < \frac{3}{8} \\ 1 & x \ge \frac{3}{8} \end{cases}$$

Determine the following probabilities:

(a) $P[X \le 1/18]$

(b) $P[X \le 1/4]$

(c) $P[X \le 5/16]$

(d) $P[X > 1/4]$

(e) $P[X \leq 1/2]$

[Montgomery and Runger, 2010, Q3-42]

Solution:

(a) $P[X \leq 1/18] = F_X(1/18) = \boxed{0}$ because $\frac{1}{18} < \frac{1}{8}$.

(b) $P[X \leq 1/4] = F_X(1/4) = \boxed{0.9}$.

(c) $P[X \leq 5/16] = F_X(5/16) = \boxed{0.9}$ because $\frac{1}{4} < \frac{5}{16} < \frac{3}{8}$.

(d) $P[X > 1/4] = 1 - P[X \leq 1/4] = 1 - F_X(1/4) = 1 - 0.9 = \boxed{0.1}$.

(e) $P[X \leq 1/2] = F_X(1/2) = \boxed{1}$ because $\frac{1}{2} > \frac{3}{8}$.

Alternatively, we can also derive the pmf first and then calculate the probabilities.

ECS 315: Probability and Random Processes **2016/1**

HW Solution 9 — Due: November 8, 5 PM

Lecturer: Prapun Suksompong, Ph.D.

Problem 1. [F2013/1] For each of the following random variables, find $P[1 < X \le 2]$.

(a) $X \sim \text{Binomial}(3, 1/3)$

(b) $X \sim \text{Poisson}(3)$

Solution:

(a) Because $X \sim \text{Binomial}(3, 1/3)$, we know that X can only take the values 0, 1, 2, 3. Only the value 2 satisfies the condition given. Therefore, $P[1 < X \le 2] = P[X = 2] = p_X(2)$. Recall that the pmf for the binomial random variable is

$$p_X(x) = \binom{n}{x} p^x (1-p)^{n-x}$$

for $x = 0, 1, 2, 3, \ldots, n$. Here, it is given that $n = 3$ and $p = 1/3$. Therefore,

$$p_X(2) = \binom{3}{2}\left(\frac{1}{3}\right)^2\left(1 - \frac{1}{3}\right)^{3-2} = 3 \times \frac{1}{9} \times \frac{2}{3} = \boxed{\frac{2}{9}}.$$

(b) Because $X \sim \text{Poisson}(3)$, we know that X can take the values $0, 1, 2, 3, \ldots$. As in the previous part, only the value 2 satisfies the condition given. Therefore, $P[1 < X \le 2] = P[X = 2] = p_X(2)$. Recall that the pmf for the Poisson random variable is

$$p_X(x) = e^{-\alpha}\frac{\alpha^x}{x!}$$

for $x = 0, 1, 2, 3, \ldots$. Here, it is given that $\alpha = 3$. Therefore,

$$p_X(2) = e^{-3}\frac{3^2}{2!} = \boxed{\frac{9}{2}e^{-3} \approx 0.2240}.$$

Problem 2. Arrivals of customers at the local supermarket are modeled by a Poisson process with a rate of $\lambda = 2$ customers per minute. Let M be the number of customers arriving between 9:00 and 9:05. What is the probability that $M < 2$?

Solution: Here, we are given that $M \sim \mathcal{P}(\alpha)$ where $\alpha = \lambda T = 2 \times 5 = 10$. Recall that, for $M \sim \mathcal{P}(\alpha)$, we have

$$P[M = m] = \begin{cases} e^{-\alpha}\frac{\alpha^m}{m!}, & m \in \{0, 1, 2, 3, \ldots\} \\ 0, & \text{otherwise} \end{cases}$$

Therefore,

$$P[M<2]=P[M=0]+P[M=1]=e^{-\alpha}\frac{\alpha^0}{0!}+e^{-\alpha}\frac{\alpha^1}{1!}$$
$$=e^{-\alpha}(1+\alpha)=e^{-10}(1+10)=11e^{-10}\approx 5\times 10^{-4}.$$

Problem 3. [M2011/1] The cdf of a random variable X is plotted in Figure 9.1.

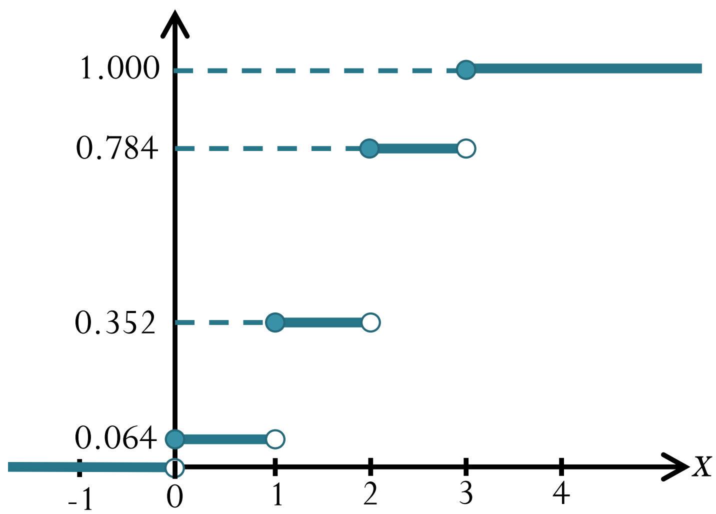


Figure 9.1: CDF of X for Problem 3

(a) Find the pmf $p_X(x)$.

(b) Find the family to which X belongs. (Uniform, Bernoulli, Binomial, Geometric, Poisson, etc.)

Solution:

(a) For discrete random variable, $P[X=x]$ is the jump size at x on the cdf plot. In this problem, there are four jumps at 0, 1, 2, 3.

- $P[X=0]=$ the jump size at $0=0.064=\frac{64}{1000}=(4/10)^3=(2/5)^3$.
- $P[X=1]=$ the jump size at $1=0.352-0.064=0.288$.
- $P[X=2]=$ the jump size at $2=0.784-0.352=0.432$.
- $P[X=3]=$ the jump size at $3=1-0.784=0.216=(6/10)^3$.

In conclusion,

$$\boxed{p_X(x) = \begin{cases} 0.064, & x = 0, \\ 0.288, & x = 1, \\ 0.432, & x = 2, \\ 0.216, & x = 3, \\ 0, & \text{otherwise.} \end{cases}}$$

(b) Among all the pmf that we discussed in class, only one can have support $= \{0, 1, 2, 3\}$ with unequal probabilities. This is the $\boxed{\text{binomial}}$ pmf. To check that it really is Binomial, recall that the pmf for binomial X is given by $p_X(x) = \binom{n}{x} p^x (1-p)^{(n-x)}$ for $x = 0, 1, 2, \ldots, n$. Here, $n = 3$. Furthermore, observe that $p_X(0) = (1-p)^n$. By comparing $p_X(0)$ with what we had in part (a), we have $1 - p = 2/5$ or $p = 3/5$. For $x = 1, 2, 3$, plugging in $p = 3/5$ and $n = 3$ in to $p_X(x) = \binom{n}{x} p^x (1-p)^{(n-x)}$ gives the same values as what we had in part (a). So, X is a binomial RV.

Problem 4. When n is large, binomial distribution Binomial(n, p) becomes difficult to compute directly . In this question, we will consider an approximation when the value of p is close to 0. In such case, the binomial can be approximated[1] by the Poisson distribution with parameter $\alpha = np$. For this approximation to work, we will see in this exercise that n does not have to be very large and p does not need to be very small.

(a) Let $X \sim$ Binomial$(12, 1/36)$. (For example, roll two dice 12 times and let X be the number of times a double 6 appears.) Evaluate $p_X(x)$ for $x = 0, 1, 2$.

(b) Compare your answers part (a) with its Poisson approximation.

Solution:

(a) For Binomial(n, p) random variable,

$$p_X(x) = \begin{cases} \binom{n}{x} p^x (1-p)^{n-x}, & x \in \{0, 1, 2, \ldots, n\}, \\ 0, & \text{otherwise.} \end{cases}$$

Here, we are given that $n = 12$ and $p = \frac{1}{36}$. Plugging in $x = 0, 1, 2$, we get $\boxed{0.7132, 0.2445, 0.0384}$, respectively

[1] More specifically, suppose X_n has a binomial distribution with parameters n and p_n. If $p_n \to 0$ and $np_n \to \alpha$ as $n \to \infty$, then

$$P[X_n = k] \to e^{-\alpha} \frac{\alpha^k}{k!}.$$

(b) A Poisson random variable with parameter $\alpha = np$ can approximate a Binomial(n,p) random variable when n is large and p is small. Here, with $n = 12$ and $p = \frac{1}{36}$, we have $\alpha = 12 \times \frac{1}{36} = \frac{1}{3}$. The Poisson pmf at $x = 0, 1, 2$ is given by $e^{-\alpha}\frac{\alpha^x}{x!} = e^{-1/3}\frac{(1/3)^x}{x!}$. Plugging in $x = 0, 1, 2$ gives $\boxed{0.7165, 0.2388, 0.0398}$, respectively.

Figure 9.2 compares the two pmfs. Note how close they are!

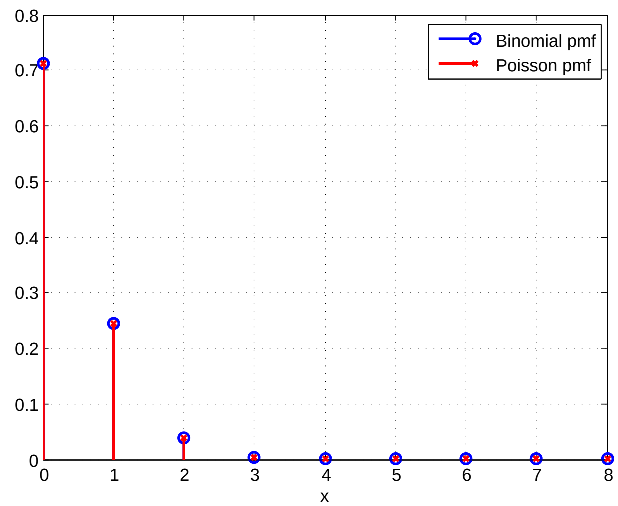


Figure 9.2: Poisson Approximation

Problem 5. You go to a party with 500 guests. What is the probability that exactly one other guest has the same birthday as you? Calculate this exactly and also approximately by using the Poisson pmf. (For simplicity, exclude birthdays on February 29.) [Bertsekas and Tsitsiklis, 2008, Q2.2.2]

Solution: Let N be the number of guests that has the same birthday as you. We may think of the comparison of your birthday with each of the guests as a Bernoulli trial. Here, there are 500 guests and therefore we are considering $n = 500$ trials. For each trial, the (success) probability that you have the same birthday as the corresponding guest is $p = \frac{1}{365}$. Then, this $N \sim$ Binomial(n,p).

(a) Binomial: $P[N = 1] = np^1(1-p)^{n-1} \approx 0.348$.

(b) Poisson: $P[N = 1] = e^{-np}\frac{(np)^1}{1!} \approx 0.348$.

Extra Questions

Here are some optional questions for those who want more practice.

Problem 6. A sample of a radioactive material emits particles at a rate of 0.7 per second. Assuming that these are emitted in accordance with a Poisson distribution, find the probability that in one second

(a) exactly one is emitted,

(b) more than three are emitted,

(c) between one and four (inclusive) are emitted

[Applebaum, 2008, Q5.27].

Solution: Let X be the number or particles emitted during the one second under consideration. Then $X \sim \mathcal{P}(\alpha)$ where $\alpha = \lambda T = 0.7 \times 1 = 0.7$.

(a) $P[X = 1] = e^{-\alpha}\frac{\alpha^1}{1!} = \alpha e^{-\alpha} = 0.7e^{-0.7} \approx 0.3477$.

(b) $P[X > 3] = 1 - P[X \leq 3] = 1 - \sum_{k=0}^{3} e^{-0.7}\frac{0.7^k}{k!} \approx 0.0058$.

(c) $P[1 \leq X \leq 4] = \sum_{k=1}^{4} e^{-0.7}\frac{0.7^k}{k!} \approx 0.5026$.

Problem 7 (M2011/1)**.** You are given an unfair coin with probability of obtaining a heads equal to $1/3,000,000,000$. You toss this coin 6,000,000,000 times. Let A be the event that you get "tails for all the tosses". Let B be the event that you get "heads for all the tosses".

(a) Approximate $P(A)$.

(b) Approximate $P(A \cup B)$.

Solution: Let N be the number of heads among the n tosses. Then, $N \sim \mathcal{B}(n, p)$. Here, we have small $p = 1/3 \times 10^9$ and large $n = 6 \times 10^9$. So, we can apply Poisson approximation. In other words, $\mathcal{B}(n, p)$ is well-approximated by $\mathcal{P}(\alpha)$ where $\alpha = np = 2$.

(a) $P(A) = P[N = 0] = e^{-2}\frac{2^0}{0!} = \frac{1}{e^2} \approx 0.1353$.

(b) $P(A \cup B) = P[N = 0] + P[N = n] = e^{-2}\frac{2^0}{0!} + e^{-2}\frac{2^{6\times 10^9}}{(6\times 10^9)!}$. The second term is extremely small compared to the first one. Hence, $P(A \cup B)$ is approximately the same as $P(A)$.

ECS 315: Probability and Random Processes	2016/1

HW Solution 10 — Due: November 15, 5 PM

Lecturer: Prapun Suksompong, Ph.D.

Problem 1. Consider a random variable X whose pmf is

$$p_X(x) = \begin{cases} 1/2, & x = -1, \\ 1/4, & x = 0, 1, \\ 0, & \text{otherwise.} \end{cases}$$

Let $Y = X^2$.

(a) Find $\mathbb{E}X$.

(b) Find $\mathbb{E}[X^2]$.

(c) Find $\operatorname{Var} X$.

(d) Find σ_X.

(e) Find $p_Y(y)$.

(f) Find $\mathbb{E}Y$.

(g) Find $\mathbb{E}[Y^2]$.

Solution:

(a) $\mathbb{E}X = \sum_x x p_X(x) = (-1) \times \frac{1}{2} + (0) \times \frac{1}{4} + (1) \times \frac{1}{4} = -\frac{1}{2} + \frac{1}{4} = \boxed{-\frac{1}{4}}$.

(b) $\mathbb{E}[X^2] = \sum_x x^2 p_X(x) = (-1)^2 \times \frac{1}{2} + (0)^2 \times \frac{1}{4} + (1)^2 \times \frac{1}{4} = \frac{1}{2} + \frac{1}{4} = \boxed{\frac{3}{4}}$.

(c) $\operatorname{Var} X = \mathbb{E}[X^2] - (\mathbb{E}X)^2 = \frac{3}{4} - \left(-\frac{1}{4}\right)^2 = \frac{3}{4} - \frac{1}{16} = \boxed{\frac{11}{16}}$.

(d) $\sigma_X = \sqrt{\operatorname{Var} X} = \boxed{\frac{\sqrt{11}}{4}}$.

(e) First, we build a table to see which values y of Y are possible from the values x of X:

x	$p_X(x)$	y
-1	1/2	$(-1)^2 = 1$
0	1/4	$(0)^2 = 0$
1	1/4	$(1)^2 = 1$

Therefore, the random variable Y can takes two values: 0 and 1. $p_Y(0) = p_X(0) = 1/4$. $p_Y(1) = p_X(-1) + p_X(1) = 1/2 + 1/4 = 3/4$. Therefore,

$$\boxed{p_Y(y) = \begin{cases} 1/4, & y = 0, \\ 3/4, & y = 1, \\ 0, & \text{otherwise.} \end{cases}}$$

(f) $\mathbb{E}Y = \sum_y y p_Y(y) = (0) \times \frac{1}{4} + (1) \times \frac{3}{4} = \boxed{\frac{3}{4}}$. Alternatively, because $Y = X^2$, we automatically have $\mathbb{E}[Y] = \mathbb{E}[X^2]$. Therefore, we can simply use the answer from part (b).

(g) $\mathbb{E}[Y^2] = \sum_y y^2 p_Y(y) = (0)^2 \times \frac{1}{4} + (1)^2 \times \frac{3}{4} = \boxed{\frac{3}{4}}$. Alternatively,

$$\mathbb{E}\left[Y^2\right] = \mathbb{E}\left[X^4\right] = \sum_x x^4 p_X(x) = (-1)^4 \times \frac{1}{2} + (0)^4 \times \frac{1}{4} + (1)^4 \times \frac{1}{4} = \frac{1}{2} + \frac{1}{4} = \frac{3}{4}.$$

Problem 2. For each of the following random variables, find $\mathbb{E}X$ and σ_X.

(a) $X \sim \text{Binomial}(3, 1/3)$

(b) $X \sim \text{Poisson}(3)$

Solution:

(a) From the lecture notes, we know that when $X \sim \text{Binomial}(n, p)$, we have $\mathbb{E}X = np$ and $\operatorname{Var} X = np(1-p)$. Here, $n = 3$ and $p = 1/3$. Therefore, $\mathbb{E}X = 3 \times \frac{1}{3} = \boxed{1}$. Also, because $\operatorname{Var} X = 3\left(\frac{1}{3}\right)\left(1 - \frac{1}{3}\right) = \frac{2}{3}$, we have $\sigma_X = \sqrt{\operatorname{Var} X} = \boxed{\sqrt{\frac{2}{3}}}$.

(b) From the lecture notes, we know that when $X \sim \text{Poisson}(\alpha)$, we have $\mathbb{E}X = \alpha$ and $\operatorname{Var} X = \alpha$. Here, $\alpha = 3$. Therefore, $\mathbb{E}X = \boxed{3}$. Also, because $\operatorname{Var} X = 3$, we have $\sigma_X = \boxed{\sqrt{3}}$.

Problem 3. Suppose X is a uniform discrete random variable on $\{-3, -2, -1, 0, 1, 2, 3, 4\}$. Find

(a) $\mathbb{E}X$

(b) $\mathbb{E}\left[X^2\right]$

(c) $\operatorname{Var} X$

(d) σ_X

Solution: All of the calculations in this question are simply plugging in numbers into appropriate formulas.

(a) $\mathbb{E}X = \boxed{0.5}$

(b) $\mathbb{E}\left[X^2\right] = \boxed{5.5}$

(c) $\operatorname{Var} X = \boxed{5.25}$

(d) $\sigma_X = \boxed{2.2913}$

Alternatively, we can find a formula for the general case of uniform random variable X on the sets of integers from a to b. Note that there are $n = b - a + 1$ values that the random variable can take. Hence, all of them has probability $\frac{1}{n}$.

(a) $\mathbb{E}X = \sum_{k=a}^{b} k\frac{1}{n} = \frac{1}{n}\sum_{k=a}^{b} k = \frac{1}{n} \times \frac{n(a+b)}{2} = \frac{a+b}{2}$.

(b) First, note that

$$\begin{aligned}\sum_{i=a}^{b} k(k-1) &= \sum_{k=a}^{b} k(k-1)\left(\frac{(k+1)-(k-2)}{3}\right)\\ &= \frac{1}{3}\left(\sum_{k=a}^{b}(k+1)k(k-1) - \sum_{k=a}^{b} k(k-1)(k-2)\right)\\ &= \frac{1}{3}\left((b+1)b(b-1) - a(a-1)(a-2)\right)\end{aligned}$$

where the last equality comes from the fact that there are many terms in the first sum that is repeated in the second sum and hence many cancellations.

Now,

$$\begin{aligned}\sum_{k=a}^{b} k^2 &= \sum_{k=a}^{b}\left(k(k-1)+k\right) = \sum_{k=a}^{b} k(k-1) + \sum_{k=a}^{b} k\\ &= \frac{1}{3}\left((b+1)b(b-1) - a(a-1)(a-2)\right) + \frac{n(a+b)}{2}\end{aligned}$$

Therefore,

$$\sum_{k=a}^{b} k^2 \frac{1}{n} = \frac{1}{3n}\left((b+1)\,b\,(b-1) - a\,(a-1)\,(a-2)\right) + \frac{a+b}{2}$$
$$= \frac{1}{3}\,a^2 - \frac{1}{6}\,a + \frac{1}{3}\,ab + \frac{1}{6}\,b + \frac{1}{3}\,b^2$$

(c) $\operatorname{Var} X = \mathbb{E}\left[X^2\right] - (\mathbb{E}X)^2 = \frac{1}{12}\,(b-a)\,(b-a+2) = \frac{1}{12}(n-1)(n+1) = \frac{n^2-1}{12}$.

(d) $\sigma_X = \sqrt{\operatorname{Var} X} = \sqrt{\frac{n^2-1}{12}}$.

Problem 4. (Expectation + pmf + Gambling + Effect of miscalculation of probability) In the eighteenth century, a famous French mathematician Jean Le Rond d'Alembert, author of several works on probability, analyzed the toss of two coins. He reasoned that because this experiment has THREE outcomes, (the number of heads that turns up in those two tosses can be 0, 1, or 2), the chances of each must be 1 in 3. In other words, if we let N be the number of heads that shows up, Alembert would say that

$$p_N(n) = 1/3 \quad \text{for} \quad N = 0, 1, 2. \tag{10.1}$$

[Mlodinow, 2008, p 50–51]

We know that Alembert's conclusion was ***wrong***. His three outcomes are not equally likely and hence classical probability formula can not be applied directly. The key is to realize that there are FOUR outcomes which are equally likely. We should not consider 0, 1, or 2 heads as the possible outcomes. There are in fact four equally likely outcomes: (heads, heads), (heads, tails), (tails, heads), and (tails, tails). These are the 4 possibilities that make up the sample space. The actual pmf for N is

$$p_N(n) = \begin{cases} 1/4, & n = 0, 2, \\ 1/2 & n = 1, \\ 0 & \text{otherwise.} \end{cases}$$

Suppose you travel back in time and meet Alembert. You could make the following bet with Alembert to gain some easy money. The bet is that if the result of a toss of two coins contains exactly one head, then he would pay you \$150. Otherwise, you would pay him \$100.

Let R be Alembert's profit from this bet and Y be the your profit from this bet.

(a) Then, $R = -150$ if you win and $R = +100$ otherwise. Use Alembert's *miscalculated* probabilities from (10.1) to determine the pmf of R (from Alembert's belief).

(b) Use Alembert's *miscalculated* probabilities from (10.1) (or the corresponding (miscalculated) pmf found in part (a)) to calculate $\mathbb{E}R$, the expected profit for Alembert.

Remark: You should find that $\mathbb{E}R > 0$ and hence Alembert will be quite happy to accept your bet.

(c) Use the *actual* probabilities, to determine the pmf of R.

(d) Use the *actual* pmf, to determine $\mathbb{E}R$.

Remark: You should find that $\mathbb{E}R < 0$ and hence Alembert should not accept your bet if he calculates the probabilities correctly.

(e) Note that $Y = +150$ if you win and $Y = -100$ otherwise. Use the *actual* probabilities to determine the pmf of Y.

(f) Use the *actual* probabilities, to determine $\mathbb{E}Y$.

Remark: You should find that $\mathbb{E}Y > 0$. This is the amount of money that you expect to gain each time that you play with Alembert. Of course, Alembert, who still believes that his calculation is correct, will ask you to play this bet again and again believing that he will make profit in the long run.

By miscalculating probabilities, one can make wrong decisions (and lose a lot of money)!
Solution:

(a) $P[R = -150] = P[N = 1]$ and $P[R = +100] = P[N \neq 1] = P[N = 0] + P[N = 2]$. So,

$$\boxed{p_R(r) = \begin{cases} p_N(1), & r = -150, \\ p_N(0) + p_N(2), & r = +100, \\ 0, & \text{otherwise.} \end{cases}}$$

Using Alembert's *miscalculated* pmf,

$$p_R(r) = \begin{cases} 1/3, & r = -150, \\ 2/3, & r = +100, \\ 0, & \text{otherwise} \end{cases}$$

(b) From $p_R(r)$ in part (a), we have $\mathbb{E}R = \sum_r p_R(r) = \frac{1}{3} \times (-150) + \frac{2}{3} \times 100 = \boxed{\frac{50}{3}} \approx 16.67$

(c) Again,

$$p_R(r) = \begin{cases} p_N(1), & r = -150, \\ p_N(0) + p_N(2), & r = +100, \\ 0, & \text{otherwise} \end{cases}$$

Using the actual pmf,

$$p_R(r) = \begin{cases} \frac{1}{2}, & r = -150, \\ \frac{1}{4} + \frac{1}{4}, & r = +100, \\ 0, & \text{otherwise} \end{cases} = \boxed{\begin{cases} \frac{1}{2}, & r = -150 \text{ or } +100, \\ 0, & \text{otherwise.} \end{cases}}$$

(d) From $p_R(r)$ in part (c), we have $\mathbb{E}R = \sum_r p_R(r) = \frac{1}{2} \times (-150) + \frac{1}{2} \times 100 = \boxed{-25}$.

(e) Observe that $Y = -R$. Hence, using the answer from part (d), we have

$$p_R(r) = \boxed{\begin{cases} \frac{1}{2}, & r = +150 \text{ or } -100, \\ 0, & \text{otherwise.} \end{cases}}$$

(f) Observe that $Y = -R$. Hence, $\mathbb{E}Y = -\mathbb{E}R$. Using the actual probabilities, $\mathbb{E}R = -25$ from part (d). Hence, $\mathbb{E}Y = \boxed{+25}$.

Extra Questions

Here are some optional questions for those who want more practice.

Problem 5. A random variables X has support containing only two numbers. Its expected value is $\mathbb{E}X = 5$. Its variance is $\operatorname{Var} X = 3$. Give an example of the pmf of such a random variable.

Solution: We first find $\sigma_X = \sqrt{\operatorname{Var} X} = \sqrt{3}$. Recall that this is the average deviation from the mean. Because X takes only two values, we can make them at exactly $\pm\sqrt{3}$ from the mean; that is

$$x_1 = 5 - \sqrt{3} \quad \text{and} \quad x_2 = 5 + \sqrt{3}.$$

In which case, we automatically have $\mathbb{E}X = 5$ and $\operatorname{Var} X = 3$. Hence, one example of such pmf is

$$\boxed{p_X(x) = \begin{cases} \frac{1}{2}, & x = 5 \pm \sqrt{3} \\ 0, & \text{otherwise} \end{cases}}$$

We can also try to find a general formula for x_1 and x_2. If we let $p = P[X = x_2]$, then $q = 1 - p = P[X = x_1]$. Given p, the values of x_1 and x_2 must satisfy two conditions: $\mathbb{E}X = m$ and $\operatorname{Var} X = \sigma^2$. (In our case, $m = 5$ and $\sigma^2 = 3$.) From $\mathbb{E}X = m$, we must have

$$x_1 q + x_2 p = m; \tag{10.2}$$

that is

$$x_1 = \frac{m}{q} - x_2\frac{p}{q}.$$

From $\operatorname{Var} X = \sigma^2$, we have $\mathbb{E}\left[X^2\right] = \operatorname{Var} X + \mathbb{E}X^2 = \sigma^2 + m^2$ and hence we must have

$$x_1^2 q + x_2^2 p = \sigma^2 + m^2. \tag{10.3}$$

Substituting x_1 from (10.2) into (10.3), we have

$$x_2^2 p - 2x_2 mp + \left(pm^2 - q\sigma^2\right) = 0$$

whose solutions are

$$x_2 = \frac{2mp \pm \sqrt{4m^2p^2 - 4p\left(pm^2 - q\sigma^2\right)}}{2p} = \frac{2mp \pm 2\sigma\sqrt{pq}}{2p} = m \pm \sigma\sqrt{\frac{q}{p}}.$$

Using (10.2), we have

$$x_1 = \frac{m}{q} - \left(m \pm \sigma\sqrt{\frac{q}{p}}\right)\frac{p}{q} = m \mp \sigma\sqrt{\frac{p}{q}}.$$

Therefore, for any given p, there are two pmfs:

$$p_X(x) = \begin{cases} 1-p, & x = m - \sigma\sqrt{\frac{p}{1-p}} \\ p, & x = m + \sigma\sqrt{\frac{1-p}{p}} \\ 0, & \text{otherwise,} \end{cases}$$

or

$$p_X(x) = \begin{cases} 1-p, & x = m + \sigma\sqrt{\frac{p}{1-p}} \\ p, & x = m - \sigma\sqrt{\frac{1-p}{p}} \\ 0, & \text{otherwise.} \end{cases}$$

Problem 6. For each of the following families of random variable X, find the value(s) of x which maximize $p_X(x)$. (This can be interpreted as the "mode" of X.)

(a) $\mathcal{P}(\alpha)$

(b) Binomial(n, p)

(c) $\mathcal{G}_0(\beta)$

(d) $\mathcal{G}_1(\beta)$

Remark [Y&G, p. 66]:

- For statisticians, the mode is the most common number in the collection of observations. There are as many or more numbers with that value than any other value. If there are two or more numbers with this property, the collection of observations is called multimodal. In probability theory, a **mode** of random variable X is a number x_{mode} satisfying
$$p_X(x_{\text{mode}}) \geq p_X(x) \quad \text{for all } x.$$
- For statisticians, the median is a number in the middle of the set of numbers, in the sense that an equal number of members of the set are below the median and above the median. In probability theory, a median, X_{median}, of random variable X is a number that satisfies
$$P[X < X_{\text{median}}] = P[X > X_{\text{median}}].$$
- Neither the mode nor the median of a random variable X need be unique. A random variable can have several modes or medians.

Solution: We first note that when $\alpha > 0$, $p \in (0,1)$, $n \in \mathbb{N}$, and $\beta \in (0,1)$, the above pmf's will be strictly positive for some values of x. Hence, we can discard those x at which $p_X(x) = 0$. The remaining points are all integers. To compare them, we will evaluate $\frac{p_X(i+1)}{p_X(i)}$.

(a) For Poisson pmf, we have
$$\frac{p_X(i+1)}{p_X(i)} = \frac{\frac{e^{-\alpha}\alpha^{i+1}}{(i+1)!}}{\frac{e^{-\alpha}\alpha^{i}}{i!}} = \frac{\alpha}{i+1}.$$

Notice that

- $\frac{p_X(i+1)}{p_X(i)} > 1$ if and only if $i < \alpha - 1$.
- $\frac{p_X(i+1)}{p_X(i)} = 1$ if and only if $i = \alpha - 1$.
- $\frac{p_X(i+1)}{p_X(i)} < 1$ if and only if $i > \alpha - 1$.

Let $\tau = \alpha - 1$. This implies that τ is the place where things change. Moving from i to $i+1$, the probability strictly increases if $i < \tau$. When $i > \tau$, the next probability value (at $i+1$) will decrease.

(i) Suppose $\alpha \in (0,1)$, then $\alpha - 1 < 0$ and hence $i > \alpha - 1$ for all i. (Note that i are are nonnegative integers.) This implies that the pmf is a strictly decreasing function and hence the maximum occurs at the first i which is $i = 0$.

(ii) Suppose $\alpha \in \mathbb{N}$. Then, the pmf will be strictly increasing until we reaches $i = \alpha - 1$. At which point, the next probability value is the same. Then, as we further increase i, the pmf is strictly decreasing. Therefore, the maximum occurs at $\alpha - 1$ and α.

(iii) Suppose $\alpha \notin \mathbb{N}$ and $\alpha \geq 1$. Then we will have have any $i = \alpha - 1$. The pmf will be strictly increasing where the last increase is from $i = \lfloor \alpha - 1 \rfloor$ to $i + 1 = \lfloor \alpha - 1 \rfloor + 1 = \lfloor \alpha \rfloor$. After this, the pmf is strictly decreasing. Hence, the maximum occurs at $\lfloor \alpha \rfloor$.

To summarize,

$$\arg\max_x p_X(x) = \boxed{\begin{cases} 0, & \alpha \in (0,1), \\ \alpha - 1 \text{ and } \alpha, & \alpha \text{ is an integer}, \\ \lfloor \alpha \rfloor, & \alpha > 1 \text{ is not an integer}. \end{cases}}$$

(b) For binomial pmf, we have

$$\frac{p_X(i+1)}{p_X(i)} = \frac{\frac{n!}{(i+1)!(n-i-1)!} p^{i+1} (1-p)^{n-i-1}}{\frac{n!}{i!(n-i)!} p^i (1-p)^{n-i}} = \frac{(n-i)\,p}{(i+1)(1-p)}.$$

Notice that

- $\frac{p_X(i+1)}{p_X(i)} > 1$ if and only if $i < np - 1 + p = (n+1)p - 1$.
- $\frac{p_X(i+1)}{p_X(i)} = 1$ if and only if $i = (n+1)p - 1$.
- $\frac{p_X(i+1)}{p_X(i)} < 1$ if and only if $i > (n+1)p - 1$.

Let $\tau = (n+1)p - 1$. This implies that τ is the place where things change. Moving from i to $i+1$, the probability strictly increases if $i < \tau$. When $i > \tau$, the next probability value (at $i+1$) will decrease.

(i) Suppose $(n+1)p$ is an integer. The pmf will strictly increase as a function of i, and then stays at the same value at $i = \tau = (n+1)p - 1$ and $i + 1 = (n+1)p - 1 + 1 = (n+1)p$. Then, it will strictly decrease. So, the maximum occurs at $(n+1)p - 1$ and $(n+1)p$.

(ii) Suppose $(n+1)p$ is not an integer. Then, there will not be any i that is $= \tau$. Therefore, we only have the pmf strictly increases where the last increase occurs when we goes from $i = \lfloor \tau \rfloor$ to $i + 1 = \lfloor \tau \rfloor + 1$. After this, the probability is strictly decreasing. Hence, the maximum is unique and occur at $\lfloor \tau \rfloor + 1 = \lfloor (n+1)p - 1 \rfloor + 1 = \lfloor (n+1)p \rfloor$.

To summarize,

$$\arg\max_x p_X(x) = \boxed{\begin{cases} (n+1)p-1 \text{ and } (n+1)p, & (n+1)p \text{ is an integer,} \\ \lfloor (n+1)p \rfloor, & (n+1)p \text{ is not an integer.} \end{cases}}$$

(c) $\frac{p_X(i+1)}{p_X(i)} = \beta < 1$. Hence, $p_X(i)$ is strictly decreasing. The maximum occurs at the smallest value of i which is $\boxed{0}$.

(d) $\frac{p_X(i+1)}{p_X(i)} = \beta < 1$. Hence, $p_X(i)$ is strictly decreasing. The maximum occurs at the smallest value of i which is $\boxed{1}$.

Problem 7. An article in Information Security Technical Report ["Malicious Software—Past, Present and Future" (2004, Vol. 9, pp. 618)] provided the data (shown in Figure 10.1) on the top ten malicious software instances for 2002. The clear leader in the number of registered incidences for the year 2002 was the Internet worm "Klez". This virus was first detected on 26 October 2001, and it has held the top spot among malicious software for the longest period in the history of virology.

Place	Name	% Instances
1	I-Worm.Klez	61.22%
2	I-Worm.Lentin	20.52%
3	I-Worm.Tanatos	2.09%
4	I-Worm.BadtransII	1.31%
5	Macro.Word97.Thus	1.19%
6	I-Worm.Hybris	0.60%
7	I-Worm.Bridex	0.32%
8	I-Worm.Magistr	0.30%
9	Win95.CIH	0.27%
10	I-Worm.Sircam	0.24%

Figure 10.1: The 10 most widespread malicious programs for 2002 (Source—Kaspersky Labs).

Suppose that 20 malicious software instances are reported. Assume that the malicious sources can be assumed to be independent.

(a) What is the probability that at least one instance is "Klez"?

(b) What is the probability that three or more instances are "Klez"?

(c) What are the expected value and standard deviation of the number of "Klez" instances among the 20 reported?

Solution: Let N be the number of instances (among the 20) that are "Klez". Then, $N \sim$binomial(n, p) where $n = 20$ and $p = 0.6122$.

(a) $P[N \geq 1] = 1 - P[N < 1] = 1 - P[N = 0] = 1 - p_N(0) = 1 - \binom{20}{0} \times 0.6122^0 \times 0.3878^{20} \approx 0.9999999941 \approx 1$.

(b)

$$\begin{aligned} P[N \geq 3] &= 1 - P[N < 3] = 1 - (P[N = 0] + P[N = 1] + P[N = 2]) \\ &= 1 - \sum_{k=0}^{2} \binom{20}{k} (0.6122)^k (0.3878)^{20-k} \approx 0.999997 \end{aligned}$$

(c) $\mathbb{E}N = np = 20 \times 0.6122 = 12.244$.
$\sigma_N = \sqrt{\operatorname{Var} N} = \sqrt{np(1-p)} = \sqrt{20 \times 0.6122 \times 0.3878} \approx 2.179$.

Y&G Q3.2.1

Wednesday, October 03, 2012 3:50 PM

In this question, you are given a pdf whose expression has an unknown constant c.

(a) To find the constant c, recall that any pdf should integrate to 1.

In this problem,

$$\int_{-\infty}^{\infty} f_X(x)\,dx = \int_0^2 cx\,dx = c\int_0^2 x\,dx = c\left.\frac{x^2}{2}\right|_0^2 = \underbrace{2c}_{\text{This should } = 1.}$$

Therefore, $c = \frac{1}{2}$.

(b) $P[0 \le X \le 1] = \int_0^1 f_X(x)\,dx = \int_0^1 \frac{1}{2}x\,dx = \frac{1}{2}\left.\frac{x^2}{2}\right|_0^1 = \frac{1}{4}$.

(c) $P[-\frac{1}{2} \le X \le \frac{1}{2}] = \int_{-1/2}^{1/2} f_X(x)\,dx = \int_0^{1/2} \frac{1}{2}x\,dx = \frac{1}{2}\left.\frac{x^2}{2}\right|_0^{1/2} = \frac{1}{16}$.

$f_X(x) = 0$ on $[-\frac{1}{2}, 0)$

(d) For $x<0$, because $f_X(t) = 0$ for $t<0$, $F_X(x) = \int_{-\infty}^{x} f_X(t)\,dt = 0$

For $0 \le x \le 2$, $f_X(t) = \frac{t}{2}$ and $F_X(x) = \int_{-\infty}^{x} f_X(t)\,dt = \int_0^x \frac{t}{2}\,dt = \left.\frac{t^2}{4}\right|_0^x = \frac{x^2}{4}$.

At $x = 2$, $F_X(2) = 1$.

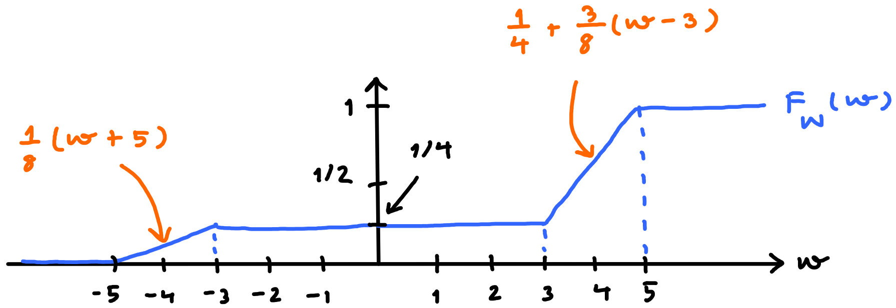
For $x>2$, $f_X(t) = 0$. Therefore, $F_X(x) = \int_{-\infty}^{x} f_X(t)\,dt = \underbrace{\int_{-\infty}^{2} f_X(t)\,dt}_{F_X(2) = 1} + \underbrace{\int_2^x \underbrace{f_X(t)}_{0}\,dt}_{0} = 1$.

Combining the three results above, we have

$$F_X(x) = \begin{cases} 0, & x<0, \\ x^2/4, & 0 \le x \le 2, \\ 1, & \text{otherwise.} \end{cases}$$

Y&G Q3.1.3

Wednesday, October 03, 2012 3:14 PM



(a) From the plot above, we see that $F_W(w)$ is a continuous function.

Because its cdf is continuous, we conclude that W is a continuous RV.

(b) $P[W \le 4] = F_W(4) = \frac{1}{4} + \frac{3}{8}(4-3) = \frac{1}{4} + \frac{3}{8} = \frac{5}{8} \approx 0.625$

by definition of cdf

(c) $P[-2 < W \le 2] = F_W(2) - F_W(-2) = \frac{1}{4} - \frac{1}{4} = 0$

For continuous RV, $P[a \le X \le b] = F_X(b) - F_X(a)$

(d) $P[W > 0] = 1 - P[W \le 0] = 1 - F_W(0) = 1 - \frac{1}{4} = \frac{3}{4}$

$P(A) = 1 - P(A^c)$

(e) $P[W \le a] = F_W(a)$. From the plot above, we know that to have $F_W(a) = \frac{1}{2}$, the value of a must be in the interval $(3,5)$.

In this interval, $F_W(a) = \frac{1}{4} + \frac{3}{8}(a-3)$.

So, we solve for "a" that satisfies

$$\frac{1}{4} + \frac{3}{8}(a-3) = \frac{1}{2} \Rightarrow a = \frac{11}{3} \approx 3.67$$

Remark: It is possible to solve this problem by finding the pdf first.
(I ask you to derive the pdf anyway in Q 3.2.3.)
However, you should also make sure that you know how to evaluate the probabilities above directly from the cdf.

Y&G Q3.2.3

Wednesday, October 03, 2012 4:18 PM

Given a cdf, we can find the pdf by taking derivative.

As discussed in class, for the location(s) where derivative does not exist, we can choose to define the pdf to be any convenient value.

In this question, the cdf is given in the form of expressions on several intervals. It is then easy to find its derivative inside each of the intervals:

$$f_W(w) = \frac{d}{dw} F_W(w) = \begin{cases} 0, & w < -5, \\ 1/8, & -5 < w < -3, \\ 0, & -3 < w < 3, \\ 3/8, & 3 < w < 5, \\ 0, & 5 < w. \end{cases}$$

It should be clear from the plot of cdf in Q3.1.3 that the derivative does not exist at $w = -5, -3, 3, 5$. We choose to assign $f_W(w) = 0$ at these points.

$$f_W(w) = \begin{cases} 1/8, & -5 < w < -3 \\ 3/8, & 3 < w < 5 \\ 0, & \text{otherwise} \end{cases}$$

ECS 315: Probability and Random Processes **2016/1**

HW Solution 12 — Due: November 29, 5 PM

Lecturer: Prapun Suksompong, Ph.D.

Problem 1 (Yates and Goodman, 2005, Q3.3.4). The pdf of random variable Y is

$$f_Y(y) = \begin{cases} y/2 & 0 \le y < 2, \\ 0, & \text{otherwise.} \end{cases}$$

(a) Find $\mathbb{E}[Y]$.

(b) Find $\operatorname{Var} Y$.

Solution:

(a) Recall that, for continuous random variable Y,

$$\mathbb{E}Y = \int_{-\infty}^{\infty} y f_Y(y) dy.$$

Note that when y is outside of the interval $[0,2)$, $f_Y(y) = 0$ and hence does not affect the integration. We only need to integrate over $[0,2)$ in which $f_Y(y) = \frac{y}{2}$. Therefore,

$$\mathbb{E}Y = \int_0^2 y\left(\frac{y}{2}\right)dy = \int_0^2 \frac{y^2}{2}dy = \left.\frac{y^3}{2 \times 3}\right|_0^2 = \boxed{\frac{4}{3}}.$$

(b) The variance of any random variable Y (discrete or continuous) can be found from

$$\operatorname{Var} Y = \mathbb{E}\left[Y^2\right] - (\mathbb{E}Y)^2$$

We have already calculate $\mathbb{E}Y$ in the previous part. So, now we need to calculate $\mathbb{E}[Y^2]$. Recall that, for continuous random variable,

$$\mathbb{E}[g(Y)] = \int_{-\infty}^{\infty} g(y) f_Y(y) dy.$$

Here, $g(y) = y^2$. Therefore,

$$\mathbb{E}\left[Y^2\right] = \int_{-\infty}^{\infty} y^2 f_Y(y) dy.$$

Again, in the integration, we can ignore the y whose $f_Y(y) = 0$:

$$\mathbb{E}\left[Y^2\right] = \int_0^2 y^2\left(\frac{y}{2}\right)dy = \int_0^2 \frac{y^3}{2}dy = \left.\frac{y^4}{2\times 4}\right|_0^2 = \boxed{2}.$$

Plugging this into the variance formula gives

$$\operatorname{Var} Y = \mathbb{E}\left[Y^2\right] - (\mathbb{E}Y)^2 = 2 - \left(\frac{4}{3}\right)^2 = 2 - \frac{16}{9} = \boxed{\frac{2}{9}}.$$

Problem 2 (Yates and Goodman, 2005, Q3.3.6)**.** The cdf of random variable V is

$$F_V(v) = \begin{cases} 0 & v < -5, \\ (v+5)^2/144, & -5 \le v < 7, \\ 1 & v \ge 7. \end{cases}$$

(a) What is $f_V(v)$?

(b) What is $\mathbb{E}[V]$?

(c) What is $\operatorname{Var}[V]$?

(d) What is $\mathbb{E}[V^3]$?

Solution: First, let's check whether V is a continuous random variable. This can be done easily by checking whether its cdf $F_V(v)$ is a continuous function. The cdf of V is defined using three expressions. Note that each expression is a continuous function. So, we only need to check whether there is/are any jump(s) at the boundaries: $v = 5$ and $v = 7$. Plugging $v = 5$ into $(v+5)^2/144$ gives 0 which matches the value of the expression for $v < -5$. Plugging $v = 7$ into $(v+5)^2/144$ gives 1 which matches the value of the expression for $v \ge 7$. SO, there is no discontinuity in $F_V(v)$. It is a continuous function and hence V itself is a continuous random variable.

(a) We can find the pdf $f_V(v)$ at almost all of the v by finding the derivative of the cdf $F_V(v)$:

$$f_V(v) = \frac{d}{dv}F_V(v) = \begin{cases} 0, & v < -5, \\ \frac{v+5}{72}, & -5 < v < 7, \\ 0, & v > 7. \end{cases}$$

Note that we still haven't specified $f_V(v)$ at $v = 5$ and $v = 7$. This is because the formula for $F_V(v)$ changes at those points and hence to actually find the derivatives, we would need to look at both the left and right derivatives at these points. The derivative may not even exist there. The good news is that we don't have to actually find them because $v = 5$ and $v = 7$ correspond to just two points on the pdf. Because V is a continuous random variable, we can "define" or "set " $f_V(v)$ to be any values there. In this case, for brevity of the expression, let's set the pdf to be 0 there. This gives

$$f_V(v) = \frac{d}{dv} F_V(v) = \boxed{\begin{cases} \frac{v+5}{72}, & -5 < v < 7, \\ 0, & \text{otherwise.} \end{cases}}$$

(b) $\mathbb{E}[V] = \int\limits_{-\infty}^{\infty} v f_V(v)\,dv = \int\limits_{-5}^{7} v\left(\frac{v+5}{72}\right)dv = \frac{1}{72}\int\limits_{-5}^{7} v^2 + 5v dv = \boxed{3}$.

(c) $\mathbb{E}[V^2] = \int\limits_{-\infty}^{\infty} v^2 f_V(v)\,dv = \int\limits_{-5}^{7} v^2\left(\frac{v+5}{72}\right)dv = 17$.

Therefore, $\operatorname{Var} V = \mathbb{E}[V^2] - (\mathbb{E}[V])^2 = 17 - 9 = \boxed{8}$.

(d) $\mathbb{E}[V^3] = \int\limits_{-\infty}^{\infty} v^3 f_V(v)\,dv = \int\limits_{-5}^{7} v^3\left(\frac{v+5}{72}\right)dv = \boxed{\frac{431}{5} = 86.2}$.

Problem 3 (Yates and Goodman, 2005, Q3.4.5)**.** X is a continuous uniform RV on the interval $(-5, 5)$.

(a) What is its pdf $f_X(x)$?

(b) What is its cdf $F_X(x)$?

(c) What is $\mathbb{E}[X]$?

(d) What is $\mathbb{E}[X^5]$?

(e) What is $\mathbb{E}\left[e^X\right]$?

Solution: For a uniform random variable X on the interval (a, b), we know that

$$f_X(x) = \begin{cases} 0, & x < a \text{ or } x > b, \\ \frac{1}{b-a}, & a \le x \le b \end{cases}$$

and

$$F_X(x) = \begin{cases} 0, & x < a \text{ or } x > b, \\ \frac{x-a}{b-a}, & a \le x \le b. \end{cases}$$

In this problem, we have $a = -5$ and $b = 5$.

(a) $f_X(x) = \boxed{\begin{cases} 0, & x < -5 \text{ or } x > 5, \\ \frac{1}{10}, & -5 \le x \le 5 \end{cases}}$

(b) $F_X(x) = \boxed{\begin{cases} 0, & x < a \text{ or } x > b, \\ \frac{x+5}{10}, & a \le x \le b. \end{cases}}$

(c) $\mathbb{E}X = \int\limits_{-\infty}^{\infty} x f_X(x)\,dx = \int\limits_{-5}^{5} x \times \frac{1}{10} dx = \frac{1}{10} \left.\frac{x^2}{2}\right|_{-5}^{5} = \frac{1}{20}\left(5^2 - (-5)^2\right) = \boxed{0}$.

In general,

$$\mathbb{E}X = \int\limits_a^b x \frac{1}{b-a} dx = \frac{1}{b-a} \int\limits_a^b x dx = \frac{1}{b-a} \left.\frac{x^2}{2}\right|_a^b = \frac{1}{b-a} \frac{b^2 - a^2}{2} = \frac{a+b}{2}.$$

With $a = -5$ and $b = 5$, we have $\mathbb{E}X = \boxed{0}$.

(d) $\mathbb{E}[X^5] = \int\limits_{-\infty}^{\infty} x^5 f_X(x)\,dx = \int\limits_{-5}^{5} x^5 \times \frac{1}{10} dx = \frac{1}{10} \left.\frac{x^6}{6}\right|_{-5}^{5} = \frac{1}{60}\left(5^6 - (-5)^6\right) = \boxed{0}$.

In general,

$$\mathbb{E}\left[X^5\right] = \int\limits_a^b x^5 \frac{1}{b-a} dx = \frac{1}{b-a} \int\limits_a^b x^5 dx = \frac{1}{b-a} \left.\frac{x^6}{6}\right|_a^b = \frac{1}{b-a} \frac{b^6 - a^6}{2}.$$

With $a = -5$ and $b = 5$, we have $\mathbb{E}[X^5] = \boxed{0}$.

(e) In general,

$$\mathbb{E}\left[e^X\right] = \int\limits_a^b e^x \frac{1}{b-a} dx = \frac{1}{b-a} \int\limits_a^b e^x dx = \frac{1}{b-a} \left. e^x \right|_a^b = \frac{e^b - e^a}{b-a}.$$

With $a = -5$ and $b = 5$, we have $\mathbb{E}\left[e^X\right] = \boxed{\frac{e^5 - e^{-5}}{10}} \approx 14.84$.

Problem 4 (Randomly Phased Sinusoid)**.** Suppose Θ is a uniform random variable on the interval $(0, 2\pi)$.

(a) Consider another random variable X defined by

$$X = 5\cos(7t + \Theta)$$

where t is some constant. Find $\mathbb{E}[X]$.

(b) Consider another random variable Y defined by

$$Y = 5\cos(7t_1 + \Theta) \times 5\cos(7t_2 + \Theta)$$

where t_1 and t_2 are some constants. Find $\mathbb{E}[Y]$.

Solution: First, because Θ is a uniform random variable on the interval $(0, 2\pi)$, we know that $f_\Theta(\theta) = \frac{1}{2\pi} 1_{(0,2\pi)}(t)$. Therefore, for "any" function g, we have

$$\mathbb{E}[g(\Theta)] = \int_{-\infty}^{\infty} g(\theta) f_\Theta(\theta) d\theta.$$

(a) X is a function of Θ. $\mathbb{E}[X] = 5\mathbb{E}[\cos(7t + \Theta)] = 5\int_0^{2\pi} \frac{1}{2\pi}\cos(7t + \theta)d\theta$. Now, we know that integration over a cycle of a sinusoid gives 0. So, $\mathbb{E}[X] = \boxed{0}$.

(b) Y is another function of Θ.

$$\begin{aligned}\mathbb{E}[Y] &= \mathbb{E}[5\cos(7t_1 + \Theta) \times 5\cos(7t_2 + \Theta)] = \int_0^{2\pi} \frac{1}{2\pi} 5\cos(7t_1 + \theta) \times 5\cos(7t_2 + \theta)d\theta \\ &= \frac{25}{2\pi}\int_0^{2\pi} \cos(7t_1 + \theta) \times \cos(7t_2 + \theta)d\theta.\end{aligned}$$

Recall[1] the cosine identity

$$\cos(a) \times \cos(b) = \frac{1}{2}\left(\cos(a + b) + \cos(a - b)\right).$$

Therefore,

$$\begin{aligned}\mathbb{E}Y &= \frac{25}{4\pi}\int_0^{2\pi} \cos(14t + 2\theta) + \cos(7(t_1 - t_2))\, d\theta \\ &= \frac{25}{4\pi}\left(\int_0^{2\pi} \cos(14t + 2\theta)\, d\theta + \int_0^{2\pi} \cos(7(t_1 - t_2))\, d\theta\right).\end{aligned}$$

The first integral gives 0 because it is an integration over two period of a sinusoid. The integrand in the second integral is a constant. So,

$$\mathbb{E}Y = \frac{25}{4\pi}\cos(7(t_1 - t_2))\int_0^{2\pi} d\theta = \frac{25}{4\pi}\cos(7(t_1 - t_2))\, 2\pi = \boxed{\frac{25}{2}\cos(7(t_1 - t_2))}.$$

[1] This identity could be derived easily via the Euler's identity:

$$\begin{aligned}\cos(a) \times \cos(b) &= \frac{e^{ja} + e^{-ja}}{2} \times \frac{e^{jb} + e^{-jb}}{2} = \frac{1}{4}\left(e^{ja}e^{jb} + e^{-ja}e^{jb} + e^{ja}e^{-jb} + e^{-ja}e^{-jb}\right) \\ &= \frac{1}{2}\left(\frac{e^{ja}e^{jb} + e^{-ja}e^{-jb}}{2} + \frac{e^{-ja}e^{jb} + e^{ja}e^{-jb}}{2}\right) \\ &= \frac{1}{2}\left(\cos(a + b) + \cos(a - b)\right).\end{aligned}$$

Extra Question

Here is an optional question for those who want more practice.

Problem 5. Let X be a uniform random variable on the interval $[0,1]$. Set

$$A = \left[0, \frac{1}{2}\right), \quad B = \left[0, \frac{1}{4}\right) \cup \left[\frac{1}{2}, \frac{3}{4}\right), \quad \text{and } C = \left[0, \frac{1}{8}\right) \cup \left[\frac{1}{4}, \frac{3}{8}\right) \cup \left[\frac{1}{2}, \frac{5}{8}\right) \cup \left[\frac{3}{4}, \frac{7}{8}\right).$$

Are the events $[X \in A], [X \in B]$, and $[X \in C]$ independent?

Solution: Note that

$$P[X \in A] = \int_0^{\frac{1}{2}} dx = \frac{1}{2},$$

$$P[X \in B] = \int_0^{\frac{1}{4}} dx + \int_{\frac{1}{2}}^{\frac{3}{4}} dx = \frac{1}{2}, \text{ and}$$

$$P[X \in C] = \int_0^{\frac{1}{8}} dx + \int_{\frac{1}{4}}^{\frac{3}{8}} dx + \int_{\frac{1}{2}}^{\frac{5}{8}} dx + \int_{\frac{3}{4}}^{\frac{7}{8}} dx = \frac{1}{2}.$$

Now, for pairs of events, we have

$$P([X \in A] \cap [X \in B]) = \int_0^{\frac{1}{4}} dx = \frac{1}{4} = P[X \in A] \times P[X \in B], \tag{12.1}$$

$$P([X \in A] \cap [X \in C]) = \int_0^{\frac{1}{8}} dx + \int_{\frac{1}{4}}^{\frac{3}{8}} dx = \frac{1}{4} = P[X \in A] \times P[X \in C], \text{ and} \tag{12.2}$$

$$P([X \in B] \cap [X \in C]) = \int_0^{\frac{1}{8}} dx + \int_{\frac{1}{2}}^{\frac{5}{8}} dx = \frac{1}{4} = P[X \in B] \times P[X \in C]. \tag{12.3}$$

Finally,

$$P([X \in A] \cap [X \in B] \cap [X \in C]) = \int_0^{\frac{1}{8}} dx = \frac{1}{8} = P[X \in A]\, P[X \in B]\, P[X \in C]. \tag{12.4}$$

From (12.1), (12.2), (12.3) and (12.4), we can conclude that the events $[X \in A], [X \in B]$, and $[X \in C]$ are $\boxed{\text{independent}}$.

ECS 315: Probability and Random Processes	2016/1

HW Solution 13 — Due: Dec 6, 5 PM

Lecturer: Prapun Suksompong, Ph.D.

Problem 1. A random variable X is a Gaussian random variable if its pdf is given by

$$f_X(x) = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{1}{2}\left(\frac{x-m}{\sigma}\right)^2},$$

for some constant m and positive number σ. Furthermore, when a Gaussian random variable has $m = 0$ and $\sigma = 1$, we say that it is a standard Gaussian random variable. There is no closed-form expression for the cdf of the standard Gaussian random variable. The cdf itself is denoted by Φ and its values (or its complementary values $Q(\cdot) = 1-\Phi(\cdot)$) are traditionally provided by a table.

Suppose Z is a standard Gaussian random variable.

(a) Use the Φ table to find the following probabilities:

- (i) $P[Z < 1.52]$
- (ii) $P[Z < -1.52]$
- (iii) $P[Z > 1.52]$
- (iv) $P[Z > -1.52]$
- (v) $P[-1.36 < Z < 1.52]$

(b) Use the Φ table to find the value of c that satisfies each of the following relation.

- (i) $P[Z > c] = 0.14$
- (ii) $P[-c < Z < c] = 0.95$

Solution:

(a)

- (i) $P[Z < 1.52] = \Phi(1.52) = \boxed{0.9357.}$
- (ii) $P[Z < -1.52] = \Phi(-1.52) = 1 - \Phi(1.52) = 1 - 0.9357 = \boxed{0.0643.}$
- (iii) $P[Z > 1.52] = 1 - P[Z < 1.52] = 1 - \Phi(1.52) = 1 - 0.9357 = \boxed{0.0643.}$
- (iv) It is straightforward to see that the area of $P[Z > -1.52]$ is the same as $P[Z < 1.52] = \Phi(1.52)$. So, $P[Z > -1.52] = \boxed{0.9357.}$
  Alternatively, $P[Z > -1.52] = 1 - P[Z \le -1.52] = 1 - \Phi(-1.52) = 1 - (1 - \Phi(1.52)) = \Phi(1.52)$.

(v) $P[-1.36 < Z < 1.52] = \Phi(1.52) - \Phi(-1.36) = \Phi(1.52) - (1 - \Phi(1.36)) = \Phi(1.52) + \Phi(1.36) - 1 = 0.9357 + 0.9131 - 1 = \boxed{0.8488.}$

(b)

(i) $P[Z > c] = 1 - P[Z \leq c] = 1 - \Phi(c)$. So, we need $1 - \Phi(c) = 0.14$ or $\Phi(c) = 1 - 0.14 = 0.86$. In the Φ table, we do not have exactly 0.86, but we have 0.8599 and 0.8621. Because 0.86 is closer to 0.8599, we answer the value of c whose $\phi(c) = 0.8599$. Therefore, $c \approx \boxed{1.08.}$

(ii) $P[-c < Z < c] = \Phi(c) - \Phi(-c) = \Phi(c) - (1 - \Phi(c)) = 2\Phi(c) - 1$. So, we need $2\Phi(c) - 1 = 0.95$ or $\Phi(c) = 0.975$. From the Φ table, we have $c \approx \boxed{1.96.}$

Problem 2. The peak temperature T, as measured in degrees Fahrenheit, on a July day in New Jersey is a $\mathcal{N}(85, 100)$ random variable.

Remark: Do not forget that, for our class, the second parameter in $\mathcal{N}(\cdot, \cdot)$ is the variance (not the standard deviation).

(a) Express the cdf of T in terms of the Φ function.

(b) Express each of the following probabilities in terms of the Φ function(s). Make sure that the arguments of the Φ functions are positive. (Positivity is required so that we can directly use the Φ/Q tables to evaluate the probabilities.)

(i) $P[T > 100]$

(ii) $P[T < 60]$

(iii) $P[70 \leq T \leq 100]$

(c) Express each of the probabilities in part (b) in terms of the Q function(s). Again, make sure that the arguments of the Q functions are positive.

(d) Evaluate each of the probabilities in part (b) using the Φ/Q tables.

(e) Observe that the Φ table ("Table 4" from the lecture) stops at $z = 2.99$ and the Q table ("Table 5" from the lecture) starts at $z = 3.00$. Why is it better to give a table for $Q(z)$ instead of $\Phi(z)$ when z is large?

Solution:

(a) Recall that when $X \sim \mathcal{N}(m, \sigma^2)$, $F_X(x) = \Phi\left(\frac{x-m}{\sigma}\right)$. Here, $T \sim \mathcal{N}(85, 10^2)$. Therefore,

$$F_T(t) = \boxed{\Phi\left(\frac{t-85}{10}\right).}$$

(b)

(i) $P[T > 100] = 1 - P[T \le 100] = 1 - F_T(100) = 1 - \Phi\left(\frac{100-85}{10}\right) = 1 - \Phi(1.5)$

(ii) $P[T < 60] = P[T \le 60]$ because T is a continuous random variable and hence $P[T = 60] = 0$. Now, $P[T \le 60] = F_T(60) = \Phi\left(\frac{60-85}{10}\right) = \Phi(-2.5) = \boxed{1 - \Phi(2.5).}$ Note that, for the last equality, we use the fact that $\Phi(-z) = 1 - \Phi(z)$.

(iii)

$$P[70 \le T \le 100] = F_T(100) - F_T(70) = \Phi\left(\frac{100-85}{10}\right) - \Phi\left(\frac{70-85}{10}\right)$$
$$= \Phi(1.5) - \Phi(-1.5) = \Phi(1.5) - (1 - \Phi(1.5)) = \boxed{2\Phi(1.5) - 1.}$$

(c) In this question, we use the fact that $Q(x) = 1 - \Phi(x)$.

(i) $1 - \Phi(1.5) = \boxed{Q(1.5).}$

(ii) $1 - \Phi(2.5) = \boxed{Q(2.5).}$

(iii) $2\Phi(1.5) - 1 = 2(1 - Q(1.5)) - 1 = 2 - 2Q(1.5) - 1 = \boxed{1 - 2Q(1.5).}$

(d)

(i) $1 - \Phi(1.5) = 1 - 0.9332 = \boxed{0.0668.}$

(ii) $1 - \Phi(2.5) = 1 - 0.99379 = \boxed{0.0062.}$

(iii) $2\Phi(1.5) - 1 = 2(0.9332) - 1 = \boxed{0.8664.}$

(e) When z is large, $\Phi(z)$ will start with 0.999... The first few significant digits will all be the same and hence not quite useful to be there.

Problem 3. Suppose that the time to failure (in hours) of fans in a personal computer can be modeled by an exponential distribution with $\lambda = 0.0003$.

(a) What proportion of the fans will last at least 10,000 hours?

(b) What proportion of the fans will last at most 7000 hours?

[Montgomery and Runger, 2010, Q4-97]

Solution: See handwritten solution.

HW13 Q3: Exponential RV

Friday, November 21, 2014 9:02 AM

Let T be the time to failure (in hours)
We know that $T \sim \mathcal{E}(\lambda)$ where $\lambda = 3 \times 10^{-4}$.
Therefore,

$$f_T(t) = \begin{cases} \lambda e^{-\lambda t}, & t > 0, \\ 0, & \text{otherwise.} \end{cases}$$

(a) Here, we want to find $P[T > 10^4]$.
We will first provide the general formula for $P[T > t]$.
For $T \sim \mathcal{E}(\lambda)$ and $t > 0$,

$$P[T > t] = \int_t^\infty f_T(\tau)\,d\tau = \int_t^\infty \lambda e^{-\lambda \tau}\,d\tau = -e^{-\lambda\tau}\Big|_t^\infty = e^{-\lambda t}$$

Therefore, $P[T > 10^4] = e^{-3 \times 10^{-4} \times 10^4} = e^{-3} \approx 0.0498$

(b) $P[T \le 7000] = 1 - P[T > 7000] = 1 - e^{-3 \times 10^{-4} \times 7000} = 1 - e^{-2.1} \approx 0.8775$

Remark: In class, we have already shown that for $T \sim \mathcal{E}(\lambda)$,

$$F_T(t) = \begin{cases} 1 - e^{-\lambda t}, & t > 0, \\ 0, & \text{otherwise,} \end{cases}$$

and

$$P[T > t] = \begin{cases} e^{-\lambda t}, & t > 0, \\ 1, & \text{otherwise.} \end{cases}$$

These formula can be applied here directly as well.

Problem 4. Consider each random variable X defined below. Let $Y = 1 + 2X$. (i) Find and sketch the pdf of Y and (ii) Does Y belong to any of the (popular) families discussed in class? If so, state the name of the family and find the corresponding parameters.

(a) $X \sim \mathcal{U}(0, 1)$

(b) $X \sim \mathcal{E}(1)$

(c) $X \sim \mathcal{N}(0, 1)$

Solution: See handwritten solution

Problem 5. Consider each random variable X defined below. Let $Y = 1 - 2X$. (i) Find and sketch the pdf of Y and (ii) Does Y belong to any of the (popular) families discussed in class? If so, state the name of the family and find the corresponding parameters.

(a) $X \sim \mathcal{U}(0, 1)$

(b) $X \sim \mathcal{E}(1)$

(c) $X \sim \mathcal{N}(0, 1)$

Solution: See handwritten solution

HW13 Q4 Affine Transformation

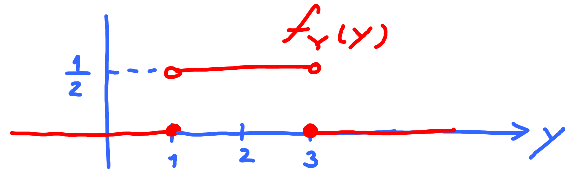
Tuesday, November 11, 2014 4:15 PM

We know that when $Y = aX+b$, we have $f_Y(y) = \frac{1}{|a|} f_X\left(\frac{y-b}{a}\right)$.

Here, $Y = \underset{a}{2}X + \underset{b}{1}$. Therefore, $f_Y(y) = \frac{1}{2} f_X\left(\frac{y-1}{2}\right)$.

(a) $X \sim \mathcal{U}(0,1) \Rightarrow f_X(x) = \begin{cases} 1, & 0<x<1, \\ 0, & \text{otherwise.} \end{cases}$

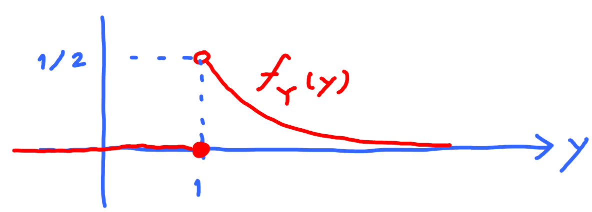
(i) Therefore, $f_Y(y) = \frac{1}{2} \times \begin{cases} 1, & 0 < \frac{y-1}{2} < 1 \\ 0, & \text{otherwise} \end{cases} = \begin{cases} 1/2, & 1<y<3, \\ 0, & \text{otherwise.} \end{cases}$



(ii) Yes. $Y \sim \mathcal{U}(1,3)$

(b) $X \sim \mathcal{E}(1) \Rightarrow f_X(x) = \begin{cases} 1 \times e^{-1\times x}, & x>0, \\ 0, & \text{otherwise} \end{cases} = \begin{cases} e^{-x}, & x>0, \\ 0, & \text{otherwise.} \end{cases}$

(i) Therefore, $f_Y(y) = \frac{1}{2} \times \begin{cases} e^{-\left(\frac{y-1}{2}\right)}, & \frac{y-1}{2} > 0, \\ 0, & \text{otherwise} \end{cases} = \begin{cases} \frac{1}{2}\sqrt{e}\, e^{-y/2}, & y>1, \\ 0, & \text{otherwise.} \end{cases}$



(ii) No. Although f_Y decays exponentially, the "exponential part" starts @ $y=1$ (not @ $y=0$). We may call this distribution a shifted exponential distribution. This distribution is quite useful for modeling output of a biological neuron with refractory period.

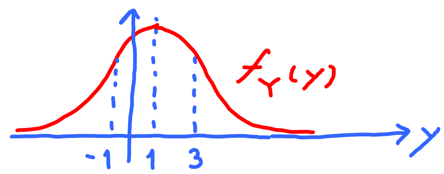
(c) We know that $X \sim \mathcal{N}(m, \sigma^2) \Rightarrow Y = \overset{2}{a}X + \overset{1}{b} \sim \mathcal{N}(am+b, a^2\sigma^2)$.

Plugging in $a = 2$ and $b = 1$, we have $Y \sim \mathcal{N}(2m+1, 4\sigma^2)$.

Here, $X \sim \mathcal{N}(\underset{m}{0}, \underset{\sigma^2}{1})$. So, $Y \sim \mathcal{N}(2 \times 0 + 1, 4 \times 1) = \mathcal{N}(1, 4)$

$\sigma = 2$

(i) $f_Y(y) = \frac{1}{\sqrt{2\pi}\, 2} e^{-\frac{1}{2}\left(\frac{y-1}{2}\right)^2}$



(ii) Yes. $Y \sim \mathcal{N}(1, 4)$

HW13 Q5 Affine Transformation

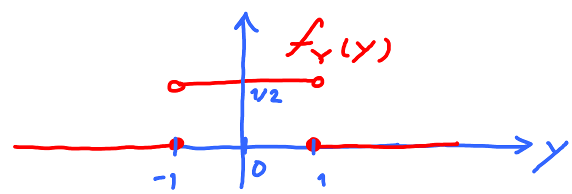
Tuesday, November 11, 2014 4:15 PM

We know that when $Y = aX+b$, we have $f_Y(y) = \frac{1}{|a|} f_X\left(\frac{y-b}{a}\right)$.

Here, $Y = \underbrace{-2}_{a}X + \underbrace{1}_{b}$. Therefore, $f_Y(y) = \frac{1}{2} f_X\left(\frac{1-y}{2}\right)$.

(a) $X \sim \mathcal{U}(0,1) \Rightarrow f_X(x) = \begin{cases} 1, & 0<x<1, \\ 0, & \text{otherwise.} \end{cases}$

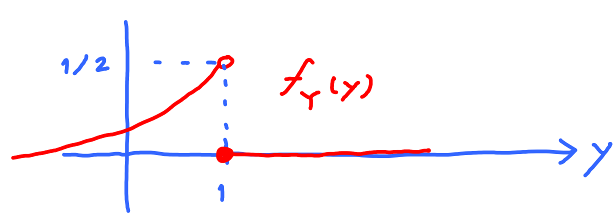
(i) Therefore, $f_Y(y) = \frac{1}{2} \times \begin{cases} 1, & 0<\frac{1-y}{2}<1 \\ 0, & \text{otherwise} \end{cases} = \begin{cases} 1/2, & -1<y<1, \\ 0, & \text{otherwise.} \end{cases}$



(ii) Yes. $Y \sim \mathcal{U}(-1,1)$

(b) $X \sim \mathcal{E}(1) \Rightarrow f_X(x) = \begin{cases} 1 \times e^{-1 \times x}, & x>0, \\ 0, & \text{otherwise} \end{cases} = \begin{cases} e^{-x}, & x>0, \\ 0, & \text{otherwise.} \end{cases}$

(i) Therefore, $f_Y(y) = \frac{1}{2} \times \begin{cases} e^{-\left(\frac{1-y}{2}\right)}, & \frac{1-y}{2} > 0, \\ 0, & \text{otherwise} \end{cases} = \begin{cases} \frac{1}{2\sqrt{e}} e^{y/2}, & y<1, \\ 0, & \text{otherwise.} \end{cases}$



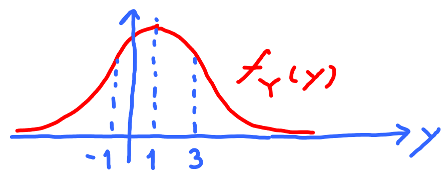
(ii) No. Although f_Y has exponential decay, its expression can not be rewritten in the form $f_Y(y) = \begin{cases} \lambda e^{-\lambda y}, & y>0, \\ 0, & \text{otherwise.} \end{cases}$

(c) We know that $X \sim \mathcal{N}(m, \sigma^2) \Rightarrow Y = \overset{-2}{a}X + \overset{1}{b} \sim \mathcal{N}(am+b,\ a^2\sigma^2)$.

Plugging in $a = -2$ and $b = 1$, we have $Y \sim \mathcal{N}(1-2m,\ 4\sigma^2)$

Here, $X \sim \mathcal{N}(\underset{m}{0}, \underset{\sigma^2}{1})$. So, $Y \sim \mathcal{N}(-2\times 0+1,\ 4\times 1) = \mathcal{N}(1, \underset{\hookrightarrow\ \sigma = 2}{4})$

(i) $f_Y(y) = \frac{1}{\sqrt{2\pi}\,2} e^{-\frac{1}{2}\left(\frac{y-1}{2}\right)^2}$



(ii) Yes. $Y \sim \mathcal{N}(1, 4)$

Problem 6. Let $X \sim \mathcal{E}(3)$.

(a) For each of the following function $g(x)$. Indicate whether the random variable $Y = g(X)$ is a continuous random variable.

(i) $g(x) = x^2$.

(ii) $g(x) = \begin{cases} 1, & x \geq 0, \\ 0, & x < 0. \end{cases}$.

(iii) $g(x) = \begin{cases} 4e^{-4x}, & x \geq 0, \\ 0, & x < 0. \end{cases}$

(iv) $g(x) = \begin{cases} x, & x \leq 5, \\ 5, & x > 5. \end{cases}$

(b) Repeat part (a), but now check whether the random variable $Y = g(X)$ is a discrete random variable.

Solution:

(a)

(i) $\boxed{\text{YES}}$.

When $y < 0$, there is no x that satisfies $g(x) = y$. When $y = 0$, there is exactly one x ($x = 0$) that satisfies $g(x) = y$. When $y > 0$, there is exactly two x ($x = \pm\sqrt{y}$) that satisfies $g(x) = y$.

Therefore, for any y, there are at most countably many x values that satisfy $g(X) = y$. Because X is a continuous random variable, we conclude that Y is also a continuous random variable.

(ii) $\boxed{\text{NO}}$. An easy way to see this is that there can be only two values out of the function $g(\cdot)$: 0 or 1. So, $Y = g(X)$ is a discrete random variable.

Alternatively, consider $y = 1$. We see that any $x \geq 0$, can make $g(x) = 1$. Therefore, $P[Y = 1] = P[X \geq 0]$. For $X \sim \mathcal{E}(3)$, $P[X \geq 0] = 1 > 0$.

Because we found a y with $P[Y = y] > 0$. Y can not be a continuous random variable.

(iii) $\boxed{\text{YES}}$.

The plot of the function $g(x)$ may help you see the following facts: When $y > 4$ or $y < 0$, there is no x that gives $y = g(x)$. When $0 < y < 4$, there is exactly one x that satisfies $y = g(x)$. Because X is a continuous random variable, we can conclude that $P[Y = y]$ is 0 for $y \neq 0$.

When $y = 0$, any $x < 0$ would satisfy $g(x) = y$. So, $P[Y = 0] = P[X < 0]$. However, because $X \sim \mathcal{E}(3)$ is always positive. $P[X < 0] = 0$.

(iv) $\boxed{\text{NO}}$. Consider $y = 5$. We see that any $x \geq 5$, can make $g(x) = 5$. Therefore,

$$P[Y = 5] = P[X \geq 5].$$

For $X \sim \mathcal{E}(3)$,

$$P[X \geq 5] = \int_5^\infty 3e^{-3x} dx = e^{-15} > 0.$$

Because $P[Y = 5] > 0$, we conclude that Y can't be a continuous random variable.

(b) To check whether a random variable is discrete, we simply check whether it has a countable support. Also, if we have already checked that a random variable is continuous, then it can't also be discrete.

(i) $\boxed{\text{NO}}$. We checked before that it is a continuous random variable.

(ii) $\boxed{\text{YES}}$ as discussed in part (a).

(iii) $\boxed{\text{NO}}$. We checked before that it is a continuous random variable.

(iv) $\boxed{\text{NO}}$. Because X is positive, $Y = g(X)$ can be any positive number in the interval $(0, 5]$. The interval is uncountable. Therefore, Y is not discrete.

We have shown previously that Y is not a continuous random variable. Here. knowing that it is not discrete means that it is of the last type: mixed random variable.

Problem 7. Cholesterol is a fatty substance that is an important part of the outer lining (membrane) of cells in the body of animals. Its normal range for an adult is 120–240 mg/dl. The Food and Nutrition Institute of the Philippines found that the total cholesterol level for Filipino adults has a mean of 159.2 mg/dl and 84.1% of adults have a cholesterol level below 200 mg/dl. Suppose that the cholesterol level in the population is normally distributed.

(a) Determine the standard deviation of this distribution.

(b) What is the value of the cholesterol level that exceeds 90% of the population?

(c) An adult is at moderate risk if cholesterol level is more than one but less than two standard deviations above the mean. What percentage of the population is at moderate risk according to this criterion?

(d) An adult is thought to be at high risk if his cholesterol level is more than two standard deviations above the mean. What percentage of the population is at high risk?

Solution: See handwritten solution

HW13 Q7: Gaussian RV - Cholesterol

Thursday, November 13, 2014 3:49 PM

Let X be the cholesterol level of a randomly chosen adult.

It is given that $X \sim \mathcal{N}(m, \sigma^2)$ where $m = 159.2$ mg/dl.

We also know that $P[X < 200] = 0.841$.

(a) $\Phi\left(\frac{200-m}{\sigma}\right) = 0.841 \Rightarrow \frac{200-m}{\sigma} \approx 1 \Rightarrow \sigma \approx 200 - m \approx 200 - 159.2 \approx 40.8$ mg/dl

From the Φ table,

$\Phi(0.99) \approx 0.8389$

$\Phi(1) \approx 0.8413$

0.841 is closer to 0.8413 than 0.8389

(b) We find x such that $\Phi\left(\frac{x-m}{\sigma}\right) = 0.9$.

From the Φ table, $\Phi(1.28) \approx 0.8997$

$\Phi(1.29) \approx 0.9015$

0.9 is closer to 0.8997

$\Rightarrow \frac{x-m}{\sigma} \approx 1.28 \Rightarrow x \approx 1.28\sigma + m \approx 211.424$ mg/dl

σ: 40.8 (from (a)); m: 159.2

(c) $P[m+\sigma < X < m+2\sigma] = F_X(m+2\sigma) - F_X(m+\sigma)$

$= \Phi\left(\frac{m+2\sigma-m}{\sigma}\right) - \Phi\left(\frac{m+\sigma-m}{\sigma}\right) = \Phi(2) - \Phi(1)$

$\approx 0.97725 - 0.8413 = 0.1359 \approx 13.59\%$

(d) $P[X > m+2\sigma] = 1 - F_X(m+2\sigma) = 1 - \Phi\left(\frac{m+2\sigma-m}{\sigma}\right) = 1 - \Phi(2)$

$\approx 1 - 0.97725 \approx 0.0228 = 2.28\%$

Problem 8 (Q3.5.6)**.** Solve this question using the Φ/Q table.

A professor pays 25 cents for each blackboard error made in lecture to the student who points out the error. In a career of n years filled with blackboard errors, the total amount in dollars paid can be approximated by a Gaussian random variable Y_n with expected value $40n$ and variance $100n$.

(a) What is the probability that Y_{20} exceeds 1000?

(b) How many years n must the professor teach in order that $P[Y_n > 1000] > 0.99$?

Solution: We are given[1] that $Y_n \sim \mathcal{N}(40n, 100n)$. Recall that when $X \sim \mathcal{N}(m, \sigma^2)$,

$$F_X(x) = \Phi\left(\frac{x-m}{\sigma}\right). \tag{13.1}$$

(a) Here $n = 20$. So, we have $Y_n \sim \mathcal{N}(40 \times 20, 100 \times 20) = \mathcal{N}(800, 2000)$. For this random variable $m = 800$ and $\sigma = \sqrt{2000}$.

We want to find $P[Y_{20} > 1000]$ which is the same as $1 - P[Y_20 \le 1000]$. Expressing this quantity using cdf, we have

$$P[Y_{20} > 1000] = 1 - F_{Y_{20}}(1000).$$

Apply (13.1) to get

$$P[Y_{20} > 1000] = 1 - \Phi\left(\frac{1000-800}{\sqrt{2000}}\right) = 1 - \Phi(4.472) \approx Q(4.47) \approx \boxed{3.91 \times 10^{-6}}.$$

(b) Here, the value of n is what we want. So, we will need to keep the formula in the general form. Again, from (13.1), for $Y_n \sim \mathcal{N}(40n, 100n)$, we have

$$P[Y_n > 1000] = 1 - F_{Y_n}(1000) = 1 - \Phi\left(\frac{1000-40n}{10\sqrt{n}}\right) = 1 - \Phi\left(\frac{100-4n}{\sqrt{n}}\right).$$

To find the value of n such that $P[Y_n > 1000] > 0.99$, we will first find the value of z which make

$$1 - \Phi(z) > 0.99. \tag{13.2}$$

At this point, we may try to solve for the value of Z by noting that (13.2) is the same as

$$\Phi(z) < 0.01. \tag{13.3}$$

[1]Note that the expected value and the variance in this question are proportional to n. This naturally occurs when we consider the sum of i.i.d. random variables. The approximation by Gaussian random variable is a result of the central limit theorem (CLT).

Unfortunately, the tables that we have start with $\Phi(0) = 0.5$ and increase to something close to 1 when the argument of the Φ function is large. This means we can't directly find 0.01 in the table. Of course, 0.99 is in there and therefore we will need to solve (13.2) via another approach.

To do this, we use another property of the Φ function. Recall that $1 - \Phi(z) = \Phi(-z)$. Therefore, (13.2) is the same as

$$\Phi\left(-z\right) > 0.99. \tag{13.4}$$

From our table, we can then conclude that (13.3) (which is the same as (13.4)) will happen when $-z > 2.33$. (If you have `MATLAB`, then you can get a more accurate answer of 2.3263.)

Now, plugging in $z = \frac{100-4n}{\sqrt{n}}$, we have $\frac{4n-100}{\sqrt{n}} > 2.33$. To solve for n, we first let $x = \sqrt{n}$. In which case, we have $\frac{4x^2-100}{x} > 2.33$ or, equivalently, $4x^2 - 2.33x - 100 > 0$. The two roots are $x = -4.717$ and $x > 5.3$. So, We need $x < -4.717$ or $x > 5.3$. Note that $x = \sqrt{n}$ and therefore can not be negative. So, we only have one case; that is, we need $x > 5.3$. Because $n = x^2$, we then conclude that we need $\boxed{n > 28.1 \text{ years}}$.

ECS 315: Probability and Random Processes **2016/1**

HW Solution 14 — Due: Not Due

Lecturer: Prapun Suksompong, Ph.D.

Problem 1. Let a continuous random variable X denote the current measured in a thin copper wire in milliamperes. Assume that the probability density function of X is

$$f_X(x) = \begin{cases} 5, & 4.9 \le x \le 5.1, \\ 0, & \text{otherwise.} \end{cases}$$

(a) Find the probability that a current measurement is less than 5 milliamperes.

(b) Find and plot the cumulative distribution function of the random variable X.

(c) Find the expected value of X.

(d) Find the variance and the standard deviation of X.

(e) Find the expected value of power when the resistance is 100 ohms?

Solution: See handwritten solution.

Problem 2. The time until a chemical reaction is complete (in milliseconds) is approximated by the cumulative distribution function

$$F_X(x) = \begin{cases} 1 - e^{-0.01x}, & x \ge 0, \\ 0, & \text{otherwise.} \end{cases}$$

(a) Determine the probability density function of X.

(b) What proportion of reactions is complete within 200 milliseconds?

Solution: See handwritten solution.

Q1: pdf and cdf - chemical reaction

Thursday, November 13, 2014 11:07 AM

$$F_X(x) = \begin{cases} 1 - e^{-0.01x}, & x \geq 0, \\ 0, & \text{otherwise.} \end{cases}$$

Note that $F_X(x)$ is a continuous function. Therefore, X is a continuous RV.

(a) $$f_X(x) = \frac{d}{dx} F_X(x) = \begin{cases} -(-0.01) e^{-0.01x}, & x > 0, \\ 0, & x < 0. \end{cases} = \begin{cases} 0.01 e^{-0.01x}, & x > 0, \\ 0, & x < 0. \end{cases}$$

At $x = 0$, the derivative does not exist. Because this is just a point, we may assign $f_X(0)$ to be any arbitrary value. Here, we set $f_X(0) = 0$:

$$f_X(x) = \begin{cases} 0.01 e^{-0.01x}, & x > 0, \\ 0, & \text{otherwise.} \end{cases}$$

(b) $P[X < 200] = P[X \leq 200] = F_X(200) = 1 - e^{-0.01 \times 200} = 1 - e^{-2} \approx 0.8647.$

Alternatively, $$P[X < 200] = \int_{-\infty}^{200} f_X(x)\,dx = \cancelto{0}{\int_{-\infty}^{0} f_X(x)\,dx} + \int_{0}^{200} f_X(x)\,dx$$

$$= \int_0^{200} 0.01 e^{-0.01x}\,dx = \frac{0.01}{(-0.01)} e^{-0.01x} \Big|_0^{200}$$

$$= \left(-e^{-0.01 \times 200}\right) - \left(-e^{-0.01 \times 0}\right) = -e^{-2} - (-1)$$

$$= 1 - e^{-2}$$

Q2: pdf, cdf, expected value, variance - current and power

Thursday, November 13, 2014 11:01 AM

$$f_X(x) = \begin{cases} 5, & 4.9 \leq x \leq 5.1, \\ 0, & \text{otherwise.} \end{cases}$$

(a) $$P[X < 5] = \int_{-\infty}^{5} f_X(x)\,dx = \cancel{\int_{-\infty}^{4.9} \underbrace{f_X(x)}_{0}\,dx}^{\,0} + \int_{4.9}^{5} \underbrace{f_X(x)}_{5}\,dx$$

$$= 5x\Big|_{4.9}^{5} = 5(5-4.9) = 5 \times 0.1 = 0.5$$

(b) $$F_X(x) = P[X \leq x] = \int_{-\infty}^{x} f_X(t)\,dt$$

For $x < 4.9$, $f_X(t) = 0$ for all t inside $(0, -\infty)$.

Therefore, $F_X(x) = \int_{-\infty}^{x} 0\,dt = 0.$

For $4.9 \leq x \leq 5.1$, $$F_X(x) = \int_{-\infty}^{x} f_X(t)\,dt = \cancel{\int_{-\infty}^{4.9} \underbrace{f_X(t)}_{0}\,dt}^{\,0} + \int_{4.9}^{x} \underbrace{f_X(t)}_{5}\,dt$$

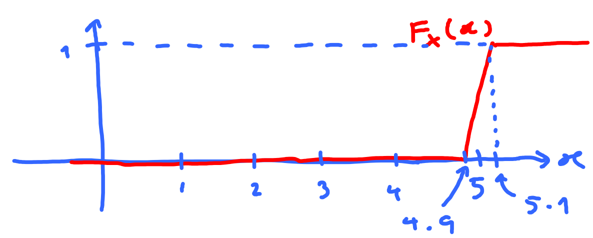
$$= 5t\Big|_{4.9}^{x} = 5 \times (x - 4.9) = 5x - 24.5.$$

For $x > 5.1$, $$F_X(x) = \int_{-\infty}^{x} f_X(t)\,dt = \cancel{\int_{-\infty}^{4.9} \underbrace{f_X(t)}_{0}\,dt}^{\,0} + \int_{4.9}^{5.1} \underbrace{f_X(t)}_{5}\,dt + \cancel{\int_{5.1}^{x} \underbrace{f_X(t)}_{0}\,dt}^{\,0}$$

$$= 5t\Big|_{4.9}^{5.1} = 5 \times (5.1 - 4.9) = 5 \times 0.2 = 1.$$

Combining the three cases above, we have the complete description of the cdf:

$$F_X(x) = \begin{cases} 0, & x < 4.9, \\ 5x - 24.5, & 4.9 \leq x \leq 5.1, \\ 1, & x > 5.1 \end{cases}$$



Note that F_X is a continuous function. This is because it is the cdf of a continuous RV.

(c) $\mathbb{E}X = \int_{-\infty}^{\infty} x f_X(x)\,dx = \cancel{\int_{0}^{4.9} x \underbrace{f_X(x)}_{0}\,dx}^{\,0} + \int_{4.9}^{5.1} x \underbrace{f_X(x)}_{5}\,dx + \cancel{\int_{5.1}^{\infty} x \underbrace{f_X(x)}_{0}\,dx}^{\,0}$

$= 5 \left.\frac{x^2}{2}\right|_{4.9}^{5.1} = \frac{5}{2}\left(5.1^2 - 4.9^2\right) = \frac{5}{2}(5.1+4.9)(5.1-4.9) = \frac{5}{2}(10)(0.2)$

$= 5$ mA

Alternatively, for $X \sim \mathcal{U}(a,b)$, we have $\mathbb{E}X = \frac{b+a}{2} = \frac{5.1+4.9}{2} = \frac{10}{2} = 5.$

(d) $\operatorname{Var} X = \mathbb{E}[X^2] - (\mathbb{E}X)^2$. From (c), we know that $\mathbb{E}X = 5$. So, to find $\operatorname{Var} X$, we need to find $\mathbb{E}[X^2]$.

$\mathbb{E}[X^2] = \int_{-\infty}^{\infty} x^2 f_X(x)\,dx = \int_{4.9}^{5.1} x^2 \times 5\,dx = 5 \left.\frac{x^3}{3}\right|_{4.9}^{5.1} = \frac{5}{3} \times (5.1^3 - 4.9^3)$

$= 25 + \frac{1}{300}.$

Therefore, $\operatorname{Var} X = \left(25 + \frac{1}{300}\right) - 5^2 = \frac{1}{300} \approx 0.0033\ (\text{mA})^2$

and $\sigma_X = \frac{1}{10\sqrt{3}}$ mA ≈ 0.0577 mA.

Alternatively, for $X \sim \mathcal{U}(a,b)$, we have $\operatorname{Var} X = \frac{(b-a)^2}{12} = \frac{(5.1-4.9)^2}{12}$

$= \frac{(0.2)^2}{12} = \frac{4}{100 \times 12} = \frac{1}{300}.$

(e) Recall that $P = IV = I \times I r = I^2 r$.

Here $I = X$. Therefore $P = X^2 r$ and

$\mathbb{E}P = \mathbb{E}[X^2 r] = r\,\mathbb{E}[X^2] = 100 \times \left(25 + \frac{1}{300}\right) = 2500 + \frac{1}{3}$

$\approx 2.50033 \times 10^3 \left[\underbrace{(\text{mA})^2\Omega}_{m^2 \underbrace{A^2\Omega}_{W}}\right] \approx 2.5$ mW.

Caution: The current is in mA.

Problem 3. Let $X \sim \mathcal{E}(5)$ and $Y = 2/X$.

(a) Check that Y is still a continuous random variable.

(b) Find $F_Y(y)$.

(c) Find $f_Y(y)$.

(d) (optional) Find $\mathbb{E}Y$. Hint: Because $\frac{d}{dy}e^{-\frac{10}{y}} = \frac{10}{y^2}e^{-\frac{10}{y}} > 0$ for $y \neq 0$. We know that $e^{-\frac{10}{y}}$ is an increasing function on our range of integration. In particular, consider $y > 10/\ln(2)$. Then, $e^{-\frac{10}{y}} > \frac{1}{2}$. Hence,

$$\int_0^\infty \frac{10}{y}e^{-\frac{10}{y}}dy > \int_{10/\ln 2}^\infty \frac{10}{y}e^{-\frac{10}{y}}dy > \int_{10/\ln 2}^\infty \frac{10}{y}\frac{1}{2}dy = \int_{10/\ln 2}^\infty \frac{5}{y}dy.$$

Remark: To be technically correct, we should be a little more careful when writing $Y = \frac{2}{X}$ because it is undefined when $X = 0$. Of course, this happens with 0 probability; so it won't create any serious problem. However, to avoid the problem, we may define Y by

$$Y = \begin{cases} 2/X, & X \neq 0, \\ 0, & X = 0. \end{cases} \tag{14.1}$$

Solution: Here, $X \sim \mathcal{E}(5)$. Therefore, X is a continuous random variable. In this question, we have $Y = g(X)$ where the function g is defined by $g(x) = \frac{2}{x}$.

(a) First, we count the number of solutions for $y = g(x)$.

- For each value of $y > 0$, there is only one x value that satisfies $y = g(x)$. (That x value is $x = \frac{2}{y}$.)
- When y $= 0$, we need $x = \infty$ or $-\infty$ to make $g(x) = 0$. However, $\pm\infty$ are not real numbers therefore they are not possible x values.
  Note that if we use (14.1), then $x = 0$ is the only solution for $y = g(x)$.
- When $y < 0$, there is no x in the support of X that satisfies $y = g(x)$.

In all three cases, for each value of y, the number of solutions for $y = g(x)$ is (at most) countable. Therefore, because X is a continuous random variable, we conclude that Y is also a continuous random variable.

(b) We consider two cases: "$y \leq 0$" and "$y > 0$".

- Because $X > 0$, we know that $Y = \frac{2}{X}$ must be > 0 and hence, $F_Y(y) = 0$ for $y \leq 0$.

- For $y > 0$,
$$F_Y(y) = P[Y \le y] = P\left[\frac{2}{X} \le y\right] = P\left[X \ge \frac{2}{y}\right].$$
Note that, for the last equality, we can freely move X and y without worrying about "flipping the inequality" or "division by zero" because both X and y considered here are strictly positive. Now, for $X \sim \mathcal{E}(\lambda)$ and $x > 0$, we have
$$P[X \ge x] = \int_x^\infty \lambda e^{-\lambda t} dt = -e^{-\lambda t}\Big|_x^\infty = e^{-\lambda x}$$
Therefore,
$$F_Y(y) = e^{-5\left(\frac{2}{y}\right)} = e^{\frac{-10}{y}}.$$

Combining the two cases above we have
$$F_Y(y) = \boxed{\begin{cases} e^{-\frac{10}{y}}, & y > 0 \\ 0, & y \le 0 \end{cases}}$$

(c) Because we have already derived the cdf in the previous part, we can find the pdf via the cdf by $f_Y(y) = \frac{d}{dy}F_Y(y)$. This gives f_Y at all points except at $y = 0$ which we will set f_Y to be 0 there. (This arbitrary assignment works for continuous RV. This is why we need to check first that the random variable is actually continuous.) Hence,
$$f_Y(y) = \boxed{\begin{cases} \frac{10}{y^2}e^{-\frac{10}{y}}, & y > 0 \\ 0, & y \le 0. \end{cases}}$$

(d) We can find $\mathbb{E}Y$ from $f_Y(y)$ found in the previous part or we can even use $f_X(x)$

Method 1:
$$\mathbb{E}Y = \int_{-\infty}^{\infty} y f_Y(y) = \int_0^\infty y\frac{10}{y^2}e^{-\frac{10}{y}}dy = \int_0^\infty \frac{10}{y}e^{-\frac{10}{y}}dy$$
From the hint, we have
$$\begin{aligned}\mathbb{E}Y &> \int_{10/\ln 2}^\infty \frac{10}{y}e^{-\frac{10}{y}}dy > \int_{10/\ln 2}^\infty \frac{10}{y}\frac{1}{2}dy = \int_{10/\ln 2}^\infty \frac{5}{y}dy \\ &= 5\ln y\Big|_{10/\ln 2}^\infty = \infty.\end{aligned}$$

Therefore, $\mathbb{E}Y = \boxed{\infty}$.

Method 2:

$$\mathbb{E}Y = \mathbb{E}\left[\frac{1}{X}\right] = \int_{-\infty}^{\infty} \frac{1}{x} f_X(x)\, dx = \int_{0}^{\infty} \frac{1}{x} \lambda e^{-\lambda x} dx > \int_{0}^{1} \frac{1}{x} \lambda e^{-\lambda x} dx$$

$$> \int_{0}^{1} \frac{1}{x} \lambda e^{-\lambda} dx = \lambda e^{-\lambda} \int_{0}^{1} \frac{1}{x} dx = \lambda e^{-\lambda} \ln x \big|_0^1 = \infty,$$

where the second inequality above comes from the fact that for $x \in (0,1)$, $e^{-\lambda x} > e^{-\lambda}$.

Problem 4. In wireless communications systems, fading is sometimes modeled by ***lognormal*** random variables. We say that a positive random variable Y is lognormal if $\ln Y$ is a normal random variable (say, with expected value m and variance σ^2).

Hint: First, recall that the ln is the natural log function (log base e). Let $X = \ln Y$. Then, because Y is lognormal, we know that $X \sim \mathcal{N}(m, \sigma^2)$. Next, write Y as a function of X.

(a) Check that Y is still a continuous random variable.

(b) Find the pdf of Y.

Solution:

Because $X = \ln(Y)$, we have $Y = e^X$. So, here, we consider $Y = g(X)$ where the function g is defined by $g(x) = e^x$.

(a) First, we count the number of solutions for $y = g(x)$. Note that for each value of $y > 0$, there is only one x value that satisfies $y = g(x)$. (That x value is $x = \ln(y)$.) For $y \le 0$, there is no x that satisfies $y = g(x)$. In both cases, the number of solutions for $y = g(x)$ is countable. Therefore, because X is a continuous random variable, we conclude that Y is also a continuous random variable.

(b) Start with $Y = e^X$. We know that exponential function gives strictly positive number. So, Y is always strictly positive. In particular, $F_Y(y) = 0$ for $y \le 0$.

Next, for $y > 0$, by definition, $F_Y(y) = P[Y \le y]$. Plugging in $Y = e^X$, we have

$$F_Y(y) = P\left[e^X \le y\right].$$

Because the exponential function is strictly increasing, the event $[e^X \le y]$ is the same as the event $[X \le \ln y]$. Therefore,

$$F_Y(y) = P[X \le \ln y] = F_X(\ln y).$$

Combining the two cases above, we have

$$F_Y(y) = \begin{cases} F_X(\ln y), & y > 0, \\ 0, & y \le 0. \end{cases}$$

Finally, we apply

$$f_Y(y) = \frac{d}{dy} F_Y(y).$$

For $y < 0$, we have $f_Y(y) = \frac{d}{dy} 0 = 0$. For $y > 0$,

$$f_Y(y) = \frac{d}{dy} F_Y(y) = \frac{d}{dy} F_X(\ln y) = f_X(\ln y) \times \frac{d}{dy} \ln y = \frac{1}{y} f_X(\ln y). \tag{14.2}$$

Therefore,

$$f_Y(y) = \begin{cases} \frac{1}{y} f_X(\ln y), & y > 0, \\ 0, & y < 0. \end{cases}$$

At $y = 0$, because Y is a continuous random variable, we can assign any value, e.g. 0, to $f_Y(0)$. Then

$$f_Y(y) = \boxed{\begin{cases} \frac{1}{y} f_X(\ln y), & y > 0, \\ 0, & \text{otherwise.} \end{cases}}$$

Here, $X \sim \mathcal{N}(m, \sigma^2)$. Therefore,

$$f_X(x) = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{1}{2}\left(\frac{x-m}{\sigma}\right)^2}$$

and

$$\boxed{f_Y(y) = \begin{cases} \frac{1}{\sqrt{2\pi}\sigma y} e^{-\frac{1}{2}\left(\frac{\ln(y)-m}{\sigma}\right)^2}, & y > 0, \\ 0, & \text{otherwise.} \end{cases}}$$

Problem 5. The input X and output Y of a system subject to random perturbations are described probabilistically by the following joint pmf matrix:

$$\begin{array}{c|c} x \backslash y & \begin{array}{ccc} 2 & 4 & 5 \end{array} \\ \begin{array}{c} 1 \\ 3 \end{array} & \begin{bmatrix} 0.02 & 0.10 & 0.08 \\ 0.08 & 0.32 & 0.40 \end{bmatrix} \end{array}$$

(a) Evaluate the following quantities:

(i) The marginal pmf $p_X(x)$

(ii) The marginal pmf $p_Y(y)$

(iii) $\mathbb{E}X$

(iv) $\operatorname{Var} X$

(v) $\mathbb{E}Y$

(vi) $\operatorname{Var} Y$

(vii) $P[XY < 6]$

(viii) $P[X = Y]$

(ix) $\mathbb{E}[XY]$

(x) $\mathbb{E}[(X-3)(Y-2)]$

(xi) $\mathbb{E}[X(Y^3 - 11Y^2 + 38Y)]$

(xii) $\operatorname{Cov}[X, Y]$

(xiii) $\rho_{X,Y}$

(b) Find $\rho_{X,X}$

(c) Calculate the following quantities using the values of $\operatorname{Var} X$, $\operatorname{Cov}[X, Y]$, and $\rho_{X,Y}$ that you got earlier.

(i) $\operatorname{Cov}[3X + 4, 6Y - 7]$

(ii) $\rho_{3X+4,6Y-7}$

(iii) $\operatorname{Cov}[X, 6X - 7]$

(iv) $\rho_{X,6X-7}$

Solution:

(a) The `MATLAB` codes are provided in the file `P_XY_EVarCov.m`.

(i) The marginal pmf $p_X(x)$ is founded by the sums along the rows of the pmf matrix:

$$p_X(x) = \boxed{\begin{cases} 0.2, & x = 1 \\ 0.8, & x = 3 \\ 0, & \text{otherwise.} \end{cases}}$$

(ii) The marginal pmf $p_Y(y)$ is founded by the sums along the columns of the pmf matrix:

$$p_Y(y) = \boxed{\begin{cases} 0.1, & y = 2 \\ 0.42, & y = 4 \\ 0.48, & y = 5 \\ 0, & \text{otherwise.} \end{cases}}$$

(iii) $\mathbb{E}X = \sum_x x p_X(x) = 1 \times 0.2 + 3 \times 0.8 = 0.2 + 2.4 = \boxed{2.6}$.

(iv) $\mathbb{E}[X^2] = \sum_x x^2 p_X(x) = 1^2 \times 0.2 + 3^2 \times 0.8 = 0.2 + 7.2 = 7.4$.

So, $\operatorname{Var} X = \mathbb{E}[X^2] - (\mathbb{E}X)^2 = 7.4 - (2.6)^2 = 7.4 - 6.76 = \boxed{0.64}$.

(v) $\mathbb{E}Y = \sum_y y p_Y(y) = 2 \times 0.1 + 4 \times 0.42 + 5 \times 0.48 = 0.2 + 1.68 + 2.4 = \boxed{4.28}$.

(vi) $\mathbb{E}[Y^2] = \sum_y y^2 p_Y(y) = 2^2 \times 0.1 + 4^2 \times 0.42 + 5^2 \times 0.48 = 19.12$.

So, $\operatorname{Var} Y = \mathbb{E}[Y^2] - (\mathbb{E}Y)^2 = 19.12 - 4.28^2 = \boxed{0.8016}$.

(vii) Among the 6 possible pairs of (x, y) shown in the joint pmf matrix, only the pairs $(1, 2)$, $(1, 4)$, $(1, 5)$ satisfy $xy < 6$. Therefore, $[XY < 6] = [X = 1]$ which implies $P[XY < 6] = P[X = 1] = \boxed{0.2}$.

(viii) Among the 6 possible pairs of (x, y) shown in the joint pmf matrix, there is no pair which has $x = y$. Therefore, $P[X = Y] = \boxed{0}$.

(ix) First, we calculate the values of $x \times y$:

$$\begin{array}{cc} x \setminus y & \begin{matrix} 2 & 4 & 5 \end{matrix} \\ \begin{matrix} 1 \\ 3 \end{matrix} & \begin{bmatrix} 2 & 4 & 5 \\ 6 & 12 & 15 \end{bmatrix} \end{array}$$

Then, each $x \times y$ is weighted (multiplied) by the corresponding probability $p_{X,Y}(x, y)$:

$$\begin{array}{cc} x \setminus y & \begin{matrix} 2 & 4 & 5 \end{matrix} \\ \begin{matrix} 1 \\ 3 \end{matrix} & \begin{bmatrix} 0.04 & 0.40 & 0.40 \\ 0.48 & 3.84 & 6.00 \end{bmatrix} \end{array}$$

Finally, $\mathbb{E}[XY]$ is sum of these numbers. Therefore, $\mathbb{E}[XY] = \boxed{11.16}$.

(x) First, we calculate the values of $(x - 3) \times (y - 2)$:

$$\begin{array}{cc} x \setminus y & \begin{matrix} 2 & 4 & 5 \end{matrix} \\ \begin{matrix} 1 \\ 3 \end{matrix} & \begin{bmatrix} 0 & -4 & -6 \\ 0 & 0 & 0 \end{bmatrix} \end{array}$$

Then, each $(x-3)\times(y-2)$ is weighted (multiplied) by the corresponding probability $p_{X,Y}(x,y)$:

$$\begin{array}{cc|ccc} & y-2 & 0 & 2 & 3 \\ x-3 & x\setminus y & 2 & 4 & 5 \\ -2 & 1 & 0 & -0.40 & -0.48 \\ 0 & 3 & 0 & 0 & 0 \end{array}$$

Finally, $\mathbb{E}\left[(X-3)(Y-2)\right]$ is sum of these numbers. Therefore,

$$\mathbb{E}\left[(X-3)(Y-2)\right] = \boxed{-0.88}.$$

(xi) First, we calculate the values of $x(y^3-11y^2+38y)$:

$$\begin{array}{l|ccc} y^3-11y^2+38y & 40 & 40 & 40 \\ x\setminus y & 2 & 4 & 5 \\ 1 & 40 & 40 & 40 \\ 3 & 120 & 120 & 120 \end{array}$$

Then, each $x(y^3-11y^2+38y)$ is weighted (multiplied) by the corresponding probability $p_{X,Y}(x,y)$:

$$\begin{array}{c|ccc} x\setminus y & 2 & 4 & 5 \\ 1 & 0.8 & 4.0 & 3.2 \\ 3 & 9.6 & 38.4 & 48.0 \end{array}$$

Finally, $\mathbb{E}\left[X(Y^3-11Y^2+38Y)\right]$ is sum of these numbers. Therefore,

$$\mathbb{E}\left[X(Y^3-11Y^2+38Y)\right] = \boxed{104}.$$

(xii) $\text{Cov}\left[X,Y\right] = \mathbb{E}\left[XY\right] - \mathbb{E}X\mathbb{E}Y = 11.16 - (2.6)(4.28) = \boxed{0.032}.$

(xiii) $\rho_{X,Y} = \frac{\text{Cov}[X,Y]}{\sigma_X\sigma_Y} = \frac{0.032}{\sqrt{0.64}\sqrt{0.8016}} = \boxed{0.044677}$

(b) $\rho_{X,X} = \frac{\text{Cov}[X,X]}{\sigma_X\sigma_X} = \frac{\text{Var}[X]}{\sigma_X^2} = \boxed{1}.$

(c)

(i) $\text{Cov}\left[3X+4, 6Y-7\right] = 3\times 6\times \text{Cov}\left[X,Y\right] \approx 3\times 6\times 0.032 \approx \boxed{0.576}.$

(ii) Note that

$$\begin{aligned} \rho_{aX+b,cY+d} &= \frac{\text{Cov}\left[aX+b, cY+d\right]}{\sigma_{aX+b}\sigma_{cY+d}} \\ &= \frac{ac\text{Cov}\left[X,Y\right]}{|a|\sigma_X|c|\sigma_Y} = \frac{ac}{|ac|}\rho_{X,Y} = \text{sign}(ac)\times\rho_{X,Y}. \end{aligned}$$

Hence, $\rho_{3X+4,6Y-7} = \text{sign}(3 \times 4)\rho_{X,Y} = \rho_{X,Y} = \boxed{0.0447}$.

(iii) $\text{Cov}[X, 6X-7] = 1 \times 6 \times \text{Cov}[X, X] = 6 \times \text{Var}[X] \approx \boxed{3.84}$.

(iv) $\rho_{X,6X-7} = \text{sign}(1 \times 6) \times \rho_{X,X} = \boxed{1}$.

Problem 6. Suppose $X \sim$ binomial$(5, 1/3)$, $Y \sim$ binomial$(7, 4/5)$, and $X \perp\!\!\!\perp Y$. Evaluate the following quantities.

(a) $\mathbb{E}[(X-3)(Y-2)]$

(b) $\text{Cov}[X, Y]$

(c) $\rho_{X,Y}$

Solution:

(a) First, because X and Y are independent, we have $\mathbb{E}[(X-3)(Y-2)] = \mathbb{E}[X-3]\,\mathbb{E}[Y-2]$. Recall that $\mathbb{E}[aX+b] = a\mathbb{E}[X]+b$. Therefore, $\mathbb{E}[X-3]\,\mathbb{E}[Y-2] = (\mathbb{E}[X]-3)(\mathbb{E}[Y]-2)$ Now, for Binomial(n, p), the expected value is np. So,

$$(\mathbb{E}[X]-3)(\mathbb{E}[Y]-2) = \left(5 \times \frac{1}{3} - 3\right)\left(7 \times \frac{4}{5} - 2\right) = -\frac{4}{3} \times \frac{18}{5} = \boxed{-\frac{24}{5}} = -4.8.$$

(b) $\text{Cov}[X, Y] = \boxed{0}$ because $X \perp\!\!\!\perp Y$.

(c) $\rho_{X,Y} = \boxed{0}$ because $\text{Cov}[X, Y] = 0$

Extra Questions

Here are some extra questions for those who want more practice.

Problem 7. Consider a random variable X whose pdf is given by

$$f_X(x) = \begin{cases} cx^2, & x \in (1,2), \\ 0, & \text{otherwise.} \end{cases}$$

Let $Y = 4|X - 1.5|$.

(a) Find $\mathbb{E}Y$.

(b) Find $f_Y(y)$.

Solution: See handwritten solution

Q7: SISO

Saturday, November 22, 2014 11:54 PM

First, we need to find the constant c.

For any pdf, we know that $\int_{-\infty}^{\infty} f_X(x)\,dx = 1$.

Therefore, $\int_1^2 c x^2\,dx = c\int_1^2 x^2\,dx = c\left.\frac{x^3}{3}\right|_1^2 = c\left(\frac{8-1}{3}\right) = c\times\frac{7}{3}$ must $= 1$.

Hence, $c = 3/7$.

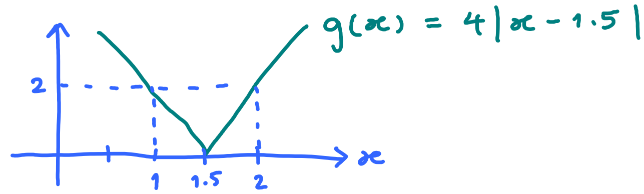
(a) $\mathbb{E}Y = \mathbb{E}\left[4|X-1.5|\right] = 4\int_1^2 |x-1.5|\,\frac{3}{7}x^2\,dx = \frac{12}{7}\int_1^2 |x-1.5|\,x^2\,dx$

$$|x-1.5| = \begin{cases} x-1.5, & x \geq 1.5 \\ 1.5-x, & x < 1.5 \end{cases}$$

$$= \frac{12}{7}\left(\int_1^{1.5}(1.5-x)\,x^2\,dx + \int_{1.5}^{2}(x-1.5)\,x^2\,dx\right) = \frac{57}{56}$$

(b) $Y = 4|X-1.5| = \left\{\begin{matrix} 4X-6, & X \geq 1.5, \\ 6-4X, & X < 1.5 \end{matrix}\right\} \equiv g(X)$

Let's plot the function $g(x)$:



First, let's check that Y is a cont. RV. This is easy to see from $g(x)$.

For each value of y, there are $\underbrace{\text{at most two}}$ values of x that satisfy $y = g(x)$.

$\Downarrow$

finite $\Rightarrow$ countable $\Rightarrow P[Y=y] = 0 \ \forall y$ ← X is a cont. RV

$\Rightarrow$ Y is a cont. RV.

Step ①: Find the cdf. Step ②: $f_Y(y) = \frac{d}{dy}F_Y(y)$

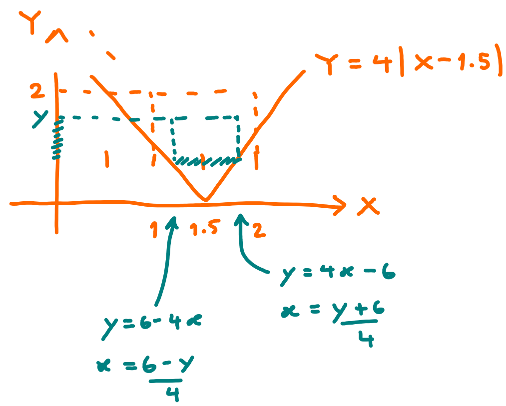
(1.1) By construction (from 1.1), we know that $Y \geq 0$. Therefore,

$F_Y(y) = 0$ for $y < 0$.

(2.1) This means $f_Y(y) = 0$ for $y < 0$. (*)

(1.2) For $y = 0$, $F_Y(0) = P[Y \leq 0] = P[X = 0] \overset{\text{for cont. X.}}{=} 0$ (**)

(1.3) For $y > 0$,



the event $[Y \le y]$ is the same as the event $\left[\frac{6-y}{4} \le X \le \frac{6+y}{4}\right]$.

Therefore,

for cont. X

$$F_Y(y) = P\left[\frac{6-y}{4} \le X \le \frac{6+y}{4}\right] = F_X\left(\frac{6+y}{4}\right) - F_X\left(\frac{6-y}{4}\right) \quad \text{when } y > 0.$$

(2.3) This implies

$$f_Y(y) = \frac{d}{dy}F_Y(y) = \frac{1}{4}f_X\left(\frac{6+y}{4}\right) + \frac{1}{4}f_X\left(\frac{6-y}{4}\right) \quad \text{when } y > 0. \quad (***)$$

Plug-in $f_X(\cdot) = \frac{3}{7}(\cdot)^2$ when $1 < (\cdot) < 2$

$$1 < \frac{6+y}{4} < 2 \qquad 1 < \frac{6-y}{4} < 2$$

$$4 < 6+y < 8 \qquad 4 < 6-y < 8$$

$$-2 < y < 2 \qquad -2 < -y < 2$$

$$-2 < y < 2$$

Note again that this analysis is valid only for $y > 0$.

Therefore,

$$f_Y(y) = \begin{cases} \frac{1}{4}\times\frac{3}{7}\left(\left(\frac{6+y}{4}\right)^2 + \left(\frac{6-y}{4}\right)^2\right), & 0 < y < 2 \\ 0, & y \ge 2 \end{cases}$$

Combining (2.1) and (2.3), we have

$$f_Y(y) = \begin{cases} \frac{1}{4}\times\frac{3}{7}\left(\left(\frac{6+y}{4}\right)^2 + \left(\frac{6-y}{4}\right)^2\right), & 0 < y < 2, \\ 0, & \text{otherwise.} \end{cases}$$

At $y = 0$, we set $f_Y(0) = 0$. This is possible because Y is a continuous RV.

$$= \begin{cases} \frac{3}{224}\left(y^2 + 36\right), & 0 < y < 2, \\ 0, & \text{otherwise.} \end{cases}$$

Check $\quad \mathbb{E}Y = \int_{-\infty}^{\infty} y f_Y(y)\,dy = \int_0^2 \frac{3}{224}\left(y^3 + 36y\right)dy = \frac{57}{56}$ ← same as part (a).

Problem 8. A webpage server can handle r requests per day. Find the probability that the server gets more than r requests at least once in n days. Assume that the number of requests on day i is $X_i \sim \mathcal{P}(\alpha)$ and that $X_1, \ldots, X_n$ are independent.

Solution: [Gubner, 2006, Ex 2.10]

$$
\begin{aligned}
P\left[\bigcup_{i=1}^{n}[X_i > r]\right] &= 1 - P\left[\bigcap_{i=1}^{n}[X_i \le r]\right] = 1 - \prod_{i=1}^{n} P\left[X_i \le r\right] \\
&= 1 - \prod_{i=1}^{n}\left(\sum_{k=0}^{r}\frac{\alpha^k e^{-\alpha}}{k!}\right) = \boxed{1 - \left(\sum_{k=0}^{r}\frac{\alpha^k e^{-\alpha}}{k!}\right)^n}.
\end{aligned}
$$

Problem 9. Suppose $X \sim$ binomial$(5, 1/3)$, $Y \sim$ binomial$(7, 4/5)$, and $X \perp\!\!\!\perp Y$.

(a) A vector describing the pmf of X can be created by the MATLAB expression:

```
x = 0:5; pX = binopdf(x,5,1/3).
```

What is the expression that would give `pY`, a corresponding vector describing the pmf of Y?

(b) Use `pX` and `pY` from part (a), how can you create the joint pmf matrix in MATLAB? Do not use "for-loop", "while-loop", "if statement". Hint: Multiply them in an appropriate orientation.

(c) Use `MATLAB` to evaluate the following quantities. Again, do not use "for-loop", "while-loop", "if statement".

(i) $\mathbb{E}X$

(ii) $P[X = Y]$

(iii) $P[XY < 6]$

Solution: The `MATLAB` codes are provided in the file `P_XY_jointfromMarginal_indp.m`.

(a) `y = 0:7; pY = binopdf(y,7,4/5);`

(b) `P = pX.'*pY;`

(c)

(i) $\mathbb{E}X = \boxed{1.667}$

(ii) $P[X = Y] = \boxed{0.0121}$

(iii) $P[XY < 6] = \boxed{0.2727}$

Problem 10. Suppose $\operatorname{Var} X = 5$. Find $\operatorname{Cov}[X, X]$ and $\rho_{X,X}$.
Solution:

(a) $\operatorname{Cov}[X, X] = \mathbb{E}[(X - \mathbb{E}X)(X - \mathbb{E}X)] = \mathbb{E}[(X - \mathbb{E}X)^2] = \operatorname{Var} X = \boxed{5}$.

(b) $\rho_{X,X} = \frac{\operatorname{Cov}[X,X]}{\sigma_X \sigma_X} = \frac{\operatorname{Var} X}{\sigma_X^2} = \frac{\operatorname{Var} X}{\operatorname{Var} X} = \boxed{1}$.

Problem 11. Suppose we know that $\sigma_X = \frac{\sqrt{21}}{10}$, $\sigma_Y = \frac{4\sqrt{6}}{5}$, $\rho_{X,Y} = -\frac{1}{\sqrt{126}}$.

(a) Find $\operatorname{Var}[X + Y]$.

(b) Find $\mathbb{E}[(Y - 3X + 5)^2]$. Assume $\mathbb{E}[Y - 3X + 5] = 1$.

Solution:

(a) First, we know that $\operatorname{Var} X = \sigma_X^2 = \frac{21}{100}$, $\operatorname{Var} Y = \sigma_Y^2 = \frac{96}{25}$, and $\operatorname{Cov}[X, Y] = \rho_{X,Y} \times \sigma_X \times \sigma_Y = -\frac{2}{25}$. Now,

$$\begin{aligned}
\operatorname{Var}[X + Y] &= \mathbb{E}\left[((X + Y) - \mathbb{E}[X + Y])^2\right] = \mathbb{E}\left[((X - \mathbb{E}X) + (Y - \mathbb{E}Y))^2\right] \\
&= \mathbb{E}\left[(X - \mathbb{E}X)^2\right] + 2\mathbb{E}[(X - \mathbb{E}X)(Y - \mathbb{E}Y)] + \mathbb{E}\left[(Y - \mathbb{E}Y)^2\right] \\
&= \operatorname{Var} X + 2\operatorname{Cov}[X, Y] + \operatorname{Var} Y \\
&= \boxed{\frac{389}{100}} = 3.89.
\end{aligned}$$

Remark: It is useful to remember that

$$\operatorname{Var}[X + Y] = \operatorname{Var} X + 2\operatorname{Cov}[X, Y] + \operatorname{Var} Y.$$

Note that when X and Y are uncorrelated, $\operatorname{Var}[X + Y] = \operatorname{Var} X + \operatorname{Var} Y$. This simpler formula also holds when X and Y are independence because independence is a stronger condition.

(b) First, we write

$$Y - aX - b = (Y - \mathbb{E}Y) - a(X - \mathbb{E}X) - \underbrace{(a\mathbb{E}X + b - \mathbb{E}Y)}_{c}.$$

Now, using the expansion

$$(u + v + t)^2 = u^2 + v^2 + t^2 + 2uv + 2ut + 2vt,$$

we have

$$\begin{aligned}(Y - aX - b)^2 = (Y - \mathbb{E}Y)^2 + a^2(X - \mathbb{E}X)^2 + c^2 \\ - 2a\,(X - \mathbb{E}X)\,(Y - \mathbb{E}Y) - 2c\,(Y - \mathbb{E}Y) + 2a\,(X - \mathbb{E}X)\,c.\end{aligned}$$

Recall that $\mathbb{E}[X - \mathbb{E}X] = \mathbb{E}[Y - \mathbb{E}Y] = 0$. Therefore,

$$\mathbb{E}\left[(Y - aX - b)^2\right] = \operatorname{Var} Y + a^2 \operatorname{Var} X + c^2 - 2a\operatorname{Cov}[X, Y]$$

Plugging back the value of c, we have

$$\boxed{\mathbb{E}\left[(Y - aX - b)^2\right] = \operatorname{Var} Y + a^2 \operatorname{Var} X + (\mathbb{E}[(Y - aX - b)])^2 - 2a\operatorname{Cov}[X, Y]}.$$

Here, $a = 3$ and $b = -5$. Plugging these values along with the given quantities into the formula gives

$$\mathbb{E}\left[(Y - aX - b)^2\right] = \boxed{\frac{721}{100}} = 7.21.$$

Problem 12. The input X and output Y of a system subject to random perturbations are described probabilistically by the joint pmf $p_{X,Y}(x, y)$, where $x = 1, 2, 3$ and $y = 1, 2, 3, 4, 5$. Let P denote the joint pmf matrix whose i,j entry is $p_{X,Y}(i, j)$, and suppose that

$$P = \frac{1}{71}\begin{bmatrix} 7 & 2 & 8 & 5 & 4 \\ 4 & 2 & 5 & 5 & 9 \\ 2 & 4 & 8 & 5 & 1 \end{bmatrix}$$

(a) Find the marginal pmfs $p_X(x)$ and $p_Y(y)$.

(b) Find $\mathbb{E}X$

(c) Find $\mathbb{E}Y$

(d) Find $\operatorname{Var} X$

(e) Find $\operatorname{Var} Y$

Solution: All of the calculations in this question are simply plugging numbers into appropriate formula. The `MATLAB` codes are provided in the file `P_XY_marginal_2.m`.

(a) The marginal pmf $p_X(x)$ is founded by the sums along the rows of the pmf matrix:

$$p_X(x) = \begin{cases} 26/71, & x = 1 \\ 25/71, & x = 2 \\ 20/71, & x = 3 \\ 0, & \text{otherwise} \end{cases} \approx \begin{cases} 0.3662, & x = 1 \\ 0.3521, & x = 2 \\ 0.2817, & x = 3 \\ 0, & \text{otherwise.} \end{cases}$$

The marginal pmf $p_Y(y)$ is founded by the sums along the columns of the pmf matrix:

$$p_Y(y) = \begin{cases} 13/71, & y = 1 \\ 8/71, & y = 2 \\ 21/71, & y = 3 \\ 15/71, & y = 4 \\ 14/71, & y = 5 \\ 0, & \text{otherwise} \end{cases} \approx \begin{cases} 0.1831, & y = 1 \\ 0.1127, & y = 2 \\ 0.2958, & y = 3 \\ 0.2113, & y = 4 \\ 0.1972, & y = 5 \\ 0, & \text{otherwise.} \end{cases}$$

(b) $\mathbb{E}X = \frac{136}{71} \approx 1.9155$

(c) $\mathbb{E}Y = \frac{222}{71} \approx 3.1268$

(d) $\operatorname{Var} X = \frac{3230}{5041} \approx 0.6407$

(e) $\operatorname{Var} Y = \frac{9220}{5041} \approx 1.8290$