

# Mathematical Induction

## Part Two

Recap from Last Time

Let  $P$  be some predicate. The ***principle of mathematical induction*** states that if

If it starts  
true...

**$P(0)$  is true**

and

...and it stays  
true...

**$\forall k \in \mathbb{N}. (P(k) \rightarrow P(k+1))$**

then

**$\forall n \in \mathbb{N}. P(n)$**

...then it's  
always true.

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**Proof:** Let  $P(n)$  be the statement “the sum of the first  $n$  powers of two is  $2^n - 1$ .” We will prove, by induction, that  $P(n)$  is true for all  $n \in \mathbb{N}$ , from which the theorem follows.

For our base case, we need to show  $P(0)$  is true, meaning that the sum of the first zero powers of two is  $2^0 - 1$ . Since the sum of the first zero powers of two is zero and  $2^0 - 1$  is zero as well, we see that  $P(0)$  is true.

For the inductive step, assume that for some arbitrary  $k \in \mathbb{N}$  that  $P(k)$  holds, meaning that

$$2^0 + 2^1 + \dots + 2^{k-1} = 2^k - 1. \quad (1)$$

We need to show that  $P(k + 1)$  holds, meaning that the sum of the first  $k + 1$  powers of two is  $2^{k+1} - 1$ . To see this, notice that

$$\begin{aligned} 2^0 + 2^1 + \dots + 2^{k-1} + 2^k &= (2^0 + 2^1 + \dots + 2^{k-1}) + 2^k \\ &= 2^k - 1 + 2^k && \text{(via (1))} \\ &= 2(2^k) - 1 \\ &= 2^{k+1} - 1. \end{aligned}$$

Therefore,  $P(k + 1)$  is true, completing the induction. ■

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New Stuff!



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# Induction in Practice

- Typically, a proof by induction will not explicitly state  $P(n)$ .
- Rather, the proof will describe  $P(n)$  implicitly and leave it to the reader to fill in the details.
- Provided that there is sufficient detail to determine
  - what  $P(n)$  is;
  - that  $P(0)$  is true; and that
  - whenever  $P(k)$  is true,  $P(k+1)$  is true,the proof is usually valid.

**Theorem:** The sum of the first  $n$  powers of two is  $2^n - 1$ .

**Proof:** By induction.

For our base case, we'll prove the theorem is true when  $n = 0$ . The sum of the first zero powers of two is zero, and  $2^0 - 1 = 0$ , so the theorem is true in this case.

For the inductive step, assume the theorem holds when  $n = k$  for some arbitrary  $k \in \mathbb{N}$ . Then

$$\begin{aligned} 2^0 + 2^1 + \dots + 2^{k-1} + 2^k &= (2^0 + 2^1 + \dots + 2^{k-1}) + 2^k \\ &= 2^k - 1 + 2^k \\ &= 2(2^k) - 1 \\ &= 2^{k+1} - 1. \end{aligned}$$

So the theorem is true when  $n = k+1$ , completing the induction. ■

Variations on Induction: ***Starting Later***

# Induction Starting at 0

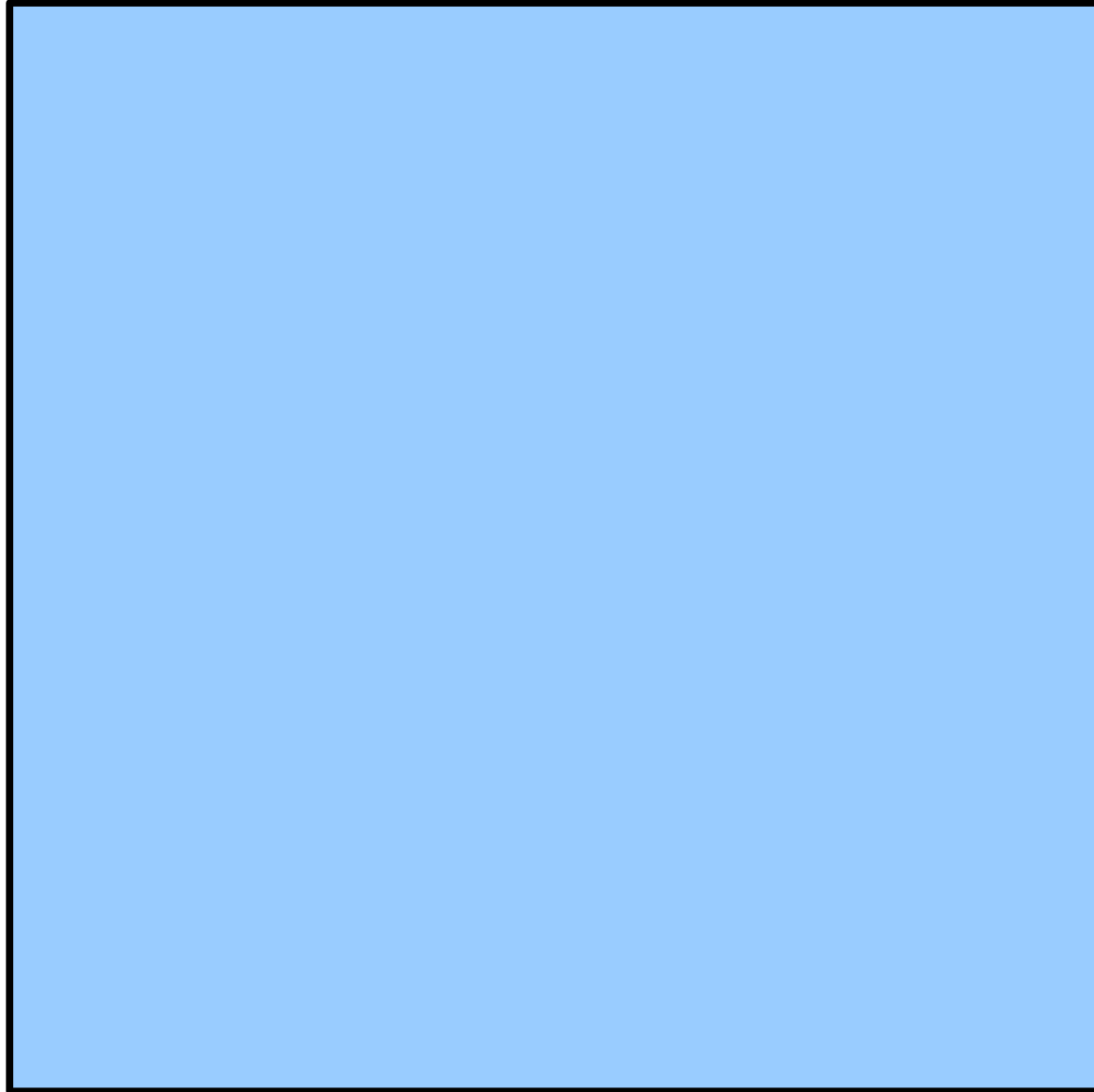
- To prove that  $P(n)$  is true for all natural numbers greater than or equal to 0:
  - Show that  $P(0)$  is true.
  - Show that for any  $k \geq 0$ , that if  $P(k)$  is true, then  $P(k+1)$  is true.
  - Conclude  $P(n)$  holds for all natural numbers greater than or equal to 0.

# Induction Starting at $m$

- To prove that  $P(n)$  is true for all natural numbers greater than or equal to  $m$ :
  - Show that  $P(m)$  is true.
  - Show that for any  $k \geq m$ , that if  $P(k)$  is true, then  $P(k+1)$  is true.
  - Conclude  $P(n)$  holds for all natural numbers greater than or equal to  $m$ .

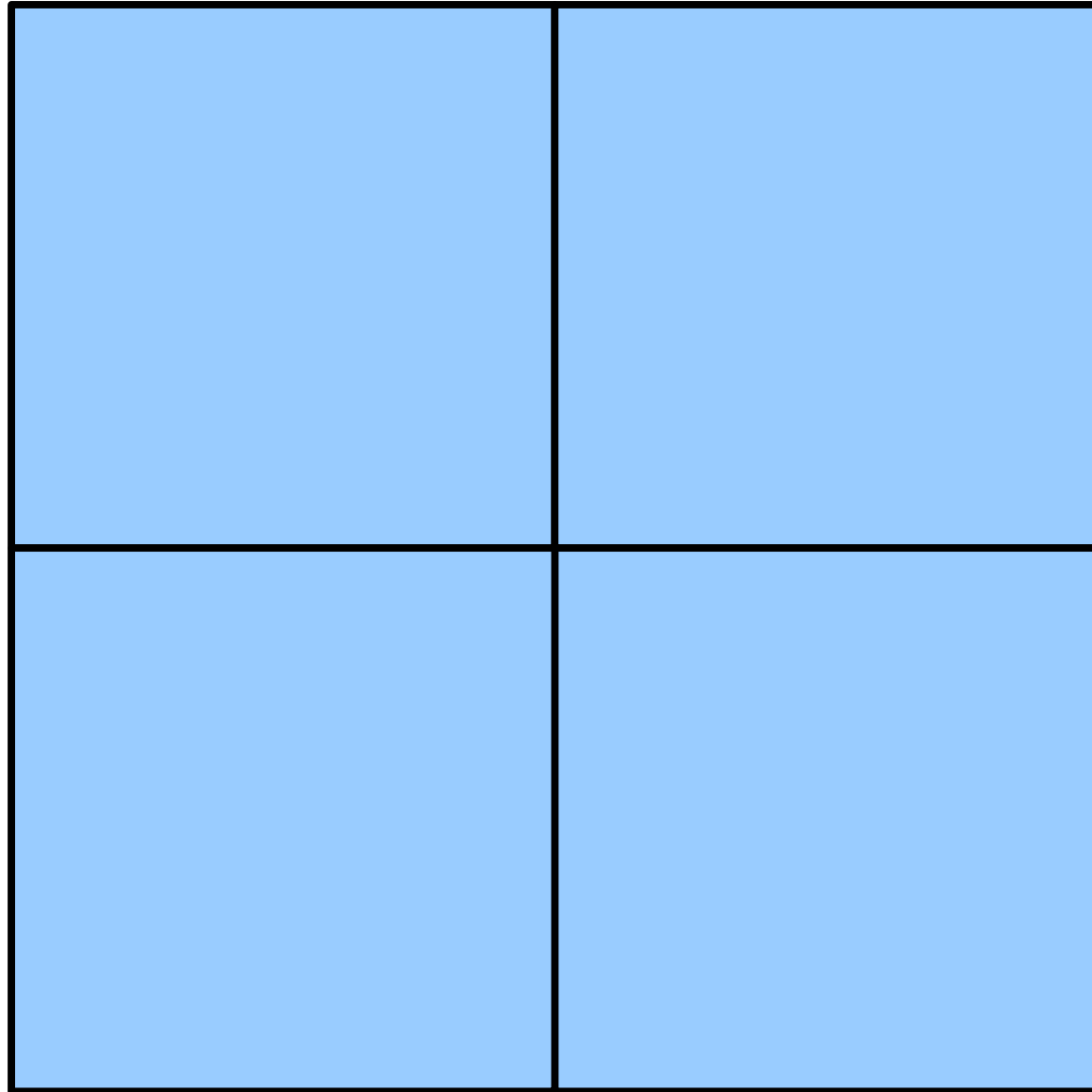
Variations on Induction: ***Bigger Steps***

# Subdividing a Square

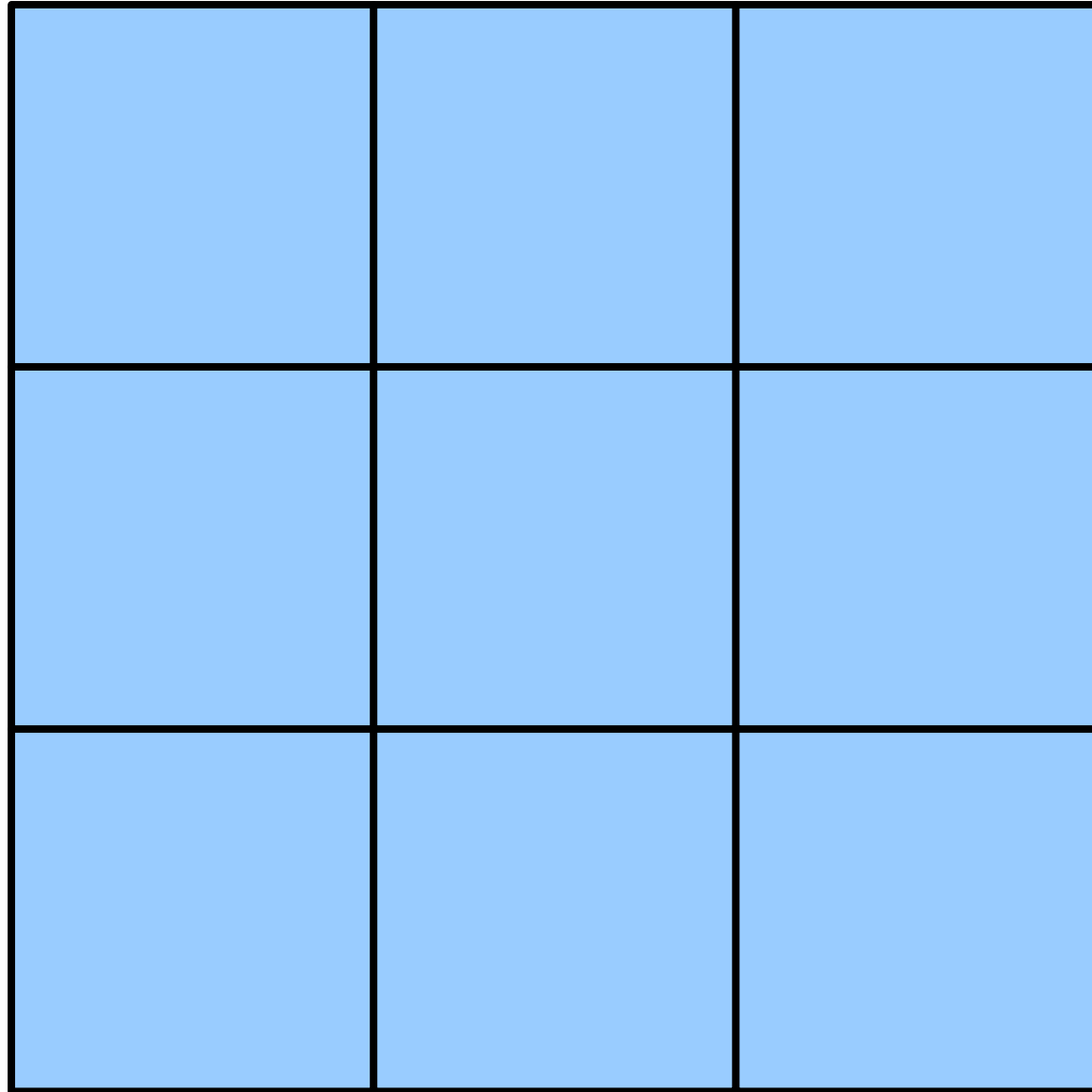




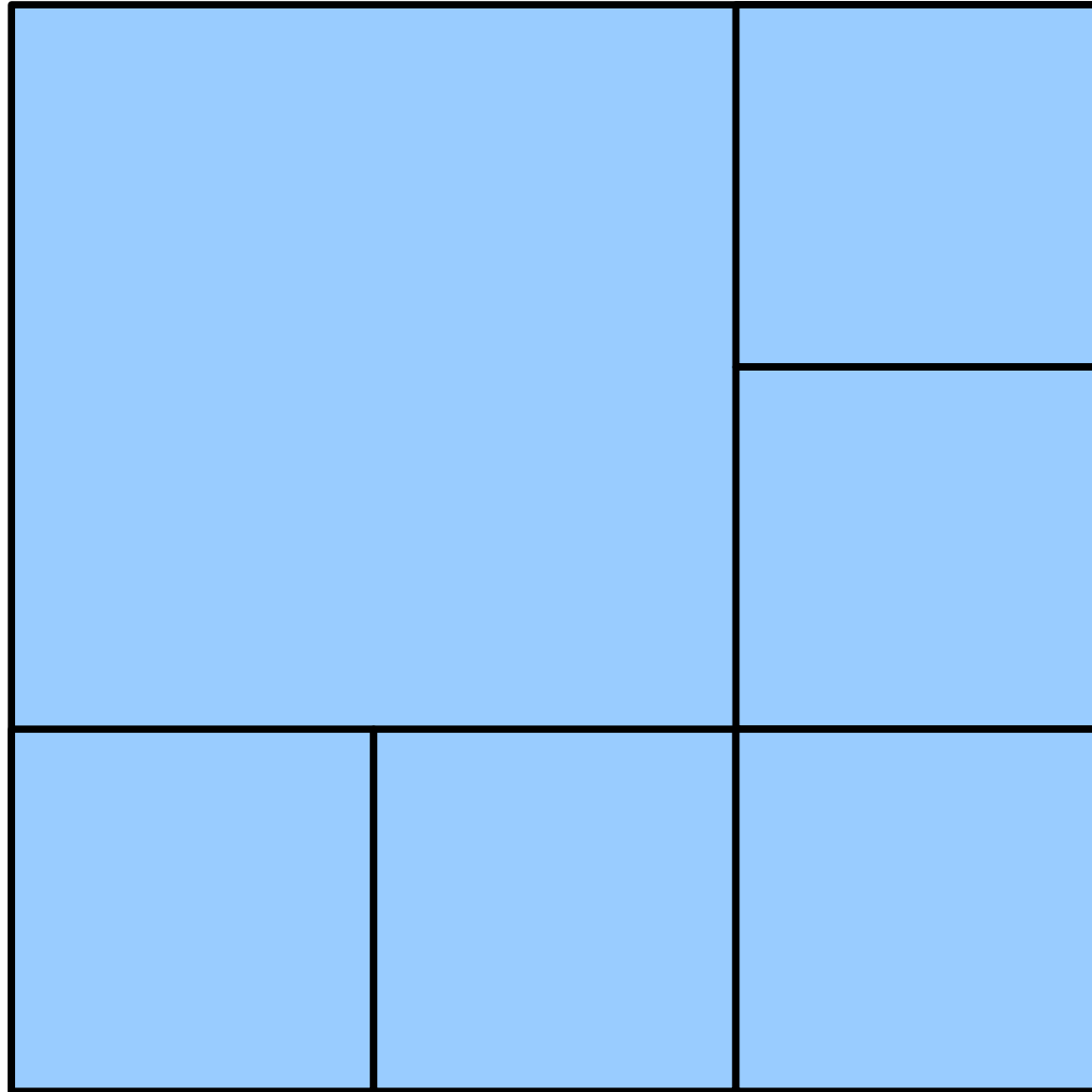
# Subdividing a Square



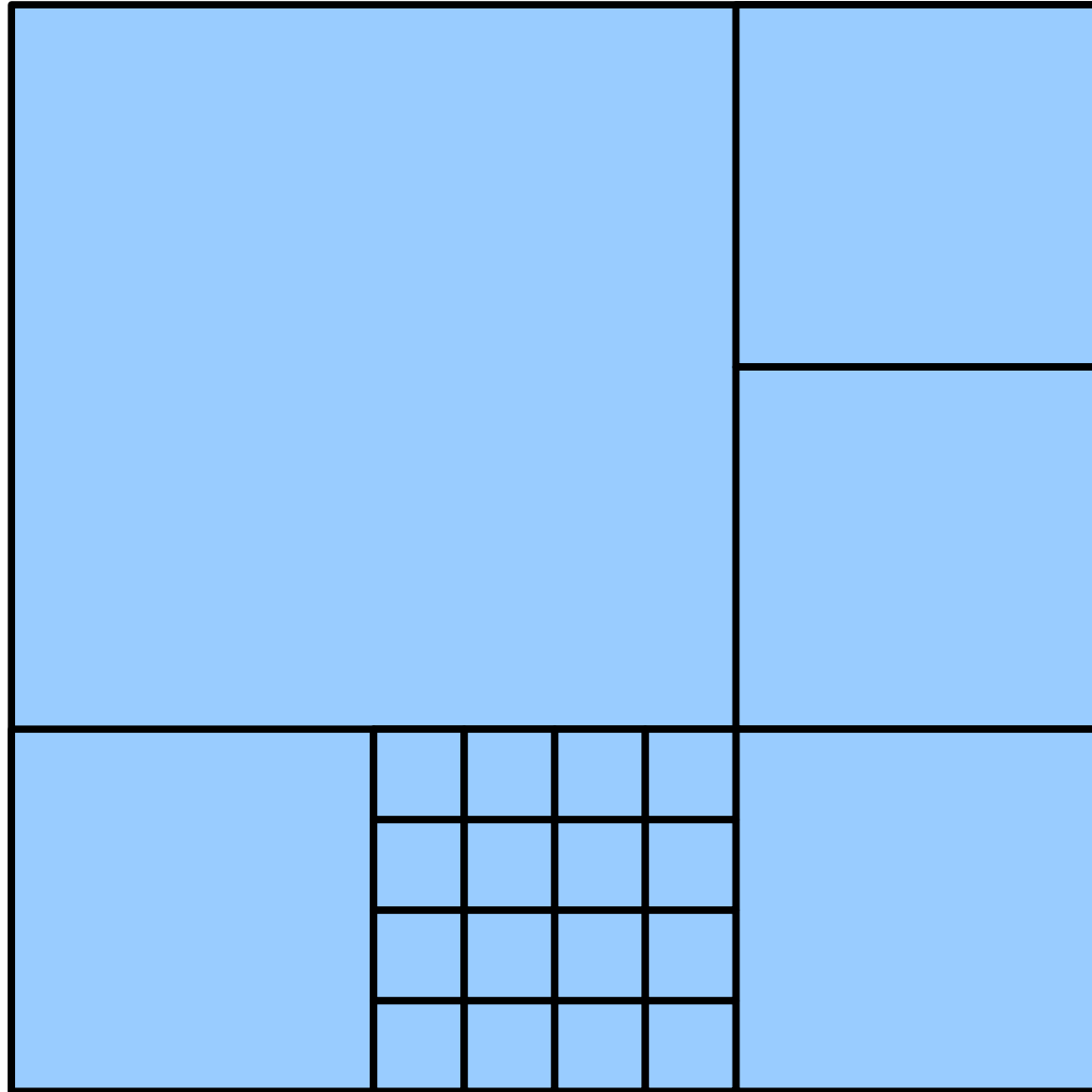
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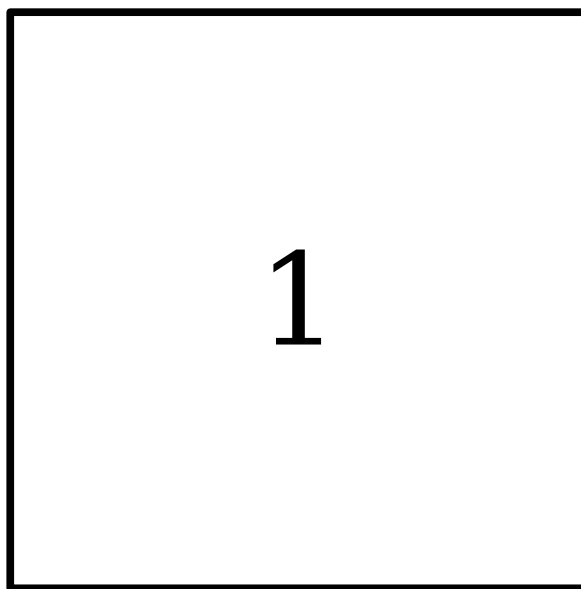


For what values of  $n$  can a square be subdivided into  $n$  squares?

1 2 3 4 5 6 7 8 9 10 11 12

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1 2 3 4 5 6 7 8 9 10 11 12





1   ~~2~~   ~~3~~   4   ~~5~~   6   7   8   9   10   11   12

1	2
4	3

1   ~~2~~   ~~3~~   4   ~~5~~   6   7   8   9   10   11   12

1		2
		3
6	5	4

1   ~~2~~   ~~3~~   4   ~~5~~   6   7   8   9   10   11   12

5	6	1
4	7	
3		2

1   ~~2~~   ~~3~~   4   ~~5~~   6   7   8   9   10   11   12

1	8		
2			
3			
4	5	6	7

1   ~~2~~   ~~3~~   4   ~~5~~   6   7   8   9   10   11   12

1	2	3
8	9	4
7	6	5

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1	2	3	
8	9		
7		10	4
		6	5

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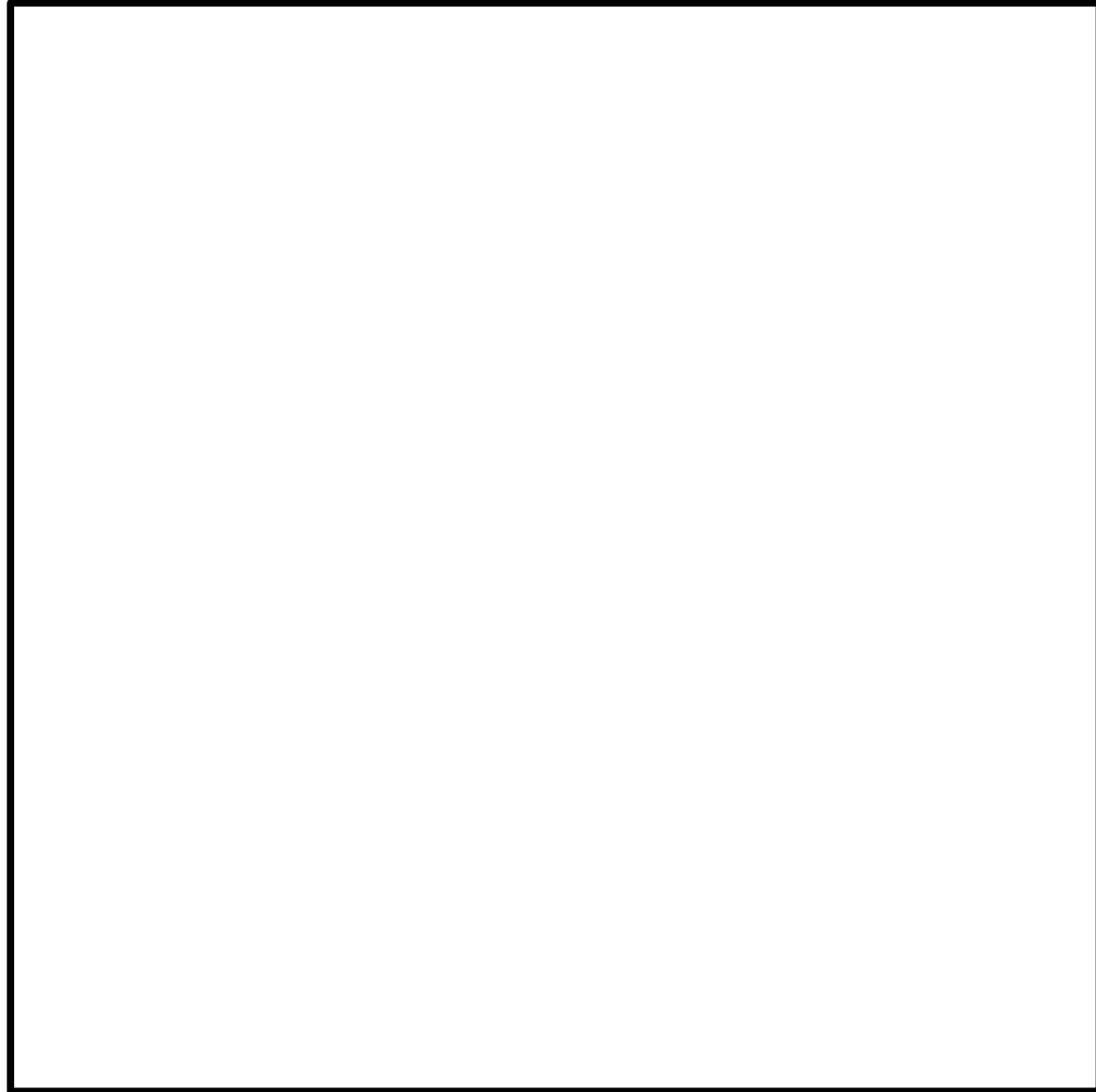
1	10		9
2	11		8
3			
4	5	6	7

1   ~~2~~   ~~3~~   4   ~~5~~   6   7   8   9   10   11   12

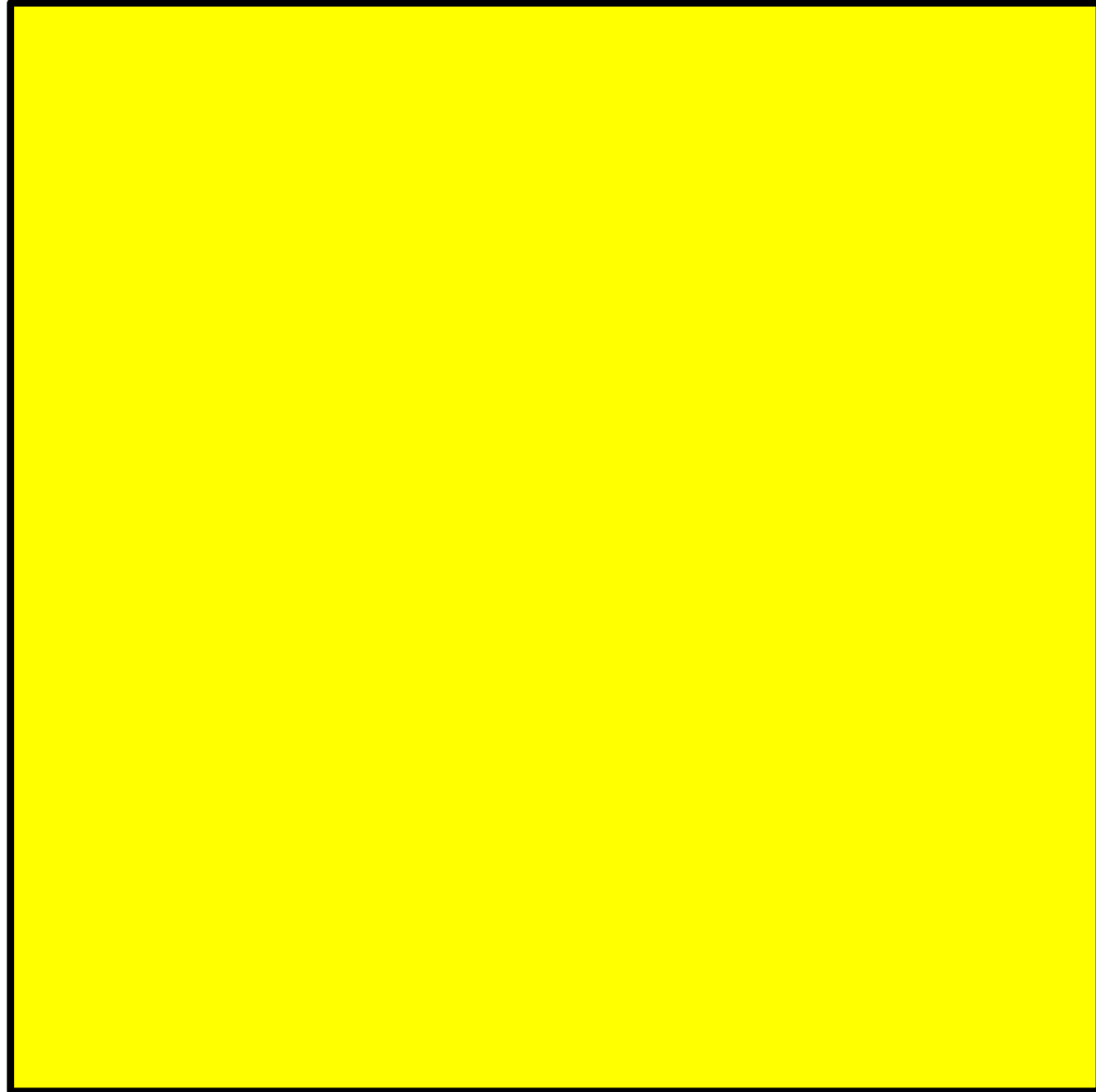
1	2		3
8	9	10	4
	12	11	
7	6		5



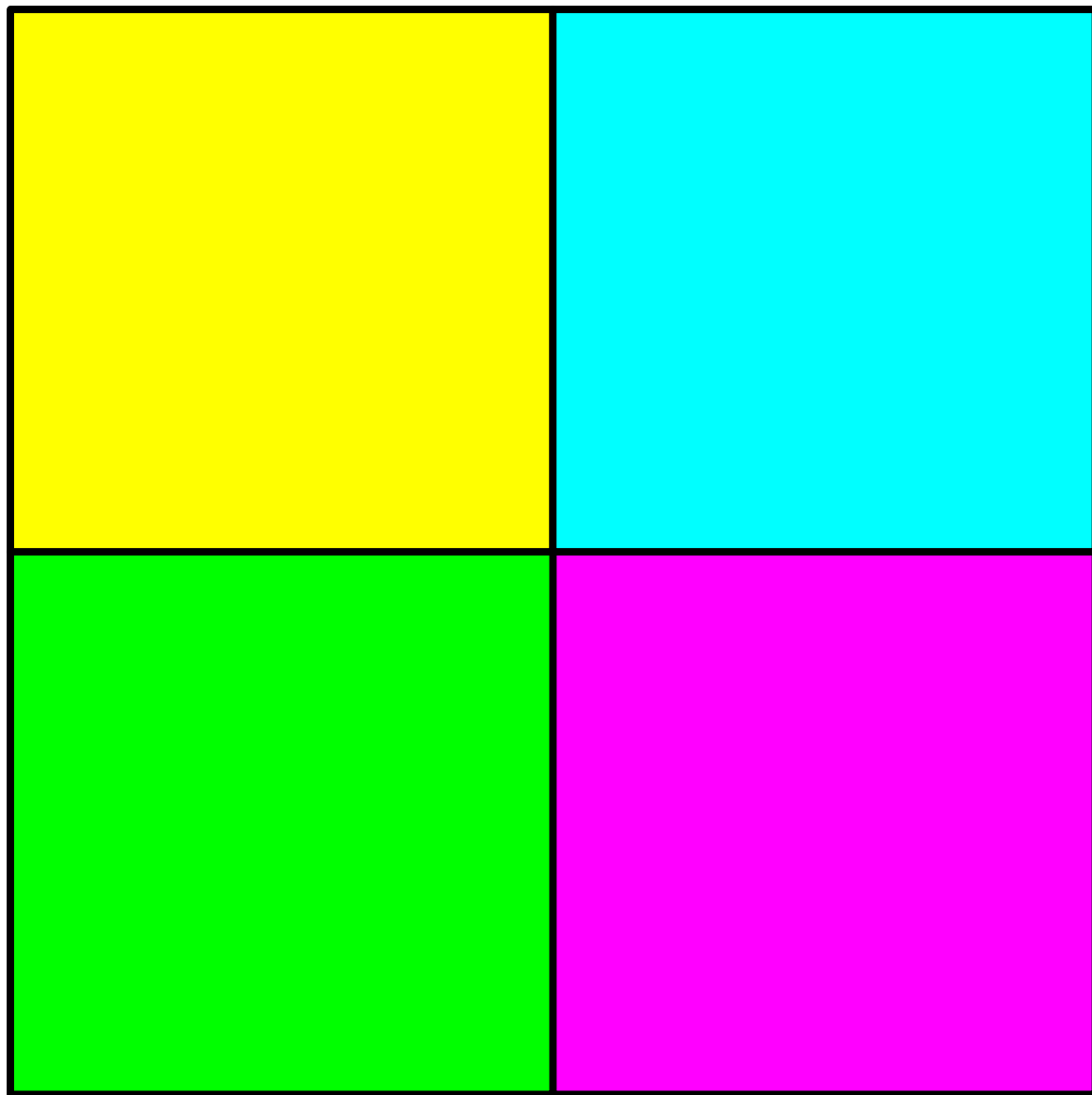
# The Key Insight



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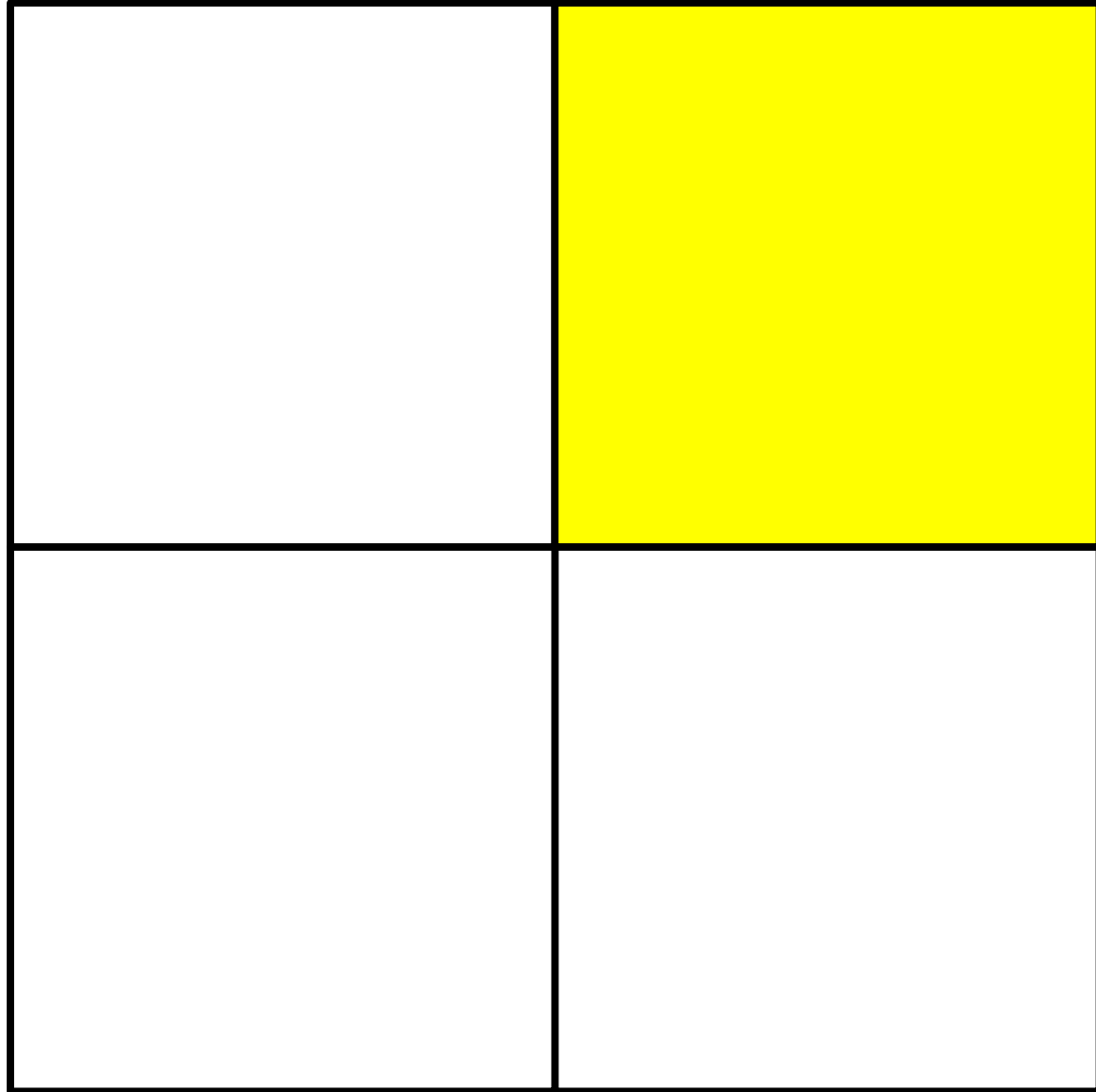


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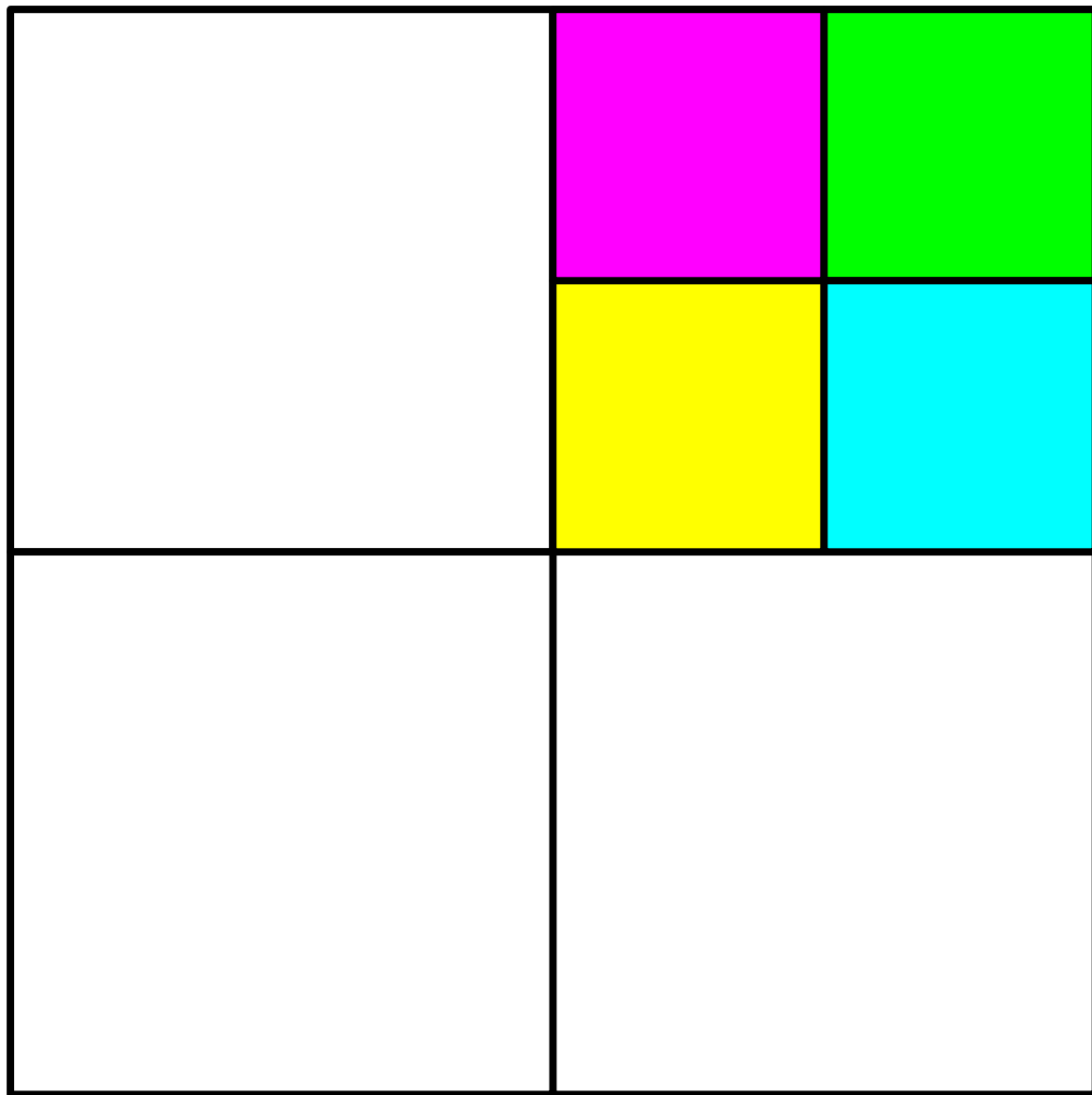


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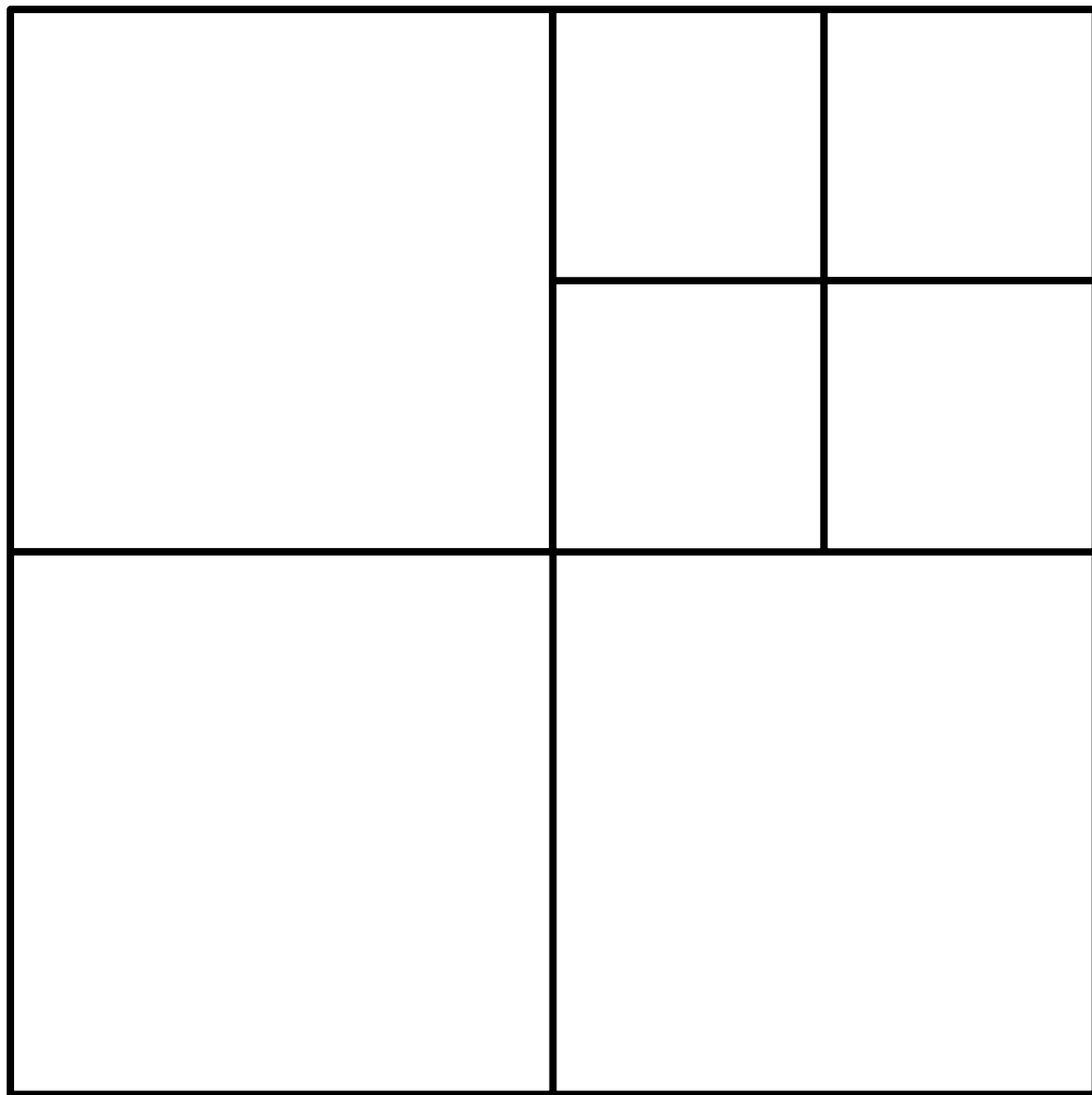

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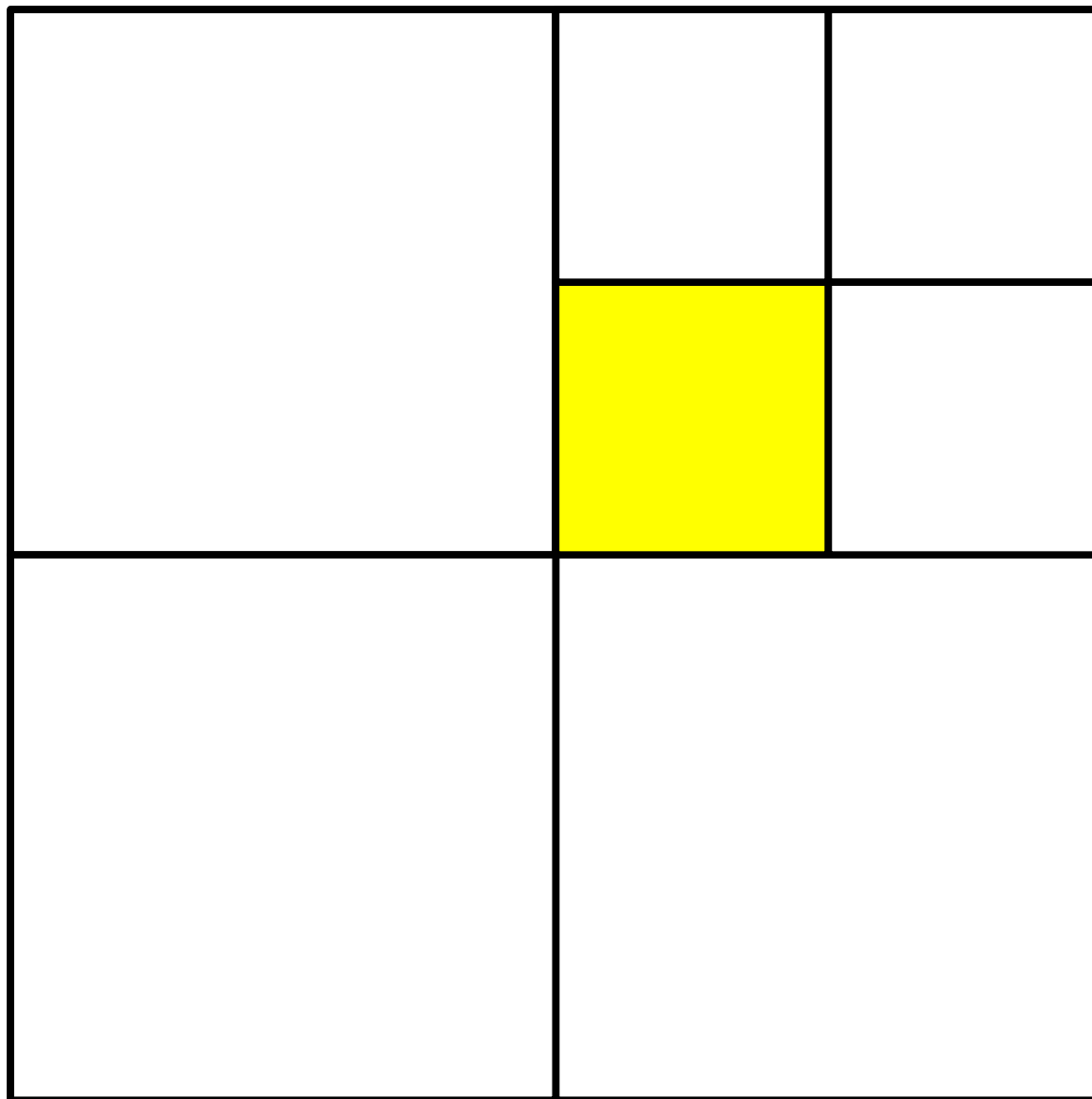
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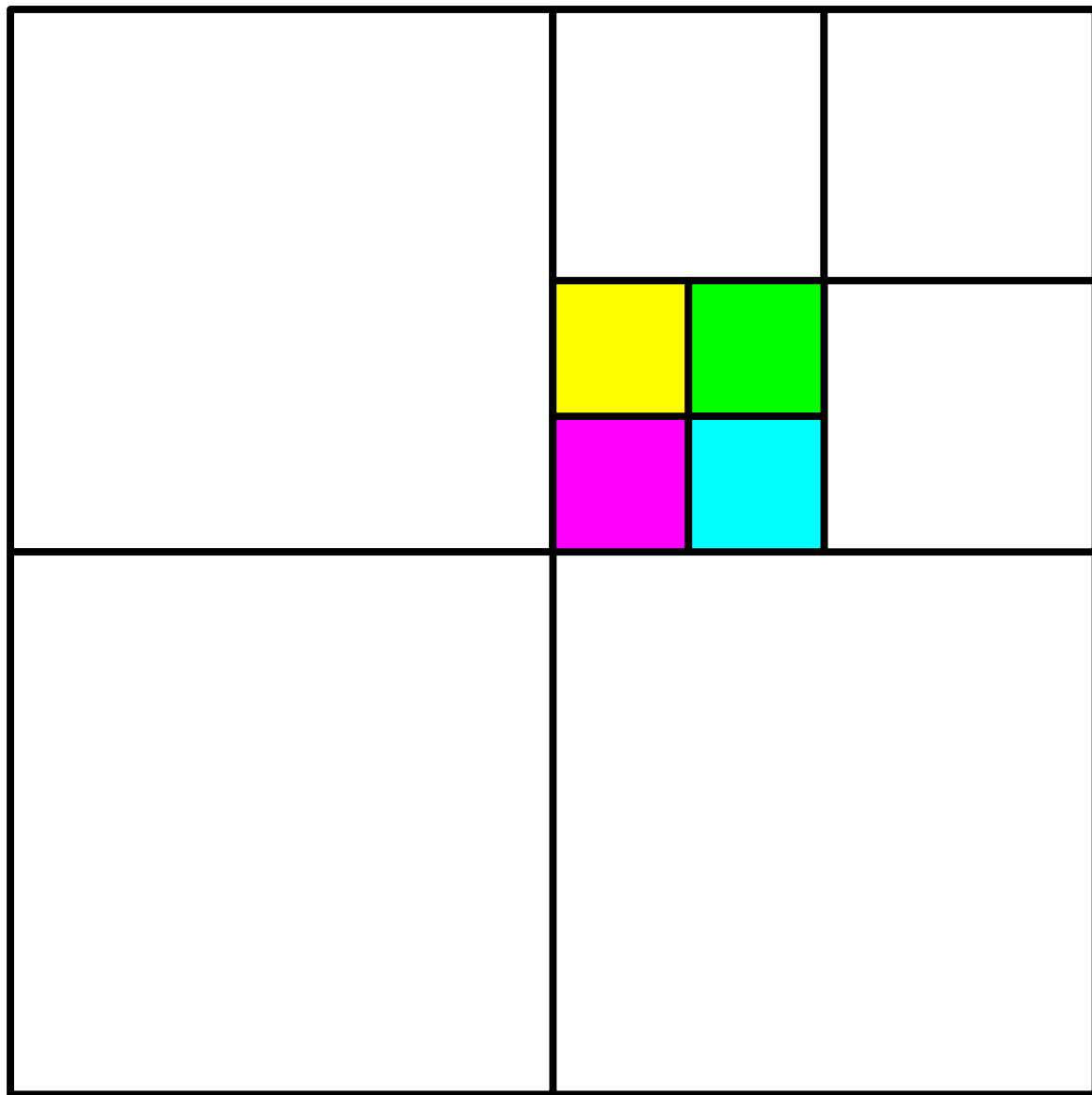


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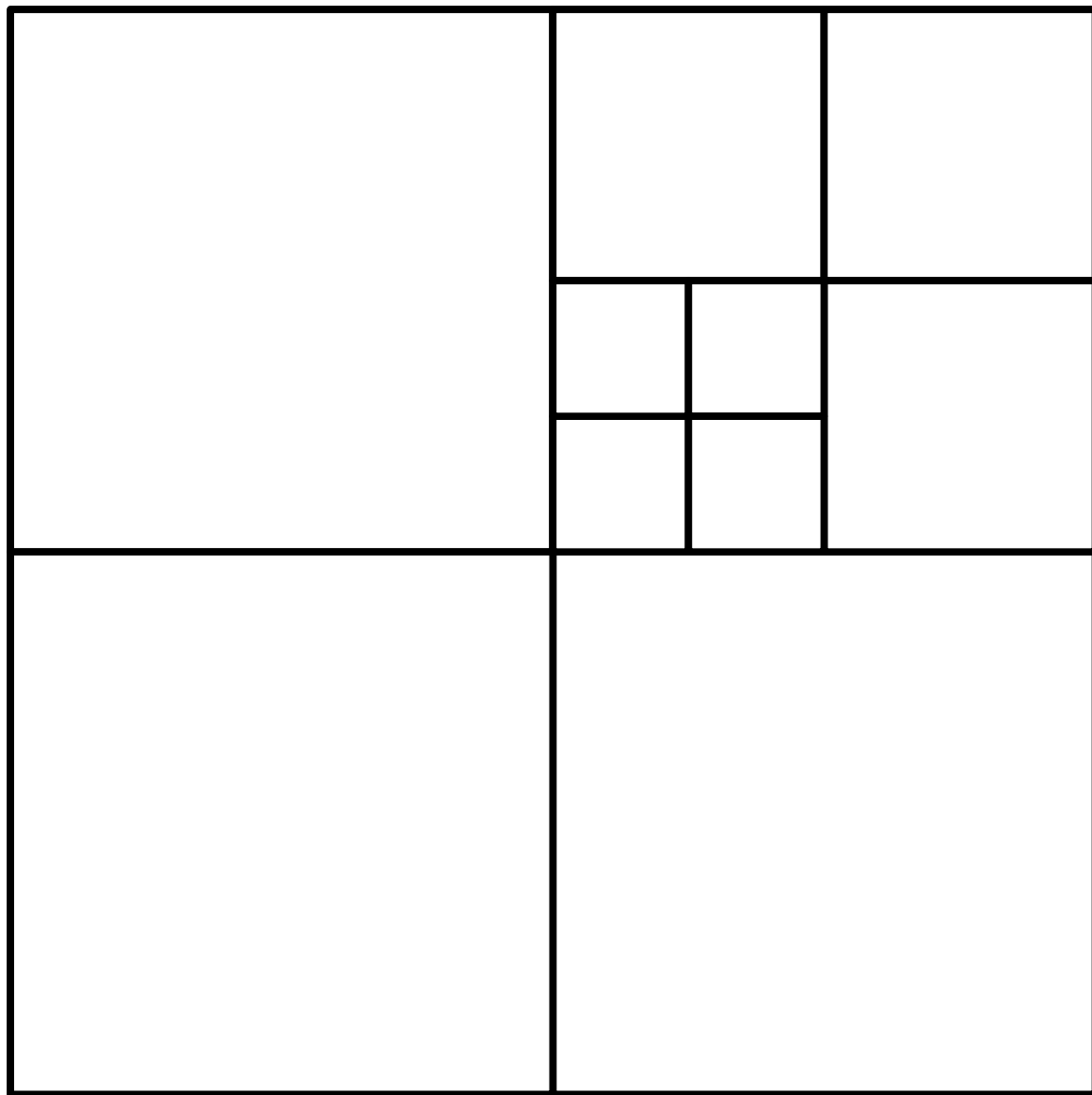




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- If we can subdivide a square into  $n$  squares, we can also subdivide it into  $n + 3$  squares.
- Since we can subdivide a bigger square into 6, 7, and 8 squares, we can subdivide a square into  $n$  squares for any  $n \geq 6$ :
  - For multiples of three, start with 6 and keep adding three squares until  $n$  is reached.
  - For numbers congruent to one modulo three, start with 7 and keep adding three squares until  $n$  is reached.
  - For numbers congruent to two modulo three, start with 8 and keep adding three squares until  $n$  is reached.

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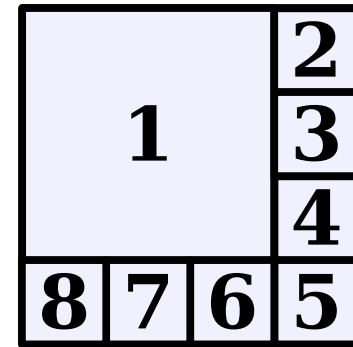
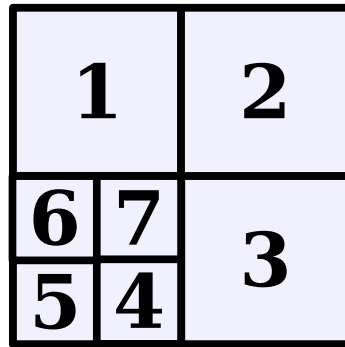
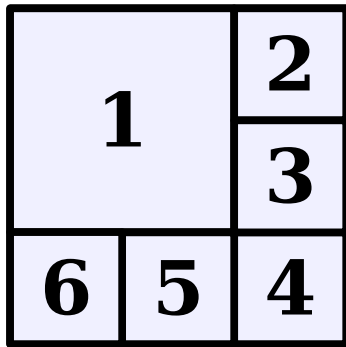
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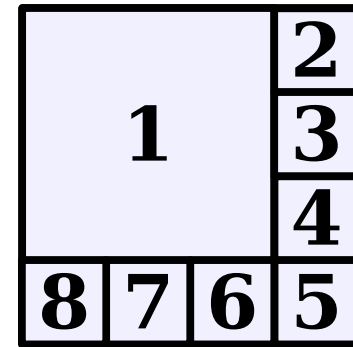
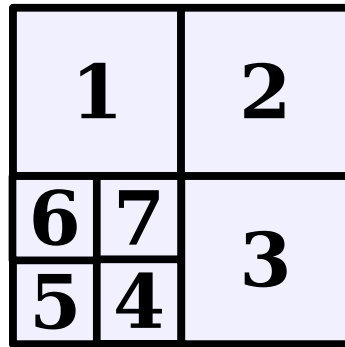
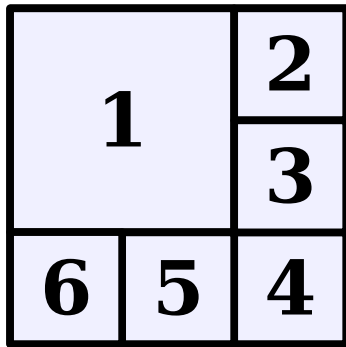
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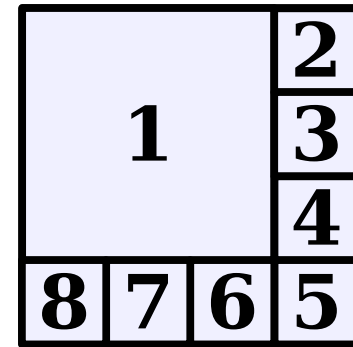
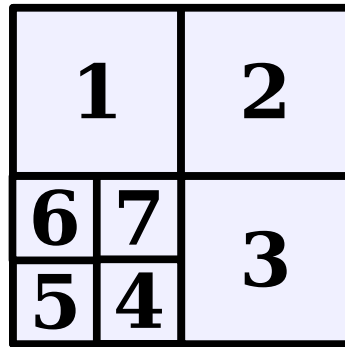
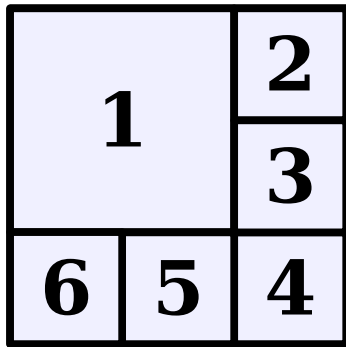


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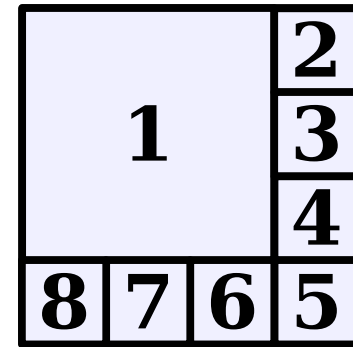
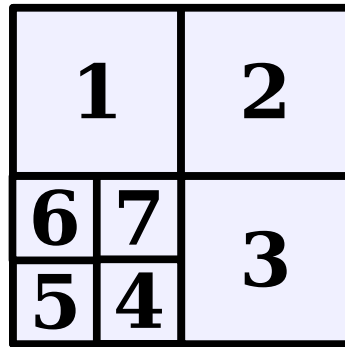
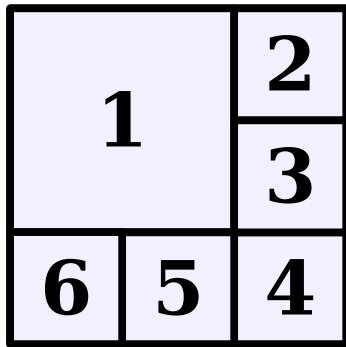


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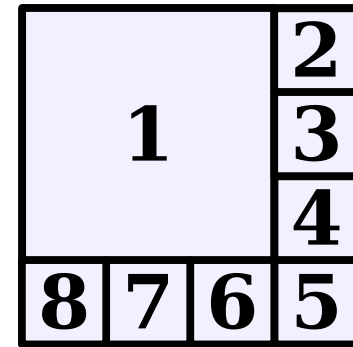
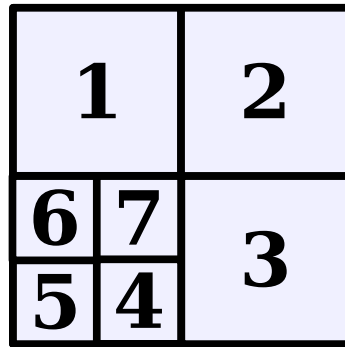
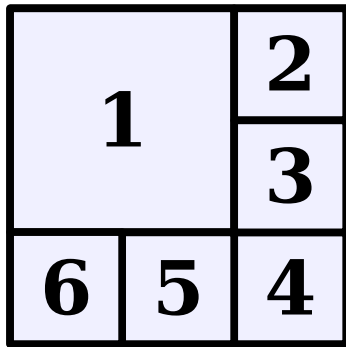


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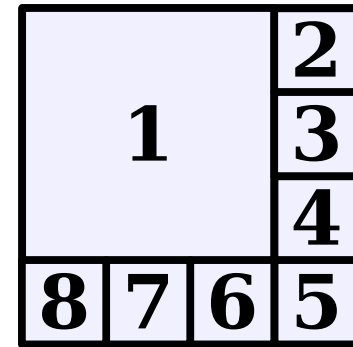
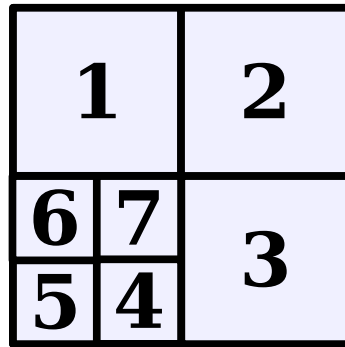
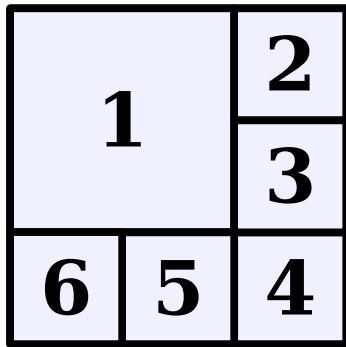


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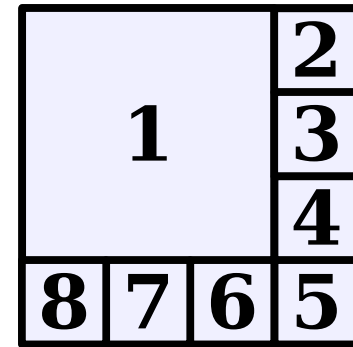
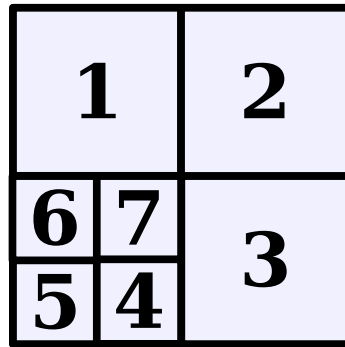
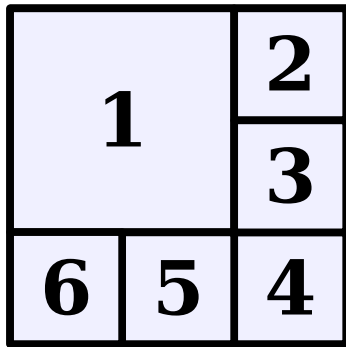


For the inductive step, assume that for some arbitrary  $k \geq 6$  that  $P(k)$  is true and that a square can be subdivided into  $k$  squares. We prove  $P(k+3)$ , that a square can be subdivided into  $k+3$  squares. To see this, start by obtaining (via the inductive hypothesis) a subdivision of a square into  $k$  squares. Then, choose any of the squares and split it into four equal squares. This removes one of the  $k$  squares and adds four more, so there will be a net total of  $k+3$  squares.

**Theorem:** For any  $n \geq 6$ , it is possible to subdivide a square into  $n$  smaller squares.

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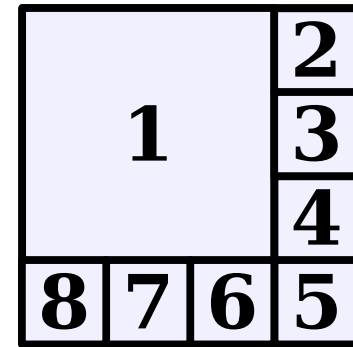
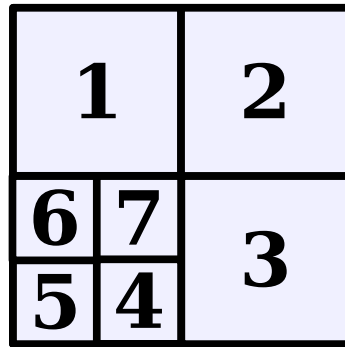
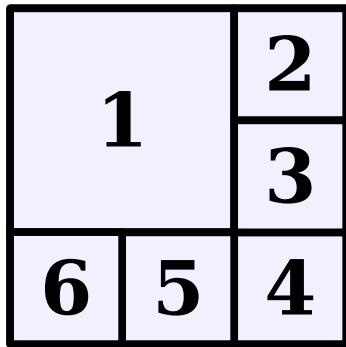


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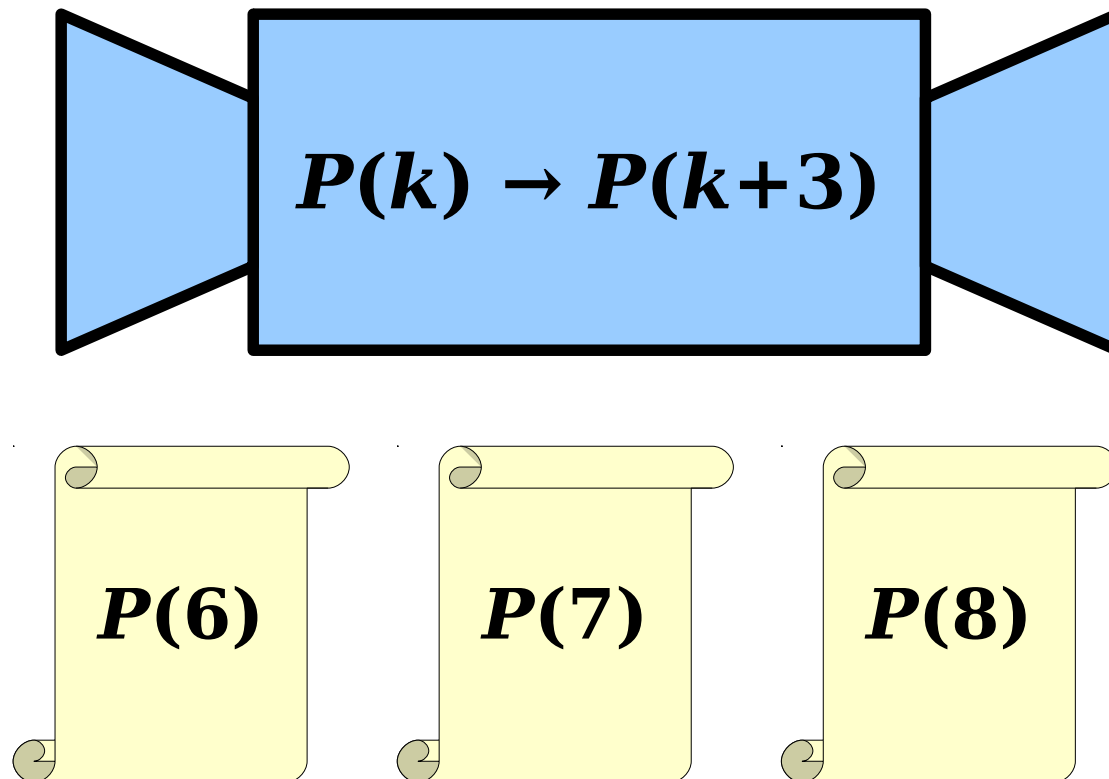


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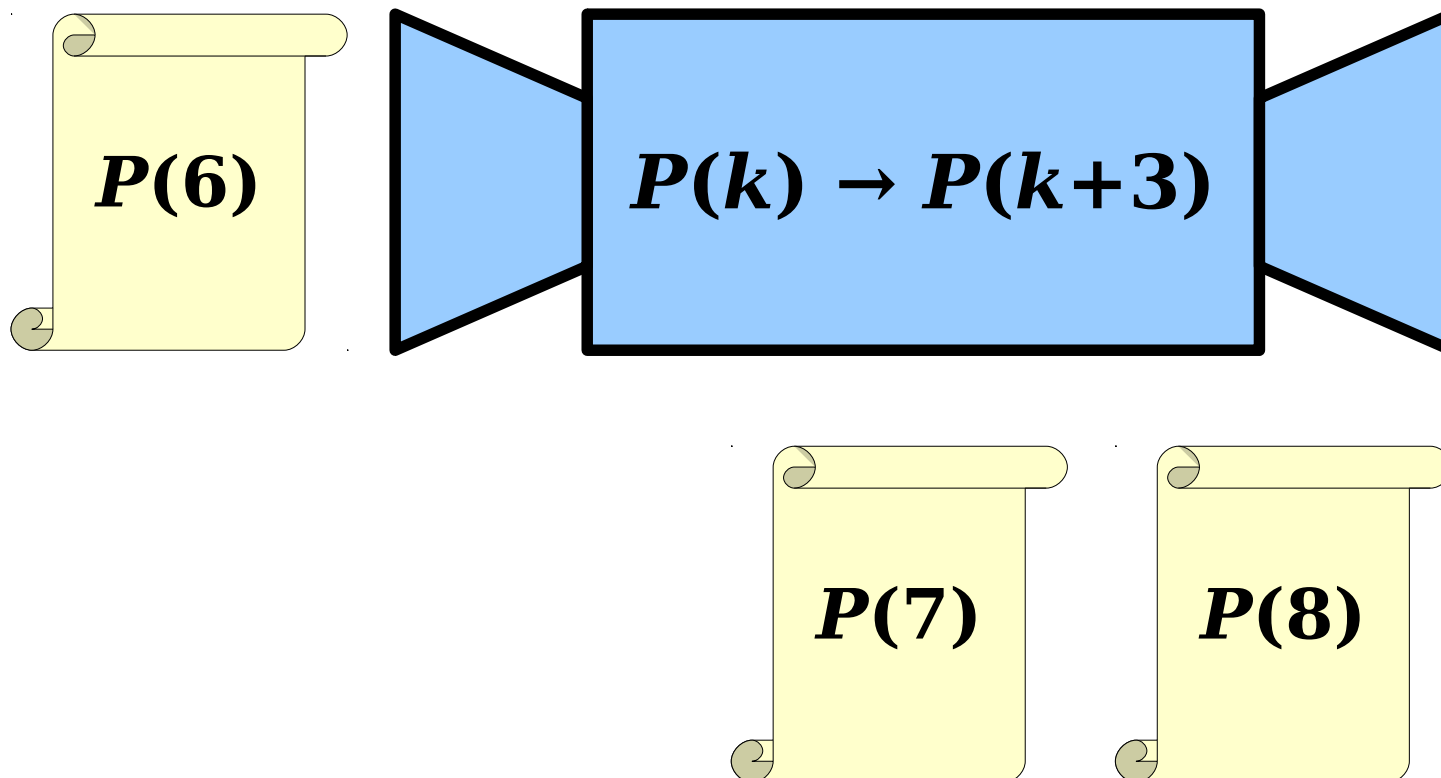
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- This induction has three consecutive base cases and takes steps of size three.
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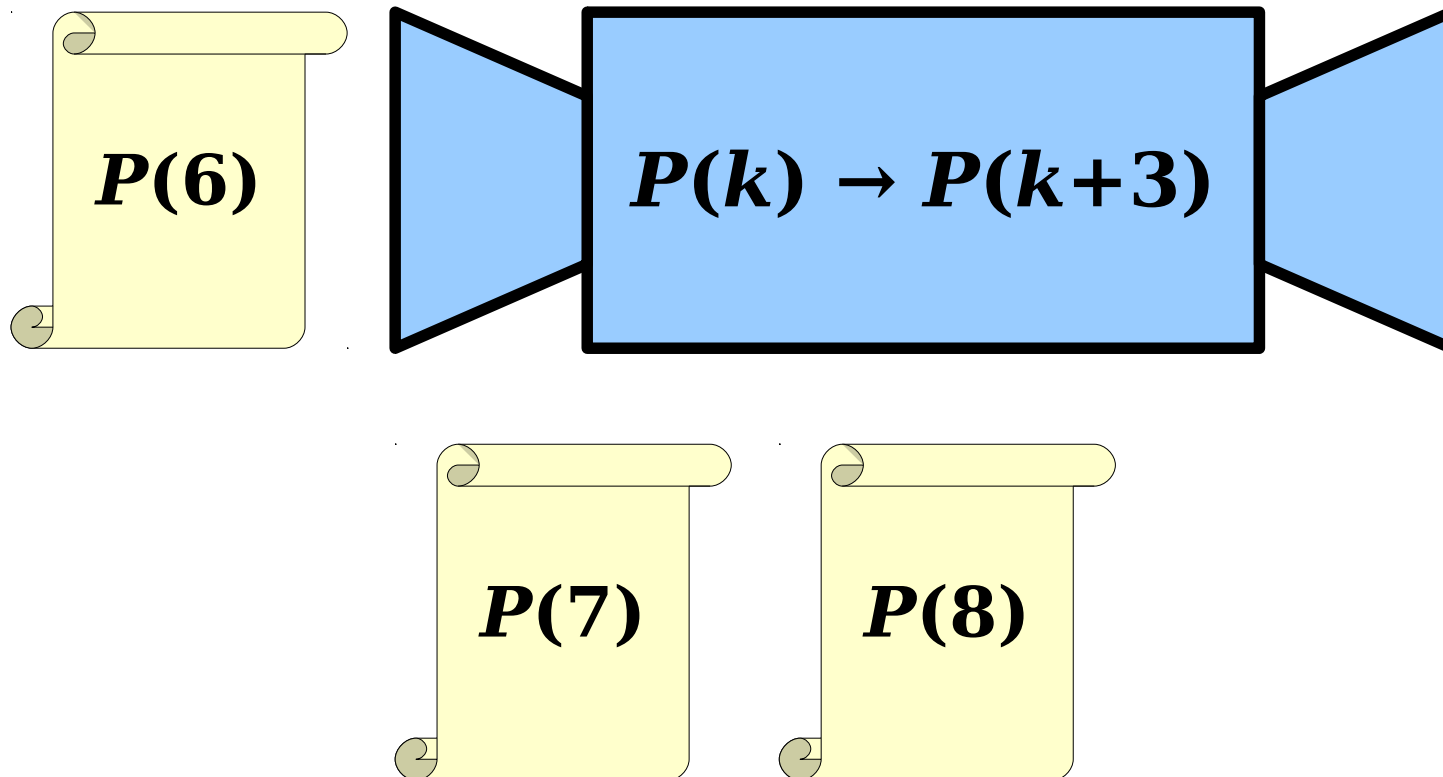
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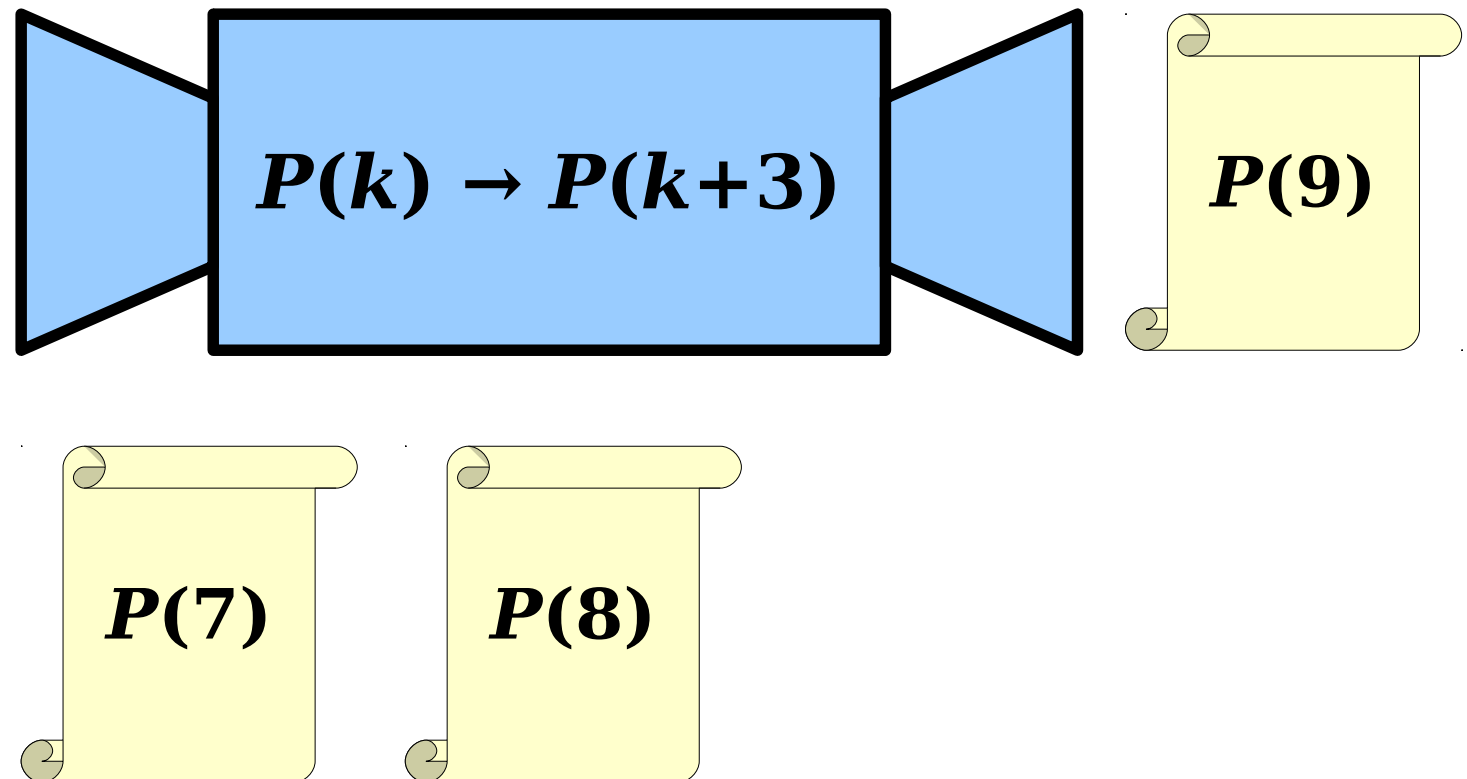
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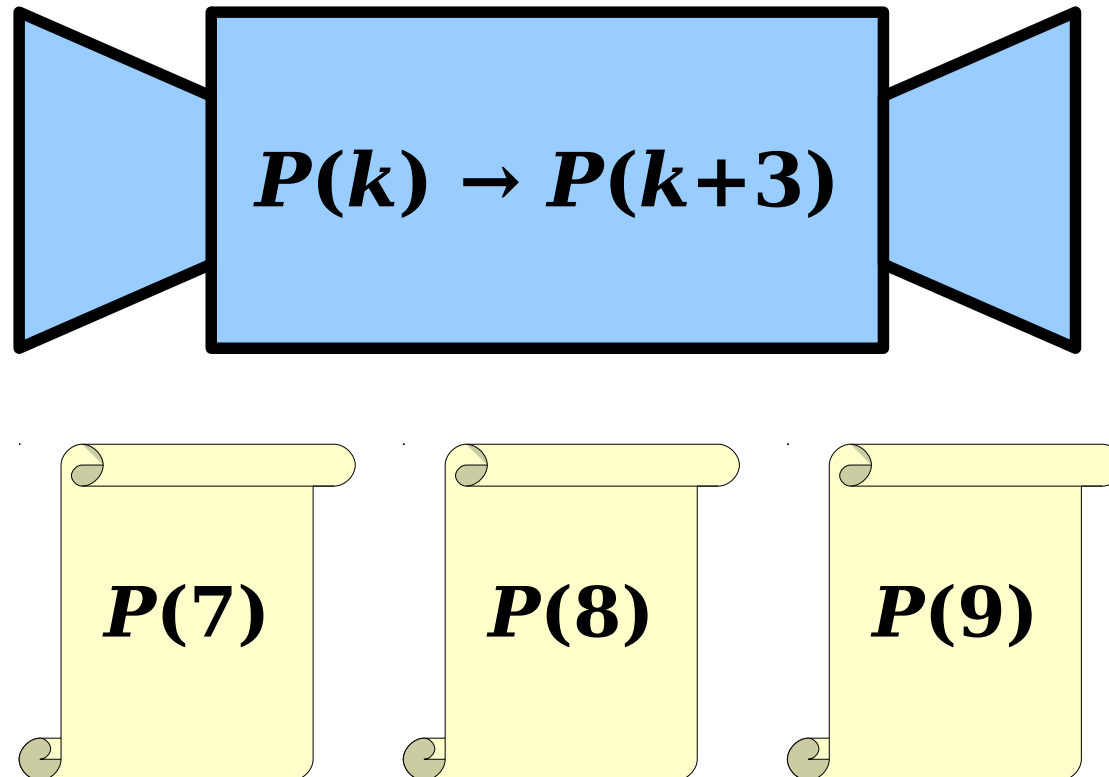
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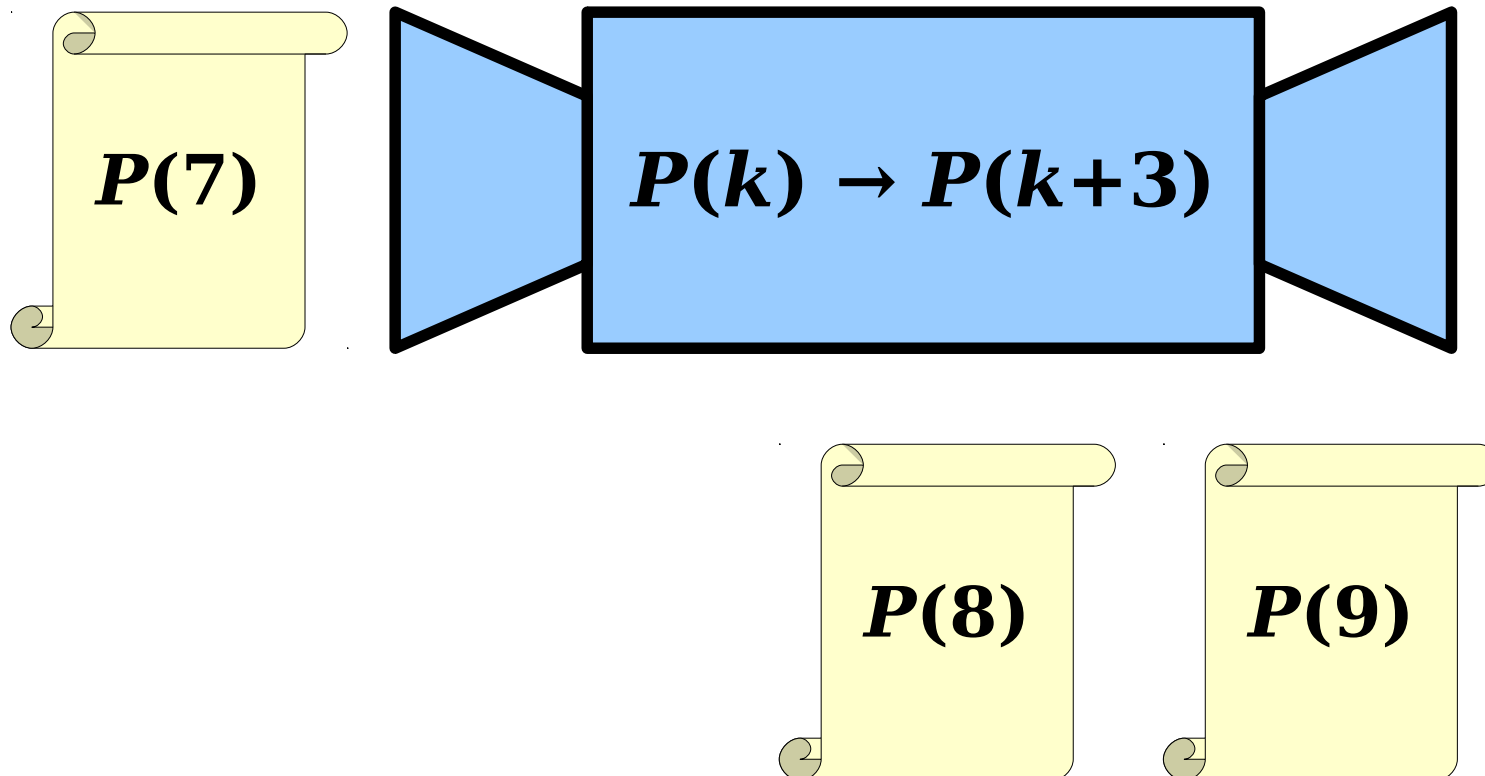
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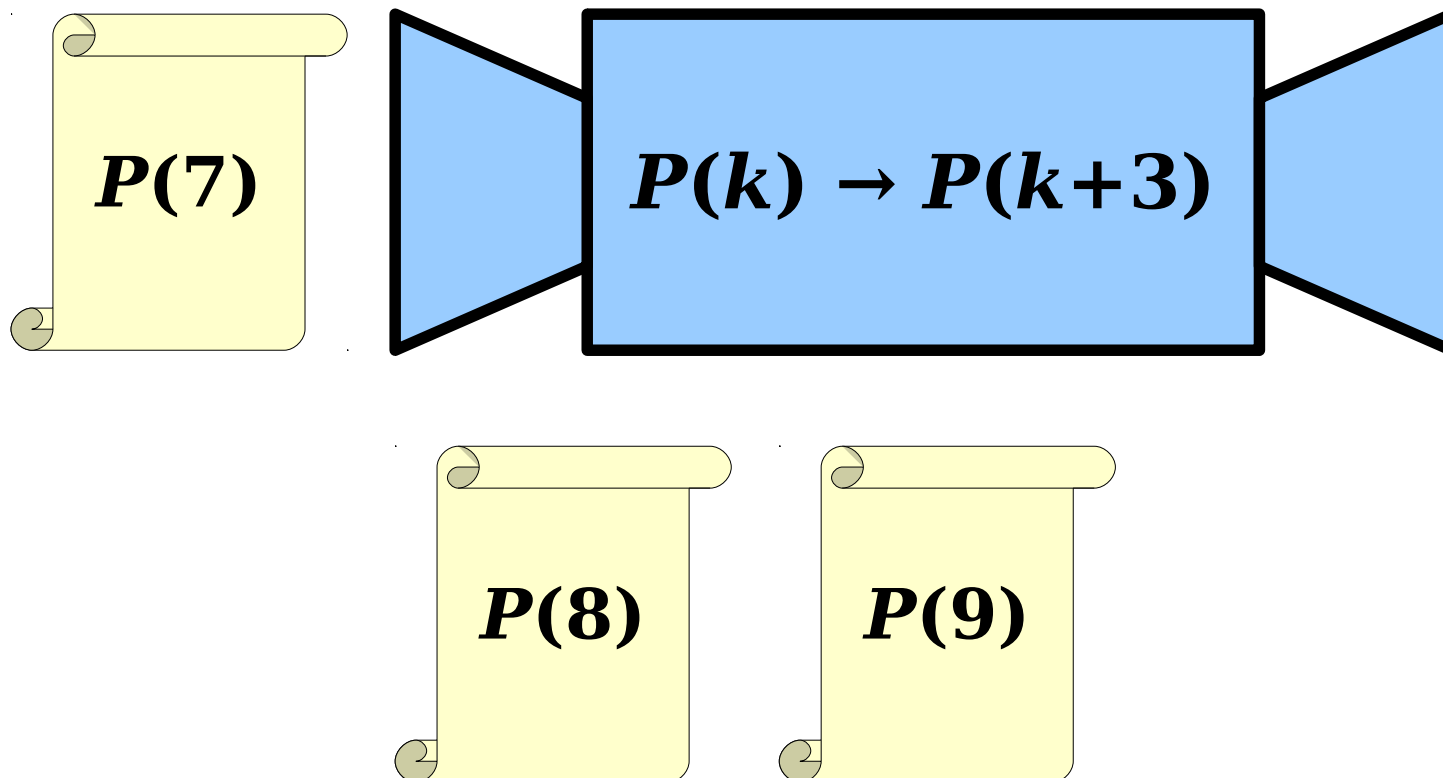
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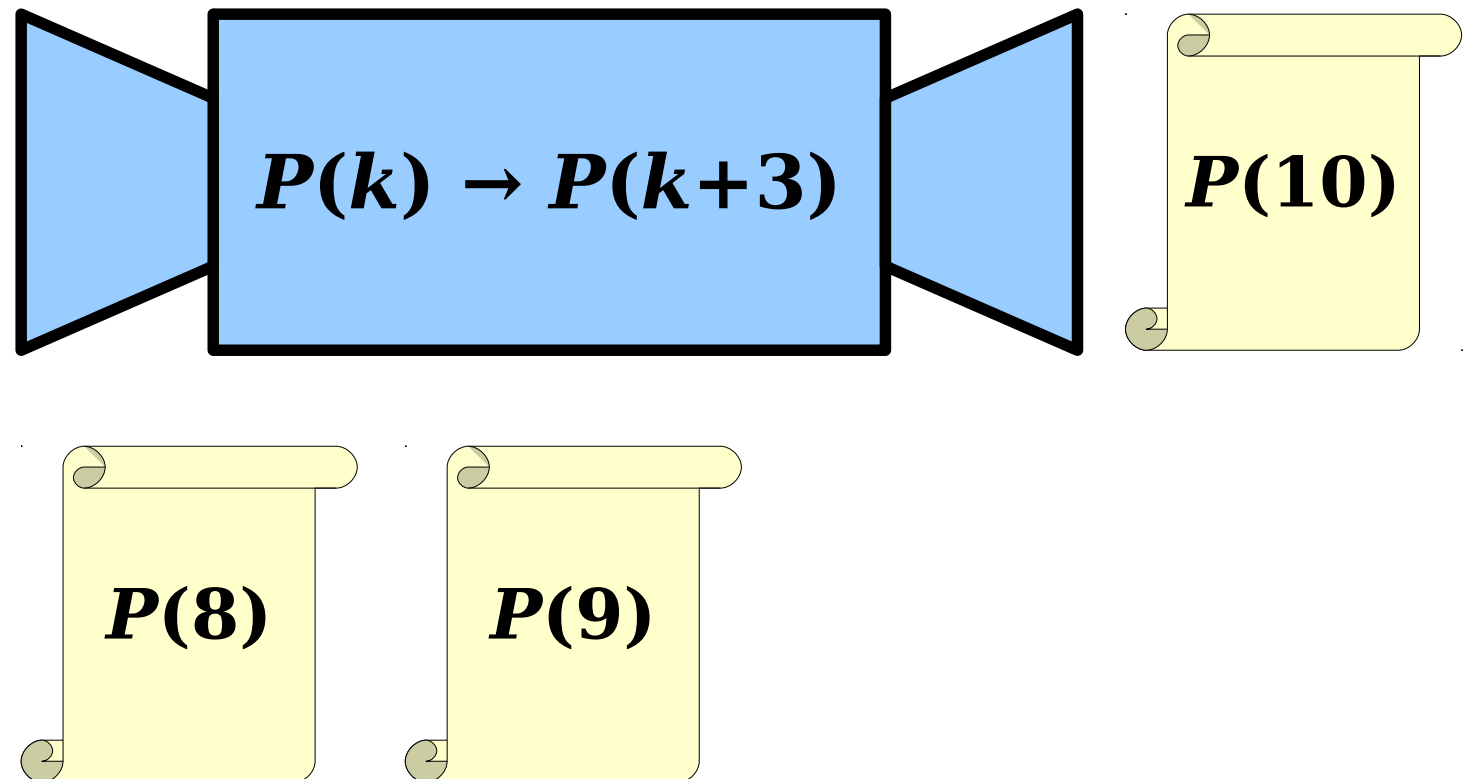
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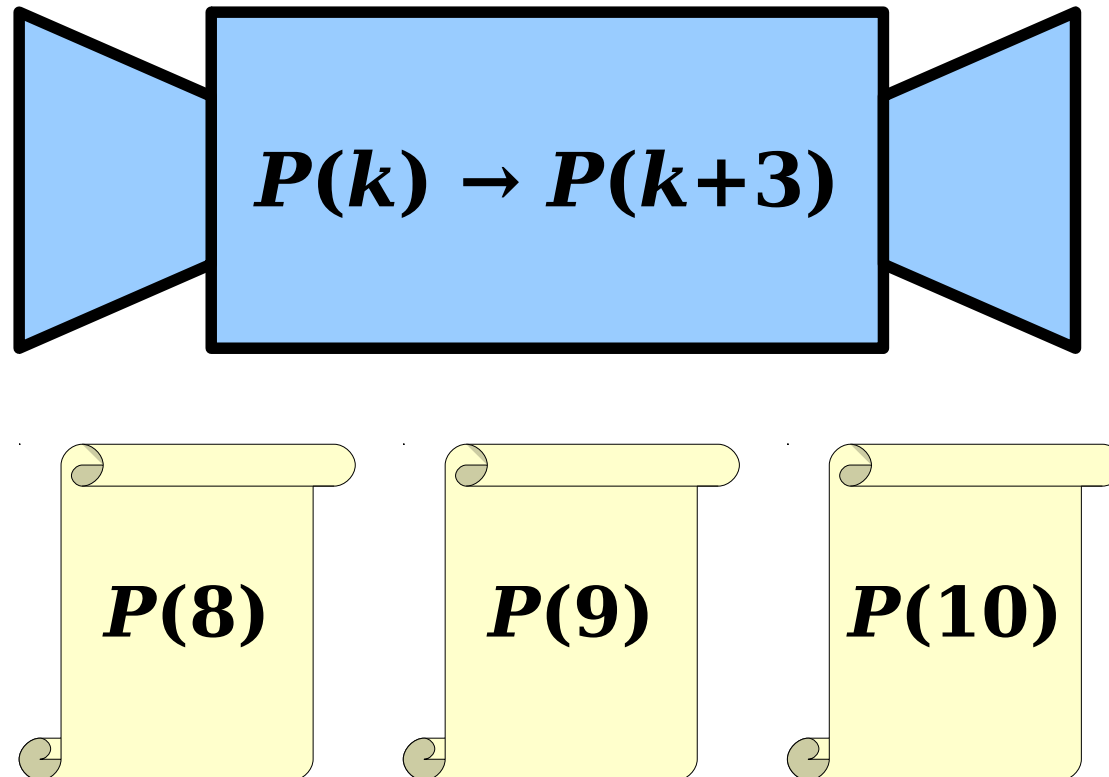
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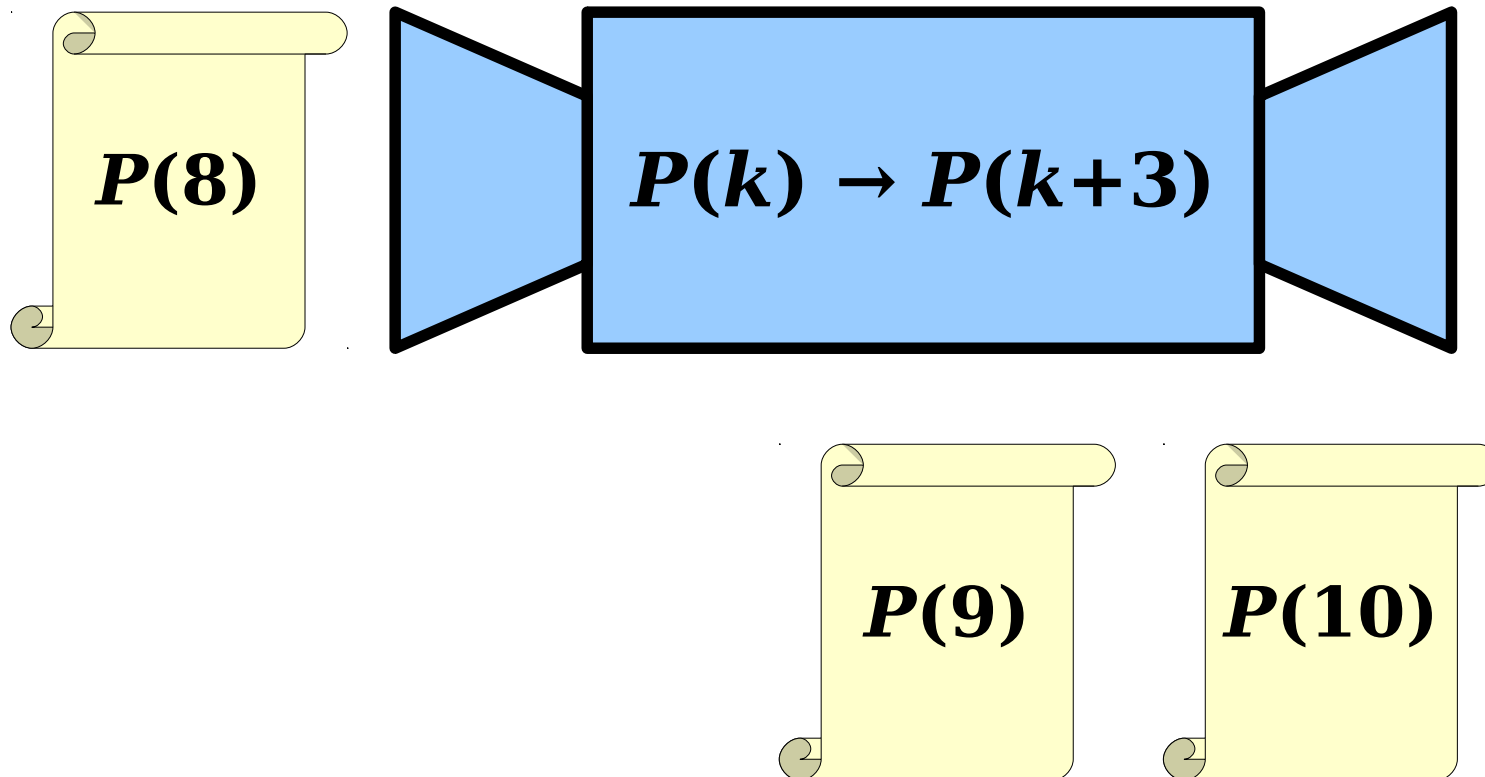
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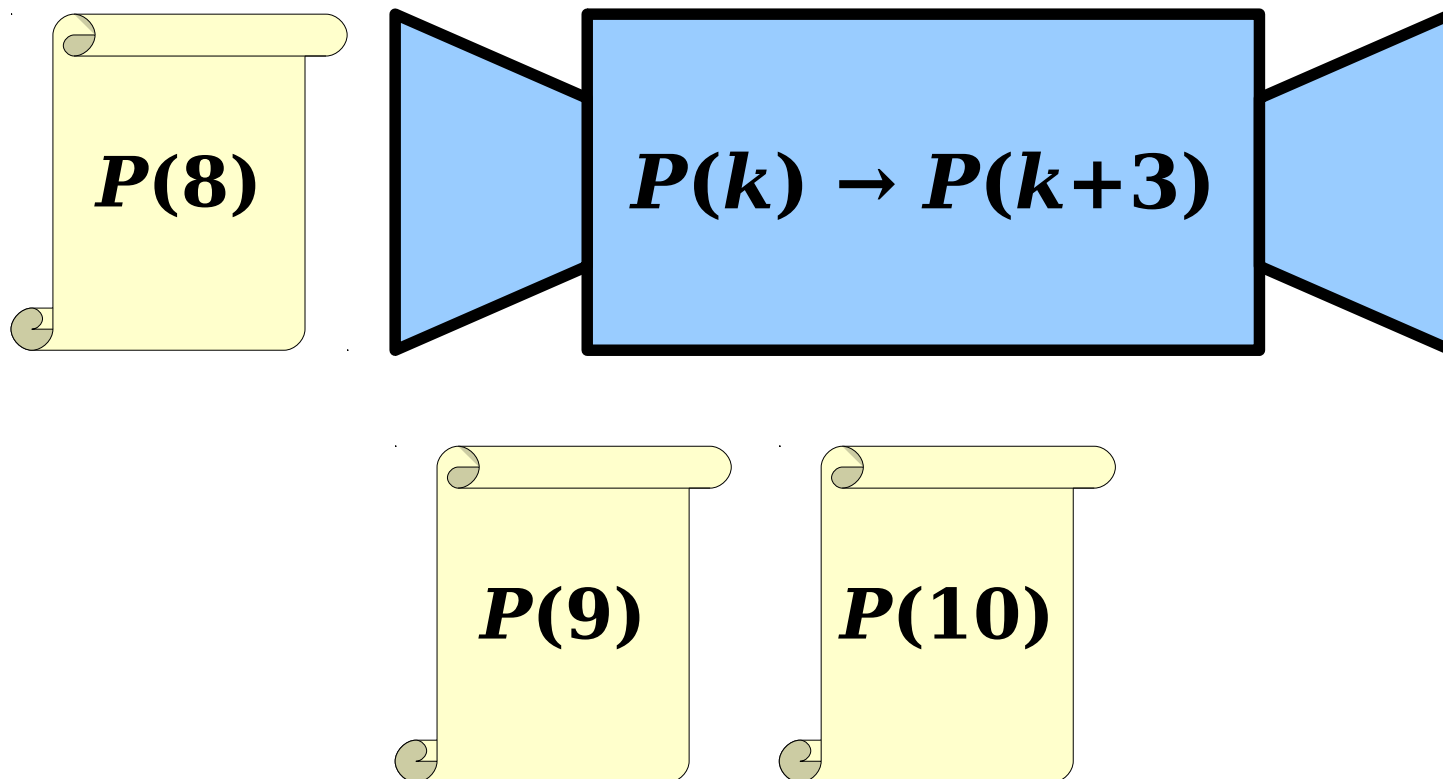
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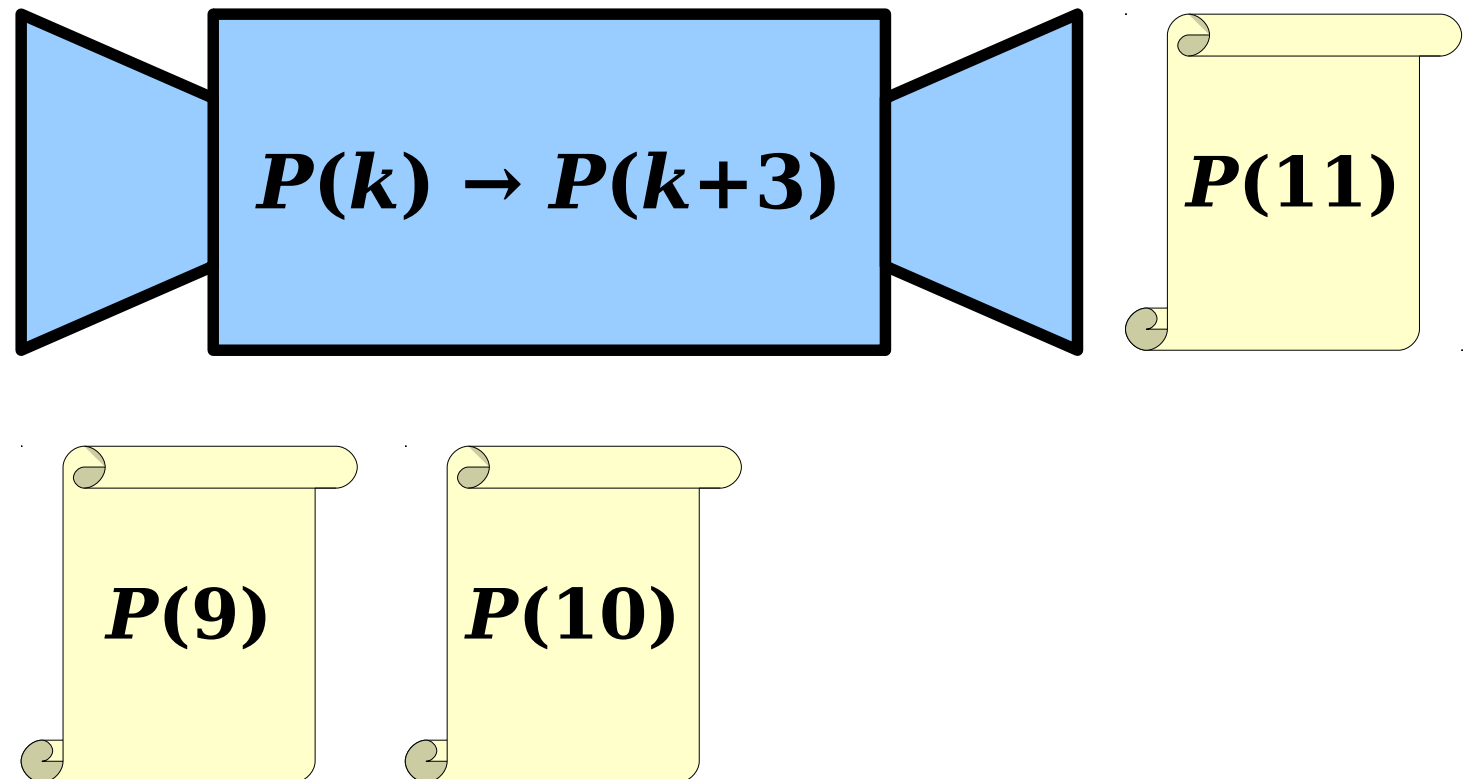
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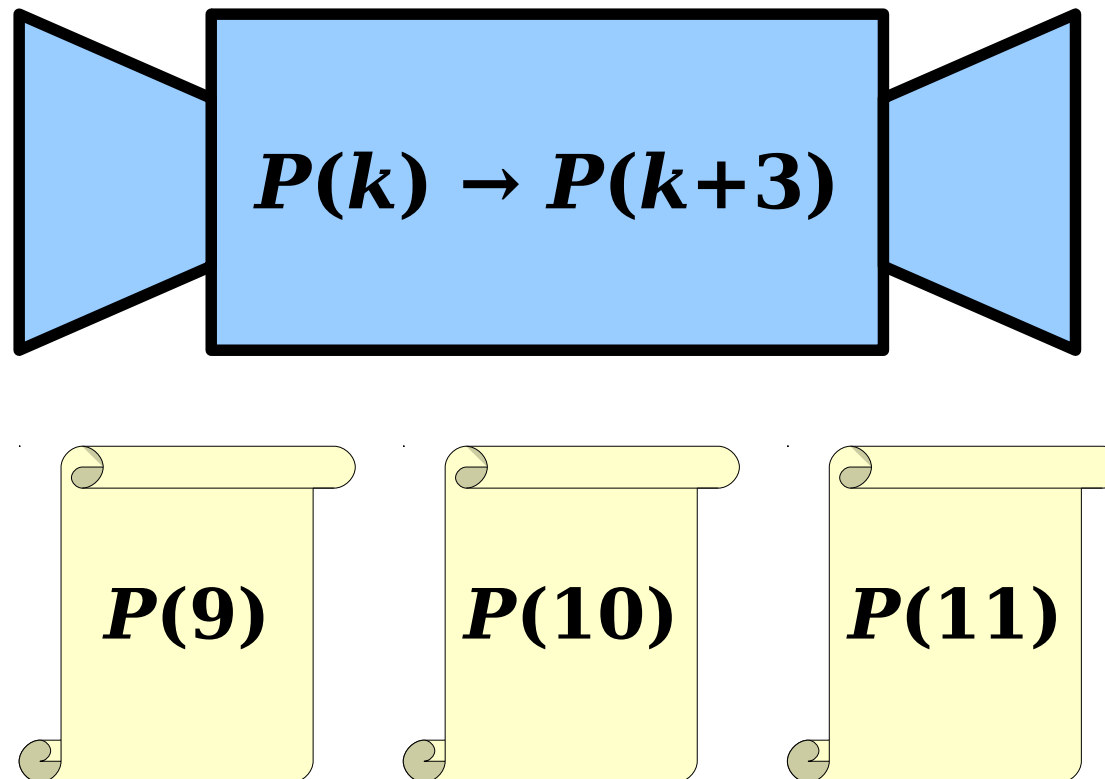
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# Generalizing Induction

- When doing a proof by induction,
  - feel free to use multiple base cases, and
  - feel free to take steps of sizes other than one.
- Just be careful to make sure you cover all the numbers you think that you're covering!
  - We won't require that you prove you've covered everything, but it doesn't hurt to double-check!

Time-Out for Announcements!

# Midterm Debrief

- You're done with the midterm! Woohoo!
- We'll be grading exams over the weekend. Once everything is graded, we'll release solutions and stats.
- Curious about any of the questions? Feel free to come and chat with us!

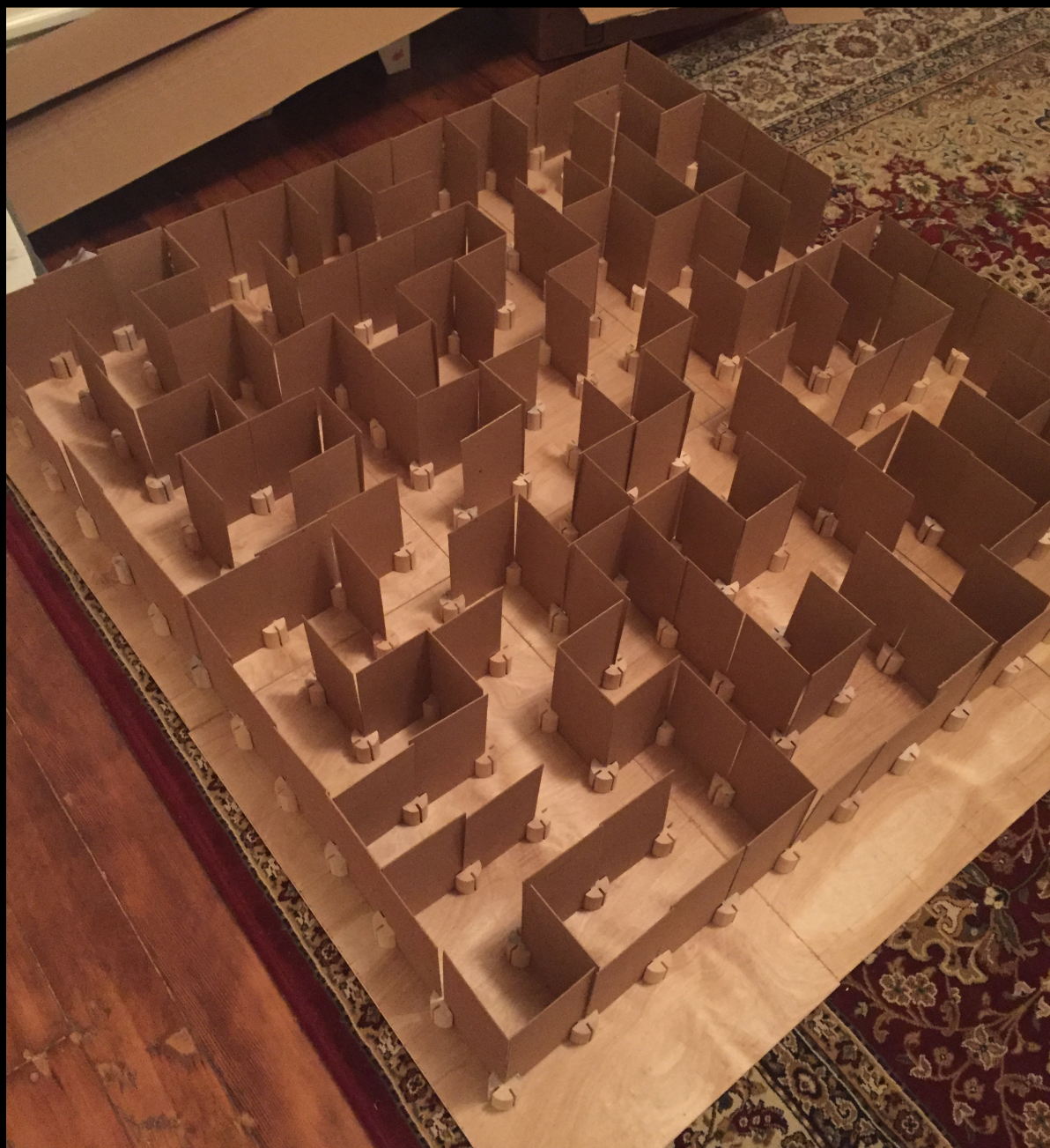


# Problem Set Four

- Problem Set Four is due this Friday at the start of class.
  - Reminder: you have three free 24-hour late days to use throughout the quarter.
  - You'll get a flat percent hit if you turn something in once you're out of late days, but it's almost certainly a good deal.
- PS4 checkpoints have been graded and returned. We'll get PS3 returned as soon as it's ready.

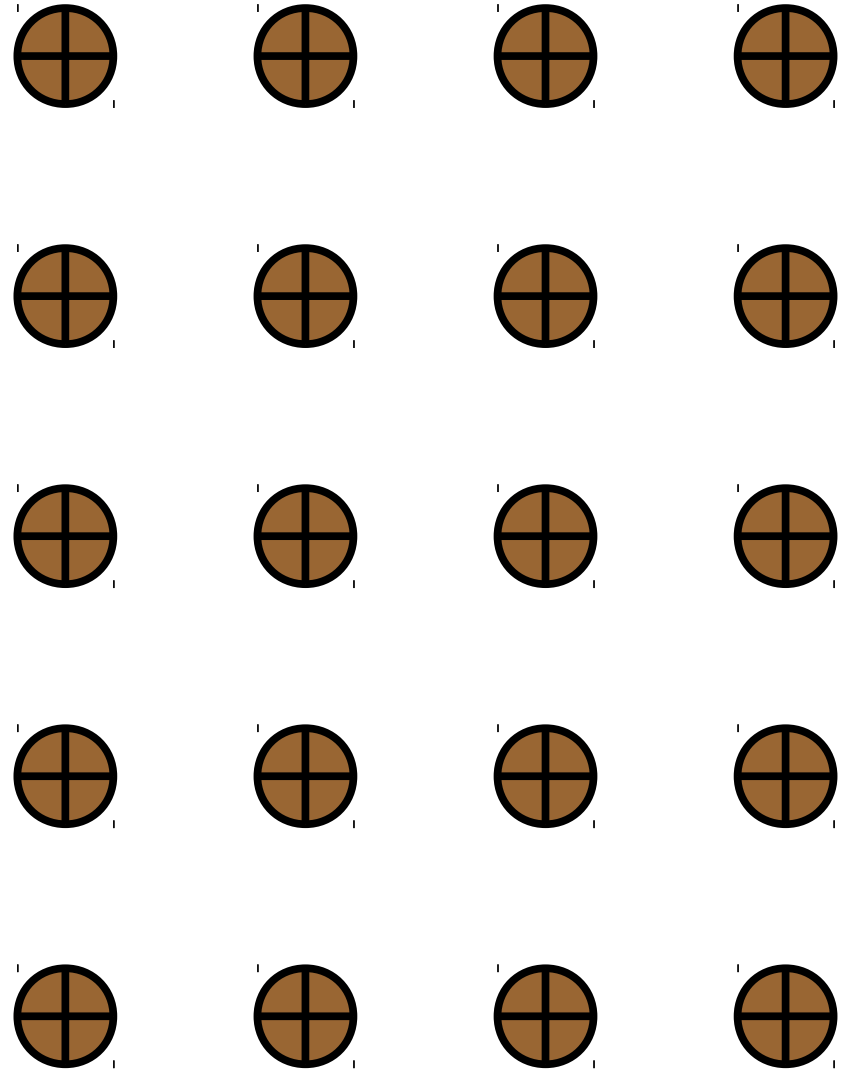
Back to CS103!

A Motivating Question: ***Rat Mazes***



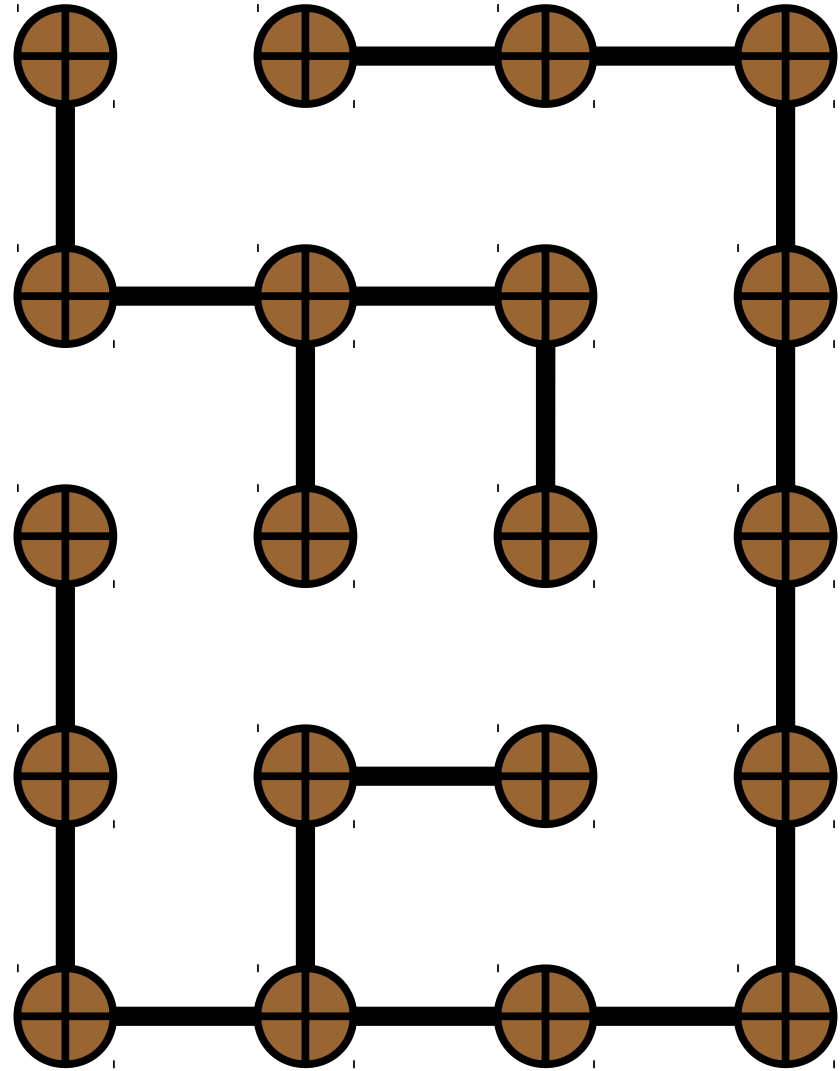
# Rat Mazes

- Suppose you want to make a rat maze consisting of an  $n \times m$  grid of pegs with slats between them.



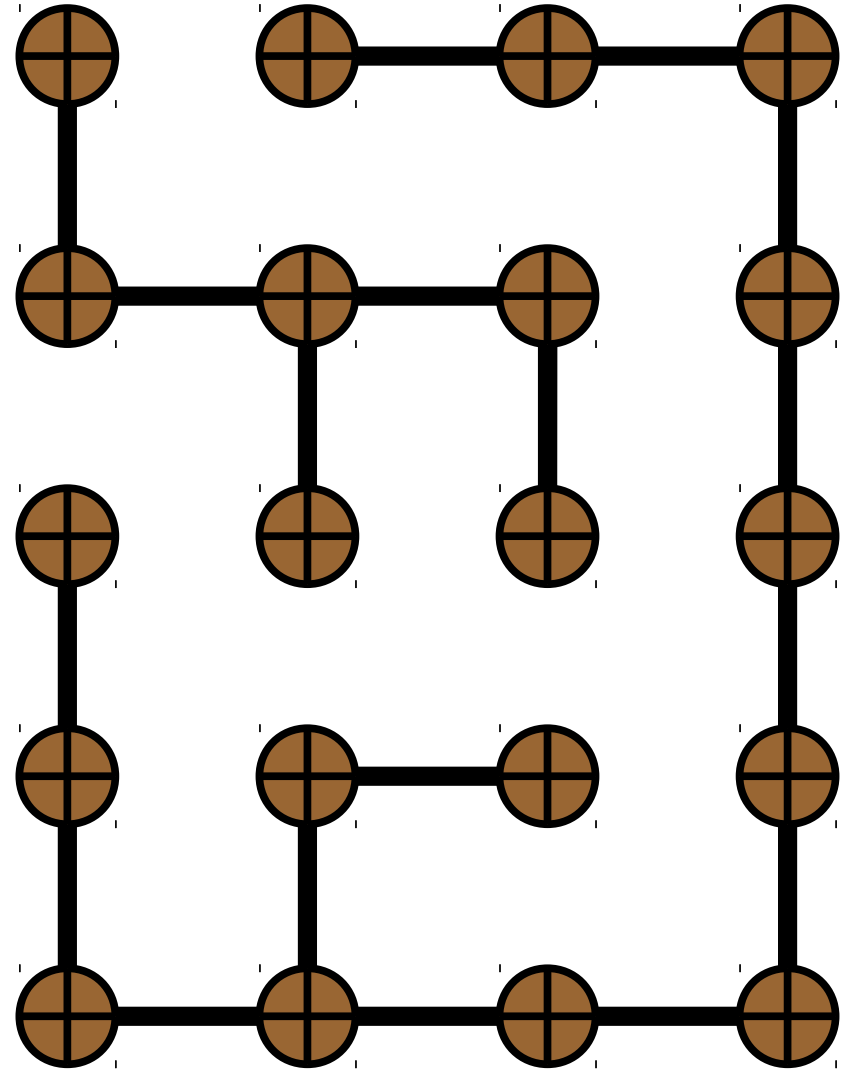
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# Rat Mazes

- Suppose you want to make a rat maze consisting of an  $n \times m$  grid of pegs with slats between them.
- The maze should have these properties:
  - There is one entrance and one exit in the border.
  - Every spot in the maze is reachable from every other spot.
  - There is exactly one path from each spot in the maze to each other spot.



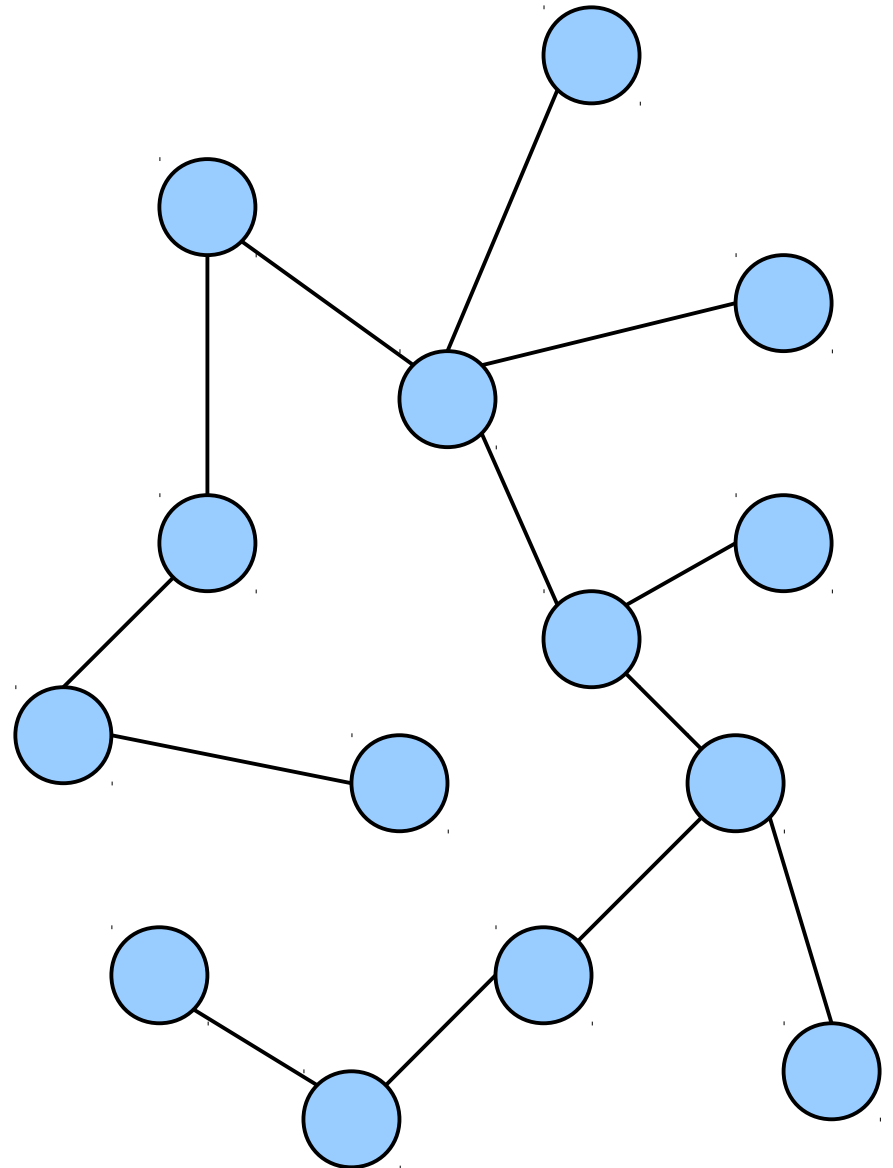
***Question:*** If you have an  $n \times m$  grid of pegs, how many slats do you need to make?



A Special Type of Graph: ***Trees***

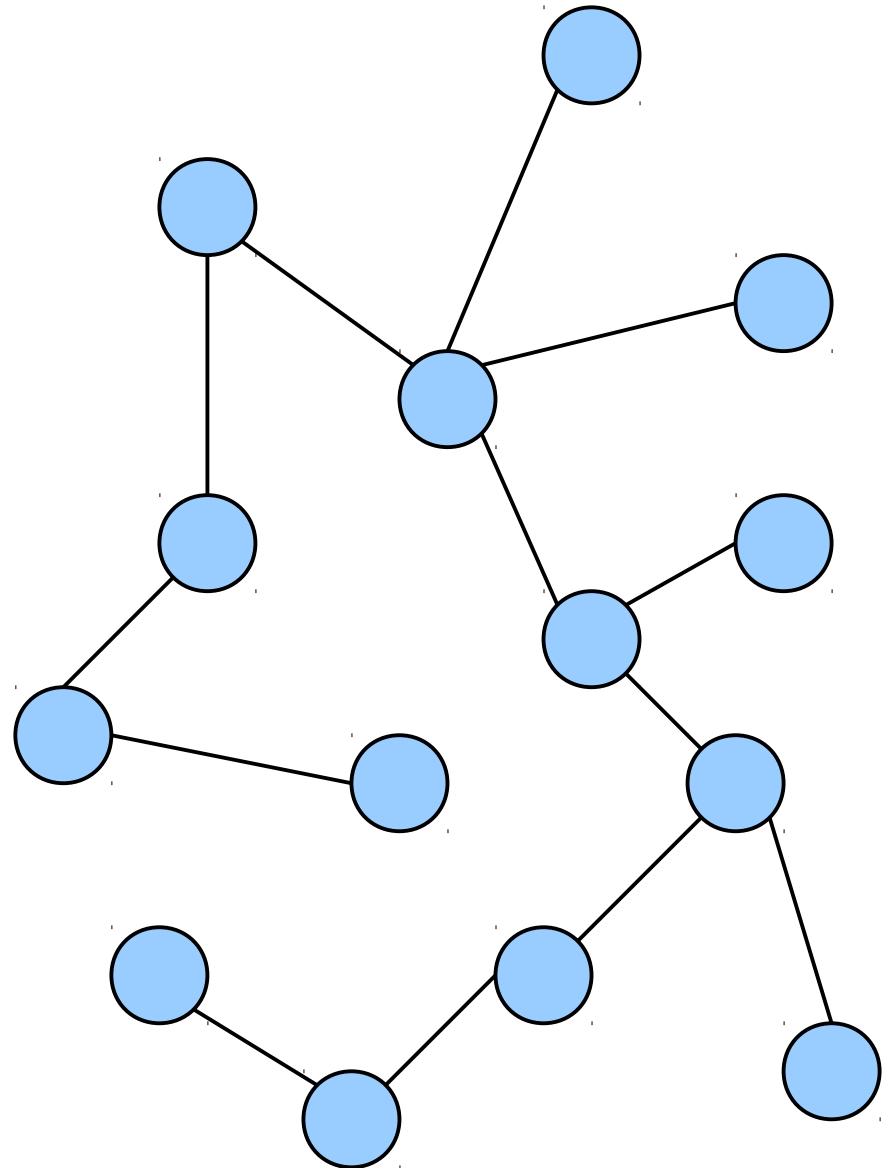
# Trees

- A **tree** is a connected, nonempty graph with no simple cycles.



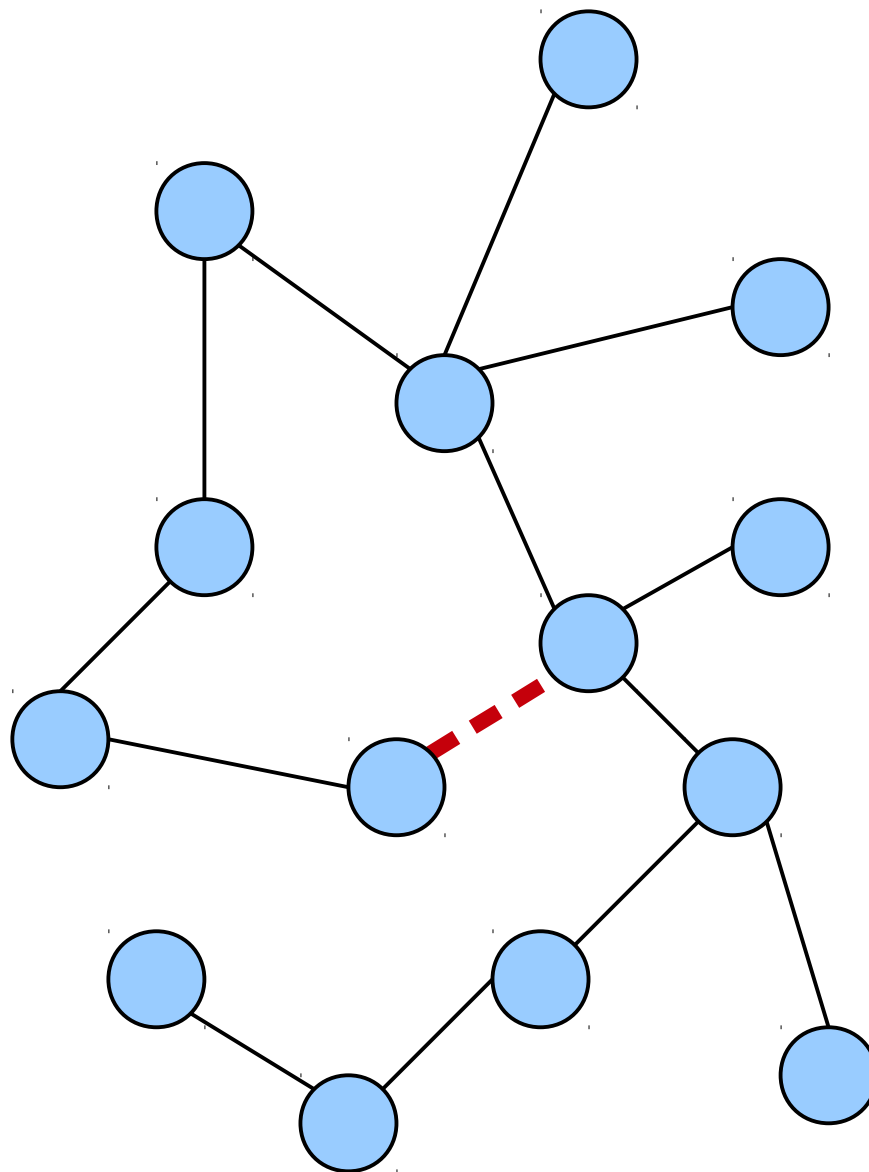
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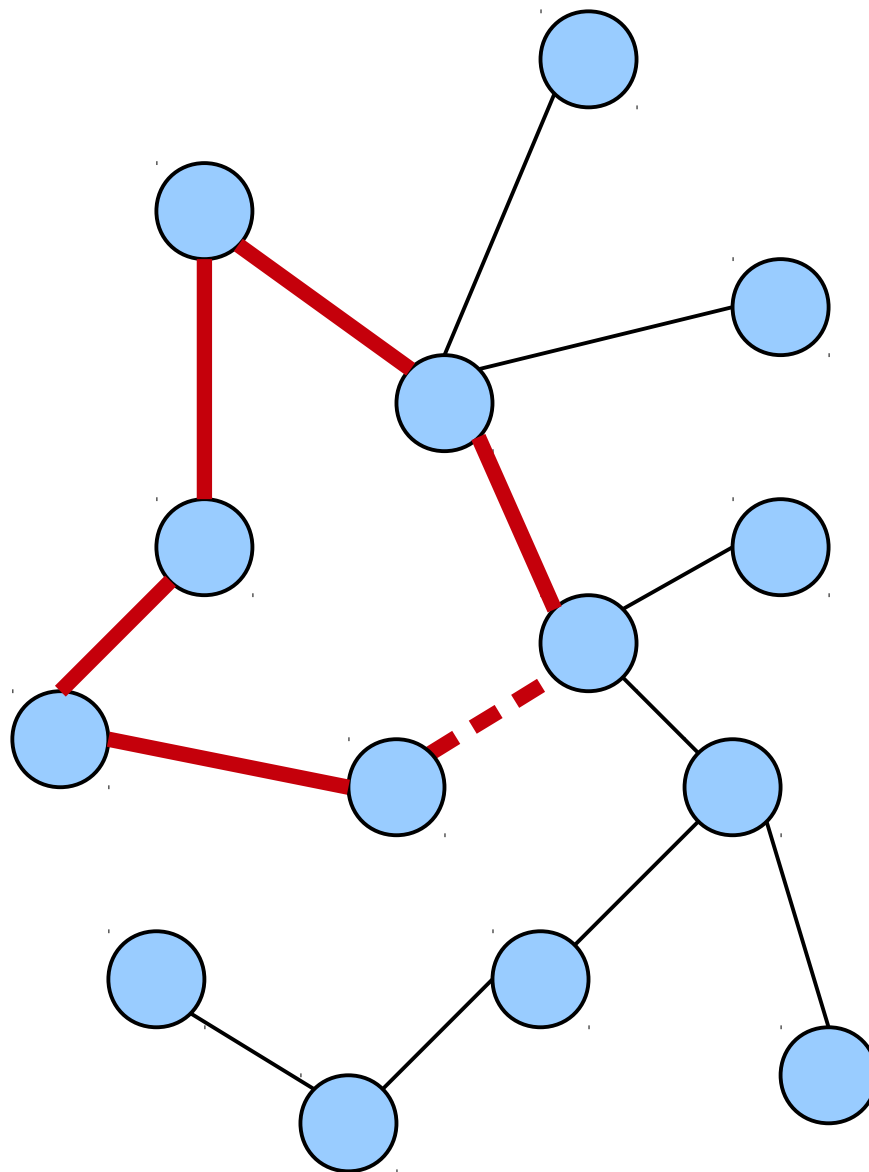
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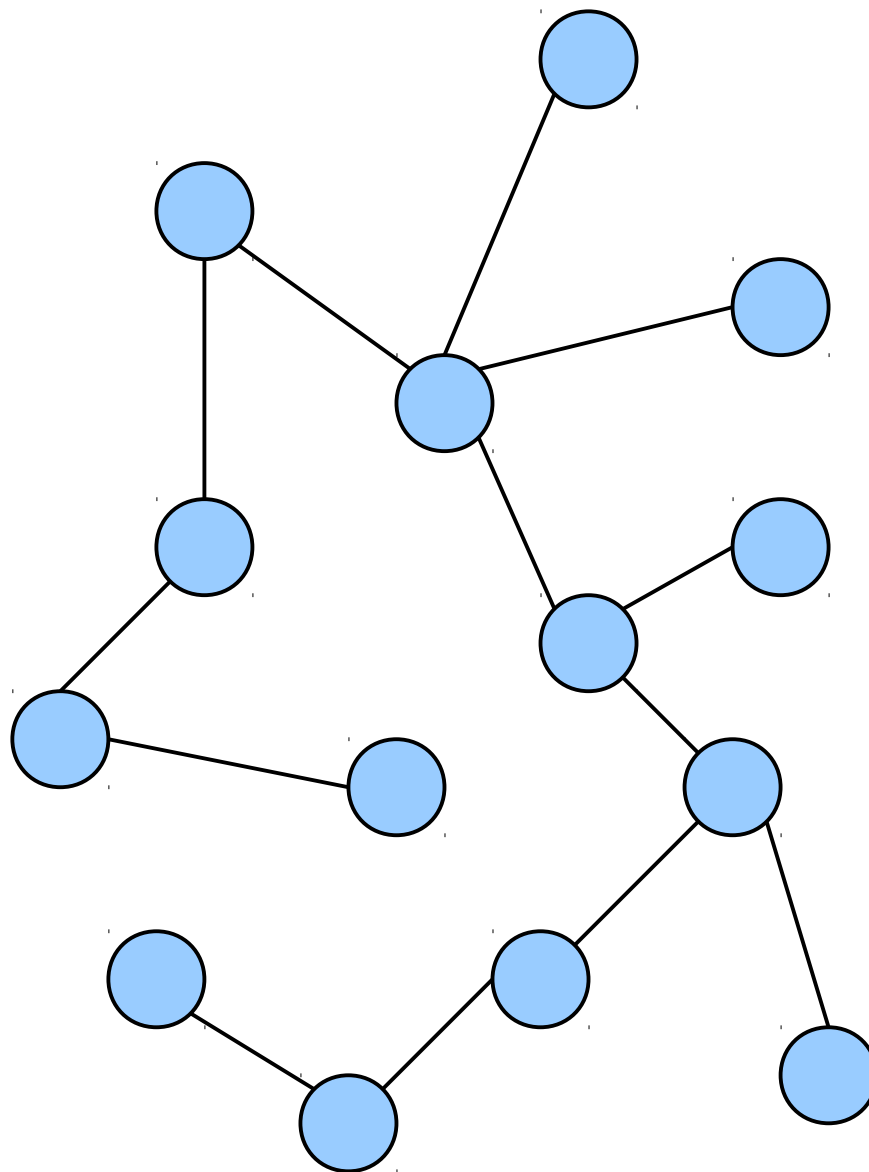
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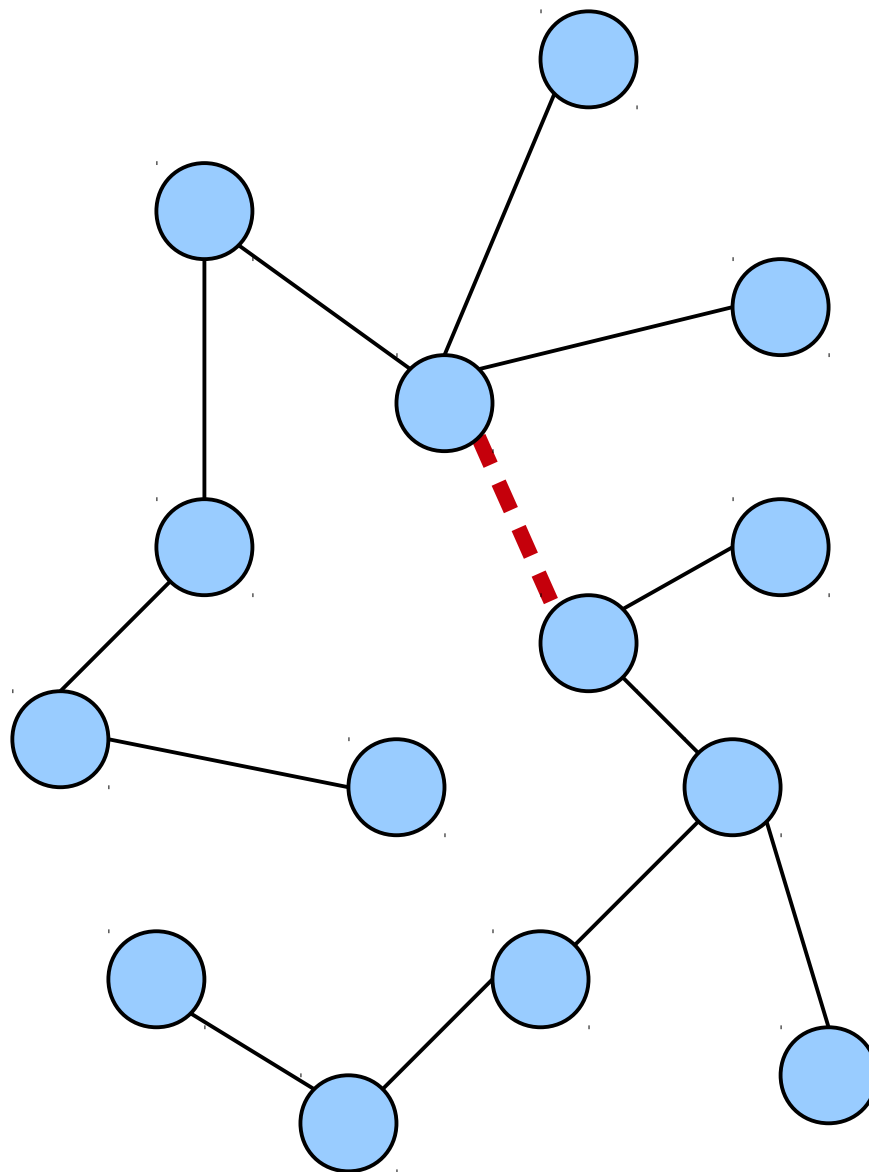
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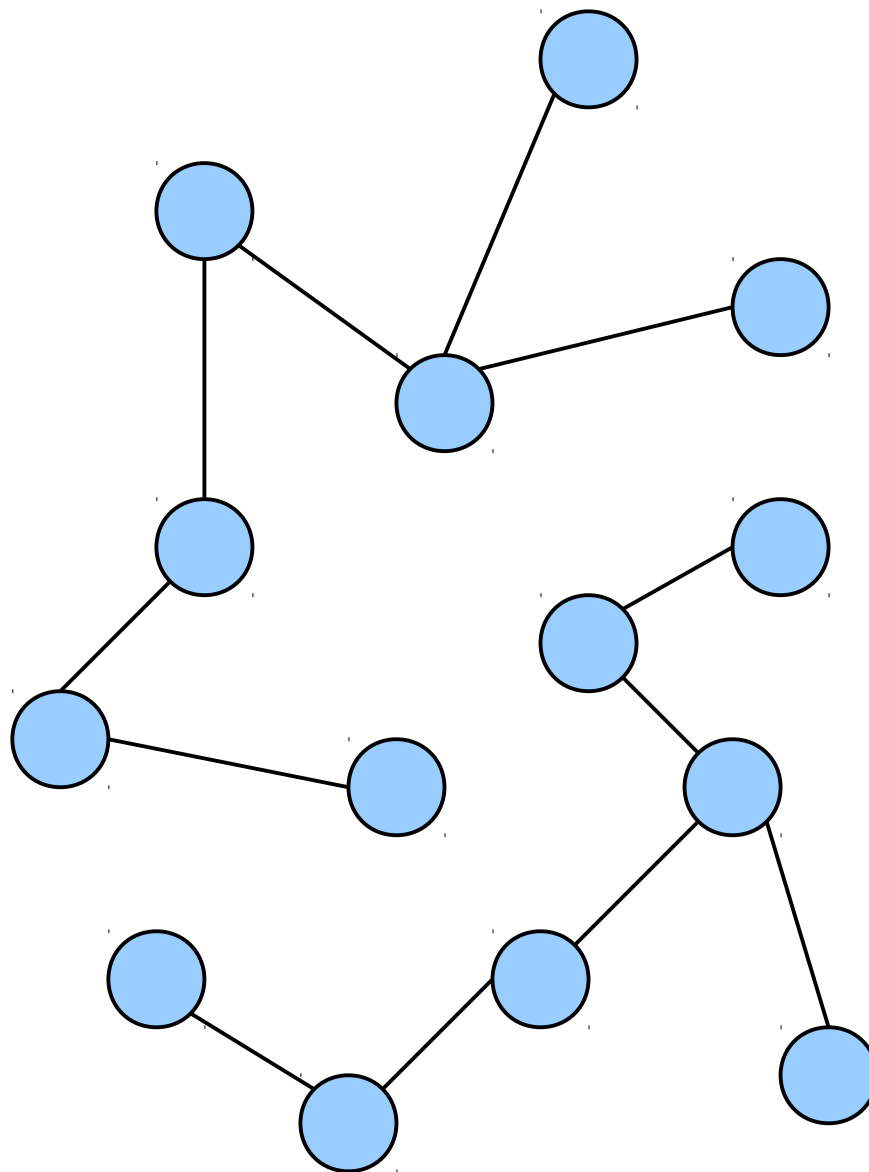
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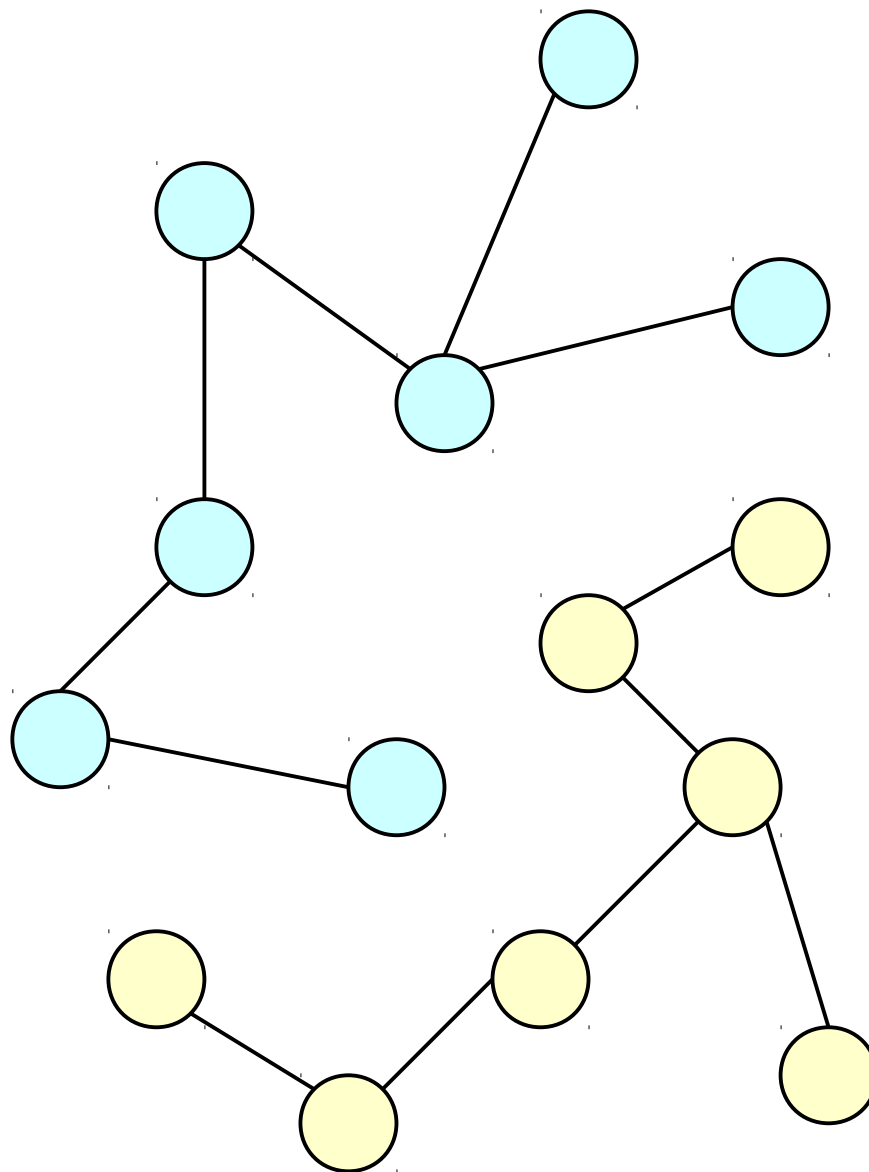
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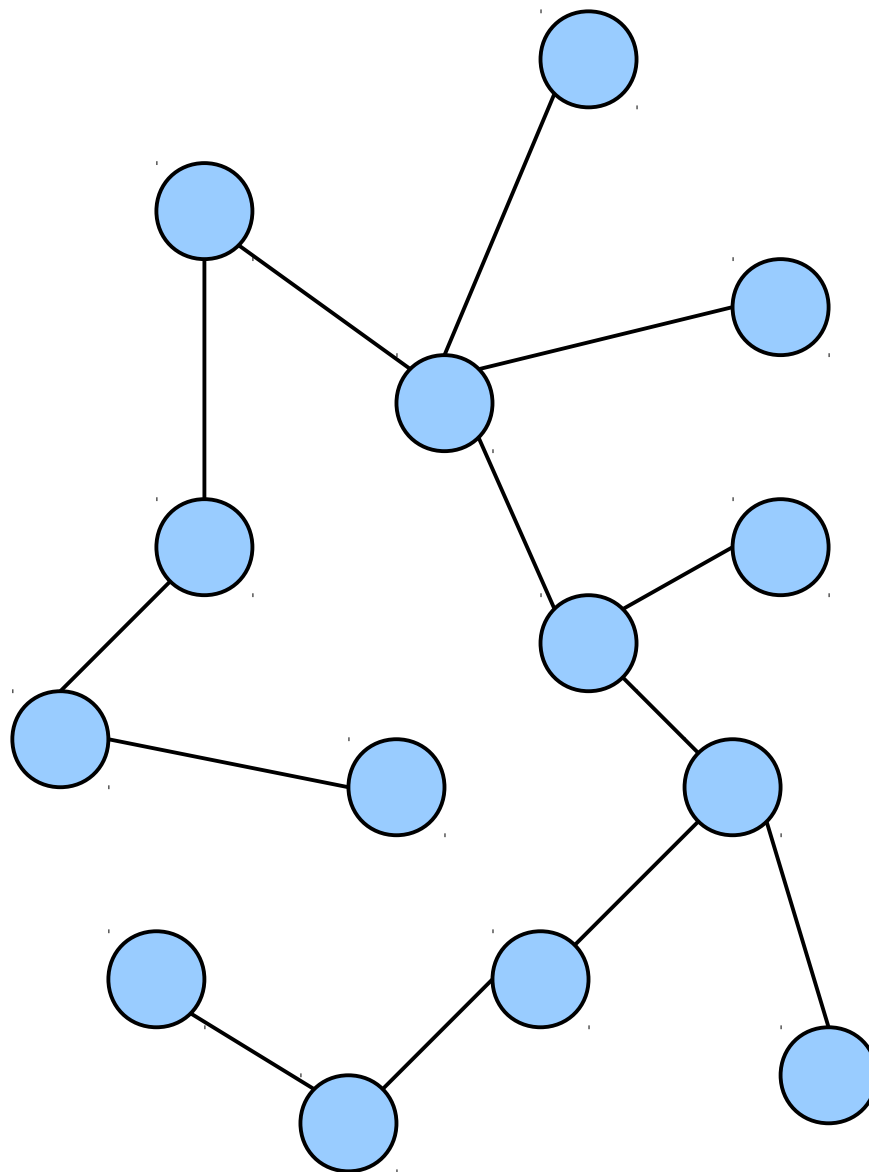
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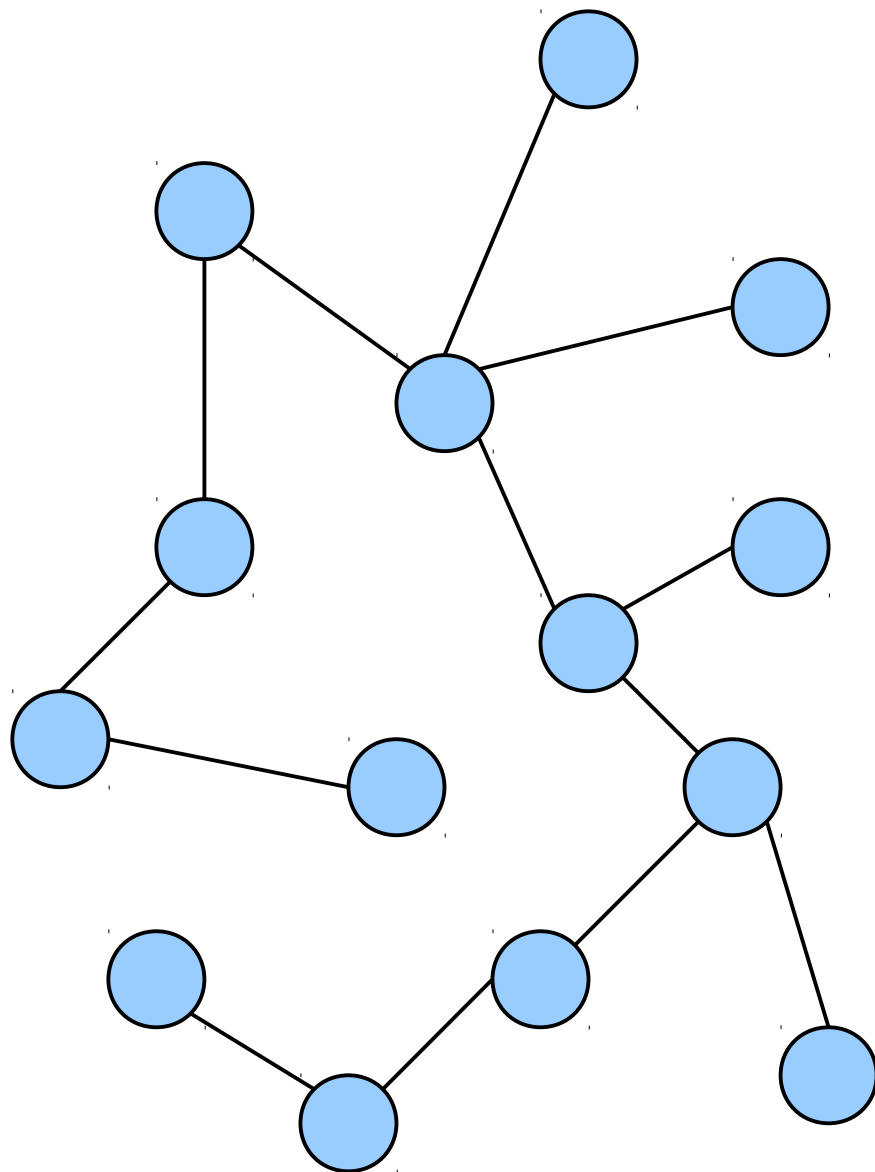
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- Proofs of these results are in the course reader if you're interested. They're also great exercises.



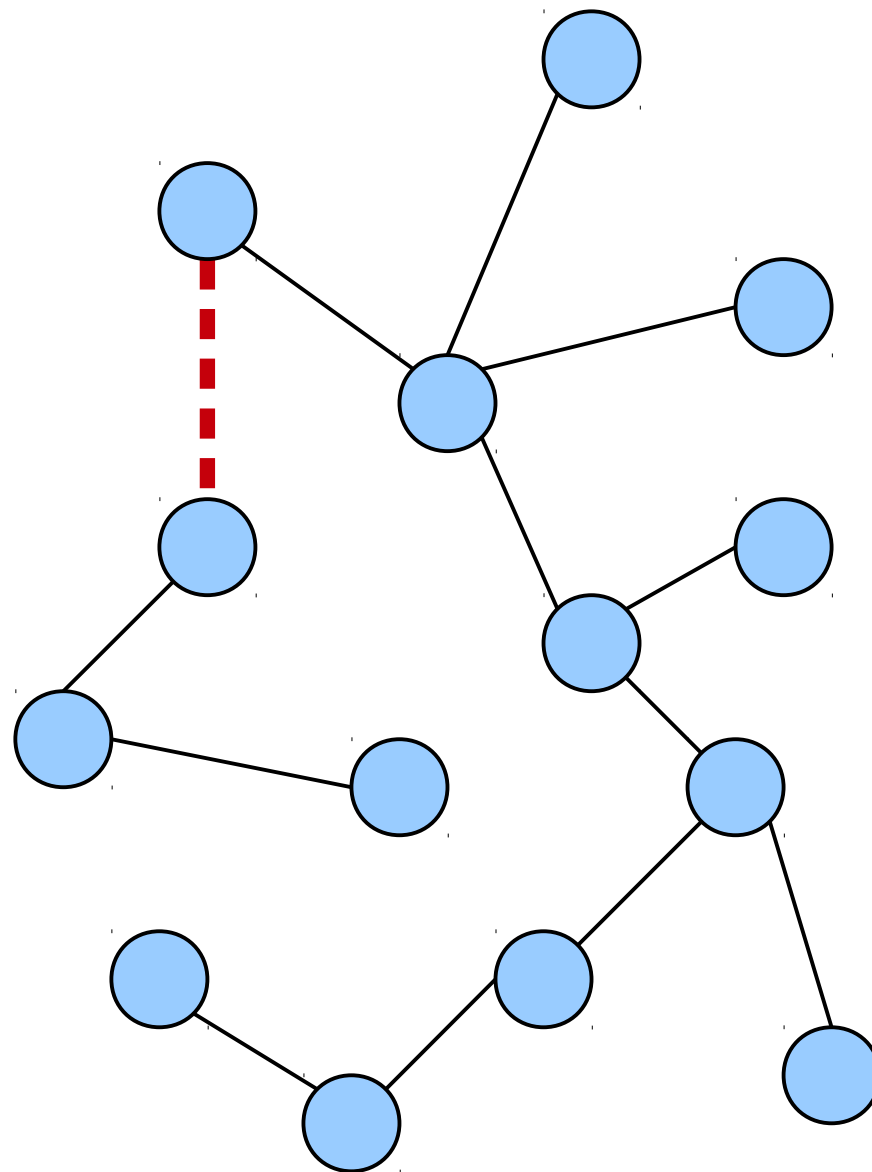
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- **Theorem:** If  $T$  is a tree with at least two nodes, then deleting any edge from  $T$  splits  $T$  into two nonempty trees  $T_1$  and  $T_2$ .
- **Proof:** Left as an exercise to the reader. ☺



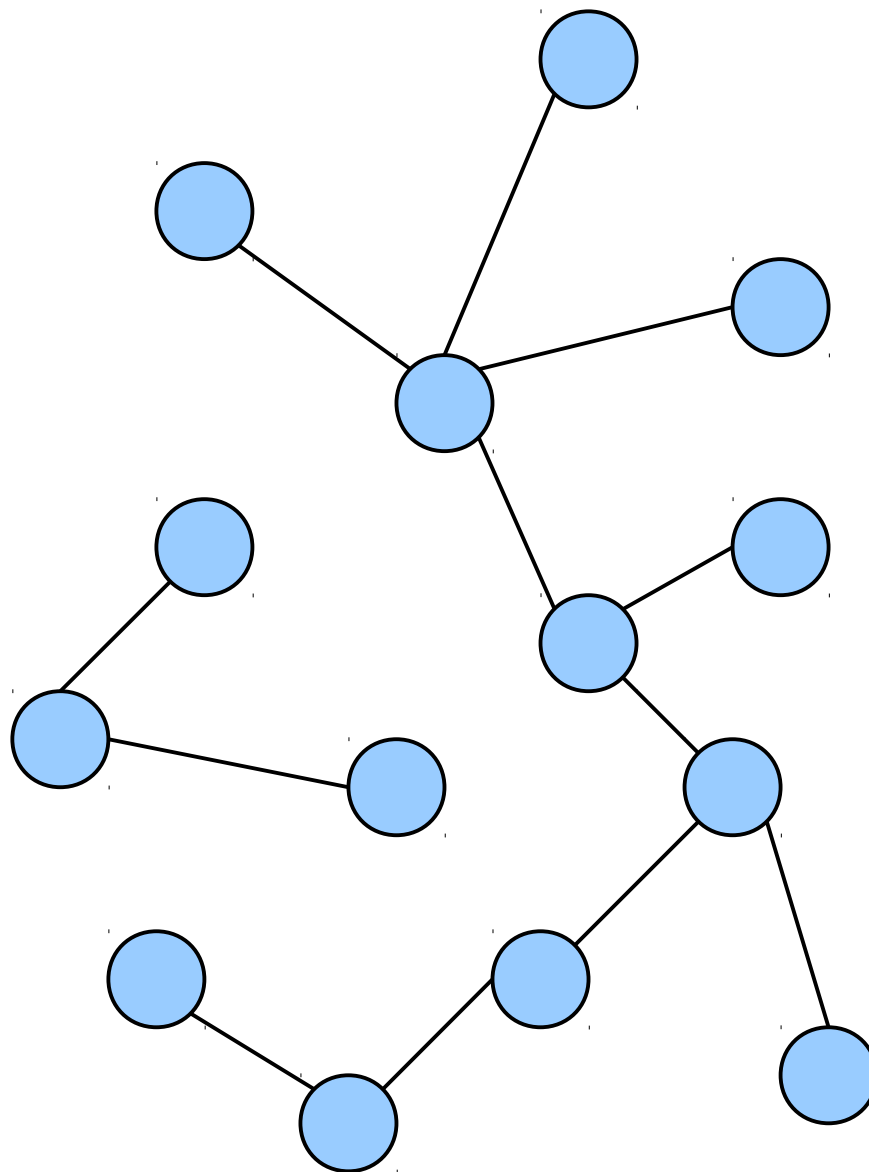
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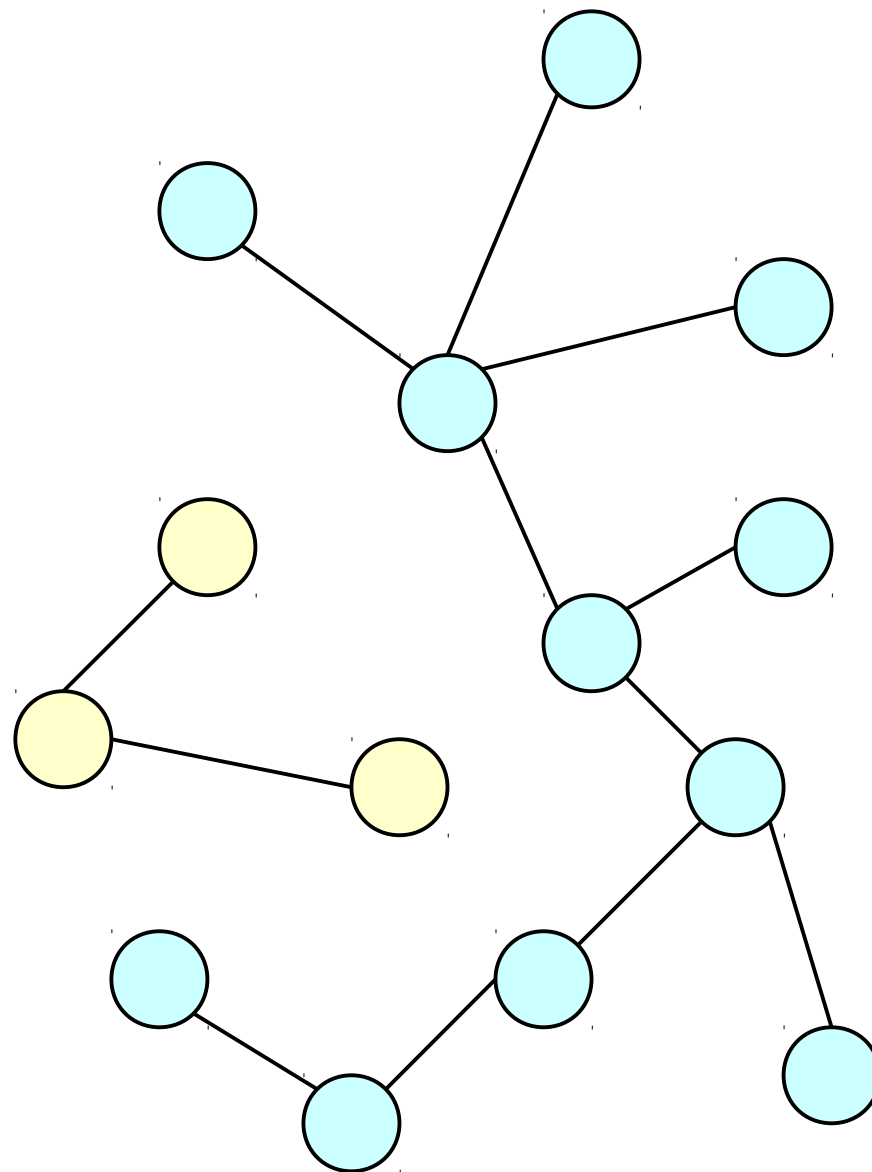
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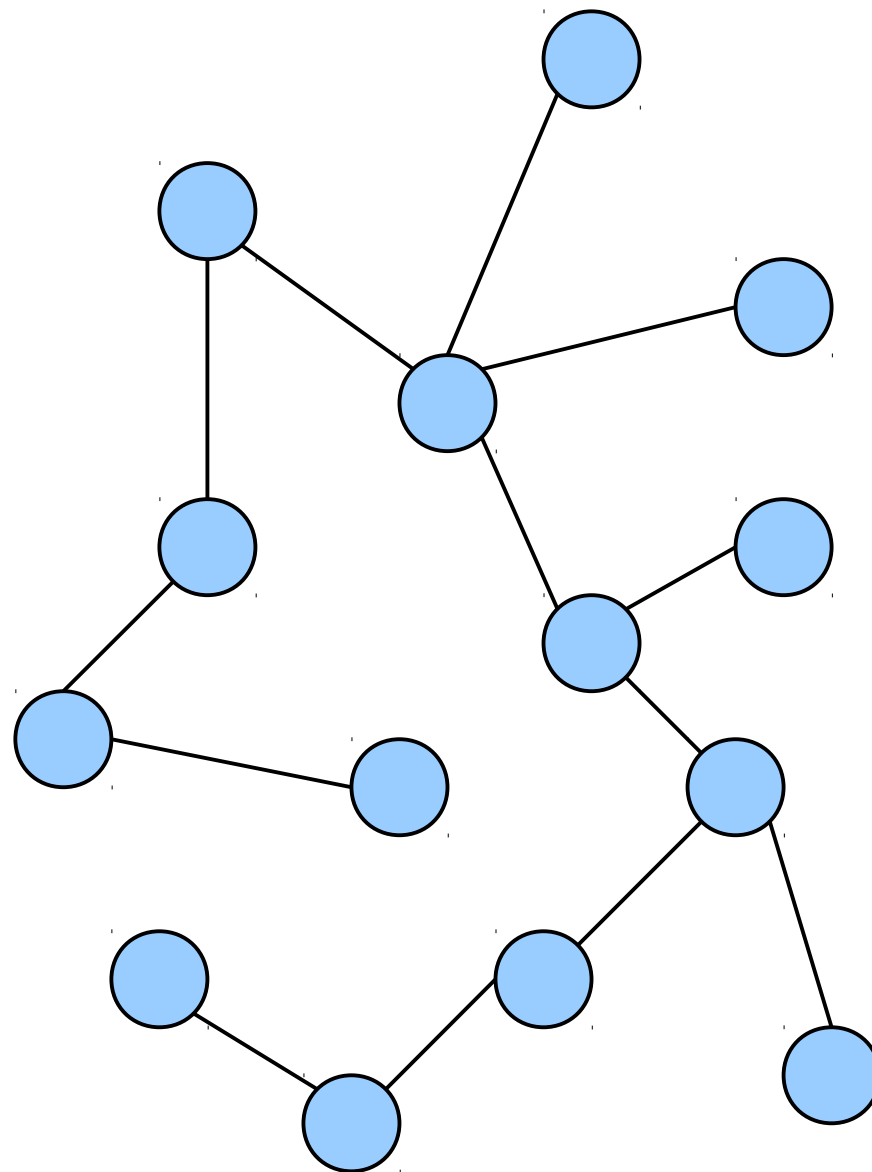
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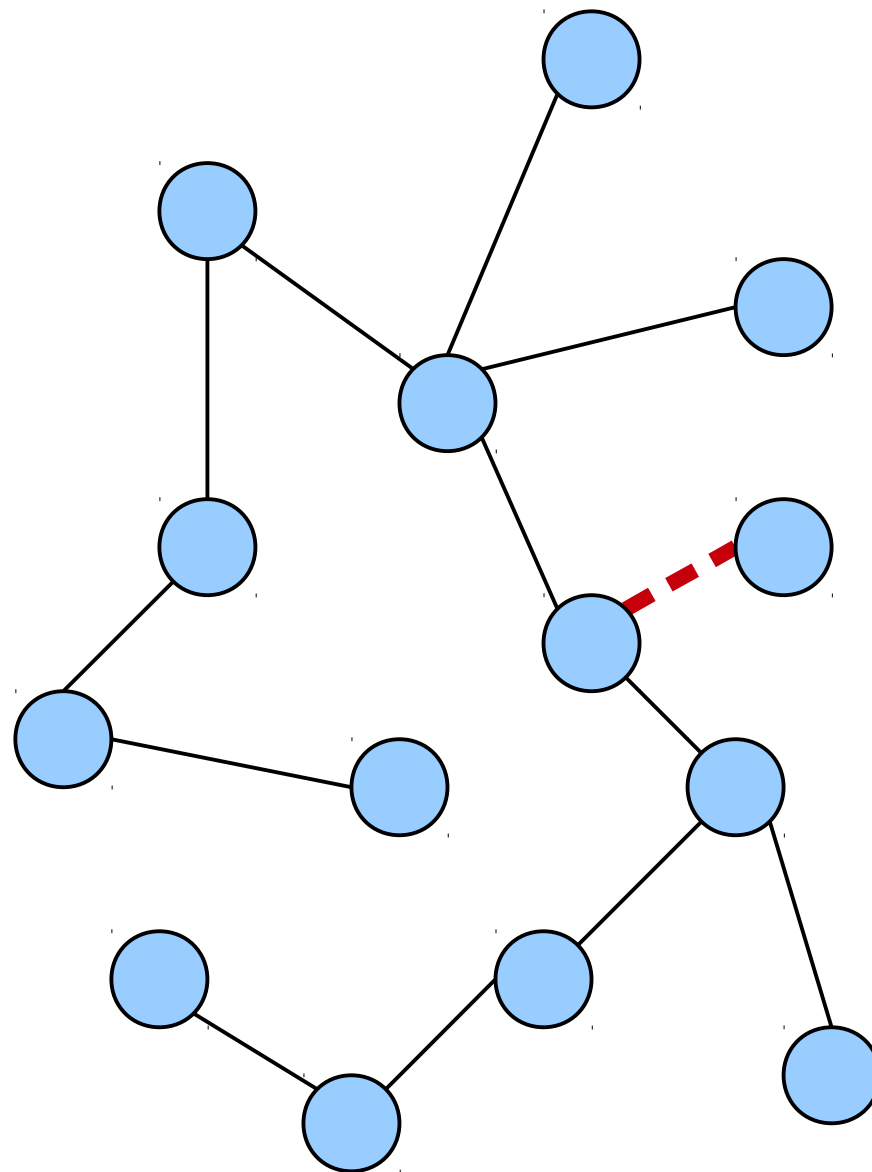
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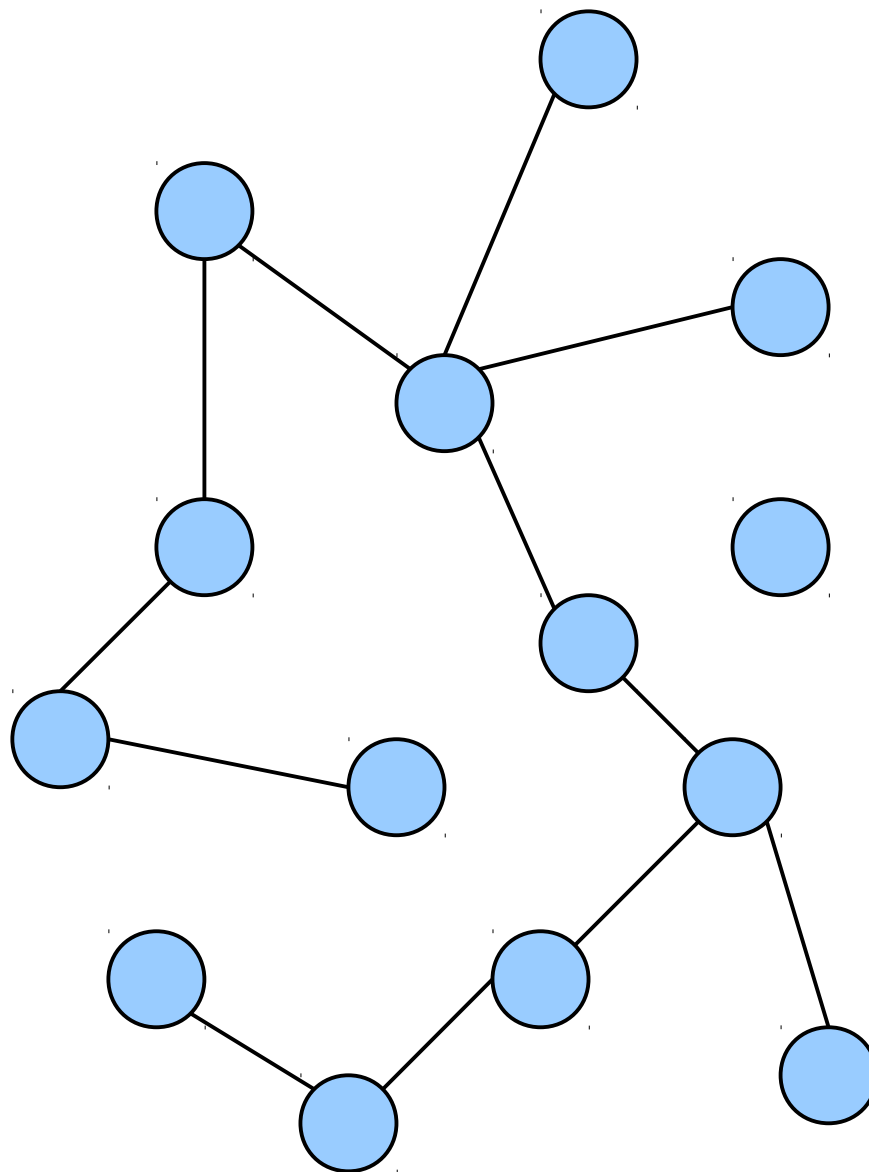
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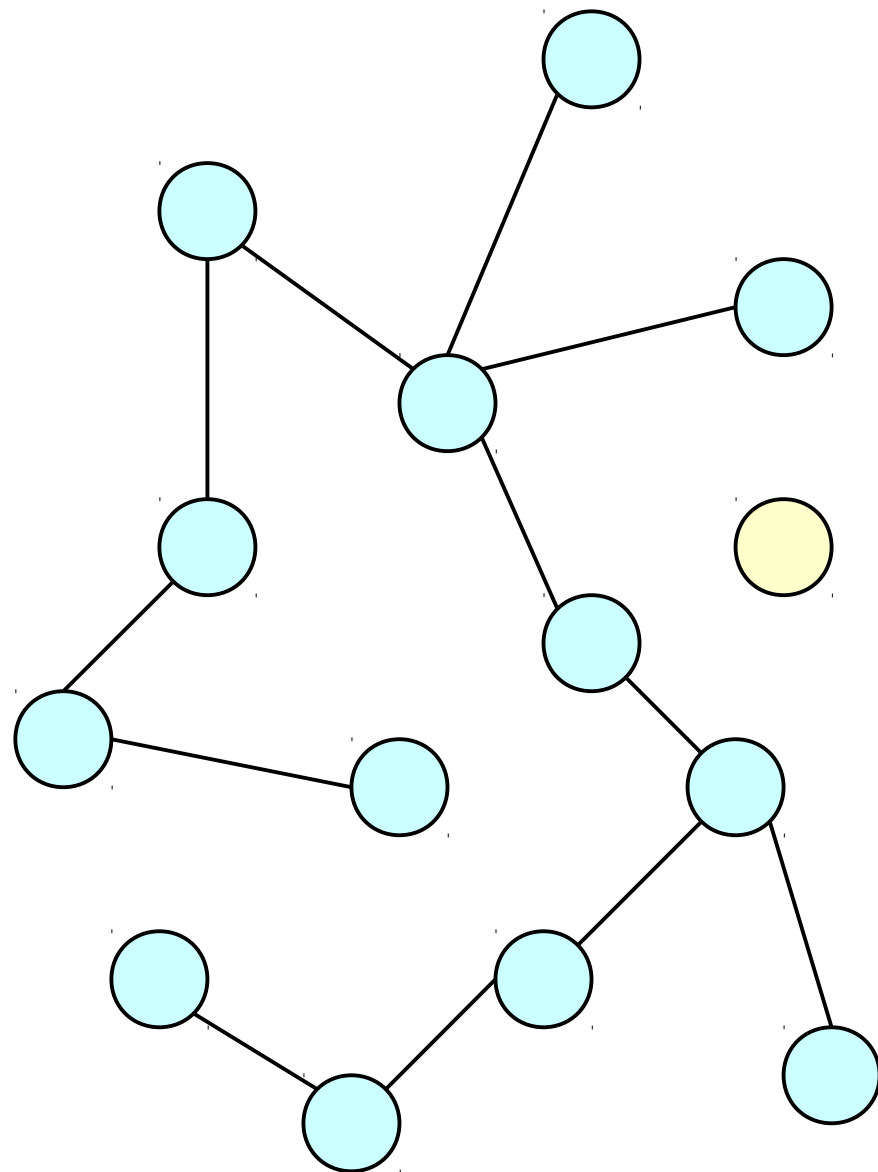
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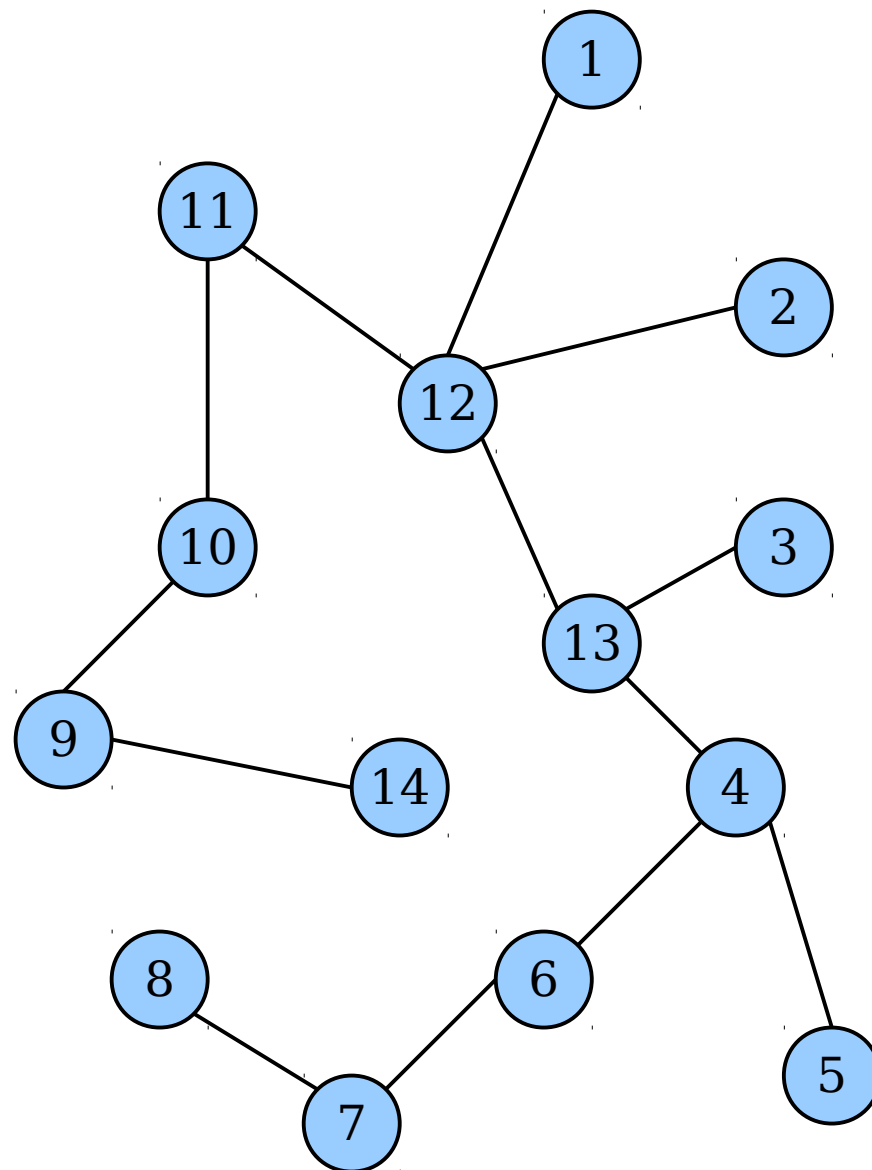
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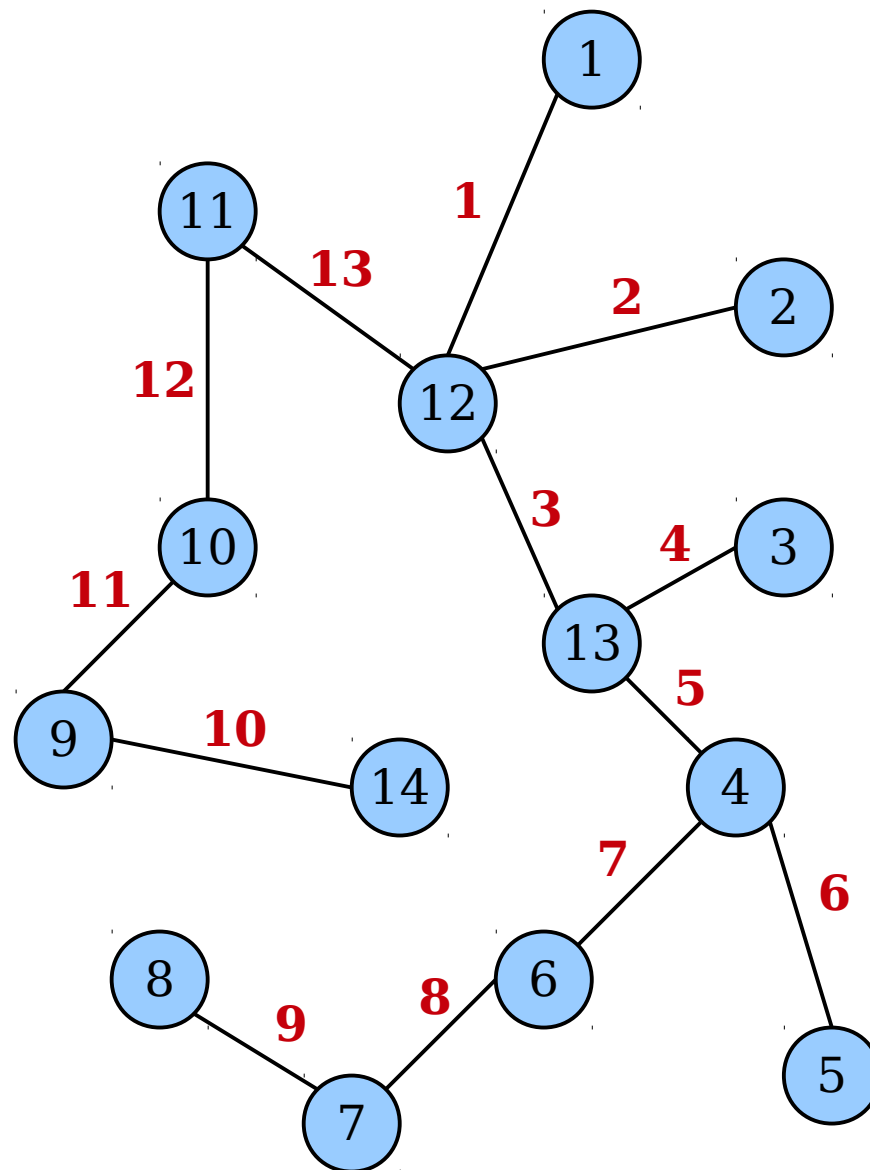
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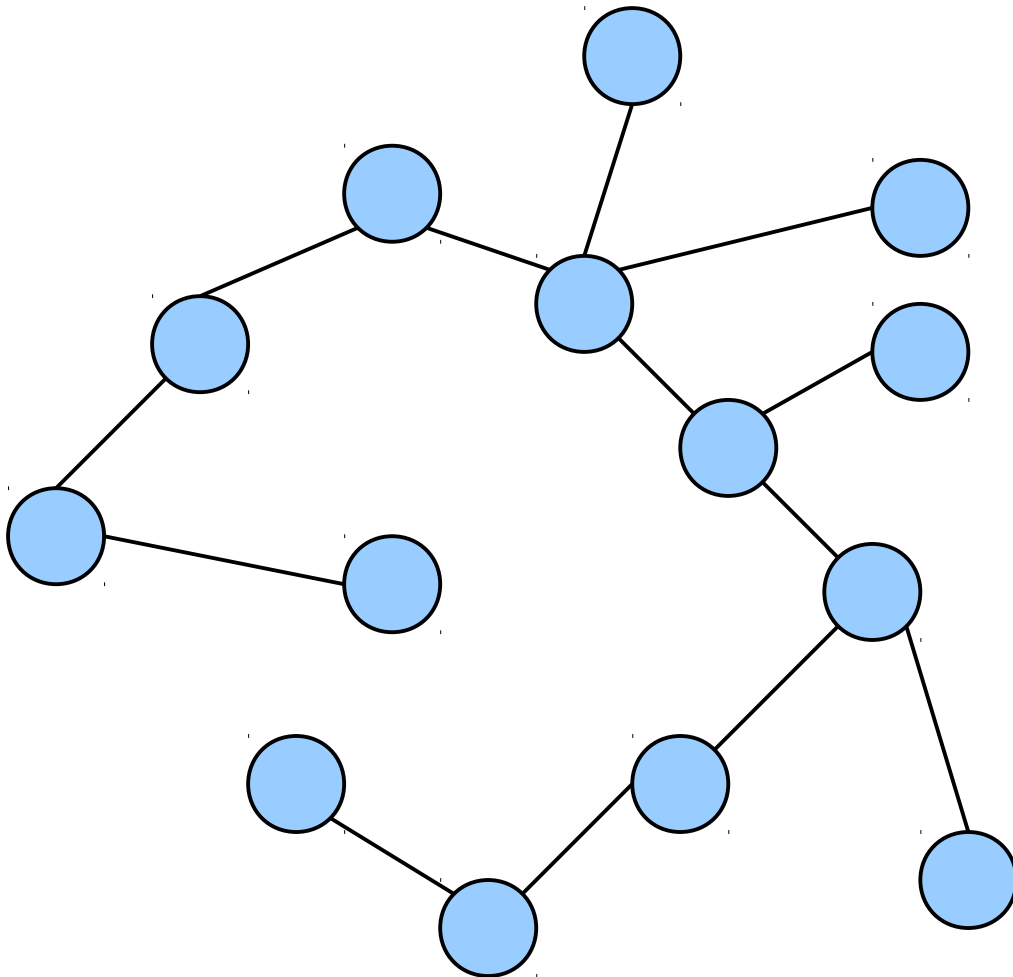
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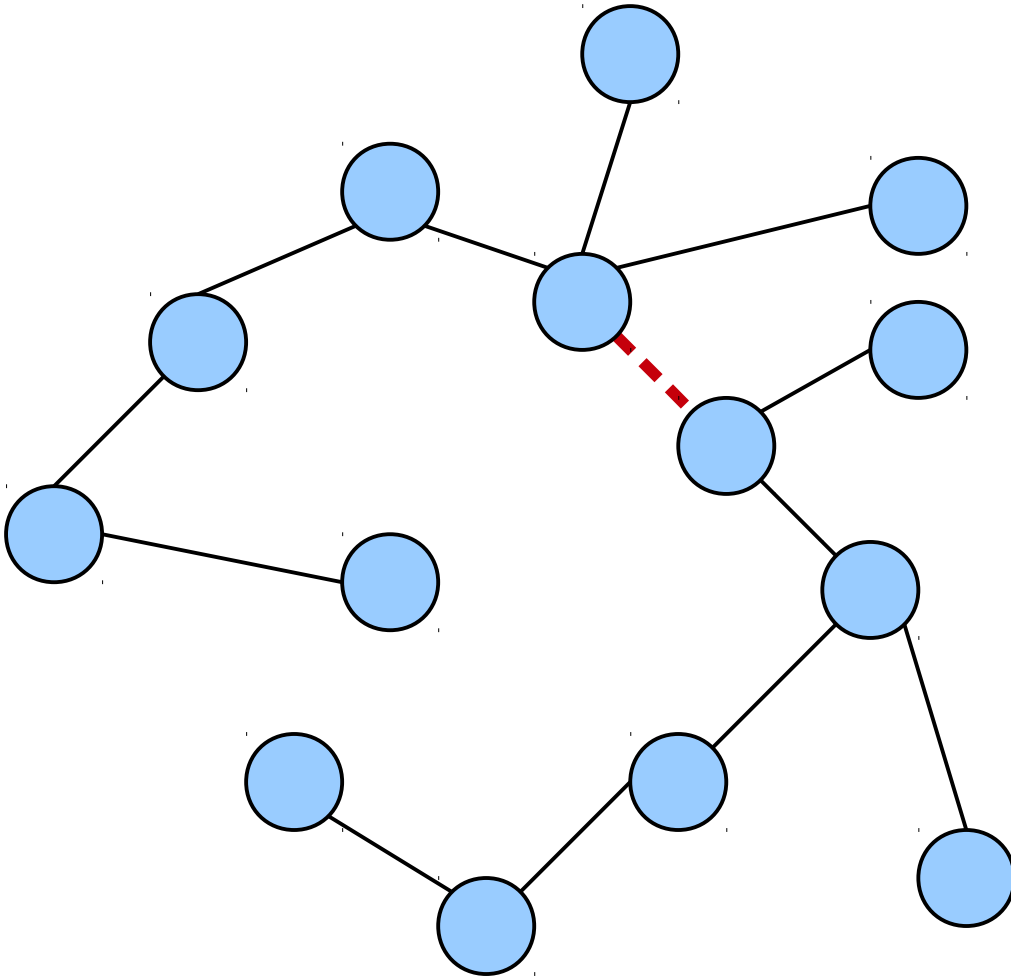
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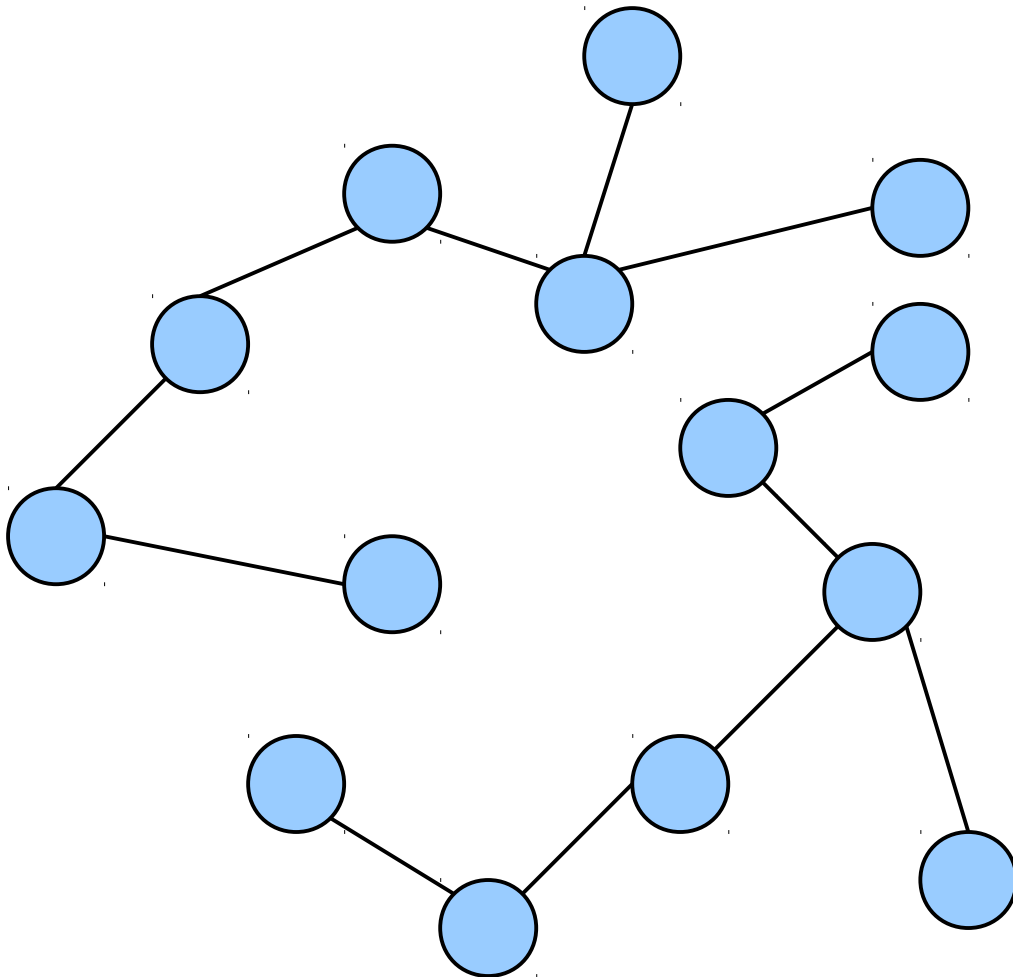
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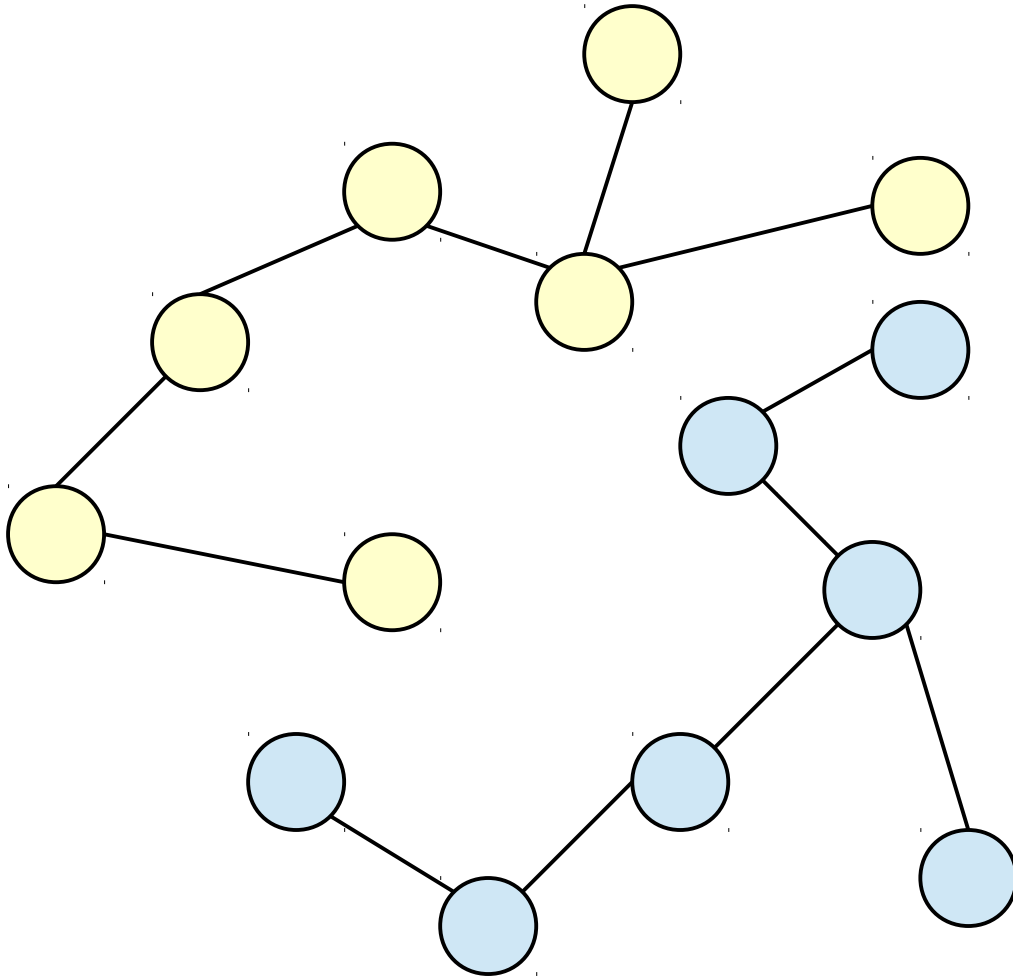
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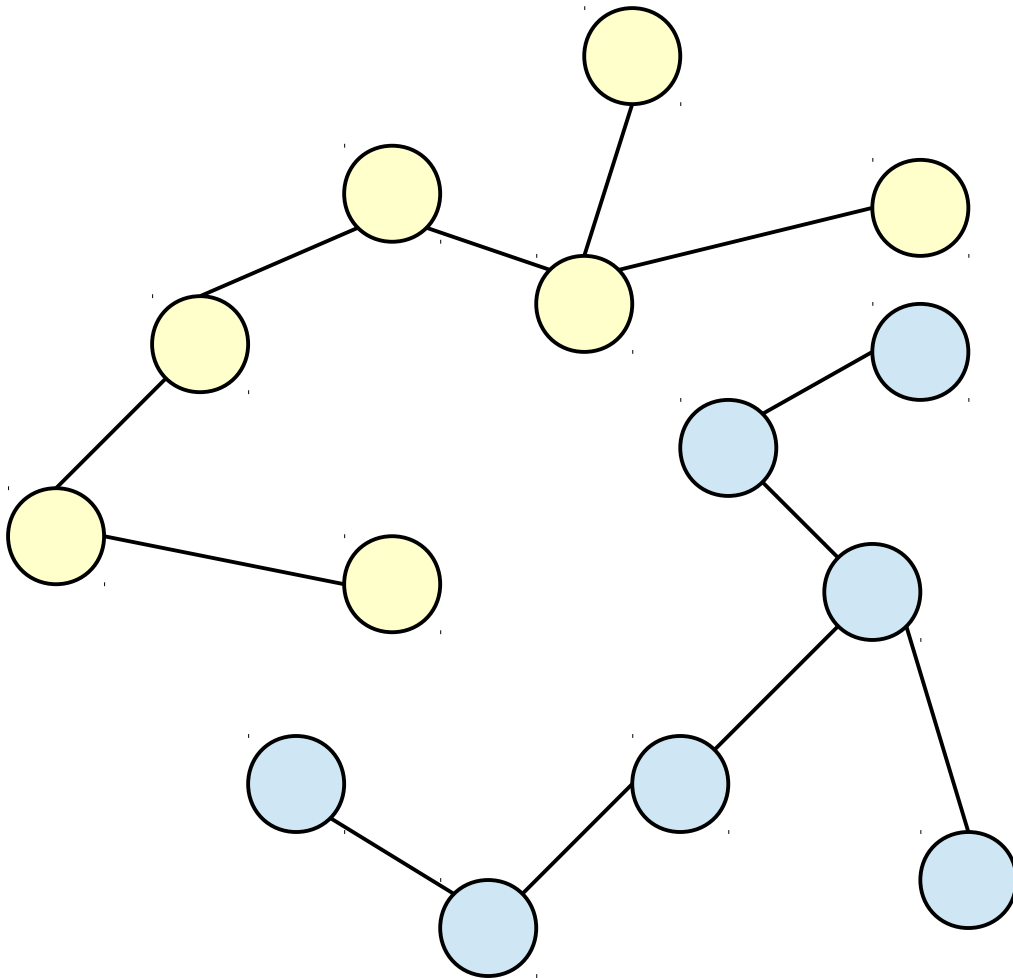
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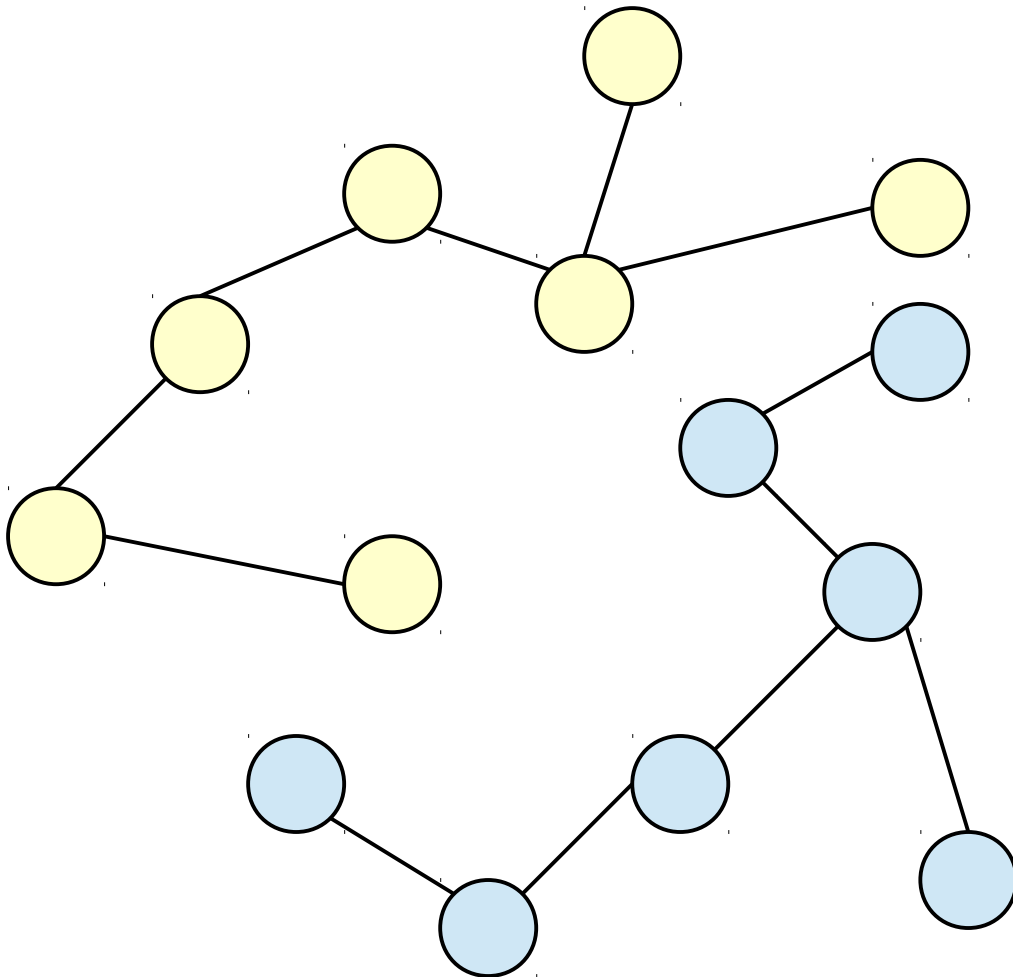


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Consider an arbitrary tree with  $k+1$  nodes.

Suppose there are  $r$  nodes in the yellow tree.

Then there are  $(k+1)-r$  nodes in the blue tree.



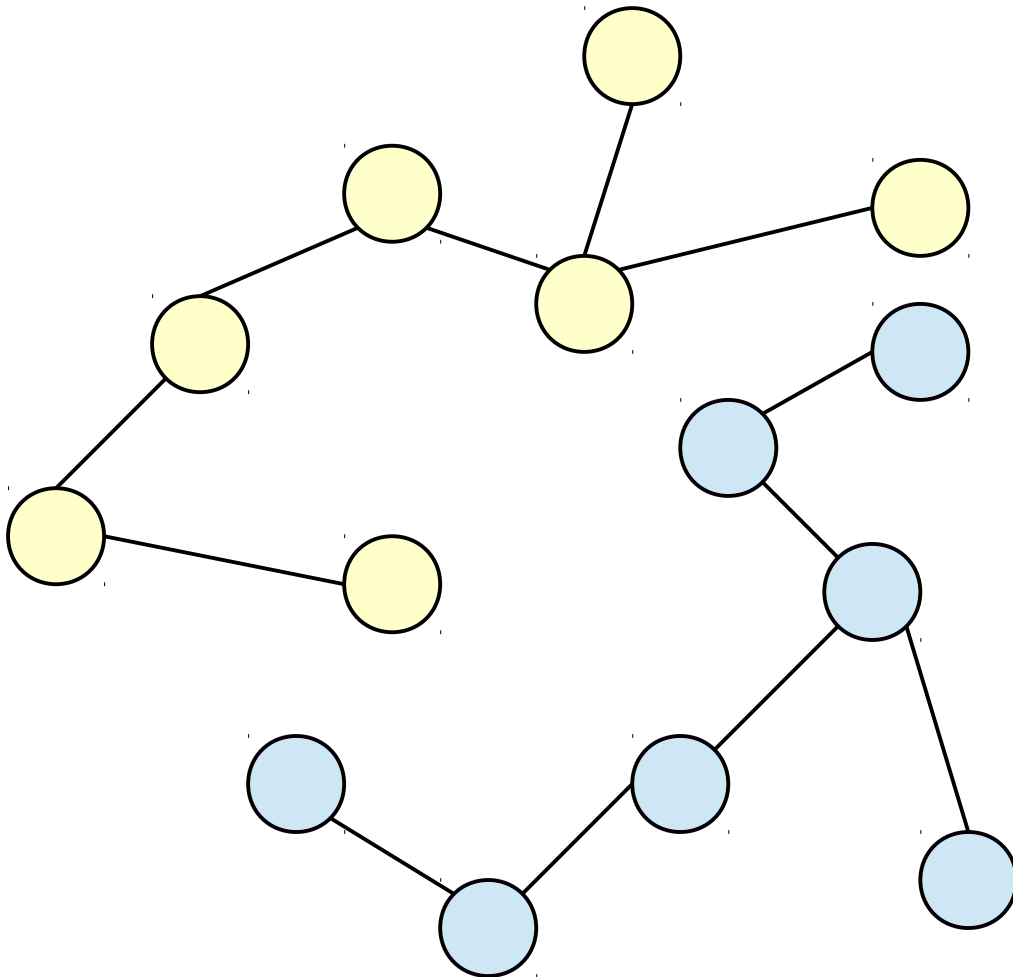
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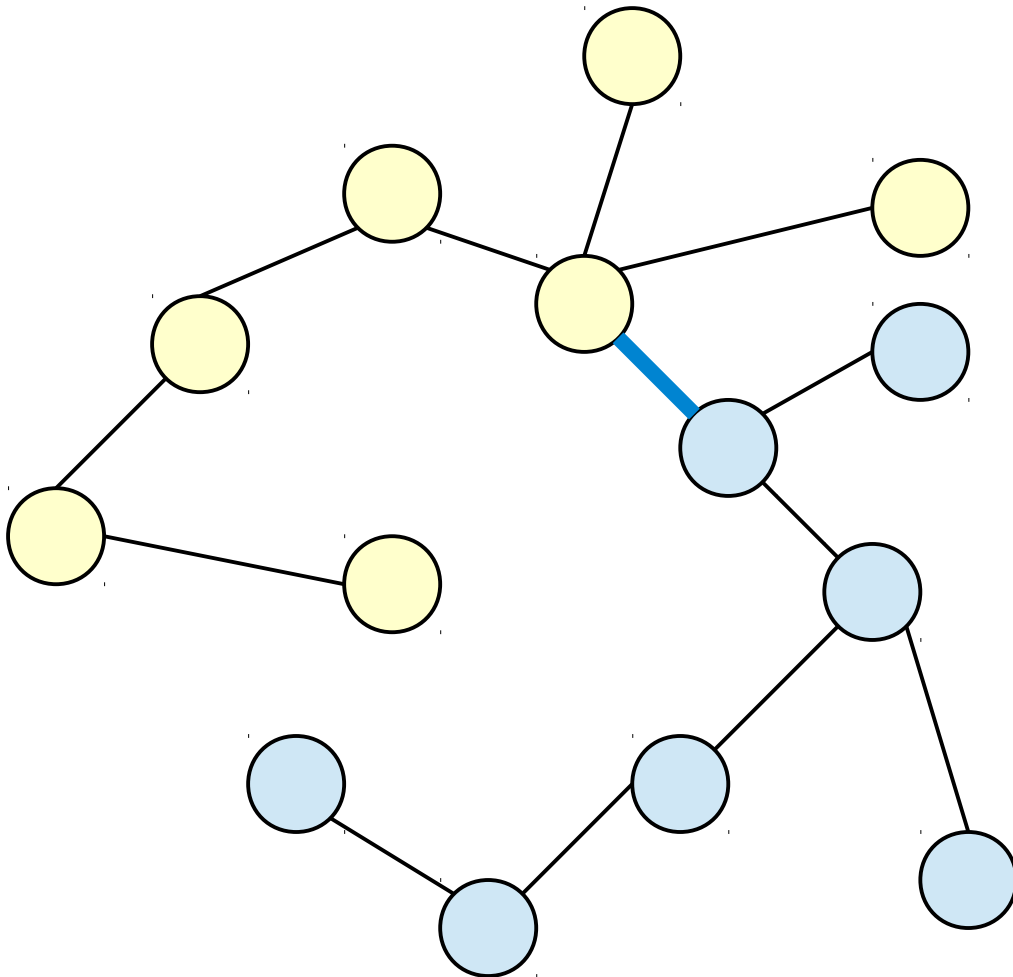
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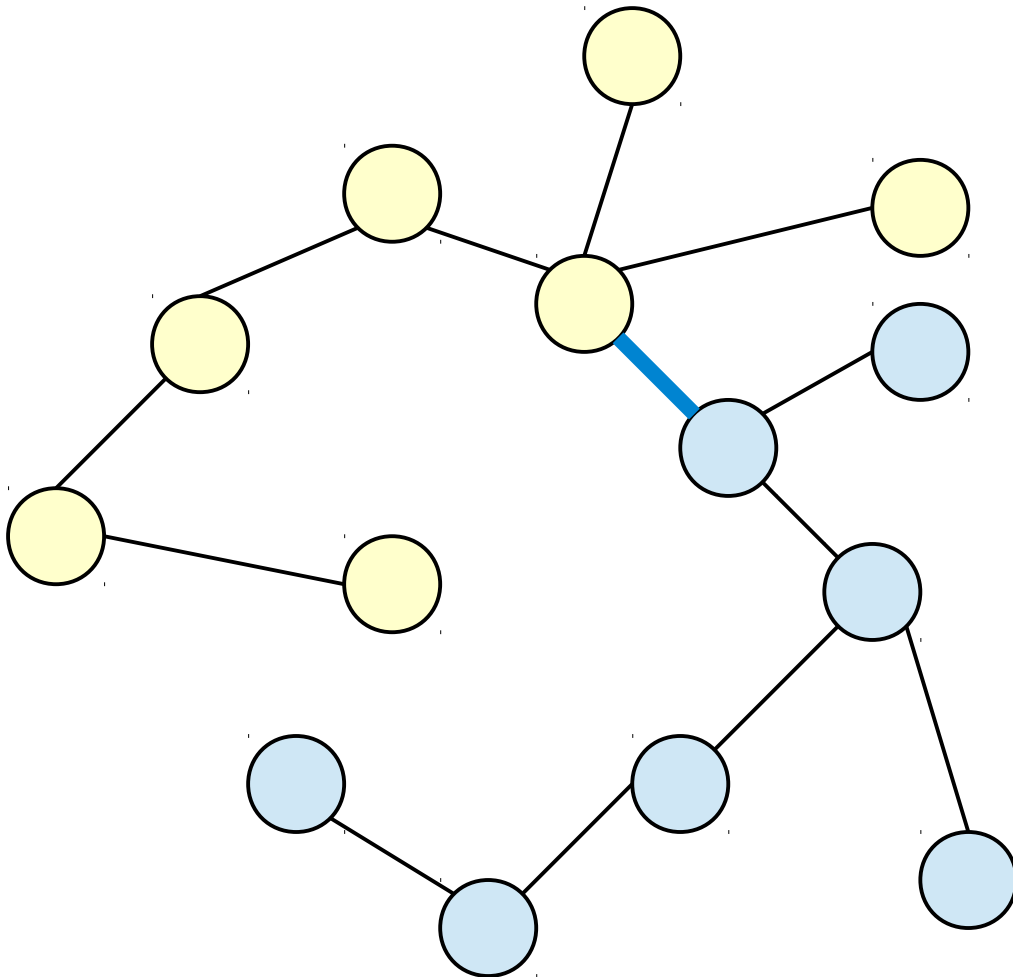
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Consider an arbitrary tree with  $k+1$  nodes.

Suppose there are  $r$  nodes in the yellow tree.

Then there are  $(k+1)-r$  nodes in the blue tree.

There are  $r-1$  edges in the yellow tree and  $k-r$  edges in the blue tree.

Adding in the initial edge we cut, there are  $r-1 + k-r + 1 = k$  edges in the original tree.



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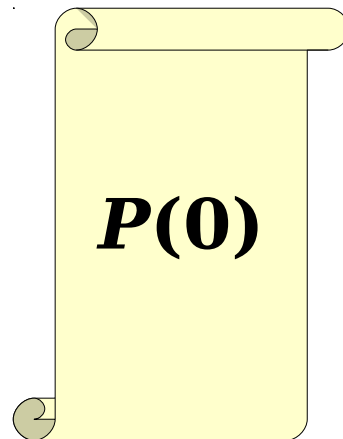
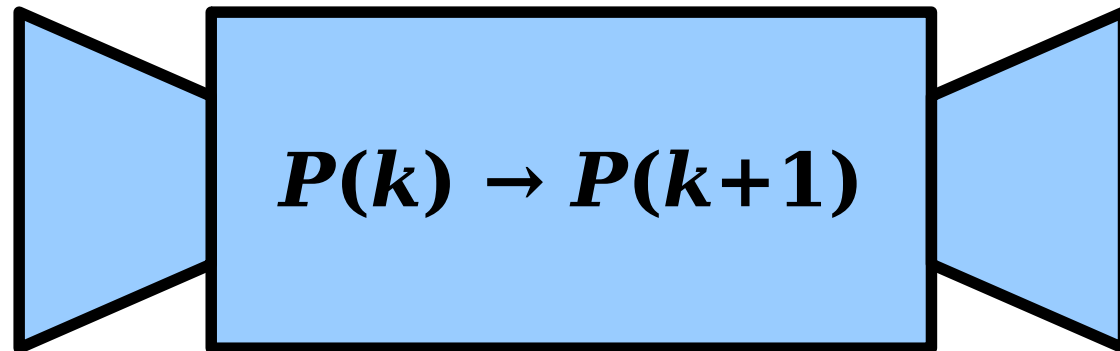
# Complete Induction

- If the following are true:
  - $P(0)$  is true, and
  - If  $P(0), P(1), P(2), \dots, P(k)$  are true, then  $P(k+1)$  is true as well.

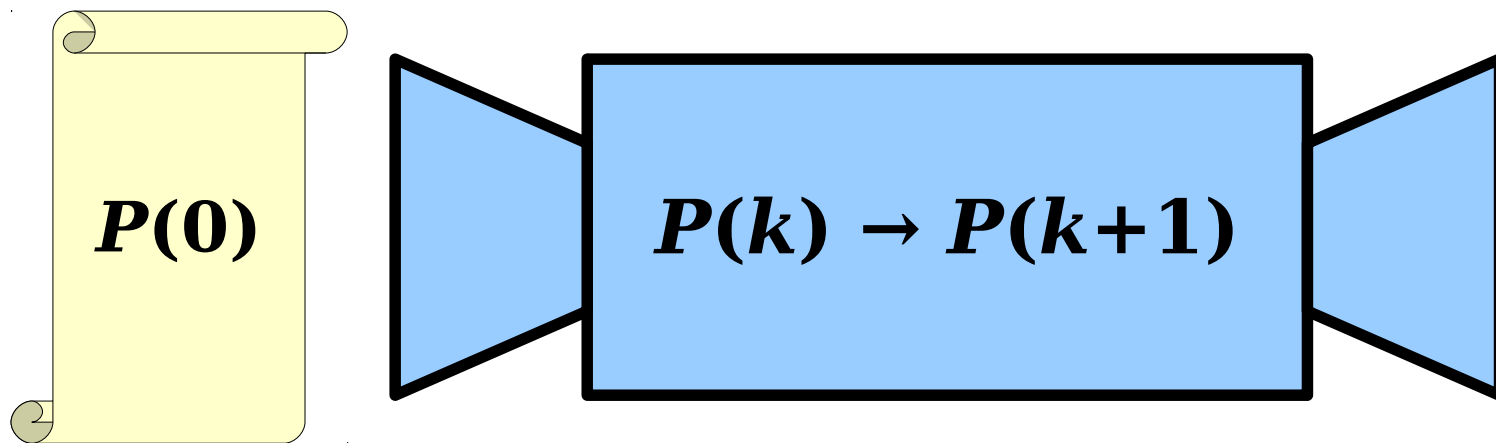
then  $P(n)$  is true for all  $n \in \mathbb{N}$ .

- This is called the ***principle of complete induction*** or the ***principle of strong induction***.
  - (This also works starting from a number other than 0; just modify what you're assuming appropriately.)

# Review: Induction as a Machine

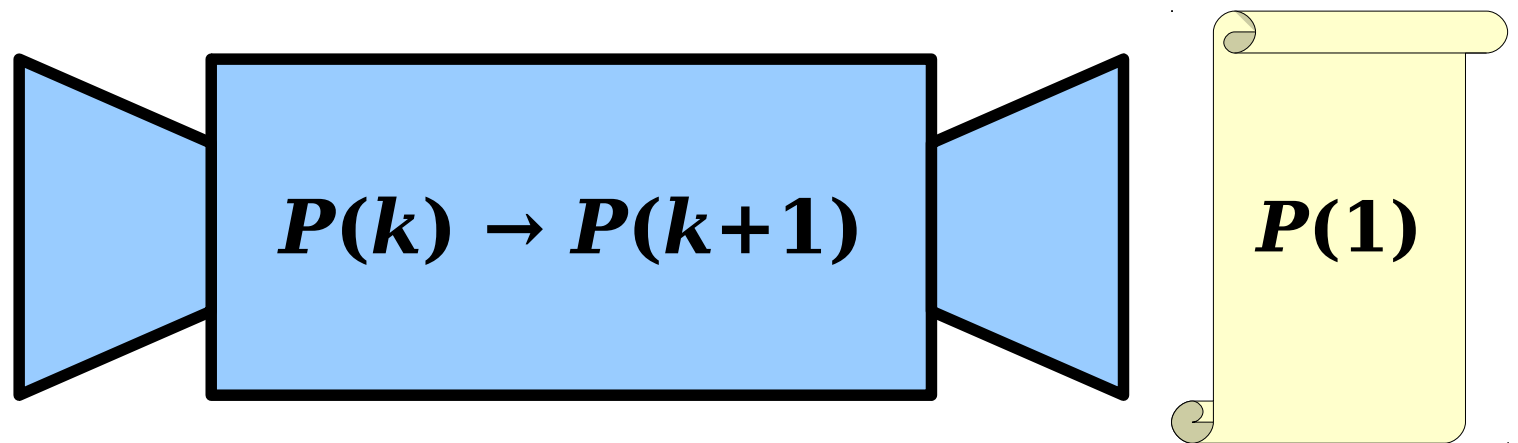


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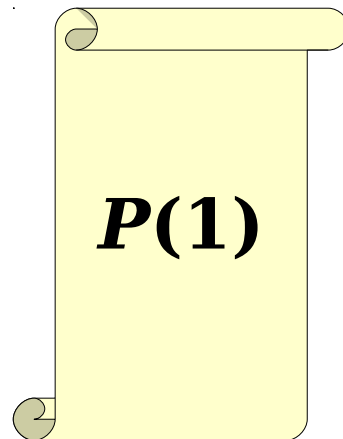
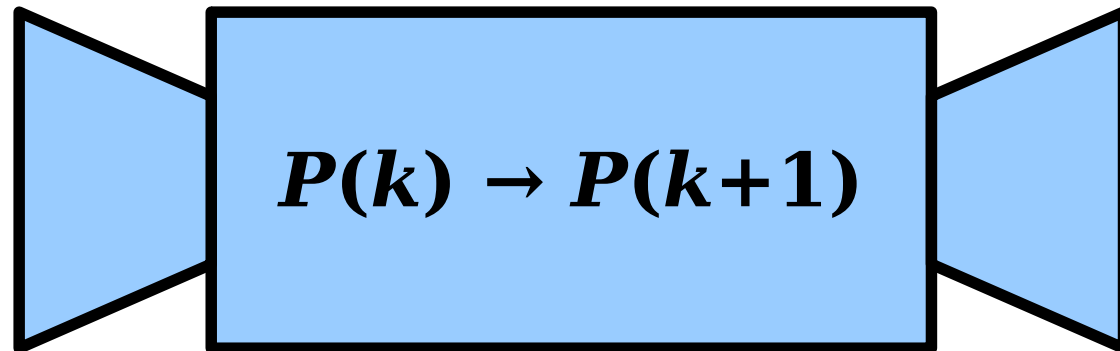




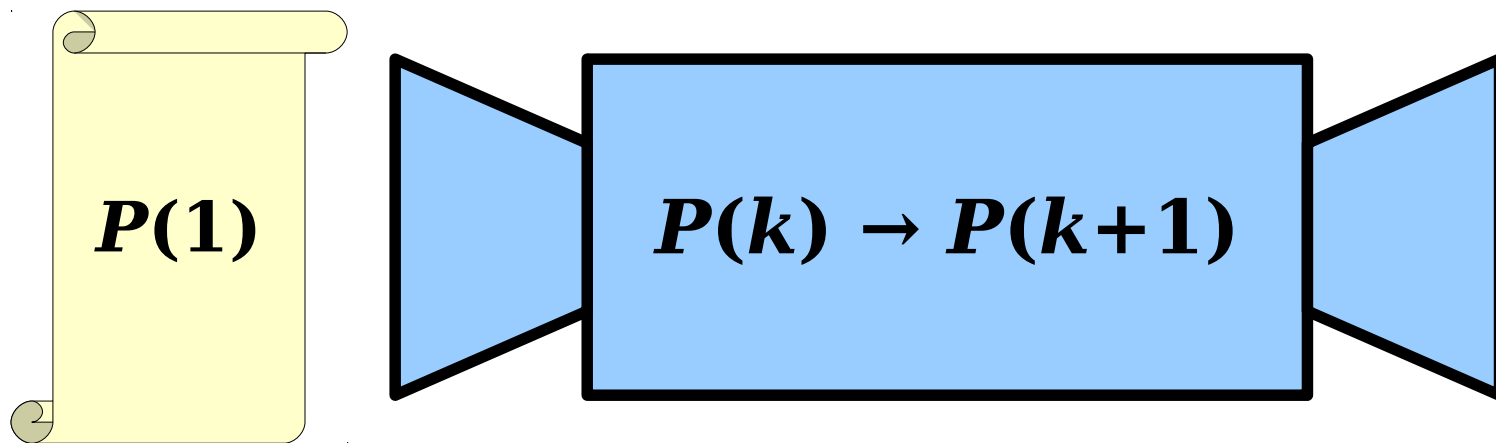
# Review: Induction as a Machine



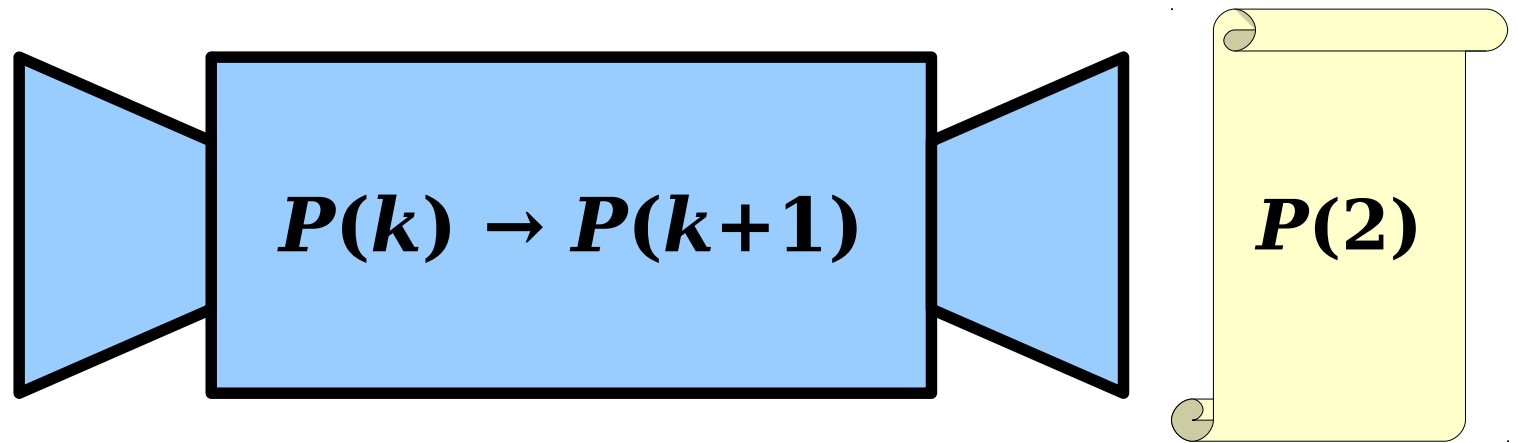
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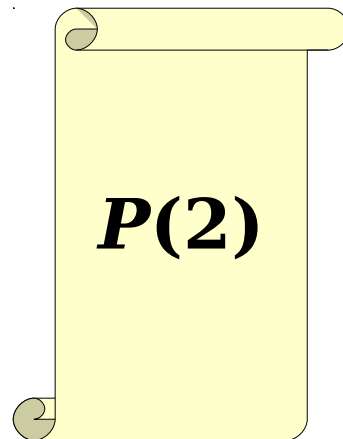
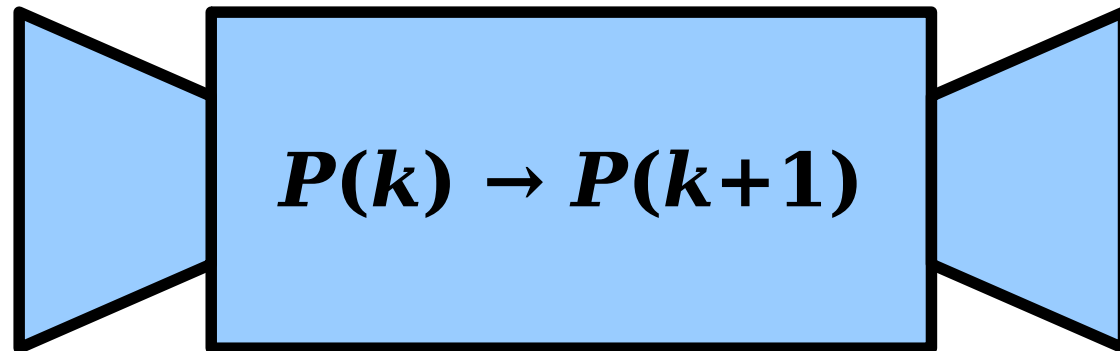
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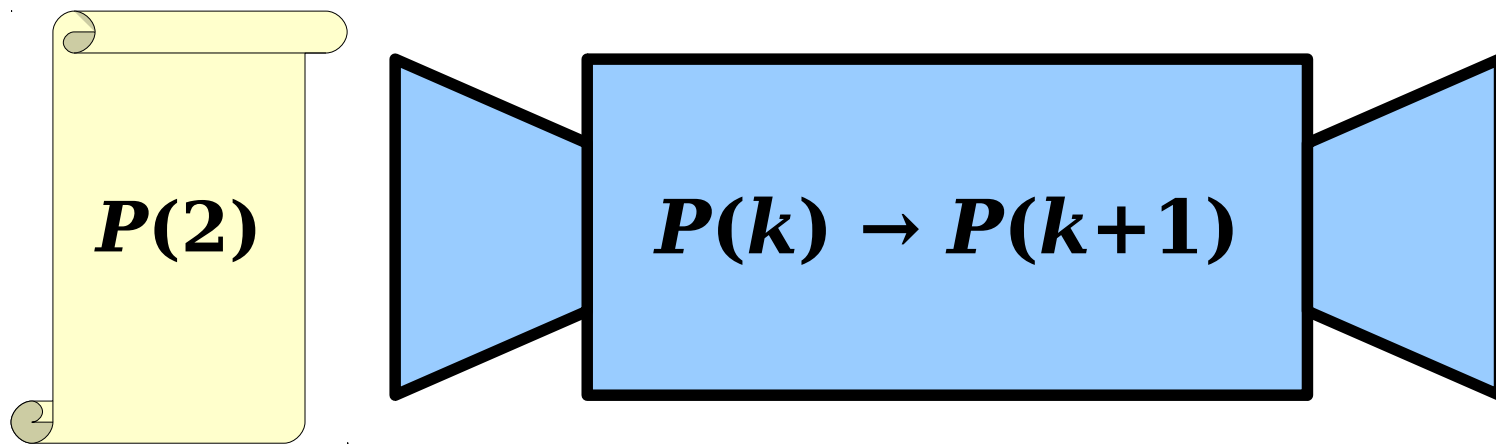
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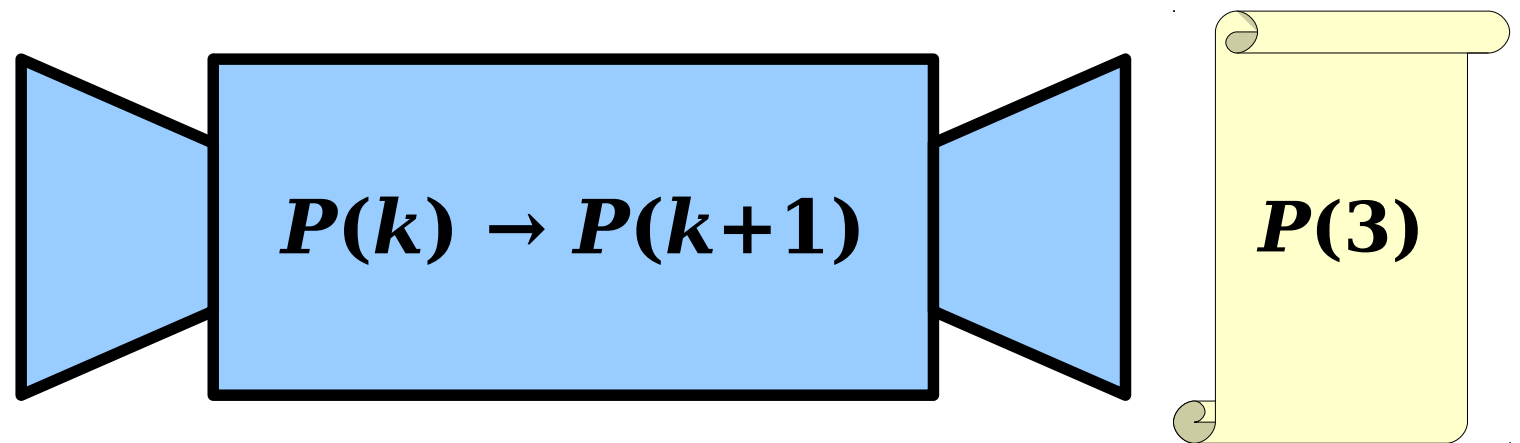
# Review: Induction as a Machine



# Review: Induction as a Machine



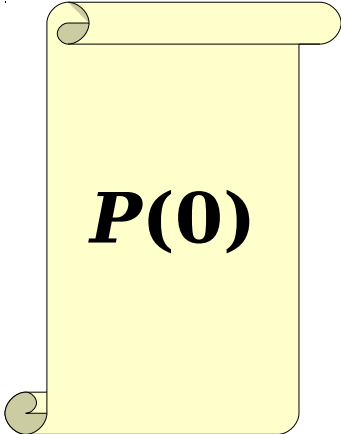
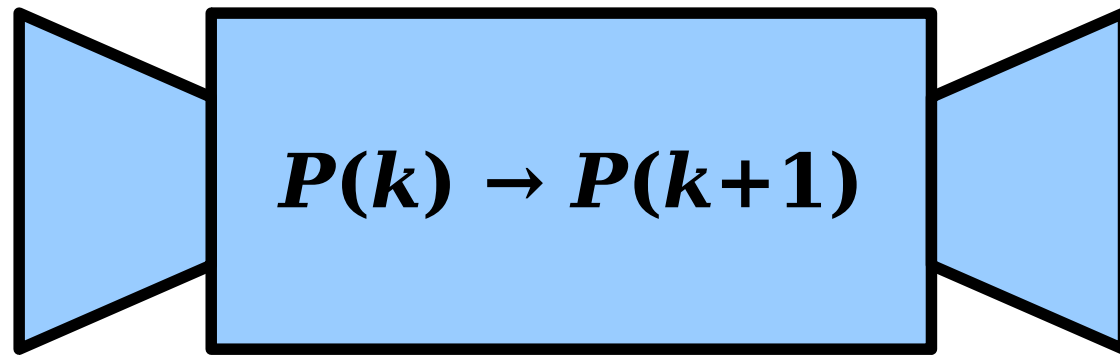
# Review: Induction as a Machine



# An Observation

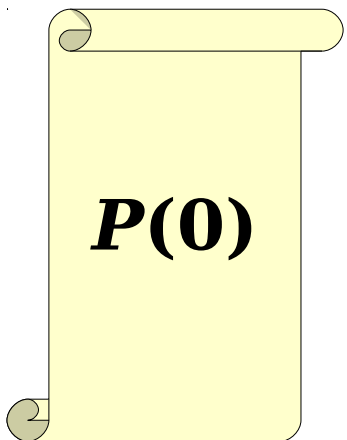
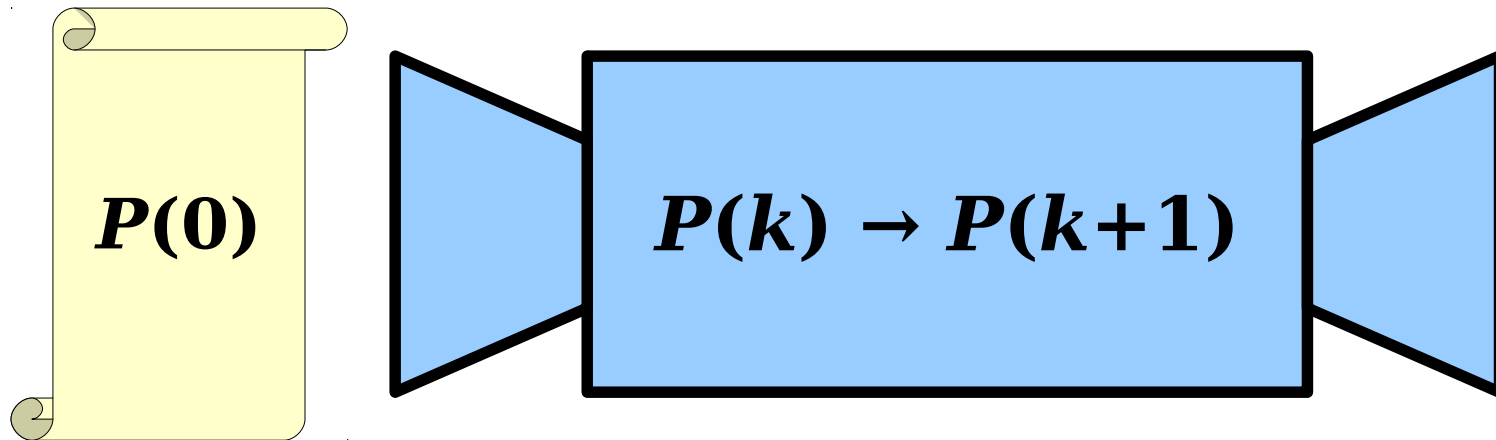


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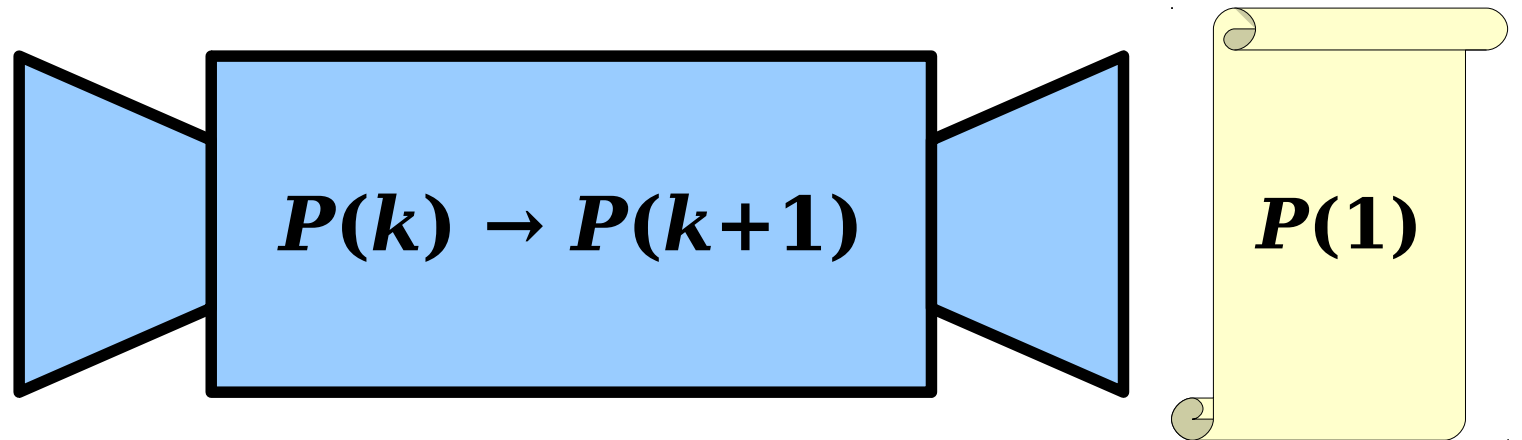


$P(0)$

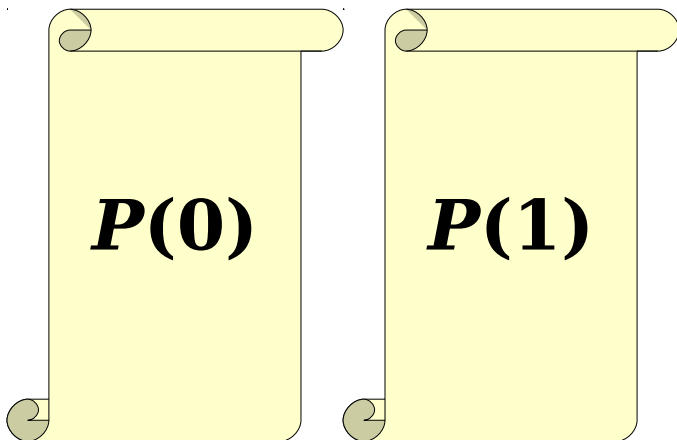
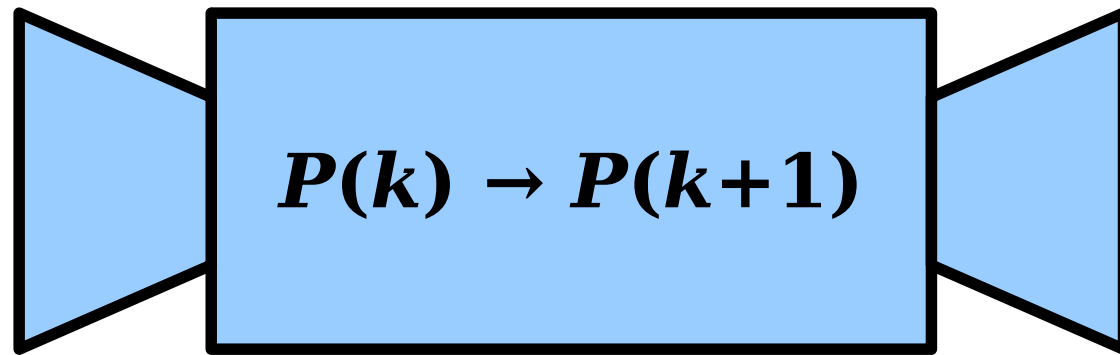
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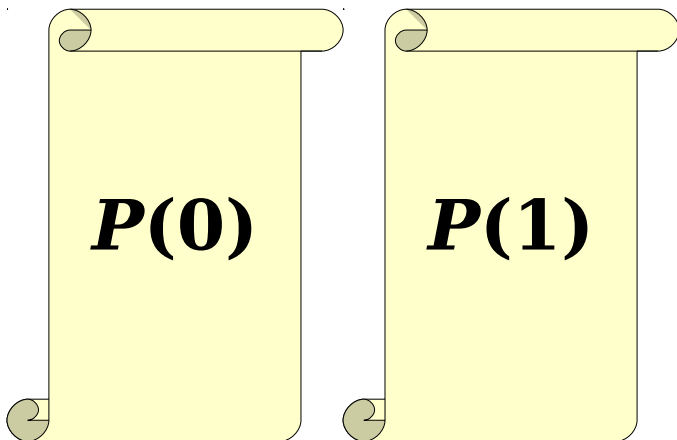
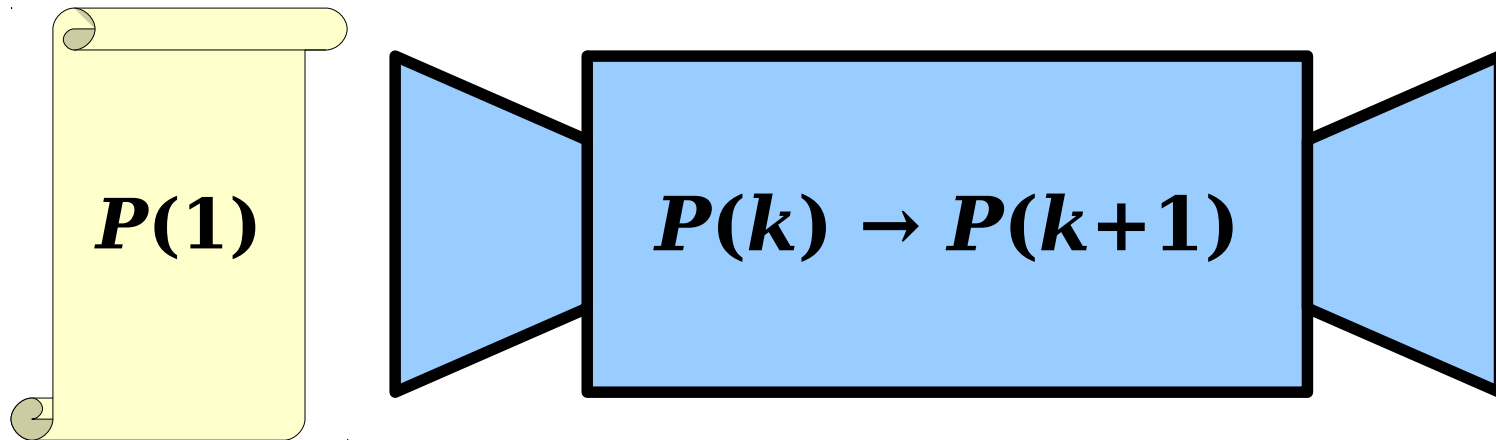
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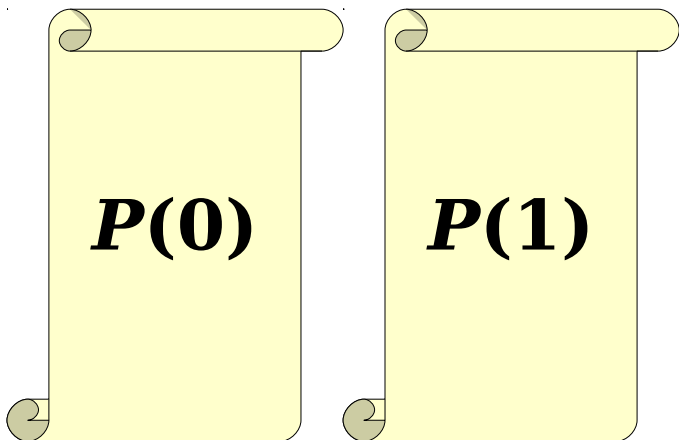
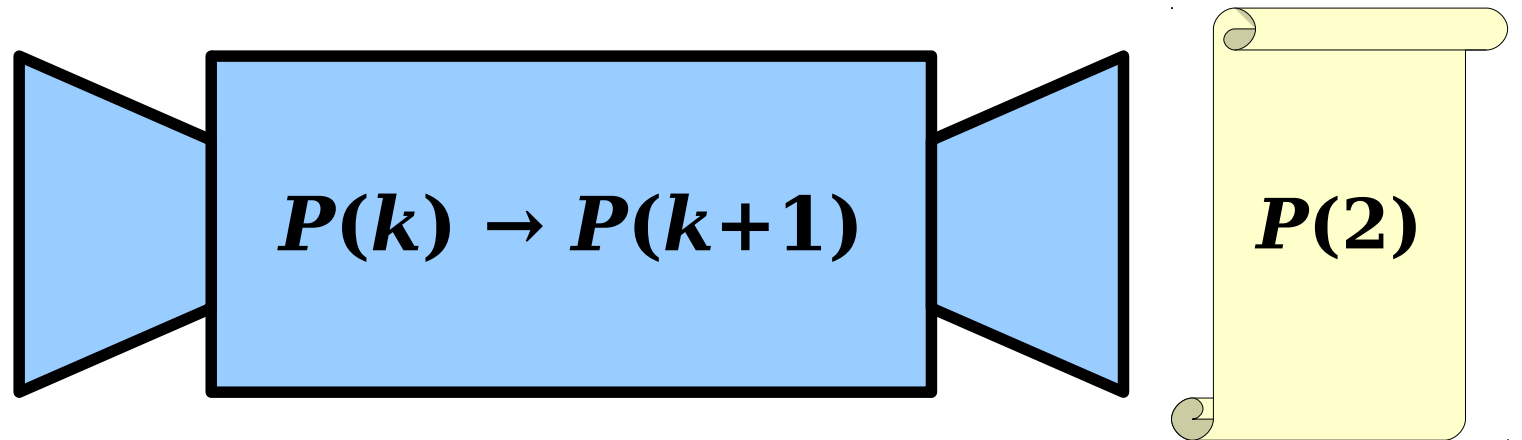
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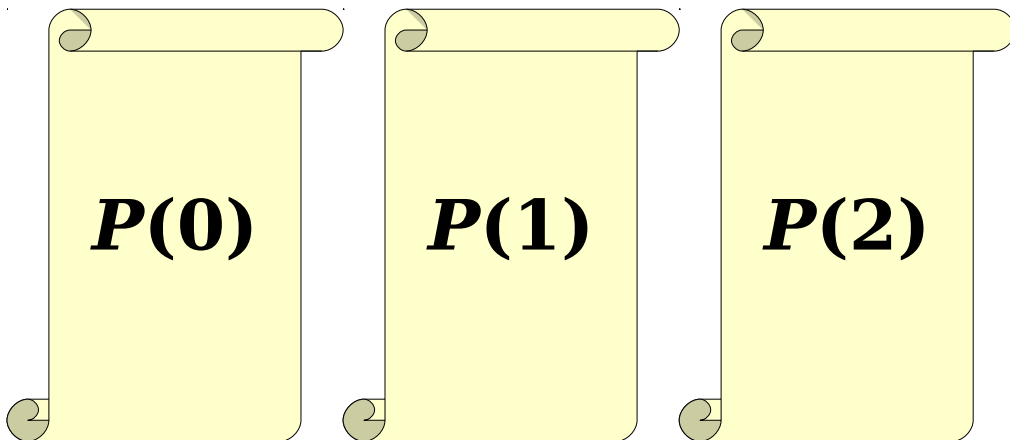
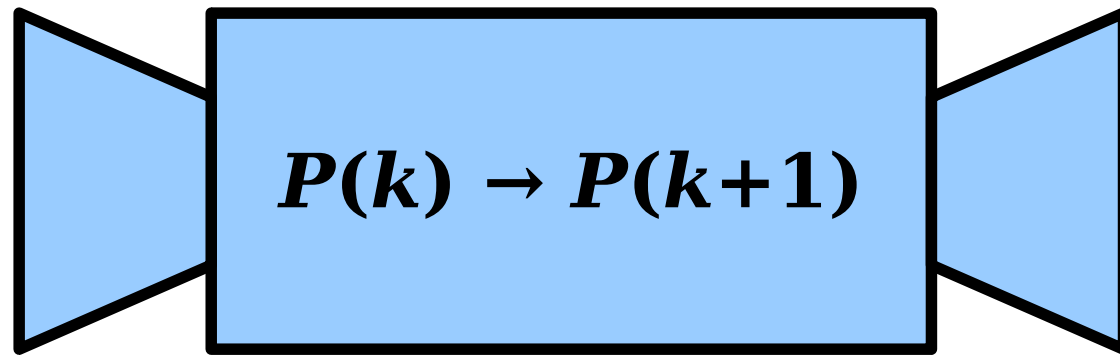
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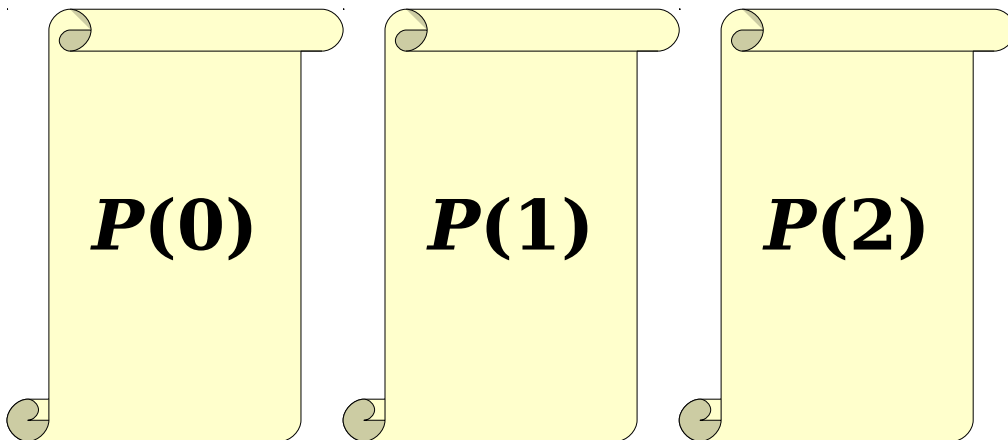
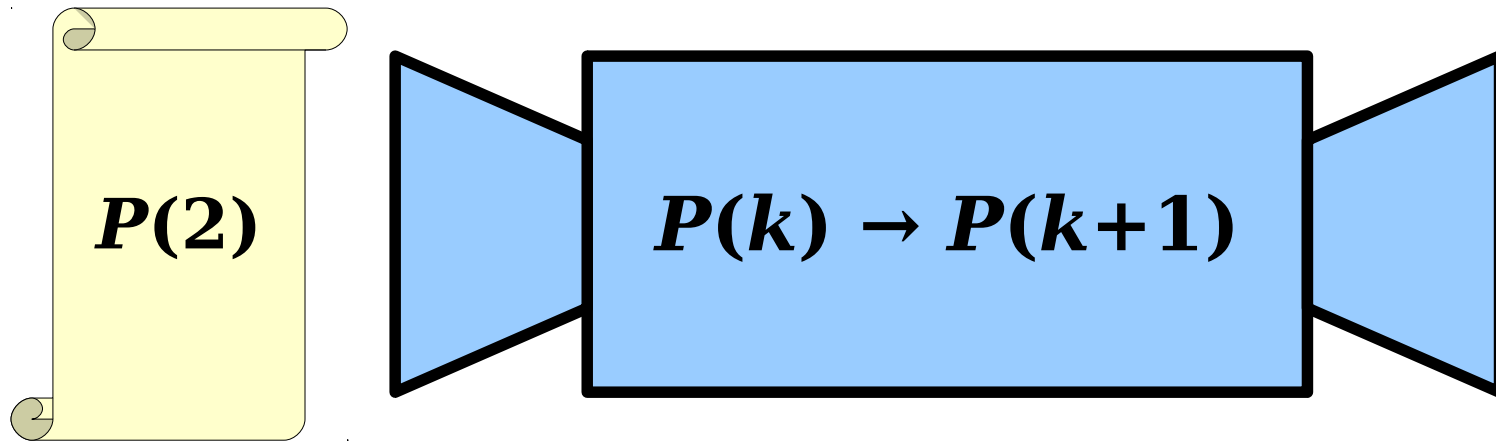
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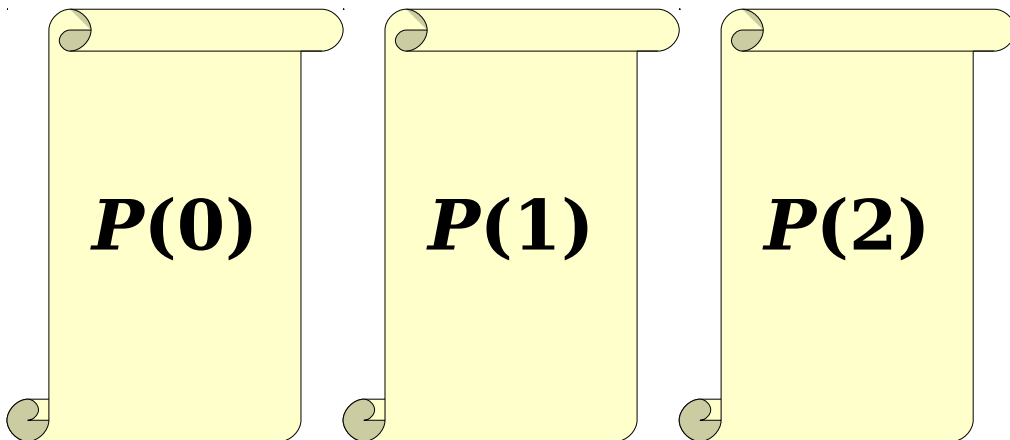
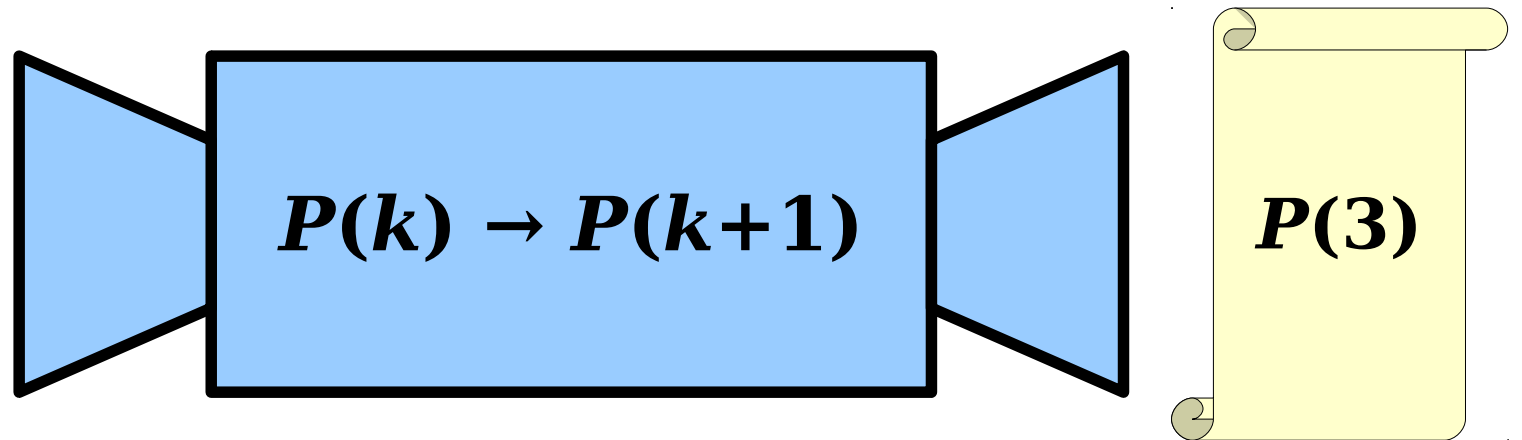


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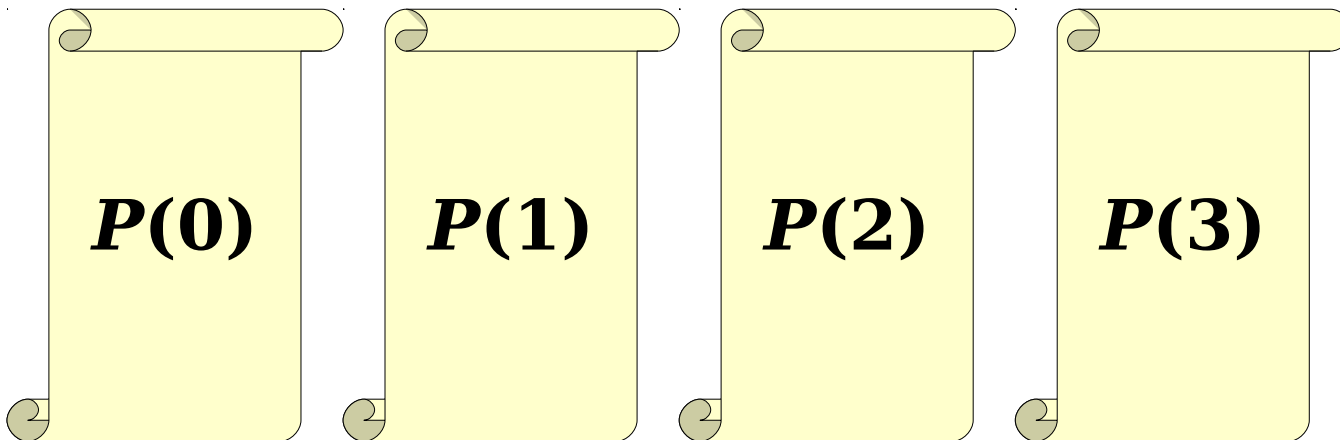
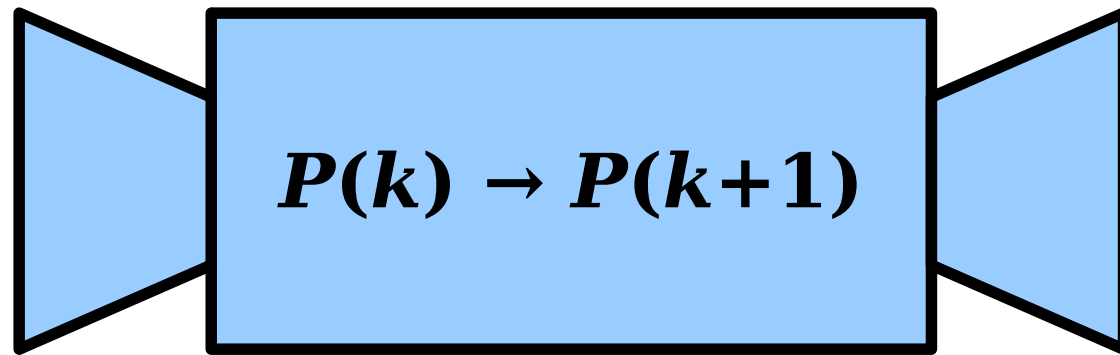




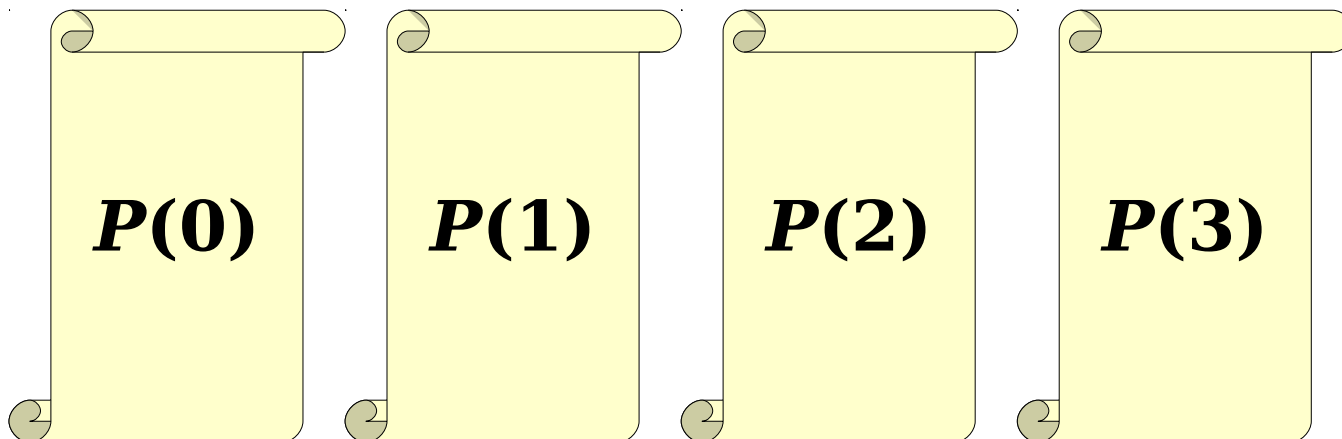
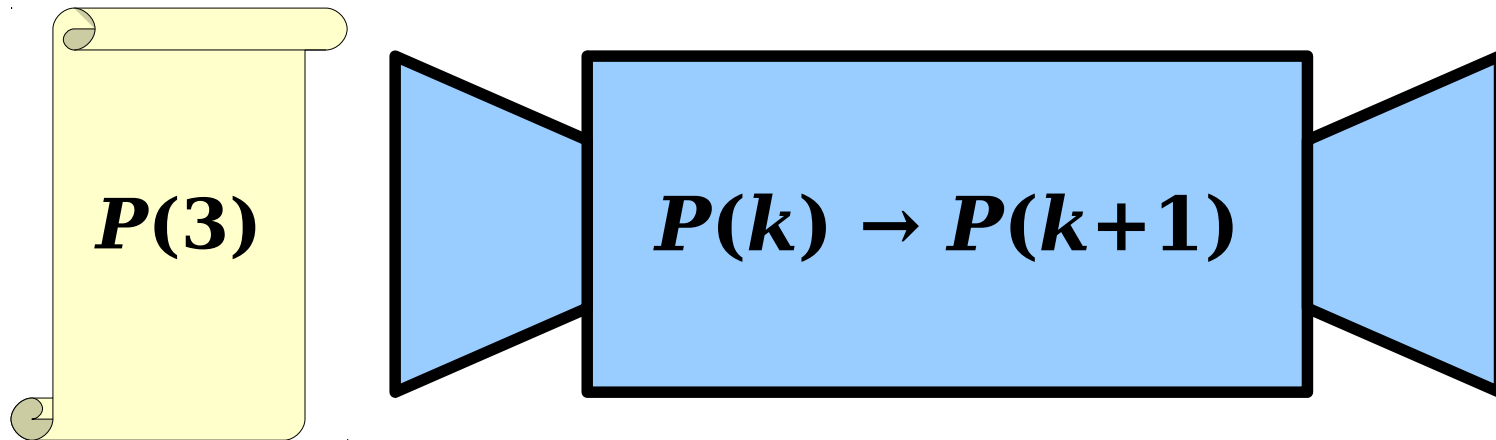
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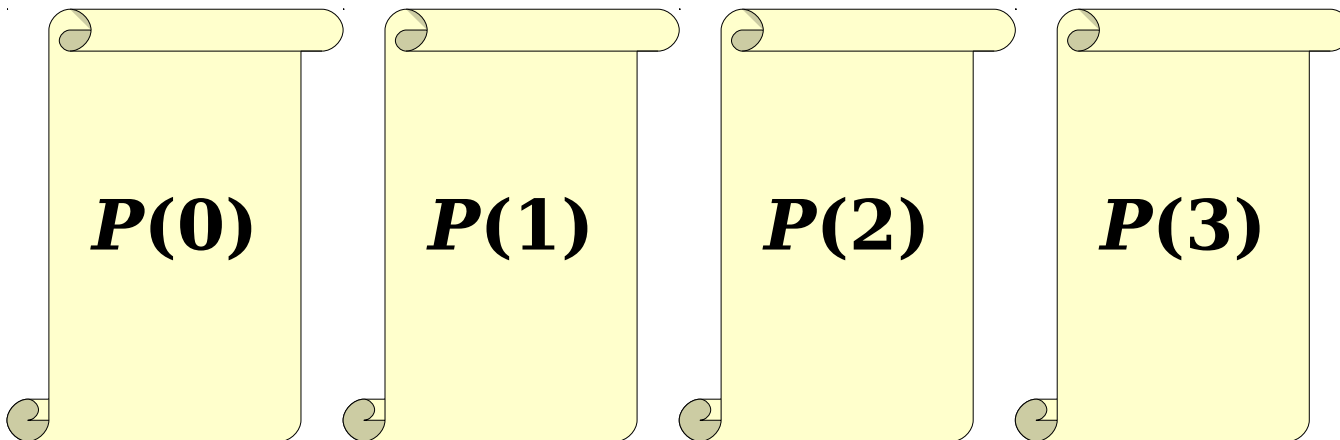
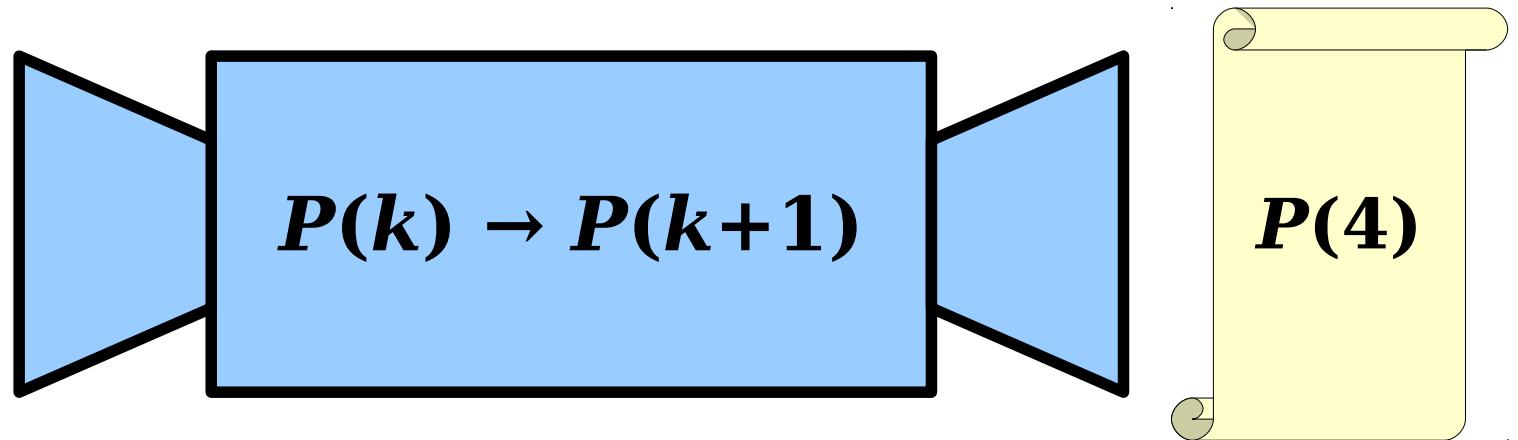
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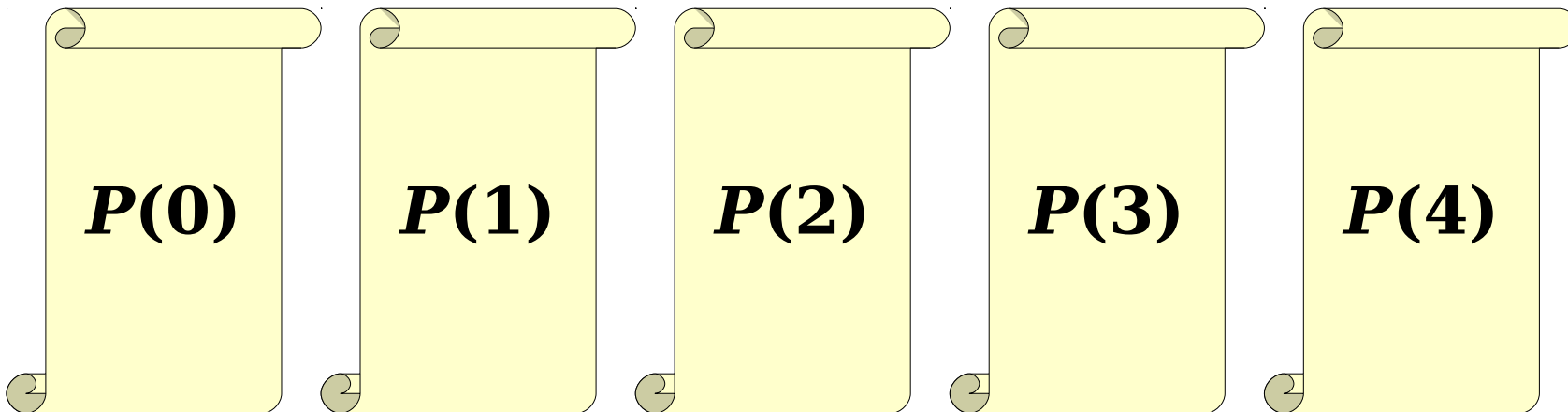
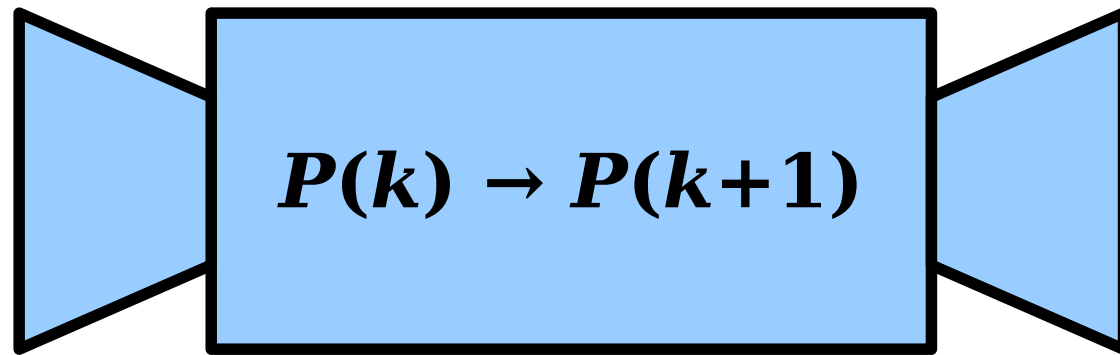
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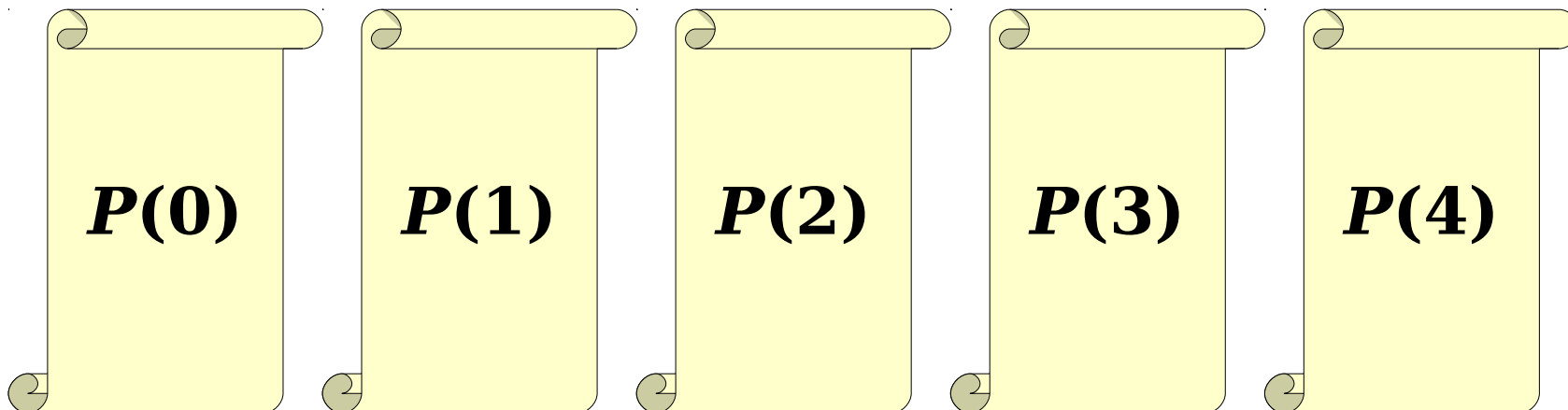
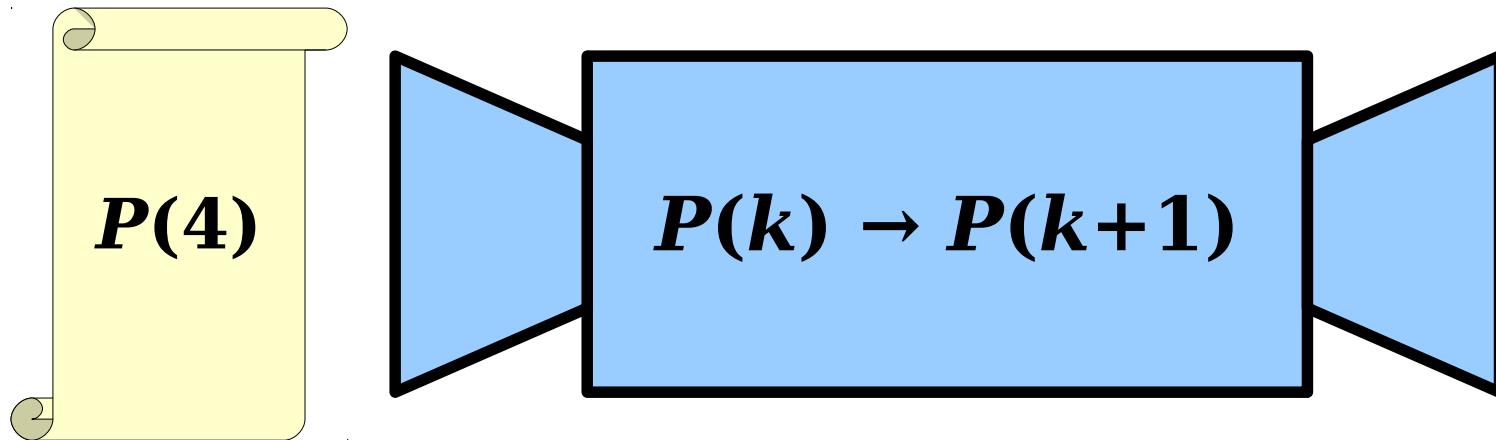
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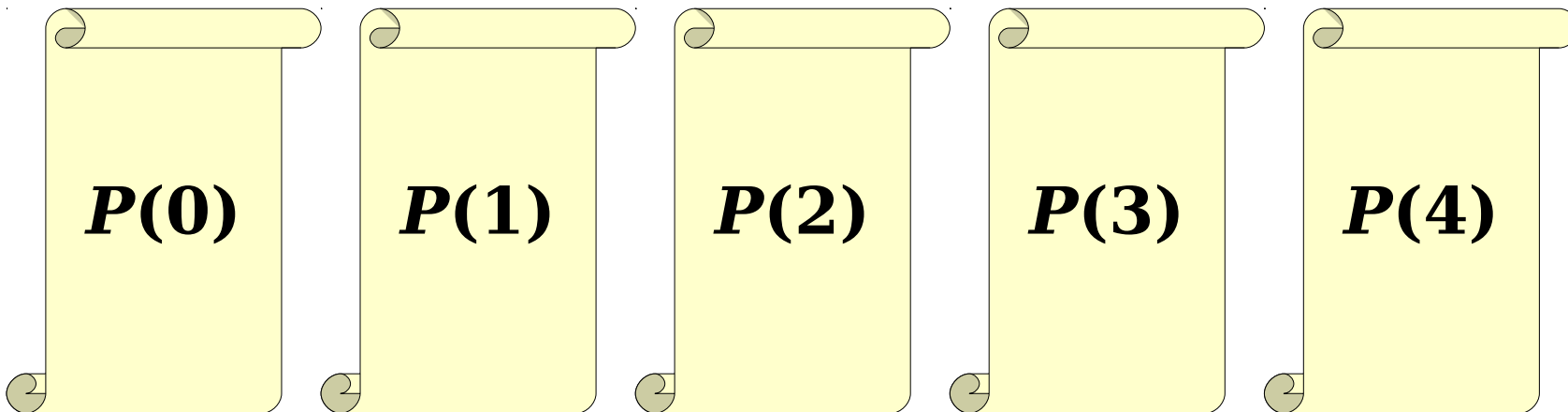
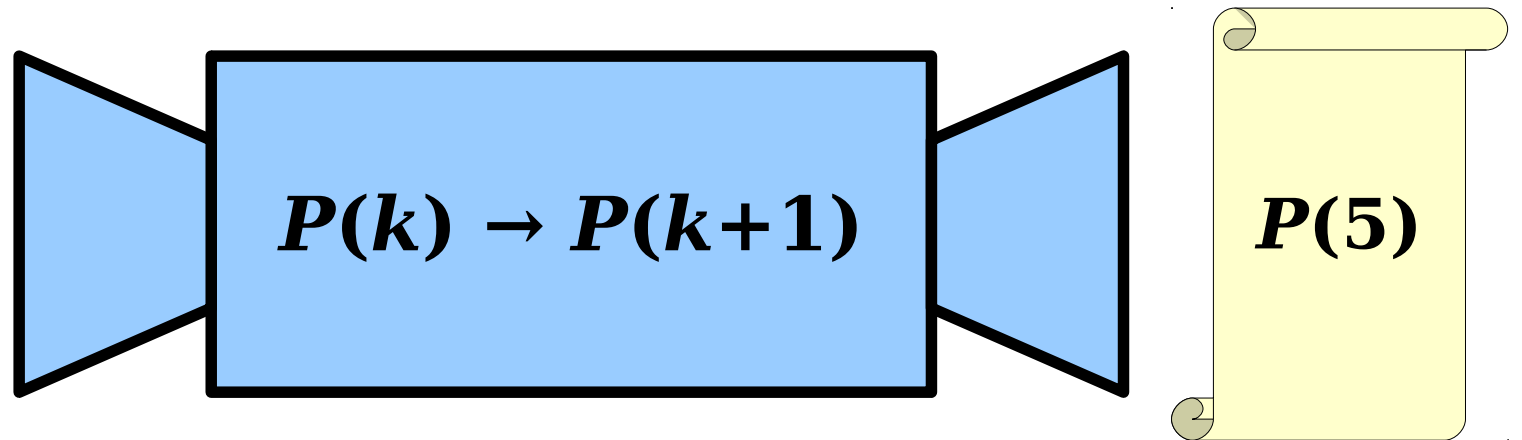
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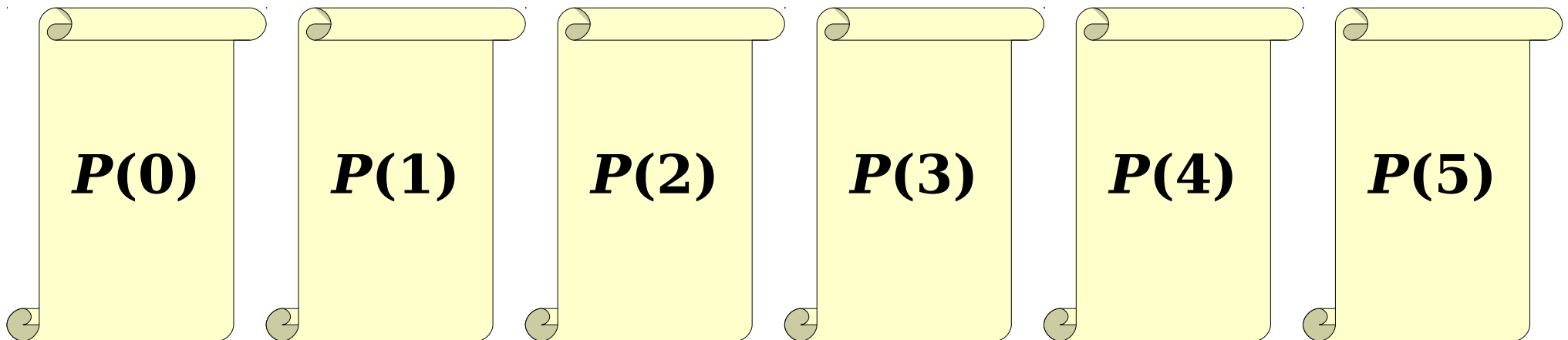
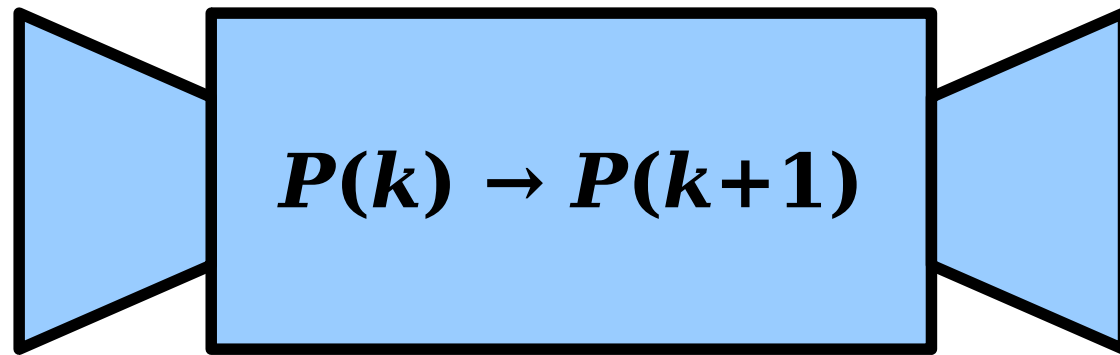
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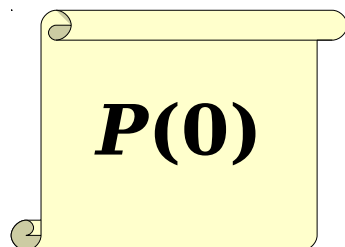
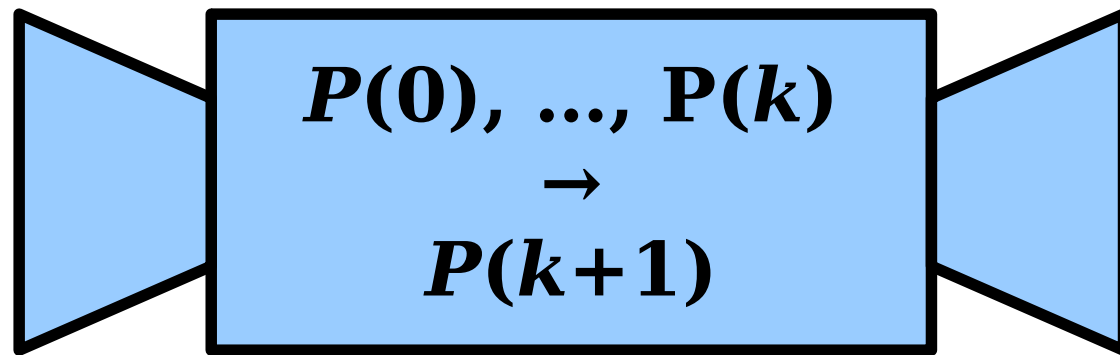


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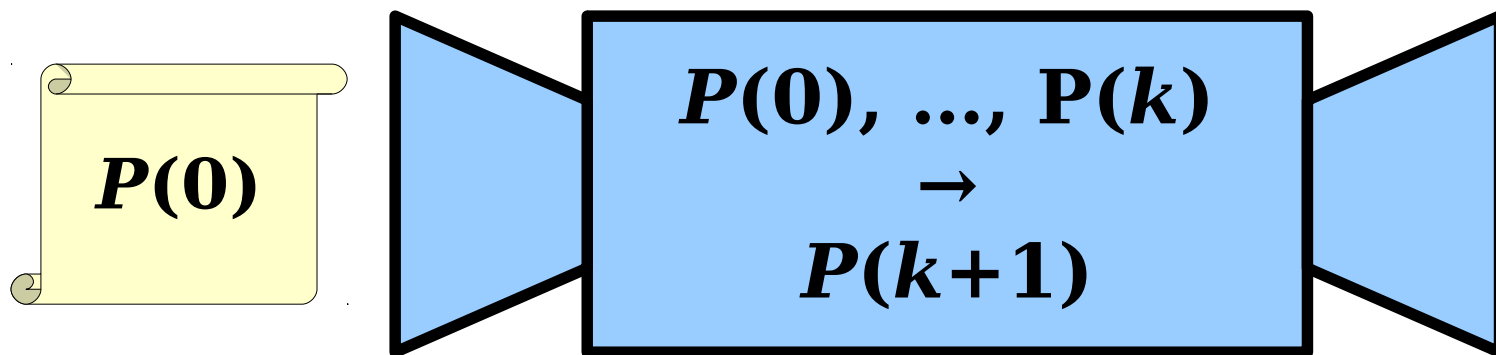




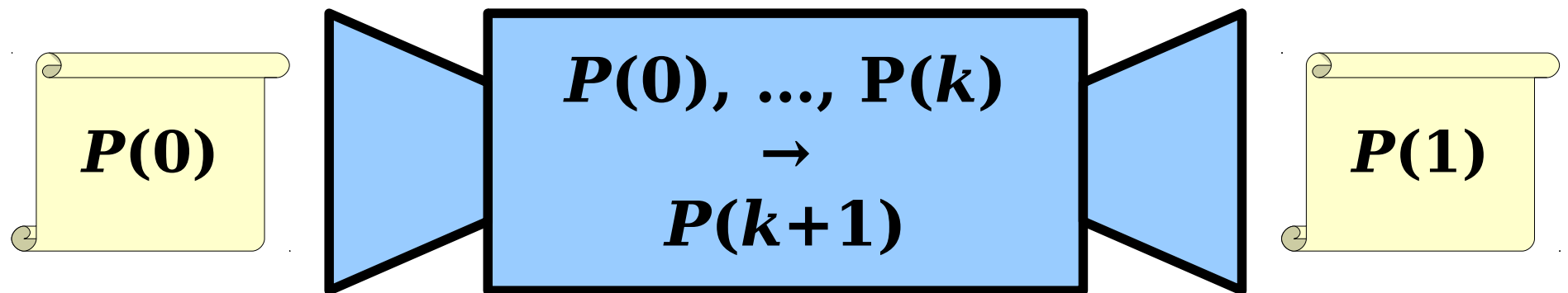
# Intuiting Complete Induction



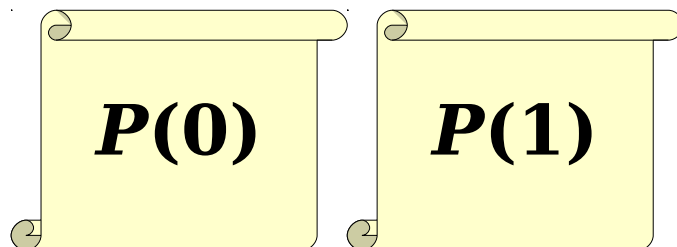
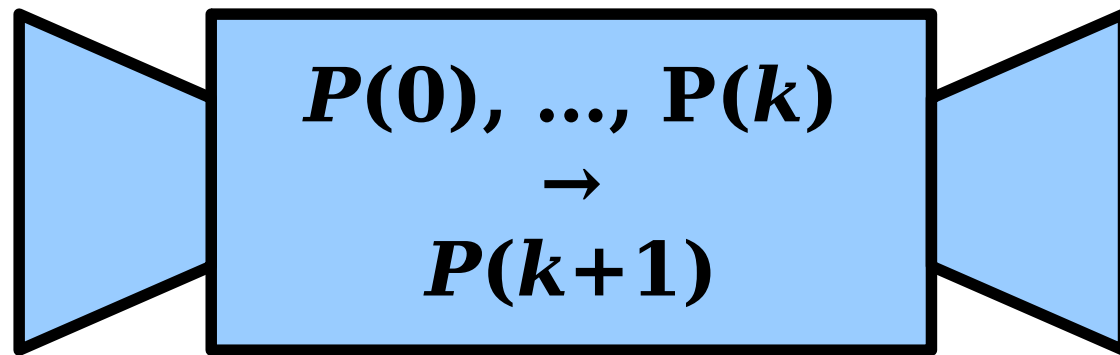
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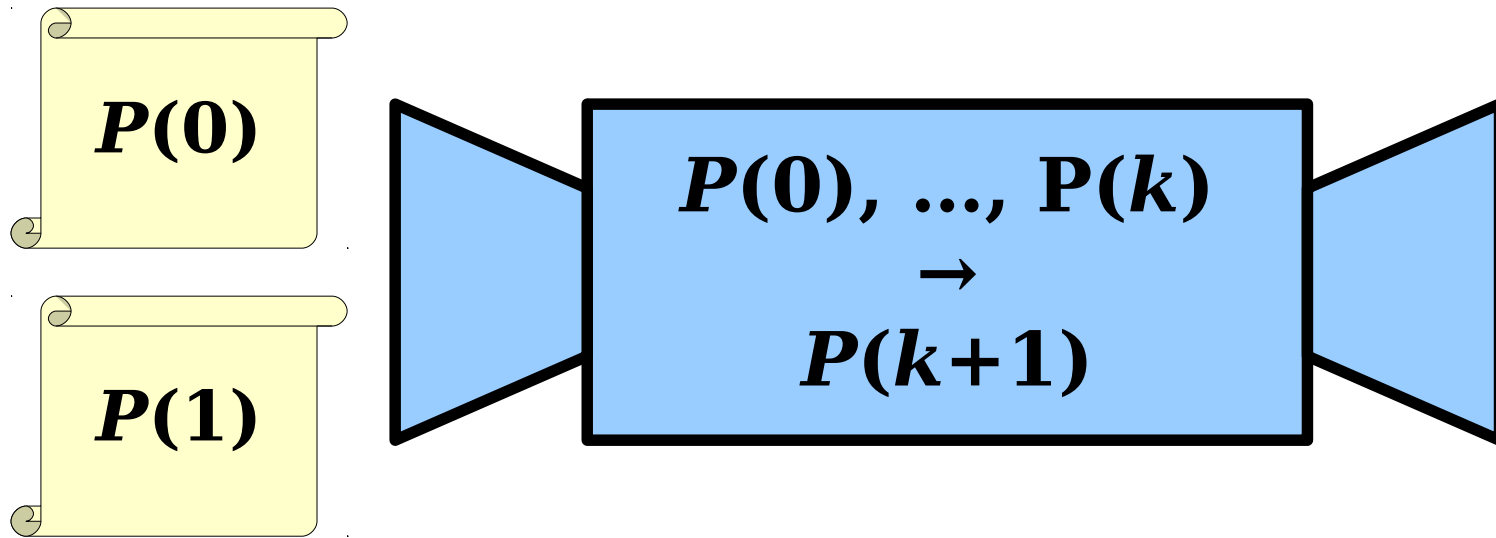
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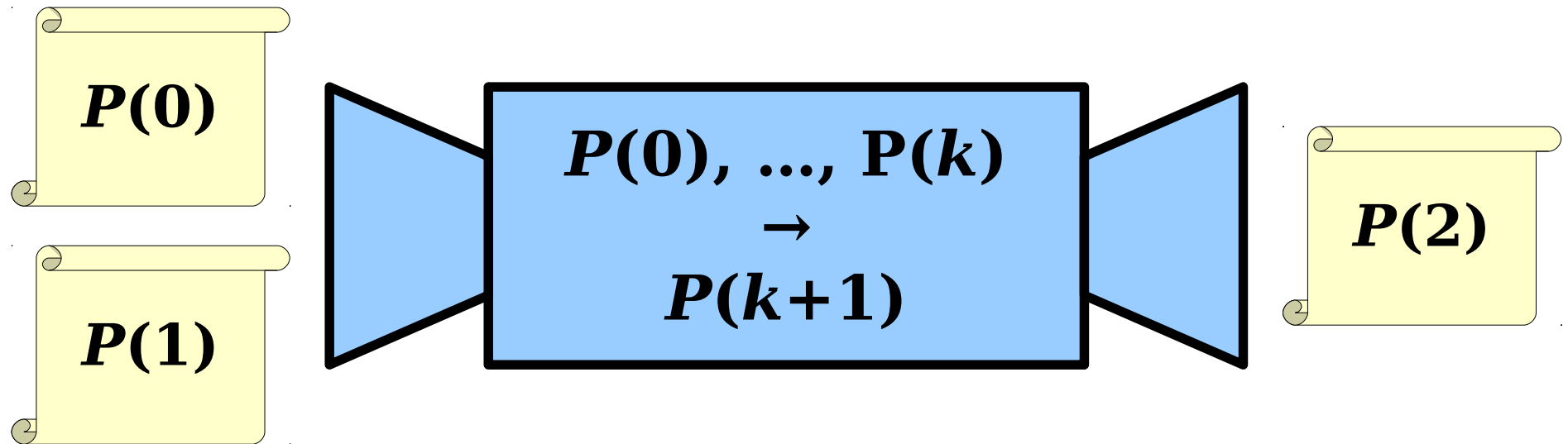
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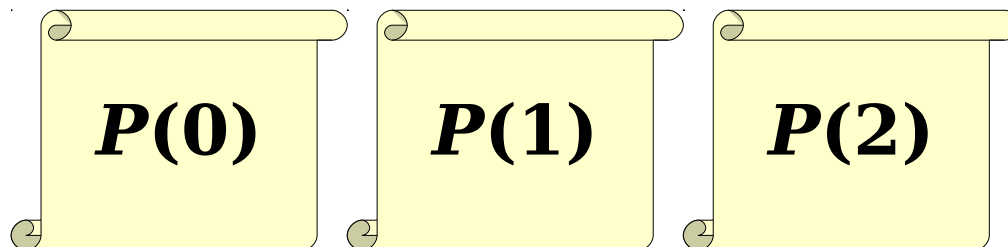
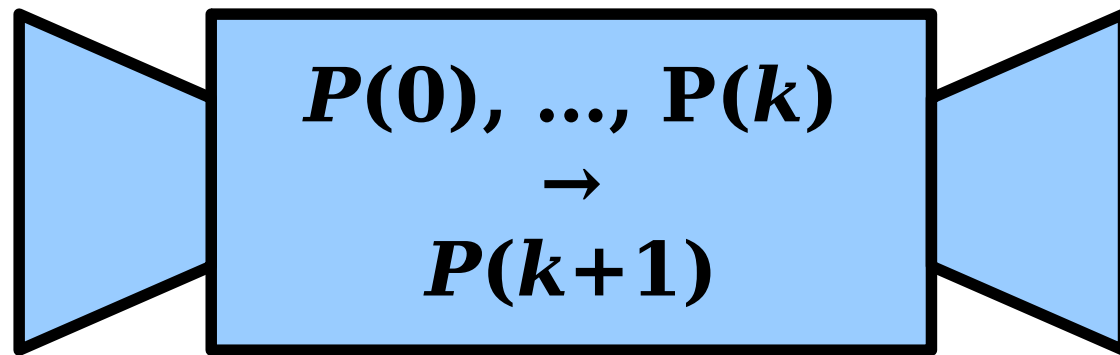
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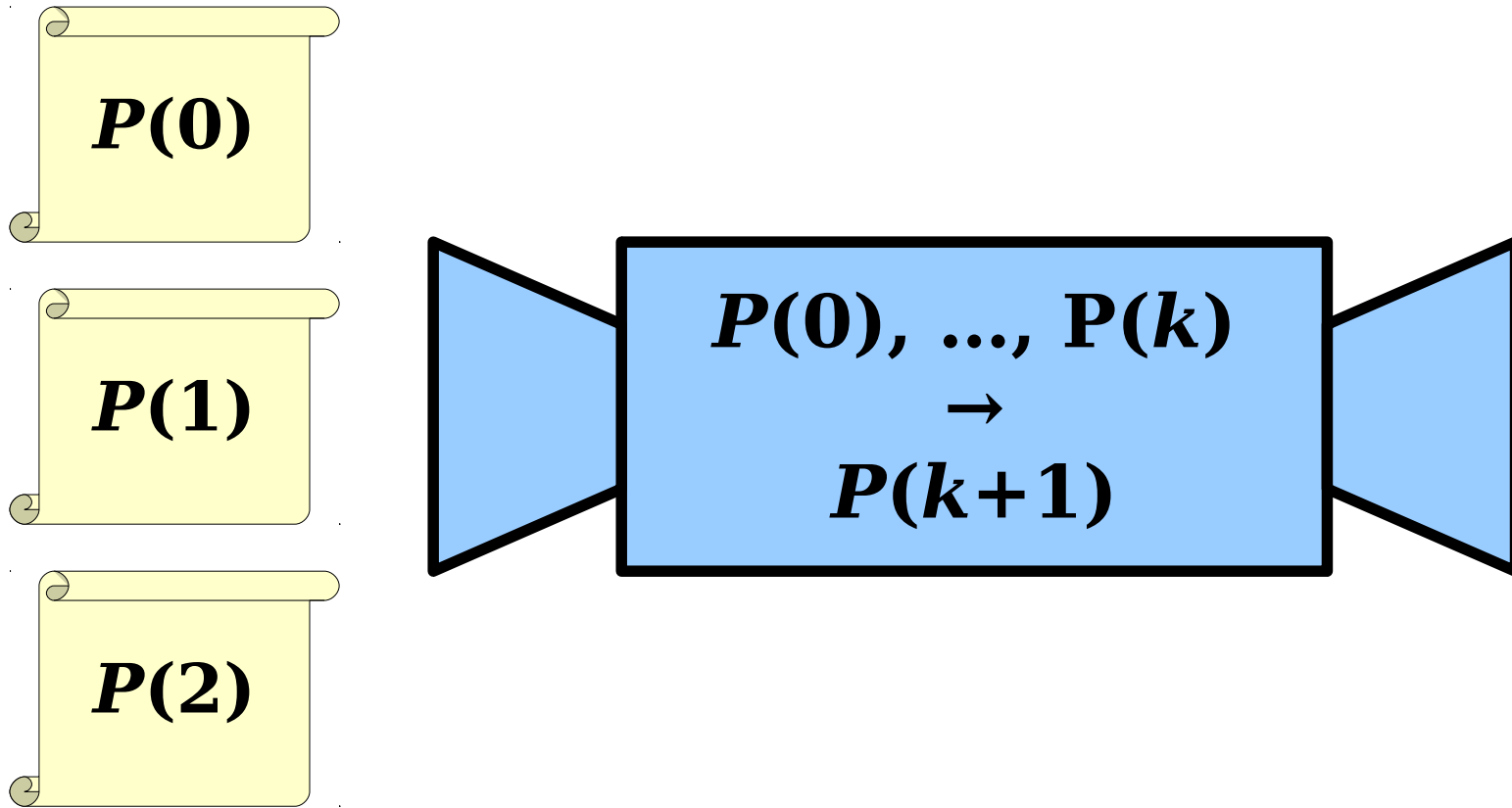
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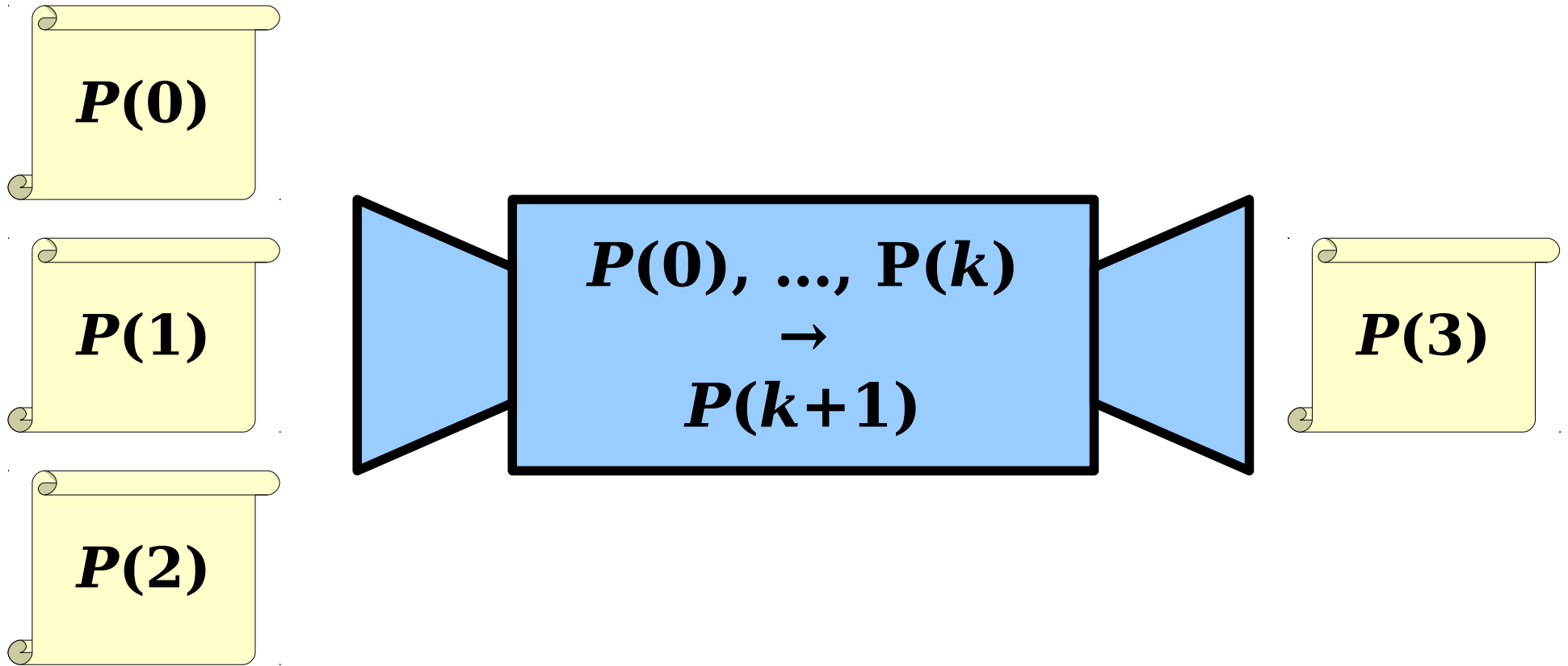


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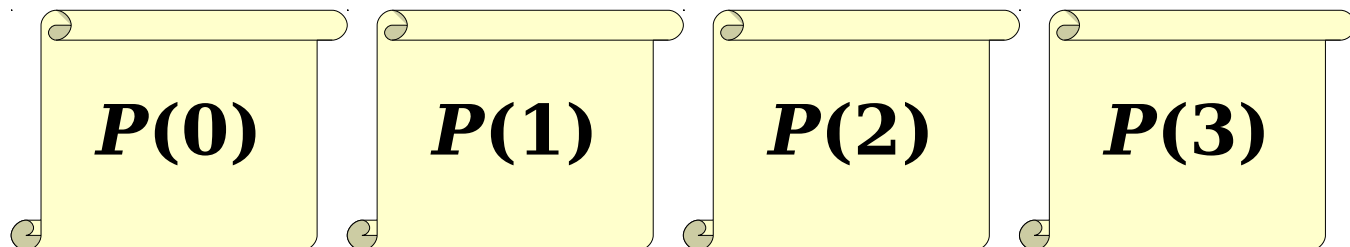
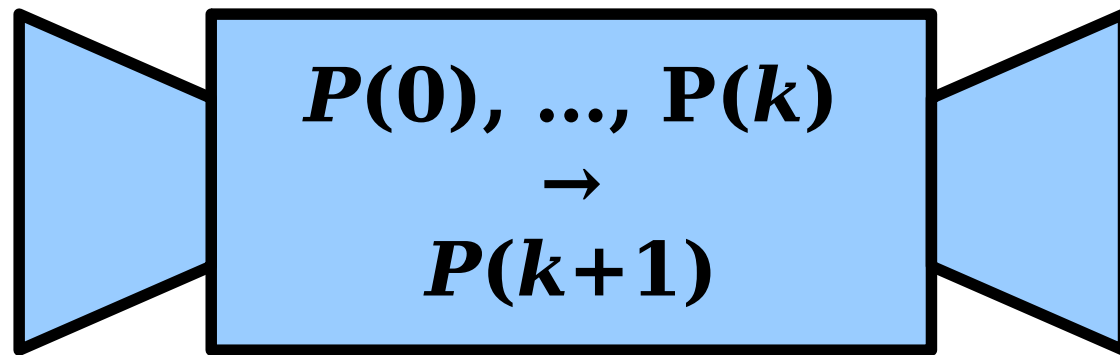




# Intuiting Complete Induction



# Intuiting Complete Induction

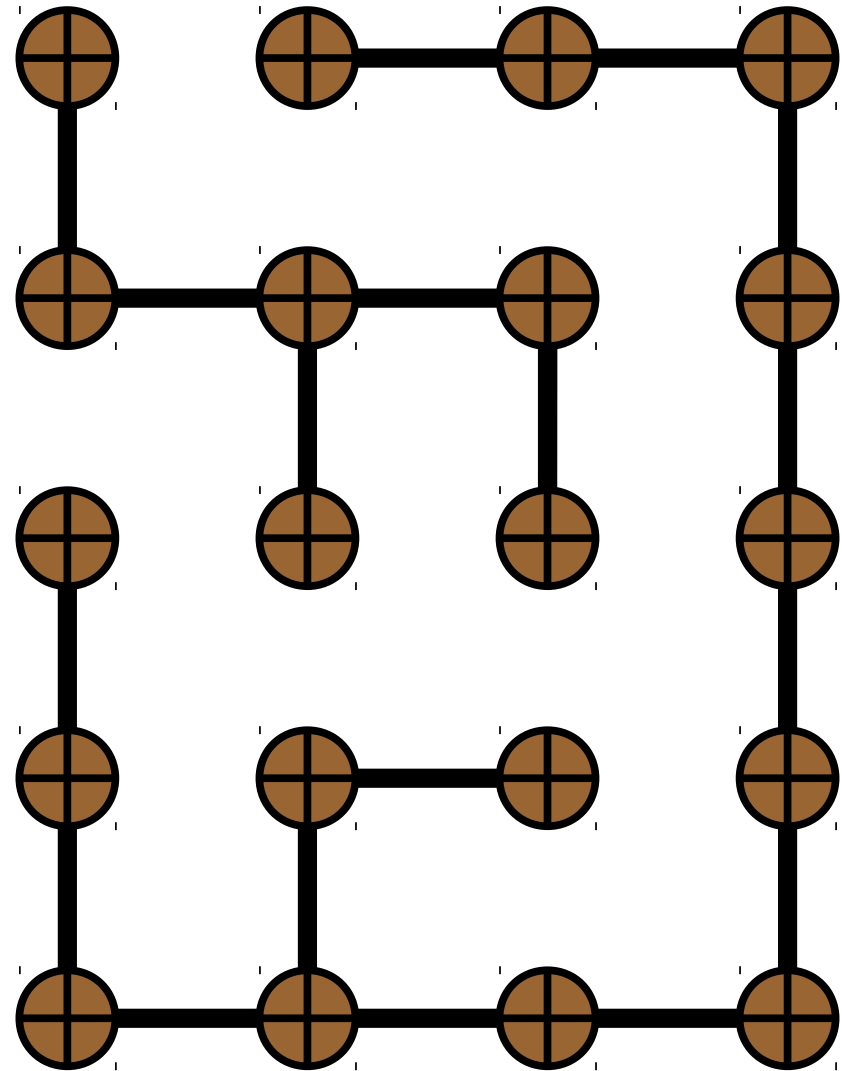


# When Use Complete Induction?

- Normal induction is good for when you are shrinking the problem size by exactly one.
  - Peeling one final term off a sum.
  - Making one weighing on a scale.
  - Considering one more action on a string.
- Complete induction is good when you are shrinking the problem, but you can't be sure by how much.
  - In the previous example, if we delete a random edge, we can't know in advance how big the resulting trees will be.

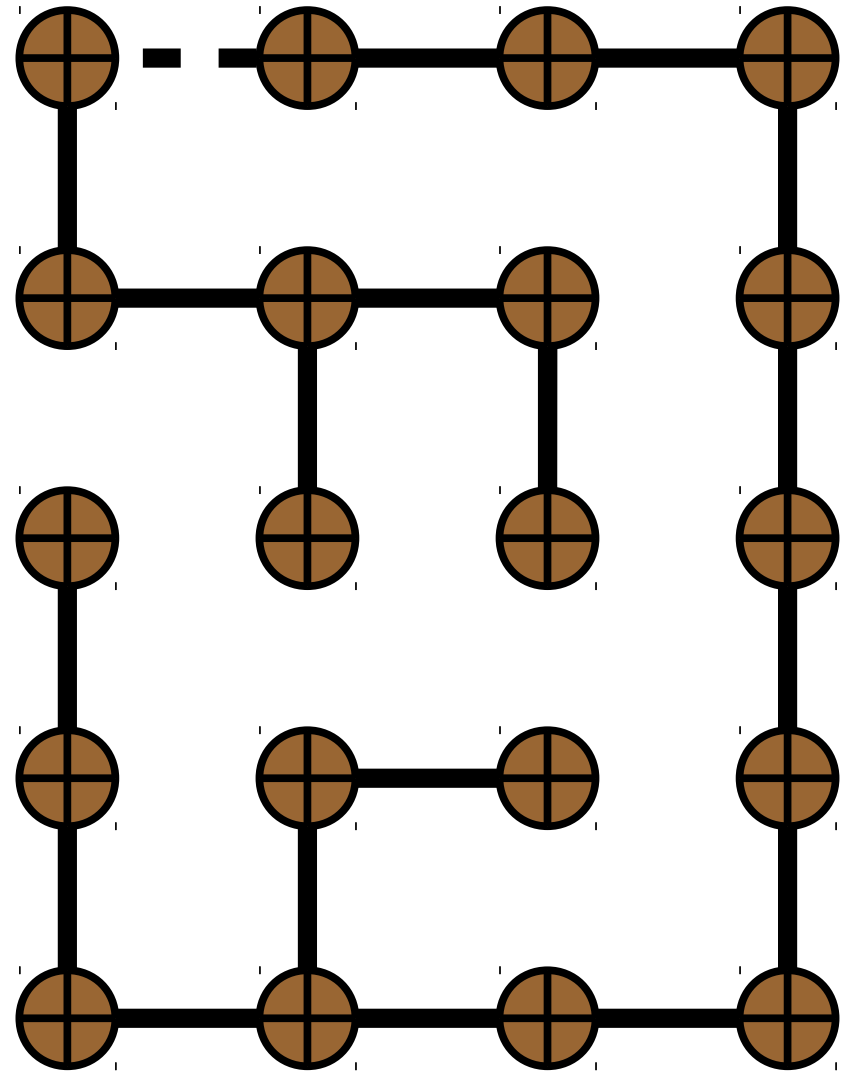
# Rat Mazes

- Suppose you want to make a rat maze consisting of an  $n \times m$  grid of pegs with slats between them.
- **Question:** How many slats do you need to create?



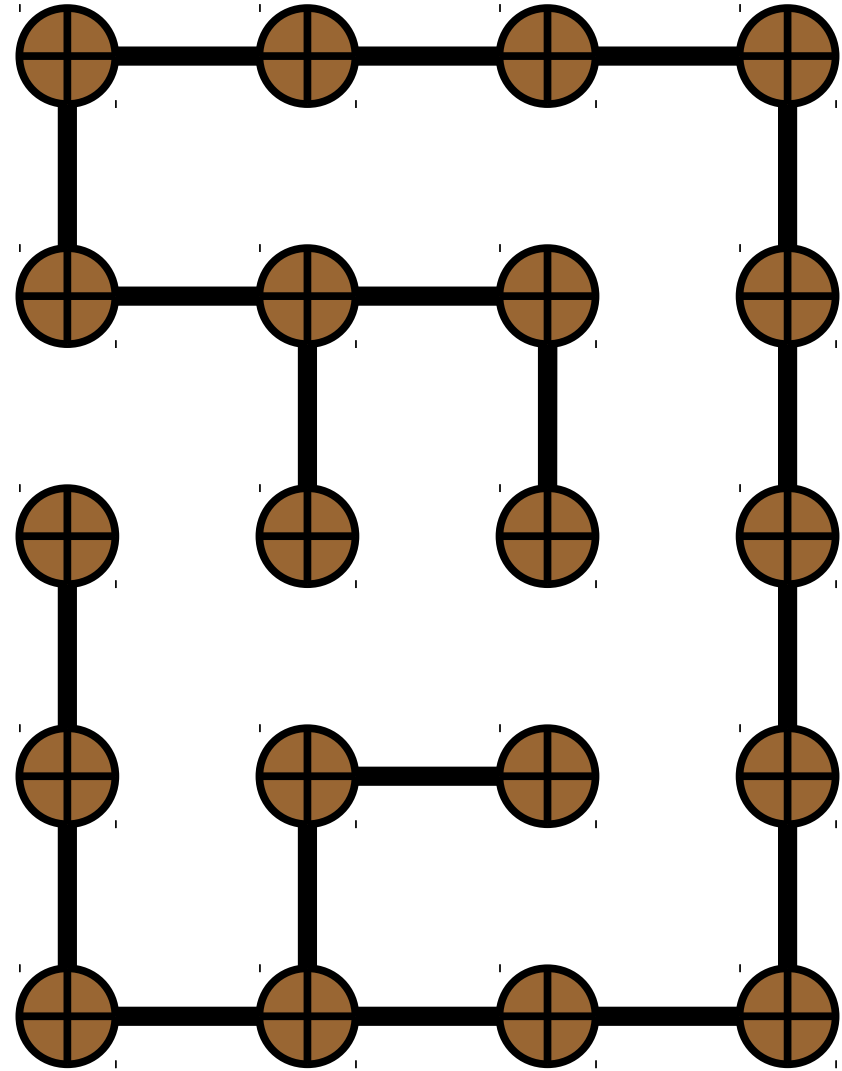
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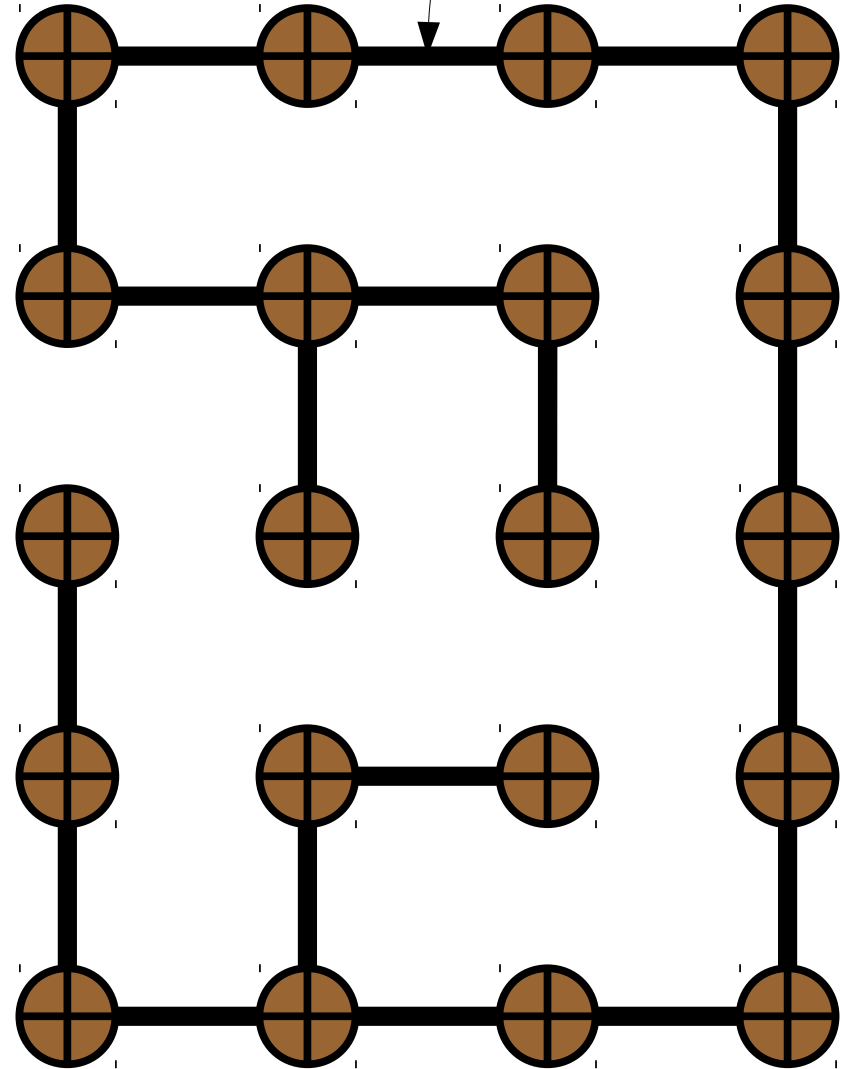
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# Rat Mazes

This is a  
tree!

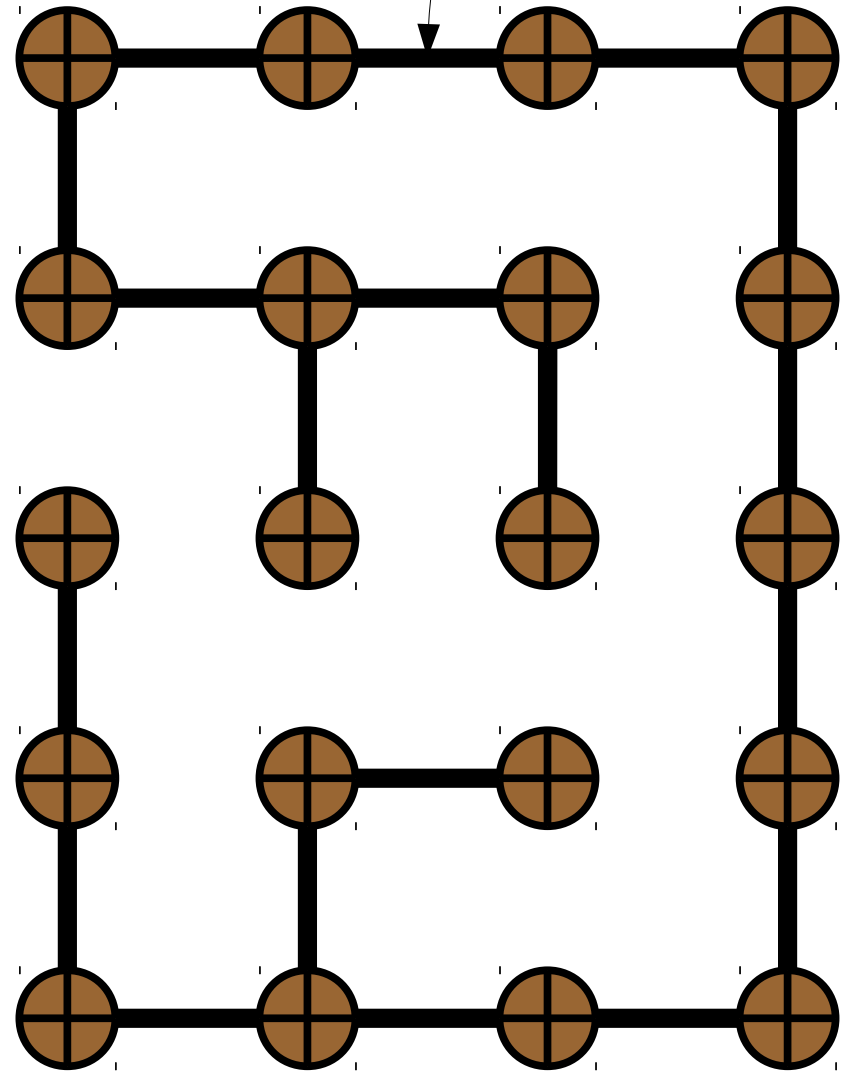
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# Rat Mazes

This is a  
tree!

- Suppose you want to make a rat maze consisting of an  $n \times m$  grid of pegs with slats between them.
- **Question:** How many slats do you need to create?
- **Answer:**  $mn - 2$ .





For more on trees, take CS161 / 261 / 267!

# Just For Funzies: The Limits of Data Compression

# Bitstrings

- A *bitstring* is a finite sequence of 0s and 1s.
- Examples:
  - 11011100
  - 010101010101
  - 0000
  - $\varepsilon$  (the *empty string*)
- There are  $2^n$  bitstrings of length  $n$ .

# Data Compression

- Inside a computer, all data are represented as sequences of 0s and 1s (bitstrings)
- To transfer data over a network (or on a flash drive, if you're still into that), it is useful to reduce the number of 0s and 1s before transferring it.
- Most real-world data can be compressed by exploiting redundancies.
  - Text repeats common patterns (“the”, “and”, etc.)
  - Bitmap images use similar colors throughout the image.
- **Idea:** Replace each bitstring with a *shorter* bitstring that contains all the original information.
  - This is called ***lossless data compression***.

101010101010101010101010101010101010

101010101010101010101010101010



Compress

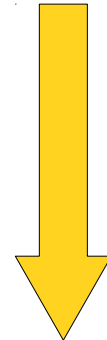
1111010

101010101010101010101010101010



Compress

1111010



Transmit

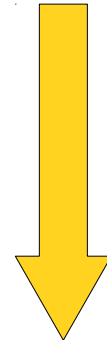
1111010

10101010101010101010101010101010



Compress

1111010



Transmit

1111010



Decompress

10101010101010101010101010101010



# Lossless Data Compression

- In order to losslessly compress data, we need two functions:
  - A **compression function**  $C$ , and
  - A **decompression function**  $D$ .
- We need to have  $D(C(x)) = x$ .
  - Otherwise, we can't uniquely encode or decode some bitstring.
- This means that  $D$  must be a left inverse of  $C$ , so (as you proved in PS3!)  $C$  must be injective.

# A Perfect Compression Function

- Ideally, the compressed version of a bitstring would always be shorter than the original bitstring.
- **Question:** Can we find a lossless compression algorithm that always compresses a string into a shorter string?
- To handle the issue of the empty string (which can't get any shorter), let's assume we only care about strings of length at least 10.

# A Counting Argument

- Let  $\mathbb{B}^n$  be the set of bitstrings of length  $n$ , and  $\mathbb{B}^{<n}$  be the set of bitstrings of length less than  $n$ .
- How many bitstrings of length  $n$  are there?
  - **Answer:**  $2^n$
- How many bitstrings of length *less than*  $n$  are there?
  - **Answer:**  $2^0 + 2^1 + \dots + 2^{n-1} = 2^n - 1$
- By the pigeonhole principle, no function from  $\mathbb{B}^n$  to  $\mathbb{B}^{<n}$  can be injective – at least two elements must collide!
- Since a perfect compression function would have to be an injection from  $\mathbb{B}^n$  to  $\mathbb{B}^{<n}$ , ***there is no perfect compression function!***

# Why this Result is Interesting

- Our result says that no matter how hard we try, it is ***impossible*** to compress every string into a shorter string.
- No matter how clever you are, you cannot write a lossless compression algorithm that always makes strings shorter.
- In practice, only highly redundant data can be compressed.
- The fields of ***information theory*** and ***Kolmogorov complexity*** explore the limits of compression; if you're interested, go explore!

***An Important Milestone***

# Recap: *Discrete Mathematics*

- The past five weeks have focused exclusively on discrete mathematics:

Induction

Functions

Graphs

The Pigeonhole Principle

Relations

Mathematical Logic

Set Theory

Cardinality

- These are building blocks we will use throughout the rest of the quarter.
- These are building blocks you will use throughout the rest of your CS career.

# Next Up: *Computability Theory*

- It's time to switch gears and address the limits of what can be computed.
- We'll explore these questions:
  - How do we model computation itself?
  - What exactly is a computing device?
  - What problems can be solved by computers?
  - What problems *can't* be solved by computers?
- ***Get ready to explore the boundaries of what computers could ever be made to do.***

# Next Time

- ***Formal Language Theory***
  - How are we going to formally model computation?
- ***Finite Automata***
  - A simple but powerful computing device made entirely of math!
- ***DFAs***
  - A fundamental building block in computing.