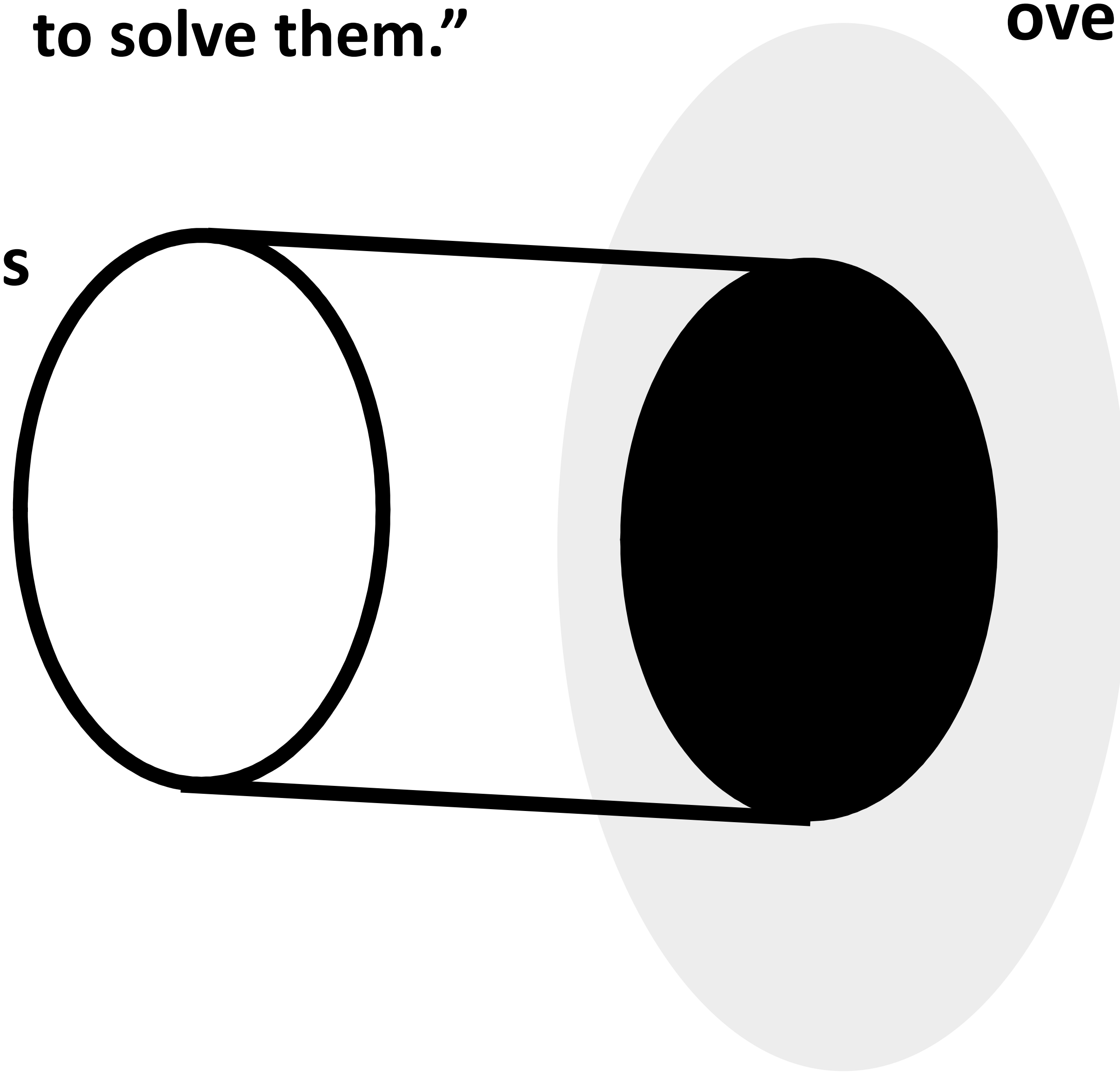
CS 154

Unrecognizability, Undecidability, Diagonalization

“There are more problems to solve than there are programs to solve them.”

Languages over {0,1}

Turing Machines



$f : A \rightarrow B$ is *not* onto $\Leftrightarrow$ $(\exists b \in B)(\forall a \in A)[f(a) \neq b]$

Let L be any set and 2^L be the power set of L

Theorem: There is *no* onto function from L to 2^L

No function from L to 2^L can "cover" all the elements in 2^L

No matter what the set L is, the power set 2^L *always* has strictly larger cardinality than L

Thm: There are *unrecognizable* languages

Suppose every language *is* recognizable.

Then for every language L' over {0,1}
there is a TM M such that L(M) = L'.

This means that the function
***f*(M) = L(M)**
from {Turing Machines} to {Languages}
is *onto*:

For every L' in {Languages}, there is an M in
{Turing Machines} such that f(M) = L'

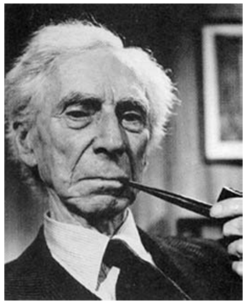
Thm: There are *unrecognizable* languages

Assuming every language is recog., there's an onto function
f: {Turing Machines} $\rightarrow$ {Languages}

{Turing Machines}	**{Languages over {0,1}}**
$\subseteq$	$\updownarrow$
{0,1}*	**{*Sets* of strings of 0s and 1s}**
$=$	$=$
Set S	**Set of all subsets of M: 2^S**

Since *f* is onto, there is also an onto *g* from S to 2^S.
But there is *no* onto function from S to 2^S. Contradiction!

This is an *extremely* generic argument!



Russell's Paradox in Set Theory

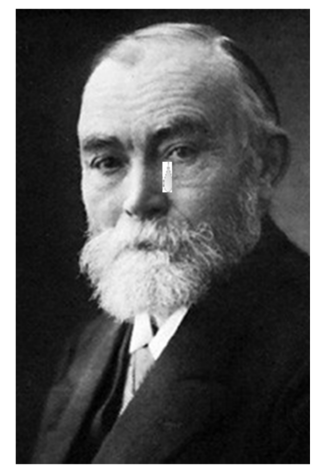
In the early 1900's, logicians were trying to define consistent foundations for mathematics.

Suppose X = "Universe of all possible sets"

Frege's Axiom: **Let $f : X \rightarrow \{0,1\}$**

Then $\{S \in X \mid f(S) = 1\}$ is a set.

Define $F = \{ S \in X \mid S \notin S \}$



Suppose $F \in F$. Then by definition, $F \notin F$.

So $F \notin F$ and by definition $F \in F$.

This logical system is inconsistent!

A Concrete Undecidable Problem: The Acceptance Problem for TMs

A_{TM} = { (M, w) | M is a TM that accepts string w }

Theorem [Turing'30s]
A_{TM} is recognizable but NOT decidable

A_{TM} = { (M,w) | M is a TM that accepts string w }

A_{TM} is undecidable: (proof by contradiction)

Suppose H is a machine that decides A_{TM}

$$H(\,(M,w)\,) = \begin{cases} \text{Accept} & \text{if M accepts w} \\ \text{Reject} & \text{if M does not accept w} \end{cases}$$

Define a new TM D as follows:

D(M): Run H on (M,M) and output the *opposite* of H

$$D(\,D\,) = \begin{cases} \text{Reject} & \text{if D accepts D} \\ \text{Accept} & \text{if D does not accept D} \end{cases}$$

Set M=D?

The table of outputs of H(x,y)

x \ y	w_1	w_2	w_3	w_4	…	D
M_1	accept	accept	accept	reject		accept
M_2	reject	accept	reject	reject		reject
M_3	accept	reject	reject	accept		accept
M_4	accept	reject	reject	reject		accept
:						
D	reject	reject	accept	accept		?

The behavior of D(x) is a *diagonal* on this table

	w_1	w_2	w_3	w_4	…	D
M_1	reject	accept	accept	reject		accept
M_2	reject	reject	reject	reject		reject
M_3	accept	reject	accept	accept		accept
M_4	accept	reject	reject	accept		accept
:						
D	reject	reject	accept	accept		?

D(x) outputs the *opposite* of H(x,x)

D(D) outputs the *opposite* of H(D,D)=D(D)

A_{TM} = { (M,w) | M is a TM that accepts string w }

A_{TM} is undecidable: (a constructive proof)

Let U be a machine that recognizes A_{TM}

$$U(\ (M,w)\ ) = \begin{cases} \text{Accept} & \text{if M accepts w} \\ \text{Rejects or loops} & \text{otherwise} \end{cases}$$

Define a new TM D_U as follows:

D_U(M): Run U on (M,M) until the simulation halts
Output the opposite answer

D_U (D_U) =

Reject if D_U accepts D_U
(i.e. if H(D_U , D_U) = Accept)

Accept if D_U rejects D_U
(i.e. if H(D_U , D_U) = Reject)

Loops if D_U loops on D_U
(i.e. if H(D_U , D_U) loops)

Note: There is no contradiction here!

D_U must loop on D_U

We have an input (D_U, D_U) which is *not* in A_{TM}
but U infinitely loops on (D_U, D_U)!

In summary:

Given the code of any machine U that *recognizes* A_{TM} (i.e. a Universal Turing Machine) we can effectively construct an input (D_U, D_U), where:

1. **(D_U, D_U) does not belong to A_{TM}**
2. **U *runs forever* on the input (D_U, D_U)**
3. **So U cannot decide A_{TM}**

Given any program that recognizes the Acceptance Problem, we can efficiently construct an input where the program hangs!

Theorem: A_{TM} is recognizable but NOT decidable

Corollary: $\neg A_{TM}$ is not recognizable

Proof: Suppose $\neg A_{TM}$ is recognizable.
Then $\neg A_{TM}$ and A_{TM} are both recognizable.
But that would mean they're both decidable…
… this is a contradiction!

The Halting Problem

HALT_{TM} = { (M,w) | M is a TM that halts on string w }

Theorem: HALT_{TM} is undecidable

Proof: Assume (for a contradiction) there is a TM H that decides HALT_{TM}

Idea: Use H to construct a TM M' that *decides* A_{TM}

M'(M,w): Run H(M,w)

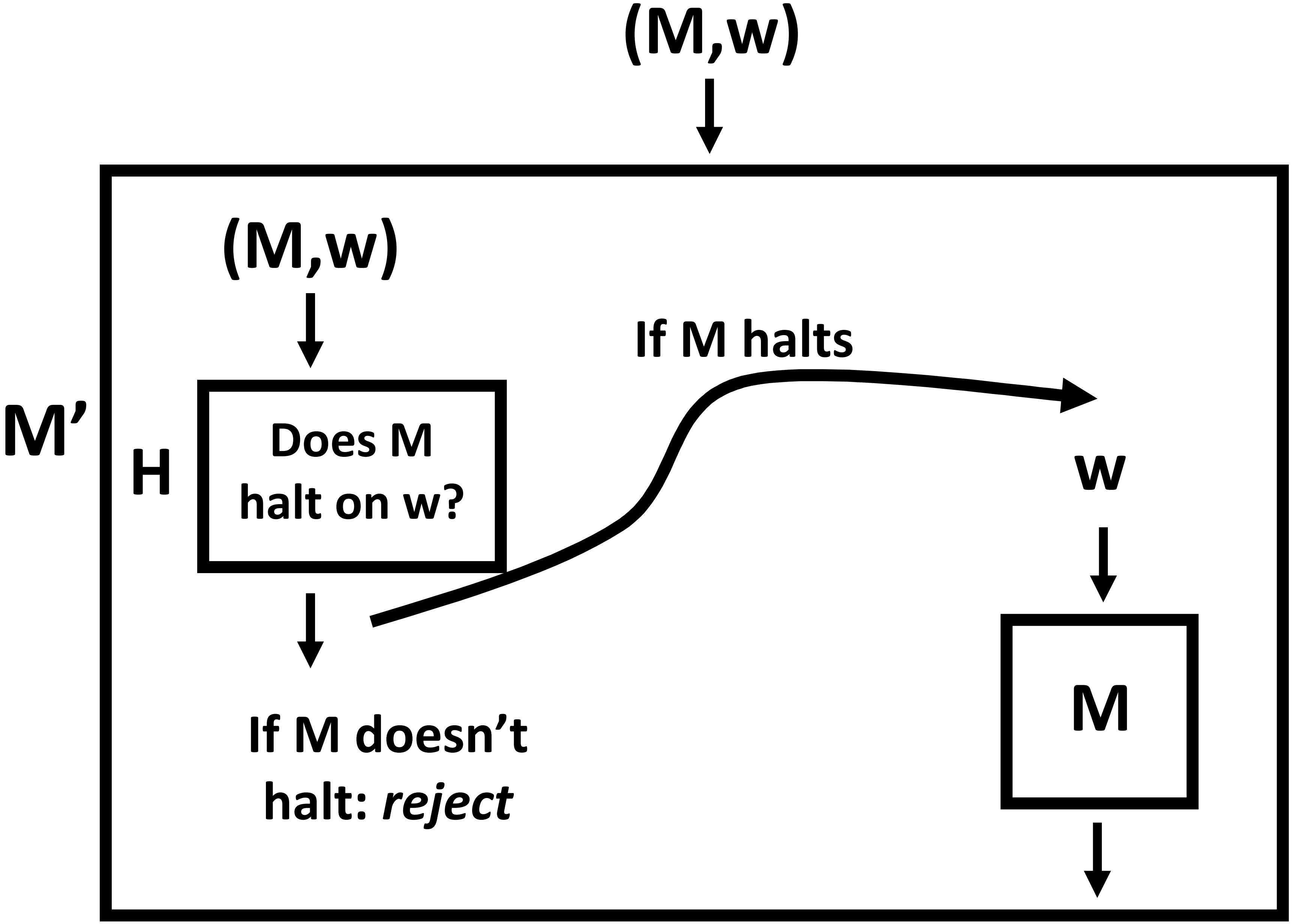
If H rejects then *reject*

If H accepts, run M on w until it halts:

If M accepts, then *accept*

If M rejects, then *reject*

Claim: If H exists, then M' decides A_{TM}


(M,w)
M'
(M,w)
H
Does M
halt on w?
If M halts
w
M
If M doesn't
halt: *reject*

Can often prove a language L is undecidable by proving: “if L is decidable, then so is A_{TM}”

We reduce A_{TM} to the language L

$$A_{TM} \leq L$$

L is “at least as difficult as” A_{TM}

Reducing from One Problem to Another

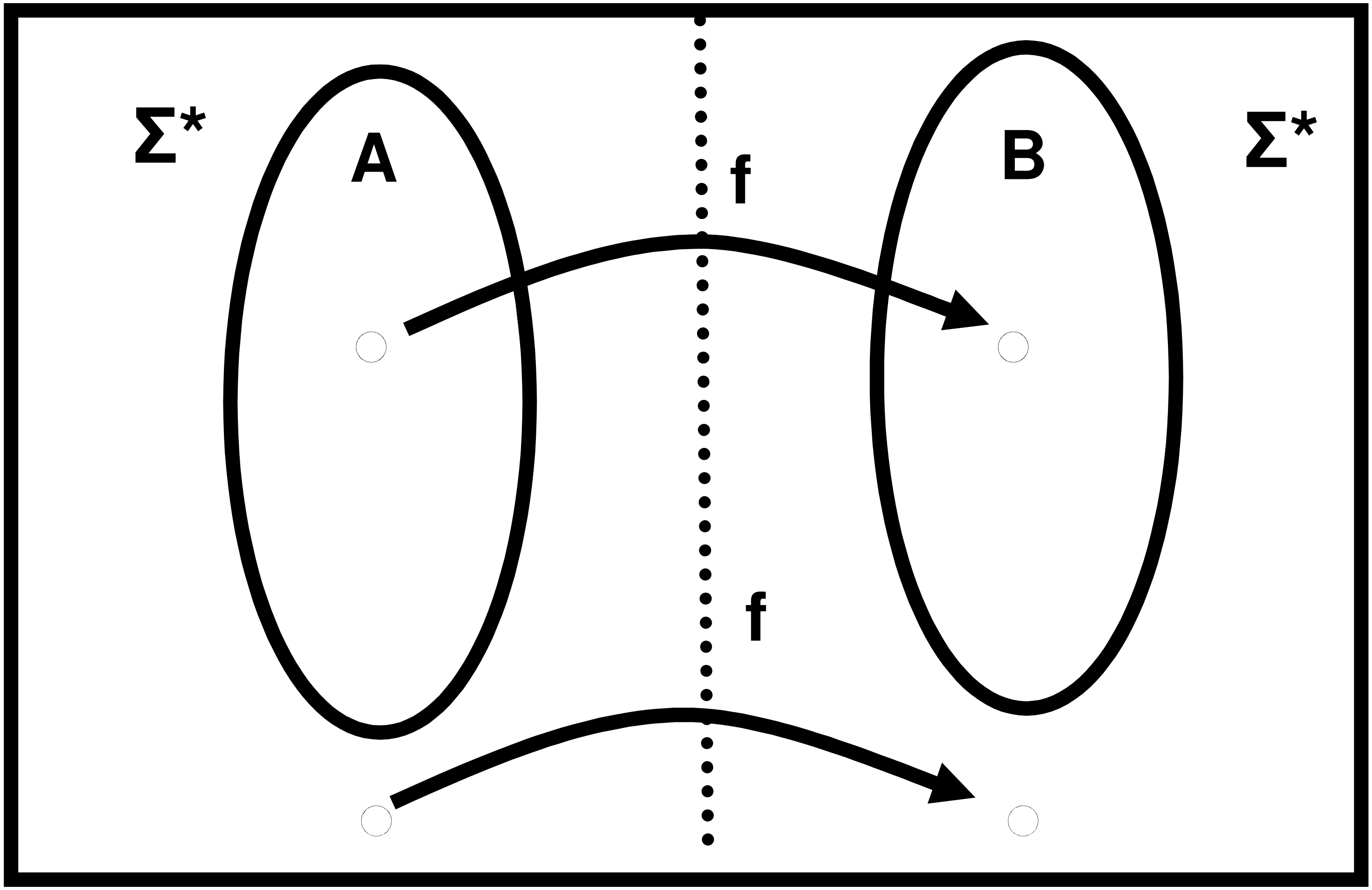
$f : \Sigma^* \rightarrow \Sigma^*$ is a computable function if
there is a Turing machine M that halts with
just f(w) written on its tape, for every input w

A language A is *mapping reducible* to language B,
written as $A \leq_m B$, if there is a computable
$f : \Sigma^* \rightarrow \Sigma^*$ such that for every w,

$$w \in A \Leftrightarrow f(w) \in B$$

f is called a mapping reduction
(or many-one reduction) from A to B

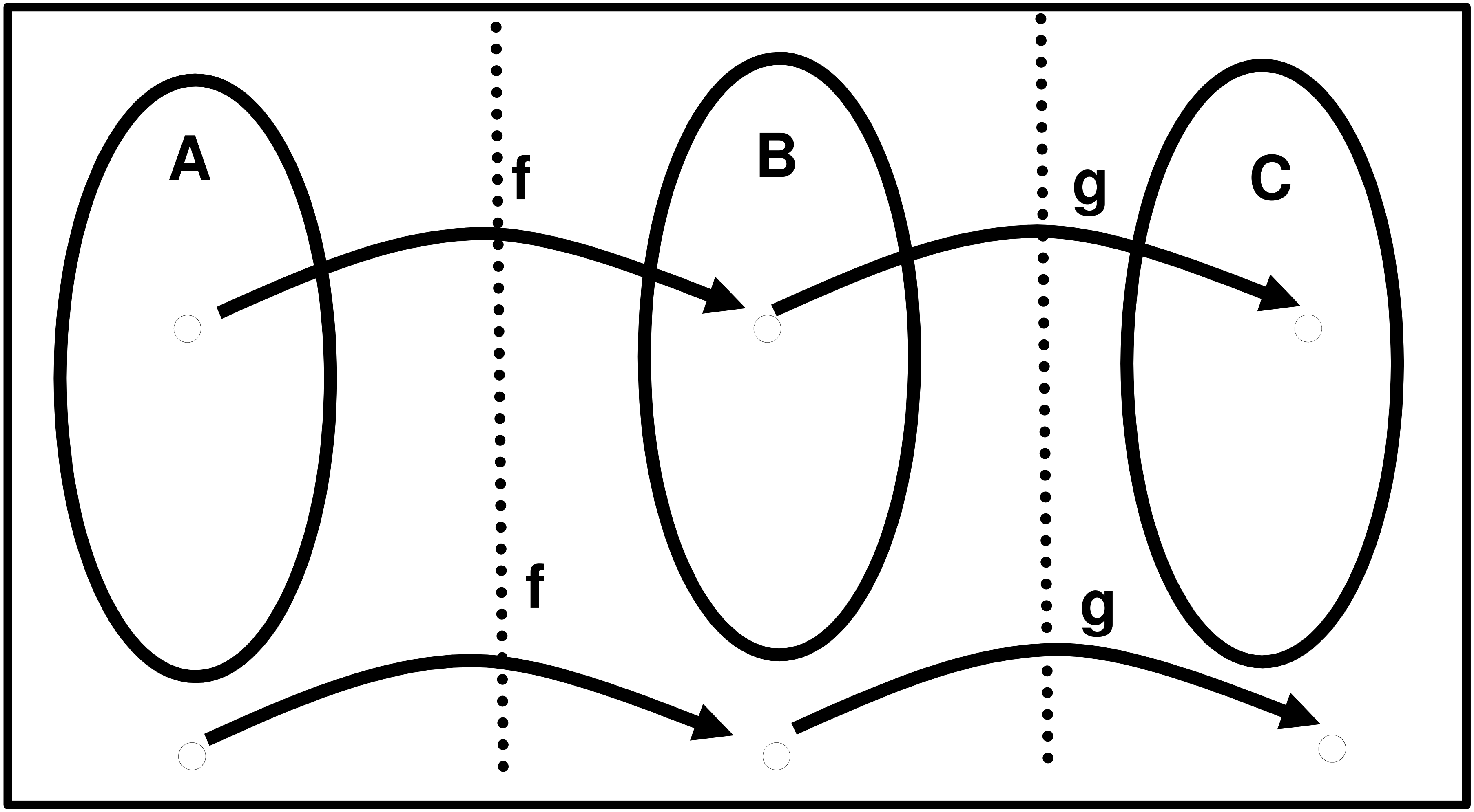
Let $f : \Sigma^* \rightarrow \Sigma^*$ be a computable function such that $w \in A \Leftrightarrow f(w) \in B$



Say: "A is mapping reducible to B"

Write: $A \leq_m B$

Theorem: If $A \leq_m B$ and $B \leq_m C$, then $A \leq_m C$



$$w \in A \Leftrightarrow f(w) \in B \Leftrightarrow g(f(w)) \in C$$

Theorem: If $A \leq_m B$ and B is decidable, then A is decidable

Proof: Suppose TM M decides B.
Let f be a mapping reduction from A to B

We build a machine M' for deciding A

M'(w):

1. Compute f(w)
2. Run M on f(w), output its answer

$w \in A \Leftrightarrow f(w) \in B$ so $w \in A \Rightarrow$ M' accepts w
$w \notin A \Rightarrow$ M' rejects w

Theorem: If A $\leq_m$ B and B is recognizable, then A is recognizable

Proof: Let M recognize B.
Let f be a mapping reduction from A to B

To *recognize* A, we build a machine M'

M'(w):
1. **Compute f(w)**
2. **Run M on f(w), output its answer if you ever receive one**

Theorem: If $A \leq_m B$ and B is decidable, then A is decidable

Corollary: If $A \leq_m B$ and A is undecidable, then B is undecidable

Theorem: If $A \leq_m B$ and B is recognizable, then A is recognizable

Corollary: If $A \leq_m B$ and A is unrecognizable, then B is unrecognizable

A mapping reduction from A_{TM} to $HALT_{TM}$

Theorem: $A_{TM} \leq_m HALT_{TM}$

f(z) := Decode z into a pair (M, w)

Construct a TM M' with the specification:

"M'(w) = Simulate M on w.

if M(w) accepts then *accept*

else *loop forever*"

Output (M', w)

We have $z \in A_{TM} \Leftrightarrow (M', w) \in HALT_{TM}$

Corollary: $HALT_{TM}$ is undecidable

Theorem: $A_{TM} \leq_m HALT_{TM}$

Corollary: $\neg A_{TM} \leq_m \neg HALT_{TM}$

Proof?

Corollary: $\neg HALT_{TM}$ **is unrecognizable!**

Proof: If $\neg HALT_{TM}$ **were recognizable, then** $\neg A_{TM}$ **would be recognizable…**

Theorem: $HALT_{TM} \leq_m A_{TM}$

Proof: Define the computable function

f(M, w) := Construct M' with the specification:
"M'(w) = If M(w) halts then *accept*
else *loop forever*"
Output (M', w)

Observe $(M, w) \in HALT_{TM} \iff (M', w) \in A_{TM}$