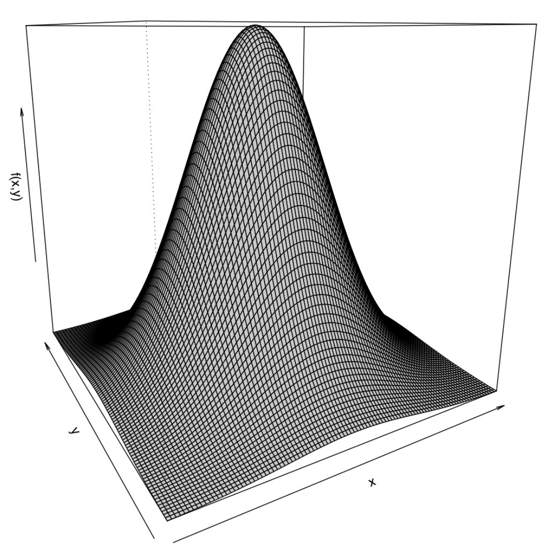
Chapter 5: JOINT PROBABILITY DISTRIBUTIONS

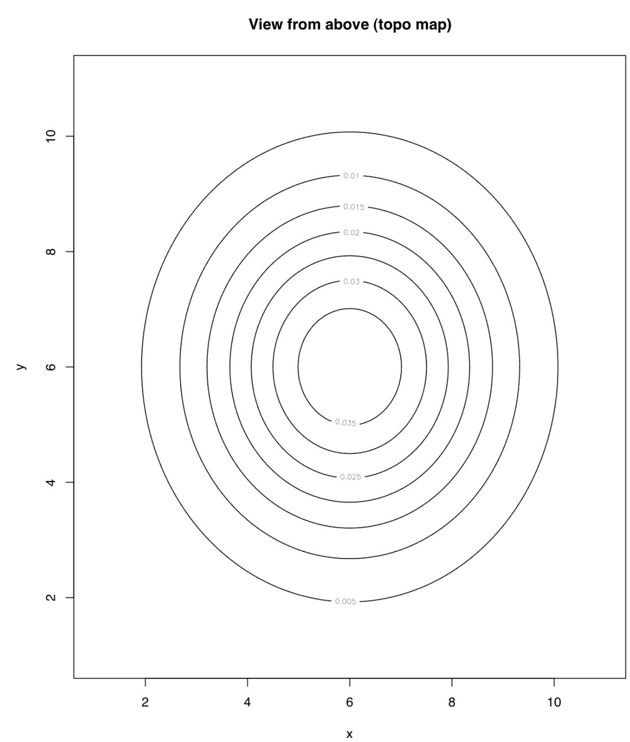
Part 3:
The Bivariate Normal

Section 5-3.2

Linear Functions of Random Variables

Section 5-4





The bivariate normal is kind of nifty because...

- The marginal distributions of X and Y are both univariate normal distributions.

- The conditional distribution of Y given X is a normal distribution.

- The conditional distribution of X given Y is a normal distribution.

- Linear combinations of X and Y (such as $Z = 2X + 4Y$) follow a normal distribution.

- It's normal almost any way you slice it.

- **Bivariate Normal Probability Density Function**

The parameters: $\mu_X, \mu_Y, \sigma_X, \sigma_Y, \rho$

$$f_{XY}(x,y) = \frac{1}{2\pi\sigma_X\sigma_Y\sqrt{(1-\rho^2)}} \times$$
$$exp\left\{\frac{-1}{2(1-\rho^2)}\left[\frac{(x-\mu_X)^2}{\sigma_X^2} - \frac{2\rho(x-\mu_X)(y-\mu_Y)}{\sigma_X\sigma_Y} + \frac{(y-\mu_Y)^2}{\sigma_Y^2}\right]\right\}$$

for $-\infty < x < \infty$ and $-\infty < y < \infty$, with

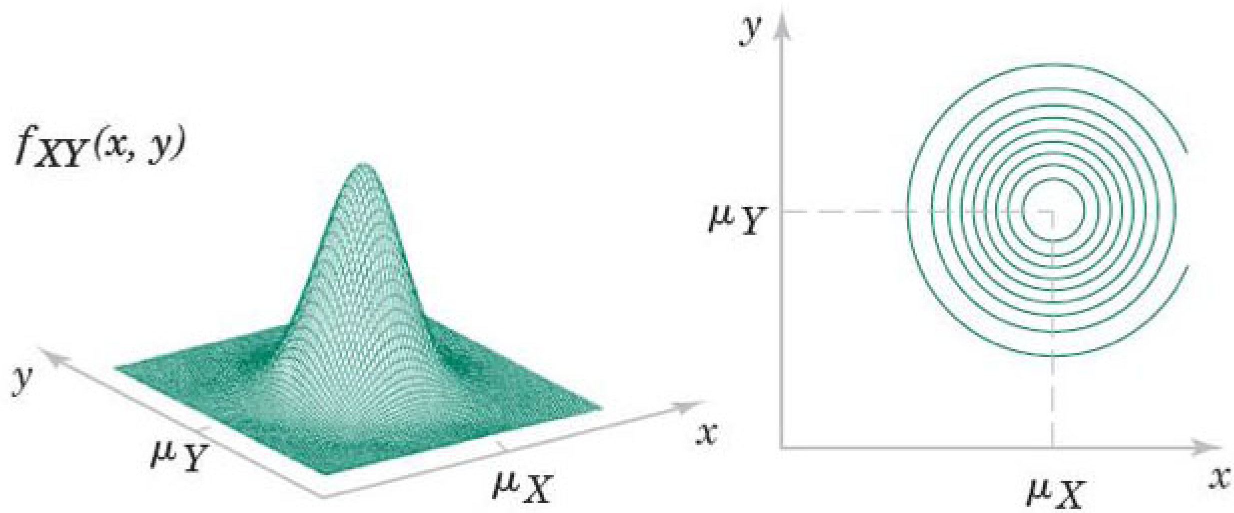
parameters $\sigma_X > 0$, $\sigma_Y > 0$,
$-\infty < \mu_X < \infty$, $-\infty < \mu_Y < \infty$,
and $-1 < \rho < 1$.

Where ρ is the correlation between X and Y.

The other parameters are the needed parameters for the marginal distributions of X and Y.

- **Bivariate Normal**

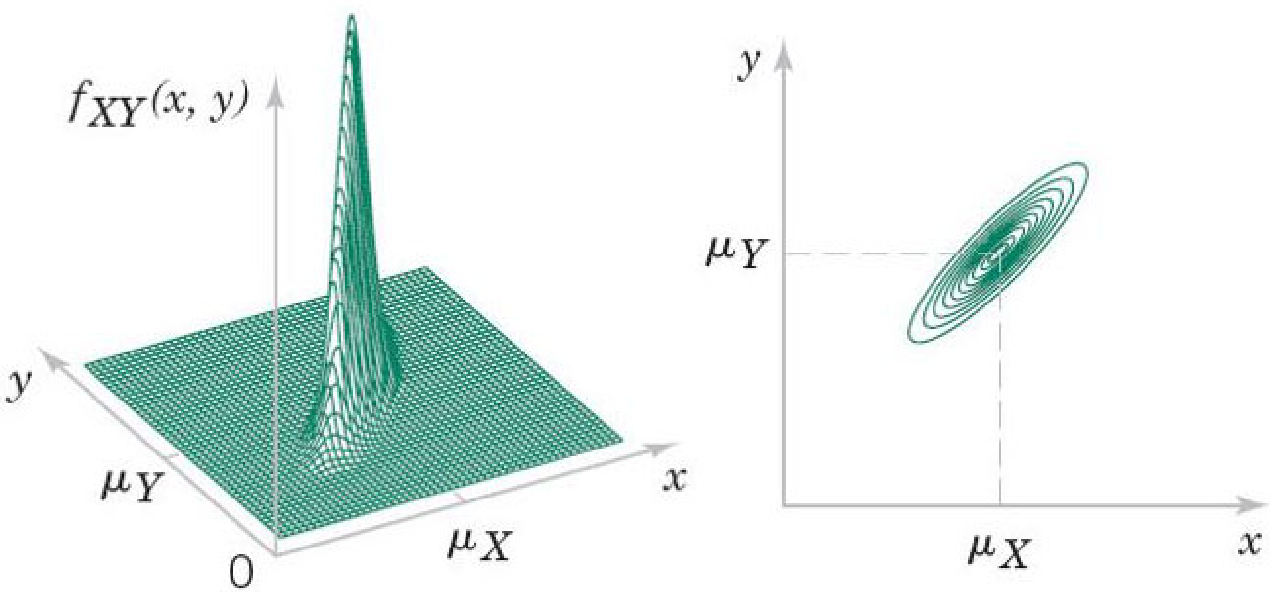
When X and Y are **independent**, the contour plot of the joint distribution looks like concentric circles (or ellipses, if they have different variances) with major/minor axes that are parallel/perpendicular to the x-axis:



The center of each circle or ellipse is at (μ_X, μ_Y).

- **Bivariate Normal**

When X and Y are **dependent**, the contour plot of the joint distribution looks like concentric diagonal ellipses, or concentric ellipses with major/minor axes that are NOT parallel/perpendicular to the x-axis:



The center of each ellipse is at (μ_X, μ_Y).

- **Marginal distributions of X and Y in the Bivariate Normal**

  Marginal distributions of X and Y are normal:

  $$X \sim N(\mu_X, \sigma_X^2) \quad \text{and} \quad Y \sim N(\mu_Y, \sigma_Y^2)$$

  Know how to take the parameters from the bivariate normal and calculate probabilities in a univariate X or Y problem.

- **Conditional distribution of $\boldsymbol{Y|x}$ in the Bivariate Normal**

  The conditional distribution of $Y|x$ is also normal:

  $$Y|x \sim N(\mu_{Y|x}, \sigma_{Y|x}^2)$$

$$Y|x \sim N(\mu_{Y|x}, \sigma^2_{Y|x})$$

where the "mean of $Y|x$" or $\mu_{Y|x}$ depends on the given x-value as

$$\mu_{Y|x} = \mu_Y + \rho \frac{\sigma_Y}{\sigma_X}(x - \mu_X)$$

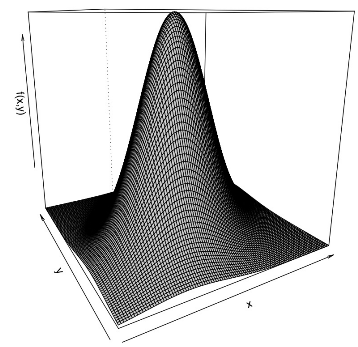
and "variance of $Y|x$" or $\sigma^2_{Y|x}$ depends on the correlation as

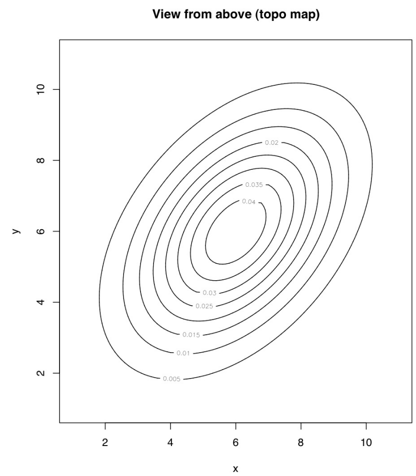
$$\sigma^2_{Y|x} = \sigma^2_Y(1 - \rho^2).$$

Know how to take the parameters from the bivariate normal and get a conditional distribution for a given x-value, and then calculate probabilities for the conditional distribution of $Y|x$ (which is a univariate distribution).

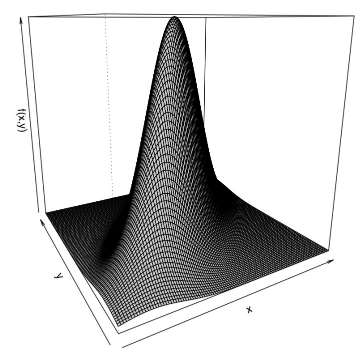
Remember that probabilities in the normal case will be found using the z-table.

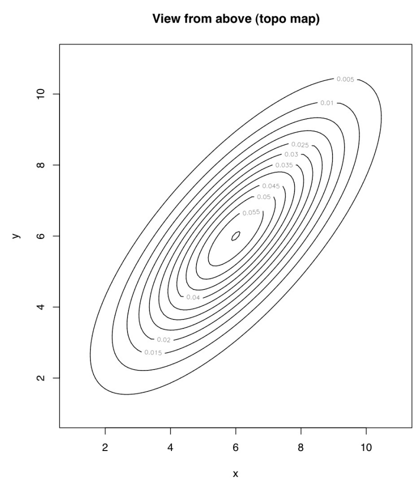
Notice what happens to the joint distribution (and conditional) as ρ gets closer to $+1$:



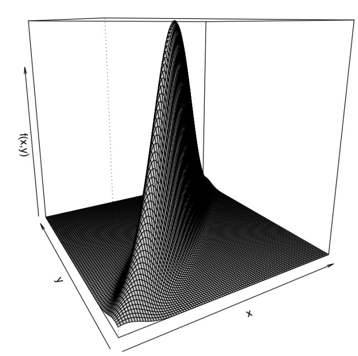


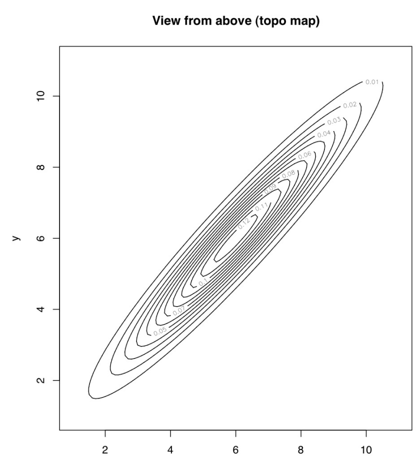
$\rho = 0.45$





$\rho = 0.75$





$\rho = 0.95$

As a last note on the bivariate normal...

Though $\rho = 0$ does not mean X and Y are independent in all cases, for the bivariate normal, this does hold.

For the Bivariate Normal, Zero Correlation Implies Independence

If X and Y have a bivariate normal distribution (so, we know the shape of the joint distribution), then with $\rho = 0$, we have X and Y as independent.

- **Example**: From book problem 5-54.

  Assume X and Y have a bivariate normal distribution with..
  $\mu_X = 120, \sigma_X = 5$
  $\mu_Y = 100, \sigma_Y = 2$
  $\rho = 0.6$

  Determine:
  (i) Marginal probability distribution of X.

  (ii) Conditional probability distribution of Y given that $X = 125$.

Linear Functions of Random Variables

Section 5-4

- **Linear Combination**

  Given random variables $X_1, X_2, \ldots, X_p$ and constants $c_1, c_2, \ldots, c_p$,

  $$Y = c_1X_1 + c_2X_2 + \cdots + c_pX_p$$

  is a <u>linear combination</u> of $X_1, X_2, \ldots, X_p$.

- **Mean of a Linear Function**

  If $Y = c_1X_1 + c_2X_2 + \cdots + c_pX_p$,

  $$E(Y) = c_1E(X_1)+c_2E(X_2)+\cdots+c_pE(X_p)$$

• Variance of a Linear Function

If $X_1, X_2, \ldots, X_p$ are random variables, and $Y = c_1X_1 + c_2X_2 + \cdots + c_pX_p$, then in general

$$V(Y) = c_1^2V(X_1) + c_2^2V(X_2) + \cdots + c_p^2V(X_p) + 2\sum_{i<j}\sum c_ic_j cov(X_i, X_j)$$

In this class, all our linear combinations of random variables will be done with independent random variables.

If $X_1, X_2, \ldots, X_p$ are **independent**, then

$$V(Y) = c_1^2V(X_1) + c_2^2V(X_2) + \cdots + c_p^2V(X_p)$$

The most common mistake for finding the variance of a linear combination is to forget to square the coefficients.

- **Example**: Semiconductor product (example 5-31)

  A semiconductor product consists of three layers. The variance of the thickness of the first, second, third layers are 25, 40, and 30 nanometers^2.

  What is the variance of the thickness of the final product?

  ANS:

- **Mean and Variance of an Average**

Suppose we randomly generate p observations from the a distribution with mean μ.

Thus, $E(X_i) = \mu$ for $i = 1, 2, \ldots, p$.

$$\text{Let } \bar{X} = \frac{(X_1 + X_2 + \cdots + X_p)}{p}$$

$$= \frac{1}{p}X_1 + \frac{1}{p}X_2 + \cdots \frac{1}{p}X_p$$

$\bar{X}$ is as mean and it is a <u>linear combination</u> of the p random variable we observed. Because $E(X_i) = \mu$ for $i = 1, 2, \ldots, p$ we have

$$E(\bar{X}) = \mu.$$

$\Rightarrow$ *The expected value of the average of p random variables, all with the same mean μ, is just μ again.*

If $X_1, X_2, \ldots, X_p$ are also <u>independent</u> and all with the same variance or $V(X_i) = \sigma^2$ then

$$V(\bar{X}) = \frac{\sigma^2}{p}$$

$\Rightarrow$ *The variance of the average of p identical random variables (i.e. $\frac{\sigma^2}{p}$) is smaller than the variance of a single random variable (i.e. σ^2).*

- **Reproductive Property of the Normal Distribution**

If $X_1, X_2, \ldots, X_p$ are <u>independent, normal random variables</u> with $E(X_i) = \mu_i$ and $V(X_i) = \sigma_i^2$ for $i = 1, 2, \ldots, p$,

$$Y = c_1X_1 + c_2X_2 + \cdots + c_pX_p$$

is a normal random variable with

$$\mu_Y = E(Y) = c_1\mu_1 + c_2\mu_2 + \cdots + c_p\mu_p$$

and

$$\sigma_Y^2 = V(Y) = c_1^2\sigma_1^2 + c_2^2\sigma_2^2 + \cdots + c_p^2\sigma_p^2$$

i.e. $Y \sim N(\mu_Y, \sigma_Y^2)$ as described above.

$\boxed{\text{A linear combination of normal r.v.'s is also normal.}}$

- **Example**: Weights of people

  Assume that the weights of individuals are independent and normally distributed with a mean of 160 pounds and a standard deviation of 30 pounds. Suppose that 25 people squeeze into an elevator that is designed to hold 4300 pounds.

  a) What is the probability that the load exceeds the design limit?

- **SPECIAL CASE: Reproductive Property of the Normal Distribution for a Random Sample**

  If the $X_1, X_2, \ldots, X_p$ are each **drawn independently from the same normal** distribution, or by notation $X_i \sim N(\mu, \sigma^2)$ for all i, then

  $$\bar{X} \sim N(\mu, \frac{\sigma^2}{p}) \quad \text{for any sample size } p.$$

  This results because $\bar{X}$ is a linear combination of normals in this situation.

- **Example**: Manufactured part (example 5-32)

  Let the random variables X_1 and X_2 denote the length and width, respectively of a manufactured part. Assume that X_1 is normal with $E(X_1) = 2$ cm and standard deviation 0.1 cm and that is X_2 is normal with $E(X_2) = 5$ cm and standard deviation 0.2 cm. We will assume X_1 andX_2 are independent.

  Find the probability that the perimeter exceed 14.5 cm.

  ANS: (next page)

Let Y represent the perimeter.

$$Y = 2X_1 + 2X_2$$

$$\mu_Y = E(Y) = 2(2) + 2(5) = 14$$

$$\begin{aligned}\sigma_Y^2 = V(Y) &= 2^2 V(X_1) + 2^2 V(X_2)\\ &= 4(0.1^2) + 4(0.2^2)\\ &= 0.04 + 0.16\\ &= 0.20\end{aligned}$$

Because both X_1 and X_2 were normal r.v.'s, the reproductive property of normal r.v.'s gives us...

$$Y \sim N(\mu_Y, \sigma_Y^2) \quad \text{or} \quad Y \sim N(14, 0.20)$$

$$\begin{aligned}P(Y > 14.5) &= P\left(\frac{Y-\mu_Y}{\sigma_Y} > \frac{14.5-14}{\sqrt{0.2}}\right)\\ &= P(Z > 1.12)\\ &= 0.13\end{aligned}$$