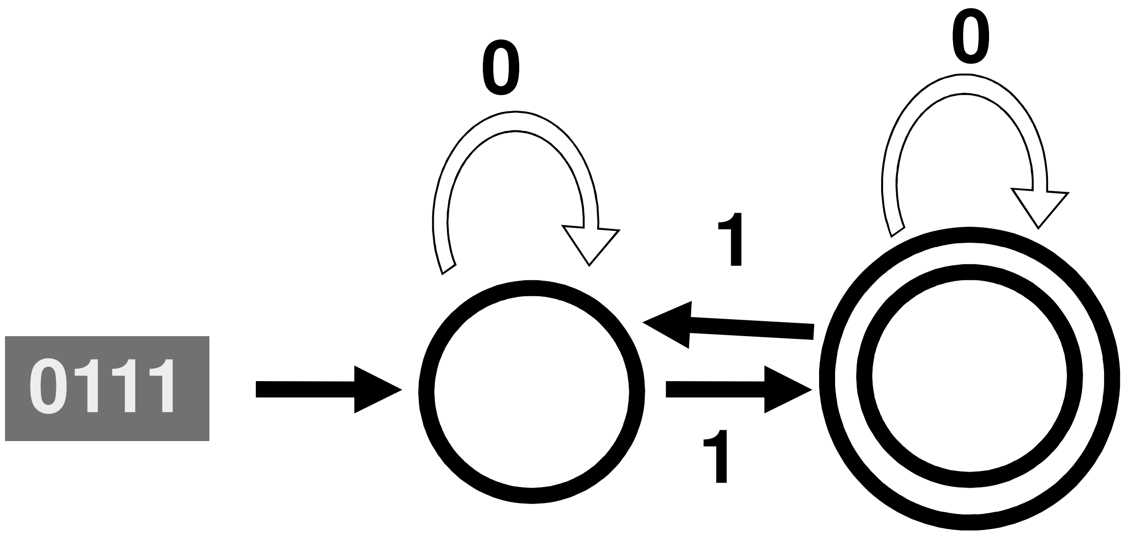
CS 154

Finite Automata, Nondeterminism, Regular Expressions

Read string left to right



The DFA accepts a string if the process ends in a double circle

A DFA is a 5-tuple $M = (Q, \Sigma, \delta, q_0, F)$

Q is the set of states (finite)

Σ is the alphabet (finite)

$\delta : Q \times \Sigma \rightarrow Q$ is the transition function

$q_0 \in Q$ is the start state

$F \subseteq Q$ is the set of accept/final states

L(M) = set of all strings that M accepts
= "the language recognized by M"

A DFA is a 5-tuple $M = (Q, \Sigma, \delta, q_0, F)$

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Definition: A language L' is **regular** if it is recognized by a DFA;
that is, there is a DFA M where L' = L(M).

Union Theorem for Regular Languages

The union of two regular languages is also a regular language

Intersection Theorem for Regular Languages

The intersection of two regular languages is also a regular language

Complement Theorem for Regular Languages

The complement of a regular language is also a regular language

The **Reverse** of a Language

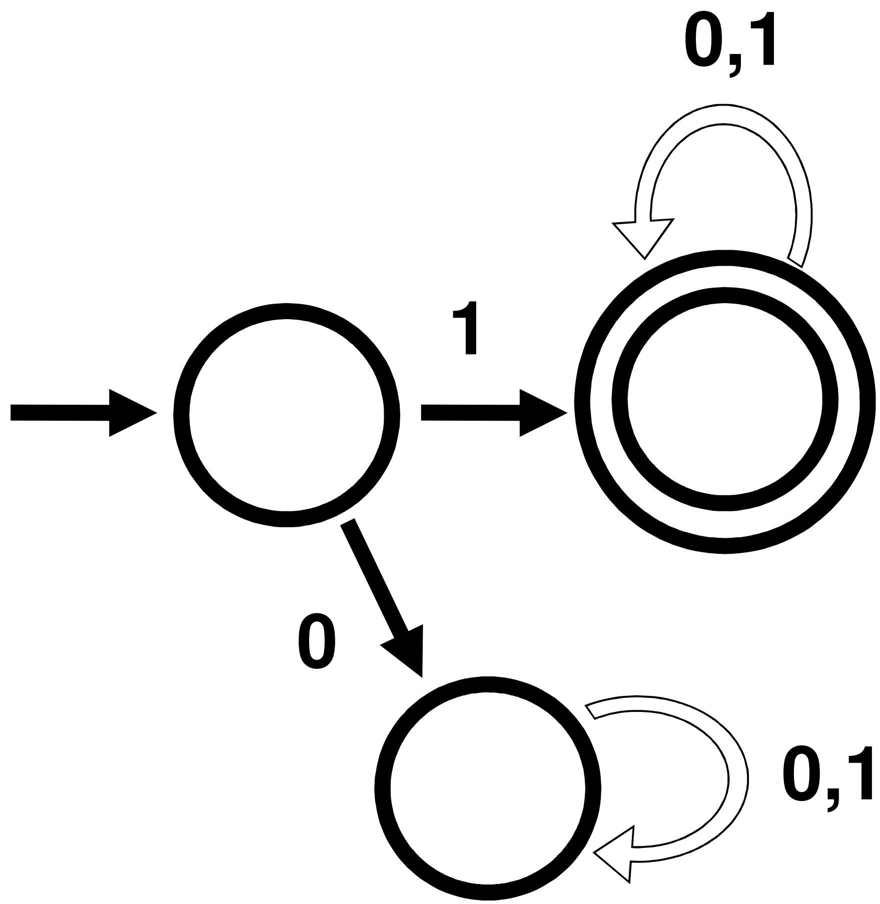
Reverse of L:

$L^R = \{ w_1 \ldots w_k \mid w_k \ldots w_1 \in L, w_i \in \Sigma\}$

If L is recognized by the usual kind of DFA,
Then L^R is recognized by a DFA that reads its strings from *right to left*!

Question: If L is regular, then is L^R also regular?

Can every "Right-to-Left" DFA be replaced by a normal "Left-to-Right" DFA?



L(M) = { w | w begins with 1}

Suppose our machine reads strings from *right* to *left*…
Then **L(M) = {w | w ends with a 1}. Is this regular?**

Reversing DFAs

Assume L is a regular language.
Let M be a DFA that recognizes L

We'll build a machine M^R that accepts L^R

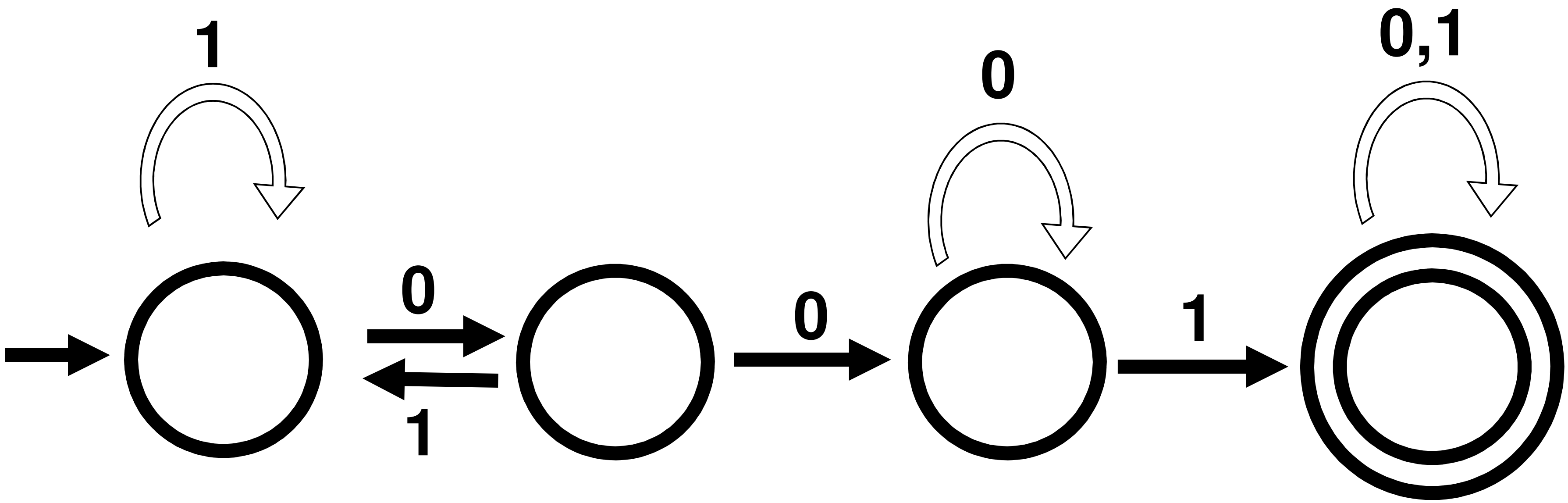
If M accepts w, then w describes a directed path in M from start to an accept

First Attempt: Try to define M^R as M with the arrows reversed, turn start state into a final state, turn final states into starts

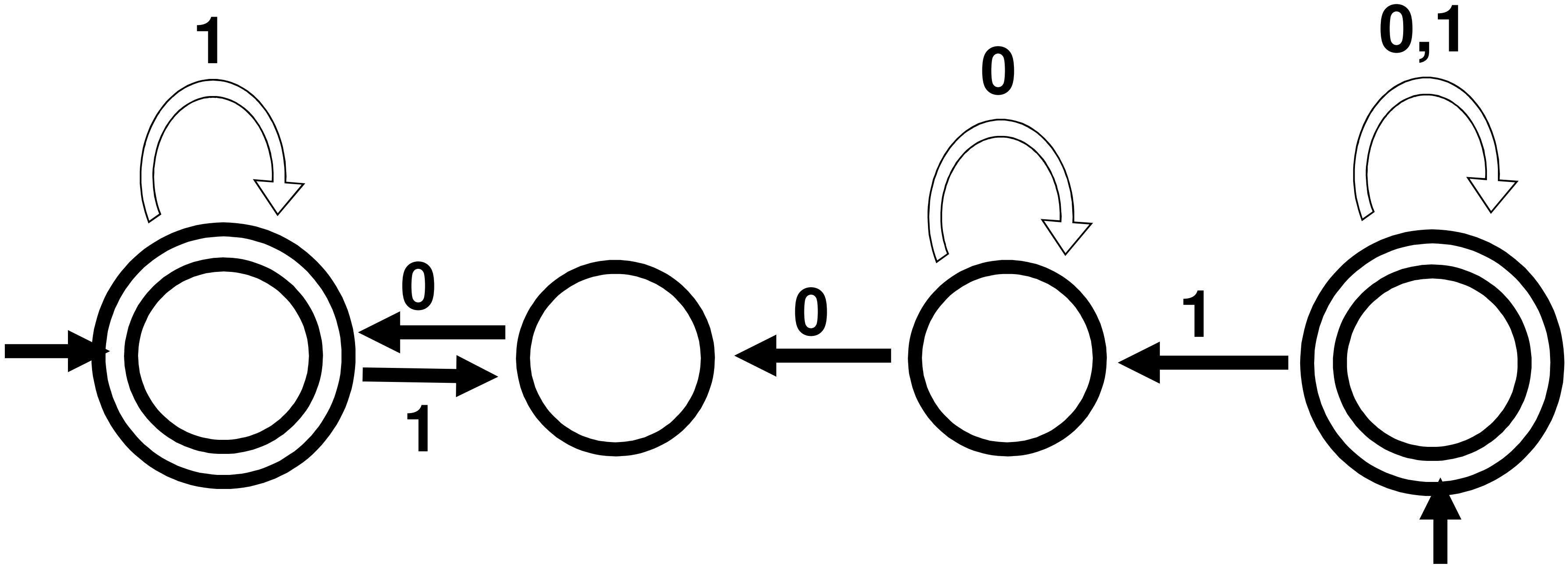
Problem: M^R IS NOT ALWAYS A DFA!

It could have many start states

Some states may have *more than one* outgoing edge, or none at all!


1
0
1
0
0
1
0,1

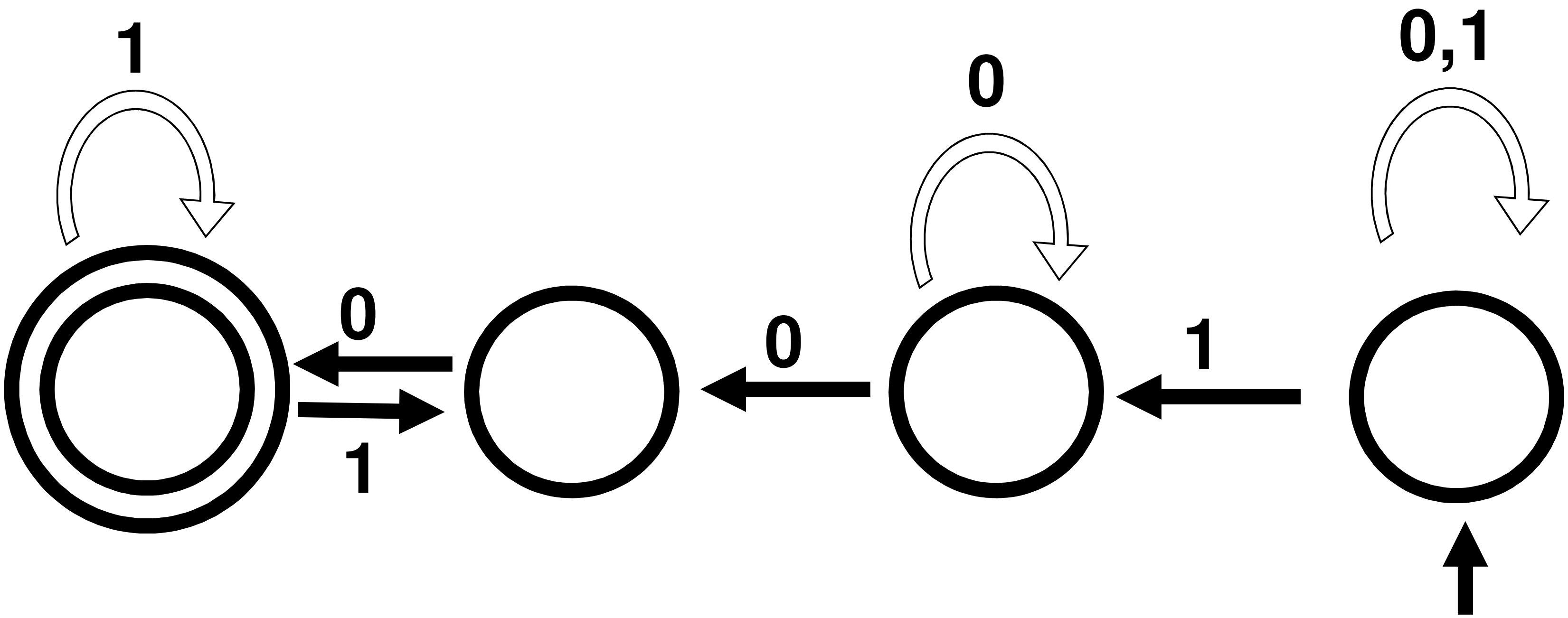
Non-deterministic Finite Automata (NFA)



What happens with 100?

We will say this new machine accepts a string x if *there is some path reading in x* that reaches *some* accept state from *some* start state

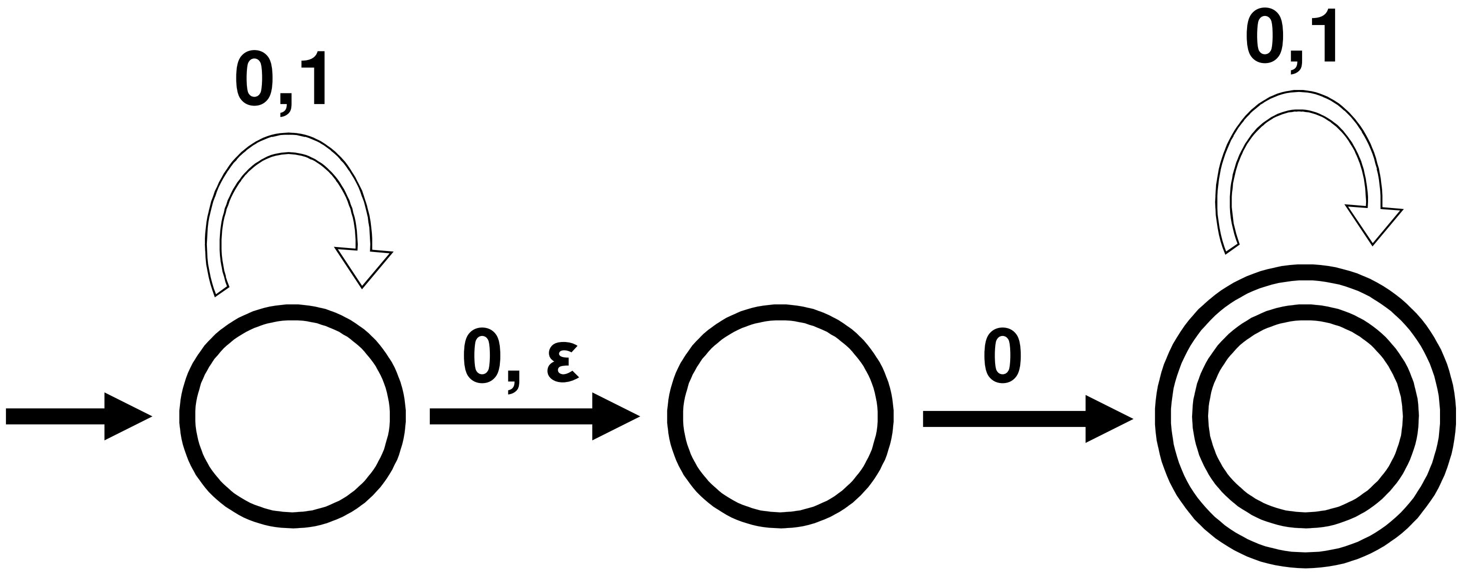
Non-deterministic Finite Automata (NFA)



Then, this machine recognizes: {w | w contains 100}

We will say this new machine accepts a string x if *there is some path reading in x* that reaches *some* accept state from *some* start state

Another Example of an NFA



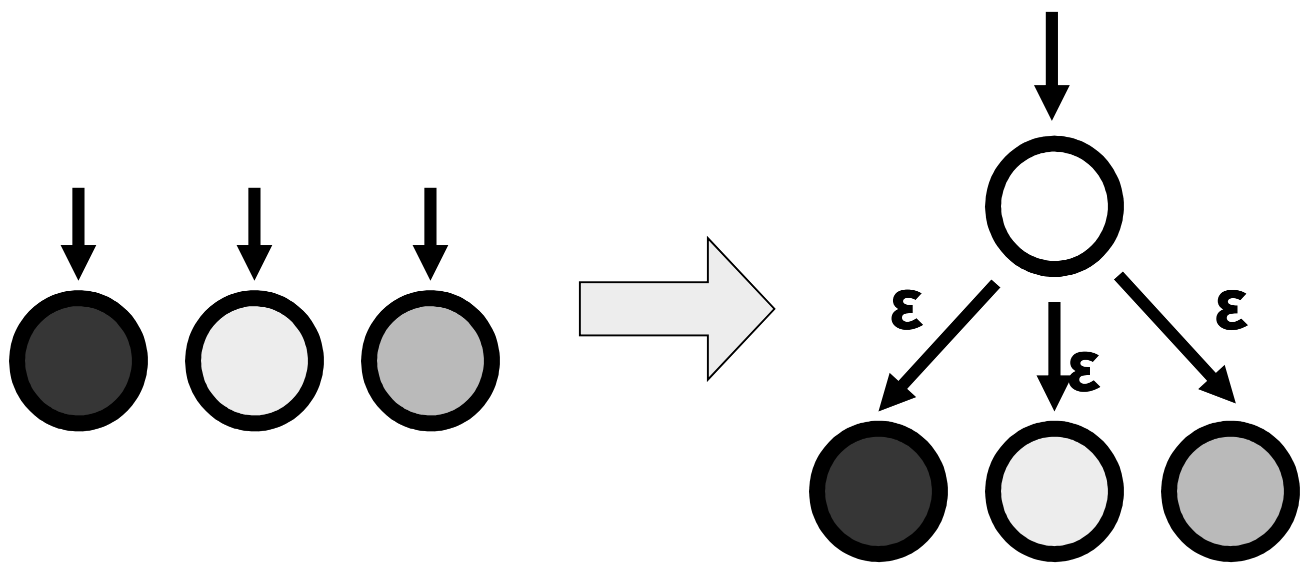
At each state, we can have ***any*** number of out arrows for a letter $\sigma \in \mathbf{\Sigma}$, **including ε**

Set of strings accepted by this NFA = {w | w contains a 0}

Multiple Start States

We allow *multiple* start states for NFAs, and Sipser allows only one

Can easily convert NFA with many start states into one with a single start state:



A *non-deterministic* finite automaton (NFA) is a 5-tuple N = (Q, Σ, δ, Q_0, F) where

Q is the set of states

Σ is the alphabet

$\delta : Q \times \Sigma_\varepsilon \to 2^Q$ **is the transition function**

$Q_0 \subseteq Q$ **is the set of start states**

$F \subseteq Q$ **is the set of accept states**

2^Q **is the set of all possible subsets of Q**

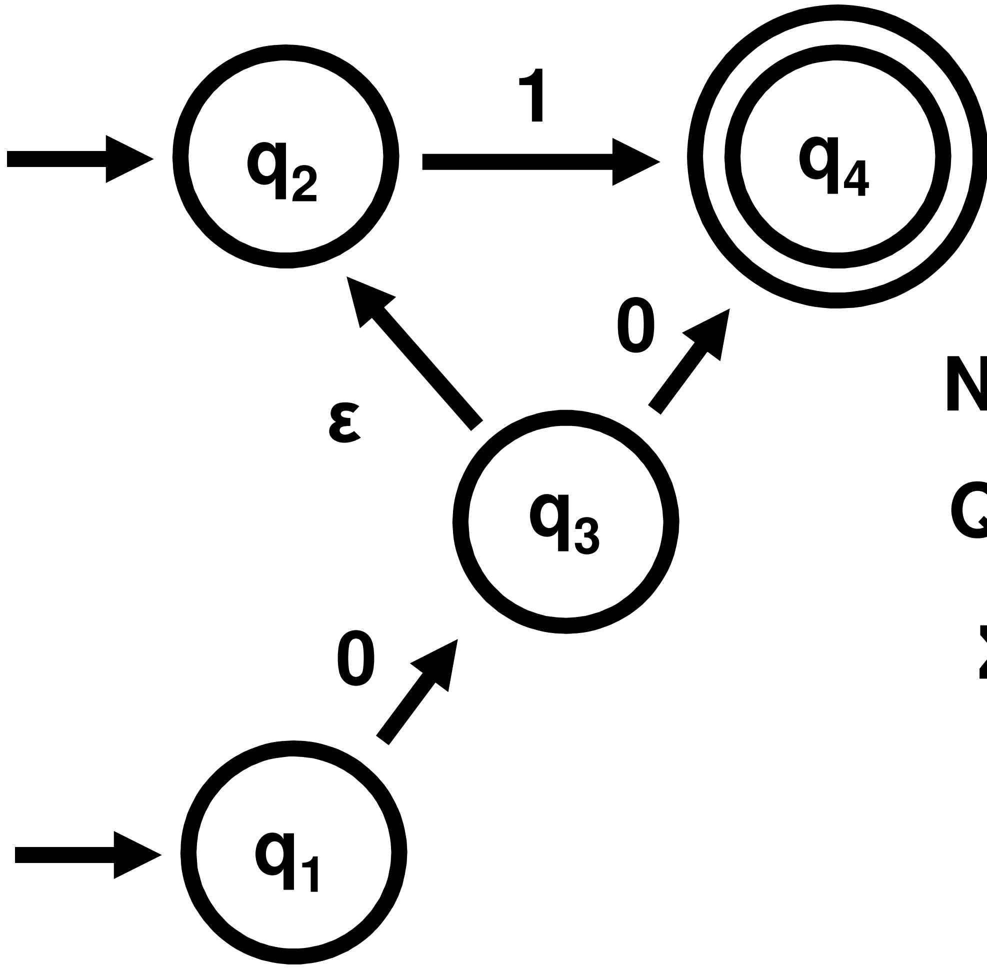
$\Sigma_\varepsilon = \Sigma \cup \{\varepsilon\}$

Def. Let $w \in \Sigma^*$. Let N be an NFA. N accepts w if there's a sequence of states $r_0, r_1, \ldots, r_k \in Q$ and w can be written as $w_1 \ldots w_k$ with $w_i \in \Sigma \cup \{\varepsilon\}$ such that

1. $r_0 \in Q_0$
2. $r_{i+1} \in \delta(r_i, w_{i+1})$ **for all i = 0, ..., k-1, and**
3. $r_k \in F$

L(N) = the language recognized by N
= set of all strings machine N accepts

A language L' is recognized by an NFA N if L' = L(N).



$N = (Q, \Sigma, \delta, Q_0, F)$

$Q = \{q_1, q_2, q_3, q_4\}$

$\Sigma = \{0,1\}$

$Q_0 = \{q_1, q_2\}$

$F = \{q_4\}$

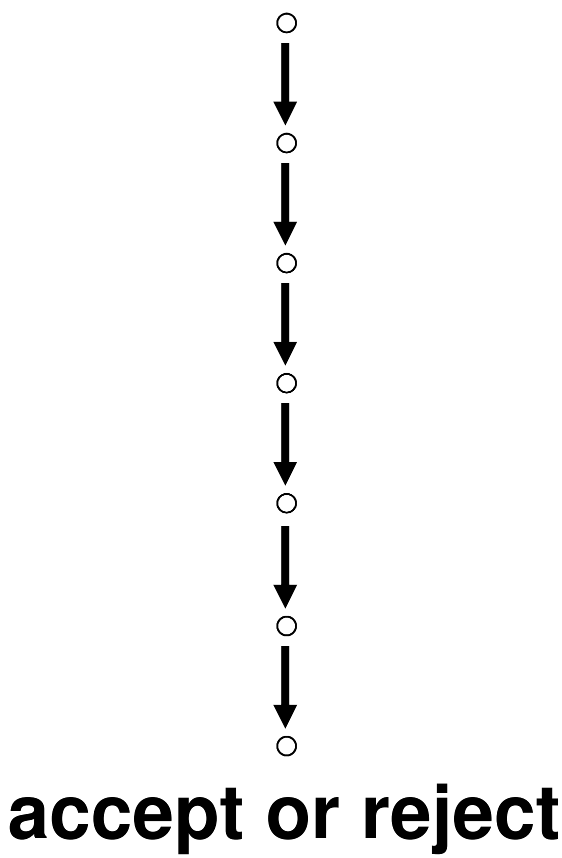
$\delta(q_2,1) = \{q_4\}$

$\delta(q_3,1) = \varnothing$

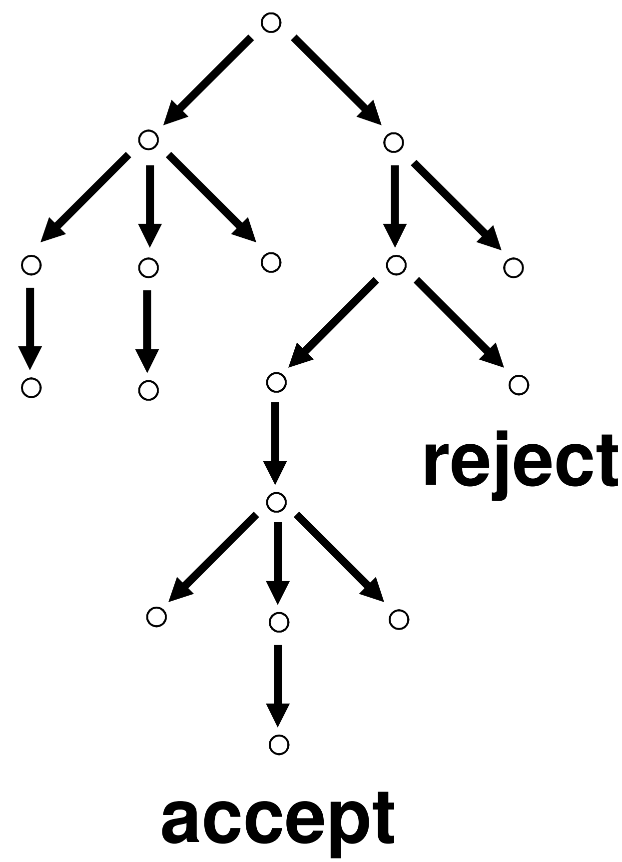
$\delta(q_1,0) = \{q_3\}$

$\boxed{L(N) = \{1,00,01\}}$

Deterministic Computation



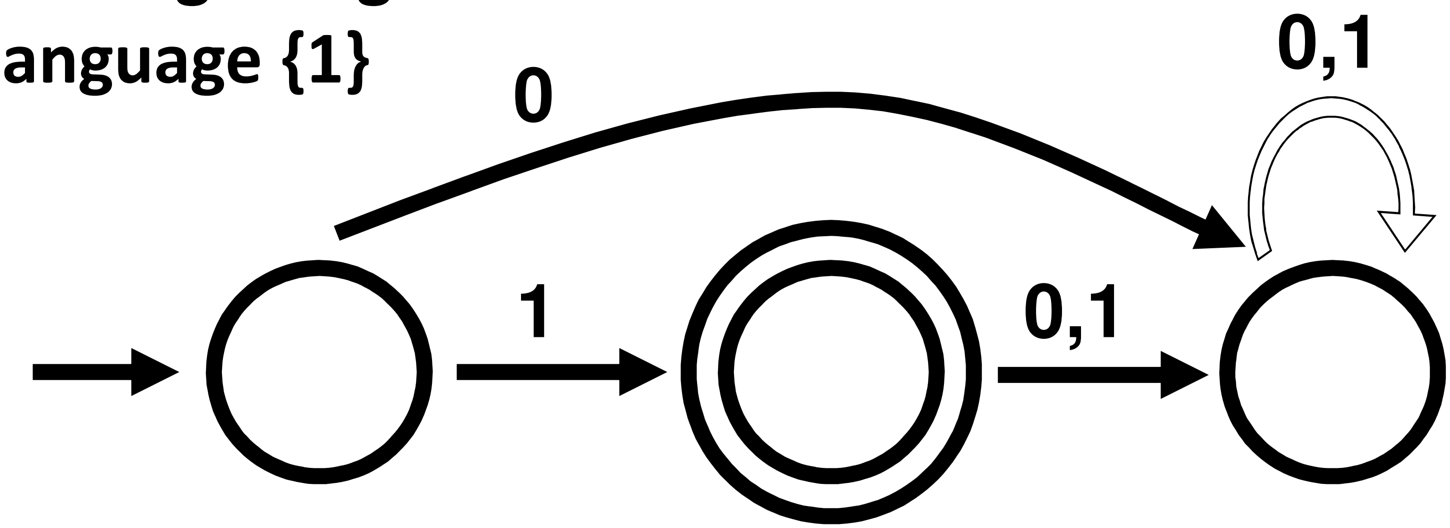
Non-Deterministic Computation



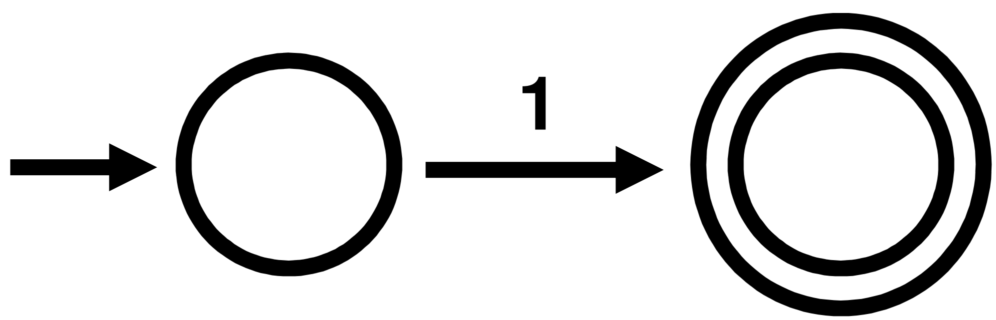
Are these equally powerful???

NFAs are generally simpler than DFAs

A DFA recognizing the language {1}



An NFA recognizing the language {1}



Every NFA can be perfectly simulated by some DFA!

Theorem: For every NFA N, there is a DFA M such that L(M) = L(N)

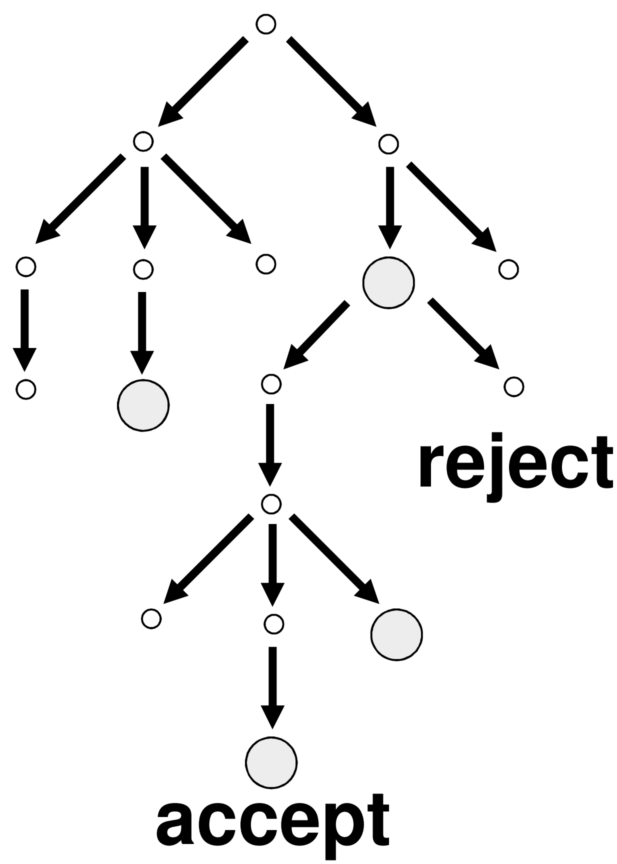
Corollary: A language L is regular if and only if L is recognized by an NFA

Corollary: L is regular iff L^R is regular

From NFAs to DFAs

Input: NFA N = $(Q, \Sigma, \delta, Q_0, F)$

Output: DFA M = $(Q', \Sigma, \delta', q_0', F')$



To learn if an NFA accepts, we could do the computation in parallel, maintaining the set of *all* possible states that can be reached

Idea:

Set $Q' = 2^Q$

From NFAs to DFAs: Subset Construction

Input: NFA $N = (Q, \Sigma, \delta, Q_0, F)$

Output: DFA $M = (Q', \Sigma, \delta', q_0', F')$

$$Q' = 2^Q$$

$$\delta' : Q' \times \Sigma \rightarrow Q'$$

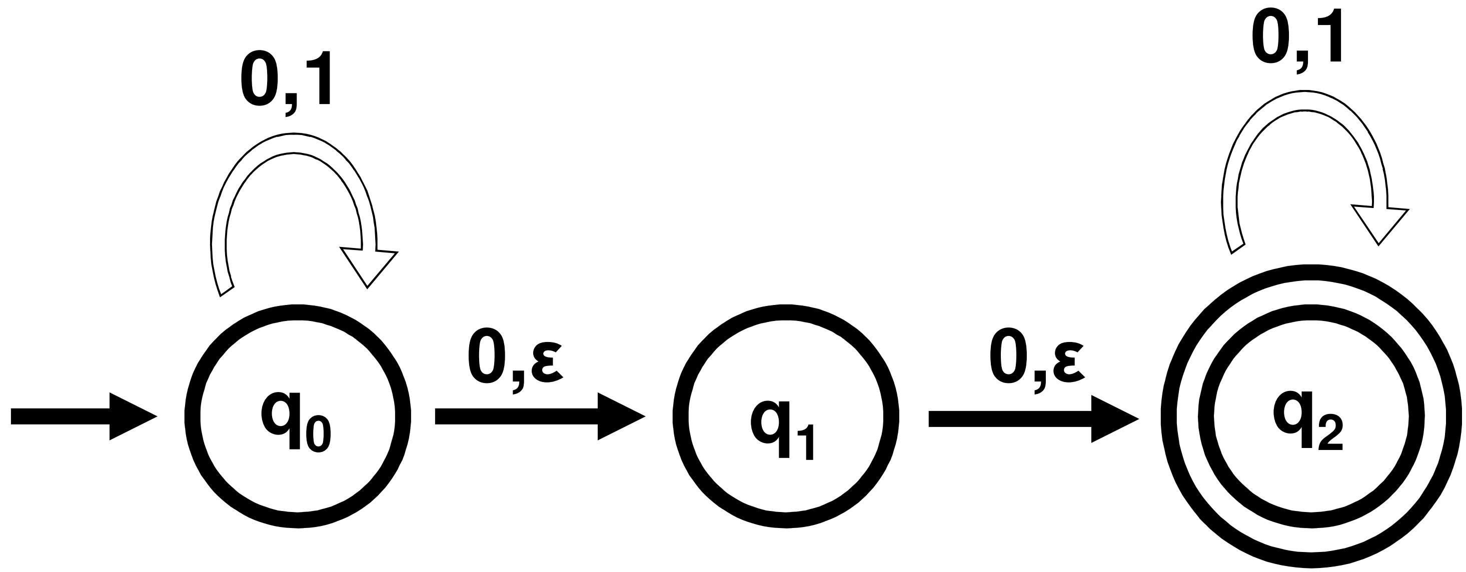
$$\delta'(R,\sigma) = \bigcup_{r \in R} \varepsilon(\,\delta(r,\sigma)\,)\ ^{*}$$

$$q_0' = \varepsilon(Q_0)$$

$$F' = \{\, R \in Q' \mid f \in R \text{ for some } f \in F \,\}$$

* For $S \subseteq Q$, the **ε-closure of S is**
$\varepsilon(S)$ = {$q \in Q$ reachable from some $s \in S$
by taking 0 or more **ε** transitions}

Example of the ε-closure



$$\varepsilon(\{q_0\}) = \{q_0, q_1, q_2\}$$

$$\varepsilon(\{q_1\}) = \{q_1, q_2\}$$

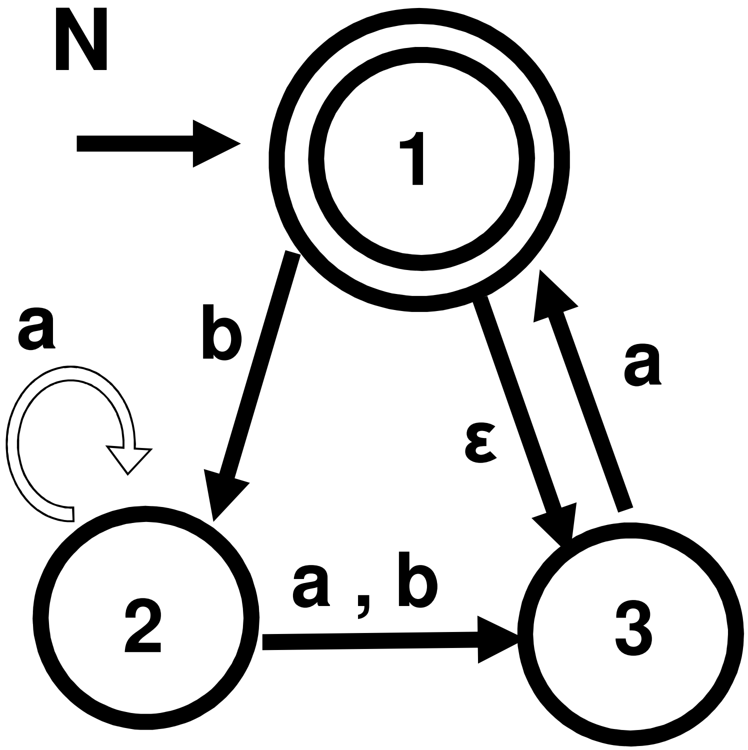
$$\varepsilon(\{q_2\}) = \{q_2\}$$

Given: NFA N = ({1,2,3}, {a,b}, δ , {1}, {1})

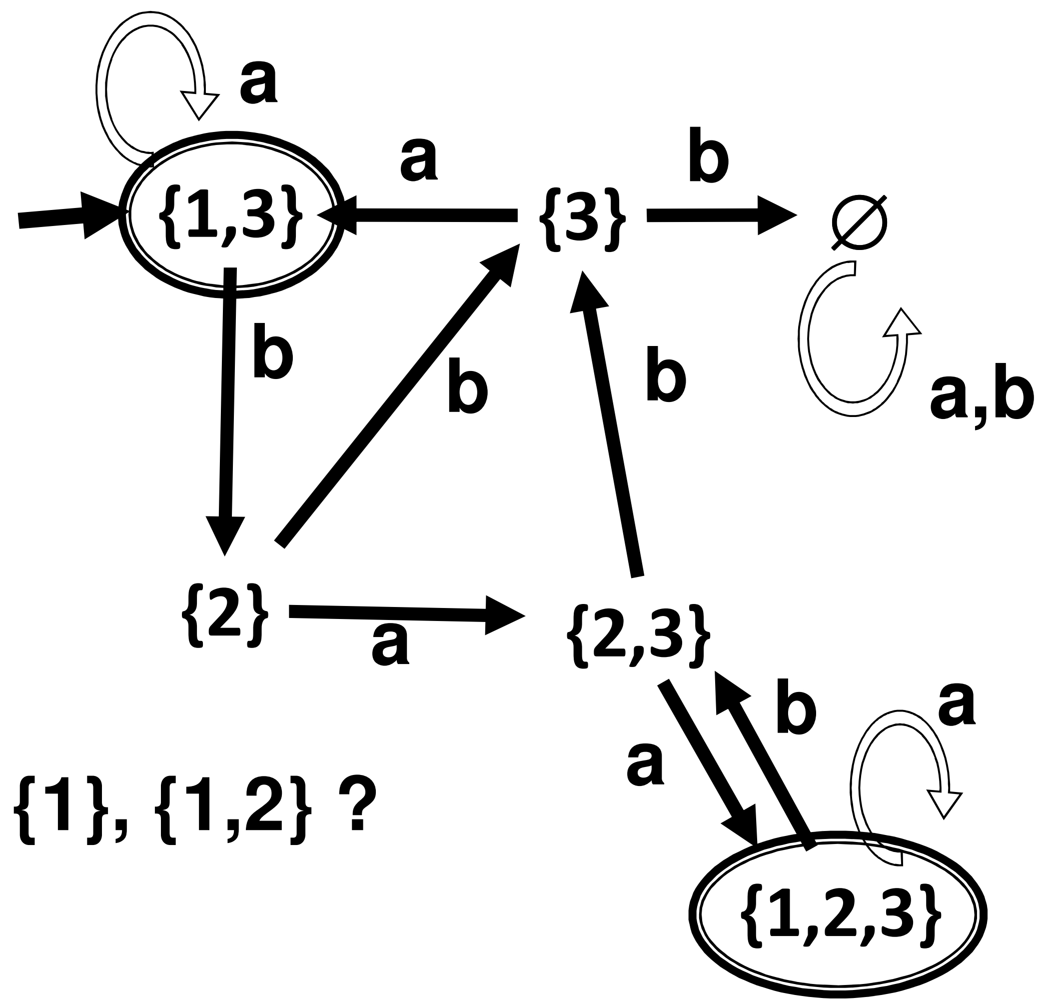
Construct: Equivalent DFA M

M = ($2^{\{1,2,3\}}$, {a,b}, δ', {1,3}, …)

N



ε({1}) = {1,3}



Reverse Theorem for Regular Languages

The reverse of a regular language is also a regular language

If **a language can be recognized by a DFA that reads strings from *right* to *left*,**
then **there is an "normal" DFA that accepts the same language**

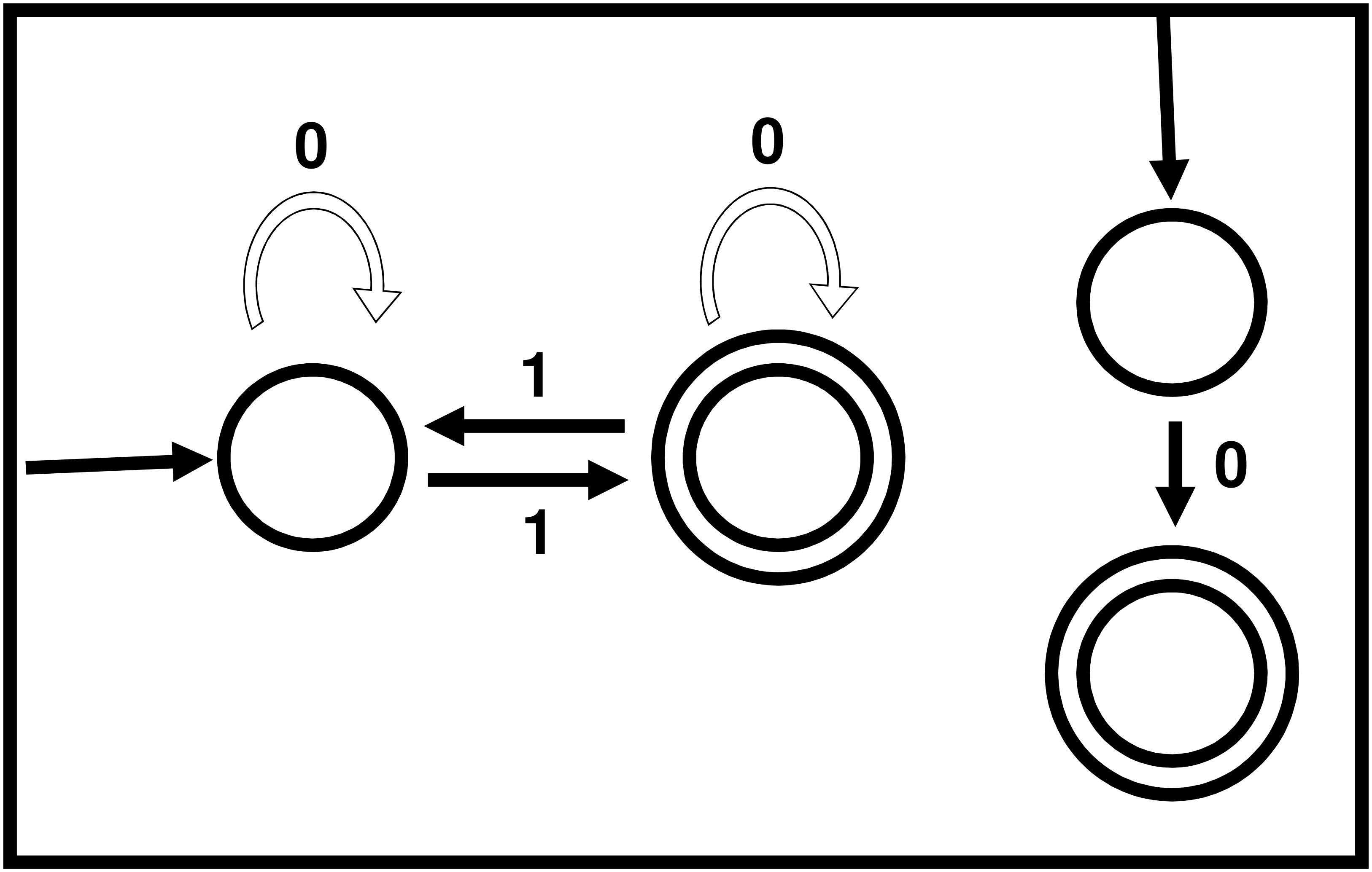
Proof?

Given a DFA for a language L, "reverse" its arrows and flip its start and accept states, getting an NFA. Convert that NFA back to a DFA.

Using NFAs in place of DFAs can make proofs about regular languages *much* easier!

Remember this on homework/exams!

Union Theorem using NFAs?

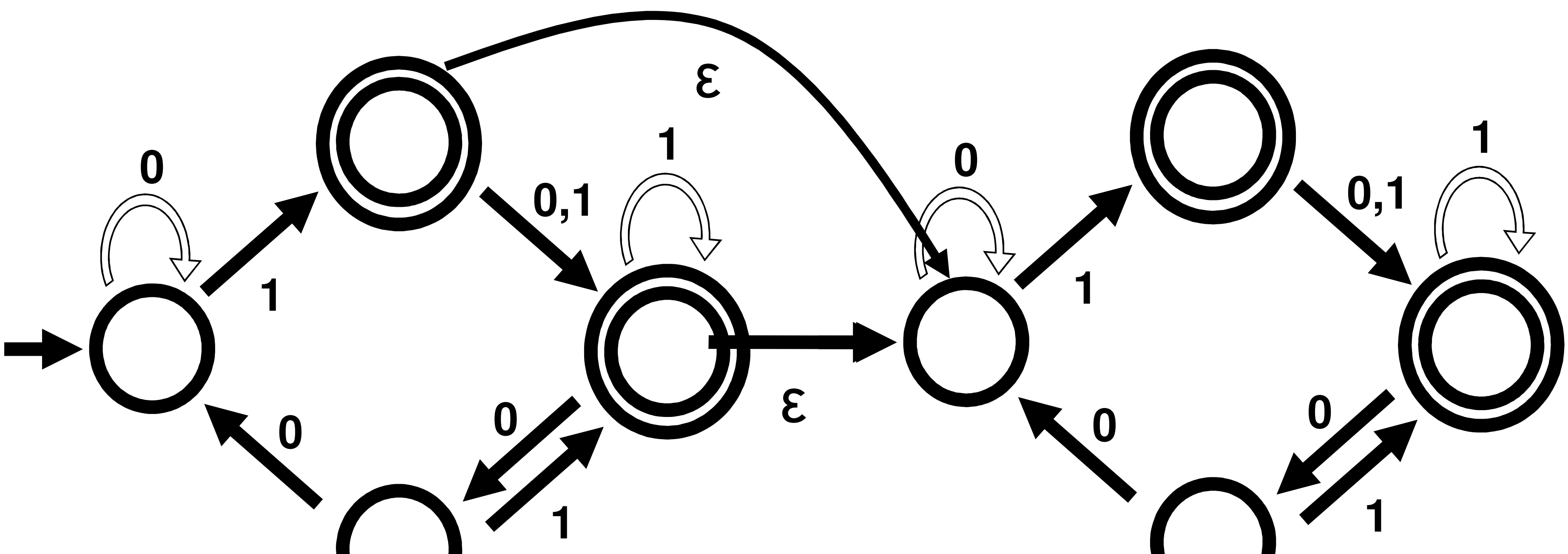


Regular Languages are closed under concatenation

Concatenation: $A \cdot B = \{ vw \mid v \in A \text{ and } w \in B \}$

Given DFAs M_1 and M_2, connect

the accept states of M_1 to the start states of M_2



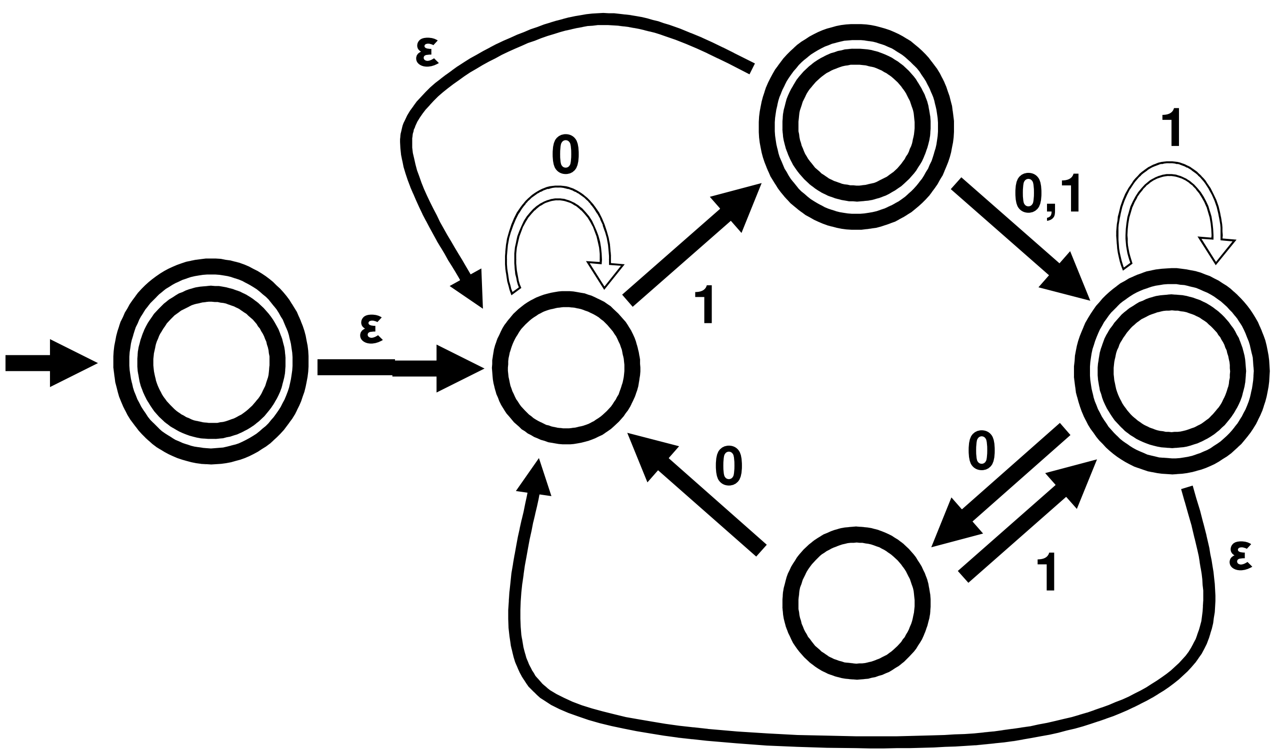
$L(N) = L(M_1) \cdot L(M_2)$

Regular Languages are closed under star

$A^* = \{ s_1 \ldots s_k \mid k \geq 0 \text{ and each } s_i \in A \}$

Let M be a DFA, and let L = L(M)

We can construct an NFA N that recognizes L*



Formally, the construction is:

Input: DFA $M = (Q, \Sigma, \delta, q_1, F)$

Output: NFA $N = (Q', \Sigma, \delta', \{q_0\}, F')$

$$Q' = Q \cup \{q_0\}$$

$$F' = F \cup \{q_0\}$$

$$\delta'(q,a) = \begin{cases} \{\delta(q,a)\} & \text{if } q \in Q \text{ and } a \neq \varepsilon \\ \{q_1\} & \text{if } q \in F \text{ and } a = \varepsilon \\ \{q_1\} & \text{if } q = q_0 \text{ and } a = \varepsilon \\ \varnothing & \text{if } q = q_0 \text{ and } a \neq \varepsilon \\ \varnothing & \text{else} \end{cases}$$

Regular Languages are closed under star

How would we *prove* that this NFA construction works?

Want to show: $\mathbf{L}(\mathrm{N}) = \mathbf{L^*}$

1. $\mathbf{L(N)} \supseteq \mathbf{L^*}$

2. $\mathbf{L(N)} \subseteq \mathbf{L^*}$

1. $L(N) \supseteq L^*$

Assume $w = w_1 \ldots w_k$ is in L^* where $w_1, \ldots, w_k \in L$

We show N accepts w by induction on k

Base Cases:

- ✓ **$k = 0$ ($w = \varepsilon$)**
- ✓ **$k = 1$ ($w \in L$)**

Inductive Step:

Assume N accepts all strings $v = v_1 \ldots v_k \in L^*$, $v_i \in L$

Let $u = u_1 \ldots u_k u_{k+1} \in L^*$, $u_j \in L$

Since N accepts $u_1 \ldots u_k$ (by induction) **and M accepts u_{k+1}, N also accepts u** (by construction)

2. L(N) $\subseteq$ L*

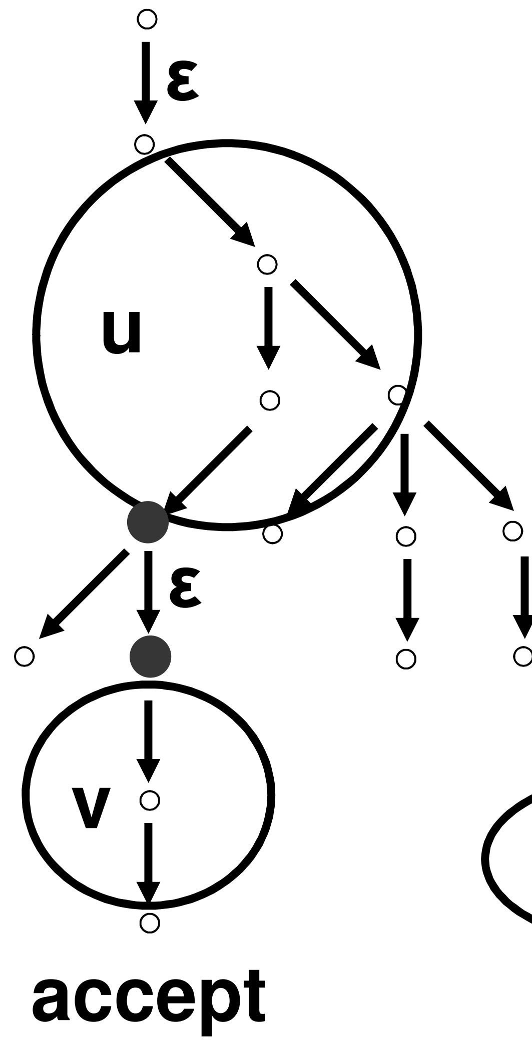
Assume w is accepted by N; we want to show w $\in$ L*

If w = ε, then w $\in$ L*

I.H. N accepts u and takes at most k ε-transitions $\Rightarrow$ u $\in$ L*

Let w be accepted by N with k+1.

Write w as w=uv, where v is the substring read after the *last* ε-transition



u$\in$L(N), so

u$\in$L*

By I.H.

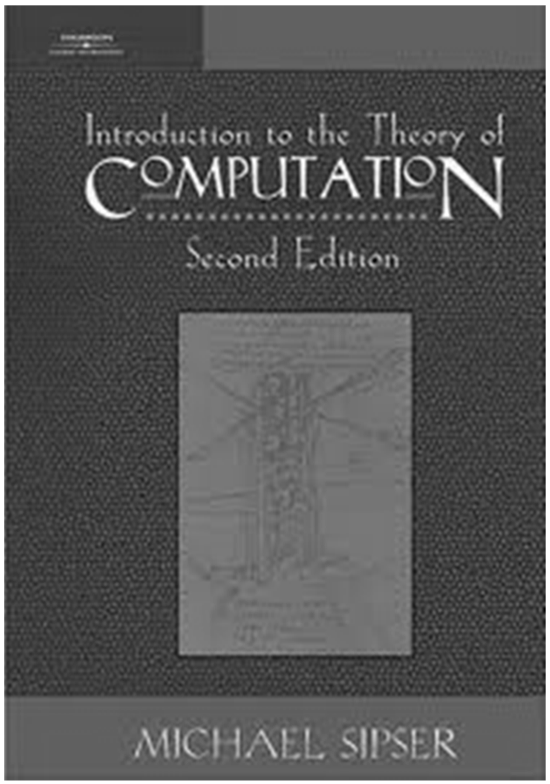
v$\in$L

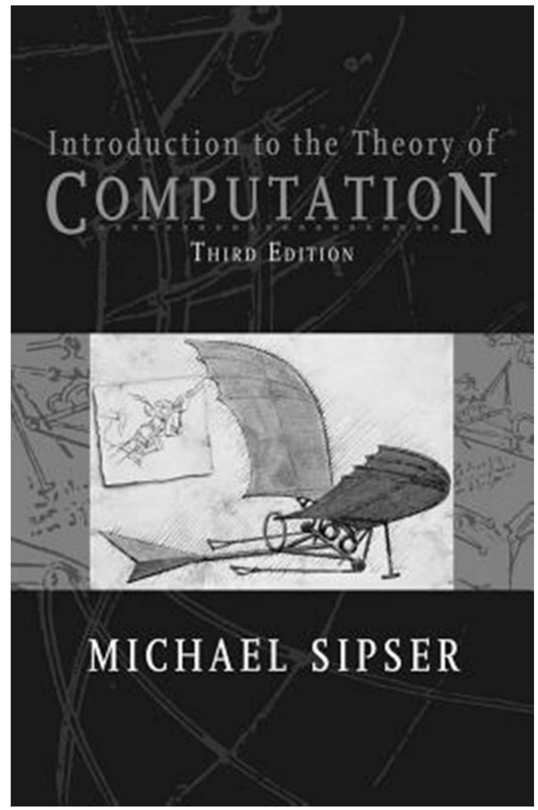
$\boxed{\textbf{w = uv} \in \textbf{L*}}$

Regular Languages are closed under all of the following operations:

- **Union:** $A \cup B = \{ w \mid w \in A \text{ or } w \in B \}$
- **Intersection:** $A \cap B = \{ w \mid w \in A \text{ and } w \in B \}$
- **Complement:** $\neg A = \{ w \in \Sigma^* \mid w \notin A \}$
- **Reverse:** $A^R = \{ w_1 \ldots w_k \mid w_k \ldots w_1 \in A \}$
- **Concatenation:** $A \cdot B = \{ vw \mid v \in A \text{ and } w \in B \}$
- **Star:** $A^* = \{ w_1 \ldots w_k \mid k \geq 0 \text{ and each } w_i \in A \}$

Homework 1 is coming out today… watch for it!





Regular Expressions

Inductive Definition of Regexp

Let Σ be an alphabet. We define the regular expressions over Σ inductively:

For all $\sigma \in \Sigma$, σ is a regexp

ε is a regexp

$\varnothing$ is a regexp

If R_1 and R_2 are both regexps, then (R_1R_2), $(R_1 + R_2)$, and $(R_1)^*$ are regexps

Precedence Order: $*$

then $\cdot$

then $+$

Example: $R_1{}^*R_2 + R_3 = ((R_1{}^*) \cdot R_2) + R_3$

Definition: Regexps Represent Languages

The regexp $\sigma \in \Sigma$ *represents* the language $\{\sigma\}$

The regexp ε represents $\{\varepsilon\}$

The regexp $\varnothing$ represents $\varnothing$

If R_1 and R_2 are regular expressions representing L_1 and L_2 then:

(R_1R_2) represents $L_1 \cdot L_2$

$(R_1 + R_2)$ represents $L_1 \cup L_2$

$(R_1)^*$ represents L_1^*

Regexps Represent Languages

For every regexp R, define L(R) to be the language that R represents

A string $w \in \Sigma^*$ is *accepted by R* (or, *w matches R)* if $w \in L(R)$

Example: 01010 matches the regexp (01)*0

end