Math 463/563
Homework #4 - Solutions

1. Let X be a continuous random variable with probability density function

$$f(x) = \begin{cases} \frac{8}{x^3} & \text{if } x \geq 2 \\ 0 & \text{otherwise} \end{cases}$$

Check that $f(x)$ is indeed a probability density function. Find $P(X > 5)$ and $E[X]$.

SOLUTION:

$$\int_{-\infty}^{\infty} f(x)dx = \int_{2}^{\infty} \frac{8dx}{x^3} = \left[\frac{8x^{-2}}{-2}\right]_2^{\infty} = 1$$

and $f(x)$ is indeed a probability density function. Now,

$$P(X > 5) = \int_5^{\infty} \frac{8dx}{x^3} = \left[\frac{8x^{-2}}{-2}\right]_5^{\infty} = \frac{4}{25} = 0.16$$

and

$$E[X] = \int_{-\infty}^{\infty} xf(x)dx = \int_2^{\infty} \frac{8dx}{x^2} = \left[\frac{8x^{-1}}{-1}\right]_2^{\infty} = 4$$

2. Let X be a continuous random variable with probability density function

$$f(x) = \begin{cases} c(x-1)^4 \text{ if } 1 < x < 2, \\ 0 \text{ otherwise} \end{cases}$$

wher c is a constant. Find c, and $E[X]$.

SOLUTION:

$$1 = \int_{-\infty}^{\infty} f(x)dx = \int_1^2 c(x-1)^4 dx = \left[\frac{c(x-1)^5}{5}\right]_1^2 = \frac{c}{5}$$

and therefore

$$c = 5$$

Now

$$E[X] = \int_{-\infty}^{\infty} xf(x)dx = \int_1^2 5x(x-1)^4 dx = \int_1^2 5[(x-1)+1](x-1)^4 dx$$

$$= \int_1^2 5(x-1)^5 dx + \int_1^2 5(x-1)^4 dx = \left[\frac{5(x-1)^6}{6}\right]_1^2 + 1 = \frac{5}{6} + 1 = \frac{11}{6}$$

3. Let X be a continuous random variable with probability density function

$$f(x) = \begin{cases} ax^2 + bx & \text{for } 0 \le x \le 1 \\ 0 & \text{otherwise} \end{cases}$$

where a and b are constants. Suppose $E[X] = 0.75$. Find a, b, $E[X^2]$ and $Var(X)$.

SOLUTION:

$$1 = \int_{-\infty}^{\infty} f(x)dx = \int_0^1 (ax^2 + bx)dx = \left[a\frac{x^3}{3} + b\frac{x^2}{2}\right]_0^1 = \frac{1}{3}a + \frac{1}{2}b$$

and

$$0.75 = E[X] = \int_{-\infty}^{\infty} xf(x)dx = \int_0^1 (ax^3 + bx^2)dx = \left[a\frac{x^4}{4} + b\frac{x^3}{3}\right]_0^1 = \frac{1}{4}a + \frac{1}{3}b$$

Thus $\left(\frac{1}{8} - \frac{1}{9}\right) a = \frac{3}{8} - \frac{1}{3} = \frac{1}{24}$ and

$$a = 3, \quad b = 0$$

Now

$$E[X^2] = \int_{-\infty}^{\infty} x^2 f(x)dx = \int_0^1 3x^4 dx = \left[\frac{3x^5}{5}\right]_0^1 = \frac{3}{5}$$

and

$$Var(X) = E[X^2] - E[X]^2 = 0.6 - 0.5625 = 0.0375$$

4. Suppose the cumulative distribution function of a random variable X is given by

$$F(x) = \begin{cases} 1 - (x+1)^{-2} \text{ if } x > 0, \\ 0 \text{ if } x \le 0. \end{cases}$$

Evaluate $P(1 < X < 3)$ and $E[X]$.

SOLUTION:

$$P(1 < X < 3) = \int_1^3 f(x)dx = \int_{-\infty}^3 f(x)dx - \int_{-\infty}^1 f(x)dx = F(3) - F(1) = \frac{15}{16} - \frac{3}{4} = \frac{3}{16}$$

5. If Y is an exponential random variable with parameter $\lambda = 3$, what is the probability that the roots of the equation

$$4x^2 + 4xY - Y + 6 = 0$$

are real?

SOLUTION: The roots $x_{1,2} = \frac{-4Y \pm \sqrt{16Y^2+16(Y-6)}}{8}$ are real if and only if

$$16Y^2 + 16(Y-6) \geq 0$$

So we need to find probability this

$$\begin{aligned} P(16Y^2 + 16(Y-6) \geq 0) &= P(Y^2 + Y - 6 \geq 0) = P((Y+3)\cdot(Y-2) \geq 0) \\ &= P(Y \leq -3) + P(Y \geq 2) = 0 + e^{-\lambda 2} = e^{-6} \end{aligned}$$

6. The *gamma function* $\Gamma(\alpha)$ is defined as

$$\Gamma(\alpha) = \int_0^\infty e^{-y} y^{\alpha-1} dy$$

for all $\alpha > 0$. Use integration by parts to prove that $\Gamma(\alpha+1) = \alpha\Gamma(\alpha)$. Compute $\Gamma(1)$ and show that $\Gamma(k) = (k-1)!$ for all positive integer k.

SOLUTION:

$$\begin{aligned} \Gamma(\alpha+1) = \int_0^\infty e^{-y} y^{\alpha} dy &= \int_0^\infty \left(-e^{-y}\right)' y^{\alpha} dy = \left(-e^{-y} y^{\alpha}\right)_0^\infty - \int_0^\infty \left(-e^{-y}\right)\left(y^{\alpha}\right)' dy \\ &= 0 + \int_0^\infty e^{-y} \alpha y^{\alpha-1} dy = \alpha\Gamma(\alpha) \end{aligned}$$

Now,

$$\Gamma(1) = \int_0^\infty e^{-y} dy = 1 = 0!$$

Thus $\Gamma(2) = 1 \cdot \Gamma(1) = 1!$, $\Gamma(3) = 2 \cdot \Gamma(2) = 2!$ and by induction, $\Gamma(k) = (k-1)!$ for all positive integer k.

7. Use the preceding exercise to show that if X is an exponential random variable with $\lambda > 0$,

$$E[X^k] = \frac{k!}{\lambda^k}$$

for all positive integer $k = 1, 2, \ldots$.

SOLUTION:

$$E[X^k] = \int_0^\infty x^k \lambda e^{-\lambda x} dx$$

Changing variable to $y = \lambda x$, get

$$E[X^k] = \int_0^\infty \left(\frac{y}{\lambda}\right)^k e^{-y} dy = \frac{1}{\lambda^k}\int_0^\infty e^{-y} y^{(k+1)-1} dy = \frac{\Gamma(k+1)}{\lambda^k} = \frac{k!}{\lambda^k}$$

8. A gamma distributed random variable with parameters (α, λ) is defined by its probability density function

$$f(x) = \begin{cases} \frac{1}{\Gamma(\alpha)}\lambda e^{-\lambda x}(\lambda x)^{\alpha-1} \text{ when } x \geq 0 \\ 0 \quad \text{otherwise,} \end{cases}$$

where $\Gamma(\alpha) = \int_0^\infty e^{-t}t^{\alpha-1}dt$. Suppose X is a gamma distributed random variable with parameters (α, λ), where $\alpha > 0$ and $\lambda > 0$. Compute $E[e^{-X}]$.

SOLUTION:

$$E[e^{-X}] = \frac{1}{\Gamma(\alpha)}\int_0^\infty \lambda e^{-x}e^{-\lambda x}(\lambda x)^{\alpha-1}dx = \frac{\lambda^\alpha}{\Gamma(\alpha)}\int_0^\infty e^{-(\lambda+1)x}x^{\alpha-1}dx$$

Let $y = (\lambda+1)x$, then

$$E[e^{-X}] = \frac{\lambda^\alpha}{\Gamma(\alpha)}\int_0^\infty e^{-y}\frac{y^{\alpha-1}}{(\lambda+1)^{\alpha-1}}\frac{dy}{\lambda+1} = \left(\frac{\lambda}{\lambda+1}\right)^\alpha \frac{1}{\Gamma(\alpha)}\int_0^\infty e^{-y}y^{\alpha-1}dy = \left(\frac{\lambda}{\lambda+1}\right)^\alpha$$

9. Let $f(t)$ be the probability density function, and $F(t)$ be the corresponding cumulative distribution function. Define the *hazard function* $h(t) = \frac{f(t)}{1-F(t)}$. Show that if X is an exponential random variable with parameter $\lambda > 0$, then its hazard function will be a constant

$$h(t) = \lambda$$

for all $t > 0$. Think of how this relates to the memorylessness property of exponential random variables.

SOLUTION: Here for $t > 0$, $f(t) = \lambda e^{-\lambda t}$ and $F(t) = 1 - e^{-\lambda t}$. Thus

$$h(t) = \frac{f(t)}{1-F(t)} = \frac{\lambda e^{-\lambda t}}{1-(1-e^{-\lambda t})} = \lambda$$

In other words, because of the memorylessness property, the hazard rate is constant at all times.