HW #6 Solutions

Exercise 3.2.10

a) Let X denote the random variable $(I_A+I_B)^2$. Then the possible values for X are 0, 1, 4 with corresponding probabilities:

$$\begin{aligned}
p(0) &= P(I_A=0 \text{ and } I_B=0) = P(I_A=0)P(I_B=0) = (1-P(A))(1-P(B)) \\
p(1) &= P(I_A=1 \text{ and } I_B=0) + P(I_A=0 \text{ and } I_B=1) \\
&= P(I_A=1)P(I_B=0) + P(I_A=0)P(I_B=1) = P(A) - 2P(A)P(B)) + P(B) \\
p(4) &= P(I_A=1 \text{ and } I_B=1) = P(I_A=1)P(I_B=1) = P(A)P(B).
\end{aligned}$$

b)

$$\begin{aligned}
E(I_A+I_B)^2 &= E(I_A^2+2I_AI_B+I_B^2) = E(I_A)+2E(I_A)E(I_B)+E(I_B) \\
&= P(A)+2P(A)P(B)+P(B)
\end{aligned}$$

Exercise 3.2.13

a) Let X_i denote the number showing up in the ith roll. Then

$$E(\sum_{i=1}^{10} X_i) = 10E(X_1) = 10(\frac{7}{2}) = 35.$$

b). Let T be the sum of the numbers in the first three rolls, S be the sum of the largest two numbers in the first three rolls, M be the minimum of the numbers in the first three rolls. Then

$$\begin{aligned}
E(S) &= E(T)-E(M) = 3E(X_1) - \sum_{j=1}^{6} P(M \geq j) = 3(\frac{7}{2}) - \sum_{j=1}^{6}(\frac{6-j+1}{6})^3 \\
&= \frac{203}{24} \approx 8.4583
\end{aligned}$$

Exercise 3.4.1

a) It follows from the binomial distribution that

$$P(\text{ exactly 5 heads appear in the first 9 tosses }) = \binom{9}{5}p^5(1-p)$$

b) It follows from the geometric distribution that

$$P(\text{ the first head appears on the 7th toss }) = (1-p)^6 p.$$

c) It follows from the negative binomial distribution that

$$P(\text{ the fifth head appears on the 12th toss }) = \binom{11}{4} p^4 (1-p)^7 p.$$

d) Let X and Y be the number of heads appear in the first 8 tosses and in the next 5 tosses, respectively. Then both X and Y have binomial distributions. By independence of the tosses, we have

$$P(X=Y) = \sum_{x=y} P(X=x, Y=y) = \sum_{x=0}^{5} \binom{8}{x} p^x (1-p)^{8-x} \binom{5}{x} p^x (1-p)^{5-x}$$

Exercise 3.4.3

Since X has the geometric distribution with the probability of success $p = \frac{1}{12}$, $E(X) = \frac{1}{p} = 12$.

Exercise 3.4.6

a) Let T_r denote the number if trials until the rth success in Bernoulli (p) trials. Then T_1 is the number of trials until the 1st success so that $T_1 - 1$ is the number of failures before the first success. Since T_1 has geometric distribution, we have

$$P(T_1 - 1 = k) = P(T_1 = k+1) = q^k p, \text{ for } k = 0, 1, 2, \cdots$$

In other words, $T_1 - 1$ has the same distribution has W.

b)

$$P(W > k) = \sum_{i=k+1}^{\infty} q^i p = p \sum_{i=k+1}^{\infty} q^i = p \frac{q^{k+1}}{1-q} = q^{k+1}$$

c) It follows from the expected value of the geometric distribution that

$$E(W) = E(T_1 - 1) = E(T_1) - 1 = \frac{1}{p} - 1 = \frac{q}{p}.$$

d) It follows from the variance of the geometric distribution as derived in the lecture that

$$Var(W) = Var(T_1) = \frac{q}{p^2}.$$

Problem # 6

Proof: By definition of conditional probability, we have

$$\begin{aligned} P(X = n+k | X > n) &= \frac{P(X = n+k \text{ and } X > n)}{P(X > n)} = \frac{P(X = n+k)}{P(X > n)} \\ &= \frac{q^{n+k-1} p}{q^n} = q^{k-1} p = P(X = k) \end{aligned}$$