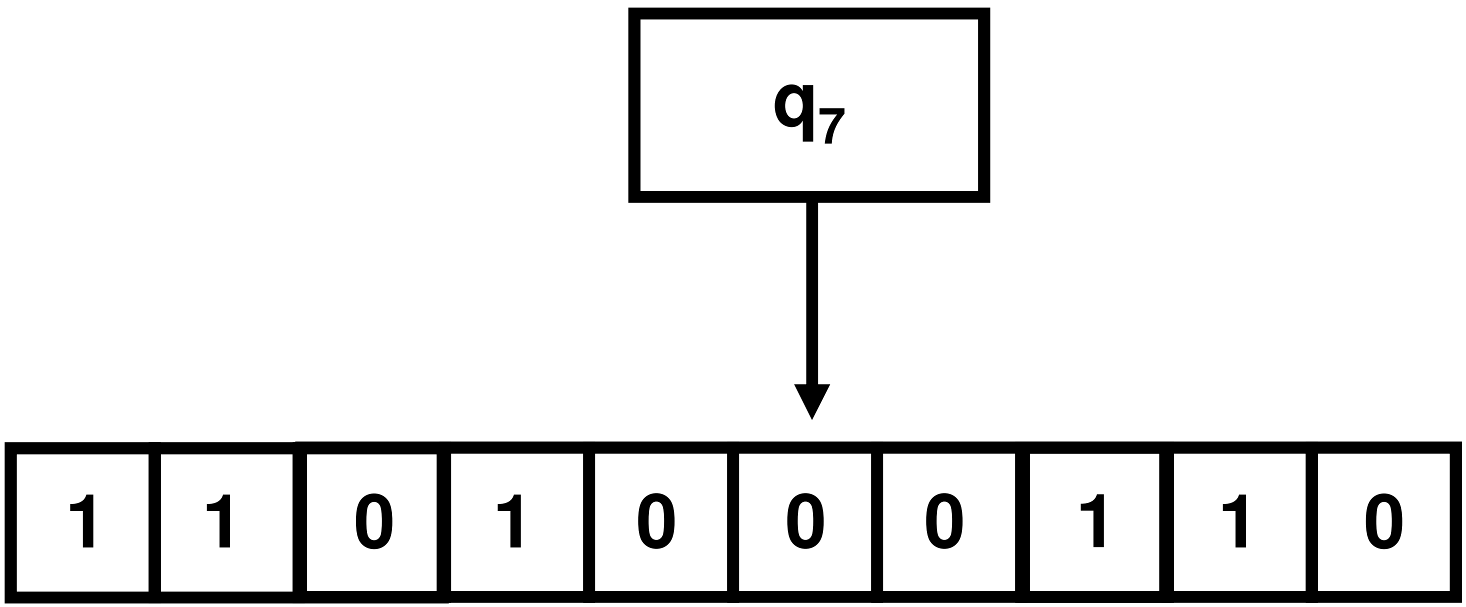
CS 154

The Church-Turing Thesis, Recognizability, Decidability, and Diagonalization

Definition: A Turing Machine is a 7-tuple $T = (Q, \Sigma, \Gamma, \delta, q_0, q_{accept}, q_{reject})$, where:

- **Q is a finite set of states**
- **Σ is the input alphabet, where $\Box \notin \Sigma$**
- **Γ is the tape alphabet, where $\Box \in \Gamma$ and $\Sigma \subseteq \Gamma$**
- **$\delta : Q \times \Gamma \rightarrow Q \times \Gamma \times \{L, R\}$**
- **$q_0 \in Q$ is the start state**
- **$q_{accept} \in Q$ is the accept state**
- **$q_{reject} \in Q$ is the reject state, and $q_{reject} \neq q_{accept}$**

Turing Machine Configurations



corresponds to the *configuration*:

$$\mathbf{11010q_7 00110} \in (Q \cup \Gamma)^*$$

Defining Acceptance and Rejection for TMs

Let C_1 and C_2 be configurations of M

Definition. C_1 *yields* C_2 if M is in configuration C_2 after running M in configuration C_1 for one step

accepting computation history of M on x

Let $w \in \Sigma^*$ and M be a Turing machine

M *accepts* w if there are configs $C_0, C_1, \ldots, C_k$, s.t.

- $C_0 = q_0w$ [the initial configuration]
- C_i yields C_{i+1} for $i = 0, \ldots, k-1$, and
- C_k contains the accept state q_{accept}

A TM *M recognizes* a language L if *M* accepts exactly those strings in L

A language L is *recognizable (a.k.a. recursively enumerable)* if some TM recognizes L

A TM *M decides* a language L if *M* accepts all strings in L and rejects all strings not in L

A language L is *decidable (a.k.a. recursive)* if some TM decides L

A Turing machine for deciding $\{ 0^{2^n} \mid n \geq 0 \}$

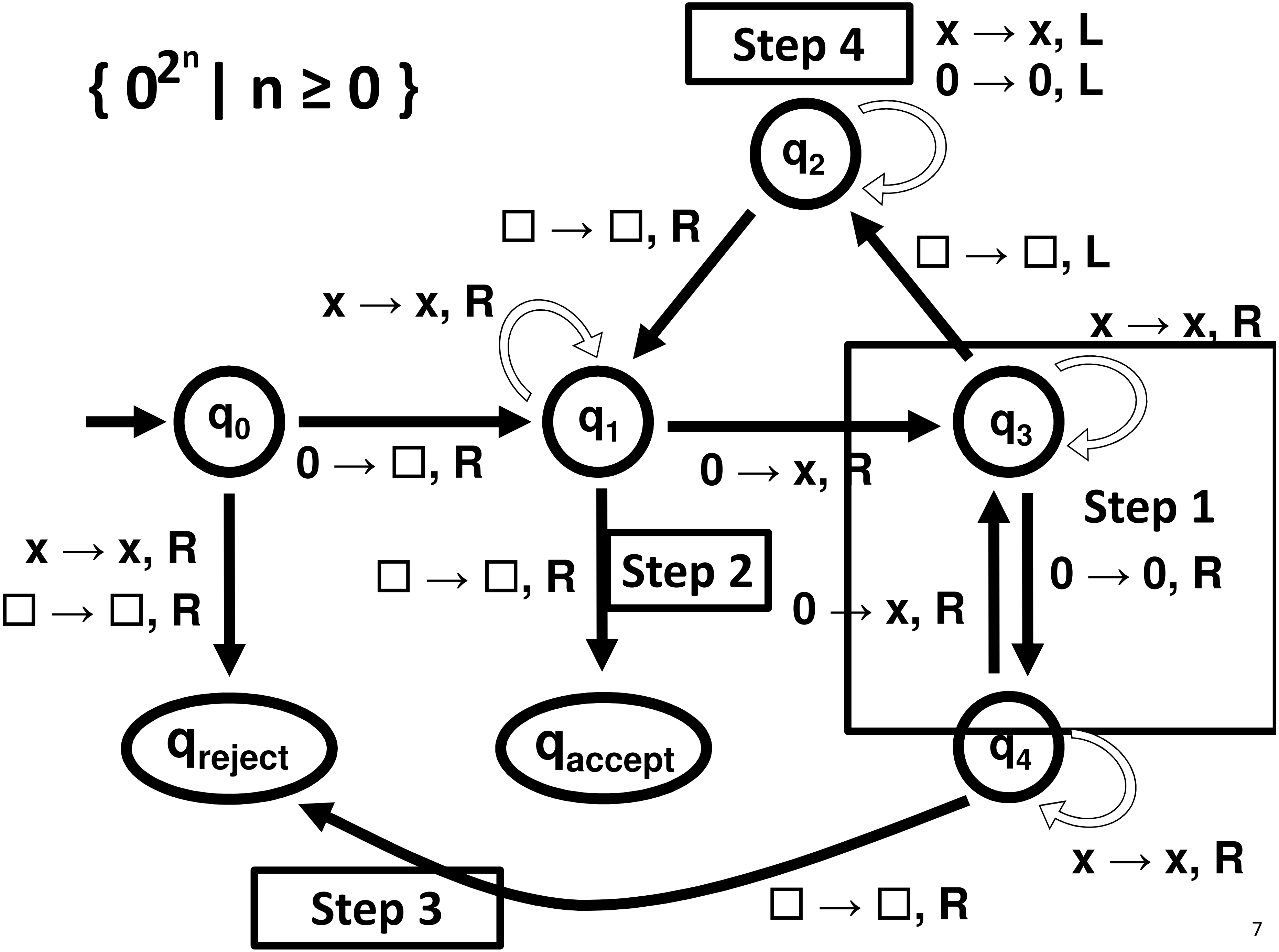
Turing Machine PSEUDOCODE:

1. Sweep from left to right, cross out every other **0**
2. If in step 1, the tape had only one **0**, *accept*
3. If in step 1, the tape had an **odd number** of **0**'s, *reject*
4. Move the head back to the first input symbol.
5. Go to step 1.

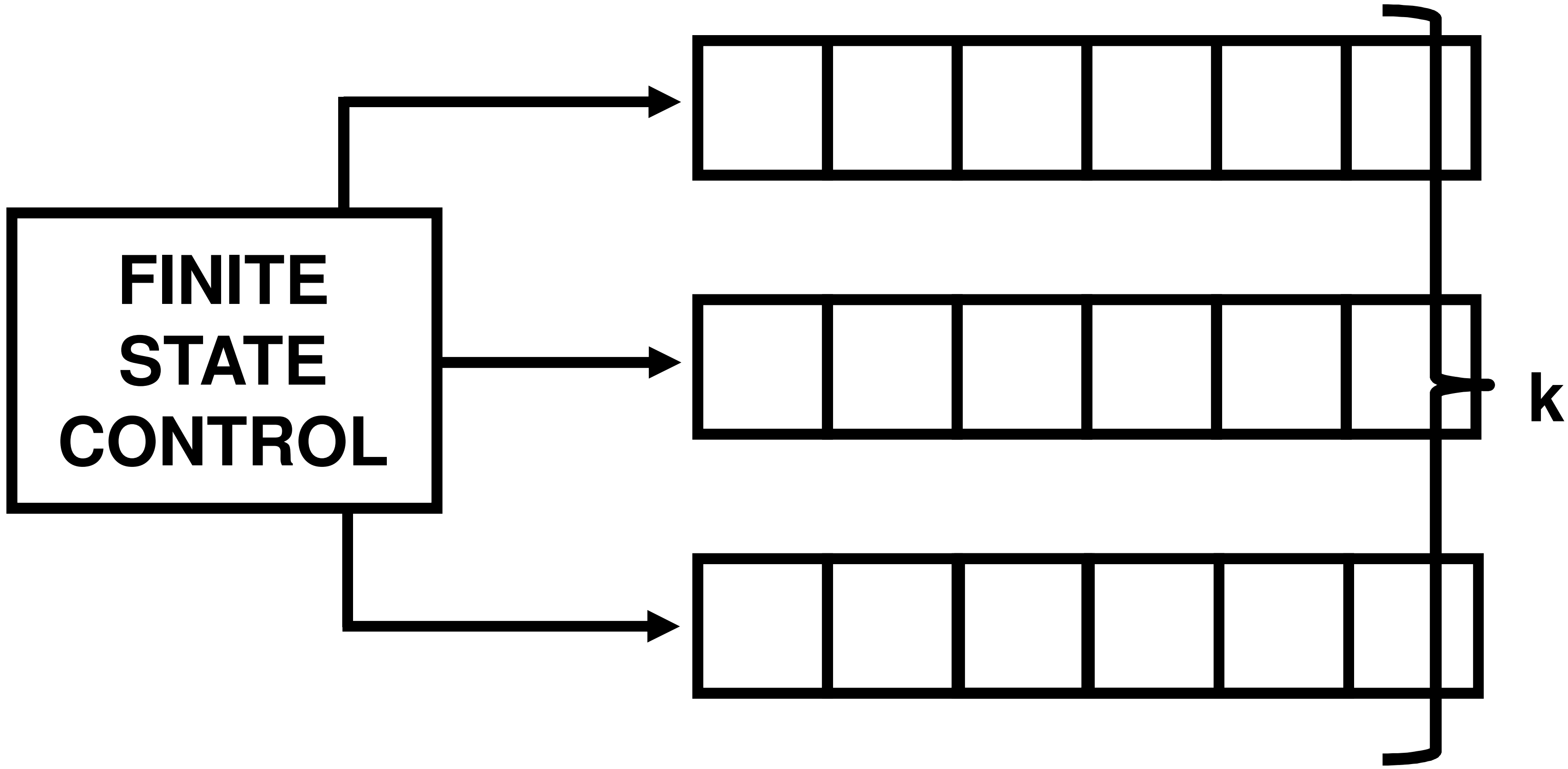
Why does this work?

Idea: Every time we return to stage 1,
the number of 0's on the tape has been halved.

$\{ 0^{2^n} \mid n \geq 0 \}$

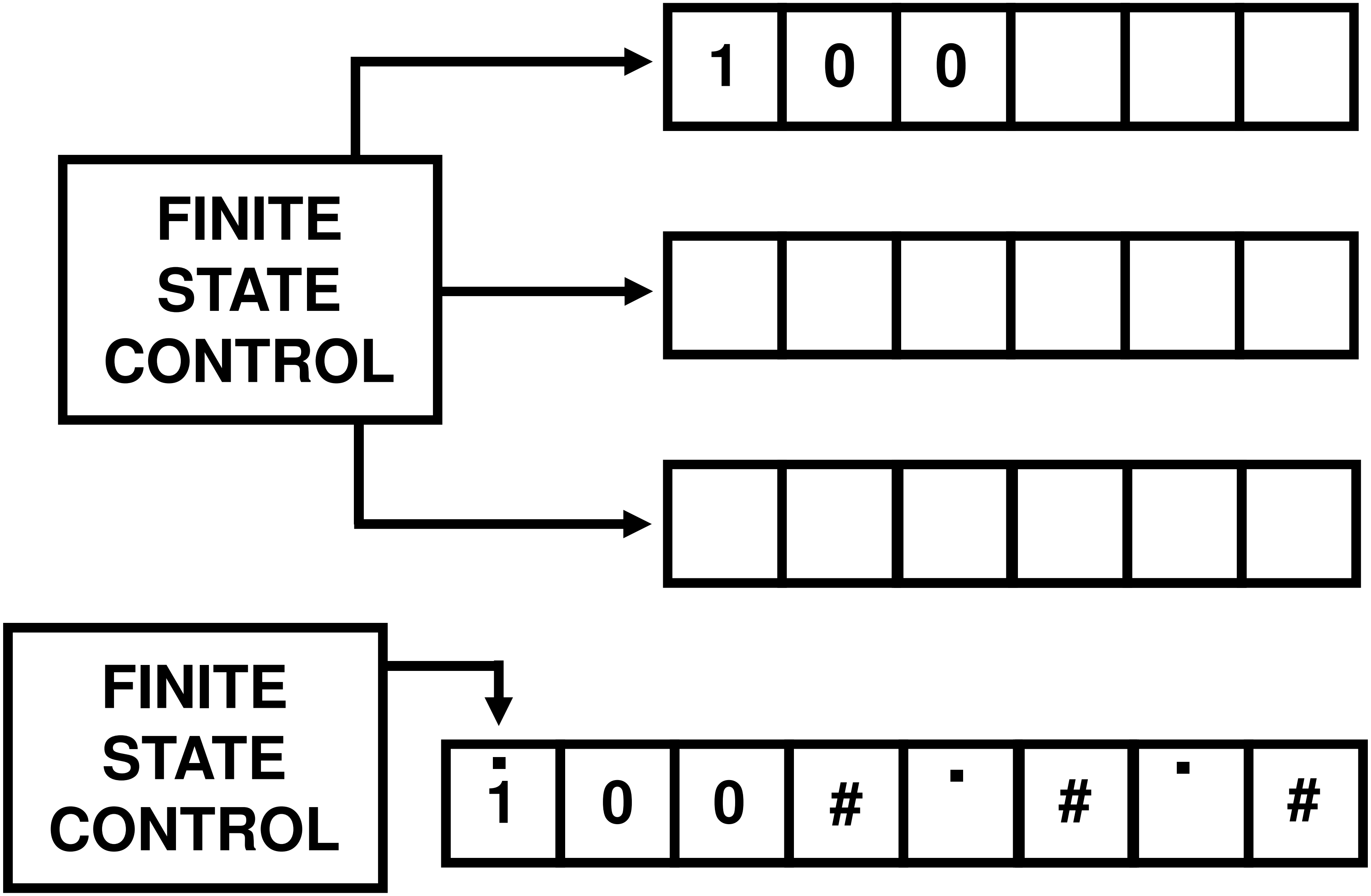


Multitape Turing Machines



$$\delta : Q \times \Gamma^k \rightarrow Q \times \Gamma^k \times \{L,R\}^k$$

Theorem: Every Multitape Turing Machine can be transformed into a single tape Turing Machine



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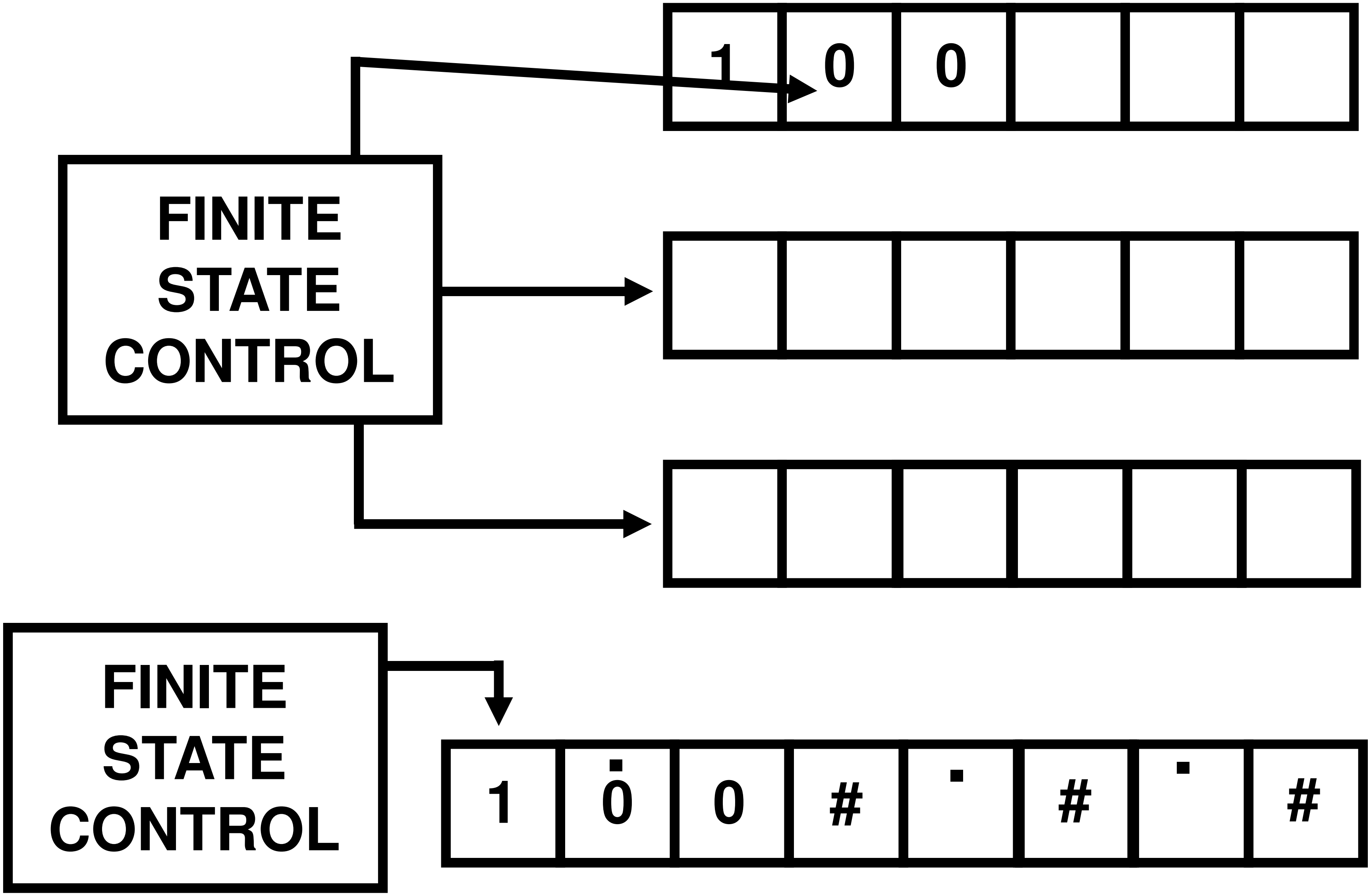

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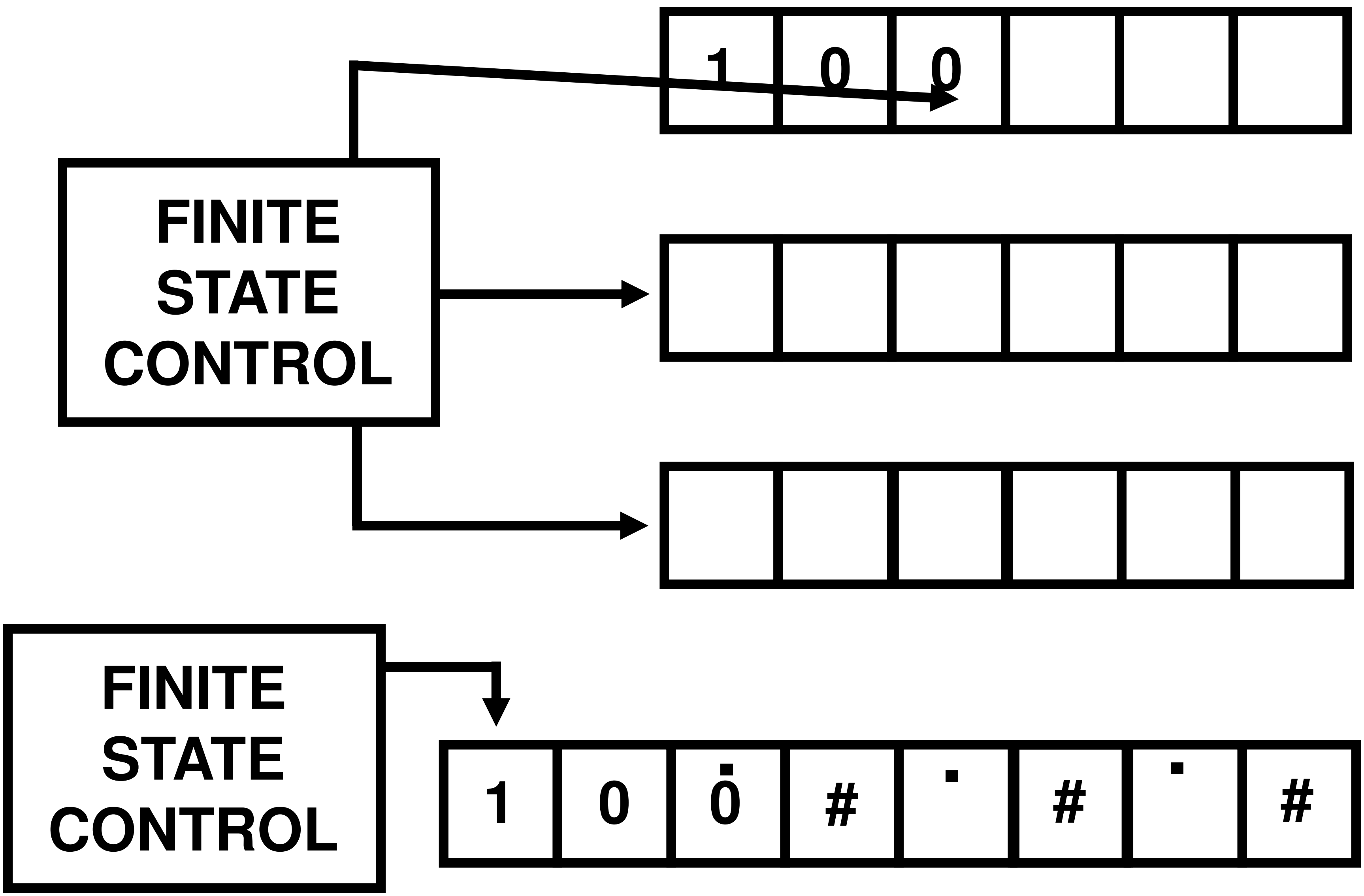

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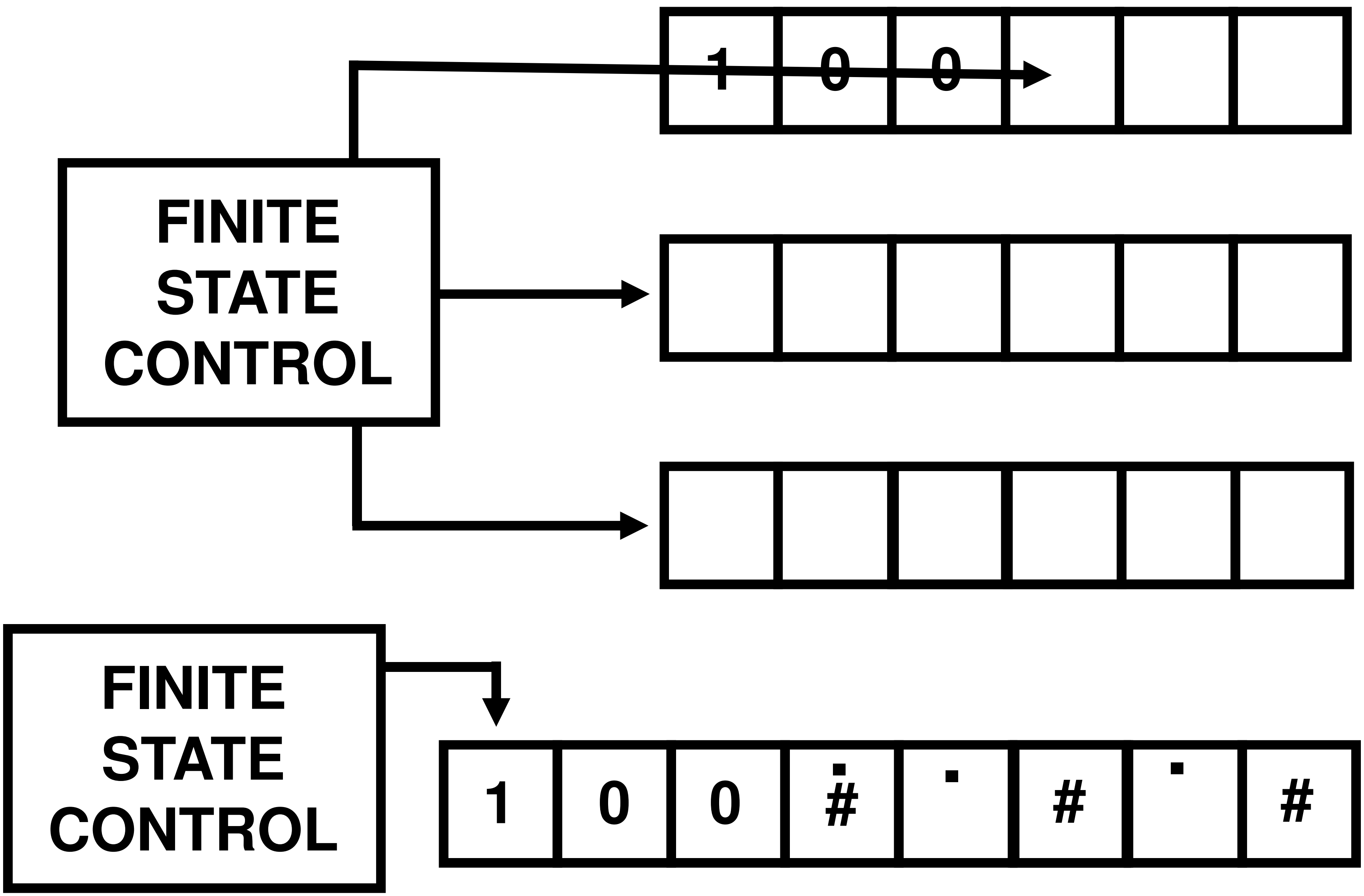

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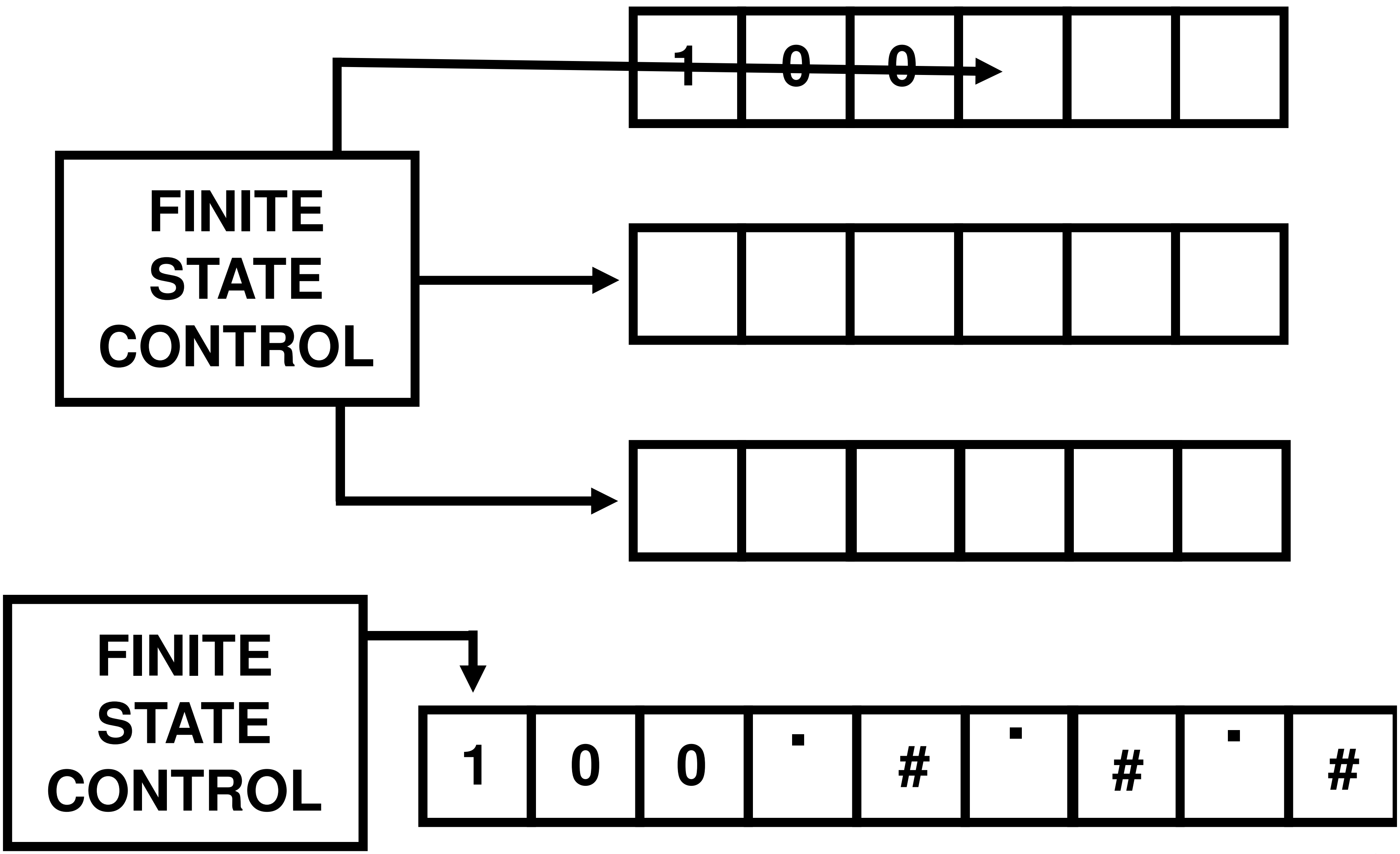
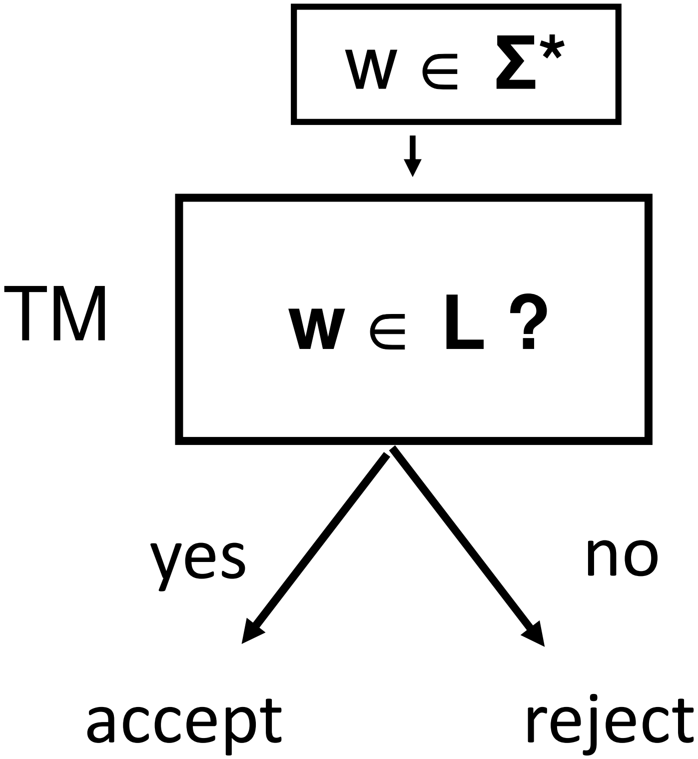
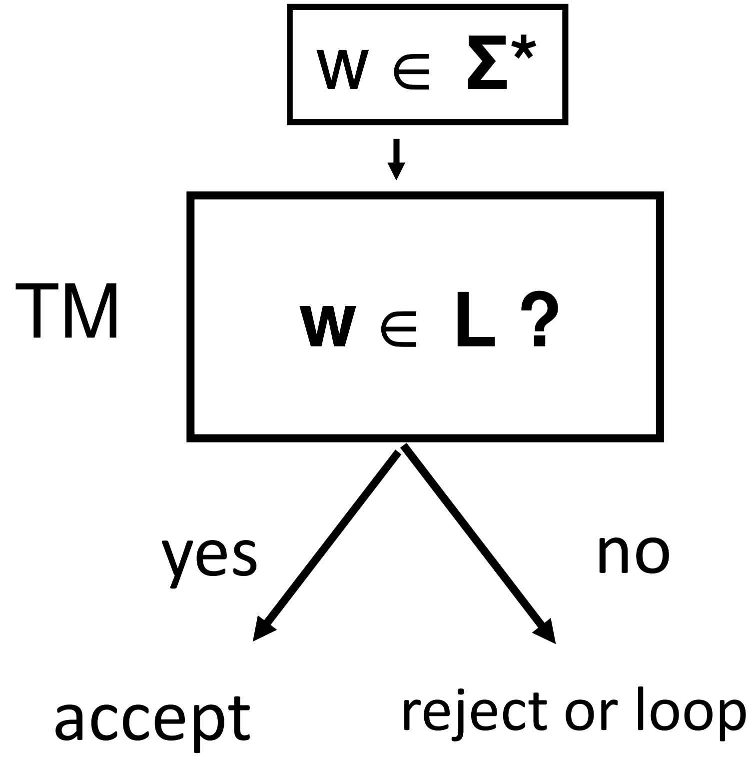




L is decidable
(recursive)



L is recognizable
(recursively enumerable)

Theorem: L is decidable iff both L and ¬L are recognizable

Recall: Given $L \subseteq \Sigma^*$, define $\neg L := \Sigma^* \setminus L$

Theorem: L is decidable

iff both L and $\neg$L are recognizable

Given: a TM M_1 that recognizes L and

a TM M_2 that recognizes $\neg$L,

we want to build a new machine M that *decides* L

How? Any ideas?

Hint: M_1 always accepts x, when x is in L

M_2 always accepts x, when x isn't in L

Recall: Given $L \subseteq \Sigma^*$, define $\neg L := \Sigma^* \setminus L$

Theorem: L is decidable
iff both L and $\neg L$ are recognizable

Given: a TM M_1 that recognizes L and
a TM M_2 that recognizes $\neg L$,
we want to build a new machine M that *decides* L

M(x): Run M_1 (x) and M_2 (x) on separate tapes.
Alternate between simulating one step
of M_1, and one step of M_2.
If M_1 ever accepts, then accept
If M_2 ever accepts, then reject

Nondeterministic Turing Machines

Have multiple transitions for a state, symbol pair

Theorem: Every nondeterministic Turing machine N can be transformed into a Turing Machine M that accepts precisely the same strings as N.

Proof Idea (more details in Sipser)
Pick a natural ordering on all strings in $(Q \cup \Gamma \cup \#)^*$

M(w): For all strings $D \in (Q \cup \Gamma \cup \#)^*$ in the ordering,

Check if D = $C_0\# \cdots \#C_k$ where $C_0, \ldots, C_k$ is *some accepting computation history* for N on w.
If so, *accept.*

Fact: We can encode Turing Machines as *bit strings*

n states

start state

reject state

$$0^n 1 0^m 1 0^k 1 0^s 1 0^t 1 0^r 1 0^u 1 \ldots$$

m tape symbols (first k are input symbols)

accept state

blank symbol

$$(\,(p, i), (q, j, L)\,) = 0^p 1 0^i 1 0^q 1 0^j 1 0$$

$$(\,(p, i), (q, j, R)\,) = 0^p 1 0^i 1 0^q 1 0^j 1 00$$

Similarly, we can encode DFAs and NFAs as *bit strings*, and $w \in \Sigma^*$ as *bit strings*

For $x \in \Sigma^*$ define $b_\Sigma(x)$ to be its binary encoding

For $x, y \in \Sigma^*$, define the *pair of x and y* to be

$$(x, y) := 0^{|b_\Sigma(x)|} 1\, b_\Sigma(x)\, b_\Sigma(y)$$

Then we define the following languages over {0,1}

A_{DFA} = { (B, w) | B encodes a DFA over some Σ, and B accepts $w \in \Sigma^*$ }

A_{NFA} = { (B, w) | B encodes an NFA, B accepts w }

A_{TM} = { (M, w) | M encodes a TM, M accepts w }

A_{TM} = { (M, w) | M encodes a TM over some Σ,
w encodes a string over Σ
and M accepts w}

Technical Note:

We'll use an decoding of pairs, TMs, and strings so that *every* binary string decodes to *some* pair (M, w)

If $z \in \{0,1\}^*$ doesn't decode to (M, w) in the usual way, then we *define* that z decodes to the pair (D, ε)
where D is a "dummy" TM that accepts nothing.

$\neg A_{TM}$ = { z | z decodes to (M, w) and
M does not accept w }

Universal Turing Machines

Theorem: There is a Turing machine U which takes as input:

- **the code of an arbitrary TM M**
- **and an input string w**

such that U accepts (M, w) $\Leftrightarrow$ M accepts w.

This is a *fundamental* property of TMs:
There is a Turing Machine that can run arbitrary Turing Machine code!

Note that DFAs/NFAs do *not* have this property.
That is, A_{DFA} and A_{NFA} are not regular.

A_{DFA} = { (D, w) | D is a DFA that accepts string w }

Theorem: A_{DFA} is decidable

Proof: A DFA is a special case of a TM.

Run the universal U on (D, w) and output its answer.

A_{NFA} = { (N, w) | N is an NFA that accepts string w }

Theorem: A_{NFA} is decidable. (Why?)

A_{TM} = { (M, w) | M is a TM that accepts string w }

Theorem: A_{TM} is recognizable

The Church-Turing Thesis

Everyone's Intuitive Notion of Algorithms = Turing Machines

This is not a theorem –
it is a falsifiable scientific hypothesis.

And it has been thoroughly tested!

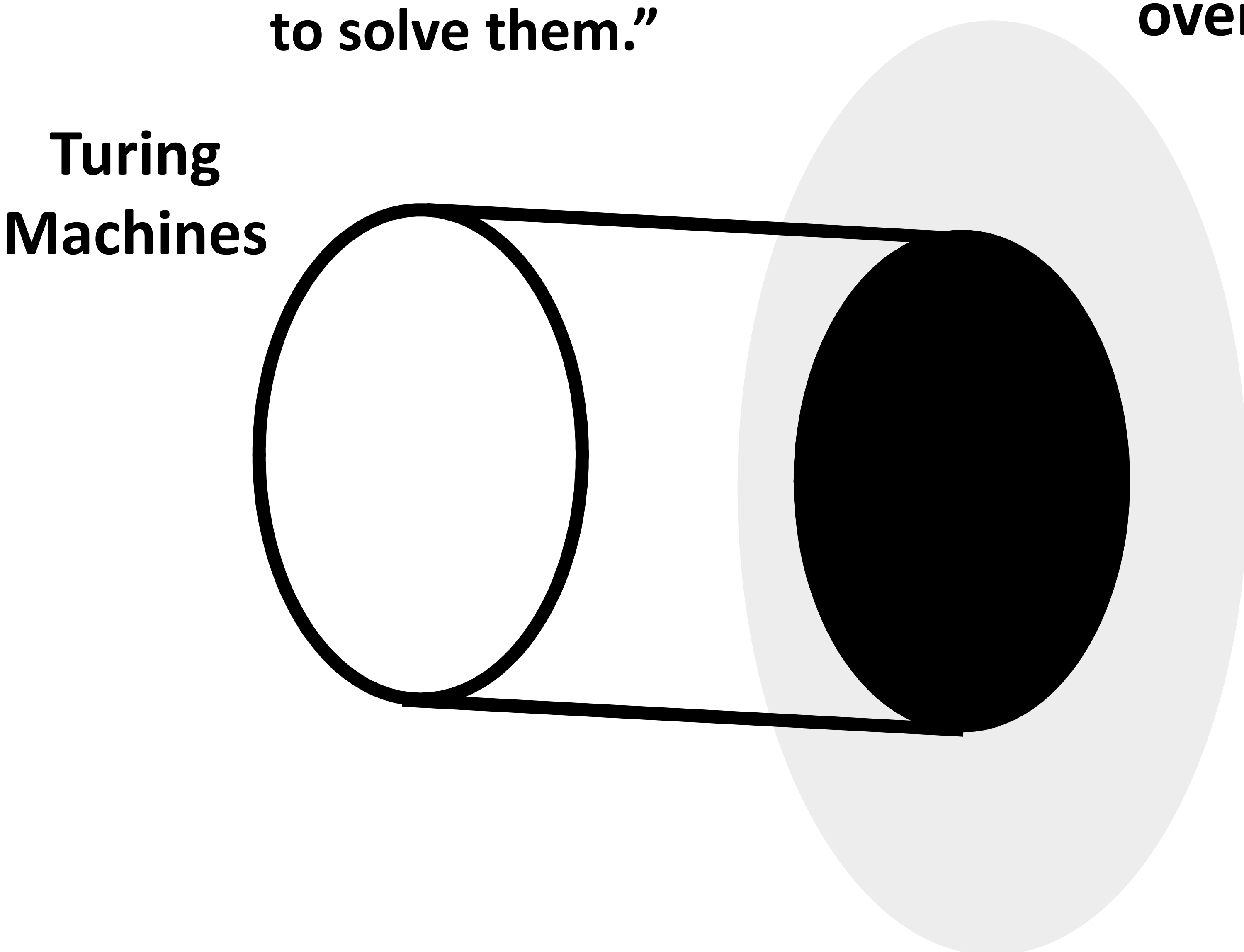
Thm: There are *unrecognizable* languages

Assuming the Church-Turing Thesis, this means there are problems that *NO* computing device can solve!

We will prove that there is no onto function from the set of all Turing Machines to the set of all languages over {0,1}.
(But the proof will work for any *finite* Σ)

That is, every mapping from Turing machines to languages *fails to cover* all possible languages

"There are more problems to solve than there are programs to solve them."



$f : A \rightarrow B$ is *not* onto $\Leftrightarrow$ $(\exists b \in B)(\forall a \in A)[f(a) \neq b]$

Let L be any set and 2^L be the power set of L

$\boxed{\text{Theorem: There is } \textit{no} \text{ onto function from L to } 2^L}$

Proof: Assume, for a contradiction,

there is an onto function $f : L \rightarrow 2^L$

Define $S = \{ x \in L \mid x \notin f(x) \} \in 2^L$

If f is onto, then there is a $y \in L$ with $f(y) = S$

Suppose $y \in S$. By definition of S, $y \notin f(y) = S$.

Suppose $y \notin S$. By definition of S, $y \in f(y) = S$.

Contradiction!

$f : A \rightarrow B$ **is *not* onto** $\Leftrightarrow$ $(\exists b \in B)(\forall a \in A)[f(a) \neq b]$

Let L be any set and 2^L be the power set of L

Theorem: There is *no* onto function from L to 2^L

Proof: **Let $f : L \rightarrow 2^L$ be an arbitrary function**

Define $S = \{ x \in L \mid x \notin f(x) \} \in 2^L$

For all $x \in L$,

If $x \in S$ then $x \notin f(x)$ [by definition of S]

If $x \notin S$ then $x \in f(x)$

In either case, we have $f(x) \neq S$. (Why?)

Therefore f is not onto!

What does this mean?

No function from L to 2^L
can “cover” all the elements in 2^L

No matter what the set L is,
the power set 2^L *always* has
strictly larger cardinality than L

Thm: There are *unrecognizable* languages

Proof: Suppose all languages are recognizable.
Then for all L, there's a Turing machine M for recognizing L.
Hence there is an onto R: {Turing Machines} $\to$ {Languages}

{Turing Machines}

$\cap\!|$

{0,1}*

||

Set M

{Languages over {0,1}}

$\updownarrow$

{*Sets* of strings of 0s and 1s}

||

Set of all subsets of M: 2^M

But there is *no* onto function from
{Turing Machines} ⊆ M to 2^M. Contradiction!