Statistics for Data Science
Unit 4 Part 1 Homework: Discrete Random Variables

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1. **Best Game in the Casinoß†**

   You flip a fair coin 3 times, and get a different amount of money depending on how many heads you get. For 0 heads, you get \$0. For 1 head, you get \$2. For 2 heads, you get \$4. Your expected winnings from the game are \$6.

   <u>Givens:</u>
   X is a binomial random variable based on n trials with success probability p.

   When X=x, let x be the number of heads among the $n = 3$ trials with the probability of a head in each trial being $p = \frac{1}{2}$.

$$b(x; n, p) = \begin{cases} \binom{n}{k} p^x (1-p)^{n-x}, & x \in \{0, 1, 2, 3, ...\} \\ 0, & otherwise. \end{cases}$$

$$\text{Probablity of 0 Heads: } P(X = 0) = b(0; 3, \frac{1}{2}) = \binom{3}{0}(\frac{1}{2})^3 = \frac{1}{8}$$

$$\text{Probablity of 1 Heads: } P(X = 1) = b(1; 3, \frac{1}{2}) = \binom{3}{1}(\frac{1}{2})^3 = \frac{3}{8}$$

$$\text{Probablity of 2 Heads: } P(X = 2) = b(2; 3, \frac{1}{2}) = \binom{3}{2}(\frac{1}{2})^3 = \frac{3}{8}$$

$$\text{Probablity of 3 Heads: } P(X = 3) = b(3; 3, \frac{1}{2}) = \binom{3}{3}(\frac{1}{2})^3 = \frac{1}{8}$$

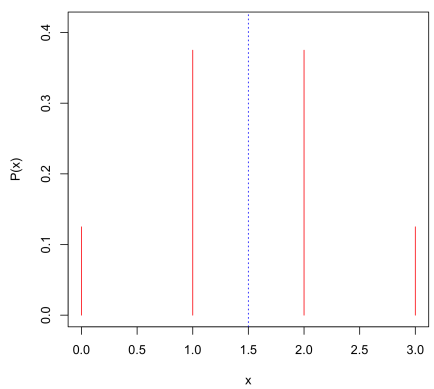

Figure 1: Plot of binomial distribution of n=3, p=$\frac{1}{2}$

(a) How much do you get paid if the coin comes up heads 3 times?

Answer:
Given fair coins, the expectation of X, $E(X) = np = 3 \cdot \frac{1}{2} = \frac{3}{2}$ (also shown in graph above)

The expected winnings from the game is \$6, such that $g(E(X)) = 6$

We also have:

$G(X = 0) = 0$
$G(X = 1) = 2$
$G(X = 2) = 4$
$G(X = 3) = A$

Using the expectation that $E(g(X)) = g(E(X))$ we can solve for A:

$$E(g(X)) = \sum_{y=0}^{x} g(y) \cdot p(y) = 6$$

$$(0 \cdot \frac{1}{8}) + (2 \cdot \frac{3}{8}) + (4 \cdot \frac{3}{8}) + (A \cdot \frac{1}{8}) = 6$$

$$(\frac{6}{8}) + (\frac{12}{8}) + (\frac{A}{8}) = 6$$

$$6 + 12 + A = 48$$

$$A = 30$$

(b) Write down a complete expression for the cumulative probability function for your winnings from the game.

Answer:

$$\text{With } g(x) = \begin{cases} 0 & ; \ x < 1 \\ \frac{16}{3} & ; \ 1 \leq x < 3 \\ 208 & ; \ x \leq 3 \end{cases} \ , \ p = \frac{1}{2}, \text{ and } n = 3$$

$$\text{Winnings } = g(P(X \leq x)) = \sum_{y=0}^{x} g(y) \cdot \tbinom{n}{y} \, p^y (1-p)^{n-y}$$

Proof:

$G(P(X \leq 0)) = (\frac{1}{8} \cdot 0) = 0$

$G(P(X \leq 1)) = (\frac{1}{8} \cdot 0) + (\frac{3}{8} \cdot \frac{16}{3}) = 2$

$G(P(X \leq 2)) = (\frac{1}{8} \cdot 0) + (\frac{3}{8} \cdot \frac{16}{3}) + (\frac{3}{8} \cdot \frac{16}{3}) = 4$

$G(P(X \leq 3)) = (\frac{1}{8} \cdot 0) + (\frac{3}{8} \cdot \frac{16}{3}) + (\frac{3}{8} \cdot \frac{16}{3}) + (\frac{1}{8} \cdot 208) = 30$

2. **Reciprocal Dice**

   Let X be a random variable representing the outcome of rolling a 6-sided die. Before the die is rolled, you are given two options:

   (a) You get $1/E(X)$ in dollars right away.

   Proof:
   Since each die roll has uniform probability we can compute $E(X)$ as:

   $$E(X) = \frac{1}{k}\sum_j x_j = \frac{1+2+3+4+5+6}{6} = 3.5$$

   $1/E(X)$ is approx .28571 dollars

   (b) You wait until the die is rolled, then get $1/X$ in dollars.

   Proof:
   Given $g(X) = 1/X$ and $E(g(X)) = \sum_x g(x) \cdot p(x)$

   $$E(g(X)) = (1 \cdot \frac{1}{6}) + (\frac{1}{2} \cdot \frac{1}{6}) + (\frac{1}{3} \cdot \frac{1}{6}) + (\frac{1}{4} \cdot \frac{1}{6}) + (\frac{1}{5} \cdot \frac{1}{6}) + (\frac{1}{6} \cdot \frac{1}{6})$$

   $E(g(X))$ is approx .408333 dollars

   (c) Which option is better for you, in expectation?

   Answer:
   .408333 is greater than .28571. The second option is better in expectation.

3. **The Baseline for Measuring Deviations**

   Given any random variable X and a real number t, we can define another random variable $Y = (X - t)^2$. In other words, for any random variable X, we can choose a real number, t, as a baseline and calculate the squared deviation of X away from t.

   You might wonder why we often square deviations (instead of taking an absolute value, or cubing them, etc.). This exercise will shed some light on why this is a natural choice.

   (a) Write down an expression for $E(Y)$ and simplify it as much as you can. Even though we haven't proved this yet, you can use the fact that for any two random variables, A and B, $E(A + B) = E(A) + E(B)$.
   Answer:

   $$\begin{aligned} E(Y) &= E[(X-t)^2] = E[(X^2 - 2tX + t^2)] \\ &= E[(X^2)] + E[(-2tX)] + E[(t^2)] \\ &= E(X^2) - 2tE(X) + t^2 \end{aligned}$$

   (b) Taking a partial derivative with respect to t, compute the value of t that minimizes $E(Y)$. (Hint: Your answer should be a very familiar value)

Answer:

$$E(Y)'_{(x,t)} = \frac{\partial}{\partial t}[E(X^2) - 2tE(X) + t^2]$$
$$= -2E(X) + 2t$$

$E(Y)$ approaches minimum when

$$t = E(X)$$

(c) What is the value of $E(Y)$ for this choice of t? (Hint: this should also be a very familiar value)

Answer:

Replacing t with $E(X)$ in $E(Y) = E(X^2) - 2tE(X) + t^2$:

$$E(X^2) - 2tE(X) + t^2 = E(X^2) - 2E(X)E(X) + [E(X)]^2$$
$$= E(X^2) - 2[E(X)]^2 + [E(X)]^2$$

The value of $E(Y)$ at the minimum will be

$$E(Y) = E(X^2) - [E(X)]^2$$

4. **Optional Advanced Exercise: Heavy Tails**

One reason to study the mathematical foundation of statistics is to recognize situations where common intuition can break down. An unusual class of distributions are those we call *heavy-tailed*. The exact definition varies, but we'll say that a heavy-tailed distribution is one for which not all moments are finite. Consider a random variable M with the following pmf:

$$p_M(x) = \begin{cases} c/x^3, & x \in \{1, 2, 3, ...\} \\ 0, & otherwise. \end{cases}$$

where c is a constant (you can calculate its value if you like, but it's not important).

(a) Is $E(M)$ finite?

Answer:

Yes it is finite because although this an infinite series, the sum of c/x^3 from 1 to ∞ will eventually converge. It converges because the equation represents a p-series ($1/n^p$) and a p-series converges when p > 1 and diverges when 0 < p < 1.

(b) Is $V(M)$ finite?

Answer:

Also finite for the same reason as (a). For $V(M)$ to be calculated the infinite series must converge.

Heavy-tailed distributions may seem odd, but they're not as rare as you might suspect. Researchers argue that the distribution of wealth is heavy-tailed; so is the distribution of computer file sizes, insurance payouts, and area burned by forest fires. These random variables are problematic in that a lot of common statistical techniques don't work on them. For this class, we'll assume that all of our variables don't have heavy-tails.