1. At a certain gas station, 40% of the customers use regular unleaded gas (A_1), 35% use extra unleaded gas (A_2), and 25% use premium unleaded gas (A_3). Of those customers using regular gas, only 30% fill their tanks (event *B*). Of those customers using extra gas, 60% fill their tanks, whereas of those using premium, 50% fill their tanks.

   a. What is the probability that the next customer will request extra unleaded gas and fill the tank?
   b. What is the probability that the next customer fills the tank?
   c. If the next customer fills the tank, what is the probability that regular gas is requested? Extra gas? Premium gas?

   ANSWER:

   $P(A_1) = .40,\ \ P(A_2) = .35,\ \ P(A_3) = .25$

   $P(B|A_1) = 0.3$

   $P(B|A_2) = 0.6$

   $P(B|A_3) = 0.5$

   Therefore,

   $P(A_1 \cap B) = P(A_1) \cdot P(B1A_1) = (.40)(.30) = .12$

   $P(A_2 \cap B) = P(A_2) \cdot P(B1A_2) = (.35)(.60) = .21$

   $P(A_3 \cap B) = P(A_3) \cdot P(B1A_3) = (.25)(.50) = .125$

   a. $(A_2 \cap B)$ = .21
   b. *P*(*B*) = $P(A_1 \cap B)$ + $P(A_2 \cap B)$ +$P(A_3 \cap B)$ = .12+.21+.125=.455
   c. P($P(A_1|B) = \dfrac{P(A_1 \cap B)}{P(B)} = \dfrac{.12}{.455} = .264$

      $P(A_2|B) = \dfrac{P(A_2 \cap B)}{P(B)} = \dfrac{.21}{.455} = .462$

      $P(A_3|B) = \dfrac{P(A_3 \cap B)}{P(B)} = \dfrac{.125}{.455} == .275$

2. The number of tickets issued by a meter reader for parking-meter violations can be modeled by a Poisson process with a rate parameter of five per hour.
   a. What is the probability that exactly three tickets are given out during a particular hour?
   b. What is the probability that at least three tickets are given out during a particular hour?
   c. How many tickets do you expect to be given during a 45-min period?

   ANSWER:

   a. $P(X = 3) = F(3;5) - F(2;5) = .265 - .125 = .140$

   b. $P(X \geq 3) = 1 - P(X \leq 2) = 1 - F(2;5) = 1 - .125 = .875$

   c. Tickets are given at the rate of 5 per hour, so for a 45 minute period the rate is $\lambda = (5)(.75) = 3.75$, which is also the expected number of tickets in a 45 minute period.

3. An aircraft can seat 220 passengers, and each of the passengers booked on the flight has a probability of 0.9 of actually arriving at the gate to board the plane, independent of the other passengers.
   a. Suppose the airline books 235 passengers on the flight. What is the probability that there will be insufficient seats to accommodate all of the passengers who wish to board the plane?
   b. If the airline wants to be 75% confident that there will be no more than 220 passengers who wish to board the plane, how many passengers can be booked on the flight?

   (a) $P(B(235, 0.9) \geq 221)$
   $\simeq P(N(235 \times 0.9, 235 \times 0.9 \times 0.1) \geq 221 - 0.5)$
   $= 1 - \Phi(1.957) = 0.025$

   (b) If n passengers are booked on the flight, it is required that
   $P(B(n, 0.9) \geq 221)$
   $\simeq P(N(n \times 0.9, n \times 0.9 \times 0.1) \geq 221 - 0.5) \leq 0.25.$

   This is satisfied at $n = 241$ but not at $n = 242$.
   Therefore, the airline can book up to 241 passengers on the flight.

4. In commuting to school, I must first get on a bus near my house and then transfer to a second bus. If the waiting time (in minutes) at each stop has a uniform distribution with $A = 0$ and $B = 5$, then it can be shown that my total waiting time Y has the pdf

$$f(y) = \begin{cases} \frac{1}{25}y & 0 \leq y < 5 \\ \frac{2}{5} - \frac{1}{25}y & 5 \leq y \leq 10 \\ 0 & y < 0 \text{ or } y > 10 \end{cases}$$

   a) Sketch a graph of the pdf of Y

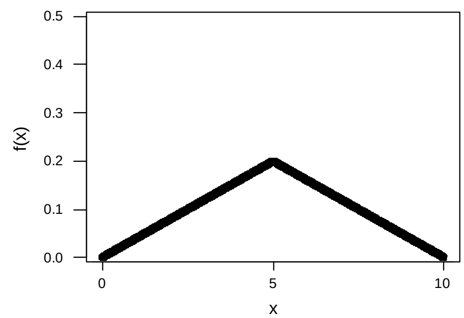
b) Verify that $\int_{-\infty}^{\infty} f(y)dy = 1$

c) What is the probability that the total waiting time is at most 3 minutes?

d) What is the probability that the total waiting time is between 3 and 8 minutes?

e) What is the probability that the total waiting time is either less than 2 minutes or more than 6 minutes?

a.



b. $\int_{-\infty}^{\infty} f(y)dy = \int_0^5 \tfrac{1}{25} y dy + \int_5^{10} (\tfrac{2}{5} - \tfrac{1}{25} y)dy$ = $\left.\frac{y^2}{50}\right]_0^5 + \left(\frac{2}{5}y - \frac{1}{50}y^2\right)\Big]_5^{10}$

= $\frac{1}{2} + \left[(4-2)-(2-\frac{1}{2})\right] = \frac{1}{2} + \frac{1}{2} = 1$

c. P(Y ≤ 3) = $\int_0^3 \tfrac{1}{25} y dy = \left.\frac{y^2}{50}\right]_0^5 = \frac{9}{50} \approx .18$

d. P(3 ≤ Y ≤ 8) = P(Y ≤ 8) - P(Y < 3) = $\frac{46}{50} - \frac{9}{50} = \frac{37}{50} = .74$

e. P(Y < 2 or Y > 6) = $\int_0^2 \frac{1}{25} y dy + \int_6^{10} \left(\frac{2}{5} - \frac{1}{25} y\right) dy = \frac{2}{5} = 0.4$

5. Consider the pdf for total waiting time *Y* for two buses

$$f(y) = \begin{cases} \frac{1}{25} y & 0 \le y < 5 \\ \frac{2}{5} - \frac{1}{25} y & 5 \le y \le 10 \\ 0 & y < 0 \text{ or } y > 10 \end{cases}$$

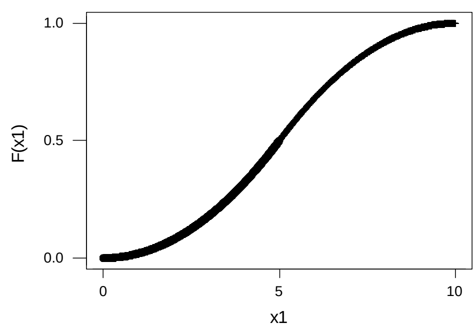
a) Compute and sketch the cdf of *Y*

b) Compute *E*(*Y*) and *Var*(*Y*)

a. For $0 \le y \le 5$, F(y) = $\int_0^y \frac{1}{25} u du = \frac{y^2}{50}$

For $5 \le y \le 10$, F(y) = $\int_0^y f(u)du = \int_0^5 f(u)du + \int_5^y f(u)du$

$$= \frac{1}{2} + \int_5^y \left(\frac{2}{5} - \frac{u}{25} \right) du = \frac{2}{5} y - \frac{y^2}{50} - 1$$



b. E(Y) = 5 by straightforward integration (or by symmetry of f(y)), and

V(Y)= $\frac{50}{12} = 4.1667$.

6. Let *X* denote the number of Sony digital cameras sold during a particular week by a certain store. The pmf of *X* is

x	0	1	2	3	4
P(*X* = *x*)	0.1	0.2	0.3	0.25	0.15

Sixty percent of all customers who purchase these cameras also buy an extended warranty. Let *Y* denote the number of purchasers during this week who buy an extended warranty.

a) What is the value of *P*(*X* = 4, *Y* = 2)?
b) Calculate *P*(*X* = *Y*)
c) Determine the joint pmf of *X* and *Y*
d) Determine the marginal pmf of *Y*

a. p(4,2) = P(Y = 2 | X = 4) · P(X = 4) = $\left[\binom{4}{2} (.6)^2 (.4)^2 \right] \cdot (.15) = .0518$

b. P(X = Y) = p(0,0) + p(1,1) + p(2,2) + p(3,3) + p(4,4) = $.1+(.2)(.6) + (.3)(.6)^2 + (.25)(.6)^3 + (.15)(.6)^4 = .4014$

c. p(x,y) = 0 unless y = 0, 1, …, x; x = 0, 1, 2, 3, 4. For any such pair,

p(x,y) = P(Y = y | X = x) · P(X = x) = $\binom{x}{y}(.6)^y(.4)^{x-y} \cdot p_x(x)$

d. $p_y(4) = p(y = 4) = p(x = 4, y = 4) = p(4,4) = (.6)^4 \cdot (.15) = .0194$

$p_y(3) = p(3,3) + p(4,3) = (.6)^3(.25) + \binom{4}{3}(.6)^3(.4)(.15) = .1058$

$p_y(2) = p(2,2) + p(3,2) + p(4,2) = (.6)^2(.3) + \binom{3}{2}(.6)^2(.4)(.25)$

$$+\binom{4}{2}(.6)^2(.4)^2(.15) = .2678$$

$p_y(1) = p(1,1) + p(2,1) + p(3,1) + p(4,1) = (.6)(.2) + \binom{2}{1}(.6)(.4)(.3)$

$$\binom{3}{1}(.6)(.4)^2(.25) + \binom{4}{1}(.6)(.4)^3(.15) = .3590$$

$p_y(0) = 1 - [.3590+.2678+.1058+.0194] = .2480$

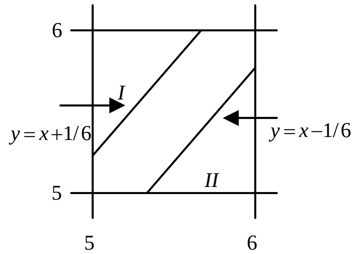
7. Somchai and Somying have agreed to meet between 5:00 PM and 6:00 PM for dinner at a local health-food restaurant. Let *X* be Somchai's arrival time and *Y* be Somying's arrival time. Suppose *X* and *Y* are independent with each uniformly distributed on the interval [5, 6].
   a) What is the joint pdf of *X* and *Y*?
   b) What is the probability that they both arrive between 5:15 and 5:45 PM?
   c) If the first one to arrive will wait only 10 minutes before leaving to eat elsewhere, what is the probability that they have dinner at the health-food restaurant?

a. f(x,y) = $\begin{cases} 1 & 5 \le x \le 6, 5 \le y \le 6 \\ 0 & otherwise \end{cases}$

since $f_x(x) = 1$, $f_y(y) = 1$ for $5 \le x \le 6$, $5 \le y \le 6$

b. $P(5.25 \le X \le 5.75, 5.25 \le Y \le 5.75) = P(5.25 \le X \le 5.75) \cdot P(5.25 \le Y \le 5.75)$ = (by independence) (.5)(.5) = .25

c.



P((X,Y) $\in$ A) = $\iint_A 1dxdy$

= area of A = 1 – (area of I + area of II)

= $1-\frac{25}{36}=\frac{11}{36}=.306$

8. A nut company markets cans of deluxe mixed nuts containing almonds, cashews, and peanuts. Suppose the net weight of each can is exactly 1 lb., but the weight contribution of each type of nut is random. Because the three weighs sum to 1, a joint probability model for any two gives all necessary information about the weight of the third type. Let the joint pdf for (*X*, *Y*) be

$$f_{X,Y}(x,y)=\begin{cases}24xy & 0\le x\le 1, 0\le y\le 1, x+y\le 1\\ 0 & \text{otherwise}\end{cases}$$

Are *X* and *Y* correlated? If not, what's the correlation coefficient of *X* and *Y*?

Cov(X,Y) = $-\frac{2}{75}$ and $\mu_x=\mu_y=\frac{2}{5}$.

E(X^2) = $\int_0^1 x^2\cdot f_x(x)dx$

$=12\int_0^1 x^3(1-x^2dx)=\frac{12}{60}=\frac{1}{5}$,

so Var (X) = $\frac{1}{5}-\frac{4}{25}=\frac{1}{25}$

Similarly, Var(Y) = $\frac{1}{25}$,

so $\rho_{X,Y}=\frac{-2/75}{\sqrt{\frac{1}{25}}\cdot\sqrt{\frac{1}{25}}}=-\frac{50}{75}=-.667$