CS154

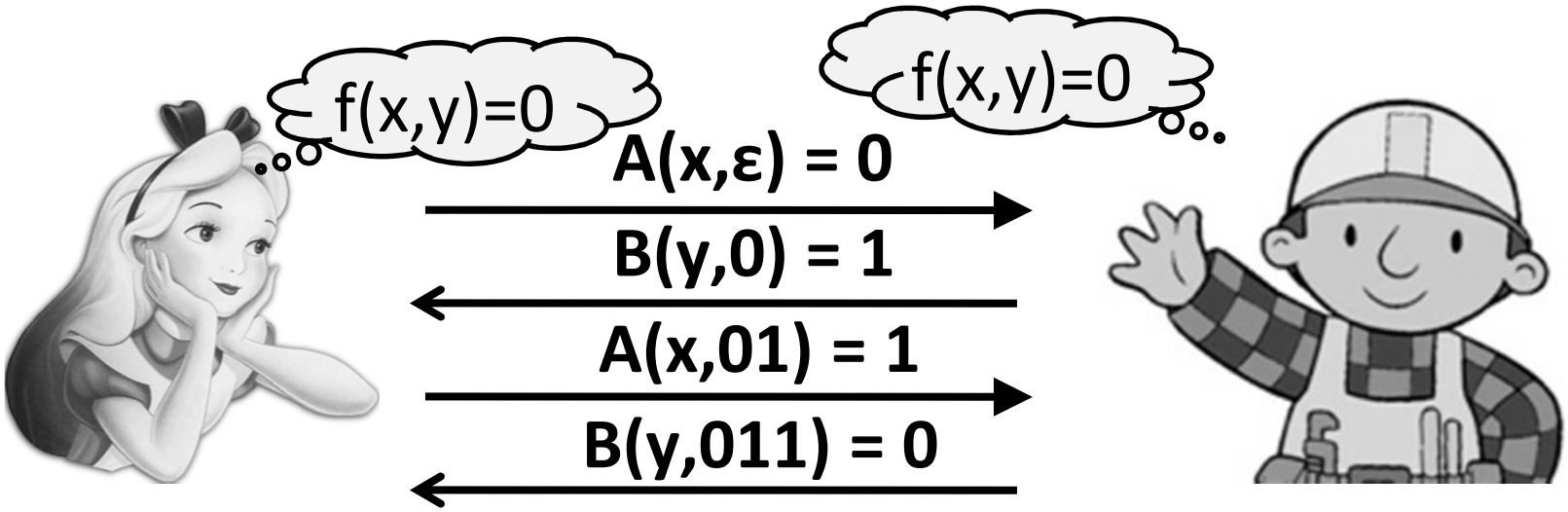
More on Communication Complexity, Start up Turing Machines

Communication Complexity

A theoretical model of distributed computing

- **Function $f : \{0,1\}^* \times \{0,1\}^* \rightarrow \{0,1\}$**
  - Two inputs, $x \in \{0,1\}^*$ and $y \in \{0,1\}^*$
  - **We assume $|x|=|y|=n$. Think of n as HUGE**
- **Two computers: Alice** and **Bob**
  - **Alice** *only* knows x, **Bob** *only* knows y
- **Goal: Compute $f(x, y)$ by communicating as few bits as possible between Alice and Bob**

We do not count computation cost. We *only* care about the number of bits communicated.



Def. A *protocol* for a function *f* is a pair of functions
***A*, *B* : {0,1}* × {0,1}* → {0, 1, STOP} with the semantics:**

On input (x, y), let r := 0, b_0 = ε.

While ($b_r \neq$ STOP),

$r++$

If r is odd, Alice sends $b_r = A(x, b_1 \cdots b_{r-1})$

else Bob sends $b_r = B(y, b_1 \cdots b_{r-1})$

Output b_{r-1}. Number of *rounds* $= r - 1$

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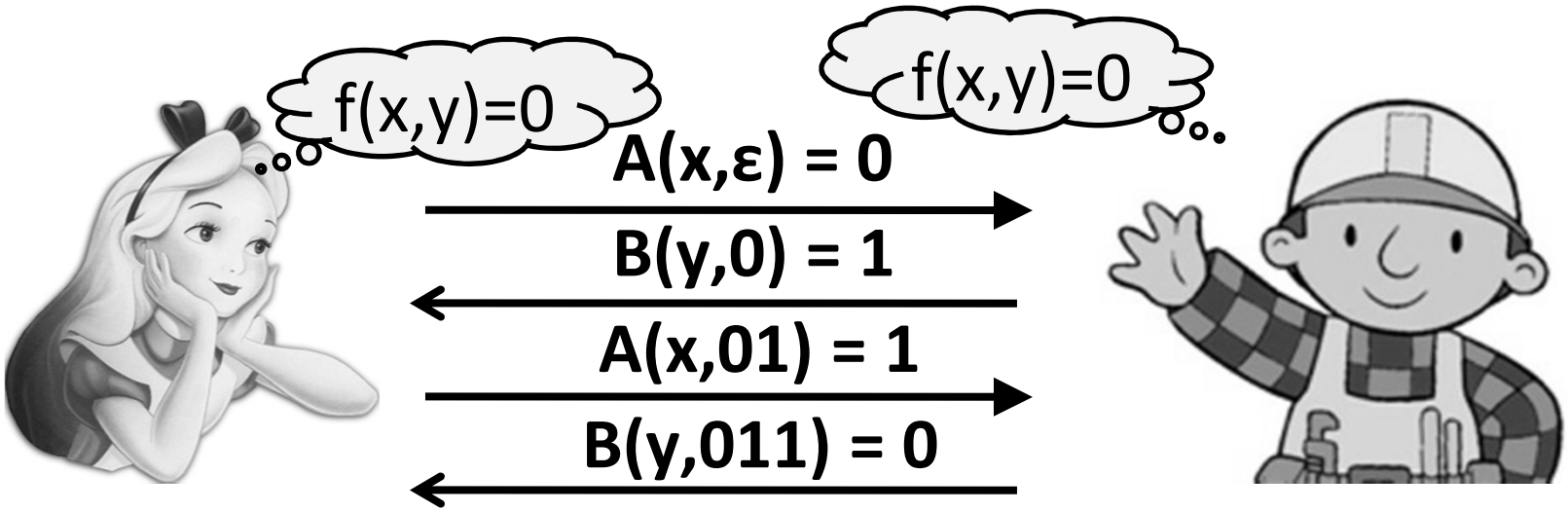

Def. The *cost of a protocol* P *for f on n-bit strings is*

$$\max_{x,y \in \{0,1\}^n} \text{[number of rounds in P to compute } f(x, y)\text{]}$$

The *communication complexity* of *f* on n-bit strings is the *minimum* cost over *all protocols* for *f* on n-bit strings

= the minimum number of rounds used by any protocol that computes $f(x, y)$, over all n-bit x, y

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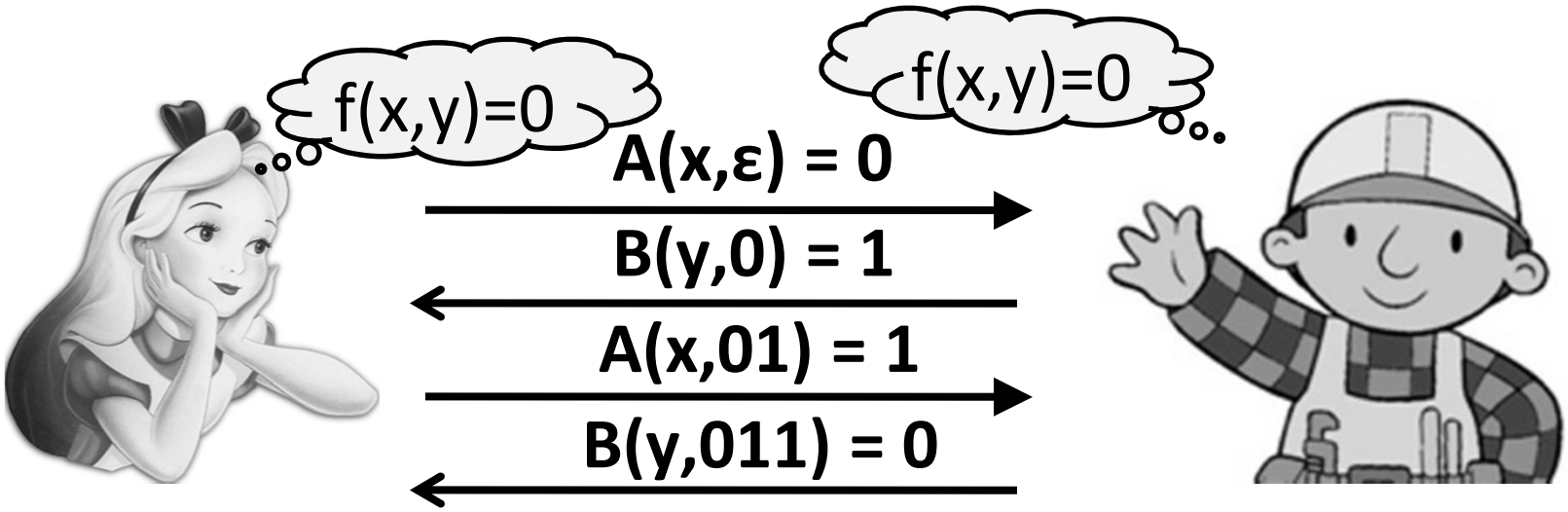


Example. Let $f : \{0,1\}^* \times \{0,1\}^* \rightarrow \{0,1\}$ be arbitrary

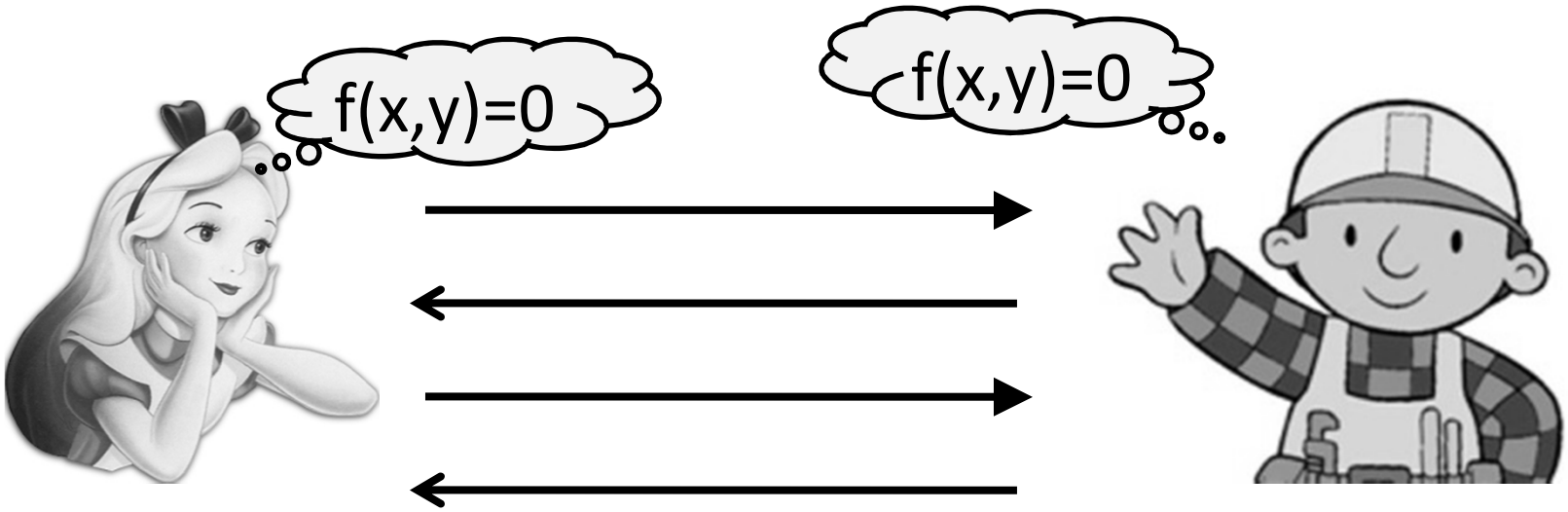
There is always a "trivial" protocol:

Alice sends the bits of her x in odd rounds

Bob sends the bits of his y in even rounds

After $2n$ rounds, they both know each other's input!

The communication complexity of every f is at most $2n$



x **y**

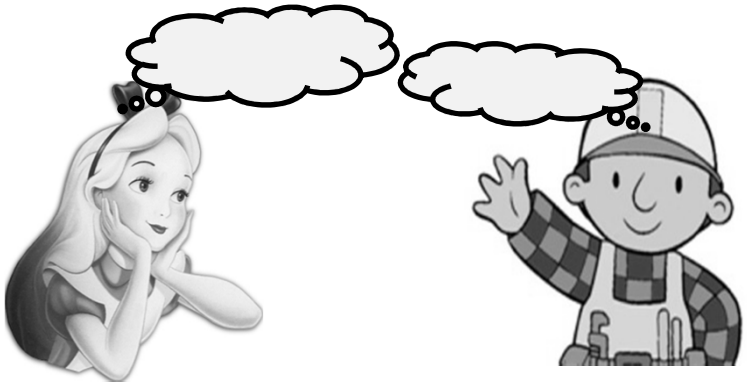
Example. EQUALS(x, y) = 1 $\Leftrightarrow$ $x = y$

What's a good protocol for computing EQUALS?

????

Communication complexity of EQUALS is at most $2n$

A Connection to Streaming Algorithms



x y

Let $L \subseteq \{0,1\}^*$

Def. f_L: $\{0,1\}^* \times \{0,1\}^* \rightarrow \{0,1\}$

for x, y with $|x|=|y|$ as:

$$f_L(x,y) = 1 \Leftrightarrow xy \in L$$

Examples:

L = { z | z has an odd number of 1s}

$\Rightarrow f_L(x,y)$ = PARITY(x,y) = $\sum_i x_i + \sum_i y_i$ mod 2

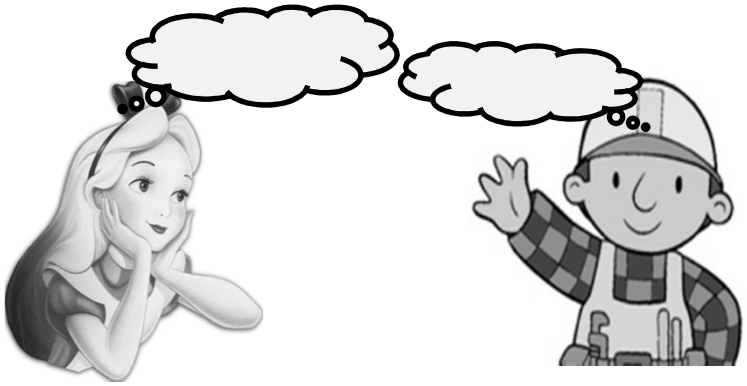
L = { z | z has more 1s than 0s}

$\Rightarrow f_L(x,y)$ = MAJORITY(x,y)

L = { zz | z $\in$ {0,1}*}

$\Rightarrow f_L(x,y)$ = EQUALS(x,y)

A Connection to Streaming Algorithms



x y

Let $L \subseteq \{0,1\}^*$

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for x, y with $|x|=|y|$ as:

$$f_L(x,y) = 1 \Leftrightarrow xy \in L$$

Theorem: If L has a streaming algorithm using $\leq s$ space, then the comm. complexity of f_L is at most $4s + 5$.

Proof: Alice runs streaming algorithm A on x.

Sends the *memory content* of A: this is s bits of space

Bob starts up A with that memory content, runs A on y.

Gets an output bit, sends to Alice.

(…why 4s+5 rounds?)

A Connection to Streaming Algorithms

Let $L \subseteq \{0,1\}^*$ Def. $f_L(x, y) = 1 \Leftrightarrow xy \in L$

Theorem: If L has a streaming algorithm using $\leq s$ space, then the comm. complexity of f_L is at most $4s + 5$.

Corollary: For every regular language L, the communication complexity of f_L is O(1).

Example: Comm. Complexity of PARITY is O(1)

Corollary: Comm. Complexity of MAJORITY is O(log n), because there's a streaming algorithm for {x : x has more 1's than 0's} with O(log n) space

What about the Comm. Complexity of EQUALS?

Communication Complexity of EQUALS

Theorem: The comm. complexity of EQUALS is $\Theta(n)$.
In particular, *every* protocol for EQUALS needs $\geq n$ bits of communication.

No communication protocol can do much better than "send your whole input"!

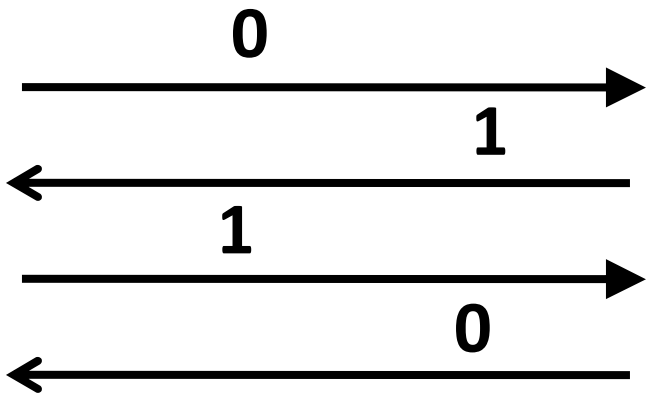
Corollary: L = {ww | w in {0,1}*} is not regular.

Moreover, every streaming algorithm for L needs $c\,n$ bits of memory, for some constant $c > 0$!

Communication Complexity of EQUALS

Theorem: The comm. complexity of EQUALS is $\Theta(n)$.
In particular, *every* protocol for EQUALS needs $\geq n$ bits of communication.

Idea: Consider all the possible ways A & B can communicate.
Definition: The *communication pattern* of a protocol on inputs (x, y) is the sequence of bits that Alice & Bob send.



x

Pattern: 0110

[redacted]

Communication Complexity of EQUALS

Theorem: The comm. complexity of EQUALS is $\Theta(n)$.
In particular, *every* protocol for EQUALS needs $\geq n$ bits of communication.

Proof: By contradiction. Suppose CC of EQUALS is $\leq n-1$.
Then there are $\leq 2^n - 1$ possible communication *patterns* of that protocol, over all pairs of inputs (x, y) with n bits each.

Claim: There are $x \neq y$ such that on both (x, x) and (y, y), the protocol of Alice and Bob uses the *same* pattern P.

Now, what is the communication pattern on (x, y)?

This pattern is also P! *(WHY?)*

So Alice & Bob *output the same bit* on (x, y) and (x, x).
But EQUALS(x, y) = 0 and EQUALS(x, x) = 1. *Contradiction!*

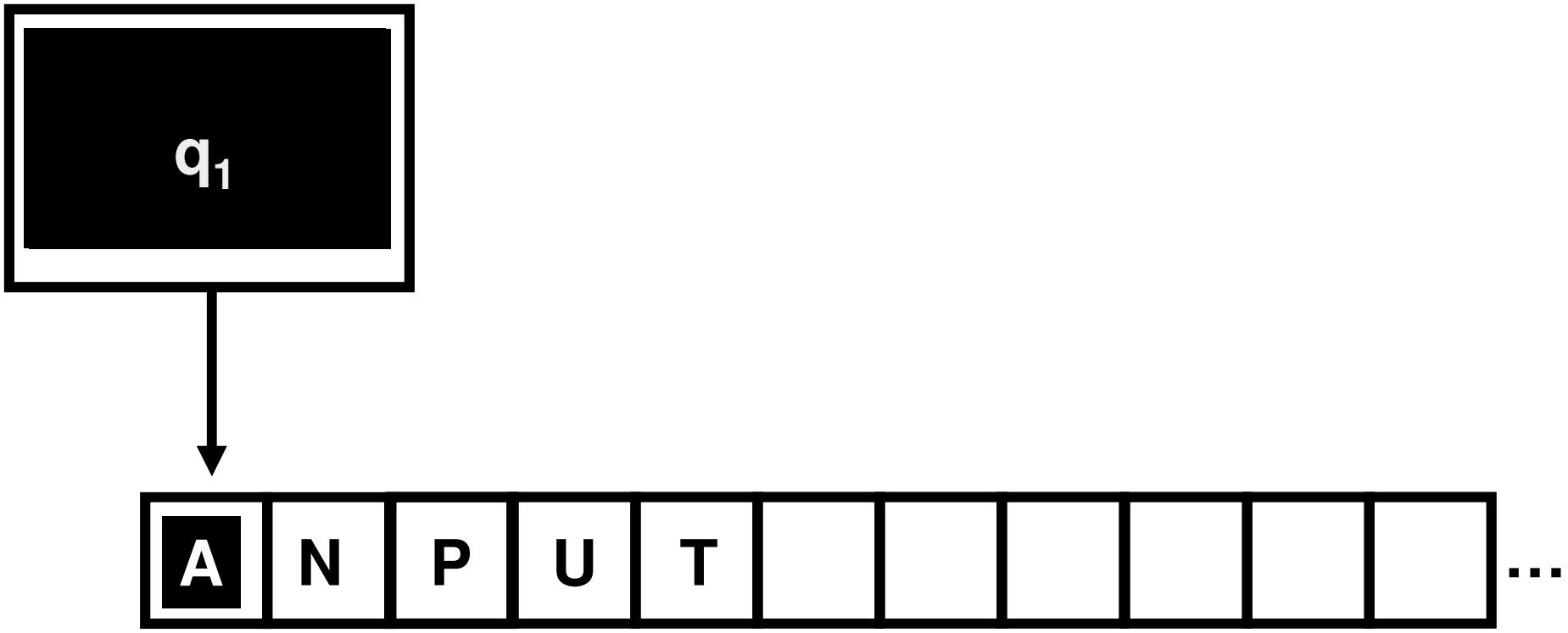
Randomized Protocols Help!

EQUALS needs $c\ n$ bits of communication, but…

Theorem: For all (x, y) of n bits each, there is a *randomized* protocol for EQUALS(x, y) using only O(log n) bits of communication, which works with probability 99.9%!

Turing Machines

Turing Machine (1936)



INFINITE REWRITABLE TAPE

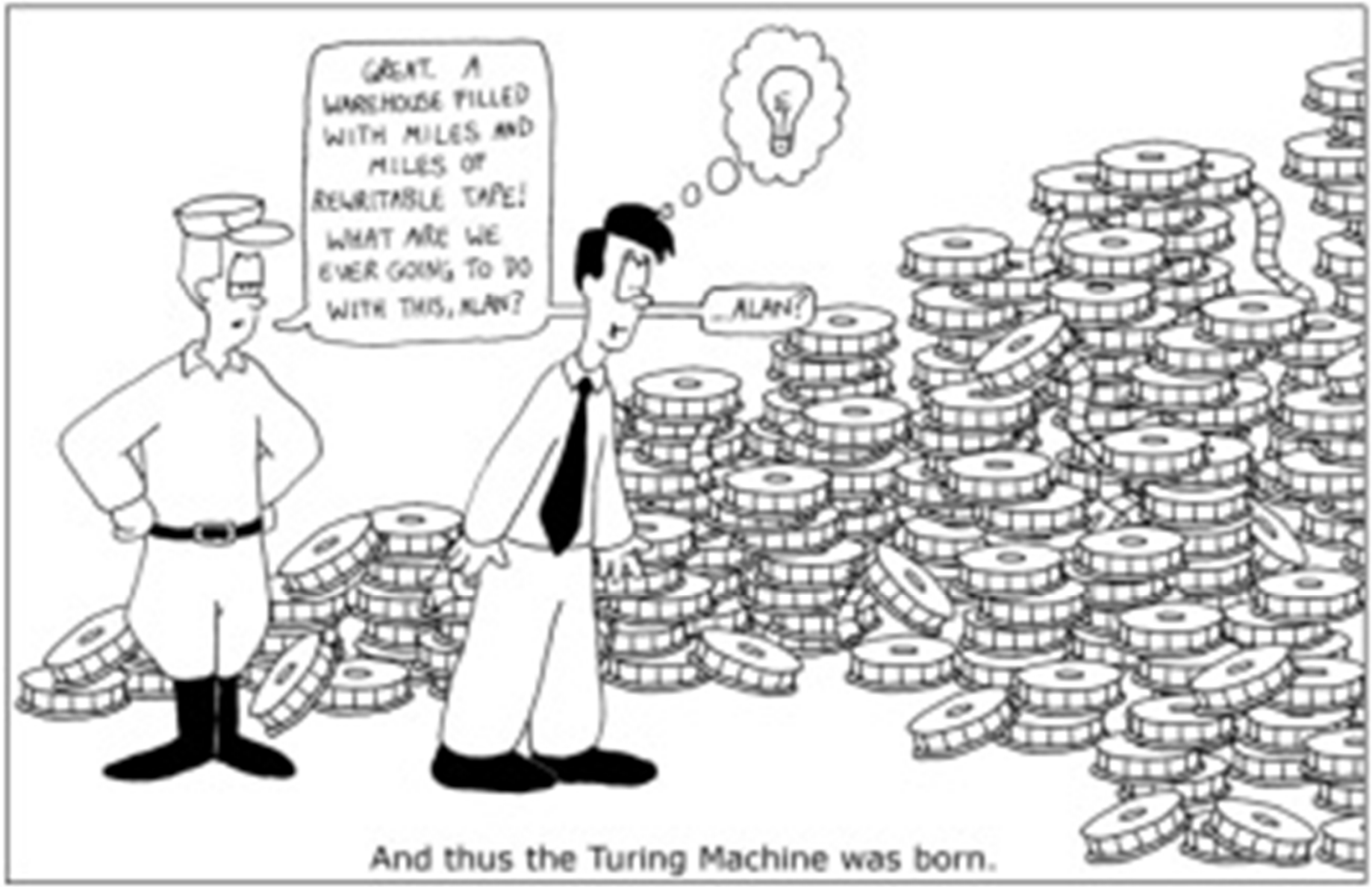
Turing Machine (1936)

ON COMPUTABLE NUMBERS, WITH AN APPLICATION TO THE ENTSCHEIDUNGSPROBLEM

By A. M. TURING.

[Received 28 May, 1936.—Read 12 November, 1936.]

The "computable" numbers may be described briefly as the real numbers whose expressions as a decimal are calculable by finite means. Although the subject of this paper is ostensibly the computable *numbers*, it is almost equally easy to define and investigate computable functions of an integral variable or a real or computable variable, computable predicates, and so forth. The fundamental problems involved are, however, the same in each case, and I have chosen the computable numbers for explicit treatment as involving the least cumbrous technique. I hope shortly to give an account of the relations of the computable numbers, functions, and so forth to one another. This will include a development


GREAT. A WAREHOUSE FILLED WITH MILES AND MILES OF REWRITABLE TAPE! WHAT ARE WE EVER GOING TO DO WITH THIS, ALAN?
...ALAN!
And thus the Turing Machine was born.

Turing Machines versus DFAs

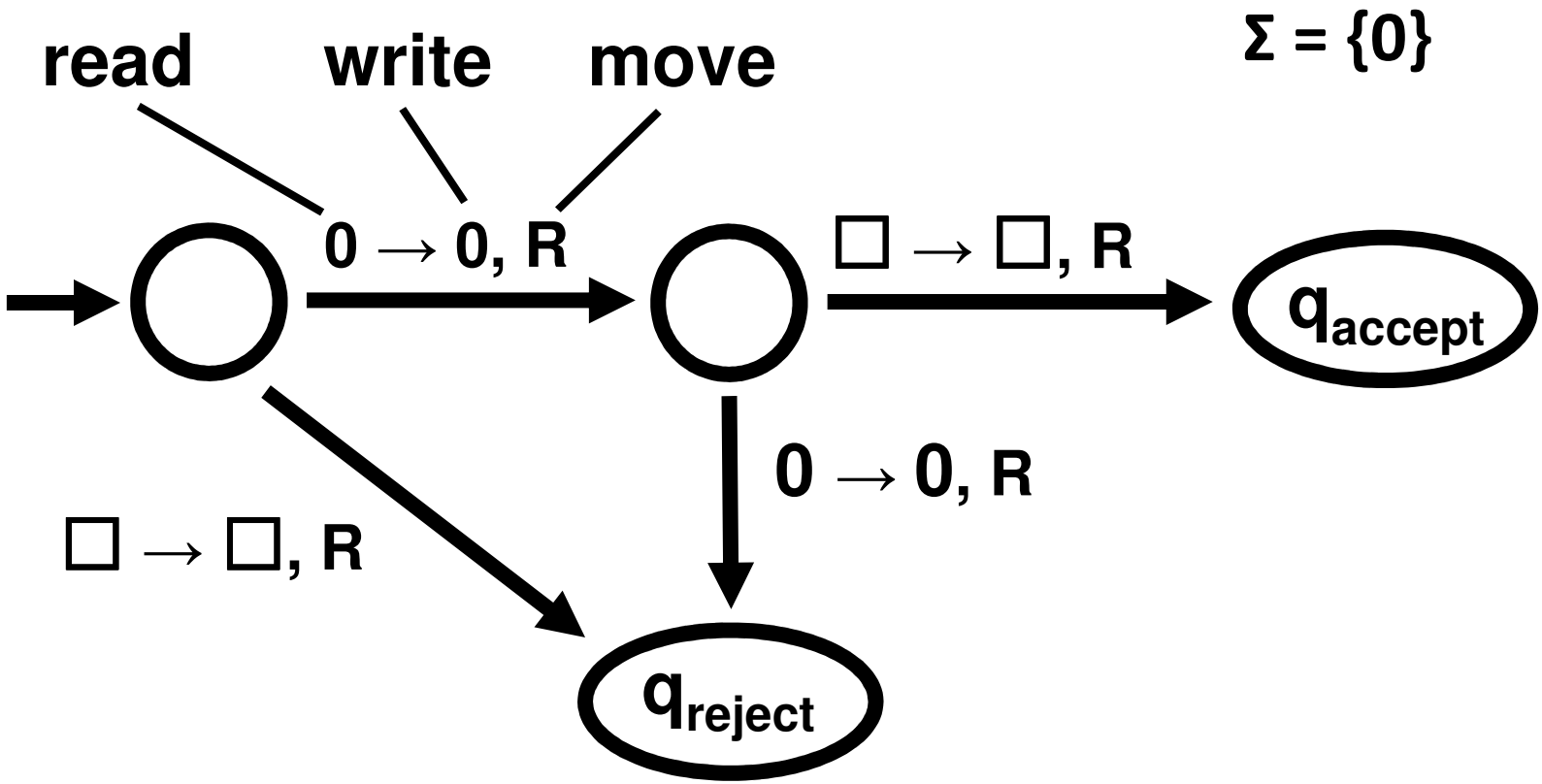
TM can both *write to* and *read from* the tape

The head can move *left and right*

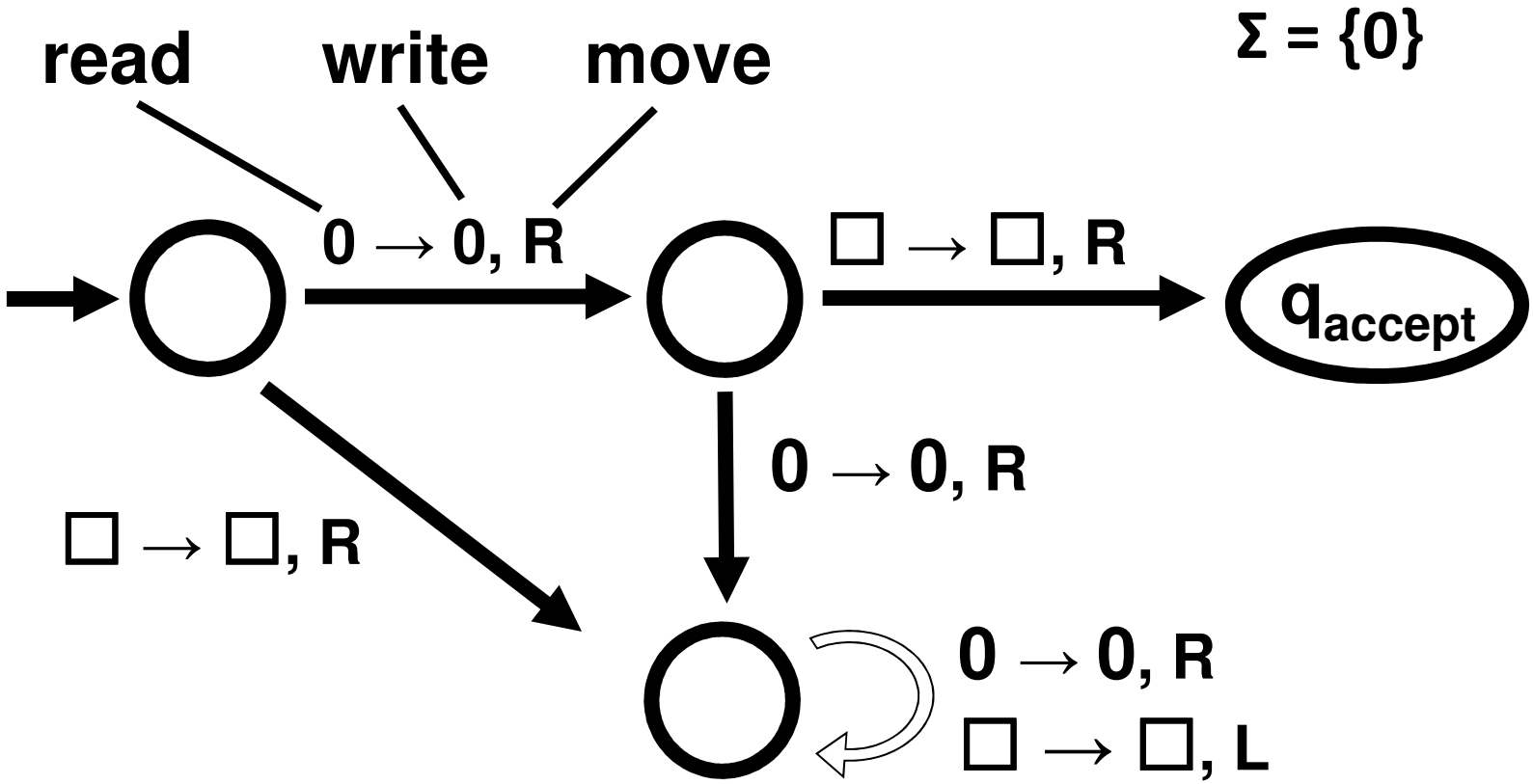
The input doesn't have to be read entirely,

and the computation can continue further (even, *forever*) after all input has been read

Accept and Reject take immediate effect

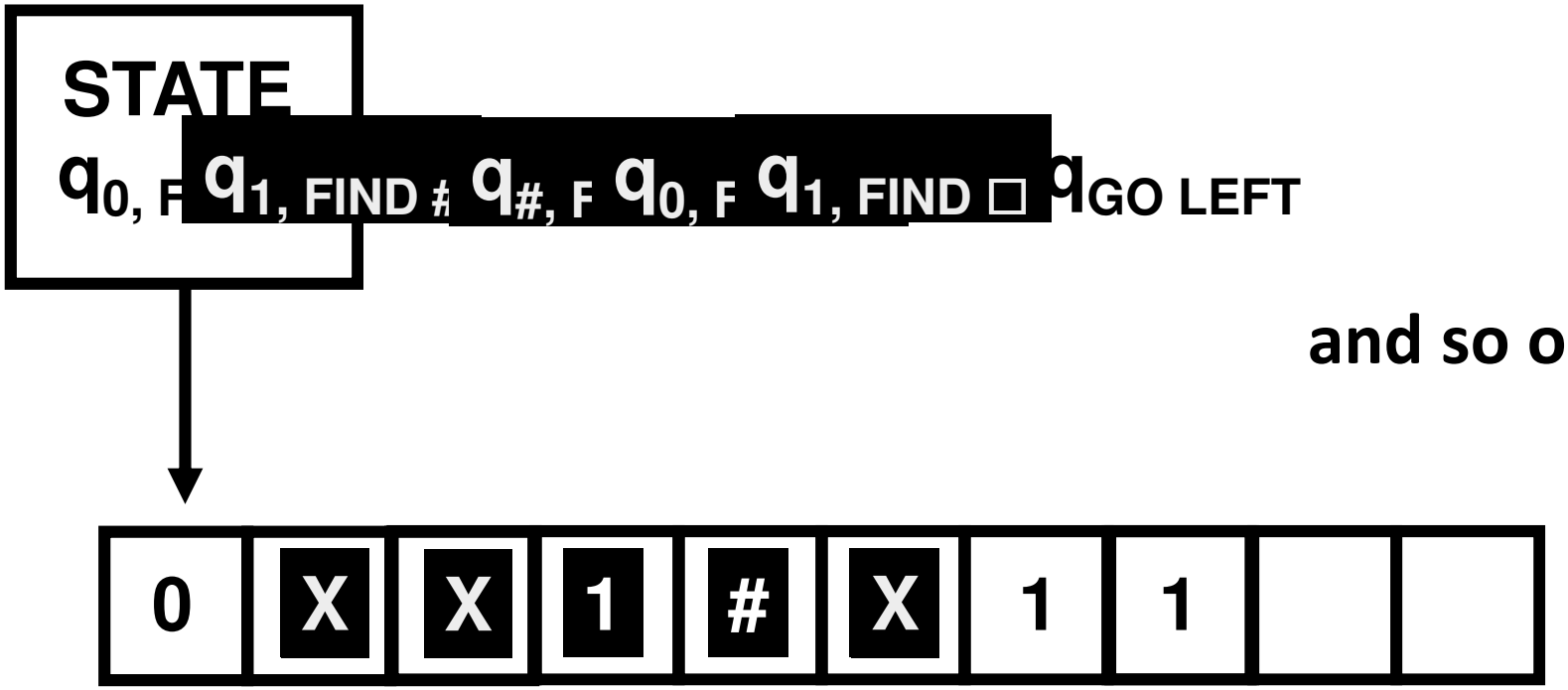


This Turing machine *decides* the language {0}



This Turing machine *recognizes* the language {0}

Deciding the language L = { w#w | w ∈ {0,1}* }



1. If there's no # on the tape (or more than one #), *reject.*
2. While there is a bit to the left of #,
   Replace the first bit with X, and check if the first bit b to the right of the # is identical. (If not, *reject*.)
   Replace that bit b with an X too.
3. If there's a bit to the right of #, then *reject* else *accept*

Definition: A Turing Machine is a 7-tuple
$T = (Q, \Sigma, \Gamma, \delta, q_0, q_{accept}, q_{reject})$, where:

Q is a finite set of states

Σ is the input alphabet, where $\square \notin \Sigma$

Γ is the tape alphabet, where $\square \in \Gamma$ and $\Sigma \subseteq \Gamma$

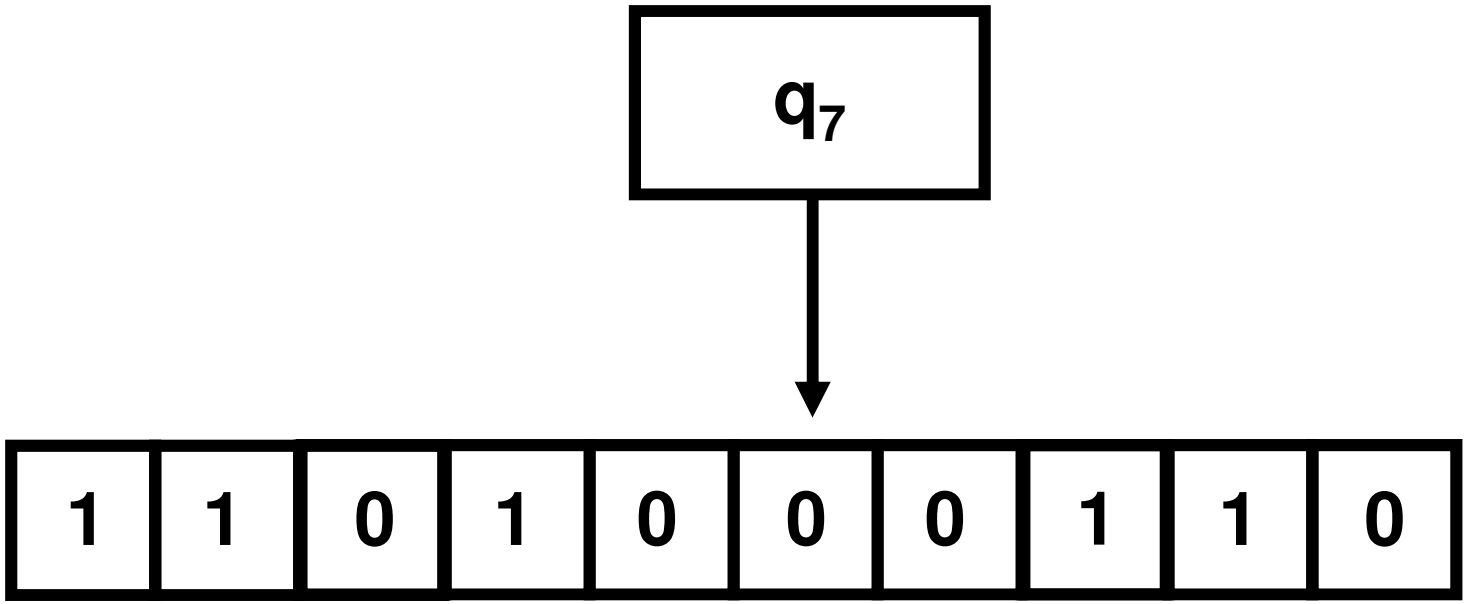
$\delta : Q \times \Gamma \rightarrow Q \times \Gamma \times \{L, R\}$

$q_0 \in Q$ is the start state

$q_{accept} \in Q$ is the accept state

$q_{reject} \in Q$ is the reject state, and $q_{reject} \neq q_{accept}$

Turing Machine Configurations



corresponds to the *configuration*:

$$11010q_700110 \in \{Q \cup \Gamma\}^*$$

Defining Acceptance and Rejection for TMs

Let C_1 and C_2 be configurations of a TM M

Definition. C_1 *yields* C_2 if M is in configuration C_2 after running M in configuration C_1 for one step

Suppose $\delta(q_1, b) = (q_2, c, L)$

Then aaq_1bb yields aq_2acb

Suppose $\delta(q_1, a) = (q_2, c, R)$

Then $cabq_1a$ yields $cabcq_2\square$

Let $w \in \Sigma^*$ and M be a Turing machine

M *accepts* w if there are configs $C_0, C_1, \ldots, C_k$, s.t.

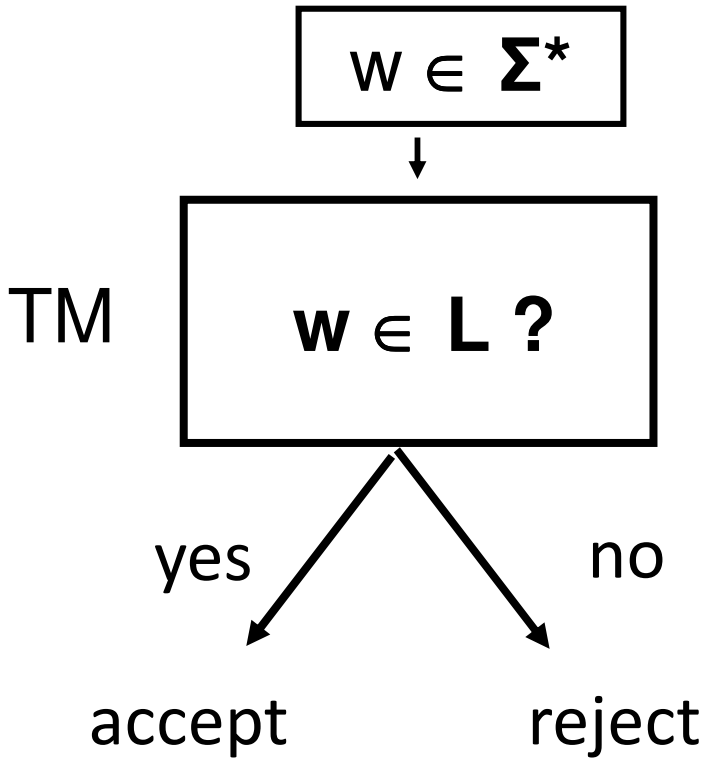
- $C_0 = q_0w$
- C_i yields C_{i+1} for i = 0, ..., k-1, and
- C_k contains the accept state q_{accept}

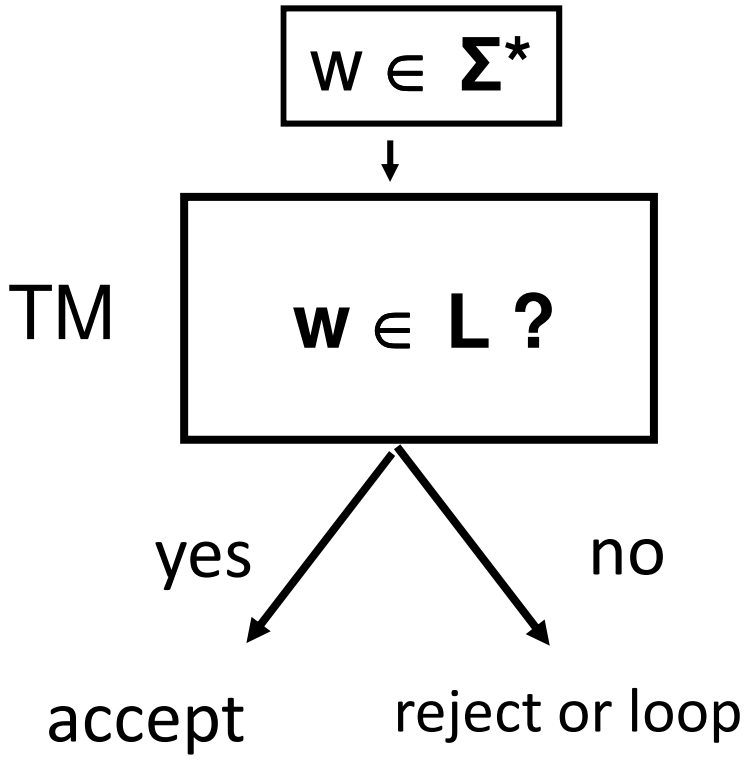
A TM *M recognizes* a language L
if *M* accepts exactly those strings in L

A language L is *recognizable*
(a.k.a. recursively enumerable)
if some TM recognizes L

A TM *M decides* a language L if *M* accepts all strings in L and rejects all strings not in L

A language L is *decidable (a.k.a. recursive)*
if some TM decides L





L is decidable
(recursive)

L is recognizable
(recursively enumerable)

A Turing machine for deciding $\{ 0^{2^n} \mid n \geq 0 \}$

Turing Machine PSEUDOCODE:

1. Sweep from left to right, cross out every other **0**
2. If in step 1, the tape had only one **0**, *accept*
3. If in step 1, the tape had an **odd number** of **0**'s, *reject*
4. Move the head back to the first input symbol.
5. Go to step 1.

Why does this work?

Idea: Every time we return to stage 1, the number of 0's on the tape has been halved.

$\{ 0^{2^n} \mid n \geq 0 \}$