# Indirect Proofs

# Outline for Today

- ***Disproofs***
  - How do you show something is *not* true?
- ***What is an Implication?***
  - Understanding a key type of mathematical statement.
- ***Proof by Contrapositive***
  - What's a contrapositive?
  - And some applications!
- ***Proof by Contradiction***
  - The basic method.
  - And some applications!

# Disproving Statements

# Proofs and Disproofs

- A ***proof*** is an argument establishing why a statement is true.
- A ***disproof*** is an argument establishing why a statement is *false*.
- Although proofs generally are more famous than disproofs, many important results in mathematics have been disproofs.
  - We'll see some later this quarter!

# Negations

- A ***proposition*** is a statement that is either true or false.
- Some examples:
  - If $n$ is an even integer, then $n^2$ is an even integer.
  - $\varnothing = \mathbb{R}$.
  - I want to swing from the chandelier.
- The ***negation*** of a proposition $X$ is a proposition that is true whenever $X$ is false and is false whenever $X$ is true.
- For example, consider the statement "it is snowing outside."
  - Its negation is "it is not snowing outside."
  - Its negation is not "it is sunny outside."
  - Its negation is not "we're in the Bay Area."

# Writing a Disproof

- The easiest way to disprove a statement is to write a proof of the *negation* of that statement.
- A typical disproof is structured as follows:
  - Start by stating that you're going to disprove some statement $X$.
  - Write out the negation of statement $X$.
  - Prove that the negation is true.

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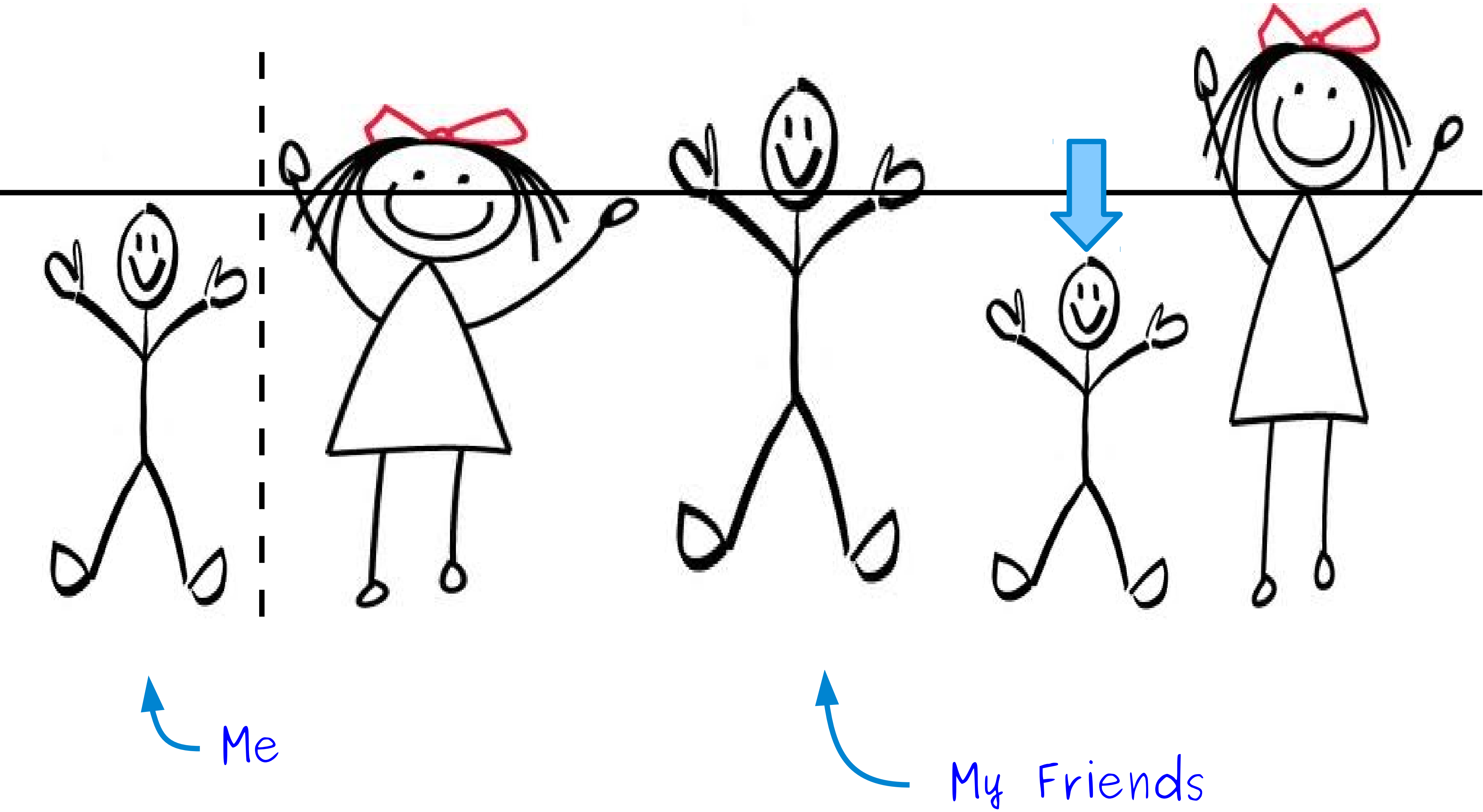
A typical disproof is structured as follows:

- Start by stating that you're going to disprove some statement $X$.
- Write out the negation of statement $X$.
- Prove that the negation is true.

How do you do this first step?

# "All My Friends Are Taller Than Me"



The negation of the *universal* statement

**Every *P* is a *Q***

is the *existential* statement

**There is a *P* that is not a *Q*.**

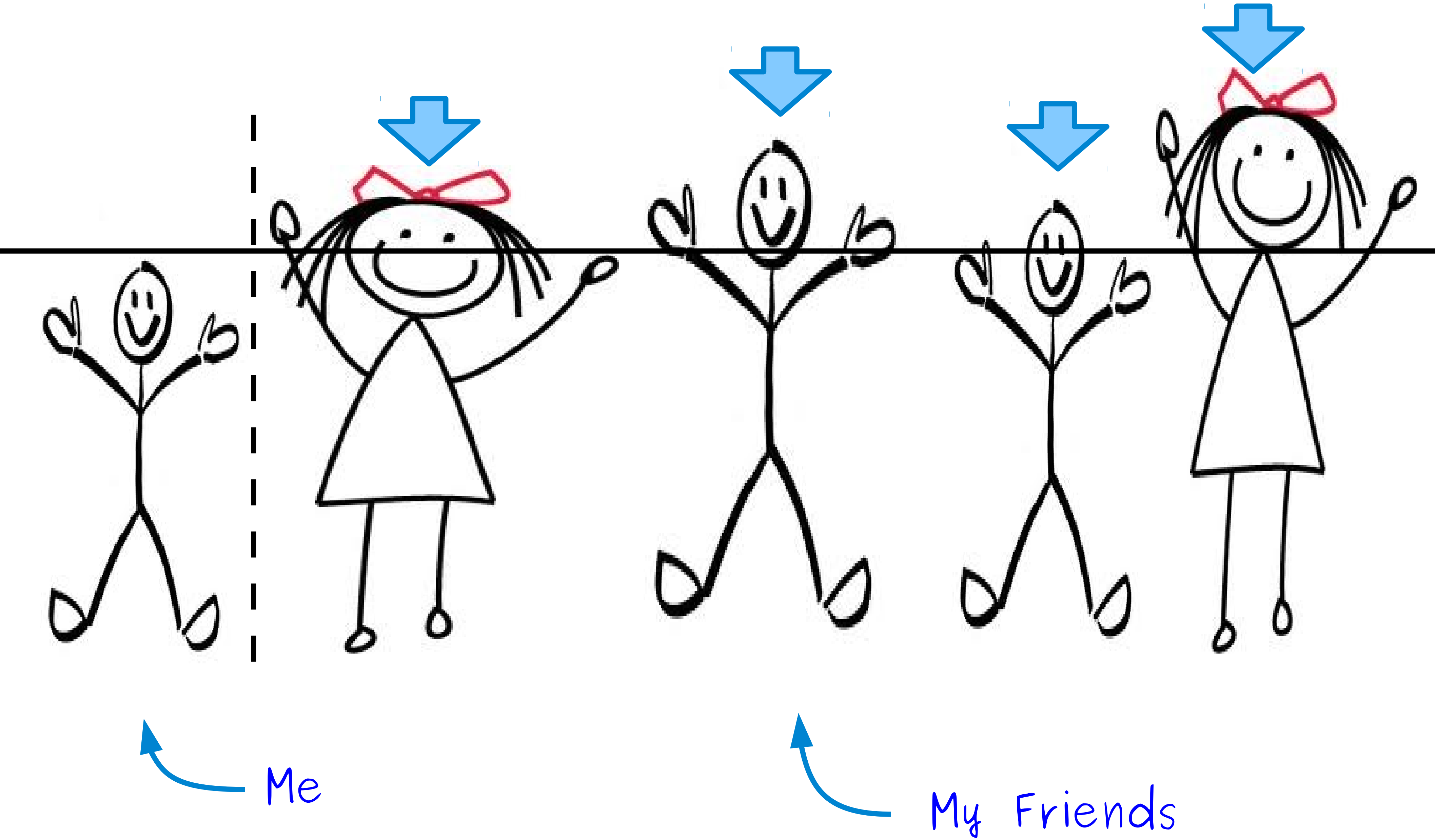
The negation of the *universal* statement

**For all *x*, *P*(*x*) is true.**

is the *existential* statement

**There exists an *x* where *P*(*x*) is false**.

# “Some Friend Is Shorter Than Me”



The negation of the *existential* statement

**There exists a *P* that is a *Q***

is the *universal* statement

**Every *P* is not a *Q*.**

The negation of the *existential* statement

**There exists an *x* where *P*(*x*) is true**

is the *universal* statement

**For all *x*, *P*(*x*) is false**.

# Logical Implication

# Implications

- An ***implication*** is a statement of the form

**If *P* is true, then *Q* is true.**

- Some examples:
  - If $n$ is an even integer, then $n^2$ is an even integer.
  - If $A \subseteq B$ and $B \subseteq A$, then $A = B$.
  - If you like the way you look that much, (ohhh baby) then you should go and love yourself.

# Implications

- An ***implication*** is a statement of the form

  **If *P* is true, then *Q* is true.**

- In the above statement, the term "*P* is true" is called the ***antecedent*** and the term "*Q* is true" is called the ***consequent***.

# What Implications Mean

- Consider the simple statement

  ***If I put fire near cotton, it will burn.***

- Some questions to consider:
  - Does this apply to all fire and all cotton, or just some types of fire and some types of cotton? *(Scope)*
  - Does the fire cause the cotton to burn, or does the cotton burn for another reason? *(Causality*)
- These are significantly deeper questions than they might seem.
- To mathematically study implications, we need to formalize what implications really mean.

# Understanding Implications

**“If there's a rainbow,**
**then it's raining somewhere.”**

- In mathematics, implication is *directional*.
  - The above statement doesn't mean that if it's raining somewhere, there has to be a rainbow.
- In mathematics, implications only apply in cases where the antecedent is true.
  - If there's no rainbow, it doesn't mean there's no rain.
- In mathematics, Implication says nothing about *causality*.
  - Rainbows do not cause rain. ☺

# Scoping Implications

- Consider the following statements:

**If $A \subseteq B$ and $B \subseteq C$, then $A \subseteq C$.**

**If $n$ is even, then $n^2$ is even.**

**If $A \subseteq B$ and $B \subseteq A$, then $A = B$.**

- In the above statements, what are $A$, $B$, $C$, and $n$? Are they *specific* objects? Or do these claims hold for all objects?

# Implications and Universals

- In discrete math, most* implications involving unknown quantities are, implicitly, universal statements.
- For example, the statement

  **If $A \subseteq B$ and $B \subseteq C$, then $A \subseteq C$**

  actually means

  **For any sets $A$, $B$, and $C$,**
  **if $A \subseteq B$ and $B \subseteq C$, then $A \subseteq C$.**

** this will become clearer on Wednesday.*

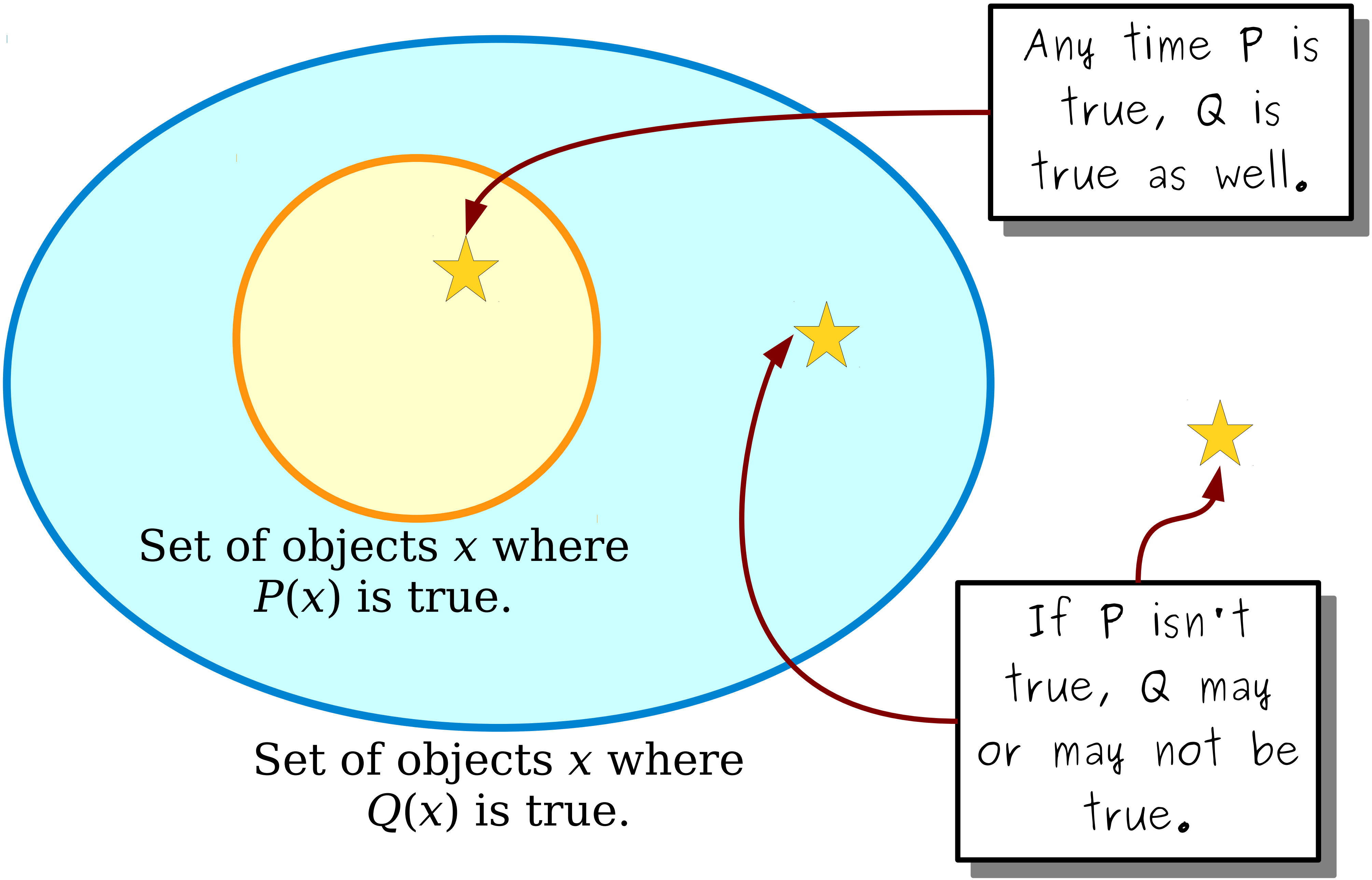
# What Implications Mean

- In mathematics, a statement of the form

  **For any $x$, if $P(x)$ is true, then $Q(x)$ is true**

  means that any time you find an object $x$ where $P(x)$ is true, you will find that $Q(x)$ is also true.
- There is no discussion of correlation or causation here. It simply means that if you find that $P(x)$ is true, you'll find that $Q(x)$ is true.

# Implication, Diagrammatically



How do you negate an implication?

# Puppy Logic

- Consider the statement

**If *x* is a puppy, then I love *x*.**

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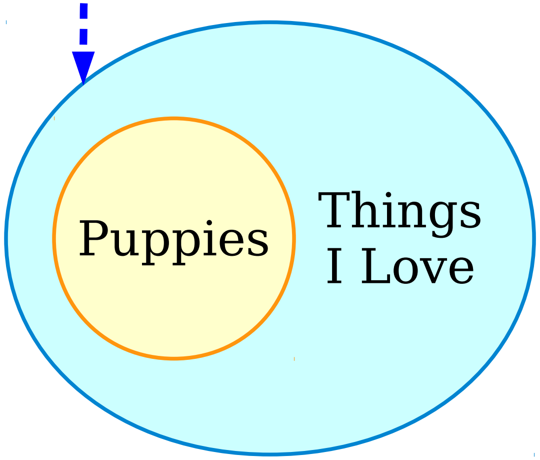



**"I love all puppies."**

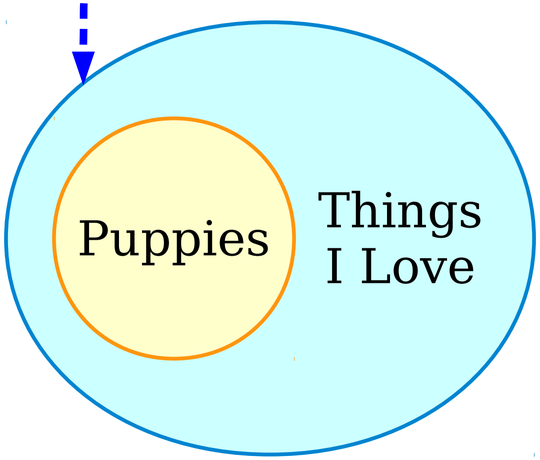
# Puppy Logic

- Consider the statement

  **If *x* is a puppy, then I love *x*.**

- The following statement is ***not*** the negation of the original statement:

  ⚠ **If *x* is a puppy, then I don't love *x*.** ⚠

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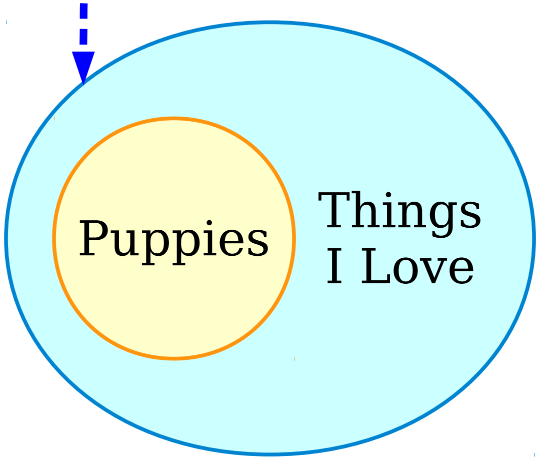
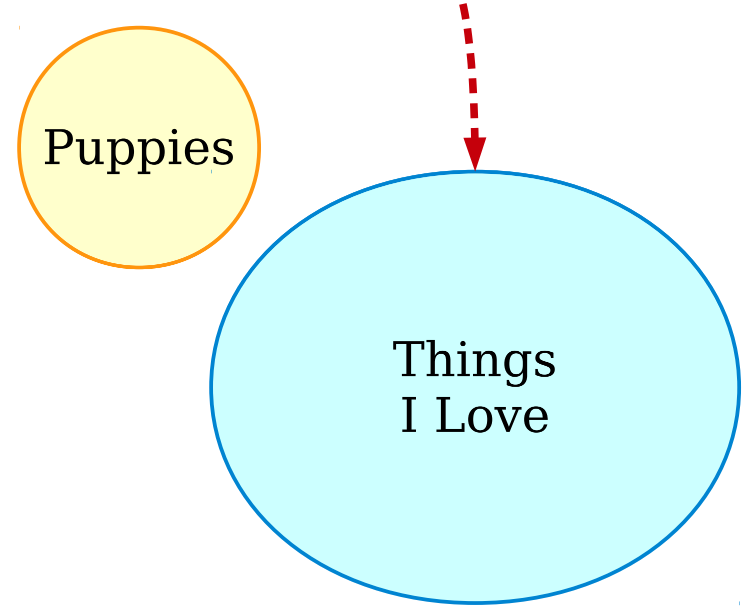



**“I love all puppies.”**




**“I don't love *any* puppies.”**

# Puppy Logic

- Consider the statement

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- The following statement is ***not*** the negation of the original statement:

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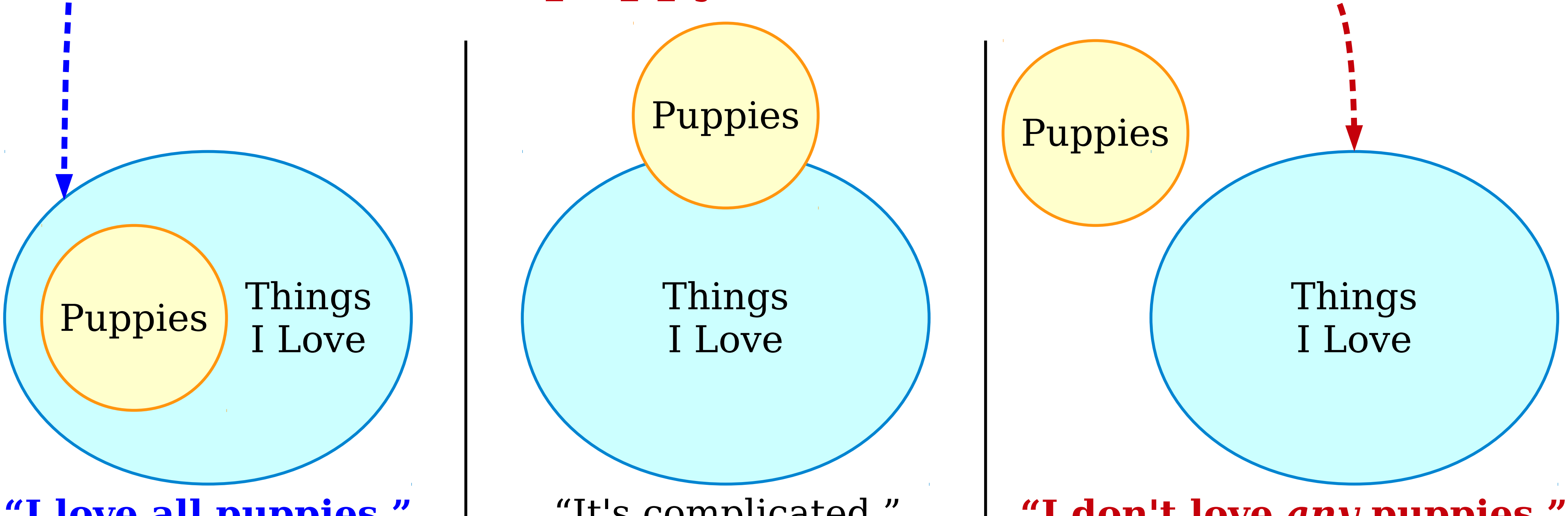



**"I love all puppies."** "It's complicated." **"I don't love *any* puppies."**

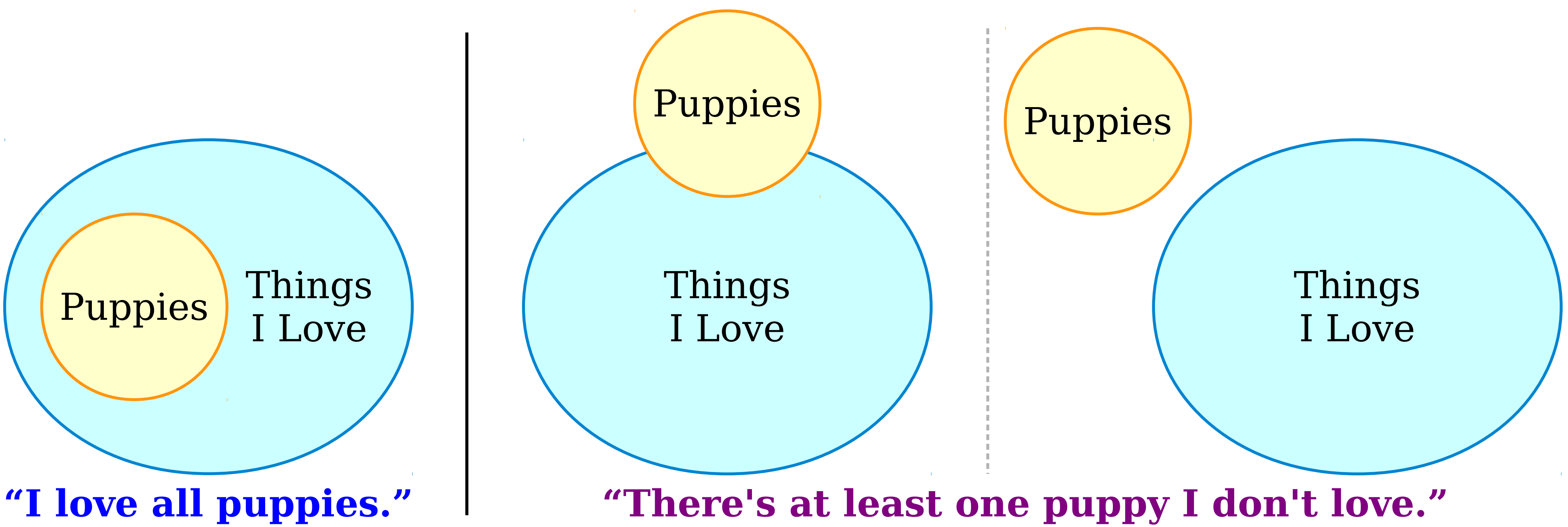
# Puppy Logic

- Consider the statement

**If *x* is a puppy, then I love *x*.**

- Here's the proper negation of our initial statement about puppies:

**There's at least one puppy I don't love.**




**"I love all puppies."**

**"There's at least one puppy I don't love."**

The negation of the statement

**"For any *x*, if *P*(*x*) is true, then *Q*(*x*) is true"**

is the statement

**"There is at least one *x* where *P*(*x*) is true and *Q*(*x*) is false."**

***The negation of an implication is not an implication!***

# Proof by Contrapositive

# The Contrapositive

- The ***contrapositive*** of the implication "If $P$, then $Q$" is the implication "If $Q$ is false, then $P$ is false."
- For example:
  - "If Harry had opened the right book, then Harry would have learned about Gillyweed."
  - Contrapositive: "If Harry didn't learn about Gillyweed, then Harry didn't open the right book."
- Another example:
  - "If I store the cat food inside, then wild raccoons will not steal my cat food."
  - Contrapositive: "If wild raccoons stole my cat food, then I didn't store it inside."

To prove the statement

**“If *P* is true, then *Q* is true,”**

you may instead prove the statement

**“If *Q* is false, then *P* is false.”**

This is called a ***proof by contrapositive***.

***Theorem:*** For any $n \in \mathbb{Z}$, if $n^2$ is even, then $n$ is even.

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***Proof:*** By contrapositive;

We're starting this proof by telling the reader that it's a proof by contrapositive. This helps cue the reader into what's about to come next.

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***Proof:*** By contrapositive; we prove that if $n$ is odd, then $n^2$ is odd.

Here, we're explicitly writing out the contrapositive. This tells the reader what we're going to prove. It also acts as a sanity check by forcing us to write out what we think the contrapositive is.

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We've said that we're going to prove this new implication, so let's go do it! The rest of this proof will look a lot like a standard direct proof.

***Theorem:*** For any $n \in \mathbb{Z}$, if $n^2$ is even, then $n$ is even.

***Proof:*** By contrapositive; we prove that if $n$ is odd, then $n^2$ is odd.

Since $n$ is odd, there is some integer $k$ such that $n = 2k + 1$.

***Theorem:*** For any $n \in \mathbb{Z}$, if $n^2$ is even, then $n$ is even.

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Since $n$ is odd, there is some integer $k$ such that $n = 2k + 1$. Squaring both sides of this equality and simplifying gives the following:

$$n^2 = (2k + 1)^2$$

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Since $n$ is odd, there is some integer $k$ such that $n = 2$
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The general pattern here is the following:

1. Start by announcing that we're going to use a proof by contrapositive so that the reader knows what to expect.

2. Explicitly state the contrapositive of what we want to prove.

3. Go prove the contrapositive.

# Biconditionals

- Combined with what we saw on Wednesday, we have proven that, if *n* is an integer:

**If *n* is even, then $n^2$ is even.**

**If $n^2$ is even, then *n* is even.**

- Therefore, if *n* is an integer:

***n* is even if and only if $n^2$ is even.**

- "If and only if" is often abbreviated ***iff***:

***n* is even iff $n^2$ is even.**

# Proving Biconditionals

- To prove ***P* iff *Q***, you need to prove that *P* implies *Q* and that *Q* implies *P*.
- You can use any proof techniques you'd like to show each of these statements.
  - In our case, we used a direct proof and a proof by contrapositive.
- Just make sure to cover both directions.

Time-Out for Announcements!

# oSTEM Mixer

- Stanford’s chapter of oSTEM (Out in STEM) will be holding a mixer even on Thursday, April 13th from 6PM – 8PM in the LGBT-CRC.
- Dinner is provided! RSVP with **this link**.

# #INCLUDE POSTER SESSION

04.07.2017
4:30 - 5:30 pm
GATES BUILDING
1ST FLOOR

30 HIGH SCHOOL STUDENTS FROM AROUND THE US WILL BE PRESENTING CS EDUCATION INITIATIVES THEY IMPLEMENTED IN THEIR COMMUNITIES

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# MENTORSHIP DINNER

04.07.2017
6:00 - 7:30 pm
TOYON LOUNGE

INTERESTED IN CS EDUCATION AND DIVERSIFYING TECHNOLOGY FIELDS?

JOIN SHE++, A NONPROFIT RUN BY STANFORD STUDENTS, SWIB, AND 30 HIGH SCHOOL STUDENTS AND 11 COLLEGE STUDENTS FROM AROUND THE WORLD FOR A FUN AND FREE MENTORSHIP DINNER AT TOYON HALL! STANFORD STUDENTS AND FACULTY INVITED.

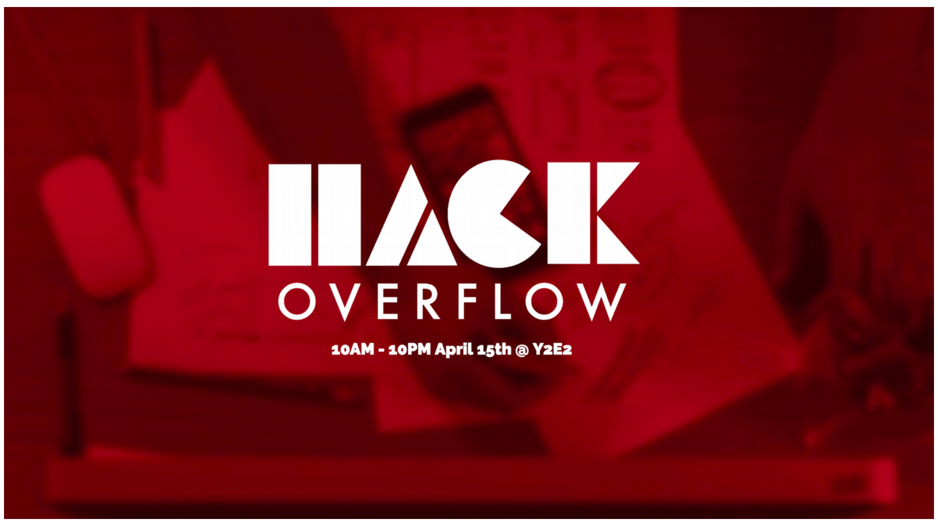
DINNER - BUCA DI BEPPO - WILL BE SERVED

#INCLUDE
FELLOWSHIP PROGRAM by she++

SHE++

STANFORD
WOMEN IN
BUSINESS

# HackOverflow




- WiCS (Women in CS) is hosting HackOverflow, their annual hackathon.
- Beginners are explicitly encouraged to attend!
- April 15th, 10AM to 10PM, in Y2E2.
- Interested? RSVP using **this link**.

# Handouts

- There are *six* (!) total handouts for today:
  - Handout 05: Problem Set Policies
  - Handout 06: Honor Code Policies
  - Handout 07: Guide to Proofs
  - Handout 08: Mathematical Vocabulary
  - Handout 09: Guide to Indirect Proofs
  - Handout 10: Problem Set One
- Be sure to read over Handouts 05 – 09; there's a lot of really important information in there!

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Be sure to read over Handouts 05 – 09; there's a lot of really important information in there!

# Announcements

- Problem Set 1 goes out today!
- ***Checkpoint*** due Monday, April 10.
  - Grade determined by attempt rather than accuracy. It's okay to make mistakes – we want you to give it your best effort, even if you're not completely sure what you have is correct.
  - We will get feedback back to you with comments on your proof technique and style.
  - The more effort you put in, the more you'll get out.
- ***Remaining problems*** due Friday, April 14.
  - Feel free to email us with questions, stop by office hours, or ask questions on Piazza!

# Submitting Assignments

- This quarter, we will be using GradeScope to handle assignment submissions. Visit www.gradescope.com and enter code **9BNNY9**.
- Summary of the late policy:
  - Everyone has *three* 24-hour late days.
  - Late days can't be used on checkpoints.
  - Nothing may be submitted more than three days past the due date.
- Because submission times are recorded automatically, we're strict about the submission deadlines.
- ***Very good idea:*** Leave at least two hours buffer time for your first assignment submission, just in case something goes wrong.
- ***Very bad idea:*** Wait until the last minute to submit.

# Working in Pairs

- You can work on the problem sets individually or in pairs.
- Each person/pair should only submit a single problem set, and should submit it only once.
- Full details about the problem sets, collaboration policy, and Honor Code can be found in Handouts 05 and 06.

# A Note on the Honor Code

Office hours start tonight!

Schedule is available
on the course website.

Back to CS103!

# Proof by Contradiction

“When you have eliminated all which is impossible, then whatever remains, however improbable, must be the truth.”

- Sir Arthur Conan Doyle, *The Adventure of the Blanched Soldier*

# Proof by Contradiction

- A ***proof by contradiction*** is a proof that works as follows:
  - To prove that $P$ is true, assume that $P$ is *not* true.
  - Beginning with this assumption, use logical reasoning to conclude something that is clearly impossible.
    - For example, that $1 = 0$, that $x \in S$ and $x \notin S$, etc.
  - This means that if $P$ is false, something that cannot possibly happen must happen.
  - Therefore, $P$ can't be false, so it must be true.

An Example: ***Set Cardinalities***

# Set Cardinalities

- We've seen sets of many different cardinalities:
  - $|\emptyset| = 0$
  - $|\{1, 2, 3\}| = 3$
  - $|\{ n \in \mathbb{N} \mid n < 137\}| = 137$
  - $|\mathbb{N}| = \aleph_0$.
- These span from the finite up through the infinite.
- ***Question:*** Is there a "largest" set? That is, is there a set that's bigger than every other set?

***Theorem:*** There is no largest set.

***Theorem:*** There is no largest set.

***Proof:***

***Theorem:*** There is no largest set.

***Proof:***

To prove this statement by contradiction,
we're going to assume its negation.

***Theorem:*** There is no largest set.

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To prove this statement by contradiction, we're going to assume its negation.

What is the negation of the statement "there is no largest set?"

***Theorem:*** There is no largest set.

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To prove this statement by contradiction, we're going to assume its negation.

What is the negation of the statement "there is no largest set?"

One option: "there *is* a largest set."

***Theorem:*** There is no largest set.

***Proof:*** Assume for the sake of contradiction that there is a largest set; call it $S$.

To prove this statement by contradiction, we're going to assume its negation.

What is the negation of the statement "there is no largest set?"

One option: "there *is* a largest set."

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Notice that we're announcing

1. that this is a proof by contradiction, and
2. what, specifically, we're assuming.

This helps the reader understand where we're going. Remember – proofs are meant to be read by other people!

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Now, consider the set $\wp(S)$.

***Theorem:*** There is no largest set.

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Now, consider the set $\wp(S)$. By Cantor's Theorem, we know that $|S| < |\wp(S)|$, so $\wp(S)$ is a larger set than $S$.

***Theorem:*** There is no largest set.

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Now, consider the set $\wp(S)$. By Cantor's Theorem, we know that $|S| < |\wp(S)|$, so $\wp(S)$ is a larger set than $S$. This contradicts the fact that $S$ is the largest set.

***Theorem:*** There is no largest set.

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Now, consider the set $\wp(S)$. By Cantor's Theorem, we know that $|S| < |\wp(S)|$, so $\wp(S)$ is a larger set than $S$. This contradicts the fact that $S$ is the largest set.

We've reached a contradiction, so our assumption must have been wrong.

***Theorem:*** There is no largest set.

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Now, consider the set $\wp(S)$. By Cantor's Theorem, we know that $|S| < |\wp(S)|$, so $\wp(S)$ is a larger set than $S$. This contradicts the fact that $S$ is the largest set.

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The three key pieces:

1. Say that the proof is by contradiction.
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In CS103, please include all these steps in your proofs!

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# Proving Implications

- To prove the implication

  **"If *P* is true, then *Q* is true."**

- you can use these three techniques:
  - ***Direct Proof.***
    - Assume $P$ and prove $Q$.
  - ***Proof by Contrapositive.***
    - Assume not $Q$ and prove not $P$.
  - ***Proof by Contradiction.***
    - … what does this look like?

# Negating Implications

- To prove the statement

  **"For any $x$, if $P(x)$, then $Q(x)$"**

  by contradiction, we do the following:

  - Assume this statement is false.
  - Derive a contradiction.
  - Conclude that the statement is true.

- What is the negation of this statement?

  **"There is an $x$ where**
  **$P(x)$ is true and $Q(x)$ is false"**

# Contradictions and Implications

- To prove the statement

  **“If *P* is true, then *Q* is true”**

  using a proof by contradiction, do the following:

  - Assume that *P* is true and that *Q* is false.
  - Derive a contradiction.
  - Conclude that if *P* is true, *Q* must be as well.

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# Rational and Irrational Numbers

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- A number $r$ is called a ***rational number*** if it can be written as

$$r = \frac{p}{q}$$

  where $p$ and $q$ are integers and $q \neq 0$.
- A number that is not rational is called ***irrational***.

# Simplest Forms

- *By definition,* if $r$ is a rational number, then $r$ can be written as $p / q$ where $p$ and $q$ are integers and $q \neq 0$.
- ***Theorem:*** If $r$ is a rational number, then $r$ can be written as $p / q$ where $p$ and $q$ are integers, $q \neq 0$, and $p$ and $q$ have no common factors other than 1 and -1.
  - That is, $r$ can be written as a fraction in simplest form.
- We're just going to take this for granted for now, though with the techniques you'll see later in the quarter you'll be able to prove it!

***Question:*** Are all real numbers rational?

***Theorem:*** $\sqrt{2}$ is irrational.

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$$\begin{aligned} p^2 &= 2q^2 \\ (2k)^2 &= 2q^2 \\ 4k^2 &= 2q^2 \\ 2k^2 &= q^2 \qquad (3) \end{aligned}$$

Equation (3) shows that $q^2$ is even. Our earlier theorem tells us that, because $q^2$ is even, $q$ must also be even. But this is not possible – we know that $p$ and $q$ have no common factors other than 1 and -1, but we've shown that $p$ and $q$ must have two as a common factor.

We have reached a contradiction, so our original assumption must have been wrong. Therefore, $\sqrt{2}$ is irrational. ■

Vi Hart on Pythagoras and
the Square Root of Two:

**http://www.youtube.com/watch?v=X1E7I7_r3Cw**

# What We Learned

- ***How do you negate formulas?***
  - It depends on the formula. There are nice rules for how to negate universal and existential statements and implications.
- ***What's an implication?***
  - It's statement of the form "if $P$, then $Q$," and states that if $P$ is true, then $Q$ is true.
- ***What is a proof by contrapositive?***
  - It's a proof of an implication that instead proves its contrapositive.
  - (The contrapositive of "if $P$, then $Q$" is "if not $Q$, then not $P$.")
- ***What's a proof by contradiction?***
  - It's a proof of a statement $P$ that works by showing that $P$ cannot be false.

# Next Time

- ***Mathematical Logic***
  - How do we formalize the reasoning from our proofs?
- ***Propositional Logic***
  - Reasoning about simple statements.
- ***Propositional Equivalences***
  - Simplifying complex statements.

# Appendix: Negating Statements

## Negating Universal Statements

**"For all $x$, P($x$) is true"**

becomes

**"There is an $x$ where P($x$) is false."**

## Negating Existential Statements

**"There exists an $x$ where P($x$) is true"**

becomes

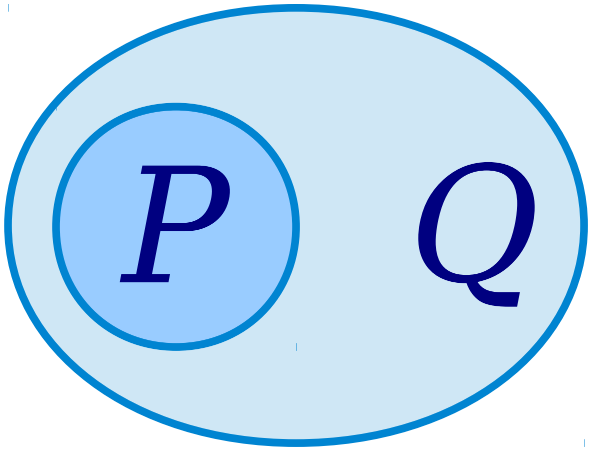
**"For all $x$, P($x$) is false."**

## Negating Implications

**"For every $x$, if $P$($x$) is true, then $Q$($x$) is true"**

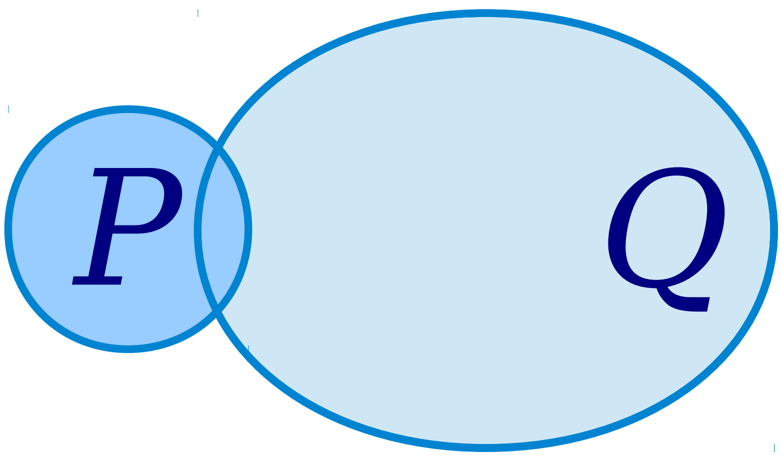
becomes

**"There is an $x$ where $P$($x$) is true and $Q$($x$) is false"**




**$P(x)$ implies $Q(x)$**

"If $P(x)$ is true, then $Q(x)$ is true."

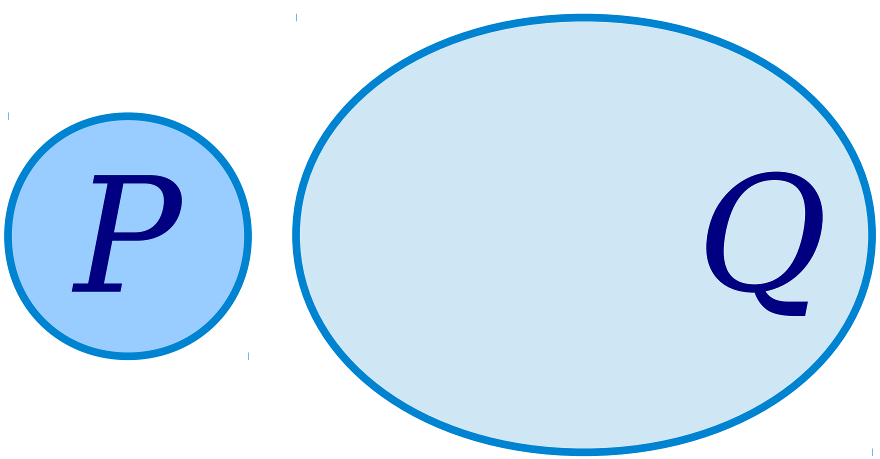



**$P(x)$ does not imply $Q(x)$**

**-and-**

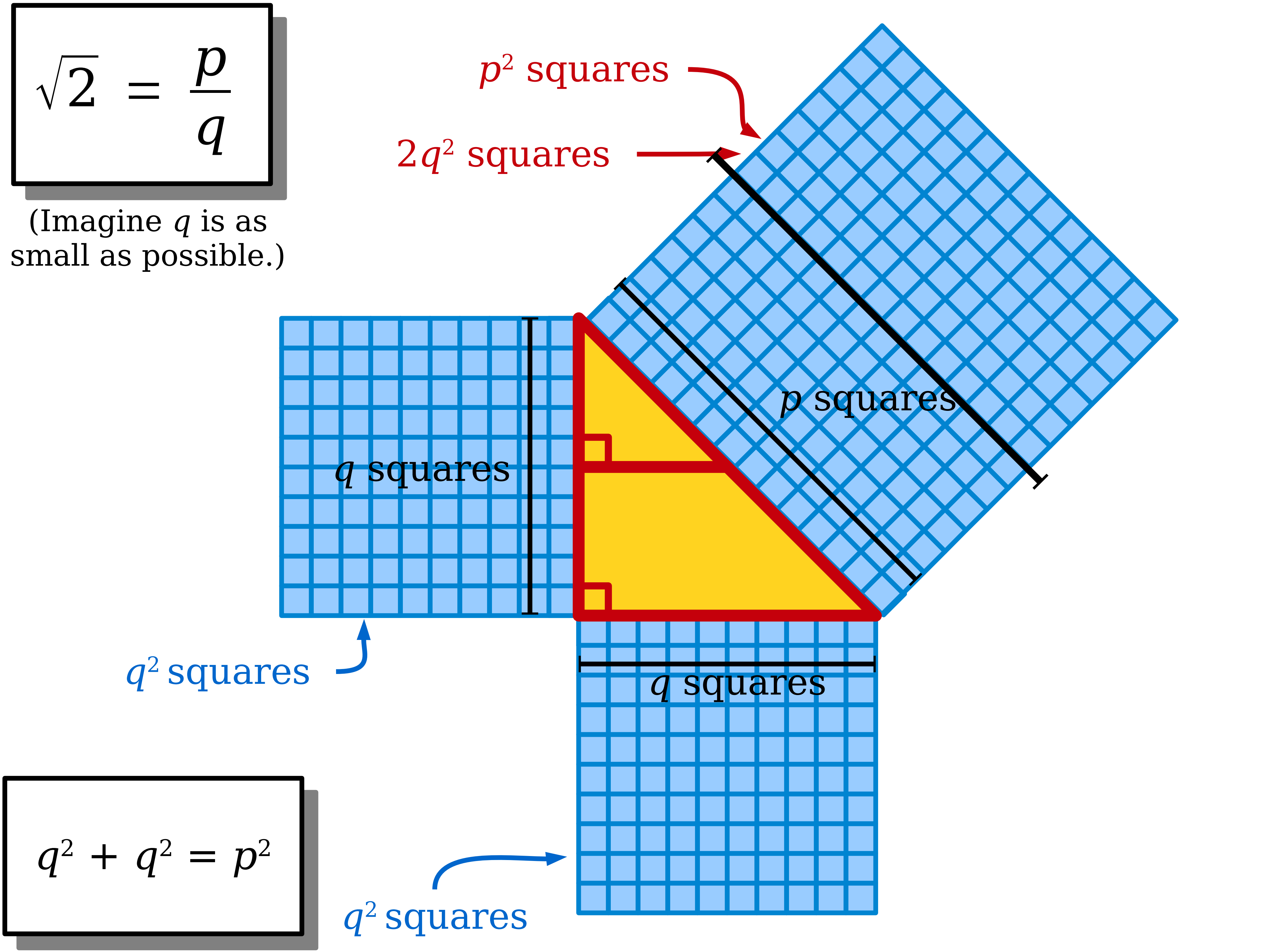
**$P(x)$ does not imply not $Q(x)$**

"Sometimes $P(x)$ is true and $Q(x)$ is true,
-and-
sometimes $P(x)$ is true and $Q(x)$ is false."




**$P(x)$ implies not $Q(x)$**

If $P(x)$ is true, then $Q(x)$ is false


$\sqrt{2} = \frac{p}{q}$
(Imagine $q$ is as small as possible.)
$p^2$ squares
$2q^2$ squares
$p$ squares
$q$ squares
$q^2$ squares
$q$ squares
$q^2$ squares
$q^2 + q^2 = p^2$