Chapter 4
Continuous Random Variables and Probability Distributions

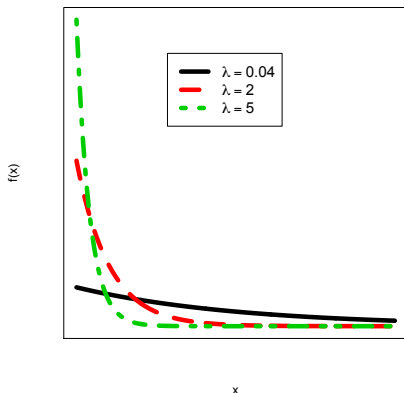**Part 3: The Exponential Distribution and the Poisson process**

Section 4.8 The Exponential Distribution

Exponential Distribution

- One place the exponential distribution arises is in the modeling of time or distance between occurrence of events.
    - Wait time between phone calls
    - Distance between recombination events on a DNA strand

- It is also used to model the distribution of component lifetime, or lifetime of a device. This is related to reliability, or length of time until a device fails.

- The shape and probabilities of an exponential distribution depends on only one parameter λ.

Exponential Distribution

- Three different exponential distributions are shown below for the $\lambda = 0.04$, $\lambda = 2$, $\lambda = 5$ values.


- The exponential distribution is connected to the Poisson distribution (through the Poisson process) and λ can be seen as a rate parameter, in terms of a long-term rate of occurrence per unit interval.

Exponential Distribution

Definition (Exponential Distribution)

The random variable X that equals the distance between successive events from a Poisson process with mean number of events $\lambda > 0$ per unit interval is an exponential random variable with parameter λ. The probability density function of X is

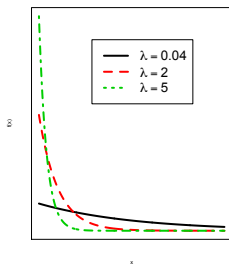$$f(x) = \lambda e^{-\lambda x} \qquad \text{for} \qquad 0 \leq x < \infty$$
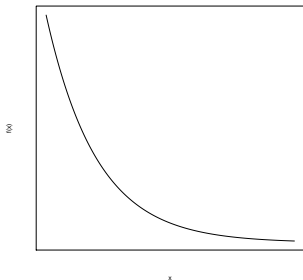
Exponential Distribution

Definition (Mean and Variance for Exponential Distribution)

For an exponential random variable with rate parameter λ,

$$\mu = E(X) = \frac{1}{\lambda} \qquad \text{and} \qquad \sigma^2 = V(X) = \frac{1}{\lambda^2}$$

- A smaller λ coincides with larger expected value or μ.
- A smaller λ coincides with more of the probability being pushed-out into the right tail (relative to the other distributions).

Exponential Distribution & the Poisson Process

- **The Exponential Distribution is connected to the Poisson process** (next slide)

- Specifically, the probability distribution of the <u>wait time</u> (continuous X) until the next event occurs in a Poisson process *IS* an exponential distribution.

$$X \equiv \text{wait time (a continuous r.v.)}$$
$$X \sim exponential(\lambda)$$
$$\text{where } \lambda \equiv \text{rate of the process}$$
$$\text{or } \lambda \equiv \text{rate of event occurrence}$$

- How long you have to wait for an event depends on how often events occur.

- A Poisson process is a certain type of situation to which the Poisson distribution applies.

The Poisson Process

Based on the name, a Poisson process must be related to counting the number of events that occur in a window of time or region in space...

In general, a Poisson process is a situation where the following conditions hold:

- **Events occur randomly**, but with a long-term average rate of λ per unit time. For example, $\lambda=10$ per hour or $\lambda=240$ per day
- One hour is just as likely as another to have an event. And **the likelihood of an event is completely independent of the past history**.
- The events are rare enough that in a very short time interval, there is a negligible chance of more than one event.

-H.B. Enderton (UCLA)

Any process that has these characteristics is called a Poisson process, and λ is called the *rate of the process*.

The Poisson Process

More on the Poisson process...

Suppose we can model the <u>number of calls</u> arriving during an **x-minute** time window with a Poisson distribution (we're modeling a count).

We assume that the calls arrive *completely at random* in time during the x-minutes.

Let the expected number of calls during a 1-minute interval be λ (a rate).

If $\lambda = 2$ (i.e. average of 2 calls per minute),
>   then the expected number of calls in 1-minute is 2 calls.
>   then the expected number of calls in 2-minutes is 4 calls.
> And generically,
>   **the expected number of calls in x-minutes is λx or $2x$ calls.**

The expected number of calls depends on two things... the length of the time interval, x, and λ, the rate of occurrence per time unit.

Exponential Distribution & the Poisson Process

- Longer fixed time intervals will have higher expected number of calls.

- **We can derive the exponential distribution as a** *wait time* **between events in a Poisson process**...

  Let N denote the number of calls in an x-minute time interval in a Poisson process with a rate parameter of λ events per minute. Then,

  $$N \sim Poisson(\lambda x).$$

  The number of calls during a <u>fixed time interval</u> of x-minutes has a Poisson distribution with mean of λx.

  We saw this in Section 3.9.

Exponential Distribution & the Poisson Process

Example (Radioactive pulses recorded by a Geiger counter)

From a time point of $t = 0$ minutes, we count the number of radioactive pulses recorded by a Geiger counter.

Let the process be a Poisson process with a rate parameter $\lambda = 6$ (i.e. 6 pulses per minute on average).

What is the **probability** that in a 0.5 minute interval, at least one pulse is received?

[THINK: How many pulses do you expect to receive in this fixed window 0.5 minutes?]

We'll need a probability distribution to calculate this probability.

Exponential Distribution & the Poisson Process

Example (Radioactive pulses recorded by a Geiger counter, cont.)

ANS:

Let N denote the number of pulses in a 0.5 minute interval.

$N \sim Poisson(0.5 * 6 = 3)$

(N is a Poisson r.v. with an expected value of 3)

$$P(N \geq 1) = 1 - P(N = 0) = 1 - \frac{e^{-3}(3)^0}{0!} = 0.950$$

- Notice the units were consistent (in minutes) for both parts of the problem... how the rate was given to you, and how the question was asked (check this).
- Notice I could have set this up using notation with Y rather than N, or ... $Y \sim Poisson(0.5 * 6 = 3)$
  because it's an arbitrary labeling.

Exponential Distribution & the Poisson Process

- **Back to deriving the exponential distribution...**

  HERE'S THE SWITCH TO WAIT TIME...

  Let X be the <u>wait time until the first call</u> from any start point in this setting.

    If you **wait** at least 3 minutes for a call, then
    NO CALL occurred in the first 3 minutes.

    If you **wait** at least 10 minutes for a call, then
    NO CALL occurred in the first 10 minutes.

    If you **wait** at least x minutes for a call, then
    NO CALL occurred in the first x minutes.

Exponential Distribution & the Poisson Process

- Let X be the <u>wait time until the first call</u> from any start point in this setting (a continuous random variable).

  Wait time random variable
    $\downarrow$

  $$P(X > x) = P(\text{you wait at least } x \text{ minutes for first call})$$
  $$= P(\text{there were no calls in the first } x \text{ minutes})$$
  $$= P(N = 0) \quad \text{where } N \sim Poisson(\lambda x)$$
  $$= \frac{e^{-(\lambda x)}(\lambda x)^0}{0!} = e^{-(\lambda x)}$$

- For the fixed time interval of λx, we can compute the probability of 0 events using the Poisson distribution.

Exponential Distribution & the Poisson Process

- We can now use this probability to derive the cumulative distribution function for X (the $r.v.$ representing the <u>wait time</u> until the first call)...

$$F(x) = P(X \leq x) = 1 - \boxed{P(X > x)} = 1 - P(N = 0) = 1 - e^{-\lambda x}$$

To get the pdf or $f(x)$ for X we simply take the derivative...

$$\begin{aligned} f(x) &= \frac{d}{dx} F(x) \\ &= \lambda e^{-\lambda x} \qquad \text{for} \qquad x \geq 0s \end{aligned}$$

It just so happens that this $f(x)$ <u>IS</u> an exponential probability density function with parameter λ, or mean $1/\lambda$ (see slides 4 & 5).

- Thus, **wait time in a Poisson process is modeled with the exponential distribution.**

Exponential Distribution & the Poisson Process

Example (Wait time in the Poisson process)

Suppose emergency alarms occur with no particular pattern (at random), but at an average rate of $\lambda = 12$ per day.

What is the expected wait time for an emergency alarm?

ANS:
Let $X \equiv$ wait time until first emergency alarm (from any start point).

X follows an exponential distribution with $\lambda = 12$.

$E(X) = \frac{1}{\lambda} = \frac{1}{12}$ day

Or we expect to wait $\frac{1}{12}$ of a day (2 hours) until the first emergency alarm.

Exponential Distribution & the Poisson Process

- This has all been phrased in terms of the 'first occurrence' from any start point, and so this holds for modeling the wait time until the 'next occurrence' from any point (same thing).

The random variable X that equals the distance between successive events of a Poisson process with mean $\lambda > 0$ is an **exponential random variable** with parameter λ. The probability density function of X is

$$f(x) = \lambda e^{-\lambda x} \quad \text{for} \quad 0 \leq x < \infty \qquad (4\text{-}14)$$

- The random variable X that represents the *distance between successive events* in a Poisson process with rate parameter λ is an exponential random variable with parameter λ, or mean $1/\lambda$.

Exponential Distribution & the Poisson Process

Example (Wait time in the Poisson process)

Suppose calls are received at a 24-hour "alcoholics anonymous" hotline according to a Poisson process with a rate $\lambda = 0.5$ calls per day.

What is the probability that they will wait more than 2 days for a call (for any given start point)?

ANS:
[We'll work it here using the exponential distribution]

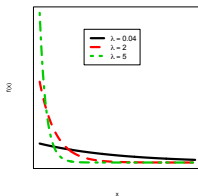[We'll work it here using the Poisson distribution]

Exponential Distribution & the Poisson Process

- The cdf, $F(x)$, for an exponential distribution...

$$P(X \leq x) = F(x) = \int_{-\infty}^{x} f(u)du = \boxed{1 - e^{-\lambda x}} \qquad \text{for } x \geq 0.$$

- Making the connection...
  For an exponential random variable X with parameter λ, we have $\mu = E(X) = \frac{1}{\lambda}$.

A smaller λ (rate of occurrence) coincides with a larger expected value of X (wait time).

A larger λ (rate of occurrence) coincides with a smaller expected value of X (wait time).

- **Lack of Memory Property of Exponential Distribution**

  For an exponential random variable X,

  $$P(X < t_1 + t_2 \mid X > t_1) = P(X < t_2)$$

  Given that you've already waited $t_1{=}5$ minutes for an event, what is the probability that you'll have an event in the next five minutes? (e.g. $t_2{=}5$)   It might seem like you should be 'due' for an event.

  But you're essentially starting from scratch at the 5 minute point...

  $$P(X < 10 \mid X > 5) = P(X < 5).$$

Exponential Distribution & the Poisson Process

Example (Particle detection p. 135 Example 4-22)

Let X denote the time between detections of a particle. Suppose detections of a particle follow a Poisson process with an average rate of occurrence of 10 detections every 14 <u>minutes</u> ($\lambda = 10/14$ per minute).

1) What is the probability that we detect a particle in the next 30 <u>seconds</u>?

ANS: Keep consistent units... 30 seconds = 0.5 minutes.

$$X \text{ is a wait time in a Poisson process.}$$

$$X \sim exponential(\lambda = 10/14)$$

$$P(X < 0.5) = F(0.5) = 1 - e^{-((10/14)*0.5)}$$

$$= 0.3003$$

Exponential Distribution & the Poisson Process

Example (Particle detection p. 135 Example 4-22, cont.)

2) Given that we have already waited for 3 minutes without a detection, what is probability that a particle is detected in the next 30 seconds?

ANS: $P(X < 3.5 | X > 3) = \frac{P(X<3.5 \text{ and } X>3)}{P(X>3)}$

$$= \frac{P(3<X<3.5)}{P(X>3)}$$

$$= \frac{F(3.5)-F(3)}{1-F(3)} = 0.3003 = P(X < 0.5)$$

Recall one of the Poisson process characteristics:
  The likelihood of an event is completely independent of the past history.