Statistics for Data Science

Unit 3 Homework: Probability Theory

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1) Gas Station Analytics

At a certain gas station, 40% of customers use regular gas (event R), 35% use mid-grade (event M), and 25% use premium (event P). Of the customers that use regular gas, 30% fill their tanks (Event F). Of the customers that use mid-grade gas, 60% fill their tanks, while of those that use premium, 50% fill their tanks. Assume that each customer is drawn independently from the entire pool of customers.

GIVENS

$P(E_R) = .4, P(E_M) = .35, P(E_P) = .25$

$P(E_F|E_R) = .3$

$P(E_F|E_M) = .6$

$P(E_F|E_P) = .5$

a) What is the probability that the next customer will request regular gas and fill the tank? e.g. E_R and E_F?

Using the multiplication rule for conditional probability,

$$P(A \cap B) = P(B) \cdot P(A|B) = P(A) \cdot P(B|A) = P(B \cap A)$$

$P(E_R \cap E_F) = P(E_R) \cdot P(E_F|E_R) = (.4)(.3) = .12$

b) What is the probability that the next customer will fill the tank? e.g. E_F?

Decomposing the probability of an event with partitioning

$$P(B) = P\big[(A_1 \cap B) \cup (A_2 \cap B) \cup \cdots \cup (A_N \cap B)\big] = \sum_{i=1}^{N} P(A_i \cap B)$$

Combined with the conditional probability rule

$$P(B|A) = \frac{P(A \cap B)}{P(A)} \iff P(A \cap B) = P(B|A)P(A)$$

Provides

$$P(B) = \sum_{i=1}^{N} P(A_i \cap B) = \sum_{i=1}^{N} P(B|A_i)P(A_i)$$

So that

$P(E_F) = P(E_R \cap E_F) + P(E_M \cap E_F) + P(E_P \cap E_F)$

Solving for the above

$P(E_R \cap E_F) = .12$

$P(E_M \cap E_F) = P(E_M) \cdot P(E_F|E_M) = (.35)(.6) = .21$

$P(E_P \cap E_F) = P(E_P) \cdot P(E_F|E_P) = (.25)(.5) = .125$

$P(E_F) = .12 + .21 + .125 = .455$

c) Given that the next customer fills the tank, what is the conditional probability that they use regular gas? e.g. What is the probability of E_R given E_F?

Using the conditional probibility rule

$$P(B|A) = \frac{P(A \cap B)}{P(A)}$$

$P(E_R|E_F) = \frac{P(E_F \cap E_R)}{P(E_F)} = \frac{.12}{.455} = .2637$

2) The Toy Bin

In a collection of toys, 1/2 are red, 1/2 are waterproof, and 1/3 are cool. 1/4 are red and waterproof. 1/6 are red and cool. 1/6 are waterproof and cool. 1/6 are neither red, nor waterproof, nor cool. Each toy has an equal chance of being selected.

GIVENS

$P(R) = \frac{1}{2}, P(W) = \frac{1}{2}, P(C) = \frac{1}{3}$

$P(R \cap W) = \frac{1}{4}, P(R \cap C) = \frac{1}{6}, P(W \cap C) = \frac{1}{6}$

$P(!R \cap !W \cap !C) = \frac{1}{6}$

$P(R \cup W \cup C) = 1 - P(!R \cap !W \cap !C) = 1 - \frac{1}{6} = \frac{5}{6}$

a) Draw an area diagram to represent these events.

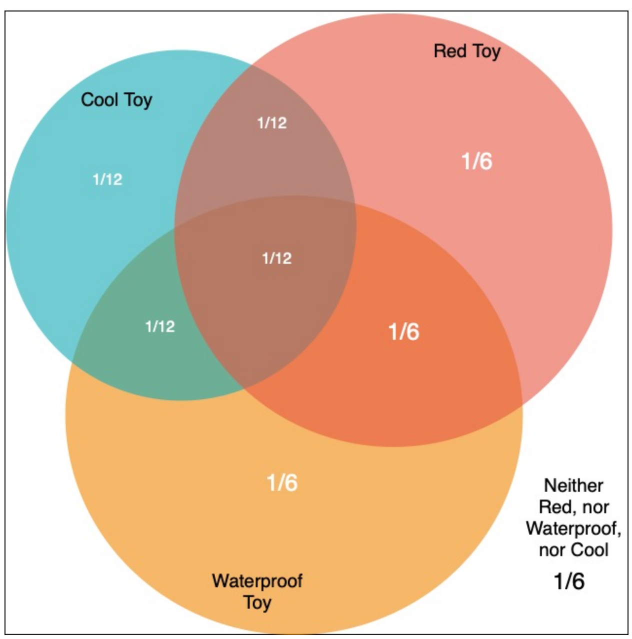

Figure 1: Venn diagram of Red Toys and Waterproof Toys and Cool Toys, oh my!

Note that 1/6 is neither red, nor waterproof, nor cool, and is therefore outside of the Venn diagram

b) What is the probability of getting a red, waterproof, cool toy? e.g. $P(R \cap W \cap C)$

Using the addition rule, for any three events A, B, and C,

$$P(A \cup B \cup C) = P(A) + P(B) + P(C) - P(A \cap B) - P(A \cap C) - P(B \cap C) + P(A \cap B \cap C)$$
$$P(R \cap W \cap C) = (1 - P(!R \cap !W \cap !C)) - P(R) - P(W) - P(C) + P(R \cap W) + P(R \cap C) + P(W \cap C)$$

$$= (1 - \frac{1}{6}) - \frac{1}{2} - \frac{1}{2} - \frac{1}{3} + \frac{1}{4} + \frac{1}{6} + \frac{1}{6}$$
$$= \frac{1}{12}$$

c) You pull out a toy at random and you observe only the color, noting that it is red. Conditional on just this information, what is the probability that the toy is not cool? e.g. $P(!C|R)$

Using the multiplication rule,

$$P(A \cap B) = P(B) \cdot P(A|B)$$

and Bayes' rule,

$$P(A|B) = \frac{P(B|A)P(A)}{P(B)}$$

and the Law of Total Probability,

$$P(B) = P(B|A) \cdot P(A) + P(B|!A) \cdot P(!A)$$

$$P(!C|R) = \frac{P(R|!C)P(!C)}{P(R)} \text{ Using Bayes' Rule}$$
$$= \frac{P(R) - P(R|C) \cdot P(C)}{P(R)} \text{ Using Law of Total Probability}$$
$$= \frac{P(R) - P(R \cap C)}{P(R)} \text{ Using Multiplication Rule}$$
$$= 1 - \frac{P(R \cap C)}{P(R)} = 1 - \frac{\frac{1}{6}}{\frac{1}{2}} = 1 - \frac{1}{3} = \frac{2}{3}$$

d) Given that a randomly selected toy is red or waterproof, what is the probability that it is cool? e.g $P(C|R \cup W)$

$$P(C|R \cup W) = \frac{P(C \cap (R \cup W))}{P(R \cup W)} = \frac{P(C \cap R) + P(C \cap W) - P(C \cap R \cap W)}{P(R) + P(W) - P(R \cap W)}$$
$$= \frac{\frac{1}{6} + \frac{1}{6} - \frac{1}{12}}{\frac{1}{2} + \frac{1}{2} - \frac{1}{4}} = \frac{\frac{1}{4}}{\frac{3}{4}} = \frac{1}{3}$$

3) On the Overlap of Two Events

Suppose for events A and B, $P(A) = 1/2$, $P(B) = 2/3$, but we have no more information about the events.

GIVENS

$P(A) = 1/2$, $P(B) = 2/3$

$$P(A \cap B) \leq P(A) \text{ and } P(A \cap B) \leq P(B)$$
$$P(A \cap B) \geq P(A) + P(B) - 1 \text{ from Boole's Inequality}$$

a) What are the maximum and minimum possible values for $P(A \cap B)$?

$$max\{P(A \cap B)\} = min\{P(A), P(B)\} = min\{\frac{1}{2}, \frac{2}{3}\} = 1/2$$

$$min\{P(A \cap B)\} = P(A) + P(B) - 1 = \frac{1}{2} + \frac{2}{3} - 1 = 1/6$$

b) What are the maximum and minimum possible values for $P(A|B)$?

$$P(A|B) = \frac{P(A \cap B)}{P(B)}$$

$$max\{P(A|B)\} = \frac{max\{P(A \cap B)\}}{P(B)} = \frac{\frac{1}{2}}{\frac{2}{3}} = 3/4$$

$$min\{P(A|B)\} = \frac{min\{P(A \cap B)\}}{P(B)} = \frac{\frac{1}{6}}{\frac{2}{3}} = 1/4$$

4) Can't Please Everyone!

Among Berkeley students who have completed w203, 3/4 like statistics. Among Berkeley students who have not completed w203, only 1/4 like statistics. Assume that only 1 out of 100 Berkeley students completes w203. Given that a Berkeley student likes statistics, what is the probability that they have completed w203?

GIVENS

$P(L|C) = 3/4, P(L|!C) = 1/4, P(C) = 1/100$

$P(!C) = 99/100$

$$P(C|L) = \frac{P(L|C) \cdot P(C)}{P(L)} \text{ (from Bayes' Rule)}$$

$$P(L) = P(L|C) \cdot P(C) + P(L|!C) \cdot P(!C) \text{ (from Law of Total Probability)}$$

$$P(C|L) = \frac{P(L|C) \cdot P(C)}{P(L|C) \cdot P(C) + P(L|!C) \cdot P(!C)} = \frac{\frac{3}{4} \cdot \frac{1}{100}}{\frac{3}{4} \cdot \frac{1}{100} + \frac{1}{4} \cdot \frac{99}{100}} = \frac{\frac{3}{400}}{\frac{102}{400}} = \frac{3}{102} = 0.0294$$