**Theorem 1 (Method of Transformations)** *Let $X$ be a continuous random variable with density function $f_X$ and the set of possible values $A$. For the invertible function $h : A \to \mathbf{R}$, let $Y = h(X)$ be a random variable with the set of possible values $B = h(A) = \{h(a) : a \in A\}$. Suppose that the inverse of $y = h(x)$ is the function $x = h^{-1}(y)$, which is differentiable for all values of $y \in B$. Then $f_Y$, the density function of $Y$, is given by*

$$f_Y(y) \quad = \quad f_X(h^{-1}(y))|(h^{-1})'(y)|, \;\; y \in B.$$

Proof: Let $F_X$ and $F_Y$ be distribution functions of $X$ and $Y = h(X)$, respectively. Differentiability of $h^{-1}$ implies that it is continuous. Since a continuous invertible function is strictly monotone, $h^{-1}$ is either strictly increasing or strictly decreasing. If it is strictly increasing, $(h^{-1})'(y) > 0$ and $(h^{-1})'(y) = |(h^{-1})'(y)|$. Moreover, in this case, $h$ is also strictly increasing so

$$F_y(y) \quad = \quad P(h(X) \leq y) = P(X \leq h^{-1}(y)) = F_X(h^{-1}(y)).$$

Differentiating,

$$F_y'(y) \quad = \quad F_X'(h^{-1}(y))(h^{-1})'(y) = f_X(h^{-1}(y))|(h^{-1})'(y)|.$$

If $h^{-1}$ is strictly decreasing, $(h^{-1})'(y) < 0$ and hence $-(h^{-1})'(y) = |(h^{-1})'(y)|$. In this case, $h$ is also strictly decreasing and we get

$$F_y(y) \quad = \quad P(h(X) \leq y) = P(X \geq h^{-1}(y)) = 1 - F_X(h^{-1}(y)).$$

Differentiating,

$$F_y'(y) \quad = \quad -F_X'(h^{-1}(y))(h^{-1})'(y) = f_X(h^{-1}(y))|(h^{-1})'(y)|.$$

**Example 1:** Let $Y = X^3$. Then $h(x) = x^3 = y$, and $h^{-1}(y) = y^{1/3} = x$. Since

$$F_Y(y) = P[Y \leq y] \quad = \quad P[X^3 \leq y] = P[X \leq y^{1/3}] = y^{1/3},$$

the density is

$$f_Y(y) = \frac{d}{dy}F_Y(y) \quad = \quad \frac{d}{dy}y^{1/3} = (1/3)y^{-2/3}.$$

Applying the formula given in the theorem, we have

$$f_Y(y) \quad = \quad f_X(h^{-1}(y))|(h^{-1})'(y)|$$

where

$$\begin{aligned} h^{-1}(y) \quad &= \quad y^{1/3} = (x^3)^{1/3} = x \\ f_X(x) \quad &= \quad \frac{d}{dx}F_X(x) = \frac{d}{dx}x = 1, \end{aligned}$$

so

$$f_X(h^{-1}(y)) = 1.$$

Next, we must evaluate $(h^{-1})'(y)$,

$$(h^{-1})'(y) = \frac{d}{dy}(h^{-1})(y) = \frac{d}{dy}y^{1/3} = (1/3)y^{-2/3},$$

and the result is

$$f_Y(y) = f_X(h^{-1}(y))|(h^{-1})'(y)| = (1)(1/3)y^{-2/3} = (1/3)y^{-2/3}.$$

**Example 2:** Let $X$ be the exponential random variable with parameter $\lambda = 2$, the density function for $X$ is

$$f_X(x) = 2e^{-2x} \text{ if } x > 0,$$

and zero otherwise. Using the method of transformations, find the density function for $Y = \sqrt{X}$.
**Solution:**

$$f_Y(y) = f_X(h^{-1}(y))|(h^{-1})'(y)|,$$

where $h(x) = \sqrt{x}$, $h^{-1}(y) = y^2 = x$, so

$$f_X(h^{-1}(y)) = f_X(y^2) = 2e^{-2y^2},$$

and

$$(h^{-1})'(y) = \frac{d}{dy}y^2 = 2y.$$

The density is given by

$$f_Y(y) = 2e^{-2y^2}(2y) = 4ye^{-2y^2} \quad y > 0.$$

We may verify that this function is a density in the usual way:

$$\begin{aligned} 1 &= \int_0^\infty f_Y(y)dy = \int_0^\infty 4ye^{-2y^2}dy \\ &= -e^{-2y^2}\Big|_0^\infty. \end{aligned}$$

**Example 3:** Let $X$ be a random variable with density function

$$f_X(x) = \frac{1}{\pi(1+x^2)}.$$

Find the density function for $Y = \tan^{-1}(X)$.

**Solution:**

$$f_Y(y) = f_X(h^{-1}(y))|(h^{-1})'(y)|,$$

where $h(x) = \tan^{-1}(x)$, $h^{-1}(y) = \tan y = x$, so

$$f_X(h^{-1}(y)) = f_X(\tan y) = \frac{1}{\pi(1+\tan^2 y)} = \frac{1}{\pi}\cos^2 y,$$

and

$$(h^{-1})'(y) = \frac{d}{dy}\tan y = \sec^2 y.$$

The density is given by

$$f_Y(y) = \frac{1}{\pi}\cos^2 y \sec^2 y = \frac{1}{\pi} \quad -\frac{\pi}{2} < y < \frac{\pi}{2},$$

that is, $y$ is uniformly distributed on $(-\frac{\pi}{2}, \frac{\pi}{2})$.