Tutorial 3 (Solution)

1. An instructor has given a short test consisting of two parts. For a randomly selected student, let X = the number of points earned on the first part and Y = the number of points earned on the second part. Suppose that the joint pmf of X and Y is given in the accompanying table.

$p(x,y)$	0	5	10	15
0	.02	.06	.02	.10
5	.04	.15	.20	.10
10	.01	.15	.14	.01

a. If the score recorded in the grade book is the total number of points earned on the two parts, what is the expected recorded score $E(X + Y)$? What's about $E|(X – Y)|$?

b. If the maximum of the two scores is recorded, what is the expected recorded score? What's about minimum?

ANSWER:

a. $E(X+Y)=\sum_x\sum_y (x+y)p(x,y)=(0+0)(.02)$ + (0 + 5)(.06) + … + (10 + 15)(.01) = 14.10

$E|(X – Y)|$ = ∑∑ |x – y| p(x,y) = |(0-0)|(0.02) + |(0-5)|(0.06) + ……+ |(10-15)|(0.01) = 5.1

b. $E[\max(X,Y)]=\sum_x\sum_y \max(x+y)\cdot p(x,y)$ = (0)(.02) + (5)(.06) +…+(15)(.01) = 9.60

E[min(x,y)] = ∑∑ min (x,y) p(x,y) = 0(0.02) + 0(0.06) + ……+ 10(0.14) + 10(0.01) = 4.5

2. Let $X_1, X_2,$ and X_3 represent the times necessary to perform three successive repair tasks at a certain service facility. Suppose they are independent normal random variables with expected values $\mu_1, \mu_2,$ and μ_3 and variances $\sigma_1^2, \sigma_2^2,$ and σ_3^2, respectively.

a. If $\mu=\mu_2=\mu_3=65$ and $\sigma_1^2=\sigma_2^2=\sigma_3^2=20$,
Calculate $P(X_1+X_2+X_3\le 210)$. What is $P(150\le X_1+X_2+X_3\le 210)$?

b. Using the μ_i's and σ_i's given in part (a), calculate $P(\bar{X}\ge 59)$ and $P(62\le\bar{X}\le 68)$.

c. Using the μ_i's and σ_i's given in part (a), calculate $P(-10\le X_1 - .5X_2 - .5X_3\le 5)$.

d. If $\mu_1=40, \mu_2=50, \mu_3=60, \sigma_1^2=10, \sigma_2^2=12,$ and $\sigma_3^2=14$, calculate
$P(X_1+X_2+X_3\le 160)$ and $P(X_1+X_2\ge 2X_3)$.

ANSWER:

a. $E(X_1+X_2+X_3)=195,\ V(X_1+X_2+X_3)=60,\ \sigma_{x_1+x_2+x_3}=7.746$

$$P(X_1+X_2+X_3\le 210)=P\left(Z\le\frac{210-195}{7.746}\right)=P(Z\le 1.94)=.9738$$

$$P\ \ 175\le X_1+X_2+X_3\le 210\ \ =P\ \ -2.58\le Z\le 1.94\ \ =.9689$$

b. $\mu_{\bar{x}}=\mu=65,\ \sigma_{\bar{x}}=\dfrac{\sigma_{\bar{x}}}{\sqrt{n}}=\dfrac{\sqrt{12}}{\sqrt{3}}=2.582$

$$P(\bar{X}\ge 59)=P\left(Z\ge\frac{59-65}{2.582}\right)=P(Z\ge -2.232)=.9898$$

$P(62 \le \bar{X} \le 68) = P(-1.16 \le Z \le 1.16) = .754$

c. $E(X_1 - .5X_2 - .5X_3) = 0;$

$V(X_1 - .5X_2 - .5X_3) = \sigma_1^2 + .25\sigma_2^2 + .25\sigma_3^2 = 30, sd = 5.4772$

$P(-10 \le X_1 - .5X_2 - .5X_3 \le 5) = P\left(\frac{-10-0}{5.4772} \le Z \le \frac{5-0}{5.4772}\right)$

$= P(-1.83 \le Z \le .91) = .8186 - .0336 = .785$

d. $E(X_1 + X_2 + X_3) = 150, V(X_1 + X_2 + X_3) = 36, \sigma_{x_1+x_2+x_3} = 6$

$P(X_1 + X_2 + X_3 \le 200) = P\left(Z \le \frac{160-150}{6}\right) = P(Z \le 1.67) = .9525$

We want $P(X_1 + X_2 \ge 2X_3)$, or written another way, $P(X_1 + X_2 - 2X_3 \ge 0)$.

$E(X_1 + X_2 - 2X_3) = 40 + 50 - 2(60) = -30,$

$V(X_1 + X_2 - 2X_3) = \sigma_1^2 + \sigma_2^2 + 4\sigma_3^2 = 78{,}36, sd = 8.832,$ so

$P(X_1 + X_2 - 2X_3 \ge 0) = P\left(Z \ge \frac{0-(-30)}{8.832}\right) = P(Z \ge 3.40) = .0003$

3. A service station has both self-service and full-service blocks. On each block, there is a single regular unleaded pump with two hoses. Let X denote the number of hoses being used on the self-service block at a particular time, and let Y denote the number of hoses on the full-service block in use at that time. The joint pmf of X and Y is given below:

P(x,y)		0	1	2	Px (row-wise sum)
	0	0.10	0.04	0.02	0.16
x	1	0.08	0.20	0.06	0.34
	2	0.06	0.14	0.30	0.50
Py (column-wise sum)		0.24	0.38	0.38	

a. Compute P(X ≤ 1 and Y ≤ 1).
b. Compute the marginal pmf of X and Y. Using Py(y), what is the P(Y ≤ 1)?
c. Are X and Y independent? Explain.

ANSWER:

a. P(X ≤ 1 and Y ≤ 1) = p(0,0) + p(0,1) + p(1,0) + p(1,1) = 0.10 + 0.04 + 0.08 + 0.20 = 0.42

b. Px(0) = 0.16, Py(0) = 0.24, Px(1) = 0.34, Py(1) = 0.38, Px(2) = 0.50, Py(2) = 0.38

P(Y ≤ 1) = Py(0) + Py(1) = 0.24 + 0.38 = 0.62

c. p(0,0) = 0.10. Px(0). Py(0) = (0.16)(0.24) = 0.0384

Since 0.10 ≠ 0.0384, so X and Y are independent.

Tutorial 3 (Solution)

4. The first assignment in a statistical computing class involves running a short program. If past experience indicates that 40% of all students will make programming errors, compute the (approximate) probability that in a class of 50 students, between 15 and 25(inclusive) will make errors. (***Hint: Normal approximation to the binomial***)

ANSWER:

Here, p = 0.40, q = 1-p = 1 - 0.40 = 0.60. n = 50.

Mean, $\mu = np = (50)(0.4) = 20$,

Variance , $\sigma^2 = npq = (50)(0.4)(0.6) = 12$, so $\sigma = \sqrt{12} = 3.464$

$$P(15 \le X \le 25) = B(25; 50, 0.4) - B(15; 50, 0.4)$$
$$= \Phi((25.5 - 20)/3.464) - \Phi((14.5 - 20)/3.464)$$
$$= \Phi(1.59) - \Phi(-1.59) = 0.9441 - 0.0559 = 0.8882$$

5. A store operates both an express and a regular checkout. On a randomly selected day, let X = the percentage of time the express checkout is in use and Y = the percentage of time that the regular checkout is in use. Suppose the joint pdf of (X,Y) is given by

$$F(x,y) = \begin{cases} 1.2(x+y^2), & 0 \le x \le 1,\ 0 \le y \le 1 \\ 0, & \text{otherwise} \end{cases}$$

a. Verify that this is a legitimate pdf.

b. What is the probability that neither checkout is busy more than one-quarter of the time?

ANSWER:

a. It is a legitimate pdf if $\int\int f(x,y)\,dx\,dy = 1$ $\quad 0 \le x \le 1, \quad 0 \le y \le 1$

$$\text{Since } \int\int 1.2\,(x + y^2)\,dx\,dy = \int\int 1.2\,x\,dx\,dy + \int\int 1.2\,y^2\,dx\,dy$$
$$= 1.2\,[\int x dx + \int y^2 dy] = 0.6 + 0.4 = 1$$

So it is legitimate.

b. $P(0 \le X \le ¼, 0 \le Y \le ¼) = \int\int 1.2\,(x + y^2)\,dx\,dy$ $\quad 0 \le x \le ¼, \quad 0 \le y \le ¼$

$$= 1.2\,[\int\int x\,dx\,dy + \int\int y^2\,dx\,dy] = 0.0109375$$

6. A college professor always finishes his lectures within 2 minutes after the bell rings to end the period and the end of the lecture. Let X = the time that elapses between the bell and the end of the lecture and suppose the pdf of X is

$$f(x) = \begin{cases} kx^2 & 0 \le x \le 2 \\ 0 & \text{otherwise} \end{cases}$$

a. Find the value of k. [Hint: Total area under the graph of $f(x)$ is 1.]
b. What is the probability that the lecture ends within 1minutes of the bell ringing?
c. What is the probability that the lecture continues beyond the bell for between 60 and 90 seconds?
d. What is the probability that the lecture continues for at least 90 seconds beyond the bell?

ANSWER:

a. $1 = \int_{-\infty}^{\infty} f(x)dx = \int_0^2 kx^2 dx = k(x^3/3)]_0^2 = k(8/3) \Rightarrow k = 3/8 = .375$

b. $P(0 \le X \le 1) = \int_0^1 .375\, x^2 dx = .125\, x^3]_0^1 = .125$

c. $P(1 \le X \le 1.5) = \int_1^{1.5} .375x^2 dx = .125\, x^3]_1^{1.5} \approx .2969$

d. $P(X \ge 1.5) = 1 - \int_0^{1.5} .375x^2 dx = 1 - .125\, x^3]_0^{1.5} \approx .5781$

7. Let $X_1, X_2, \ldots, X_{100}$ denote the actual net weights of 100 randomly selected 50-lb bags of fertilizer.

a. If the expected weight of each bag is 50 and the variance is 1, calculate $P(49.8 \le \bar{X} \le 50.3)$ (approximately) using the CLT.

b. If the expected weight of each bag is 49.8 lb rather than 50 lb so that on average bags are underfilled, calculate $P(49.8 \le \bar{X} \le 50.3)$.

ANSWER:

a. $\mu_{\bar{x}} = \mu = 50, \sigma_{\bar{x}} = \dfrac{\sigma_x}{\sqrt{n}} = \dfrac{1}{\sqrt{100}} = .10$

$$P(49.8 \le \bar{X} \le 50.3) = P\left(\frac{49.8-50}{.10} \le Z \le \frac{50.3-50}{.10}\right)$$
$$= P(-2.0 \le Z \le 3.0) = .9759$$

b. $$P(49.8 \le \bar{X} \le 50.3) \approx P\left(\frac{49.8-49.8}{.10} \le Z \le \frac{50.3-49.8}{.10}\right)$$
$$= P(0 \le Z \le 5) = .5000$$

8. Suppose your waiting time for a bus in the morning is uniformly distributed on [0,5], whereas waiting time in the evening is uniformly distributed on [0,10] independent of morning waiting time.

a. If you take the bus each morning and evening for a week, what is your total expected waiting time? [*Hint:* Define random variables $X_1, \ldots X_{10}$ and use a rule of expected value.)

b. What is the variance of your total waiting time?

c. What are the expected value and variance of the difference between morning and evening waiting times on a given day?

d. What are the expected value and variance of the difference between morning waiting time and total evening waiting time for a particular week?

ANSWER:

Let $X_1, \ldots, X_5$ denote morning times and $X_6, \ldots, X_{10}$ denote evening times.

a. $E(X_1 + \ldots + X_{10}) = E(X_1) + \ldots + E(X_{10}) = 5E(X_1) + 5E(X_6) = 5(2.5) + 5(5) = 37.5$

b. $\text{Var}(X_1 + \ldots + X_{10}) = \text{Var}(X_1) + \ldots + \text{Var}(X_{10}) = 5\,\text{Var}(X_1) + 5\text{Var}(X_6)$

$= 5[\dfrac{25}{12} + \dfrac{100}{12}] = \dfrac{625}{12} = 52.083$

c. $E(X_1 - X_6) = E(X_1) - E(X_6) = 2.5 - 5 = -2.5$

$$\text{Var}(X_1 - X_6) = \text{Var}(X_1) + \text{Var}(X_6) = \frac{25}{12} + \frac{100}{12} = \frac{125}{12} = 10.417$$

d. $E[(X_1 + \ldots + X_5) - (X_6 + \ldots + X_{10})] = 5(2.5) - 5(5) = -12.5$

$\text{Var}[(X_1 + \ldots + X_5) - (X_6 + \ldots + X_{10})]$

$= \text{Var}(X_1 + \ldots + X_5) + \text{Var}(X_6 + \ldots + X_{10})] = 52.083$

9. Let X denote the number of brand X VCRs sold during a particular week by a certain store. The pmf of X is

x	0	1	2	3	4
$p_x(x)$	.1	.2	.3	.25	.15

Seventy percent of all customers who purchase brand X VCRs also buy an extended warranty. Let Y denote the number of purchasers during this week who buy an extended warranty.

a. What is $P(X = 4, Y = 2)$? [*Hint:* This probability equals $P(Y = 2/X = 4) \cdot P(X = 4)$; now think of the four purchases as four trials of a binomial experiment, with success on a trial corresponding to buying an extended warranty.]

b. Calculate $P(X = Y)$.

c. Determine the joint pmf of X and Y and then the marginal pmf of Y.

ANSWER:

a. $p(4,2) = P(Y = 2 \mid X = 4) \cdot P(X = 4) = [\binom{4}{2}(.7)^2(.4)^2] \cdot (.15) = .0397$

b. $P(X = Y) = p(0,0) + p(1,1) + p(2,2) + p(3,3) + p(4,4) = .1 + (.2)(.7) + (.3)(.7)^2 +$
$(.25)(.7)^3 + (.15)(.7)^4 = .5088$

c. $p(x,y) = 0$ unless $y = 0,1,\ldots,x;\, x = 0,1,2,3,4.$ For any such pair,

$$p(x,y) = P(Y = y \mid X = x) \cdot P(X = x) = \binom{x}{y}(.7)^y(.3)^{x-y} \cdot p_x(x)$$

$$p_y(4) = p(y = 4) = p(x = 4, y = 4) = p(4,4) = (.7)^4 \cdot (.15) = .0360$$

$$p_y(3) = p(3,3) + p(4,3) = (.7)^3(.25) + \binom{4}{3}(.7)^3(.3)(.15) = .1475$$

$$p_y(2) = p(2,2) + p(3,2) + p(4,2) = (.7)^2(.3) + \binom{3}{2}(.7)^2(.3)(.25)$$
$$+ \binom{4}{2}(.7)^2(.3)^2(.15) = .2969$$

$$P_y(1) = p(1,1) + p(2,1) + p(3,1) + p(4,1) = (.7)(.2) + \binom{2}{1}(.7)(.3)(.3)$$
$$\binom{3}{1}(.7)(.3)^2(.25) + \binom{4}{1}(.7)(.3)^3(.15) = .3246$$

$$p_y(0) = 1 - [.3246 + .2969 + .1475 + .0360] = .1950$$