CS154

Streaming Algorithms and Communication Complexity

Streaming Algorithms

Streaming Algorithms

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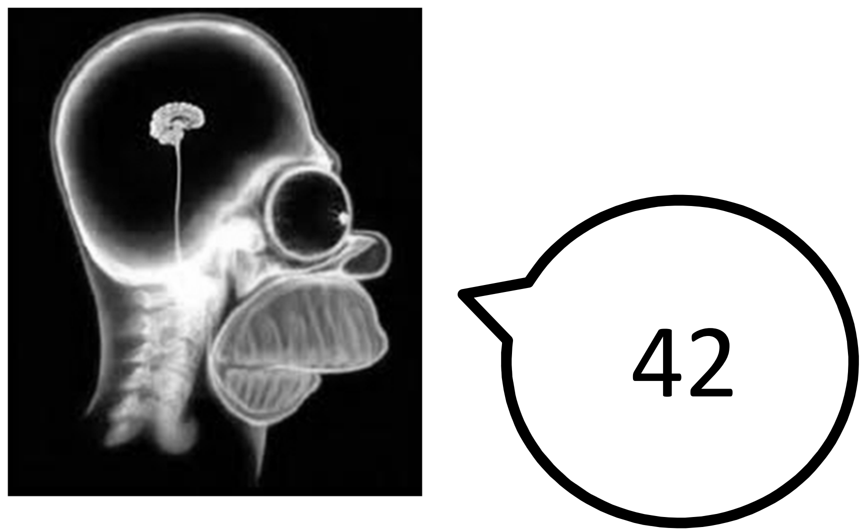

$L = \{x \mid x$ has more 1's than 0's$\}$

Initialize: $C := 0$ and $B := 0$
When the next symbol x is read,
If $(C = 0)$ then $B := x$, $C := 1$
If $(C \neq 0)$ and $(B = x)$ then $C := C + 1$
If $(C \neq 0)$ and $(B \neq x)$ then $C := C - 1$
When the stream stops,
accept if $B=1$ and $C > 0$, else _reject_

B = the majority bit
C = how many more times that B appears

On all strings of length n, the algorithm uses $1+\log_2(n+1)$ bits of space _(to store B & C)_

Streaming Algorithms

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Can recognize non-regular languages

Streaming algorithms differ from DFAs in several significant ways:

1. **Streaming algorithms can output more than one bit**
2. **The "memory" or "space" of a streaming algorithm can (slowly) *increase* as it reads longer strings**
3. **Could also make multiple passes over the data, could be randomized**

DFAs and Streaming

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Theorem: Suppose a language L can be recognized by a DFA M with ≤ 2^p states. Then L is computable by a streaming algorithm A using ≤ p bits of space.

Proof Idea: Can define algorithm A as follows:
Initialize: Encode the *start state* of M in memory.
When the next symbol σ is read: Use the transition function of M to update the state of M.
When the string ends: Output *accept* if the current state of M is a final state, *reject* otherwise.

DFAs and Streaming

For any $L \subseteq \Sigma^*$ define $L_n = \{x \text{ in } L : |x| = n\}$

Theorem: Suppose L' is computable by a streaming algorithm A using $f(n)$ bits of space, on all strings of length up to n.
Then for all n, there is a DFA M with $\leq 2^{f(n)}$ states such that $L'_n = L(M)_n$

Proof Idea: States of M = $2^{f(n)}$ possible settings of A's memory, on strings of length up to n

Start state of M = Initial memory configuration of A

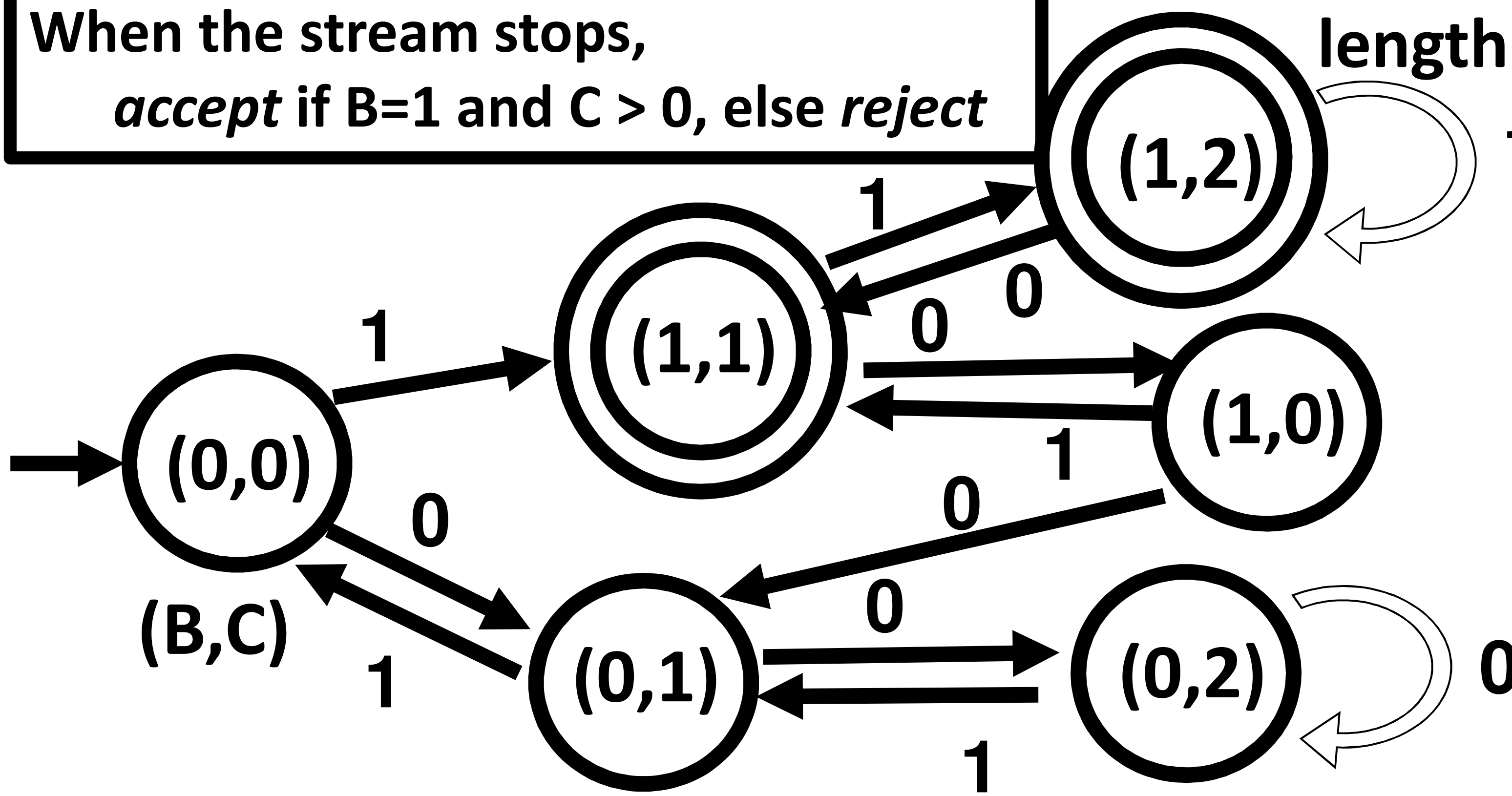
Transition function = Mimic how A updates its memory

Final states of M = Memory configurations in which A would accept, if the string ends

Example: L = {x | x has more 1's than 0's}

Initialize: C := 0 and B := 0
When the next symbol x is read,
If (C = 0) then B := x, C := 1
If (C $\neq$ 0) and (B = x) then C := C + 1
If (C $\neq$ 0) and (B $\neq$ x) then C := C – 1
When the stream stops,
accept if B=1 and C > 0, else *reject*

Want: A DFA that agrees with L on all strings of length ≤ 2



L = {x | x has more 1's than 0's}

Is there a streaming algorithm for L using much *less than* ($\log_2$ n) space?

Theorem: Every streaming algorithm for L requires at least ($\log_2$ n)-1 bits of space (for infinitely many n)

We will use:
- Myhill-Nerode Theorem
- The connection between DFAs and streaming

L = {x | x has more 1's than 0's}

Theorem: Every streaming algorithm for L requires at least ($\log_2$ n)-1 bits of space

Proof Idea: Let n be even, let L_n = {x in L : |x| = n}

We will give a set S_n of n/2+1 strings such that each pair in S_n is *distinguishable* in L_n

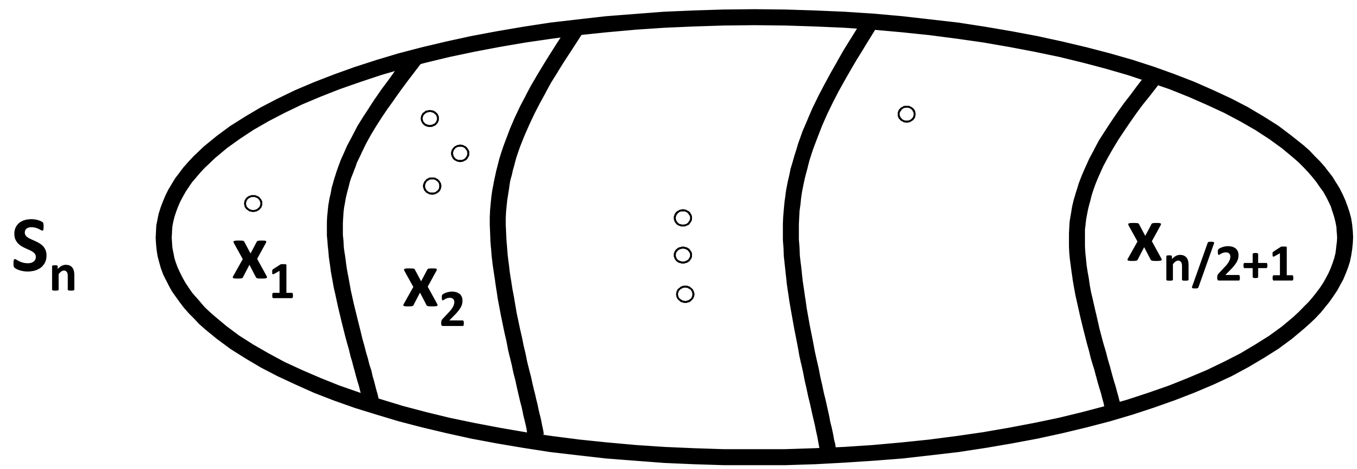
Myhill-Nerode Thm $\Rightarrow$ Every DFA recognizing L_n needs at least n/2+1 states

$\Rightarrow$ Every streaming algorithm for L needs at least (log n)-1 bits of memory on strings of length n

L = {x | x has more 1's than 0's}

Theorem: Every streaming algorithm for L requires at least ($\log_2$ n)-1 bits of space

Suppose we partition all strings into their equivalence classes under $\equiv_{L_n}$



But the number of states in a DFA recognizing L_n is at least the number of equivalence classes under $\equiv_{L_n}$

$L = \{x \mid x \text{ has more 1's than 0's}\}$

Theorem: Every streaming algorithm for L requires at least $(\log_2 n)-1$ bits of space

Proof (Slide 1): Let $S_n = \{0^{n/2-i}\, 1^i \mid i = 0,\ldots,n/2\}$
Let $x = 0^{n/2-k}\, 1^k$ and $y = 0^{n/2-j}\, 1^j$ be from S_n, with $k > j$

Claim: $z = 0^{k-1} 1^{n/2-(k-1)}$ distinguishes x and y in L_n

xz has n/2-1 zeroes and n/2+1 ones $\Rightarrow$ $xz \in L_n$

yz has n/2+(k-j-1) zeroes and n/2-(k-j-1) ones
But $k-j-1 \geq 0$, so $yz \notin L_n$

So the string z distinguishes x and y, and $x \not\equiv_{L_n} y$

$L = \{x \mid x$ has more 1's than 0's$\}$

Theorem: Every streaming algorithm for L requires at least $(\log_2 n)-1$ bits of space

Proof (Slide 2):

All pairs of strings in S_n are distinguishable in L_n

$\Rightarrow$ There are at least $|S_n|$ equiv classes of $\equiv_{L_n}$

By the Myhill-Nerode Theorem:

$\Rightarrow$ All DFAs recognizing L_n need $\geq |S_n|$ states

$\Rightarrow$ Every streaming algorithm for L requires at least $(\log_2 |S_n|)$ bits of space.

Recall $|S_n|=n/2+1$ and we're done!

Number of Distinct Elements

The DE problem
Input: $x \in \{0,1,\ldots,2^k\}^*$, $2^k > |x|^2$

Output: The number of distinct elements appearing in x

Note: There is a streaming algorithm for DE using $O(k\ n)$ space

Theorem: Every streaming algorithm for DE requires $\Omega(k\ n)$ space

Randomized Algorithms Help!

The DE problem
Input: $x \in \{0,1,\ldots,2^k\}^*$, $2^k > |x|^2$

Output: The number of distinct elements appearing in x

Theorem: There is a *randomized* streaming algorithm that can approximate DE to within 0.1% error, using O(k + log n) space!

See the lecture notes for more details.

Communication Complexity

Communication Complexity

A theoretical model of distributed computing

- **Function $f : \{0,1\}^* \times \{0,1\}^* \to \{0,1\}$**
  - Two inputs, $x \in \{0,1\}^*$ and $y \in \{0,1\}^*$
  - **We assume $|x|=|y|=n$. Think of n as HUGE**
- **Two computers: Alice** and **Bob**
  - **Alice** *only* knows x, **Bob** *only* knows y
- **Goal: Compute $f(x, y)$ by communicating as few bits as possible between Alice and Bob**

We do not count computation cost. We *only* care about the number of bits communicated.

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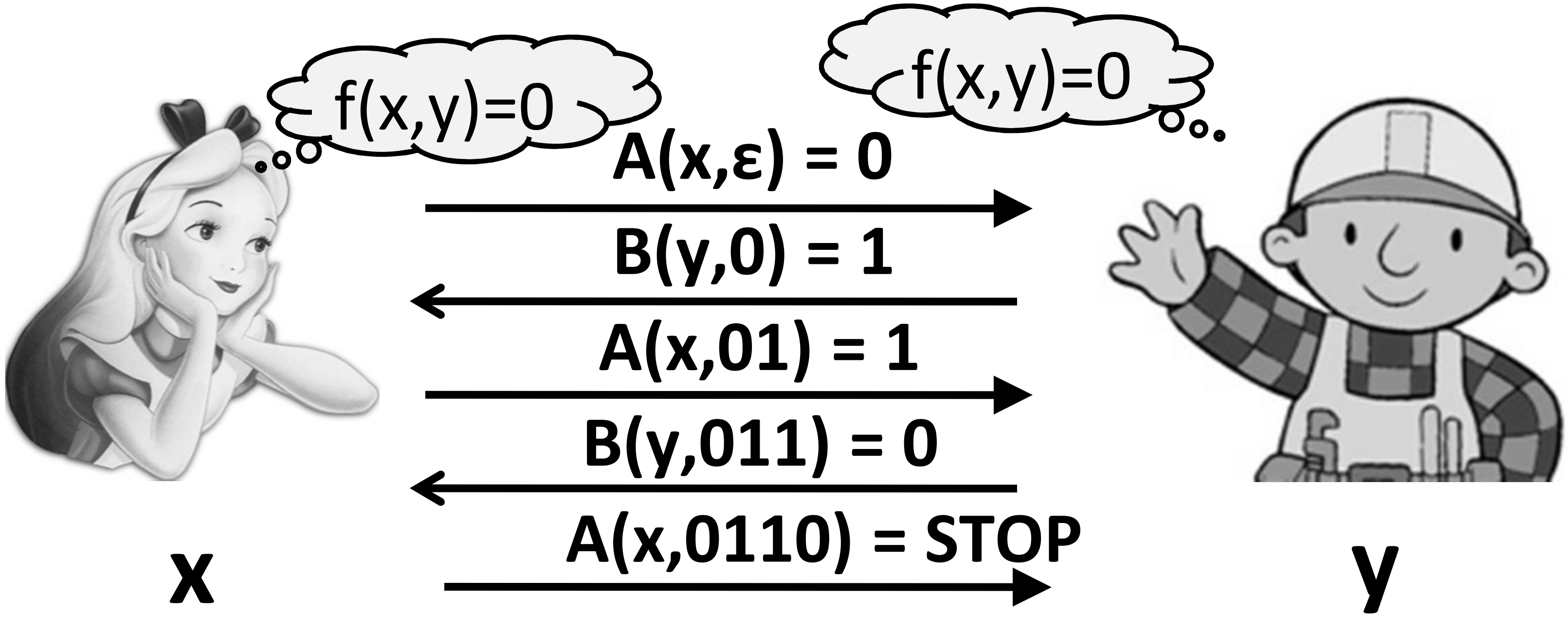
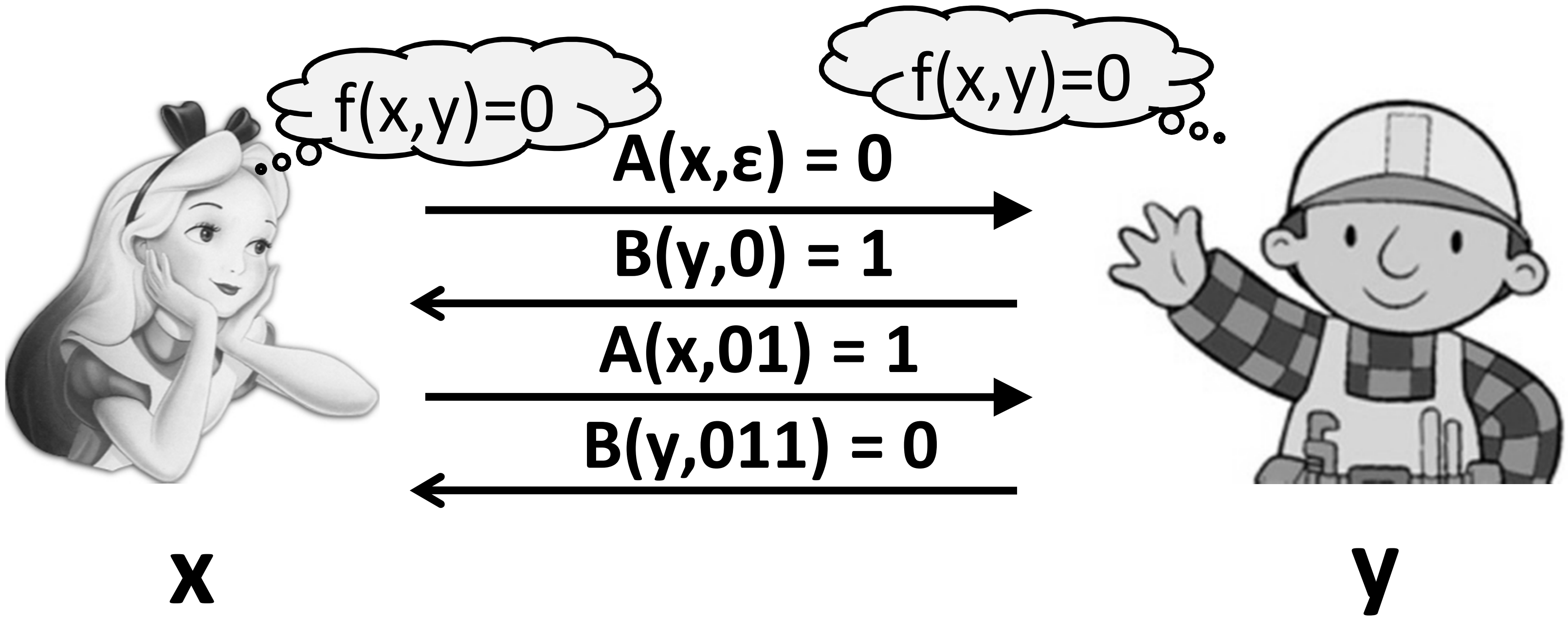
Alice and Bob Have a Conversation



In every step: A bit is sent, which is a function of the party's input and all the bits communicated so far.

Communication cost = number of bits communicated
= 4 (in the example)

We assume Alice and Bob alternate in communicating, and the last bit sent is the value of f([redacted]



Def. A *protocol* for a function f is a pair of functions

A, B : {0,1}* × {0,1}* → {0, 1, STOP} with the semantics:

On input (x, y), let r := 0, b_0 = ε.

While ($b_r \neq$ STOP),

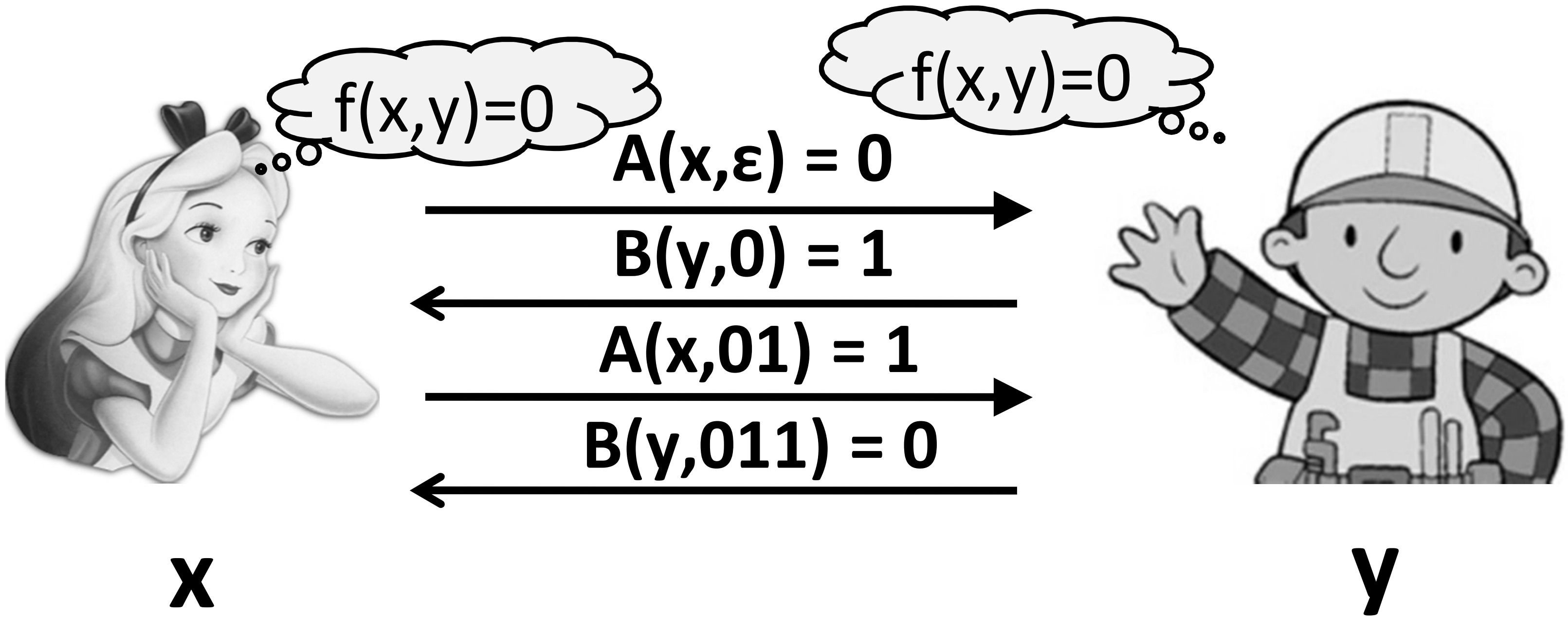
$r++$

If r is odd, Alice sends $b_r = A(x, b_1 \cdots b_{r-1})$

else Bob sends $b_r = B(y, b_1 \cdots b_{r-1})$

Output b_{r-1}. Number of *rounds* $= r - 1$

[redacted]



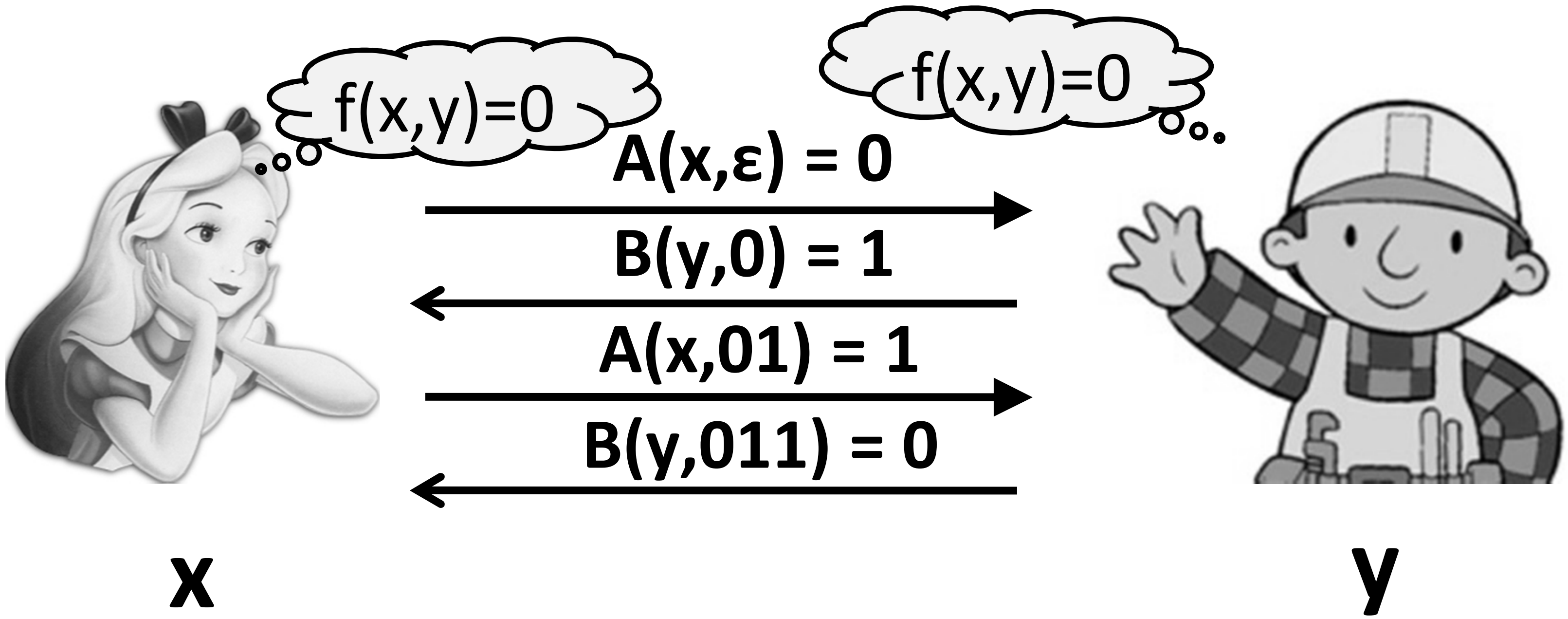
Def. The *cost of a protocol* P *for f on n-bit strings is*

$$\max_{x,y\,\in\{0,1\}^n} \text{[number of rounds in P to compute } f(x,y)\text{]}$$

The *communication complexity* of *f* on n-bit strings is the *minimum* cost over *all protocols* for *f* on n-bit strings

= the minimum number of rounds used by any protocol that computes *f(x, y),* over all n-bit x, y

[redacted]



Example. Let f : {0,1}* $\times$ {0,1}* → {0,1} be arbitrary

There is always a "trivial" protocol:

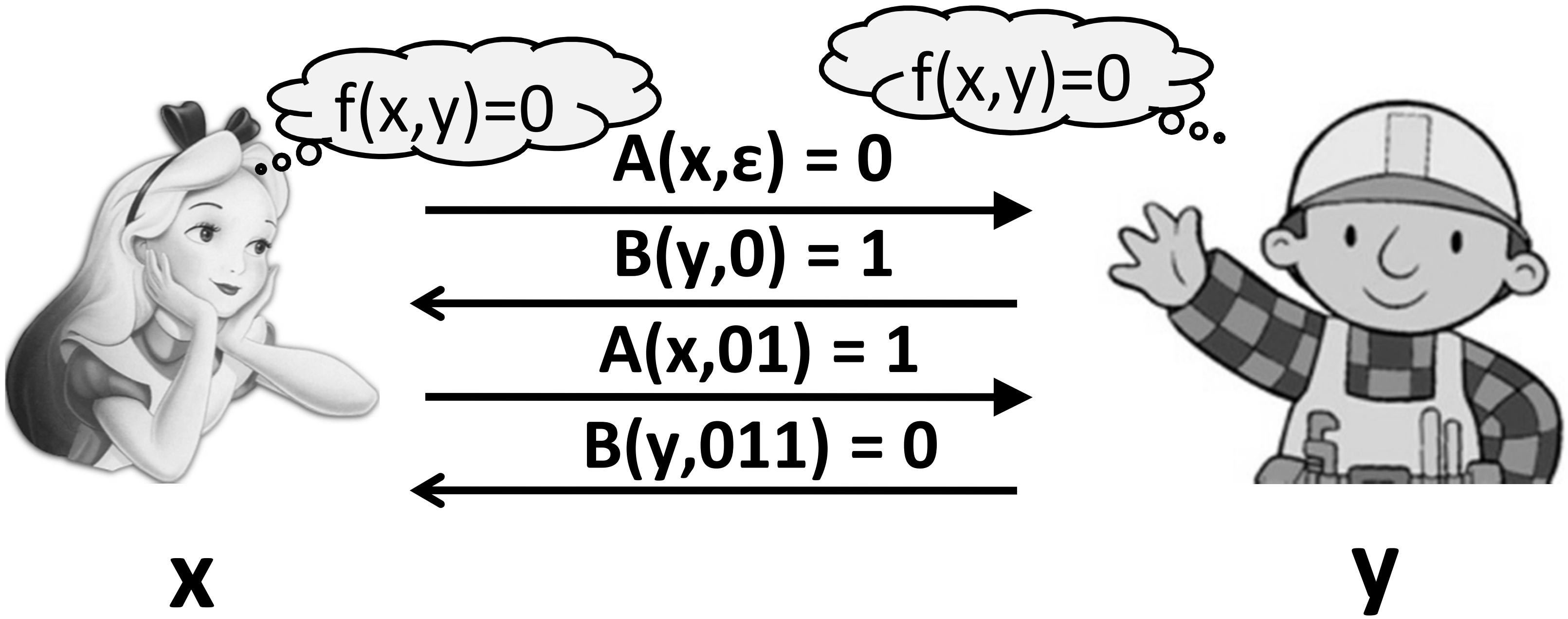
Alice sends the bits of her x in odd rounds

Bob sends the bits of his y in even rounds

After $2n$ rounds, they both know each other's input!

The communication complexity of every f is at most $2n$

[redacted]



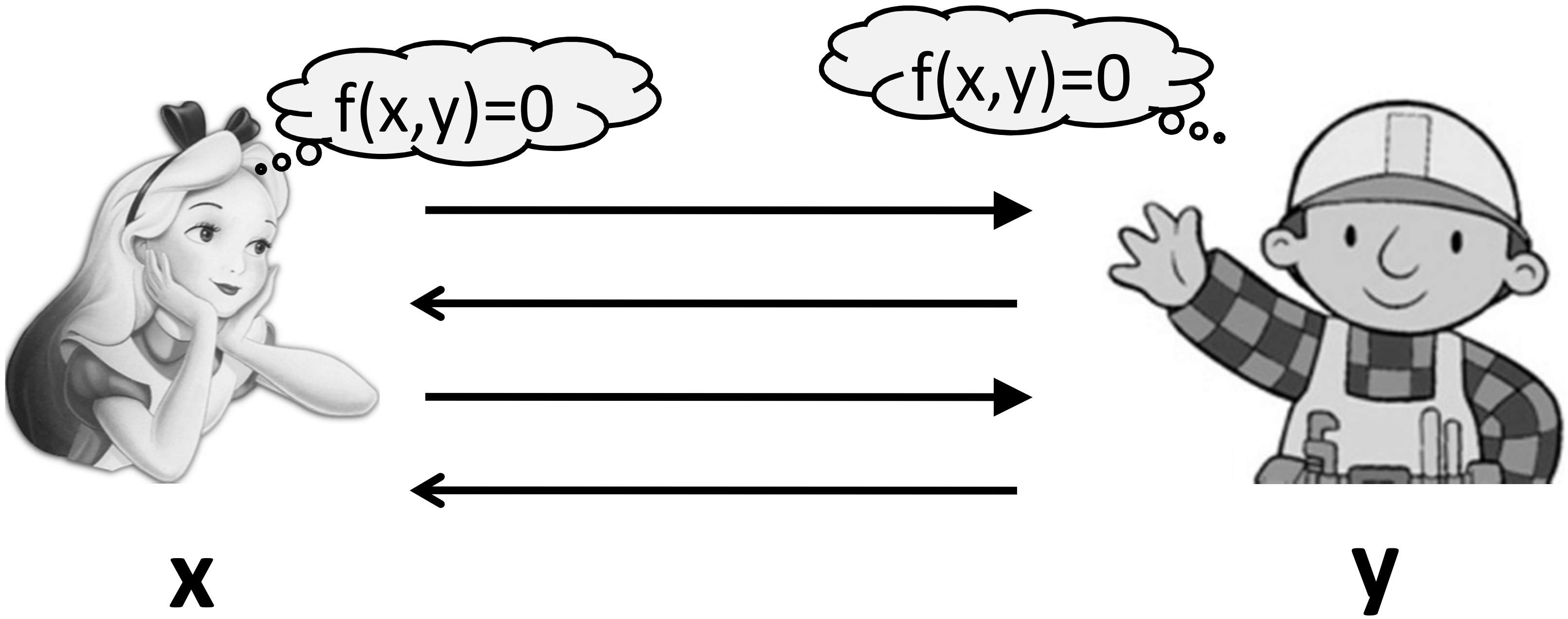
Example. **PARITY(x, y) = $\sum_i x_i + \sum_i y_i$ mod 2.**

What's a good protocol for computing PARITY?

Alice sends b_1 = ($\sum_i x_i$ mod 2)
Bob sends b_2 = ($b_1 + \sum_i y_i$ mod 2). Alice stops.

The communication complexity of PARITY is 2

[redacted]

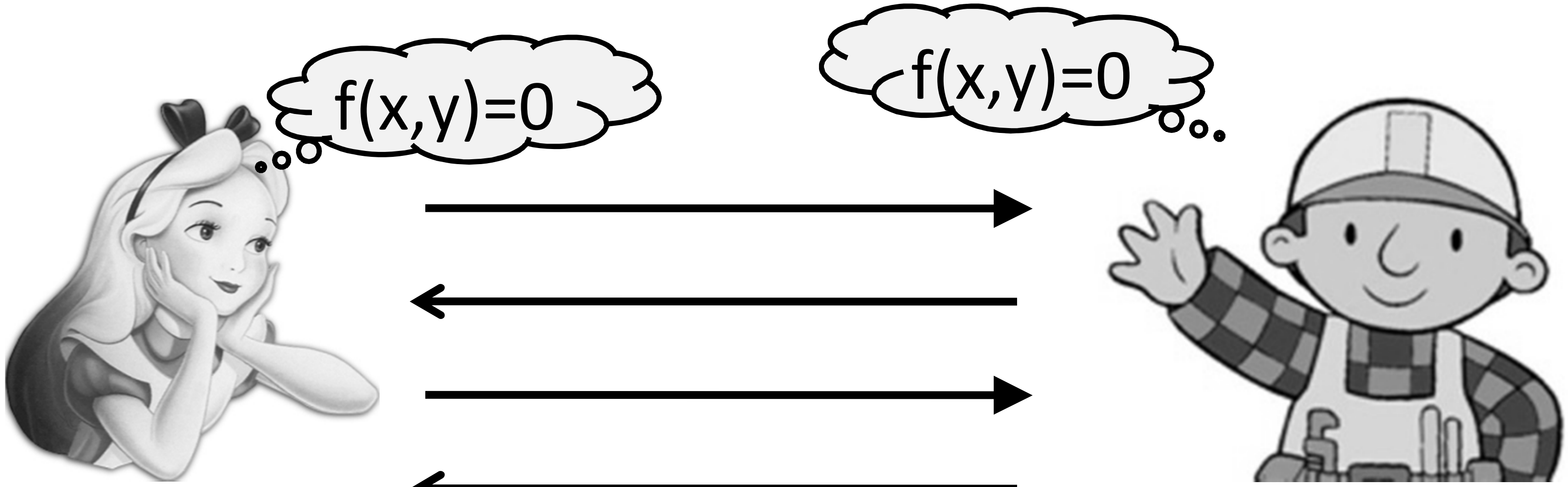


Example. MAJORITY(x, y) = most frequent bit in xy

What's a good protocol for computing MAJORITY?

Alice sends N_x $=$ number of 1s in x
Bob computes N_y $=$ number of 1s in y,
sends 1 iff $N_x + N_y$ is greater than (|x|+|y|)/2 = n

Communication complexity of MAJORITY is [redacted]



x **y**

Example. EQUALS(x, y) = 1 $\Leftrightarrow$ x = y

What's a good protocol for computing EQUALS?

????

Communication complexity of EQUALS is at most 2n

[redacted]