"A pinch of **probability** is worth a pound of perhaps"

James Thurber

Lecture 17: Practice Multivariate

Probability Theory and Applications

Fall 2008

October 27

Announcements

- Assignment due Friday 10/31
- Exam Monday 11/3
  - Emphasis: Cont. RV, named cont. RV, Multivariate RV
  - Two pages of notes
  - Simple calculator
  - Will supply tables and names distributions like last time
- Practice Classes between now and then
- No office hours on October 30.
- Added office hours

  Friday 10/31 from 11-12 and 2-3.

  Monday 11/3 from 12 -2

Outline

- Warm Up
- Conditional Expectation Review
- Conditional Expectation Examples

Warm UP

A soft drink machine has a random amount X in supply at the beginning of a given day and dispense a random amount Y during the day (amount measured in gallons). Note the machine is not resupplied so Y≤X. The joint density of X and Y is

$$f(x, y) = \begin{cases} 1/2 & 0 \le y \le x, 0 \le x \le 2 \\ 0 & o.w. \end{cases}$$

What is the marginal distribution of Y given X?

What is the probability that less than ½ gallon is sold given that the machine contains 1 gallon at the start of the day.

Worksheet

The marginal of X is

$$f_X(x) = \int_{?}^{?} f(x,y)\,dy$$

Note the E(X)=4/3

The conditional distribution of Y given X is

$$f_{Y|X}(y\,|\,x) = \frac{f(x,y)}{f_X(x)}$$

P(Y≤1/2|X=2)= $\int_{0}^{.5} f(Y\,|\,X = 1/2)\,dy$

Notes

The conditional expectation of X given Y=y

$$E[X \mid Y = y]$$

can be viewed as a function of Y because its value depends on the value y of Y.

Is E[X|Y=y] a probability distribution?

Warm-Up Problem

Recall

$$f_{Y|X}(y|x)=\begin{cases}1/x & x\le y\le 2\\ 0 & o.w.\end{cases} \quad with\, x>0$$

E[Y|X]=

$$E(Y|X)=\int_0^x y f_{Y|X}(y|x)dy=\int_0^x y/x dy=\frac{x}{2}$$

$$E(Y|X=2)=\frac{1}{2}$$

Conditional Expectation in Practice

- Daily electricity consumption depends on temperature
- Time to failure is dependent on temperature
- Parts produced by different subcontractors with different failure rates.
- Response time to web query depends on number of jobs arriving during query.

Recall

- Law of Iterated Expectation

$$E[X] = E\left[E[X \mid Y]\right]$$

- Law of Total Variance

$$Var[X] = E\left[\mathrm{var}[X \mid Y]\right] + \mathrm{var}\left[E[X \mid Y]\right]$$

WARM UP

E[Y]=E[E(Y|X)]

$$E[Y] = E[E[Y \mid X]] = E[X / 2] = 1/2 * (4/3)] = 2/3$$

You check by computing marginal of Y directly and then finding expectation.

Also see how it works for variance.

Stick Problem

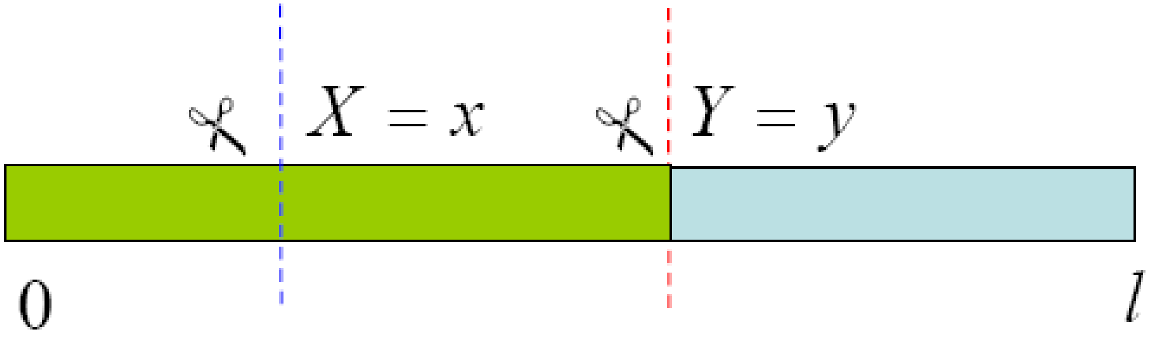
We start with a stick of length l. We break it at a point which is chosen randomly and uniformly over its length, and keep the piece that contains the left end of the stick. Then we repeat the process on the stick we were left with.

What is the expected length of the stick that we are left with after breaking it twice?

Set Up

Let Y be the length of the stick after we break it the first time.

Let X be the length of the stick after we break it the second time.



solution

Let Y be the length of the stick after we break it the first time.

Let X be the length of the stick after we break it the second time.

$$f_Y(y) = \begin{cases} \frac{1}{l} & 0 < y < l \\ 0 & o.w. \end{cases}$$

$$f_{X|Y}(x \mid y) = \begin{cases} \frac{1}{y} & 0 < x < y \\ 0 & o.w. \end{cases}$$

$$E[Y] = \frac{\ell}{2}$$

$$E_{X|Y}[X \mid Y = y] = \frac{y}{2}$$

Both are uniformly distributed

Expected Value of X

Using Law of Iterated Expectations

$$E[X] = E\left[E[X \mid Y]\right]$$

$$= E\left[\frac{Y}{2}\right]$$

$$= \frac{\ell}{4}$$

What is Variance of X?

- Can use Law of Total Variance

$$Var\left[X\right] = E\left[\mathrm{var}\left[X \mid Y\right]\right] + \mathrm{var}\left[E\left[X \mid Y\right]\right]$$

We know variance of uniforms

Let Y be the length of the stick after we break it the first time.

Let X be the length of the stick after we break it the second time.

$$f_Y(y) = \begin{cases} \frac{1}{l} & 0 < y < l \\ 0 & o.w. \end{cases}$$

$$f_{X|Y}(x \mid y) = \begin{cases} \frac{1}{y} & 0 < x < y \\ 0 & o.w. \end{cases}$$

$$E_{X|Y}\left[X \mid Y = y\right] = \frac{y}{2}$$

$$E\left[Y\right] = \frac{\ell}{2}$$

$$\mathrm{var}_{X|Y}\left[X \mid Y = y\right] = \frac{y^2}{12}$$

$$\mathrm{var}(Y) = \frac{\ell^2}{12}$$

Variance of X

- By Law of Total Variance

$$Var[X] = E[\operatorname{var}[X \mid Y]] + \operatorname{var}[E[X \mid Y]]$$

$$= E\left[\frac{Y^2}{12}\right] + \operatorname{var}\left[\frac{Y}{2}\right]$$

$$= \frac{1}{12}E\left[Y^2\right] + \frac{1}{4}\operatorname{var}[Y]$$

$$= \frac{1}{12}\left[\operatorname{var}(Y) + E[Y]^2\right] + \frac{1}{4}\operatorname{var}[Y]$$

$$= \frac{1}{12}\left[\frac{\ell^2}{12} + \frac{\ell^2}{4}\right] + \frac{1}{4}\left[\frac{\ell^2}{12}\right] = \frac{7\ell^2}{144}$$

Problem 2

In the die-coin experiment, a fair die is rolled and then a fair coin is tossed the number of times showing on the die. Let N denote the die score and *X* the number of heads. Find E(X)?

Simulation:

http://www.math.uni-konstanz.de/~kohlmann/ftp/applets/DieCoinExperiment.html

Problem 2

In the die-coin experiment, a fair die is rolled and then a fair coin is tossed the number of times showing on the die. Let N denote the die score and X the number of heads.

- Find the conditional distribution of X given N.
- Find $E(X \mid N)$.
- Find $\mathrm{var}(X \mid N)$.
- Find $E(X)$.
- Find $\mathrm{var}(X)$.

Problem 3

- Roll a die until 6 appears.
- Let Y = # 1's and X = # of Rolls.

What is the Expect value of Y?

Set up

Use $E[X]=E[E(Y|X)]$.

Find marginal of Y