# CS 154

## More on Reductions, Rice's Theorem

# Reducing One Problem to Another

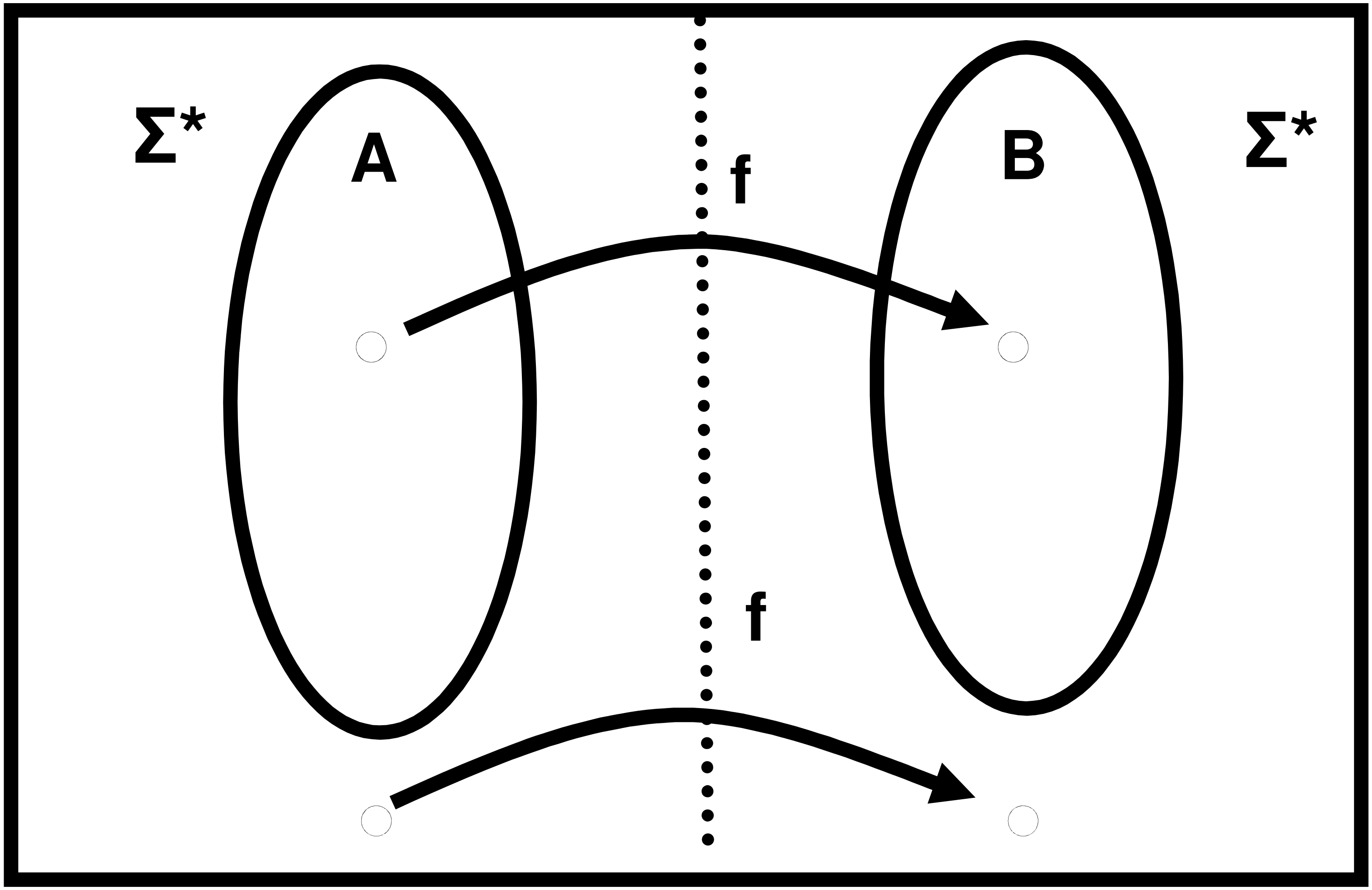
$f : \Sigma^* \rightarrow \Sigma^*$ **is a computable function if there is a Turing machine M that halts with just $f$(w) written on its tape, for every input w**

**A language A is *mapping reducible* to language B, written as A $\leq_m$ B, if there is a computable $f : \Sigma^* \rightarrow \Sigma^*$ such that for every w,**

$$w \in A \Leftrightarrow f(w) \in B$$

***f*** **is called a mapping reduction (or many-one reduction) from A to B**

Let $f : \Sigma^* \rightarrow \Sigma^*$ be a computable function such that $w \in A \Leftrightarrow f(w) \in B$




Say: "A is mapping reducible to B"

Write: $A \leq_m B$

# Examples

$A_{DFA}$ = { (D, w) | D encodes a DFA over some Σ, and D accepts w $\in$ Σ* }

$A_{NFA}$ = { (N, w) | N encodes an NFA, D accepts w }

**Theorem: $A_{DFA} \leq_m A_{NFA}$**

Every DFA can be trivially written as an NFA.

So one mapping reduction f from $A_{DFA}$ to $A_{NFA}$ is:

f(D,w) := Construct NFA N which is equivalent to D
Output (N,w)

**Theorem: $A_{NFA} \leq_m A_{DFA}$**

f(N,w) := Use the subset construction to convert NFA N into an equivalent DFA D. Output (D,w)

**Theorem: If A $\leq_m$ B and B is decidable, then A is decidable**

**Corollary: If A $\leq_m$ B and A is undecidable, then B is undecidable**

**Theorem: If A $\leq_m$ B and B is recognizable, then A is recognizable**

**Corollary: If A $\leq_m$ B and A is unrecognizable, then B is unrecognizable**

**Theorem: $A_{TM} \leq_m HALT_{TM}$**

**Define**

**f(z) := Decode z into a pair (M, w)**

**Construct M' with the specification:**

**"M'(w) = Simulate M on w.**

**if M(w) accepts then *accept***

**else *loop forever*"**

**Output (M', w)**

**We have $z \in A_{TM} \Leftrightarrow (M', w) \in HALT_{TM}$**

**Theorem: $A_{TM} \leq_m HALT_{TM}$**

**Corollary: $\neg A_{TM} \leq_m \neg HALT_{TM}$**

**Corollary: $\neg HALT_{TM}$ is unrecognizable!**

**Proof: If $\neg HALT_{TM}$ were recognizable, then**
**$\neg A_{TM}$ would be recognizable…**

# Theorem: $\mathrm{HALT_{TM}} \leq_m \mathrm{A_{TM}}$

**Proof: Define the computable function:**

**f(z) := Decode z into a pair (M, w)**

**Construct M' with the specification:**

**"M'(w) = Simulate M on w.**

**If M(w) halts then *accept***

**else *loop forever*"**

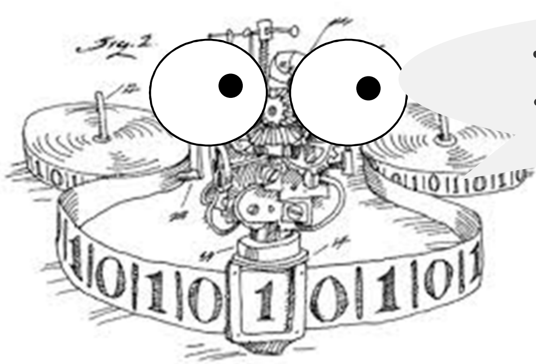
**Output (M', w)**

**Observe $(M, w) \in \mathrm{HALT_{TM}} \Leftrightarrow (M', w) \in \mathrm{A_{TM}}$**

# Corollary: $HALT_{TM} \equiv_m A_{TM}$

Yo, T.M.! I can give you the magical power to either compute the halting problem, or the acceptance problem. Which do you want?

Wow, hm, so hard to choose…




I can't decide!

# The Emptiness Problem for TMs

**$\text{EMPTY}_{\text{TM}}$ = { M | M is a TM such that $L(M) = \varnothing$}**

***Given a program, does it reject or loop on every input?***

**Theorem: $\text{EMPTY}_{\text{TM}}$ is *not recognizable***

**Proof: Show that $\neg A_{\text{TM}} \leq_m \text{EMPTY}_{\text{TM}}$**

**f(z) := Decode z into a pair (M, w).**

**Output a TM M' with the behavior:**

***"M'(x) := if (x = w) then output answer of M(w), else reject"***

**$z \notin A_{\text{TM}} \Leftrightarrow$ M doesn't accept w**

**$\Leftrightarrow L(M') = \varnothing \Leftrightarrow M' \in \text{EMPTY}_{\text{TM}}$**

**$\Leftrightarrow f(z) \in \text{EMPTY}_{\text{TM}}$**

# The Emptiness Problem for Other Stuff

**$\text{EMPTY}_{\text{DFA}}$ = { M | M is a DFA such that L(M) = $\varnothing$}**

***Given a DFA, does it reject every input?***

**Theorem: $\text{EMPTY}_{\text{DFA}}$ is decidable**

## Why?

**$\text{EMPTY}_{\text{NFA}}$ = { M | M is a NFA such that L(M) = $\varnothing$}**

**$\text{EMPTY}_{\text{REX}}$ = { R | M is a regexp such that L(M) = $\varnothing$}**

# The Equivalence Problem

$EQ_{TM}$ = {(M, N) | M, N are TMs and L(M) = L(N)}

***Do two programs compute the same function?***

**Theorem:** $EQ_{TM}$ is ***unrecognizable***

**Proof:** Reduce $EMPTY_{TM}$ to $EQ_{TM}$

**Let $M_\varnothing$ be a TM that always loops forever, so $L(M_\varnothing) = \varnothing$**

**Define $f(M) := (M, M_\varnothing)$**

$$M \in EMPTY_{TM} \iff L(M) = L(M_\varnothing)$$
$$\iff (M, M_\varnothing) \in EQ_{TM}$$

# Moral: Analyzing Programs is Really, Really Hard.

## How can we more easily tell when some “program analysis” problem is undecidable?

## Problem 1 Undecidable

{ (M, w) | M is a TM that on input w, tries to move its head past the left end of the input }

## Problem 2 Decidable

{ (M, w) | M is a TM that on input w, moves its head left at least once, at some point}

# Problem 1 Undecidable

L' = { (M, w) | M is a TM that on input w, tries to move its head past the left end of the input }

Proof: Reduce $A_{TM}$ to L'

On input (M,w), make a TM N that shifts w over one cell, marks a special symbol $ on the leftmost cell, then simulates M(w) on the tape.

If M's head moves to the cell with $ but has not yet accepted, N moves the head back to the right.

If M accepts, N tries to move its head past the $.

(M,w) is in $A_{TM}$ if and only if (N,w) is in L'

# Problem 2 Decidable

**{ (M, w) | M is a TM that on input w, moves its head left at least once, at some point}**

**On input (M,w), run M on w for**

**$|Q| + |w| + 1$ steps,**

**where $|Q|$ = number of states of M.**

| | |
|---|---|
| **Accept** | **If M's head moved left at all** |
| **Reject** | **Otherwise** |

***(Why does this work?)***

## Problem 3

REVERSE = { M | M is a TM with the property:
for all w, M(w) accepts $\Leftrightarrow$ M($w^R$) accepts}.

Decidable or not?

REVERSE is undecidable.

# Rice's Theorem

**Let P : {Turing Machines} $\rightarrow$ {0,1}.**
**(Think of 0=false, 1=true) Suppose P satisfies:**

**1. (Nontrivial) There are TMs $M_{YES}$ and $M_{NO}$ where $P(M_{YES}) = 1$ and $P(M_{NO}) = 0$**

**2. (Semantic) For all TMs $M_1$ and $M_2$, If $L(M_1) = L(M_2)$ then $P(M_1) = P(M_2)$**

**Then, $L = \{M \mid P(M) = 1\}$ is undecidable.**

**A Huge Hammer for Undecidability!**

# Some Examples and Non-Examples

**Semantic Properties *P(M)***

- M accepts 0
- for all w, M(w) accepts iff M($w^R$) accepts
- L(M) = {0}
- L(M) is empty
- L(M) = $\Sigma^*$
- M accepts 154 strings

**Not Semantic!**

- M halts and rejects 0
- M tries to move its head off the left end of the tape, on input 0
- M never moves its head left on input 0
- M has exactly 154 states
- M halts on all inputs

L = {M | *P*(M) is true} is undecidable

There are $M_1$ and $M_2$ such that $L(M_1) = L(M_2)$ and $P(M_1) \neq P(M_2)$

**Rice's Theorem: If P is nontrivial and semantic, then $L = \{M \mid P(M) = 1\}$ is undecidable.**

**Proof:** Either reduce $A_{TM}$ or $\neg A_{TM}$ to the language L

Define $M_{\varnothing}$ to be a TM such that $L(M_{\varnothing}) = \varnothing$

Case 1: $P(M_{\varnothing}) = 0$

Since P is nontrivial, there's $M_{YES}$ such that $P(M_{YES}) = 1$

Reduction from $A_{TM}$ to L   On input (M,w), output:

"$M_w(x)$ := If ((M accepts w) & ($M_{YES}$ accepts x)) then ACCEPT, else REJECT"

If M accepts w, then $L(M_w) = L(M_{YES})$

Since $P(M_{YES}) = 1$, we have $P(M_w) = 1$ and $M_w \in L$

If M does not accept w, then $L(M_w) = L(M_{\varnothing}) = \varnothing$

Since $P(M_{\varnothing}) = 0$, we have $M_w \notin L$

Rice's Theorem: If P is nontrivial and semantic, then L = $\{M \mid P(M) = 1\}$ is undecidable.

Proof: Either reduce $A_{TM}$ or $\neg A_{TM}$ to the language L

Define $M_{\varnothing}$ to be a TM such that $L(M_{\varnothing}) = \varnothing$

Case 2: $P(M_{\varnothing}) = 1$

Since P is nontrivial, there's $M_{NO}$ such that $P(M_{NO}) = 0$

Reduction from $\neg A_{TM}$ to L On input (M,w), output:

"$M_w$(x) := If ((M accepts w) & ($M_{NO}$ accepts x)) then ACCEPT, else REJECT"

If M does not accept w, then $L(M_w) = L(M_{\varnothing}) = \varnothing$

Since $P(M_{\varnothing}) = 1$, we have $M_w \in L$

If M accepts w, then $L(M_w) = L(M_{NO})$

Since $P(M_{NO}) = 0$, we have $M_w \notin L$

# The Regularity Problem for Turing Machines

**$\text{REGULAR}_{\text{TM}}$ = { M | M is a TM and L(M) is regular}**

***Given a program, is it equivalent to some DFA?***

**Theorem: $\text{REGULAR}_{\text{TM}}$ is *not recognizable***

**Proof: Use Rice's Theorem!**

**P(M) := "L(M) is regular" is nontrivial:**

**- there's an $M_{\varnothing}$ which never halts: $P(M_{\varnothing}) = 1$**

**- there's an M' deciding $\{0^n1^n \mid n \geq 0\}$: P(M') = 0**

**P is also semantic:**

**If L(M) = L(M') then L(M) is regular iff L(M') is regular, so P(M) = 1 iff P(M') = 1, so P(M) = P(M')**

**By Rice's Thm, we have $\neg A_{\text{TM}} \leq_m \text{REGULAR}_{\text{TM}}$**

# Recognizability via Logic

**Def. A decidable predicate R(x,y) is a proposition about the input strings x and y, such that some TM M implements R. That is,**

**for all x, y, R(x,y) is TRUE $\Rightarrow$ M(x,y) accepts**
**R(x,y) is FALSE $\Rightarrow$ M(x,y) rejects**

**Can think of R as a function from $\Sigma^* \times \Sigma^* \rightarrow \{T,F\}$**

**EXAMPLES: R(x,y) = "xy has at most 100 zeroes"**
**R(N,y) = "TM N halts on y in at most 99 steps"**

**Theorem: A language A is *recognizable* if and only if there is a decidable predicate R(x, y) such that:**

$$A = \{ x \mid \exists y\ R(x, y) \}$$

**Proof: (1) If $A = \{ x \mid \exists y\ R(x,y) \}$ then A is recognizable**

**Define the TM M(x): For all finite-length strings y,**
**If R(x,y) is true, accept.**
**Then, M accepts exactly those x s.t. $\exists y\ R(x,y)$ is true**

**(2) If A is recognizable, then $A = \{ x \mid \exists y\ R(x,y) \}$**

**Suppose TM M recognizes A.**
**Let R(x,y) be TRUE iff M accepts x in |y| steps**
**Then, M accepts x $\Leftrightarrow \exists y\ R(x,y)$**