MATH 477 – Section E1

Summary of Jointly Distributed Random Variables

- *Definition.* For two random variables X and Y, the *joint cumulative probability distribution function* of X and Y is

$$F(a,b) = \mathbb{P}\{X \le a, Y \le b\} \qquad -\infty < a, b < \infty$$

- *Definition.* If X and Y have joint c.d.f. F, then the *marginal cumulative distribution function* of X is

$$F_X(a) = \lim_{b\to\infty} F(a,b) \stackrel{\text{def}}{=} F(a,\infty).$$

  Similarly, the marginal c.d.f. of Y is $F_Y(b) = \lim_{a\to\infty} F(a,b) \stackrel{\text{def}}{=} F(\infty,b)$.

- *Definition.* If X and Y are discrete random variables, the *joint probability mass function* of X and Y is

$$p(x,y) = \mathbb{P}\{X = x, Y = y\}\ .$$

  The probability mass function of X is

$$p_X(x) = \mathbb{P}\{X = x\} = \sum_{y:p(x,y)>0} p(x,y)$$

  and, similarly, the probability mass function of Y is $p_Y(y) = \mathbb{P}\{Y = y\} = \sum_{x:p(x,y)>0} p(x,y)$.

- *Definition.* We say that X and Y are *jointly continuous* if there exists a function $f(x,y)$ defined for all $x, y \in \mathbb{R}$ with the property that for all $C \subseteq \mathbb{R}^2$ (that is, for all subsets C of the plane),

$$\mathbb{P}\left\{(X,Y) \in C\right\} = \int\int_{(x,y)\in C} f(x,y)\,dx\,dy\ .$$

  The function $f(x,y)$ is called the *joint probability density function* of X and Y. In particular, if $A, B \subseteq \mathbb{R}$, then

$$\mathbb{P}\{X \in A, Y \in B\} = \int_B \int_A f(x,y)\,dx\,dy.$$

- If $f(x,y)$ is the joint p.d.f. of X and Y then the following are true:

  1. $\mathbb{P}\{X \in A, Y \in B\} = \int_B \int_A f(x,y)\,dx\,dy.$

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2. If F is the joint c.d.f. of X and Y, then

$$F(a,b) = \int_{-\infty}^{b}\int_{-\infty}^{a} f(x,y)\,dx\,dy.$$

3. If F is the joint c.d.f. of X and Y, then

$$f(a,b) = \frac{\partial^2}{\partial a \partial b} F(a,b)$$

wherever the partial derivatives are defined.

4. $f_X(x) = \int_{-\infty}^{\infty} f(x,y)\,dy$ and $f_Y(y) = \int_{-\infty}^{\infty} f(x,y)\,dx$ where f_X and f_Y are the marginal densities of X and Y, respectively.

- *Definition.* The *joint c.d.f.* of random variables $X_1, X_2, \ldots, X_n$, denoted $F(a_1, a_2, \ldots, a_n)$, is defined by
$$F(a_1, a_2, \ldots, a_n) = \mathbb{P}\left\{X_1 \le a_1, X_2 \le a_2, \ldots, X_n \le a_n\right\}.$$

- *Definition.* The random variables $X_1, X_2, \ldots, X_n$ are said to be *jointly continuous* if there exists a function $f(x_1, x_2, \ldots, x_n)$ called the *joint p.d.f.*, such that for all $C \subseteq \mathbb{R}^n$,
$$\mathbb{P}\left\{(X_1, X_2, \ldots, X_n) \in C\right\} = \int\int\cdots\int_{(x_1,x_2,\ldots,x_n)\in C} f(x_1, x_2, \ldots, x_n)\, dx_1\, dx_2 \cdots dx_n\ .$$
In particular for all $A_1, A_2, \ldots, A_n \in \mathbb{R}$,
$$\mathbb{P}\left\{X_1 \in A_1, X_2 \in A_2, \ldots, X_n \in A_n\right\} = \int_{A_n}\cdots\int_{A_2}\int_{A_1} f(x_1, x_2, \ldots, x_n)\, dx_1\, dx_2 \cdots dx_n\ .$$

- *Definition.* The random variables X and Y are said to be *independent* if for any $A, B \subseteq \mathbb{R}$,
$$\mathbb{P}\{X \in A, Y \in B\} = \mathbb{P}\{X \in A\}\mathbb{P}\{Y \in B\}\ .$$

- X and Y are independent if and only if for all $a, b \in \mathbb{R}$,
$$\mathbb{P}\{X \le A, Y \le B\} = \mathbb{P}\{X \le a\}\mathbb{P}\{Y \le b\}\ .$$

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- X and Y are independent if and only if, for all $a, b \in \mathbb{R}$,
$$F(a,b) = F_X(a)F_Y(b) \ ,$$
where F is the joint c.d.f. of X and Y and F_X and F_Y denote the marginal c.d.f.'s of X and Y, respectively.

- If X and Y are discrete, then X and Y are independent if and only if, for all $x, y \in \mathbb{R}$,
$$p(x,y) = p_X(x)p_Y(y) \ ,$$
where p is the joint p.m.f. of X and Y and p_X and p_Y denote the marginal p.m.f.'s of X and Y, respectively.

- If X and Y are jointly continuous, then X and Y are independent if and only if, for all $x, y \in \mathbb{R}$,
$$f(x,y) = f_X(x)f_Y(y) \ ,$$
where f is the joint p.d.f. of X and Y and f_X and f_Y denote the marginal p.d.f.'s of X and Y, respectively.

- *Definition.* The random variables $X_1, X_2, \ldots, X_n$ are said to be *jointly independent* if for any $A_1, A_2, \ldots, A_n \subseteq \mathbb{R}$,
$$\mathbb{P}\{X_1 \in A_1, X_2 \in A_2, \ldots, X_n \in A_n\} = \prod_{i=1}^{n} \mathbb{P}\{X_i \in A_i\} \ .$$
As in the case of two random variables: (1) the c.d.f.'s multiply, (2) if the r.v.'s are discrete, the p.m.f.'s multiply, and (3) if the r.v.'s are jointly continuous, the p.d.f.'s multiply as well.

- Note that if X and Y are independent and individually continuous with p.d.f.'s f_X and f_Y, respectively, then they are necessarily jointly continuous with p.d.f. $f(x,y) = f_X(x)f_Y(y)$.

- Let X and Y be independent, jointly continuous random variables. Then, if F_{X+Y} denotes the c.d.f. and f_{X+Y} the p.d.f. of the random variable $X+Y$,
$$f_{X+Y}(a) = \int_{\infty}^{\infty} f_X(a-y)f_Y(y)\,dy \ ,$$
and
$$F_{X+Y}(a) = \int_{\infty}^{\infty} F_X(a-y)f_Y(y)\,dy \ , \tag{1}$$
where f_X and f_Y denote the marginal p.d.f.'s of X and Y, respectively, and F_X and F_Y denote the marginal c.d.f.'s of X and Y, respectively. Equation (1) is called the *convolution* of F_X and F_Y.

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- *Proposition 3.1.* If X and Y are independent gamma random variables with respective parameters (s, λ) and (t, λ), then $X + Y$ is a gamma random variable with parameters $(s + t, \lambda)$.

- *Proposition 3.2.* If X_i, $i = 1, \ldots, n$, are independent random variables that are normally distributed with respective parameters μ_i, σ_i^2, $i = 1, \ldots, n$, then $\sum_{i=1}^{n} X_i$ is normally distributed with parameters $\sum_{i=1}^{n} \mu_i$ and $\sum_{i=1}^{n} \sigma_i^2$.

- If X and Y are independent Poisson r.v.'s with respective parameters λ_1 and λ_2, then $X_1 + X_2$ is a Poisson r.v. with parameter $\lambda_1 + \lambda_2$.

- If X and Y are binomial r.v.'s with respective parameters (n, p) and (m, p) then $X + Y$ is a binomial r.v. with parameters $(n + m, p)$.

- If X and Y are discrete random variables with a joint p.m.f. of p and with marginal p.m.f.'s of p_X and p_Y, respectively, then we denote $p_{X|Y}$, the conditional p.m.f., as follows:

$$p_{X|Y}(x \mid y) = I\!P\{X = x \mid Y = y\} = \frac{p(x, y)}{p_Y(y)}.$$

- If X and Y are discrete with conditional p.m.f. $p_{X|Y}$, then we denote $F_{X|Y}$, the conditional c.d.f., as follows:

$$F_{X|Y}(x \mid y) = I\!P\{X \leq x \mid Y = y\} = \sum_{a \leq x} p_{X|Y}(a \mid y) .$$

- If X and Y are independent and discrete, then $p_{X|Y}(x \mid y) = p_X(x)$.

- If X and Y are jointly continuous random variables with a joint p.d.f. of f and marginal p.d.f.'s of f_X and f_Y, respectively, then we define

$$f_{X|Y}(x \mid y) = \frac{f(x, y)}{f_Y(y)} .$$

Summary of Jointly Distributed Random Variables

- If X and Y are jointly continuous with conditional p.d.f. $f_{X|Y}$, then we denote $F_{X|Y}$, the conditional c.d.f., as follows:

$$F_{X|Y}(a \mid y) = I\!P\{X \leq a \mid Y = y\} = \int_{-\infty}^{a} f_{X|Y}(x \mid y)\, dx \,.$$

- If X and Y are independent and continuous, then $f_{X|Y}(x \mid y) = f_X(x)$.