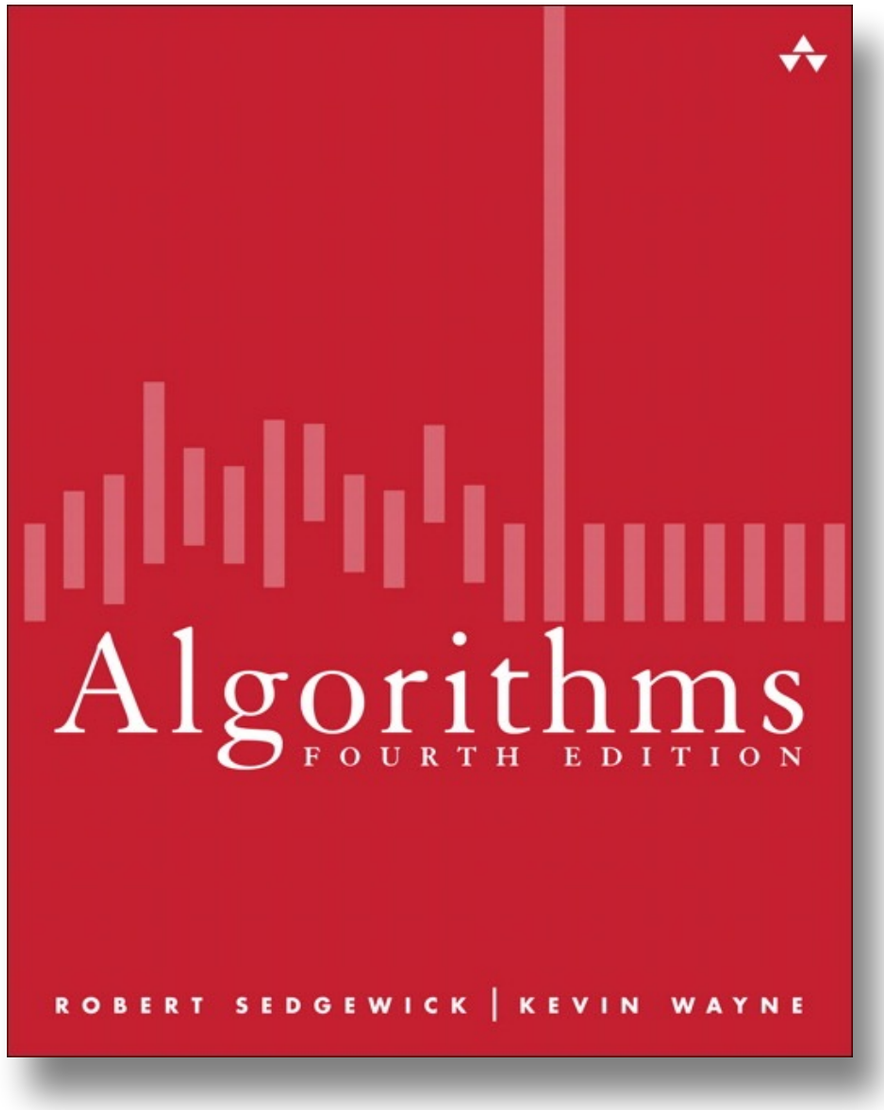
CSC 402

Data Structures I
Lecture 5

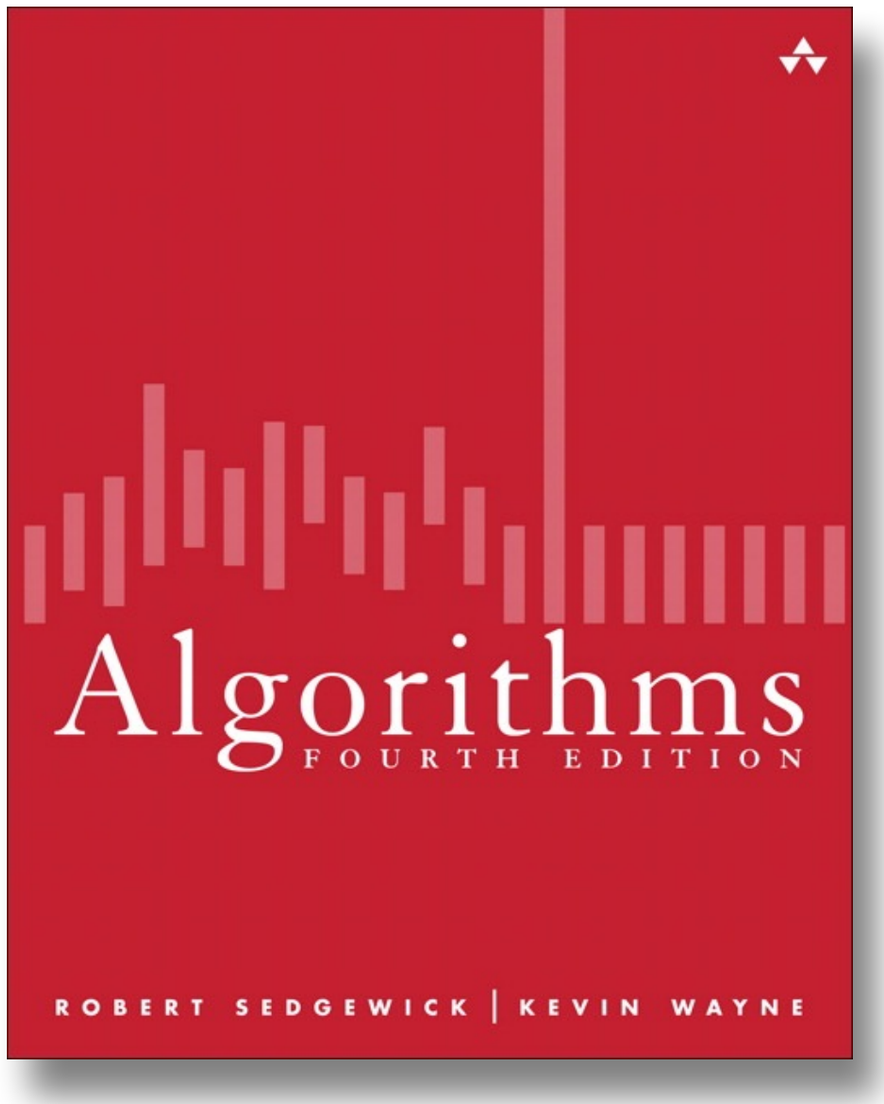
Lecture Overview



2.1 Elementary Sorts

- Rules of the game
- Selection sort
- Insertion sort
- Shellsort
- Shuffling

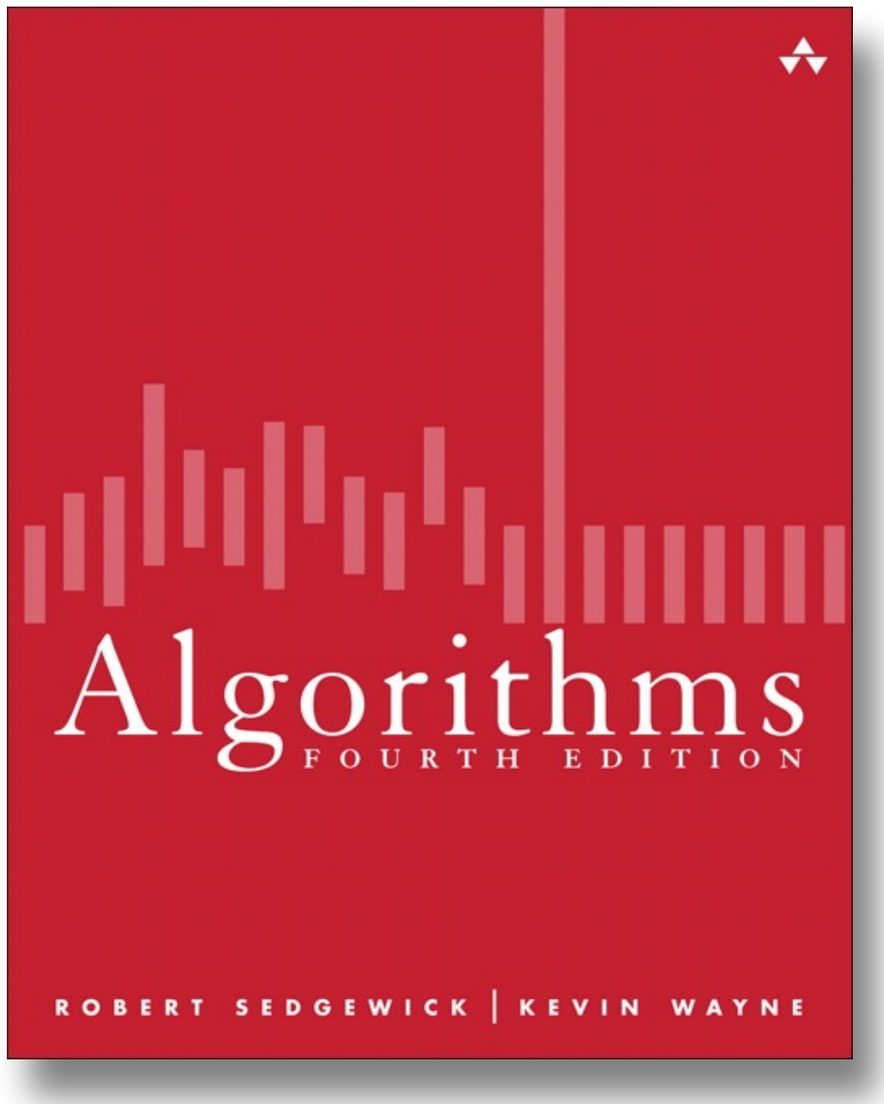
Lecture Overview



2.2 Mergesort

- Mergesort
- Bottom-up Mergesort
- Sorting complexity
- Comparators
- Stability

Lecture Overview



2.2 Mergesort

- Mergesort
- Bottom-up Mergesort
- Sorting complexity
- Comparators
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Mergesort

The algorithms that we consider in this section is based on a simple operation known as merging: combining two ordered arrays to make one larger ordered array. This operation immediately lends itself to a simple recursive sort method known as **mergesort**: to sort an array, divide it into two halves, sort the two halves (recursively), and then merge the results.

Mergesort guarantees to sort an array of N items in time proportional to N log N, no matter what the input. Its prime disadvantage is that it uses extra space proportional to N.

input	M	E	R	G	E	S	O	R	T	E	X	A	M	P	L	E
sort left half	E	E	G	M	O	R	R	S	T	E	X	A	M	P	L	E
sort right half	E	E	G	M	O	R	R	S	A	E	E	L	M	P	T	X
merge results	A	E	E	E	E	G	L	M	M	O	P	R	R	S	T	X

Mergesort overview

Mergesort

Basic plan.

- Divide array into two halves.
- Recursively sort each half.
- Merge two halves.

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sort left half	E	E	G	M	O	R	R	S	T	E	X	A	M	P	L	E
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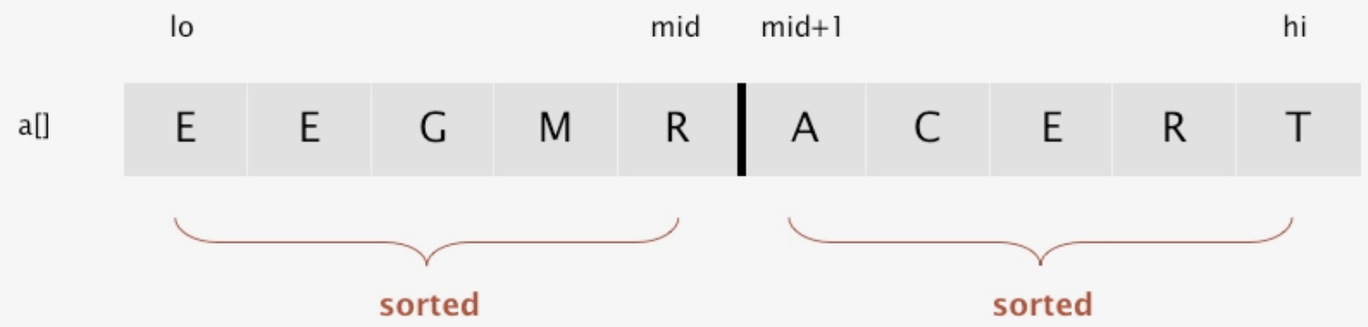
merge results	A	E	E	E	E	G	L	M	M	O	P	R	R	S	T	X

Mergesort overview

Merge Demo

Merging demo

Goal. Given two sorted subarrays `a[lo]` to `a[mid]` and `a[mid+1]` to `a[hi]`, replace with sorted subarray `a[lo]` to `a[hi]`.



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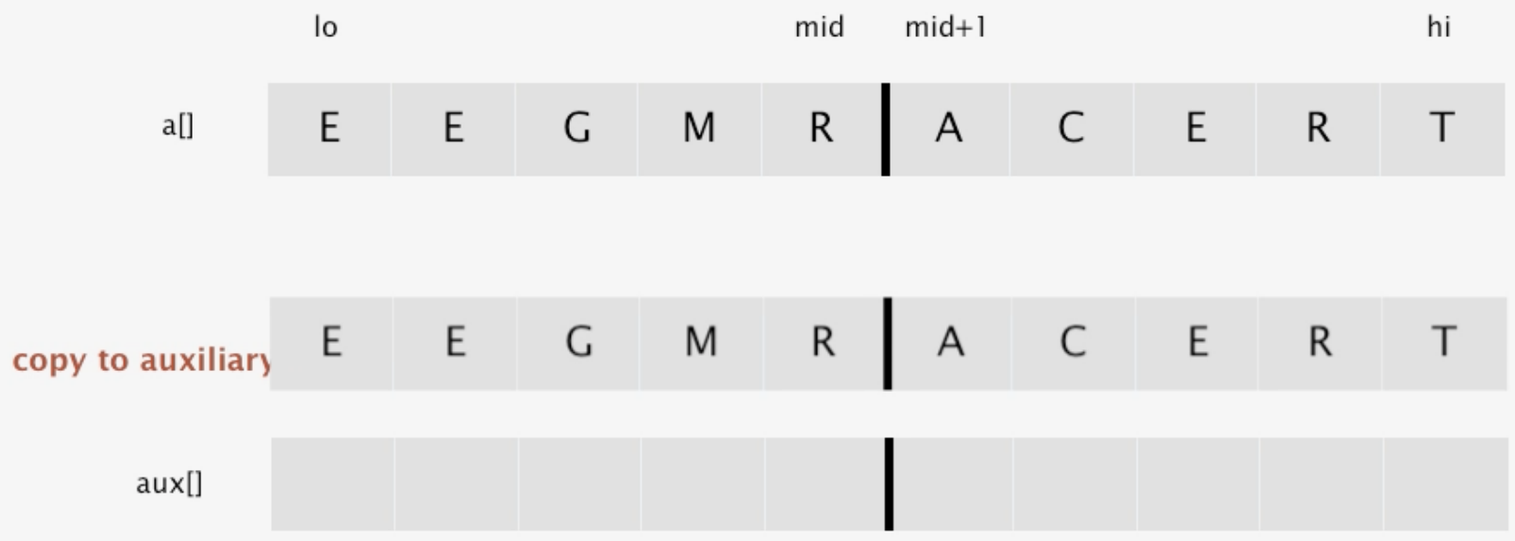

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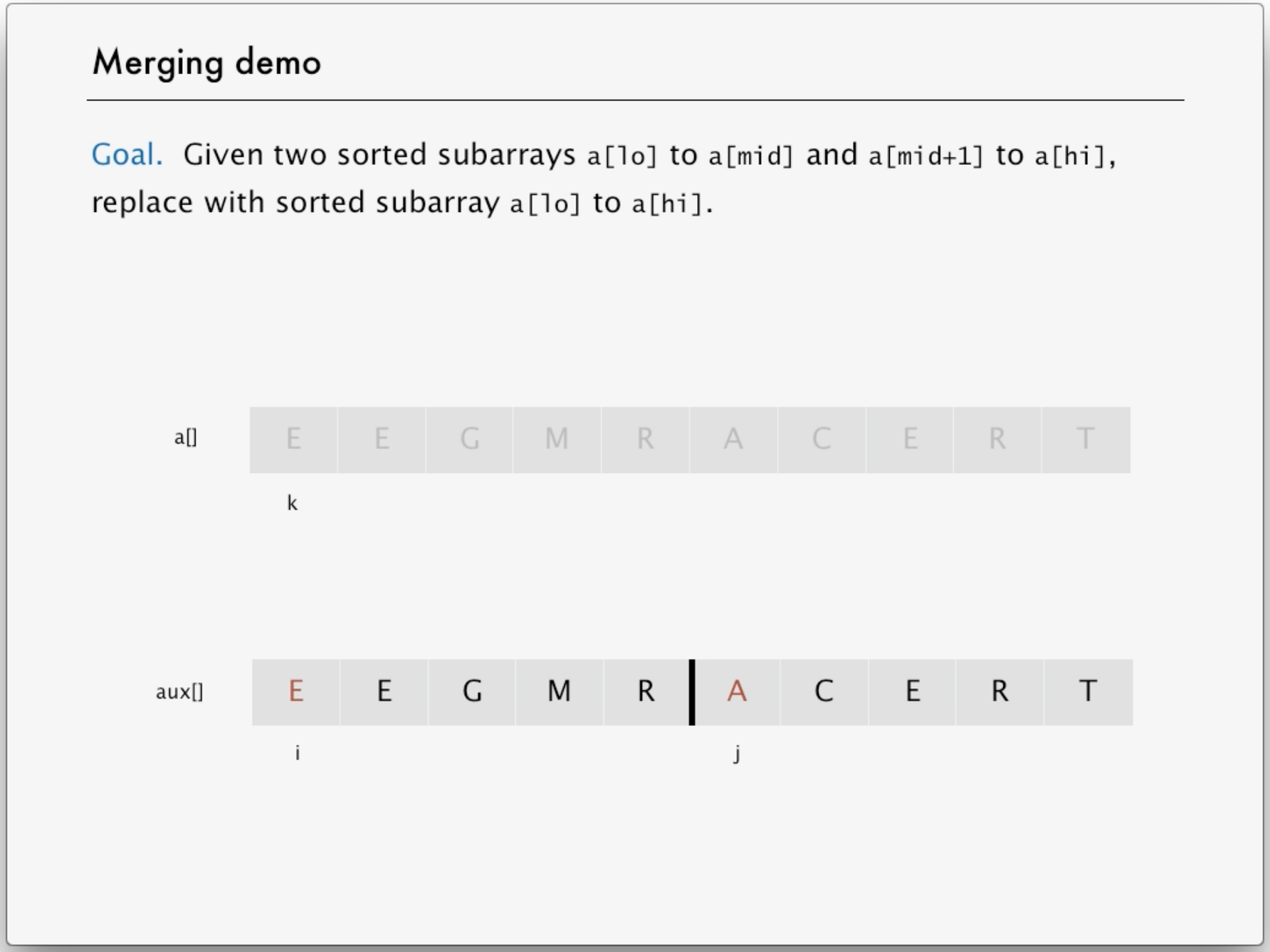
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k
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i
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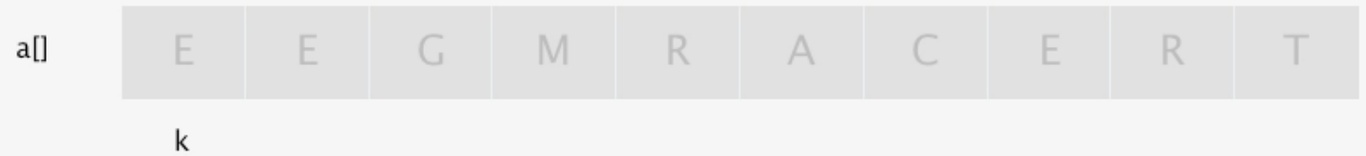
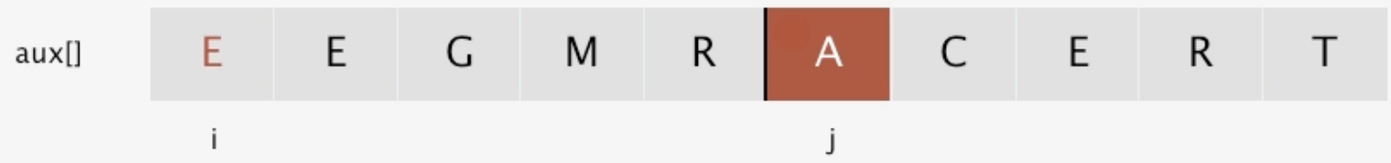


compare minimum in each subarray



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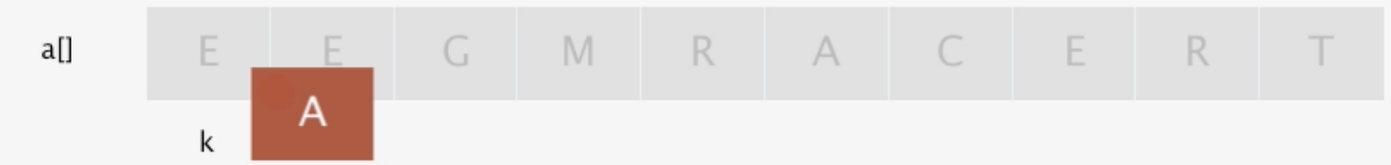

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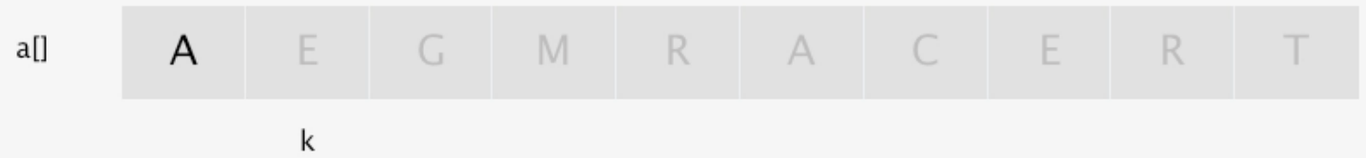

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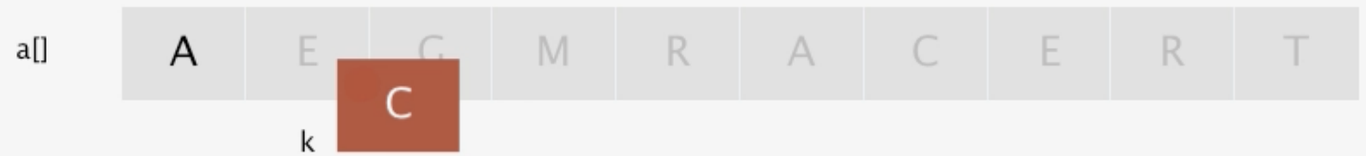

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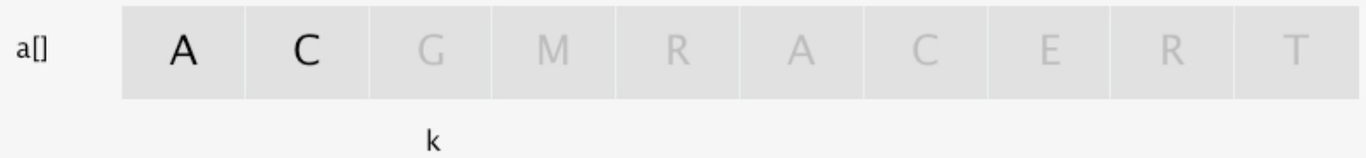

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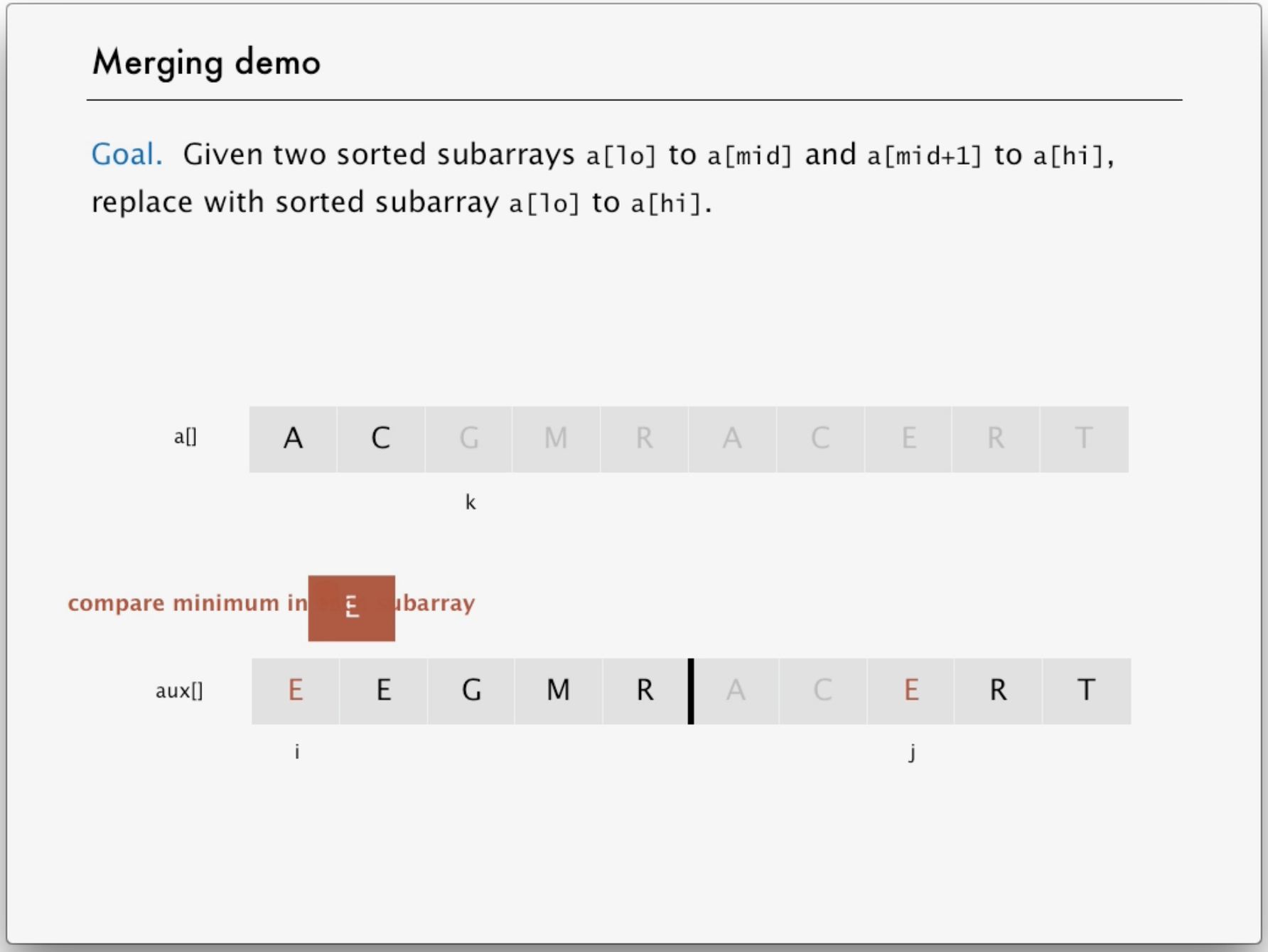
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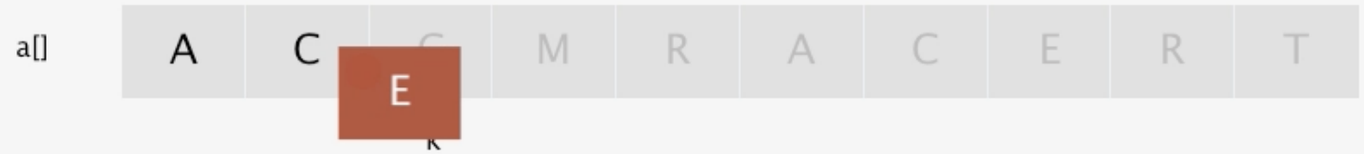

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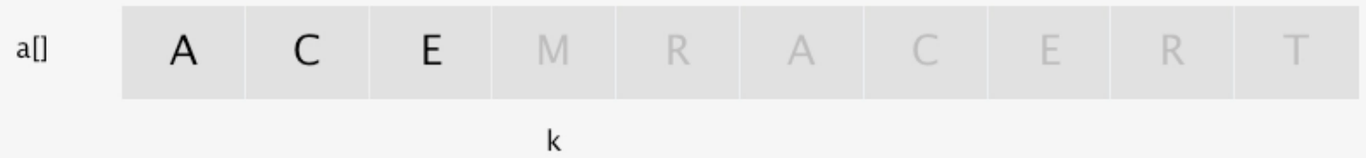

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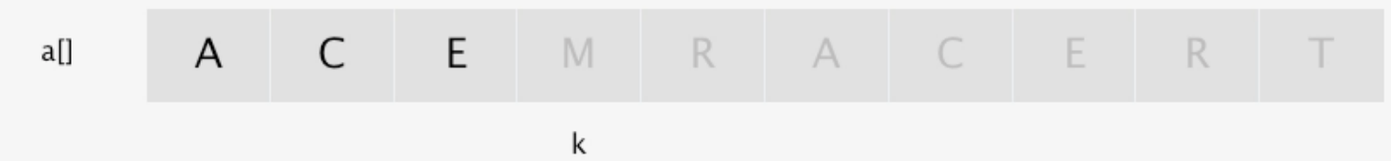

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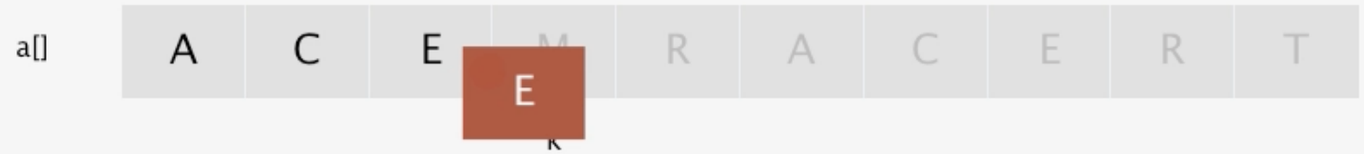

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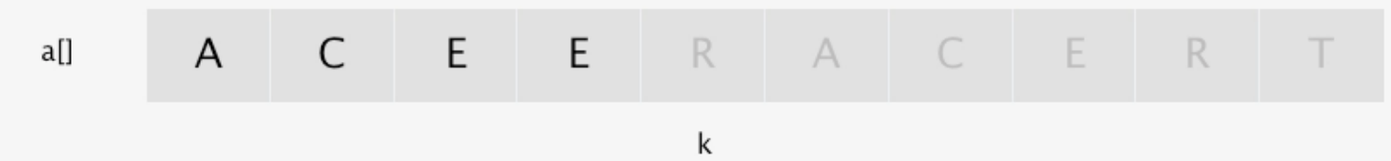

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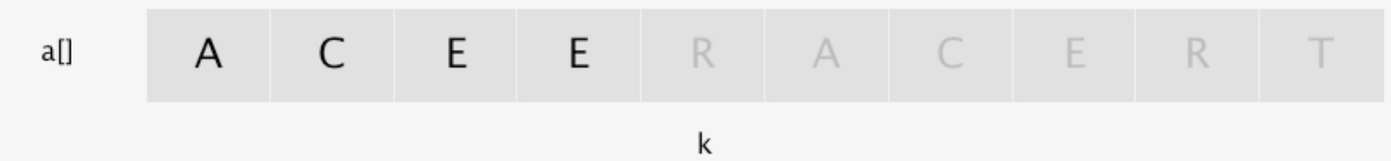

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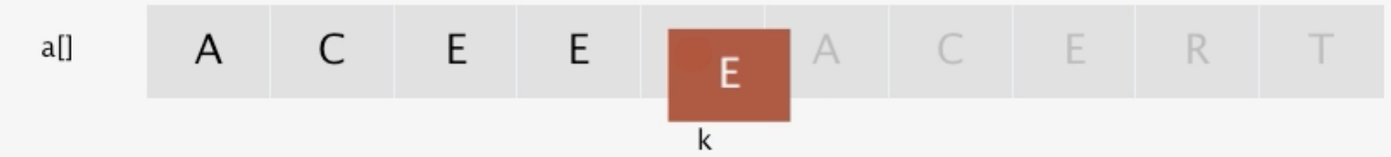

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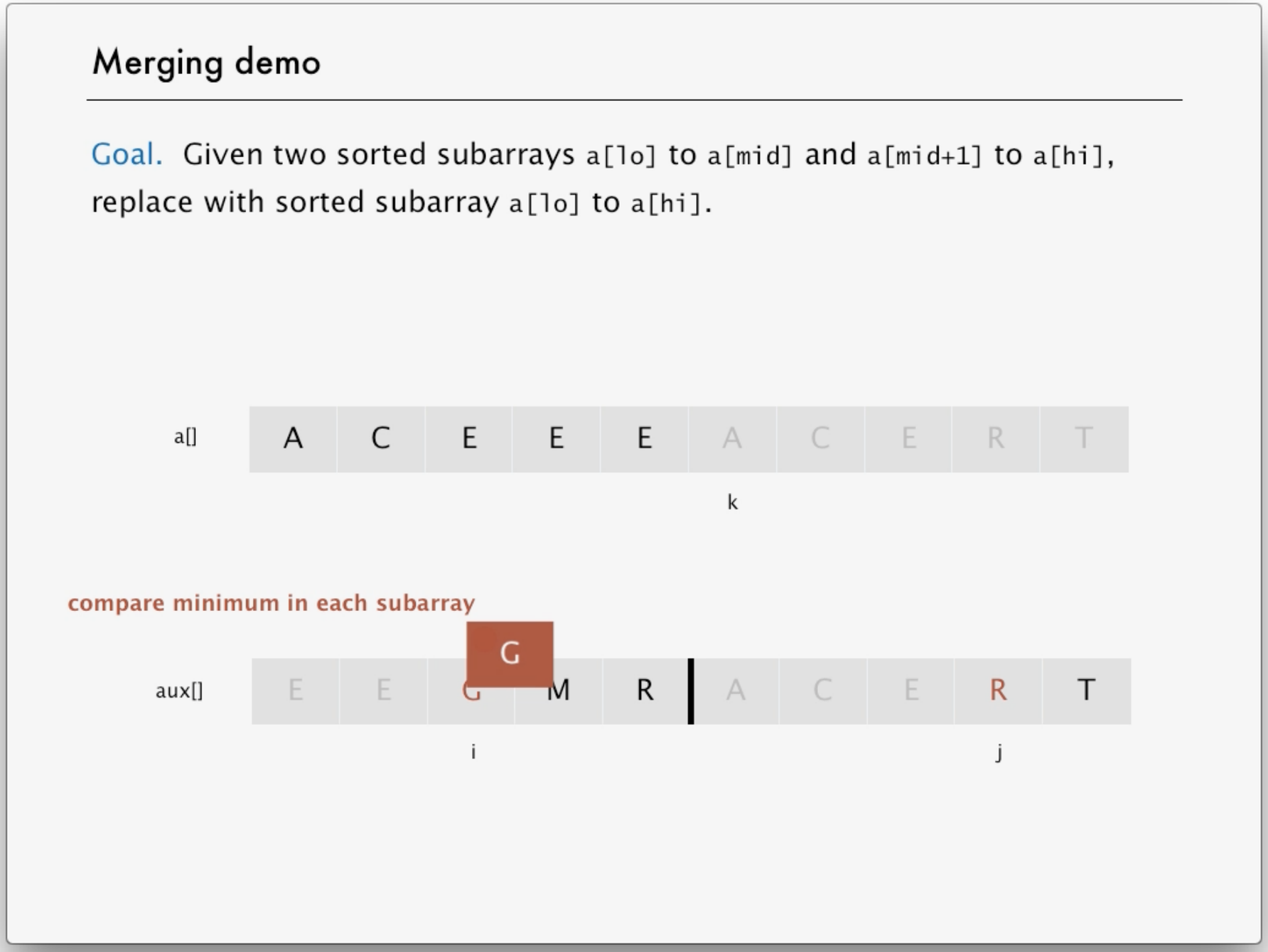
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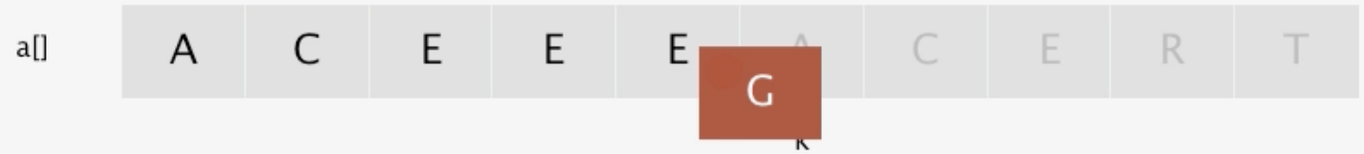

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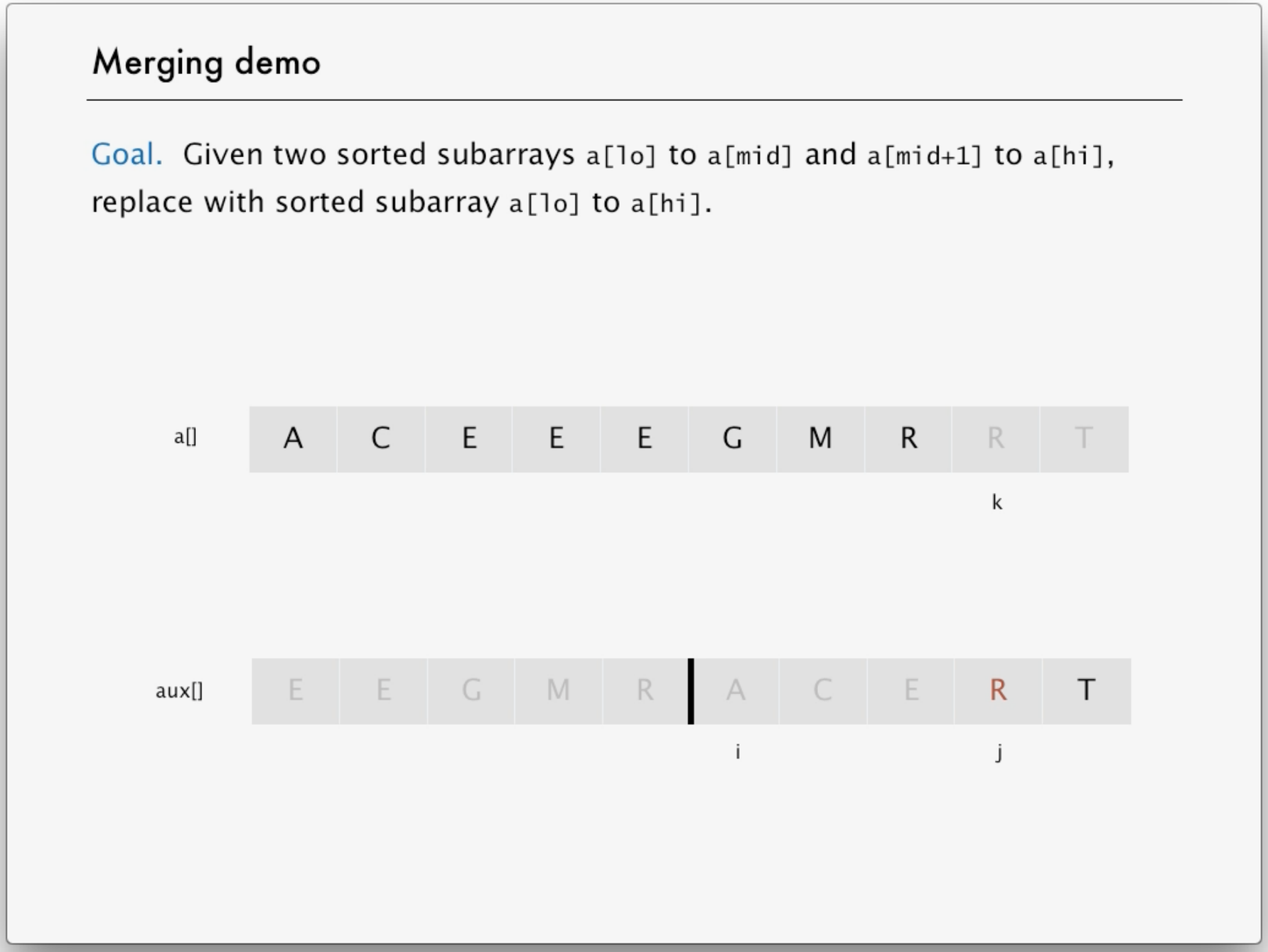
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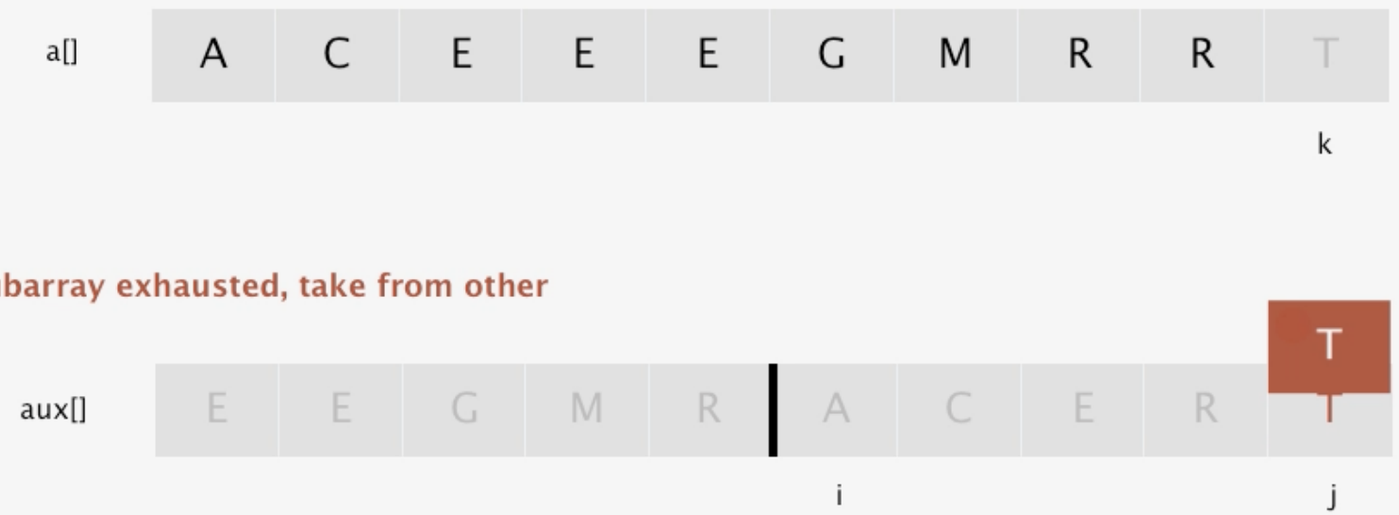

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both subarrays exhausted, done

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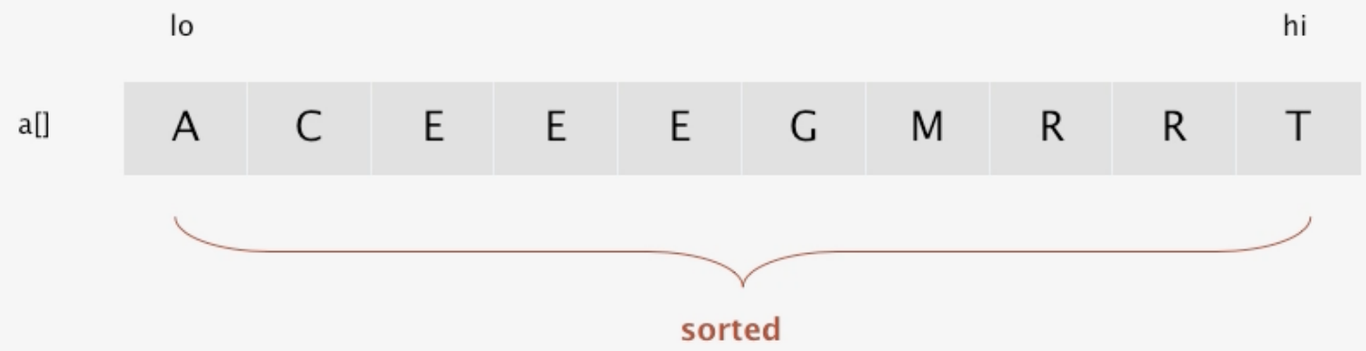
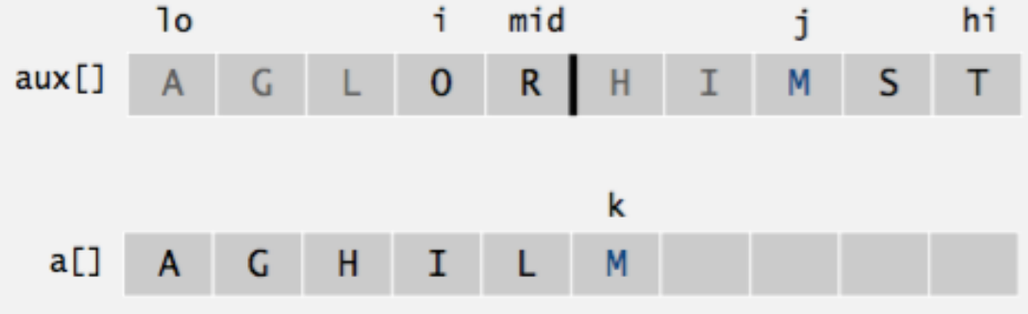


Merge Demo

Merging: Java implementation

```
private static void merge(Comparable[] a, Comparable[] aux, int lo, int mid, int hi)
{
   for (int k = lo; k <= hi; k++)
      aux[k] = a[k];                                                    copy

   int i = lo, j = mid+1;
   for (int k = lo; k <= hi; k++)
   {
      if      (i > mid)                  a[k] = aux[j++];                merge
      else if (j > hi)                   a[k] = aux[i++];
      else if (less(aux[j], aux[i])) a[k] = aux[j++];
      else                               a[k] = aux[i++];
   }
}
```



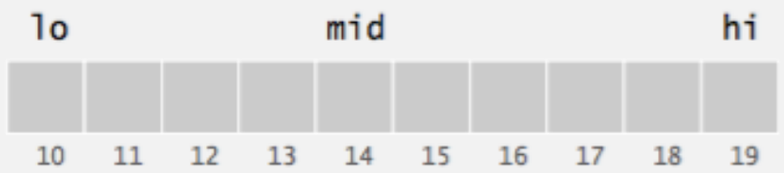
Merge Demo

Mergesort: Java implementation

```
public class Merge
{
   private static void merge(...)
   {  /* as before */  }

   private static void sort(Comparable[] a, Comparable[] aux, int lo, int hi)
   {
      if (hi <= lo) return;
      int mid = lo + (hi - lo) / 2;
      sort(a, aux, lo, mid);
      sort(a, aux, mid+1, hi);
      merge(a, aux, lo, mid, hi);
   }

   public static void sort(Comparable[] a)
   {
      Comparable[] aux = new Comparable[a.length];
      sort(a, aux, 0, a.length - 1);
   }
}
```



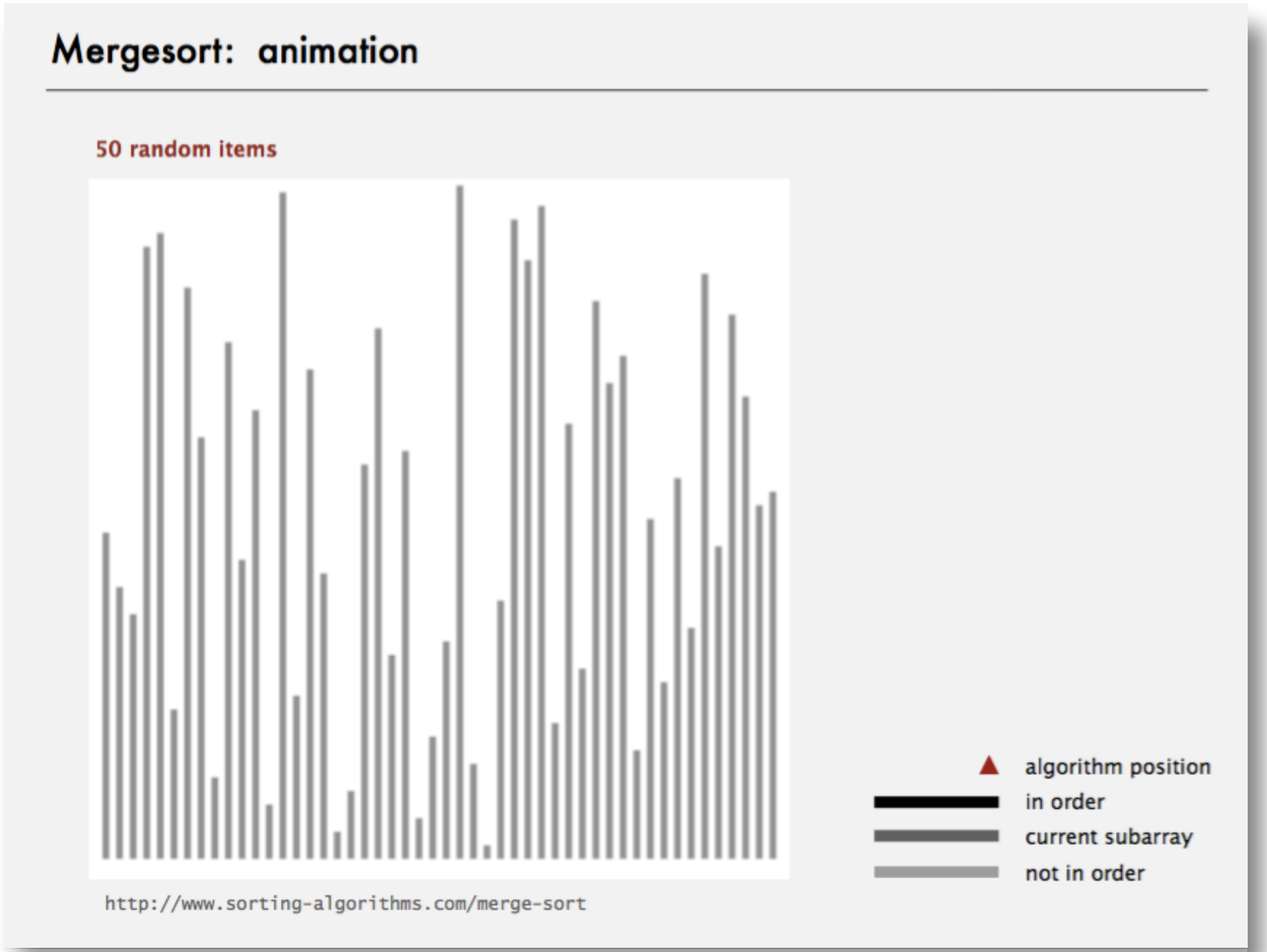
Merge Demo

Mergesort: trace

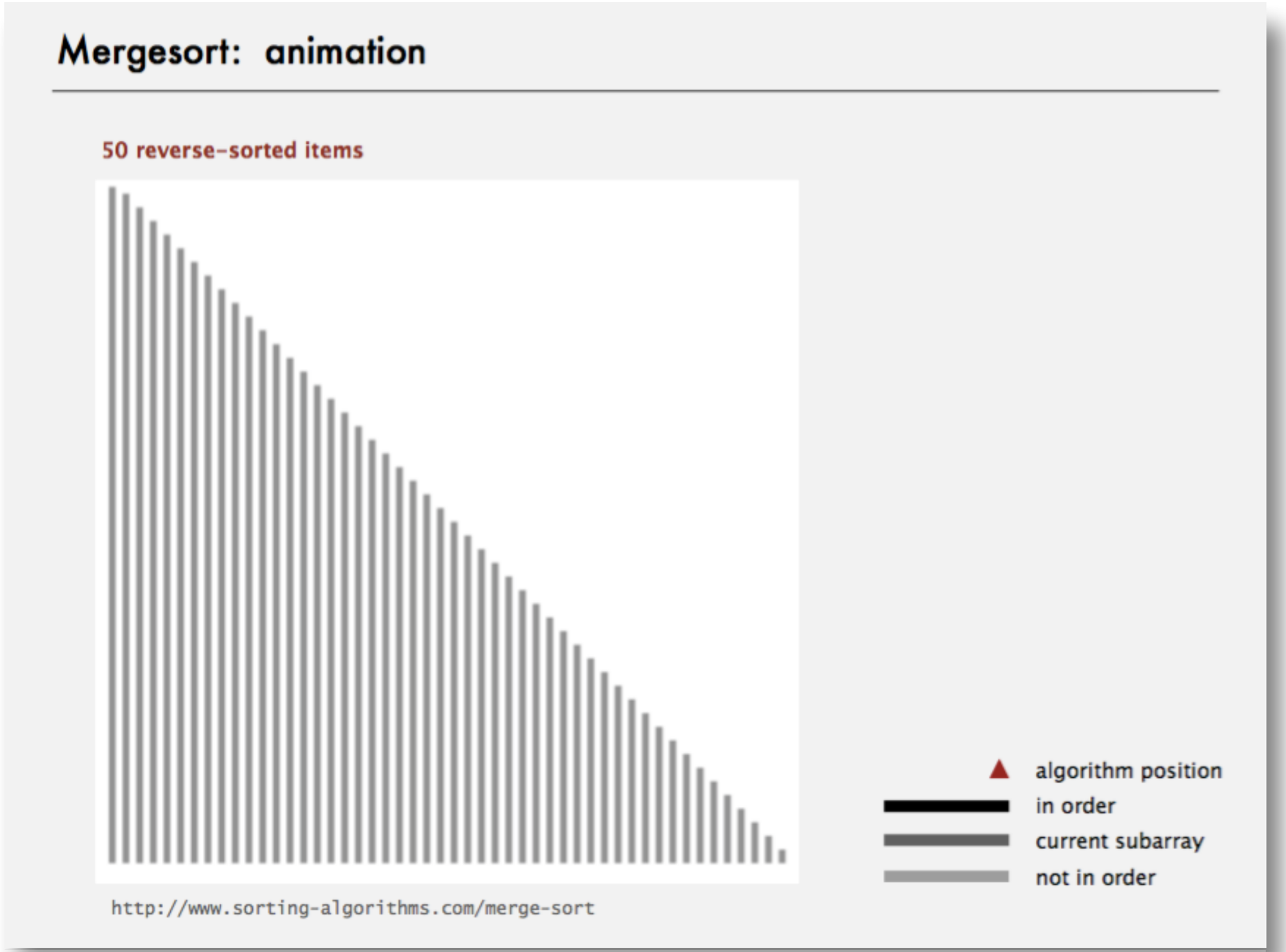
	lo	hi	a[] 0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
			M	E	R	G	E	S	O	R	T	E	X	A	M	P	L	E
merge(a, aux, 0, 0, 1)	0	1	E	M	R	G	E	S	O	R	T	E	X	A	M	P	L	E
merge(a, aux, 2, 2, 3)	2	3	E	M	G	R	E	S	O	R	T	E	X	A	M	P	L	E
merge(a, aux, 0, 1, 3)	0	3	E	G	M	R	E	S	O	R	T	E	X	A	M	P	L	E
merge(a, aux, 4, 4, 5)	4	5	E	G	M	R	E	S	O	R	T	E	X	A	M	P	L	E
merge(a, aux, 6, 6, 7)	6	7	E	G	M	R	E	S	O	R	T	E	X	A	M	P	L	E
merge(a, aux, 4, 5, 7)	4	7	E	G	M	R	E	O	R	S	T	E	X	A	M	P	L	E
merge(a, aux, 0, 3, 7)	0	7	E	E	G	M	O	R	R	S	T	E	X	A	M	P	L	E
merge(a, aux, 8, 8, 9)	8	9	E	E	G	M	O	R	R	S	E	T	X	A	M	P	L	E
merge(a, aux, 10, 10, 11)	10	11	E	E	G	M	O	R	R	S	E	T	A	X	M	P	L	E
merge(a, aux, 8, 9, 11)	8	11	E	E	G	M	O	R	R	S	A	E	T	X	M	P	L	E
merge(a, aux, 12, 12, 13)	12	13	E	E	G	M	O	R	R	S	A	E	T	X	M	P	L	E
merge(a, aux, 14, 14, 15)	14	15	E	E	G	M	O	R	R	S	A	E	T	X	M	P	E	L
merge(a, aux, 12, 13, 15)	12	15	E	E	G	M	O	R	R	S	A	E	T	X	E	L	M	P
merge(a, aux, 8, 11, 15)	8	15	E	E	G	M	O	R	R	S	A	E	E	L	M	P	T	X
merge(a, aux, 0, 7, 15)	0	15	A	E	E	E	E	G	L	M	M	O	P	R	R	S	T	X

result after recursive call

Merge Demo


Mergesort: animation
50 random items
algorithm position
in order
current subarray
not in order
http://www.sorting-algorithms.com/merge-sort

Merge Demo


Mergesort: animation
50 reverse-sorted items
algorithm position
in order
current subarray
not in order
http://www.sorting-algorithms.com/merge-sort

Merge Demo

Mergesort: empirical analysis

Running time estimates:

- Laptop executes 10^8 compares/second.
- Supercomputer executes 10^{12} compares/second.

	insertion sort (N^2)			mergesort (N log N)		
computer	thousand	million	billion	thousand	million	billion
home	instant	2.8 hours	317 years	instant	1 second	18 min
super	instant	1 second	1 week	instant	instant	instant

Bottom line. Good algorithms are better than supercomputers.

Merge Demo

Mergesort: number of compares

Proposition. Mergesort uses $\leq N \lg N$ compares to sort an array of length N.

Pf sketch. The number of compares $C(N)$ to mergesort an array of length N satisfies the recurrence:

$$C(N) \leq C(\lceil N/2 \rceil) + C(\lfloor N/2 \rfloor) + N \quad \text{for } N > 1, \text{ with } C(1) = 0.$$

left half — right half — merge

We solve the recurrence when N is a power of 2: ← result holds for all N (analysis cleaner in this case)

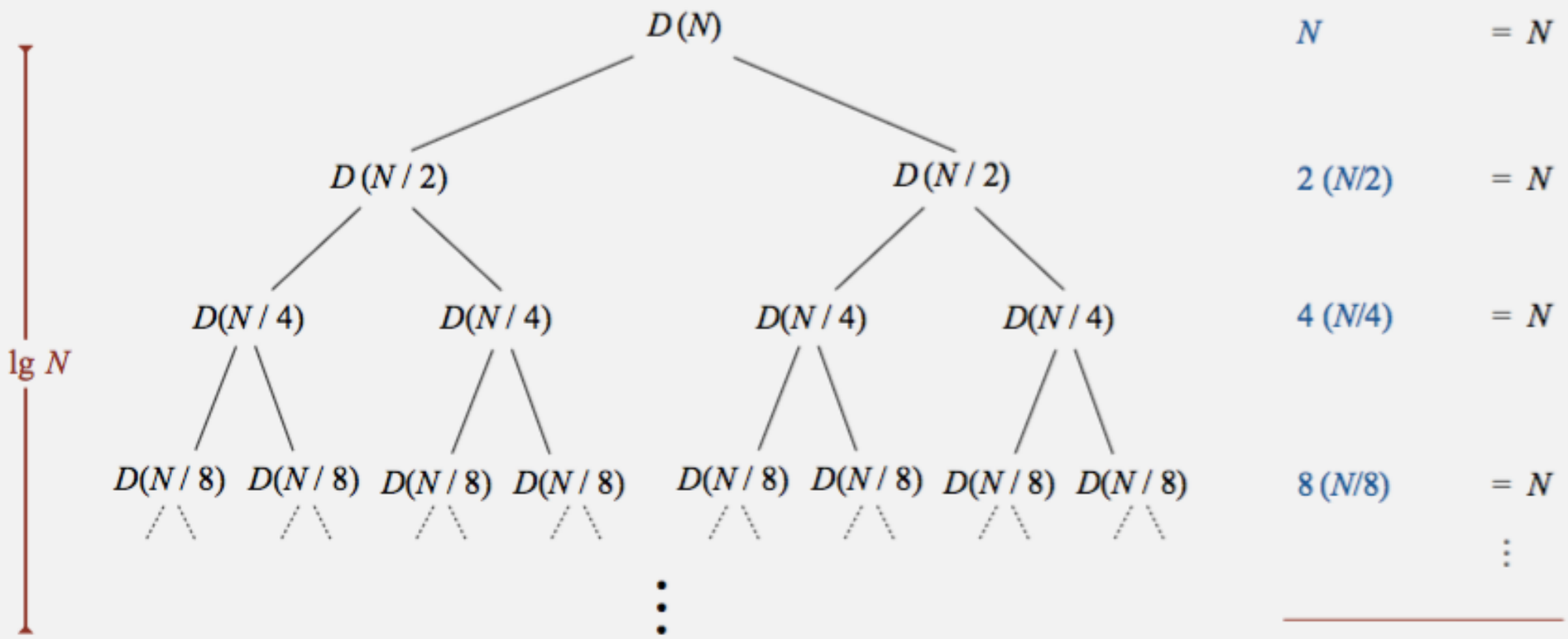
$$D(N) = 2\,D(N/2) + N, \text{ for } N > 1, \text{ with } D(1) = 0.$$

Merge Demo

Divide-and-conquer recurrence: proof by picture

Proposition. If $D(N)$ satisfies $D(N) = 2\,D(N/2) + N$ for $N > 1$, with $D(1) = 0$, then $D(N) = N \lg N$.

Pf 1. [assuming N is a power of 2]



Merge Demo

Divide-and-conquer recurrence: proof by induction

Proposition. If $D(N)$ satisfies $D(N) = 2\,D(N/2) + N$ for $N > 1$, with $D(1) = 0$, then $D(N) = N \lg N$.

Pf 2. [assuming N is a power of 2]

- Base case: $N = 1$.
- Inductive hypothesis: $D(N) = N \lg N$.
- Goal: show that $D(2N) = (2N) \lg (2N)$.

$$
\begin{aligned}
D(2N) &= 2\,D(N) + 2N && \text{given} \\
&= 2\,N \lg N + 2N && \text{inductive hypothesis} \\
&= 2\,N\,(\lg (2N) - 1) + 2N && \text{algebra} \\
&= 2\,N \lg (2N) && \text{QED}
\end{aligned}
$$

Merge Demo

Mergesort: number of array accesses

Proposition. Mergesort uses $\leq 6 N \lg N$ array accesses to sort an array of length N.

Pf sketch. The number of array accesses $A(N)$ satisfies the recurrence:

$$A(N) \leq A(\lceil N/2 \rceil) + A(\lfloor N/2 \rfloor) + 6N \text{ for } N > 1, \text{ with } A(1) = 0.$$

Key point. Any algorithm with the following structure takes $N \log N$ time:

```
public static void linearithmic(int N)
{
    if (N == 0) return;
    linearithmic(N/2);   <-- solve two problems
    linearithmic(N/2);   <-- of half the size
    linear(N);           <-- do a linear amount of work
}
```

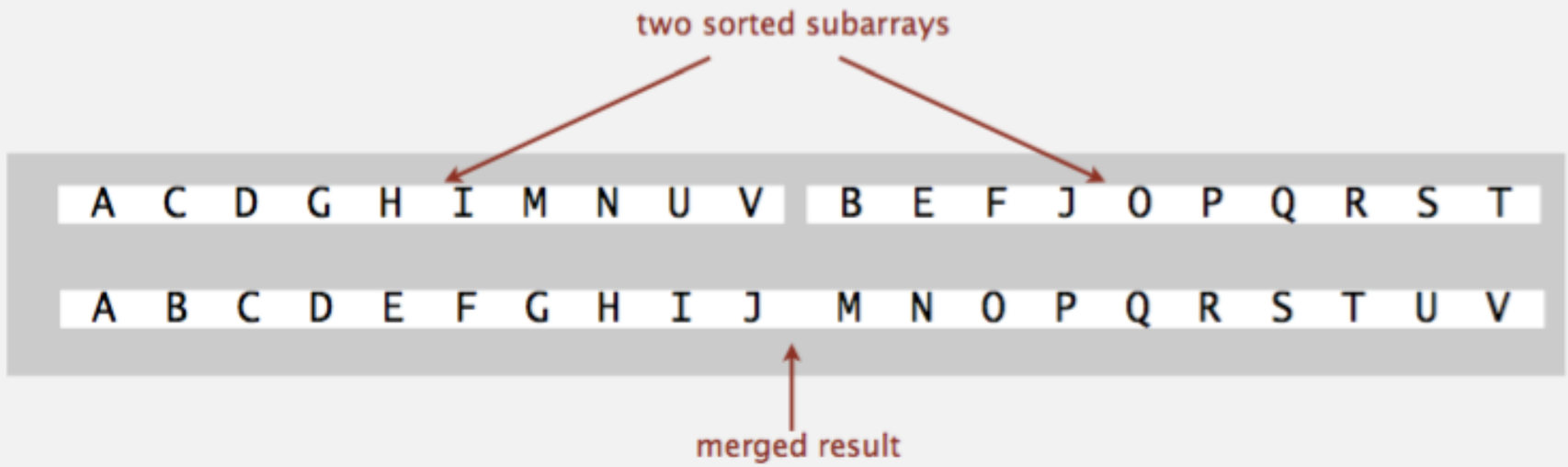
Notable examples. FFT, hidden-line removal, Kendall-tau distance, ...

Merge Demo

Mergesort analysis: memory

Proposition. Mergesort uses extra space proportional to N.

Pf. The array `aux[]` needs to be of length N for the last merge.



Def. A sorting algorithm is in-place if it uses $\leq c \log N$ extra memory.

Ex. Insertion sort, selection sort, shellsort.

Challenge 1 (not hard). Use `aux[]` array of length $\sim \frac{1}{2} N$ instead of N.

Challenge 2 (very hard). In-place merge. [Kronrod 1969]

Merge Demo

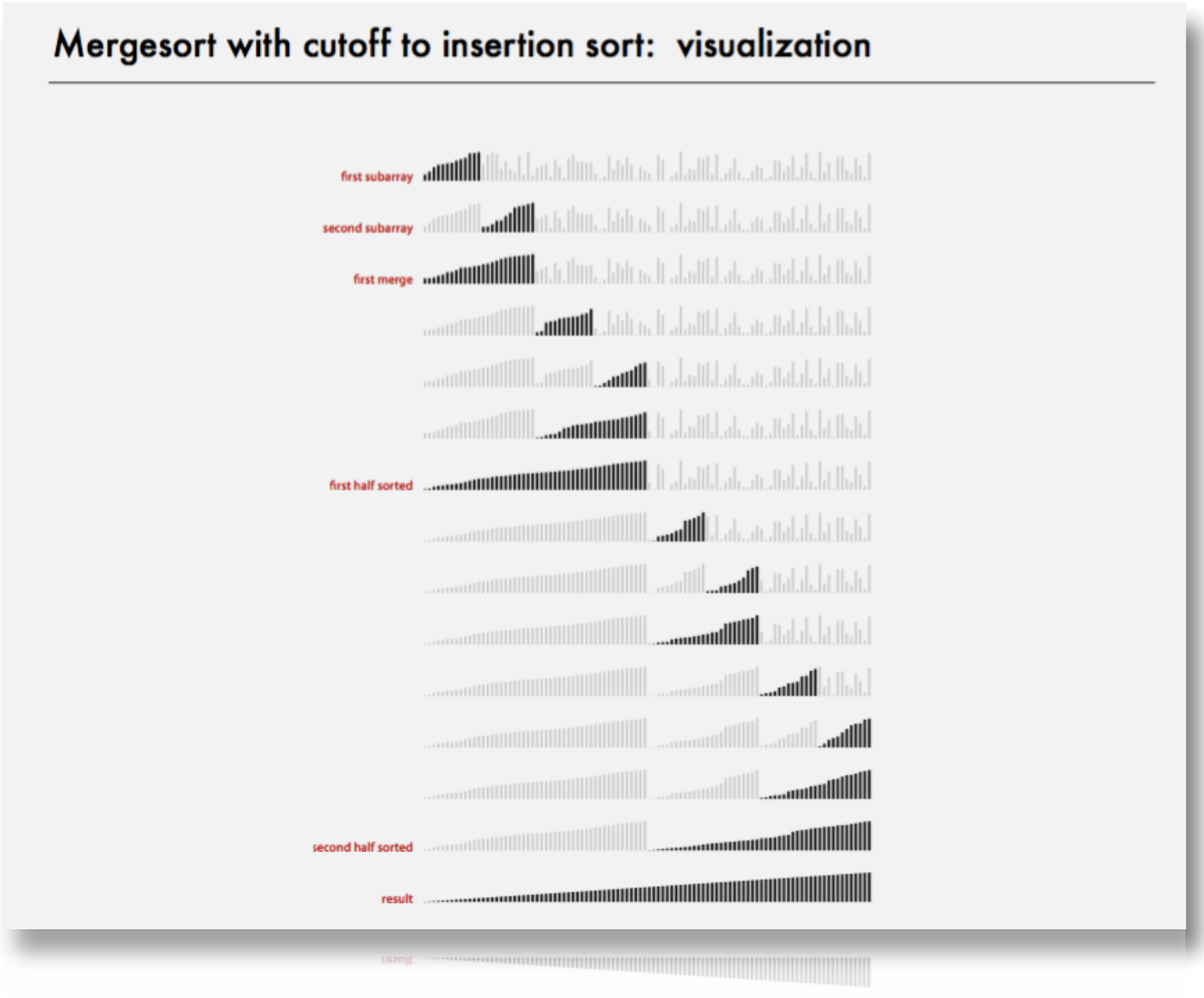
Mergesort: practical improvements

Use insertion sort for small subarrays.

- Mergesort has too much overhead for tiny subarrays.
- Cutoff to insertion sort for ≈ 10 items.

```
private static void sort(Comparable[] a, Comparable[] aux, int lo, int hi)
{
   if (hi <= lo + CUTOFF - 1)
   {
      Insertion.sort(a, lo, hi);
      return;
   }
   int mid = lo + (hi - lo) / 2;
   sort (a, aux, lo, mid);
   sort (a, aux, mid+1, hi);
   merge(a, aux, lo, mid, hi);
}
```

Merge Demo

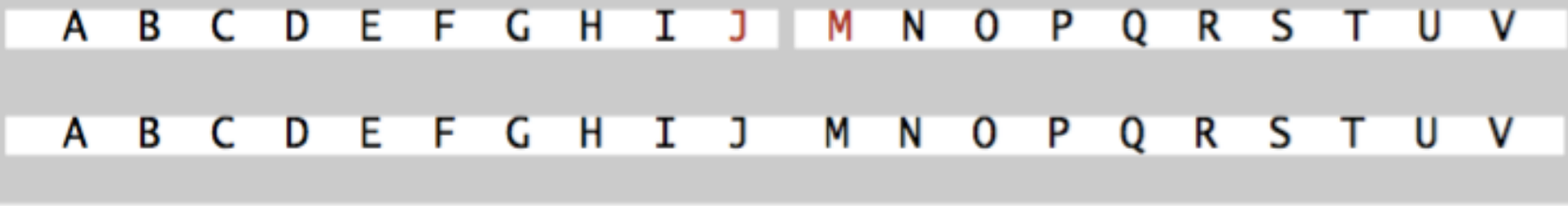

Mergesort with cutoff to insertion sort: visualization
first subarray
second subarray
first merge
first half sorted
second half sorted
result

Merge Demo

Mergesort: practical improvements

Stop if already sorted.

- Is largest item in first half ≤ smallest item in second half?
- Helps for partially-ordered arrays.




```
private static void sort(Comparable[] a, Comparable[] aux, int lo, int hi)
{
   if (hi <= lo) return;
   int mid = lo + (hi - lo) / 2;
   sort (a, aux, lo, mid);
   sort (a, aux, mid+1, hi);
   if (!less(a[mid+1], a[mid])) return;
   merge(a, aux, lo, mid, hi);
}
```

Merge Demo

Mergesort: practical improvements

Eliminate the copy to the auxiliary array. Save time (but not space)
by switching the role of the input and auxiliary array in each recursive call.

```
private static void merge(Comparable[] a, Comparable[] aux, int lo, int mid, int hi)
{
   int i = lo, j = mid+1;
   for (int k = lo; k <= hi; k++)
   {
      if      (i > mid)           aux[k] = a[j++];
      else if (j > hi)            aux[k] = a[i++];
      else if (less(a[j], a[i])) aux[k] = a[j++];
      else                        aux[k] = a[i++];
   }
}
```

← merge from a[] to aux[]

```
private static void sort(Comparable[] a, Comparable[] aux, int lo, int hi)
{
   if (hi <= lo) return;
   int mid = lo + (hi - lo) / 2;
   sort (aux, a, lo, mid);
   sort (aux, a, mid+1, hi);
   merge(a, aux, lo, mid, hi);
}
```

↑ assumes aux[] is initialize to a[] once, before recursive calls

↑ switch roles of aux[] and a[]

Merge Demo

Java 6 system sort

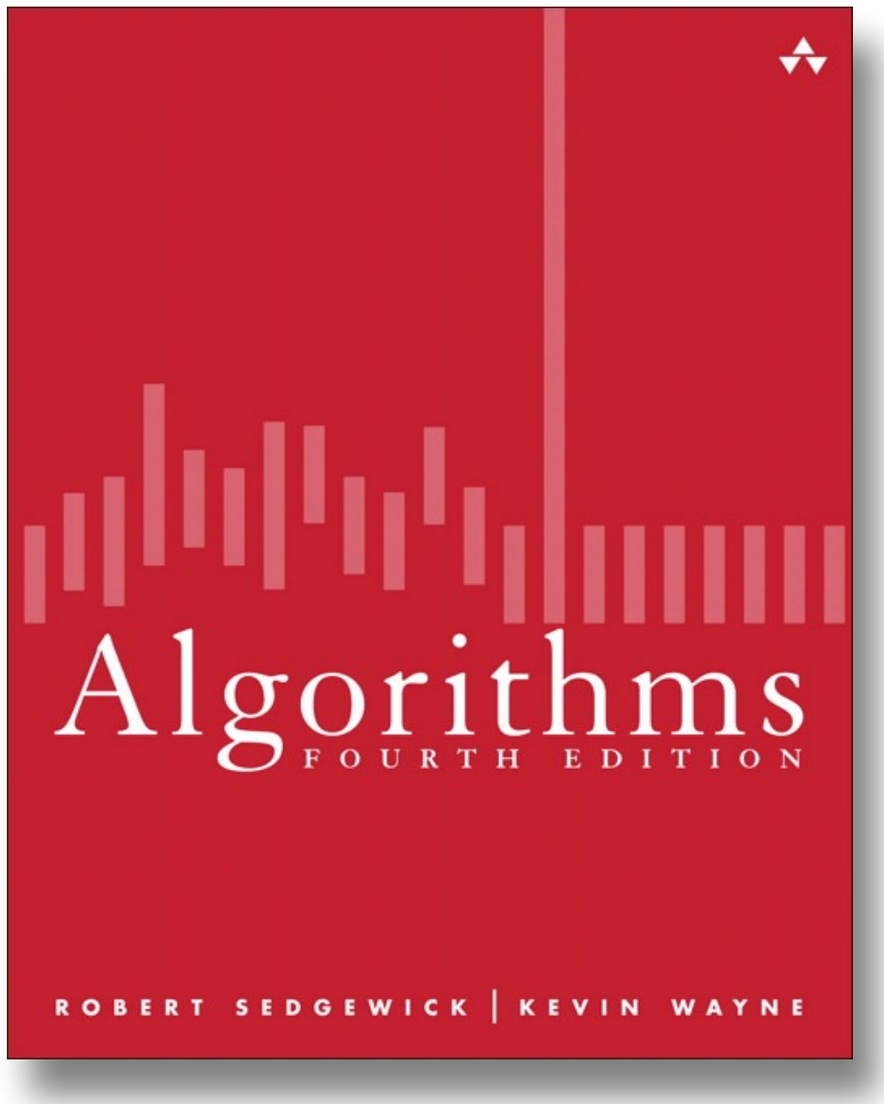
Basic algorithm for sorting objects = mergesort.

- Cutoff to insertion sort = 7.
- Stop-if-already-sorted test.
- Eliminate-the-copy-to-the-auxiliary-array trick.

Arrays.sort(a)

http://www.java2s.com/Open-Source/Java/6.0-JDK-Modules/j2me/java/util/Arrays.java.html

Lecture Overview



2.2 Mergesort

- Mergesort
- Bottom-up Mergesort
- Sorting complexity
- Comparators
- Stability

Bottom-up Mergesort

Bottom-up mergesort. Even though we are thinking in terms of merging together two large subarrays, the fact is that most merges are merging together tiny subarrays. Another way to implement mergesort is to organize the merges so that we do all the merges of tiny arrays on one pass, then do a second pass to merge those arrays in pairs, and so forth, continuing until we do a merge that encompasses the whole array. This method requires even less code than the standard recursive implementation. We start by doing a pass of 1-by-1 merges (considering individual items as subarrays of size 1), then a pass of 2-by-2 merges (merge subarrays of size 2 to make subarrays of size 4), then 4-by-4 merges, and so forth. MergeBU.java is an implementation of bottom-up mergesort.

Merge Demo

Bottom-up mergesort

Basic plan.

- Pass through array, merging subarrays of size 1.
- Repeat for subarrays of size 2, 4, 8, ...

	a[i]															
	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
	M	E	R	G	E	S	O	R	T	E	X	A	M	P	L	E
sz = 1																
merge(a, aux, 0, 0, 1)	E	M	R	G	E	S	O	R	T	E	X	A	M	P	L	E
merge(a, aux, 2, 2, 3)	E	M	G	R	E	S	O	R	T	E	X	A	M	P	L	E
merge(a, aux, 4, 4, 5)	E	M	G	R	E	S	O	R	T	E	X	A	M	P	L	E
merge(a, aux, 6, 6, 7)	E	M	G	R	E	S	O	R	T	E	X	A	M	P	L	E
merge(a, aux, 8, 8, 9)	E	M	G	R	E	S	O	R	E	T	X	A	M	P	L	E
merge(a, aux, 10, 10, 11)	E	M	G	R	E	S	O	R	E	T	A	X	M	P	L	E
merge(a, aux, 12, 12, 13)	E	M	G	R	E	S	O	R	E	T	A	X	M	P	L	E
merge(a, aux, 14, 14, 15)	E	M	G	R	E	S	O	R	E	T	A	X	M	P	E	L
sz = 2																
merge(a, aux, 0, 1, 3)	E	G	M	R	E	S	O	R	E	T	A	X	M	P	E	L
merge(a, aux, 4, 5, 7)	E	G	M	R	E	O	R	S	E	T	A	X	M	P	E	L
merge(a, aux, 8, 9, 11)	E	G	M	R	E	O	R	S	A	E	T	X	M	P	E	L
merge(a, aux, 12, 13, 15)	E	G	M	R	E	O	R	S	A	E	T	X	E	L	M	P
sz = 4																
merge(a, aux, 0, 3, 7)	E	E	G	M	O	R	R	S	A	E	T	X	E	L	M	P
merge(a, aux, 8, 11, 15)	E	E	G	M	O	R	R	S	A	E	E	L	M	P	T	X
sz = 8																
merge(a, aux, 0, 7, 15)	A	E	E	E	E	G	L	M	M	O	P	R	R	S	T	X

Merge Demo

Bottom-up mergesort: Java implementation

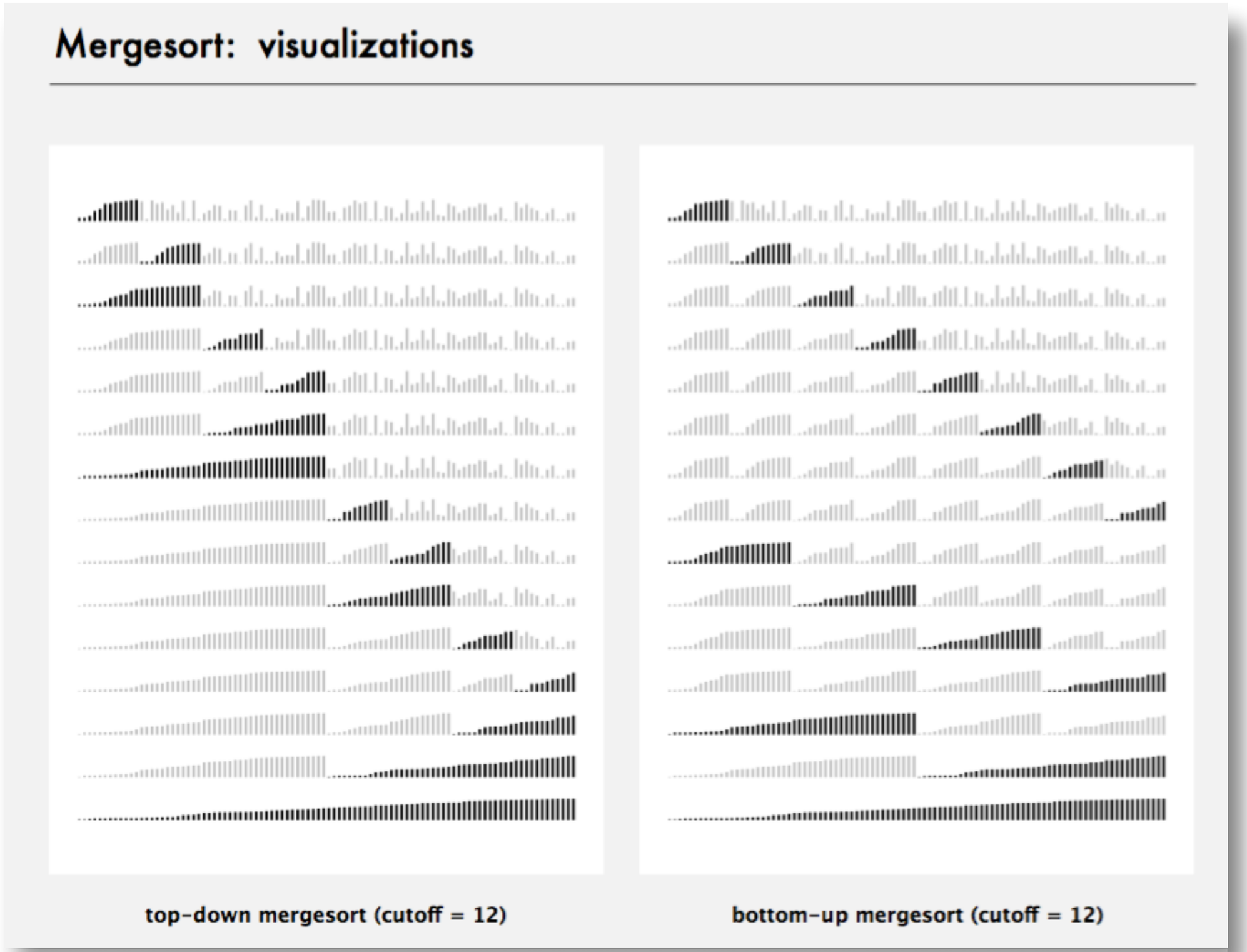
```
public class MergeBU
{
   private static void merge(...)
   {  /* as before */  }

   public static void sort(Comparable[] a)
   {
      int N = a.length;
      Comparable[] aux = new Comparable[N];
      for (int sz = 1; sz < N; sz = sz+sz)
         for (int lo = 0; lo < N-sz; lo += sz+sz)
            merge(a, aux, lo, lo+sz-1, Math.min(lo+sz+sz-1, N-1));
   }
}
```

but about 10% slower than recursive, top-down mergesort on typical systems

Bottom line. Simple and non-recursive version of mergesort.

Merge Demo


Mergesort: visualizations
top-down mergesort (cutoff = 12)
bottom-up mergesort (cutoff = 12)

Merge Demo

Natural mergesort

Idea. Exploit pre-existing order by identifying naturally-occurring runs.

input

1	5	10	16	3	4	23	9	13	2	7	8	12	14

first run

1	5	10	16	3	4	23	9	13	2	7	8	12	14

second run

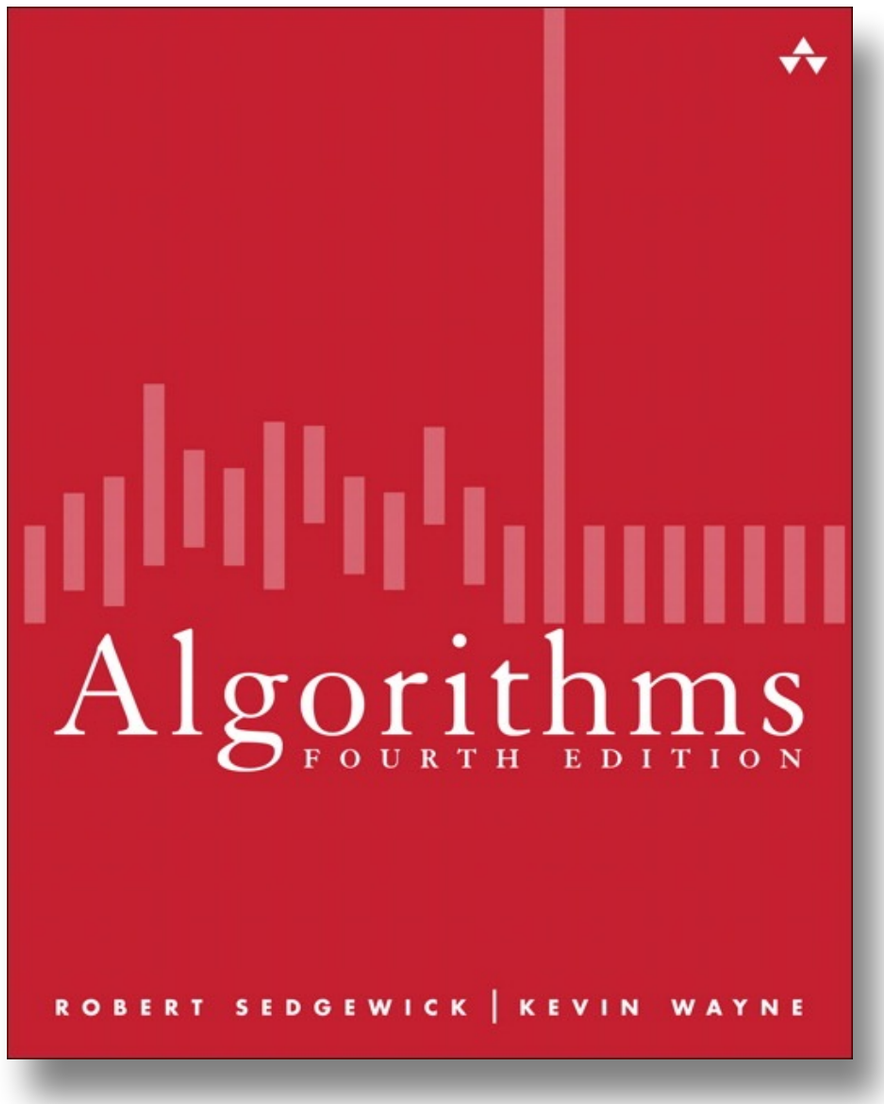
1	5	10	16	3	4	23	9	13	2	7	8	12	14

merge two runs

1	3	4	5	10	16	23	9	13	2	7	8	12	14

Tradeoff. Fewer passes vs. extra compares per pass to identify runs.

Lecture Overview



2.2 Mergesort

- Mergesort
- Bottom-up Mergesort
- Sorting complexity
- Comparators
- Stability

Sorting Complexity

Complexity of sorting

Computational complexity. Framework to study efficiency of algorithms for solving a particular problem X.

Model of computation. Allowable operations.

Cost model. Operation count(s).

Upper bound. Cost guarantee provided by some algorithm for X.

Lower bound. Proven limit on cost guarantee of all algorithms for X.

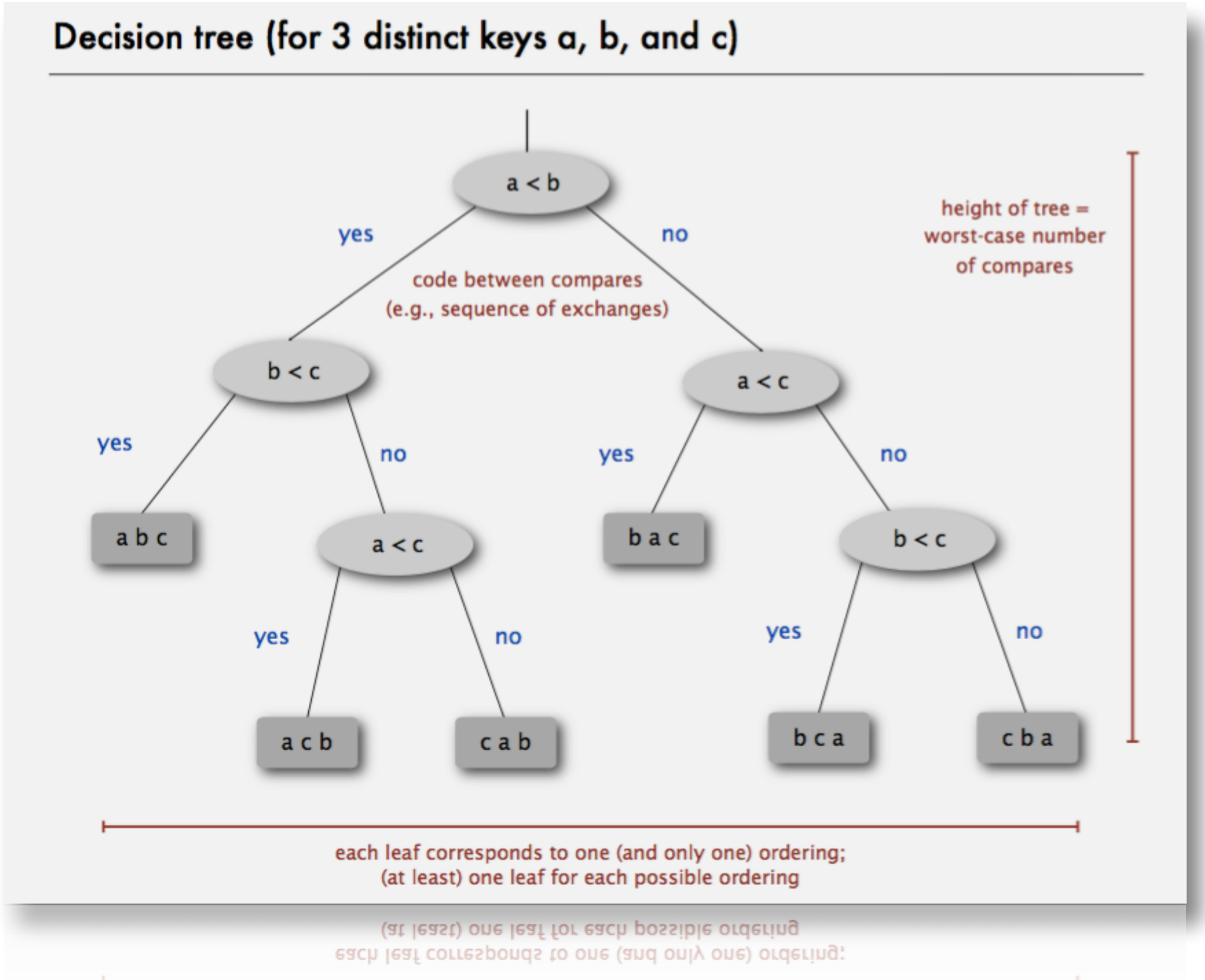
Optimal algorithm. Algorithm with best possible cost guarantee for X.

lower bound ~ upper bound

Example: sorting.

- Model of computation: decision tree. ← can access information only through compares (e.g., Java Comparable framework)
- Cost model: # compares.
- Upper bound: $\sim N \lg N$ from mergesort.
- Lower bound:
- Optimal algorithm:

Sorting Complexity



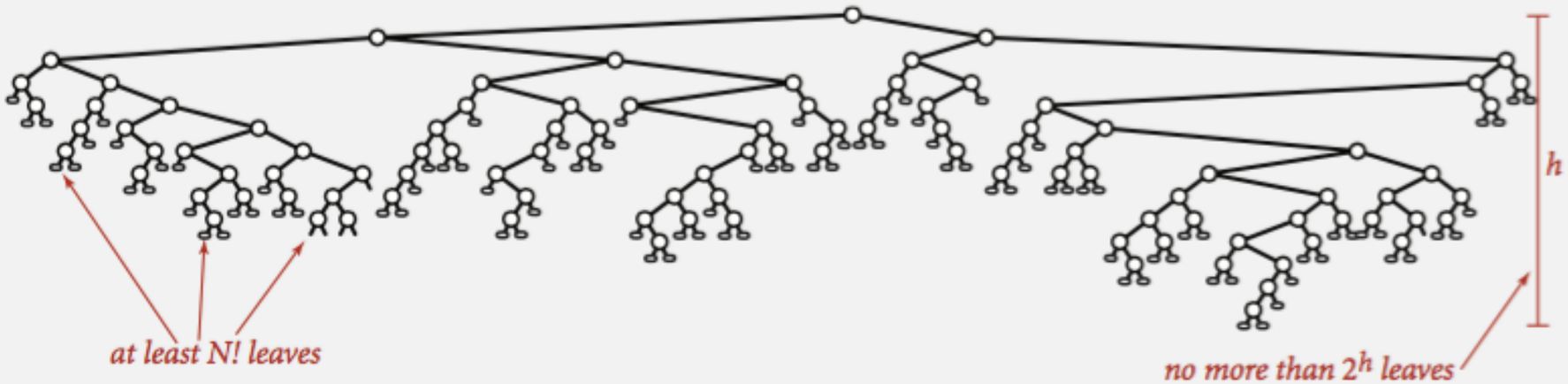
Sorting Complexity

Compare-based lower bound for sorting

Proposition. Any compare-based sorting algorithm must use at least $\lg(N!) \sim N \lg N$ compares in the worst-case.

Pf.

- Assume array consists of N distinct values a_1 through a_N.
- Worst case dictated by height h of decision tree.
- Binary tree of height h has at most 2^h leaves.
- $N!$ different orderings $\Rightarrow$ at least $N!$ leaves.



Sorting Complexity

Compare-based lower bound for sorting

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Pf.

- Assume array consists of N distinct values a_1 through a_N.
- Worst case dictated by height h of decision tree.
- Binary tree of height h has at most 2^h leaves.
- $N!$ different orderings $\Rightarrow$ at least $N!$ leaves.

$$2^h \geq \text{\# leaves} \geq N!$$

$$\Rightarrow h \geq \lg(N!) \sim N \lg N$$

Stirling's formula

Sorting Complexity

Complexity of sorting

Model of computation. Allowable operations.

Cost model. Operation count(s).

Upper bound. Cost guarantee provided by some algorithm for X.

Lower bound. Proven limit on cost guarantee of all algorithms for X.

Optimal algorithm. Algorithm with best possible cost guarantee for X.

Example: sorting.

- Model of computation: decision tree.
- Cost model: # compares.
- Upper bound: $\sim N \lg N$ from mergesort.
- Lower bound: $\sim N \lg N$.
- Optimal algorithm = mergesort.

First goal of algorithm design: optimal algorithms.

Sorting Complexity

Complexity results in context

Compares? Mergesort is optimal with respect to number compares.

Space? Mergesort is not optimal with respect to space usage.

Lessons. Use theory as a guide.

Ex. Design sorting algorithm that guarantees $\frac{1}{2} N \lg N$ compares?

Ex. Design sorting algorithm that is both time- and space-optimal?

Sorting Complexity

Complexity results in context (continued)

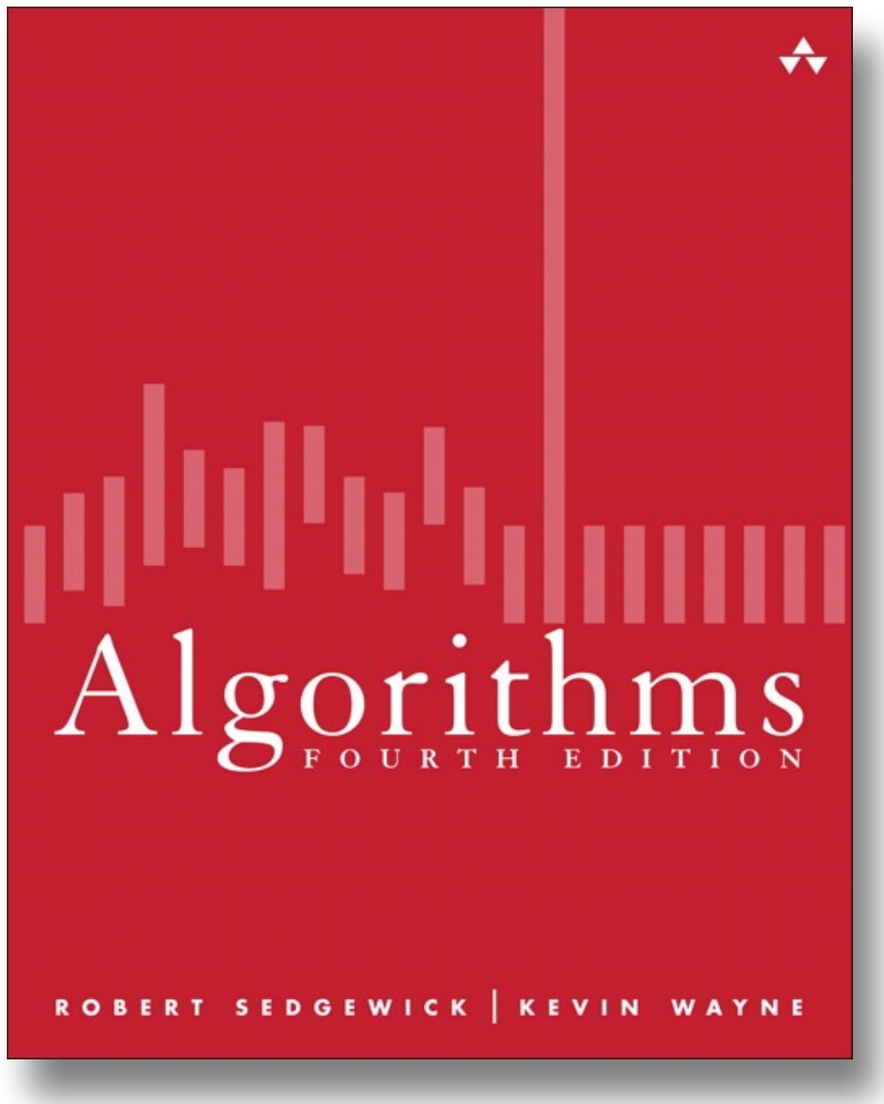
Lower bound may not hold if the algorithm can take advantage of:

- The initial order of the input.
  Ex: insert sort requires only a linear number of compares on partially-sorted arrays.

- The distribution of key values.
  Ex: 3-way quicksort requires only a linear number of compares on arrays with a constant number of distinct keys. [stay tuned]

- The representation of the keys.
  Ex: radix sort requires no key compares — it accesses the data via character/digit compares.

Lecture Overview



2.2 Mergesort

- Mergesort
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Comparators

Comparable interface: review

Comparable interface: sort using a type's natural order.

```
public class Date implements Comparable<Date>
{
   private final int month, day, year;

   public Date(int m, int d, int y)
   {
      month = m;
      day   = d;
      year  = y;
   }
   ...
   public int compareTo(Date that)
   {
      if (this.year  < that.year ) return -1;
      if (this.year  > that.year ) return +1;
      if (this.month < that.month) return -1;
      if (this.month > that.month) return +1;
      if (this.day   < that.day  ) return -1;
      if (this.day   > that.day  ) return +1;
      return 0;
   }
}
```

natural order

Comparators

Comparator interface

Comparator interface: sort using an alternate order.

```
public interface Comparator<Key>
```

`int compare(Key v, Key w)`	*compare keys v and w*

Required property. Must be a total order.

string order	example
natural order	Now is the time
case insensitive	is Now the time
Spanish language	café cafetero cuarto churro nube ñoño
British phone book	McKinley Mackintosh

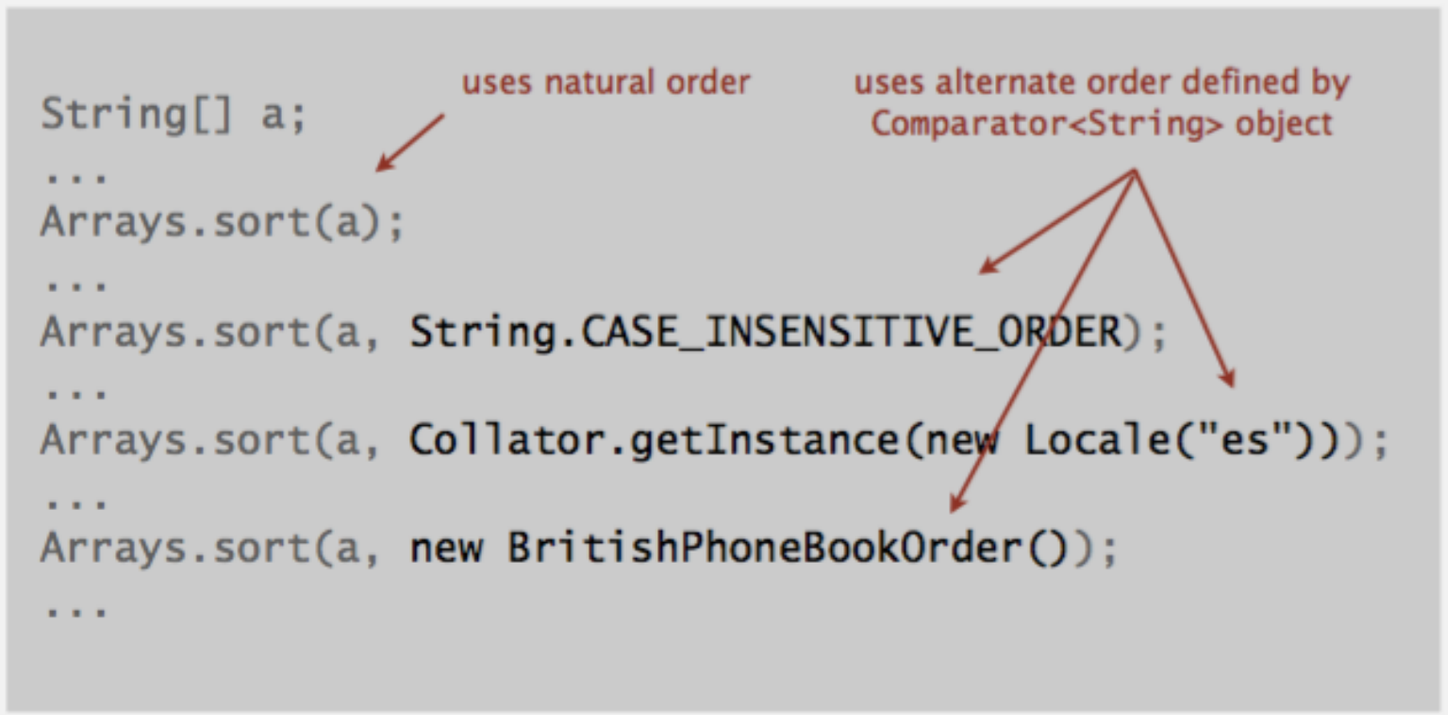
pre-1994 order for digraphs ch and ll and rr

Comparators

Comparator interface: system sort

To use with Java system sort:

- Create `Comparator` object.
- Pass as second argument to `Arrays.sort()`.



Bottom line. Decouples the definition of the data type from the definition of what it means to compare two objects of that type.

Comparators

Comparator interface: using with our sorting libraries

To support comparators in our sort implementations:

- Use `Object` instead of `Comparable`.
- Pass `Comparator` to `sort()` and `less()` and use it in `less()`.

insertion sort using a Comparator

```
public static void sort(Object[] a, Comparator comparator)
{
   int N = a.length;
   for (int i = 0; i < N; i++)
      for (int j = i; j > 0 && less(comparator, a[j], a[j-1]); j--)
         exch(a, j, j-1);
}

private static boolean less(Comparator c, Object v, Object w)
{  return c.compare(v, w) < 0;   }

private static void exch(Object[] a, int i, int j)
{  Object swap = a[i]; a[i] = a[j]; a[j] = swap;  }
```

Comparators

Comparator interface: implementing

To implement a comparator:

- Define a (nested) class that implements the `Comparator` interface.
- Implement the `compare()` method.

```
public class Student
{
   private final String name;
   private final int section;
   ...

   public static class ByName implements Comparator<Student>
   {
     public int compare(Student v, Student w)
      {  return v.name.compareTo(w.name);  }
   }

   public static class BySection implements Comparator<Student>
   {
      public int compare(Student v, Student w)
      {  return v.section - w.section;  }
   }
}
```

this trick works here
since no danger of overflow

Comparators

Comparator interface: implementing

To implement a comparator:

- Define a (nested) class that implements the `Comparator` interface.
- Implement the `compare()` method.

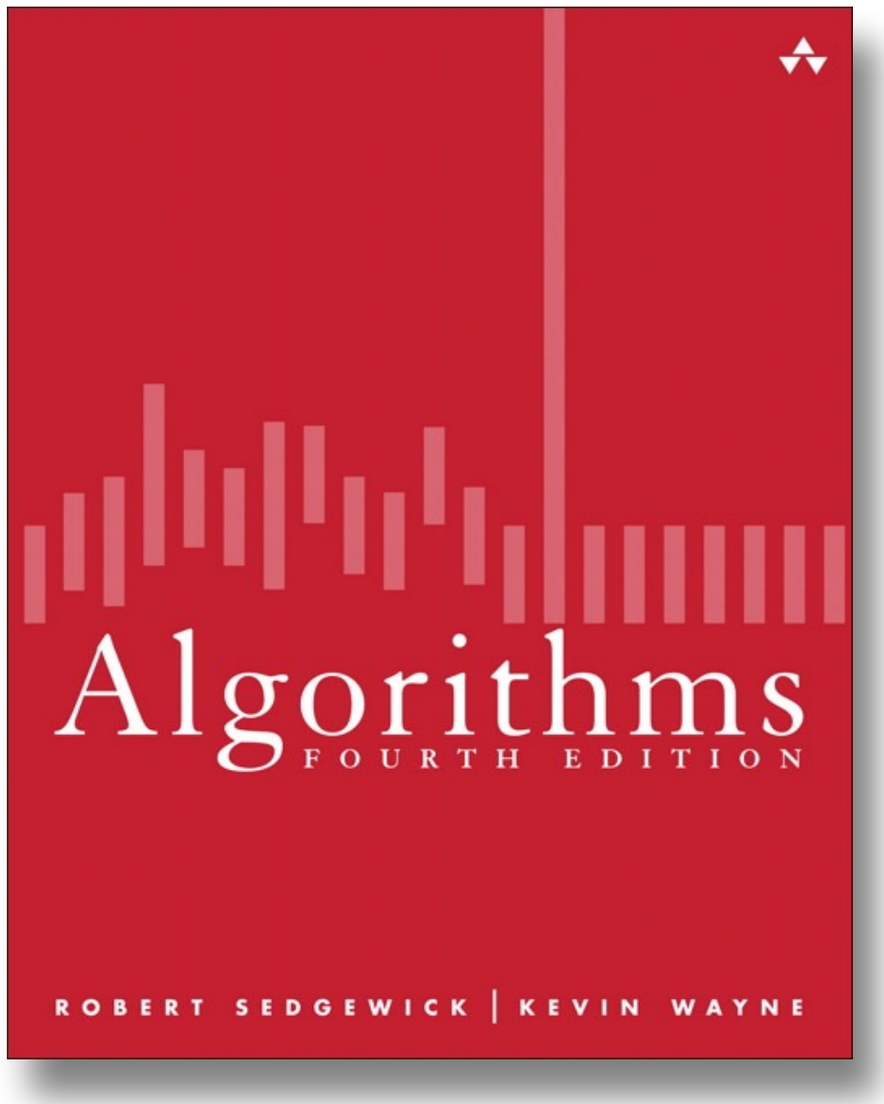
```
Arrays.sort(a, new Student.ByName());
```

Andrews	3	A	664-480-0023	097 Little
Battle	4	C	874-088-1212	121 Whitman
Chen	3	A	991-878-4944	308 Blair
Fox	3	A	884-232-5341	11 Dickinson
Furia	1	A	766-093-9873	101 Brown
Gazsi	4	B	766-093-9873	101 Brown
Kanaga	3	B	898-122-9643	22 Brown
Rohde	2	A	232-343-5555	343 Forbes

```
Arrays.sort(a, new Student.BySection());
```

Furia	1	A	766-093-9873	101 Brown
Rohde	2	A	232-343-5555	343 Forbes
Andrews	3	A	664-480-0023	097 Little
Chen	3	A	991-878-4944	308 Blair
Fox	3	A	884-232-5341	11 Dickinson
Kanaga	3	B	898-122-9643	22 Brown
Battle	4	C	874-088-1212	121 Whitman
Gazsi	4	B	766-093-9873	101 Brown

Lecture Overview



2.2 Mergesort

- Mergesort
- Bottom-up Mergesort
- Sorting complexity
- Comparators
- Stability

Stability

Stability

A typical application. First, sort by name; then sort by section.

```
Selection.sort(a, new Student.ByName());
```

Andrews	3	A	664-480-0023	097 Little
Battle	4	C	874-088-1212	121 Whitman
Chen	3	A	991-878-4944	308 Blair
Fox	3	A	884-232-5341	11 Dickinson
Furia	1	A	766-093-9873	101 Brown
Gazsi	4	B	766-093-9873	101 Brown
Kanaga	3	B	898-122-9643	22 Brown
Rohde	2	A	232-343-5555	343 Forbes

```
Selection.sort(a, new Student.BySection());
```

Furia	1	A	766-093-9873	101 Brown
Rohde	2	A	232-343-5555	343 Forbes
Chen	3	A	991-878-4944	308 Blair
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Kanaga	3	B	898-122-9643	22 Brown
Gazsi	4	B	766-093-9873	101 Brown
Battle	4	C	874-088-1212	121 Whitman

@#%&@! Students in section 3 no longer sorted by name.

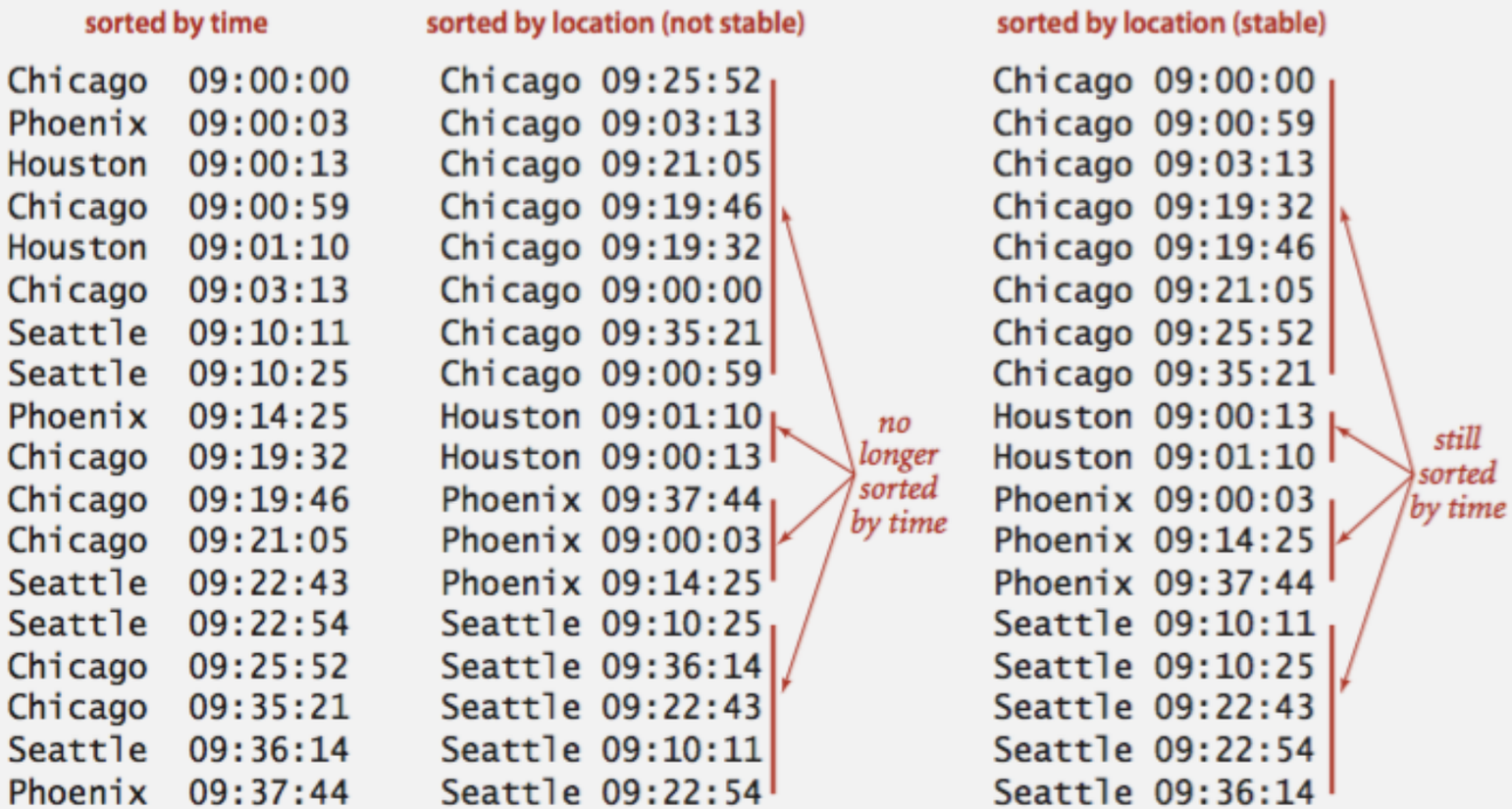
A stable sort preserves the relative order of items with equal keys.

Stability

Stability

Q. Which sorts are stable?

A. Need to check algorithm (and implementation).



Stability

Stability: insertion sort

Proposition. Insertion sort is stable.

```
public class Insertion
{
   public static void sort(Comparable[] a)
   {
      int N = a.length;
      for (int i = 0; i < N; i++)
         for (int j = i; j > 0 && less(a[j], a[j-1]); j--)
            exch(a, j, j-1);
   }
}
```

i	j	0	1	2	3	4
0	0	B_1	A_1	A_2	A_3	B_2
1	0	A_1	B_1	A_2	A_3	B_2
2	1	A_1	A_2	B_1	A_3	B_2
3	2	A_1	A_2	A_3	B_1	B_2
4	4	A_1	A_2	A_3	B_1	B_2
		A_1	A_2	A_3	B_1	B_2

Pf. Equal items never move past each other.

Stability

Stability: selection sort

Proposition. Selection sort is not stable.

```
public class Selection
{
   public static void sort(Comparable[] a)
   {
      int N = a.length;
      for (int i = 0; i < N; i++)
      {
         int min = i;
         for (int j = i+1; j < N; j++)
            if (less(a[j], a[min]))
               min = j;
         exch(a, i, min);
      }
   }
}
```

i	min	0	1	2
0	2	B_1	B_2	A
1	1	A	B_2	B_1
2	2	A	B_2	B_1
		A	B_2	B_1

Pf by counterexample. Long-distance exchange can move one equal item past another one.

Stability

Stability: shellsort

Proposition. Shellsort sort is not stable.

```
public class Shell
{
   public static void sort(Comparable[] a)
   {
      int N = a.length;
      int h = 1;
      while (h < N/3) h = 3*h + 1;
      while (h >= 1)
      {
         for (int i = h; i < N; i++)
         {
            for (int j = i; j > h && less(a[j], a[j-h]); j -= h)
               exch(a, j, j-h);
         }
         h = h/3;
      }
   }
}
```

h	0	1	2	3	4
	B_1	B_2	B_3	B_4	A_1
4	A_1	B_2	B_3	B_4	B_1
1	A_1	B_2	B_3	B_4	B_1
	A_1	B_2	B_3	B_4	B_1

Pf by counterexample. Long-distance exchanges.

Stability

Stability: mergesort

Proposition. Mergesort is stable.

```
public class Merge
{
   private static void merge(...)
   {  /* as before */  }

   private static void sort(Comparable[] a, Comparable[] aux, int lo, int hi)
   {
      if (hi <= lo) return;
      int mid = lo + (hi - lo) / 2;
      sort(a, aux, lo, mid);
      sort(a, aux, mid+1, hi);
      merge(a, aux, lo, mid, hi);
   }

   public static void sort(Comparable[] a)
   {  /* as before */  }
}
```

Pf. Suffices to verify that merge operation is stable.

Stability

Stability: mergesort

Proposition. Merge operation is stable.

```
private static void merge(...)
{
   for (int k = lo; k <= hi; k++)
      aux[k] = a[k];

   int i = lo, j = mid+1;
   for (int k = lo; k <= hi; k++)
   {
      if      (i > mid)                  a[k] = aux[j++];
      else if (j > hi)                   a[k] = aux[i++];
      else if (less(aux[j], aux[i])) a[k] = aux[j++];
      else                               a[k] = aux[i++];
   }
}
```

0	1	2	3	4	5	6	7	8	9	10
A_1	A_2	A_3	B	D	A_4	A_5	C	E	F	G

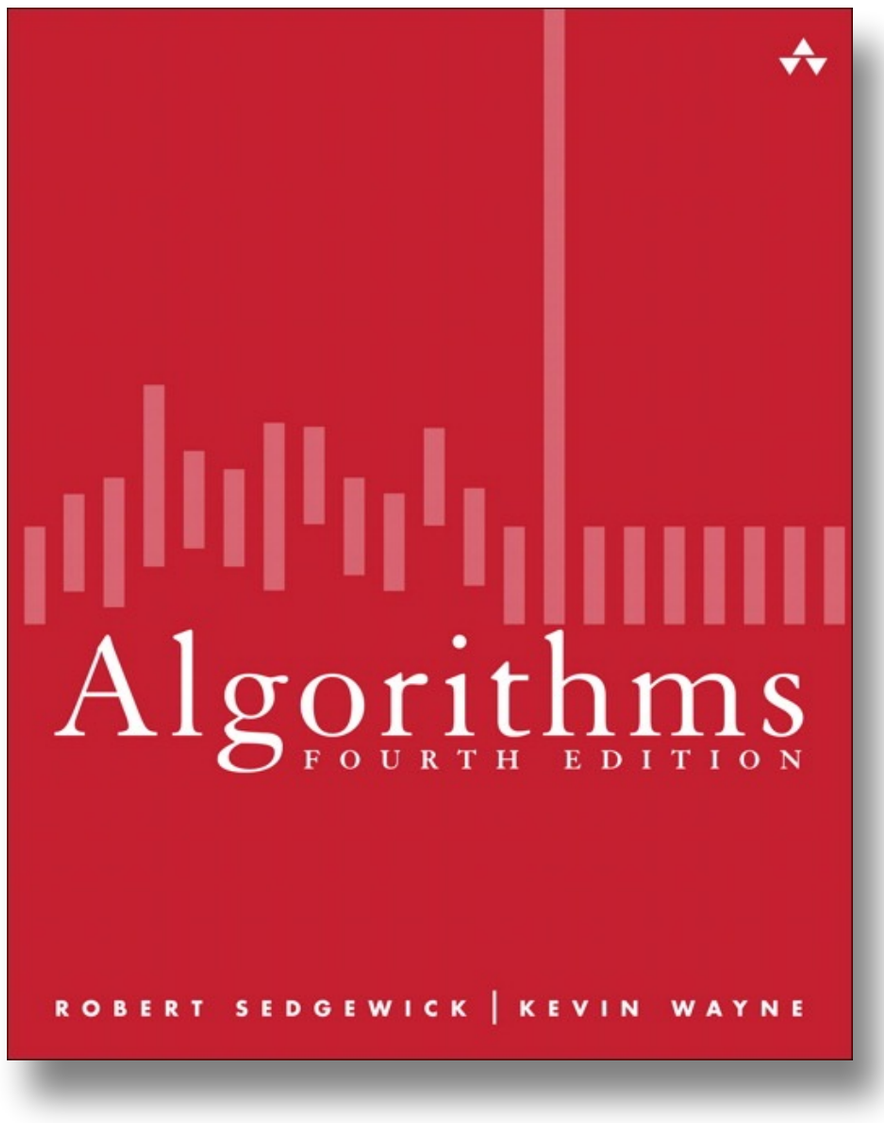
Pf. Takes from left subarray if equal keys.

Stability

Sorting summary

	inplace?	stable?	best	average	worst	remarks
selection	✔		$\frac{1}{2} N^2$	$\frac{1}{2} N^2$	$\frac{1}{2} N^2$	N exchanges
insertion	✔	✔	N	$\frac{1}{4} N^2$	$\frac{1}{2} N^2$	use for small N or partially ordered
shell	✔		$N \log_3 N$	?	$c N^{3/2}$	tight code; subquadratic
merge		✔	$\frac{1}{2} N \lg N$	$N \lg N$	$N \lg N$	$N \log N$ guarantee; stable
timsort		✔	N	$N \lg N$	$N \lg N$	improves mergesort when preexisting order
?	✔	✔	N	$N \lg N$	$N \lg N$	holy sorting grail

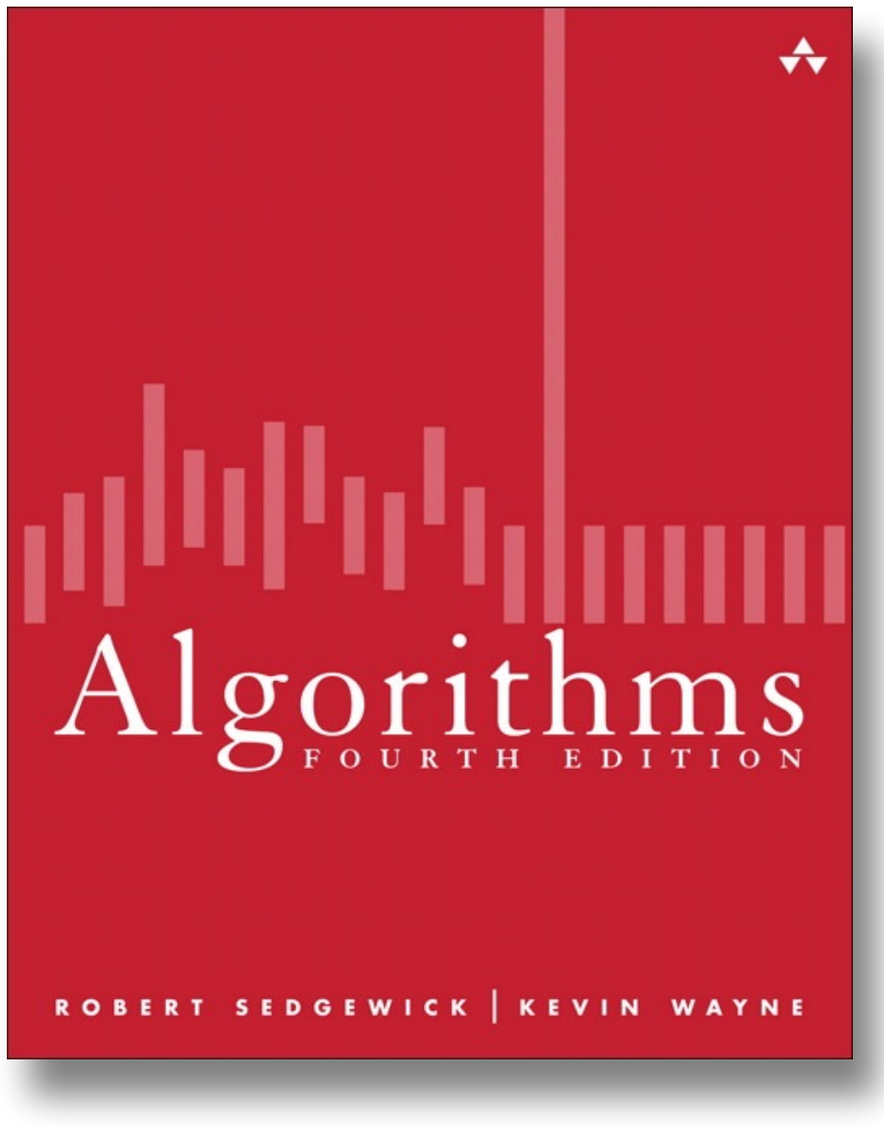
Lecture Overview



2.3 Quicksort

- Quicksort
- Selection
- Duplicate keys
- System sorts

Lecture Overview



2.3 Quicksort

- Quicksort
- Selection
- Duplicate keys
- System sorts

Quicksort

Quicksort is popular because it is not difficult to implement, works well for a variety of different kinds of input data, and is substantially faster than any other sorting method in typical applications. It is in-place (uses only a small auxiliary stack), requires time proportional to N log N on the average to sort N items, and has an extremely short inner loop.

Quicksort is a divide-and-conquer method for sorting. It works by partitioning an array into two parts, then sorting the parts independently.

input	M	E	R	G	E	S	O	R	T	E	X	A	M	P	L	E
sort left half	E	E	G	M	O	R	R	S	T	E	X	A	M	P	L	E
sort right half	E	E	G	M	O	R	R	S	A	E	E	L	M	P	T	X
merge results	A	E	E	E	E	G	L	M	M	O	P	R	R	S	T	X

Mergesort overview

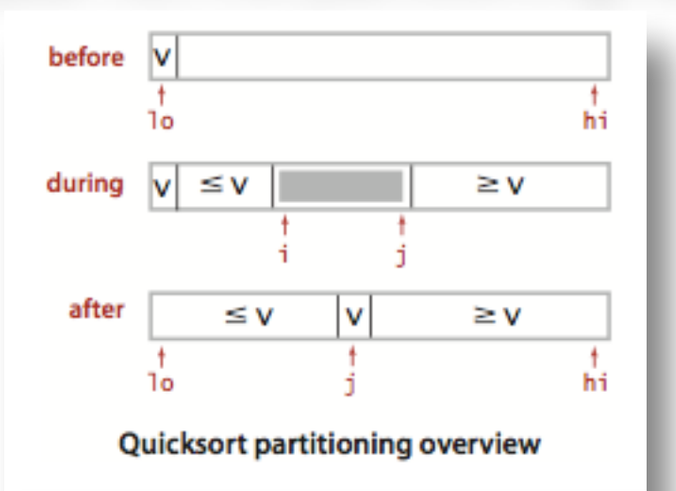
Quicksort

The crux of the method is the partitioning process, which rearranges the array to make the following three conditions hold:

- The entry a[j] is in its final place in the array, for some j.
- No entry in a[lo] through a[j-1] is greater than a[j].
- No entry in a[j+1] through a[hi] is less than a[j].

We achieve a complete sort by partitioning, then recursively applying the method to the subarrays. It is a randomized algorithm, because it randomly shuffles the array before sorting it.

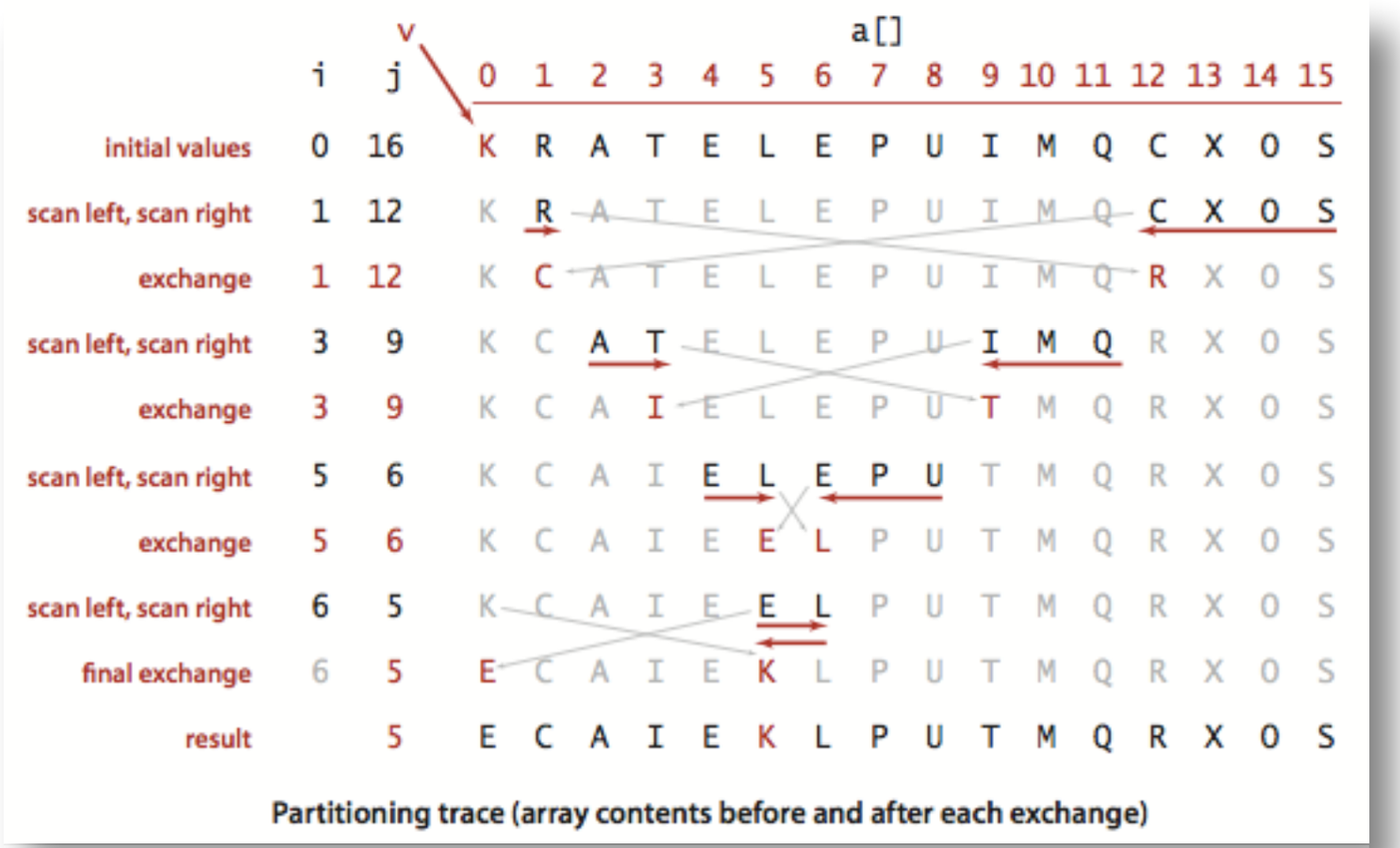
Partitioning. To complete the implementation, we need to implement the partitioning method. We use the following general strategy: First, we arbitrarily choose a[lo] to be the partitioning item—the one that will go into its final position. Next, we scan from the left end of the array until we find an entry that is greater than (or equal to) the partitioning item, and we scan from the right end of the array until we find an entry less than (or equal to) the partitioning item.



Quicksort partitioning overview

Partitioning

The two items that stopped the scans are out of place in the final partitioned array, so we exchange them. When the scan indices cross, all that we need to do to complete the partitioning process is to exchange the partitioning item a[lo] with the rightmost entry of the left subarray (a[j]) and return its index j.

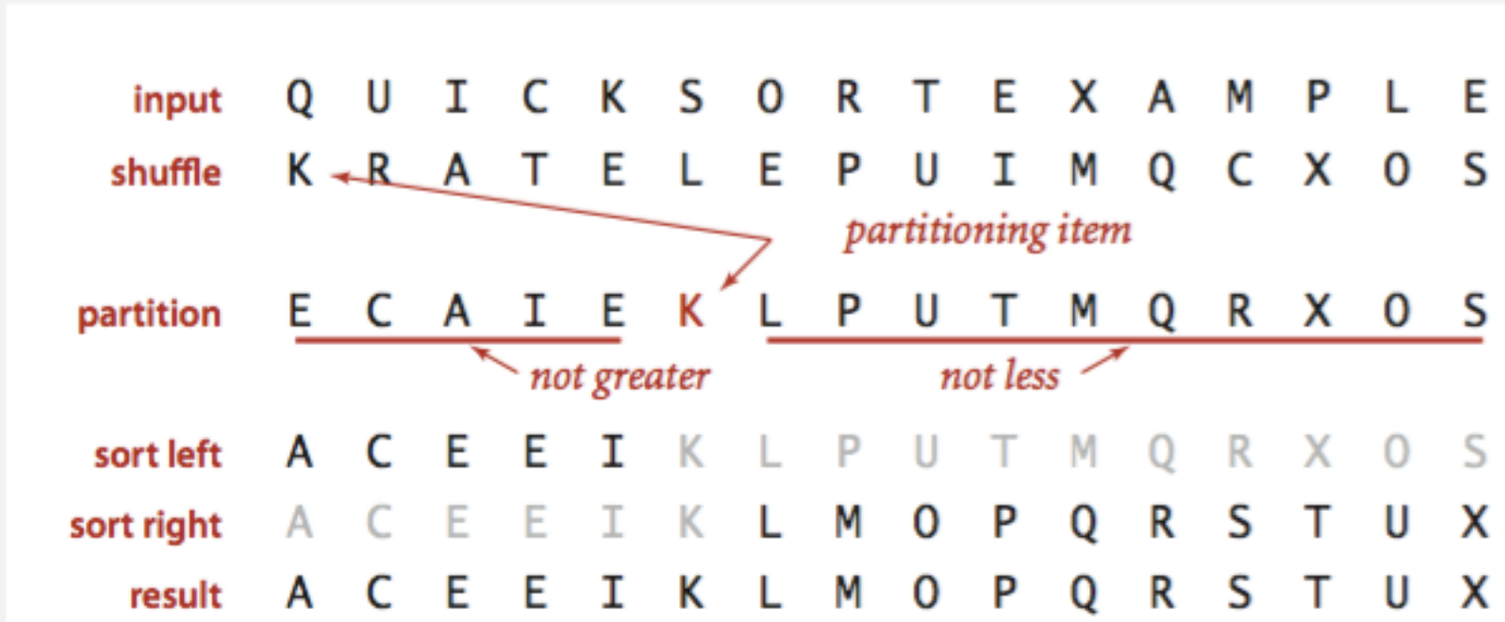


Partitioning trace (array contents before and after each exchange)

Partitioning

Basic plan.

- Shuffle the array.
- Partition so that, for some `j`
  - entry `a[j]` is in place
  - no larger entry to the left of `j`
  - no smaller entry to the right of `j`
- Sort each subarray recursively.



Partitioning

- Invented quicksort to translate Russian into English.
  [but couldn't explain his algorithm or implement it!]
- Learned Algol 60 (and recursion).
- Implemented quicksort.

Tony Hoare
1980 Turing Award

Algorithms

ALGORITHM 64
QUICKSORT
C. A. R. Hoare
Elliott Brothers Ltd., Borehamwood, Hertfordshire, Eng.

```
procedure  quicksort (A,M,N); value M,N;
             array A; integer M,N;
comment  Quicksort is a very fast and convenient method of
sorting an array in the random-access store of a computer. The
entire contents of the store may be sorted, since no extra space is
required. The average number of comparisons made is 2(M−N) ln
(N−M), and the average number of exchanges is one sixth this
amount. Suitable refinements of this method will be desirable for
its implementation on any actual computer;
begin      integer I,J;
           if M < N then begin partition (A,M,N,I,J);
                               quicksort (A,M,J);
                               quicksort (A, I, N)
                         end
end        quicksort
```

Communications of the ACM (July 1961)

Partitioning

- Invented quicksort to translate Russian into English.
  [but couldn't explain his algorithm or implement it!]
-
-

> *" There are two ways of constructing a software design: One way is to make it so simple that there are obviously no deficiencies, and the other way is to make it so complicated that there are no obvious deficiencies. The first method is far more difficult. "*

> *" I call it my billion-dollar mistake. It was the invention of the null reference in 1965… This has led to innumerable errors, vulnerabilities, and system crashes, which have probably caused a billion dollars of pain and damage in the last forty years. "*

Communications of the ACM (July 1961)

Partitioning

- Refined and popularized quicksort.
- Analyzed many versions of quicksort.

Bob Sedgewick

Programming Techniques — S. L. Graham, R. L. Rivest Editors

Implementing Quicksort Programs

Robert Sedgewick
Brown University

This paper is a practical study of how to implement the Quicksort sorting algorithm and its best variants on real computers, including how to apply various code optimization techniques. A detailed implementation combining the most effective improvements to Quicksort is given, along with a discussion of how to implement it in assembly language. Analytic results describing the performance of the programs are summarized. A variety of special situations are considered from a practical standpoint to illustrate Quicksort's wide applicability as an internal sorting method which requires negligible extra storage.

Key Words and Phrases: Quicksort, analysis of algorithms, code optimization, sorting
CR Categories: 4.0, 4.6, 5.25, 5.31, 5.5

Acta Informatica 7, 327—355 (1977)

The Analysis of Quicksort Programs*

Robert Sedgewick

Received January 19, 1976

Summary. The Quicksort sorting algorithm and its best variants are presented and analyzed. Results are derived which make it possible to obtain exact formulas describing the total expected running time of particular implementations on real computers of Quicksort and an improvement called the median-of-three modification. Detailed analysis of the effect of an implementation technique called loop unwrapping is presented. The paper is intended not only to present results of direct practical utility, but also to illustrate the intriguing mathematics which arises in the complete analysis of this important algorithm.

Partitioning

Quicksort partitioning demo

Repeat until `i` and `j` pointers cross.

- Scan `i` from left to right so long as `(a[i] < a[lo])`.
- Scan `j` from right to left so long as `(a[j] > a[lo])`.
- Exchange `a[i]` with `a[j]`.

Partitioning

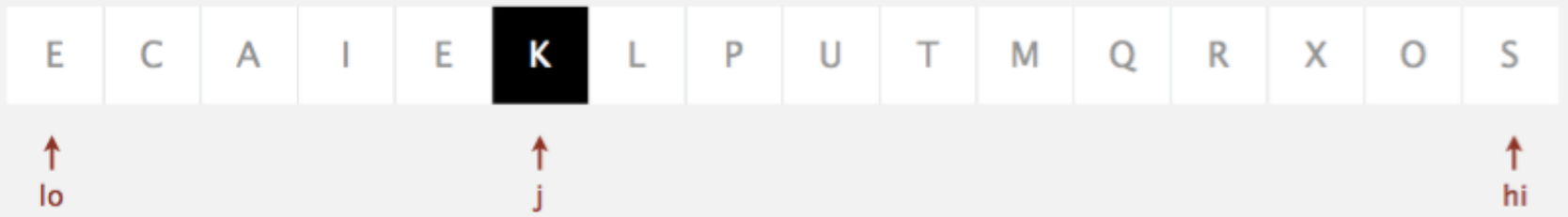
Quicksort partitioning demo

Repeat until `i` and `j` pointers cross.

- Scan `i` from left to right so long as `(a[i] < a[lo])`.
- Scan `j` from right to left so long as `(a[j] > a[lo])`.
- Exchange `a[i]` with `a[j]`.

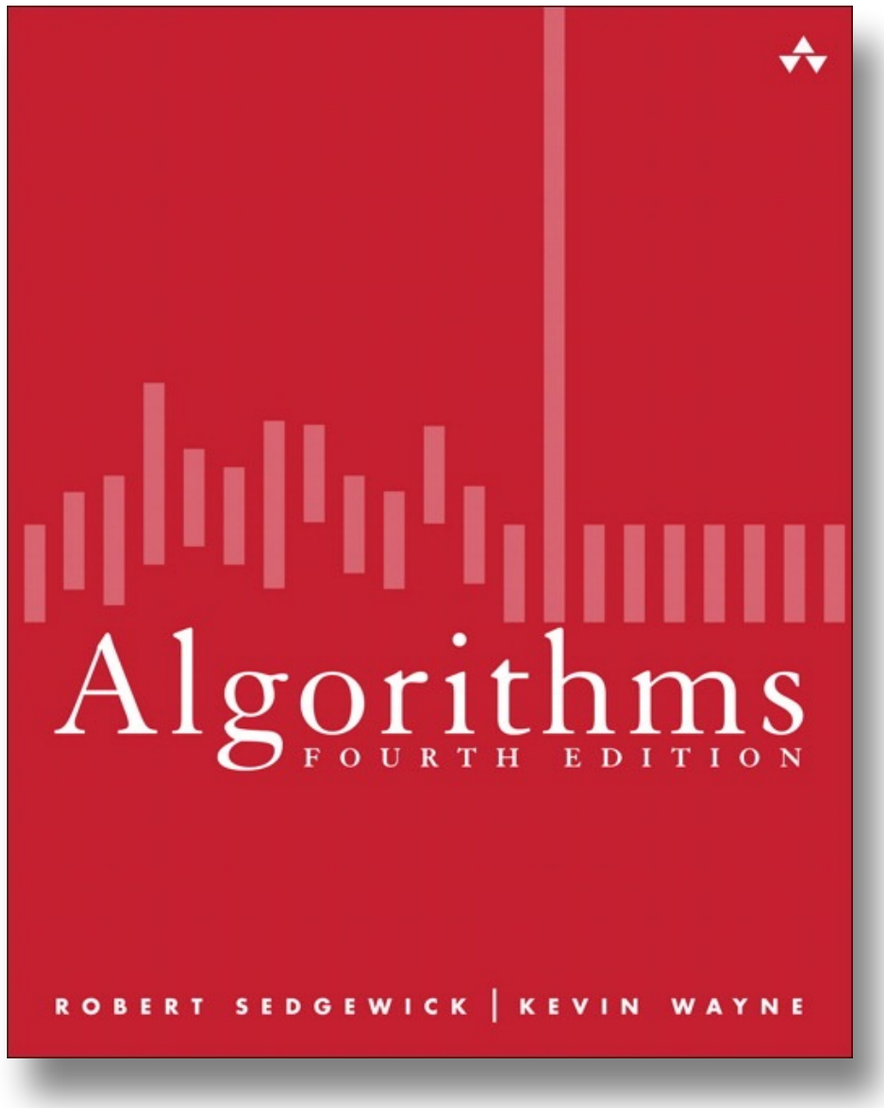
When pointers cross.

- Exchange `a[lo]` with `a[j]`.



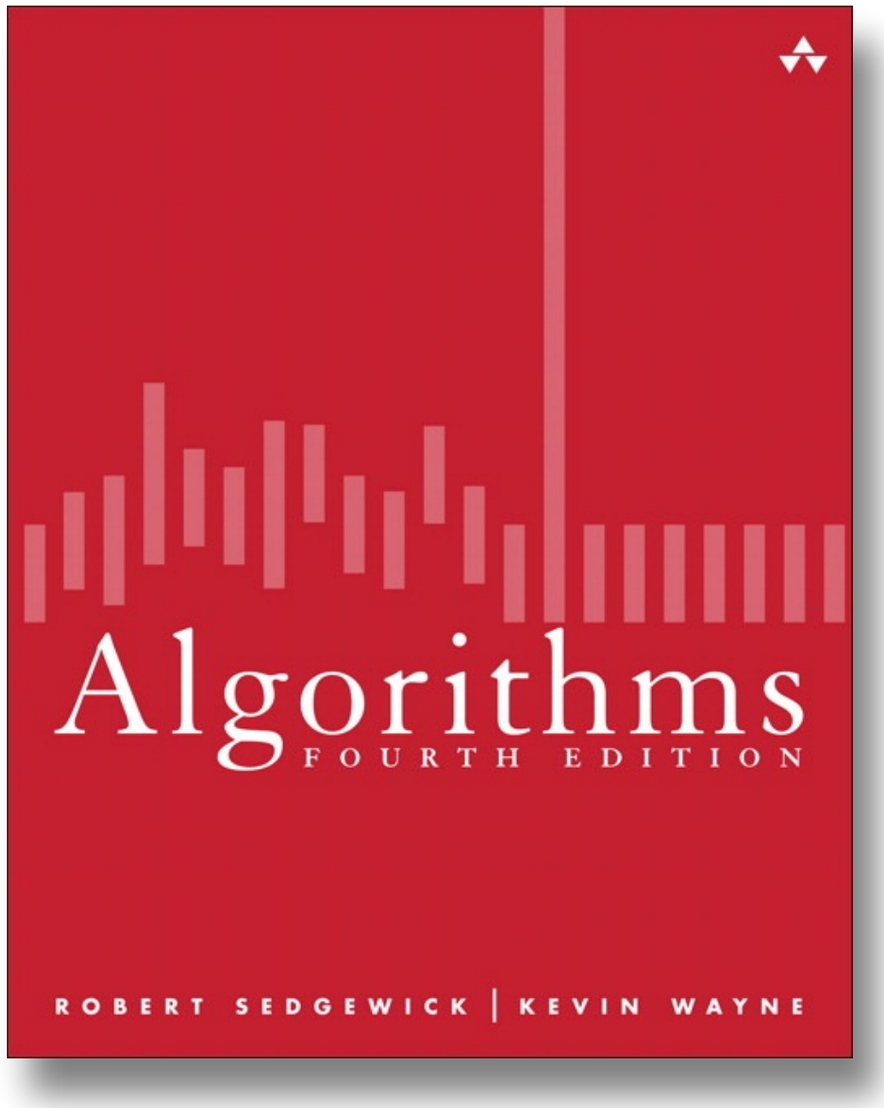
partitioned!

Lecture Overview



2.3 Partitioning Demo

- Sedgewick 2-way partitioning
- Dijkstra 3-way partitioning
- Bentley-McIlroy 3-way partitioning
- Dual-pivot partitioning



2.3 Partitioning Demo

- Sedgewick 2-way partitioning
- Dijkstra 3-way partitioning
- Bentley-McIlroy 3-way partitioning
- Dual-pivot partitioning

Partitioning: Sedgewick 2-way partitioning

Quicksort partitioning demo

Repeat until `i` and `j` pointers cross.

- Scan `i` from left to right so long as `(a[i] < a[lo])`.
- Scan `j` from right to left so long as `(a[j] > a[lo])`.
- Exchange `a[i]` with `a[j]`.

stop i scan because a[i] >= a[lo]

Partitioning: Sedgewick 2-way partitioning

Quicksort partitioning demo

Repeat until `i` and `j` pointers cross.

- Scan `i` from left to right so long as `(a[i] < a[lo])`.
- Scan `j` from right to left so long as `(a[j] > a[lo])`.
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Partitioning: Sedgewick 2-way partitioning

Quicksort partitioning demo

Repeat until `i` and `j` pointers cross.

- Scan `i` from left to right so long as `(a[i] < a[lo])`.
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Partitioning: Sedgewick 2-way partitioning

Quicksort partitioning demo

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stop j scan and exchange a[i] with a[j]

Partitioning: Sedgewick 2-way partitioning

Quicksort partitioning demo

Repeat until `i` and `j` pointers cross.

- Scan `i` from left to right so long as `(a[i] < a[lo])`.
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Partitioning: Sedgewick 2-way partitioning

Quicksort partitioning demo

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Partitioning: Sedgewick 2-way partitioning

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- Exchange `a[i]` with `a[j]`.

stop i scan because a[i] >= a[lo]

Partitioning: Sedgewick 2-way partitioning

Quicksort partitioning demo

Repeat until `i` and `j` pointers cross.

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Partitioning: Sedgewick 2-way partitioning

Quicksort partitioning demo

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Partitioning: Sedgewick 2-way partitioning

Quicksort partitioning demo

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Partitioning: Sedgewick 2-way partitioning

Quicksort partitioning demo

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Partitioning: Sedgewick 2-way partitioning

Quicksort partitioning demo

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stop i scan because a[i] >= a[lo]

Partitioning: Sedgewick 2-way partitioning

Quicksort partitioning demo

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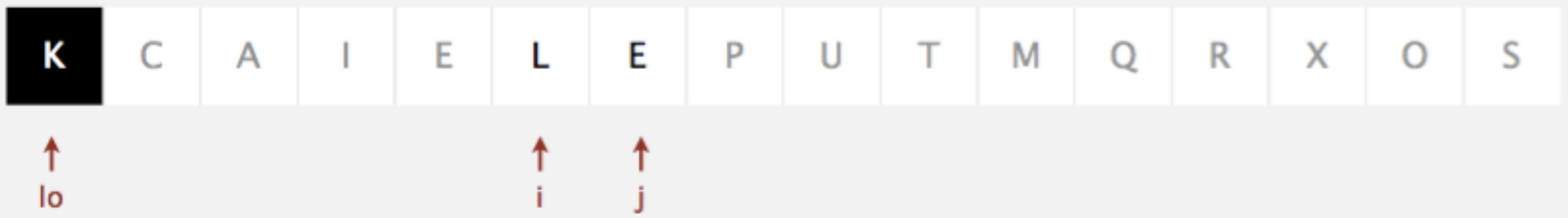

stop j scan and exchange a[i] with a[j]

Partitioning: Sedgewick 2-way partitioning

Quicksort partitioning demo

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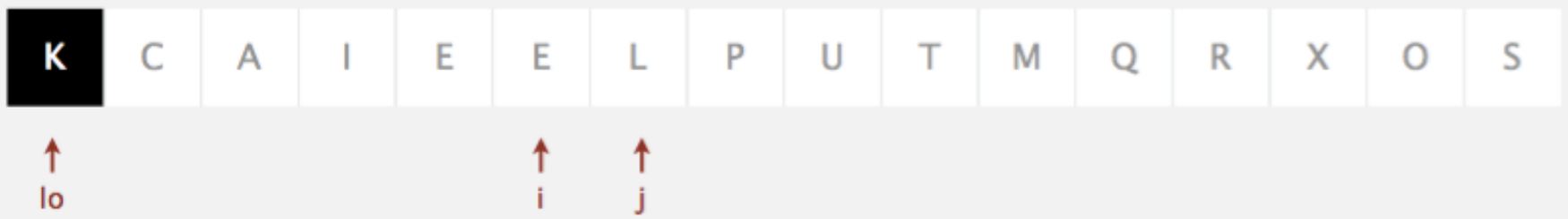

Partitioning: Sedgewick 2-way partitioning

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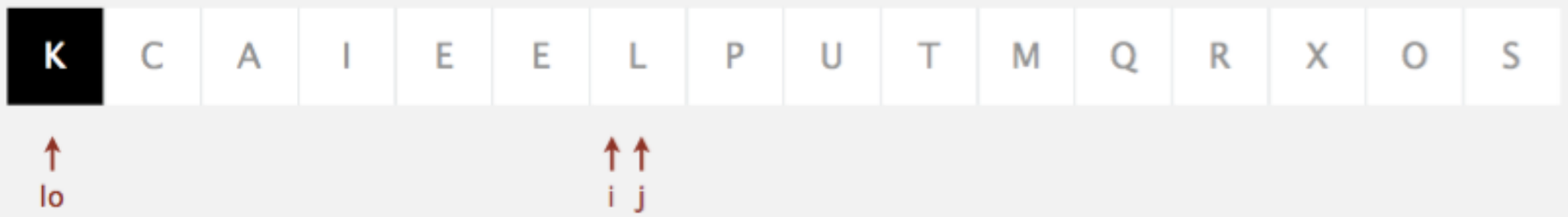

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Quicksort partitioning demo

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- Scan `i` from left to right so long as `(a[i] < a[lo])`.
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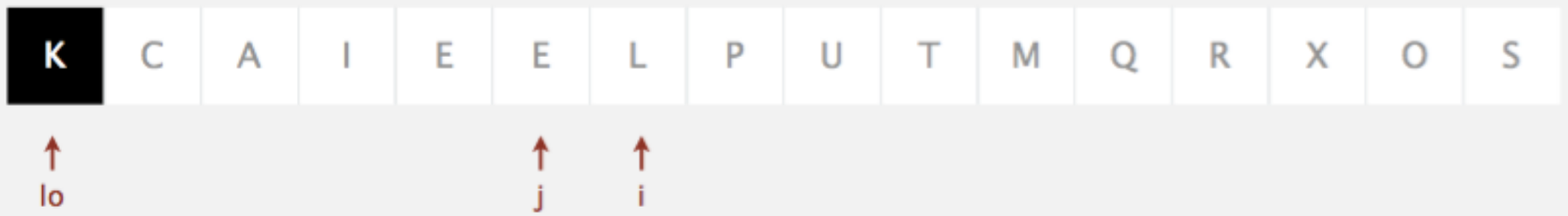
- Exchange `a[i]` with `a[j]`.



stop j scan because a[j] <= a[lo]

Partitioning: Sedgewick 2-way partitioning

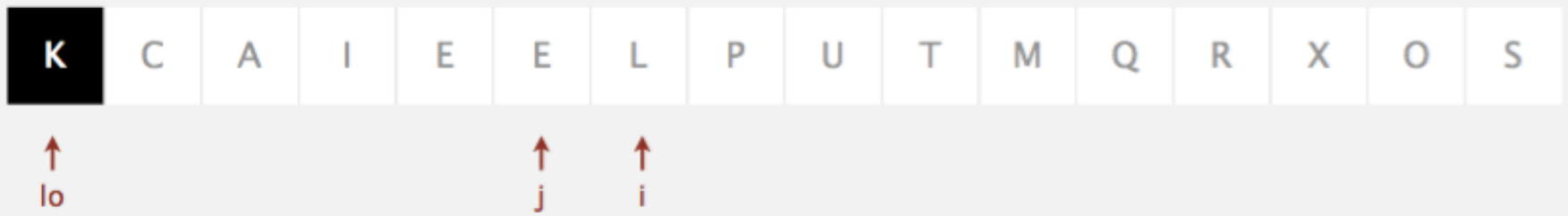
Quicksort partitioning demo

Repeat until `i` and `j` pointers cross.

- Scan `i` from left to right so long as `(a[i] < a[lo])`.
- Scan `j` from right to left so long as `(a[j] > a[lo])`.
- Exchange `a[i]` with `a[j]`.

When pointers cross.

- Exchange `a[lo]` with `a[j]`.



pointers cross: exchange a[lo] with a[j]

Partitioning: Sedgewick 2-way partitioning

Quicksort partitioning demo

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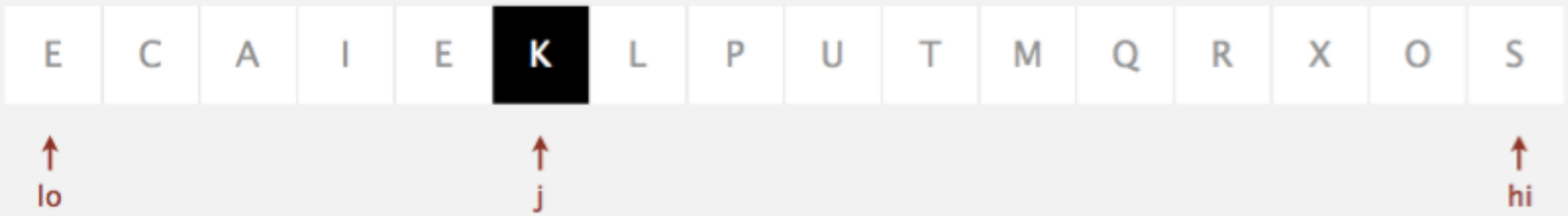
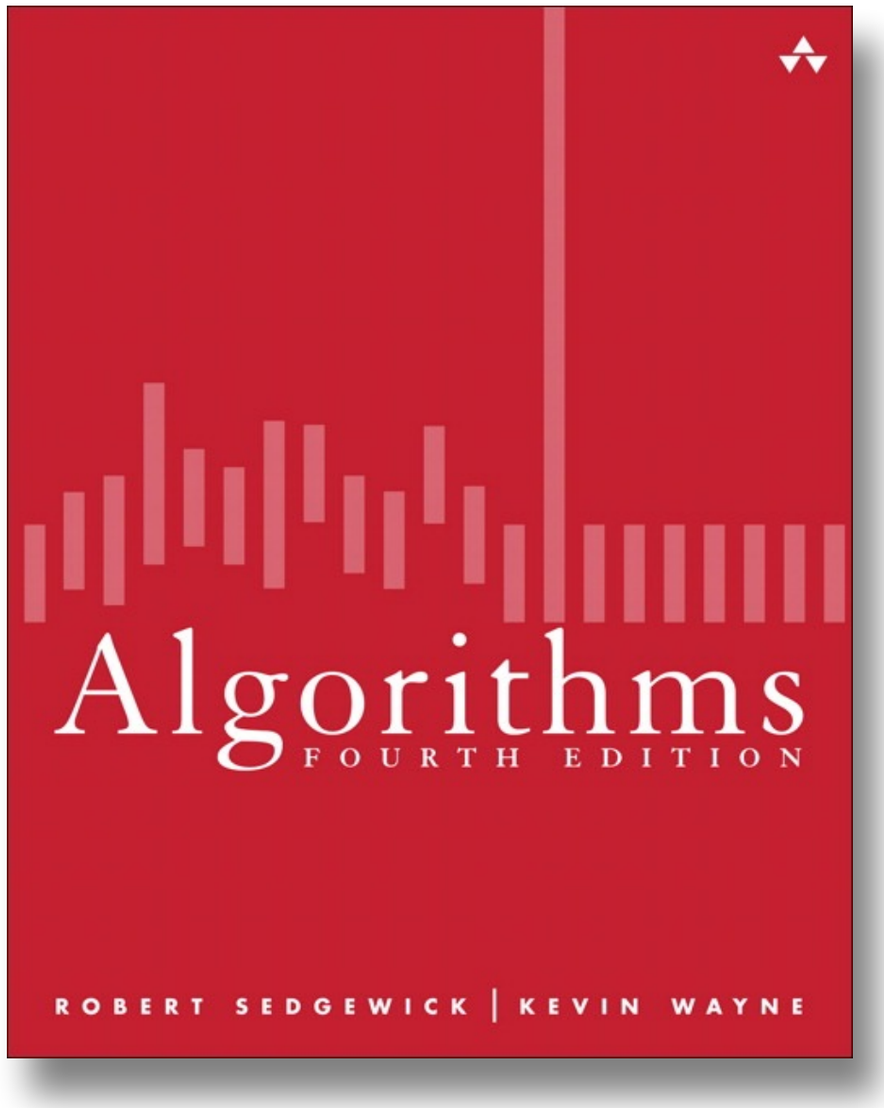


partitioned!



2.3 Partitioning Demo

- Sedgewick 2-way partitioning
- Dijkstra 3-way partitioning
- Bentley-McIlroy 3-way partitioning
- Dual-pivot partitioning

Partitioning: Dijkstra 3-way partitioning

Dijkstra 3-way partitioning demo

- Let `v` be partitioning item `a[lo]`.
- Scan `i` from left to right.
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Dijkstra 3-way partitioning demo

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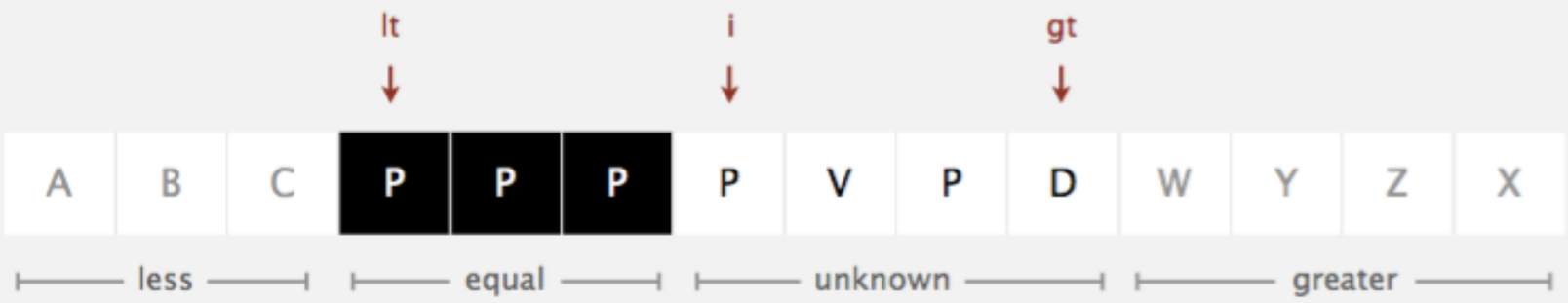

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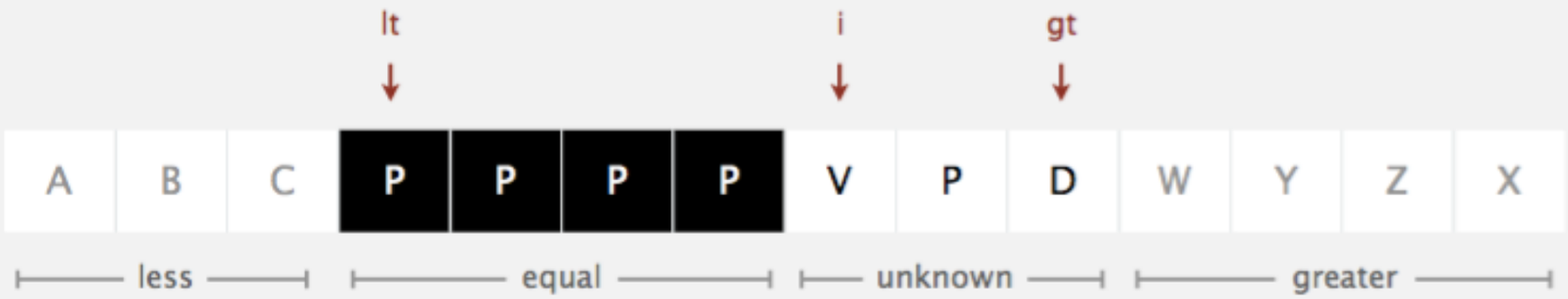

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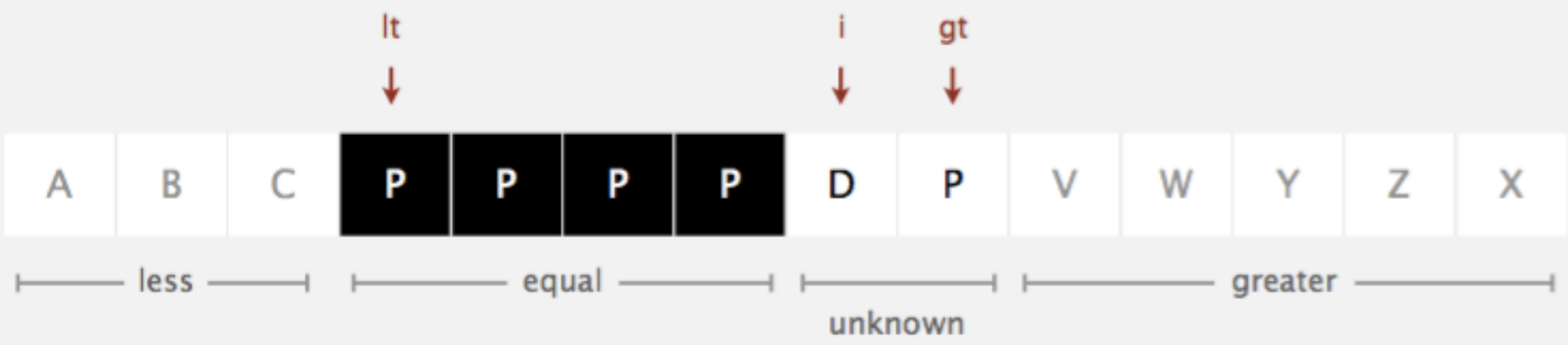

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Dijkstra 3-way partitioning demo

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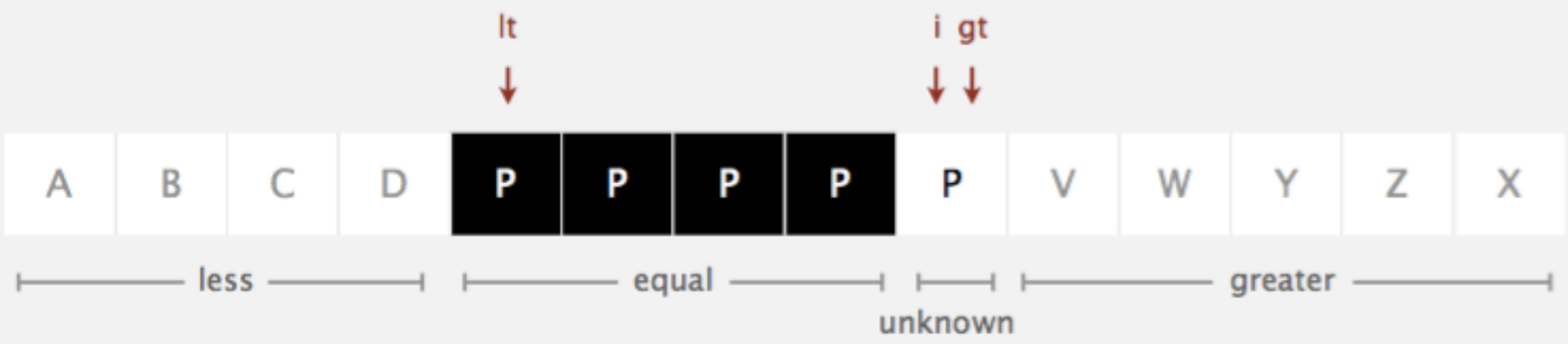

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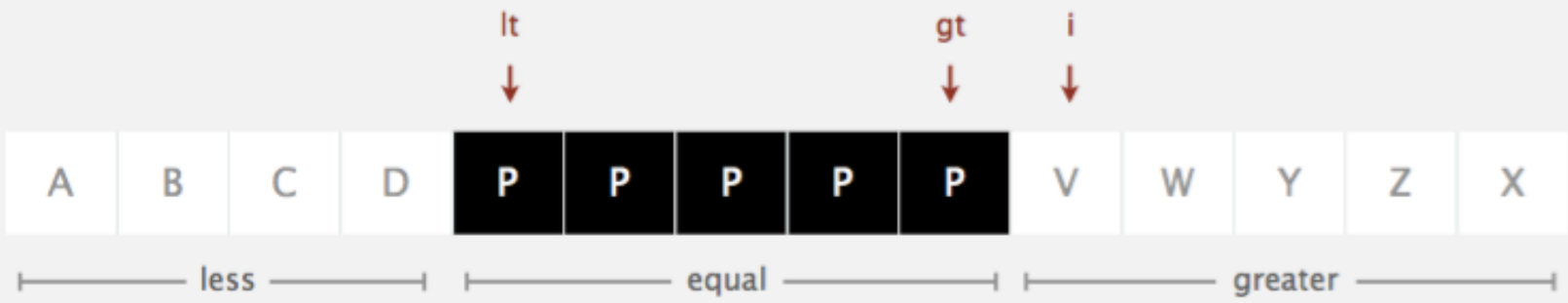

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Dijkstra 3-way partitioning demo

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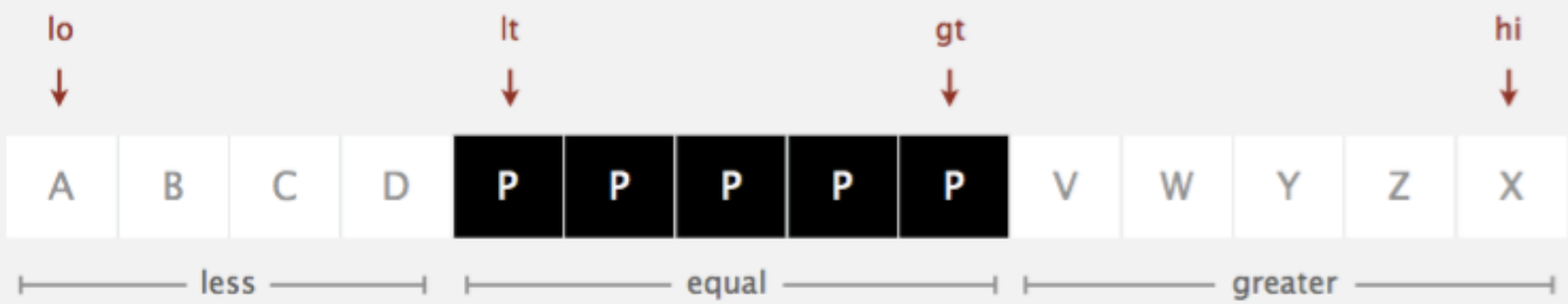

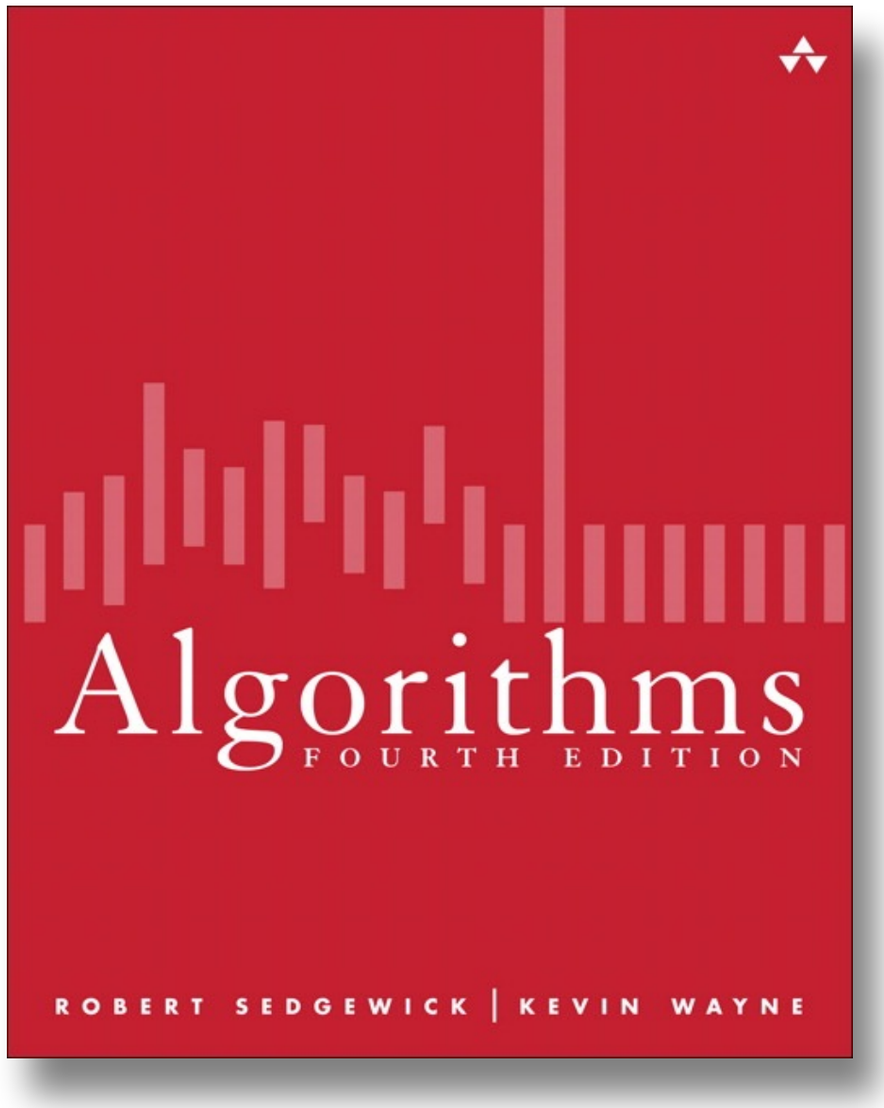

2.3 Partitioning Demo

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- Dijkstra 3-way partitioning
- Bentley-McIlroy 3-way partitioning
- Dual-pivot partitioning

Partitioning: Bentley-McIlroy 3-way partitioning

Bentley-McIlroy 3-way partitioning demo

Phase I. Repeat until `i` and `j` pointers cross.

- Scan `i` from left to right so long as `(a[i] < a[lo])`.
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Partitioning: Bentley-McIlroy 3-way partitioning

Bentley-McIlroy 3-way partitioning demo

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Bentley-McIlroy 3-way partitioning demo

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Partitioning: Bentley-McIlroy 3-way partitioning

Bentley-McIlroy 3-way partitioning demo

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Partitioning: Bentley-McIlroy 3-way partitioning

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Partitioning: Bentley-McIlroy 3-way partitioning

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- If `(a[j] == a[lo])`, exchange `a[j]` with `a[q]` and decrement `q`.

Partitioning: Bentley-McIlroy 3-way partitioning

Bentley-McIlroy 3-way partitioning demo

Phase I. Repeat until `i` and `j` pointers cross.

- Scan `i` from left to right so long as `(a[i] < a[lo])`.
- Scan `j` from right to left so long as `(a[j] > a[lo])`.
- Exchange `a[i]` with `a[j]`.
- If `(a[i] == a[lo])`, exchange `a[i]` with `a[p]` and increment `p`.
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Bentley-McIlroy 3-way partitioning demo

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pointers cross

Partitioning: Bentley-McIlroy 3-way partitioning

Bentley-McIlroy 3-way partitioning demo

Phase II. Swap equal keys to the center.

- Scan `j` and `p` from right to left and exchange `a[j]` with `a[p]`.
- Scan `i` and `q` from left to right and exchange `a[i]` with `a[q]`.

exchange `a[j]` with `a[p]`

Partitioning: Bentley-McIlroy 3-way partitioning

Bentley-McIlroy 3-way partitioning demo

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Partitioning: Bentley-McIlroy 3-way partitioning

Bentley-McIlroy 3-way partitioning demo

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Partitioning: Bentley-McIlroy 3-way partitioning

Bentley-McIlroy 3-way partitioning demo

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Partitioning: Bentley-McIlroy 3-way partitioning

Bentley-McIlroy 3-way partitioning demo

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Partitioning: Bentley-McIlroy 3-way partitioning

Bentley-McIlroy 3-way partitioning demo

Phase II. Swap equal keys to the center.

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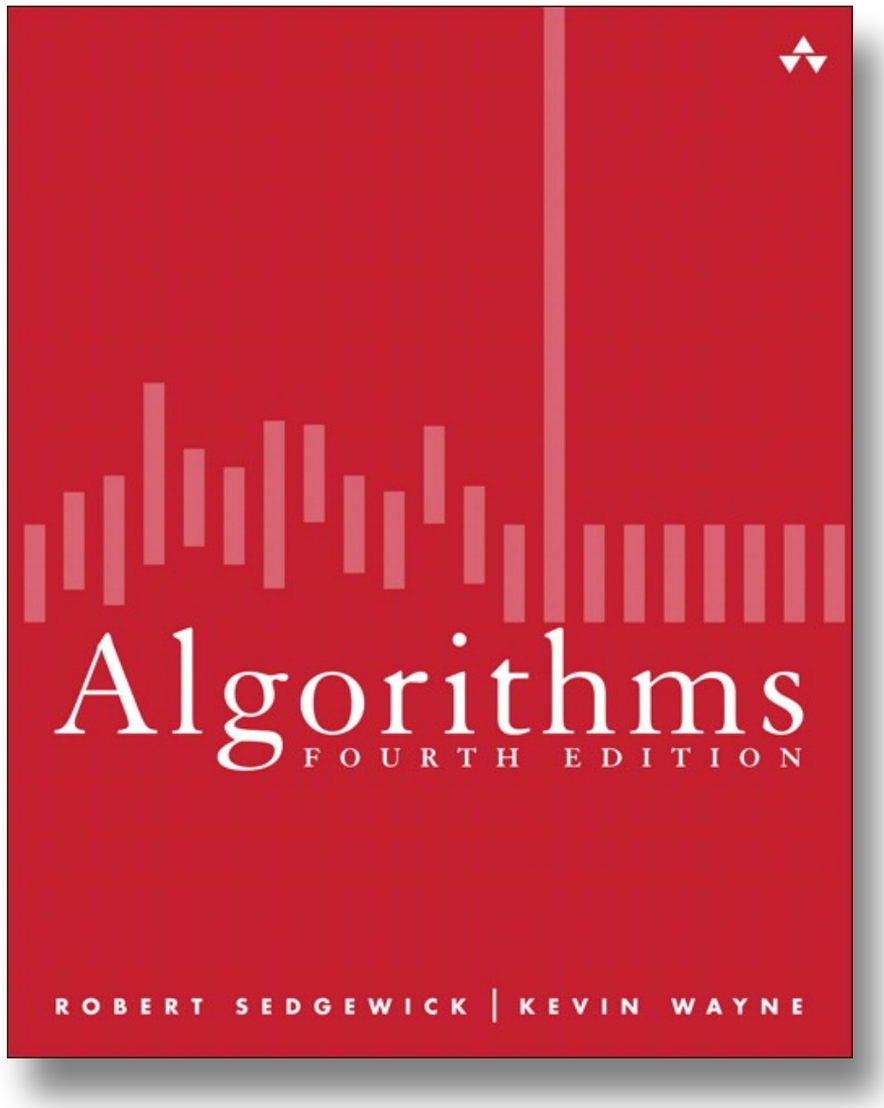
3-way partitioned



2.3 Partitioning Demo

- Sedgewick 2-way partitioning
- Dijkstra 3-way partitioning
- Bentley-McIlroy 3-way partitioning
- Dual-pivot partitioning

Partitioning: Dual-pivot partitioning

Dual-pivot partitioning demo

Initialization.

- Choose `a[lo]` and `a[hi]` as partitioning items.
- Exchange if necessary to ensure `a[lo]` ≤ `a[hi]`.

exchange a[lo] and a[hi]

Partitioning: Dual-pivot partitioning

Dual-pivot partitioning demo

Initialization.

- Choose `a[lo]` and `a[hi]` as partitioning items.
- Exchange if necessary to ensure `a[lo]` ≤ `a[hi]`.

Partitioning: Dual-pivot partitioning

Dual-pivot partitioning demo

Main loop. Repeat until `i` and `gt` pointers cross.

- If `(a[i] < a[lo])`, exchange `a[i]` with `a[lt]` and increment `lt` and `i`.
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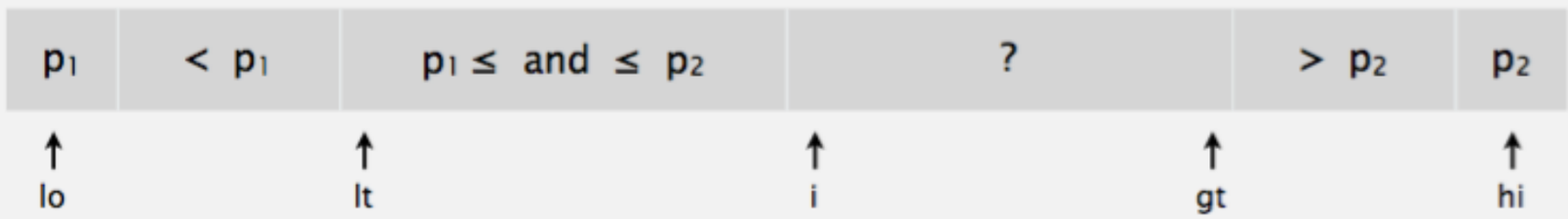
- Else, increment `i`.



exchange a[i] and a[lt]; increment lt and i

Partitioning: Dual-pivot partitioning

Dual-pivot partitioning demo

Main loop. Repeat until `i` and `gt` pointers cross.

- If `(a[i] < a[lo])`, exchange `a[i]` with `a[lt]` and increment `lt` and `i`.
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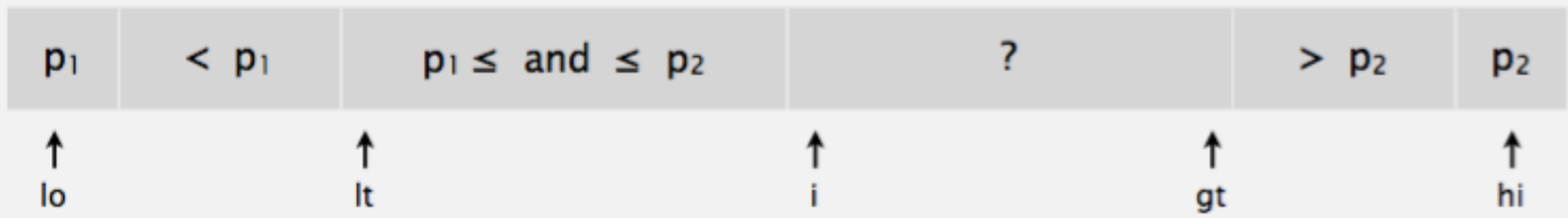

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Partitioning: Dual-pivot partitioning

Dual-pivot partitioning demo

Main loop. Repeat until `i` and `gt` pointers cross.

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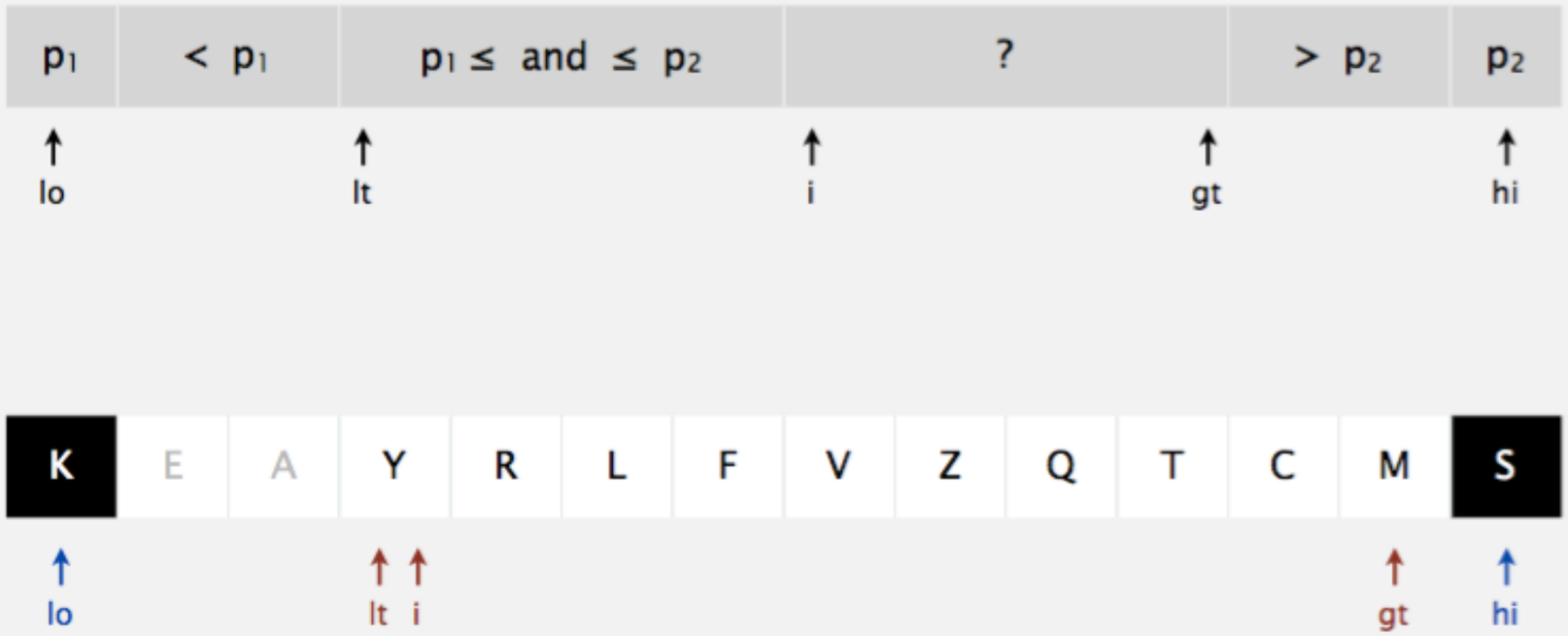

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Partitioning: Dual-pivot partitioning

Dual-pivot partitioning demo

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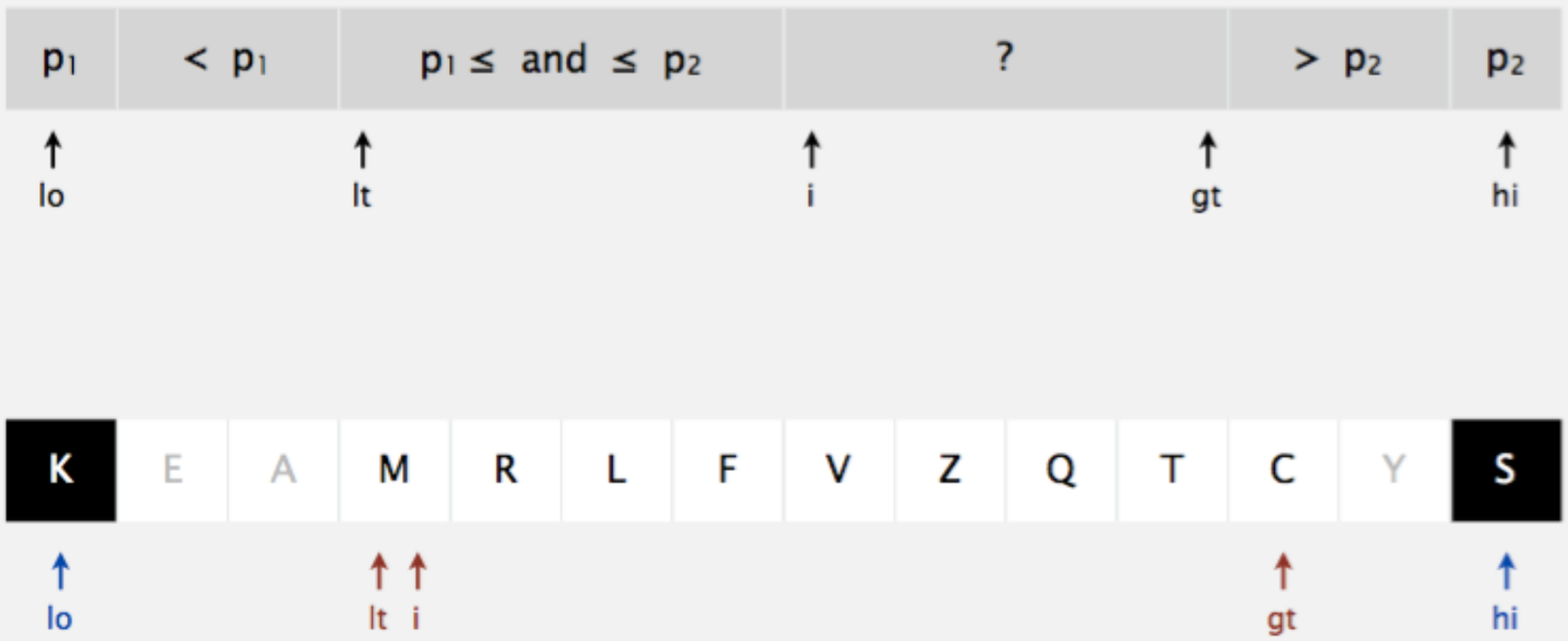

Partitioning: Dual-pivot partitioning

Dual-pivot partitioning demo

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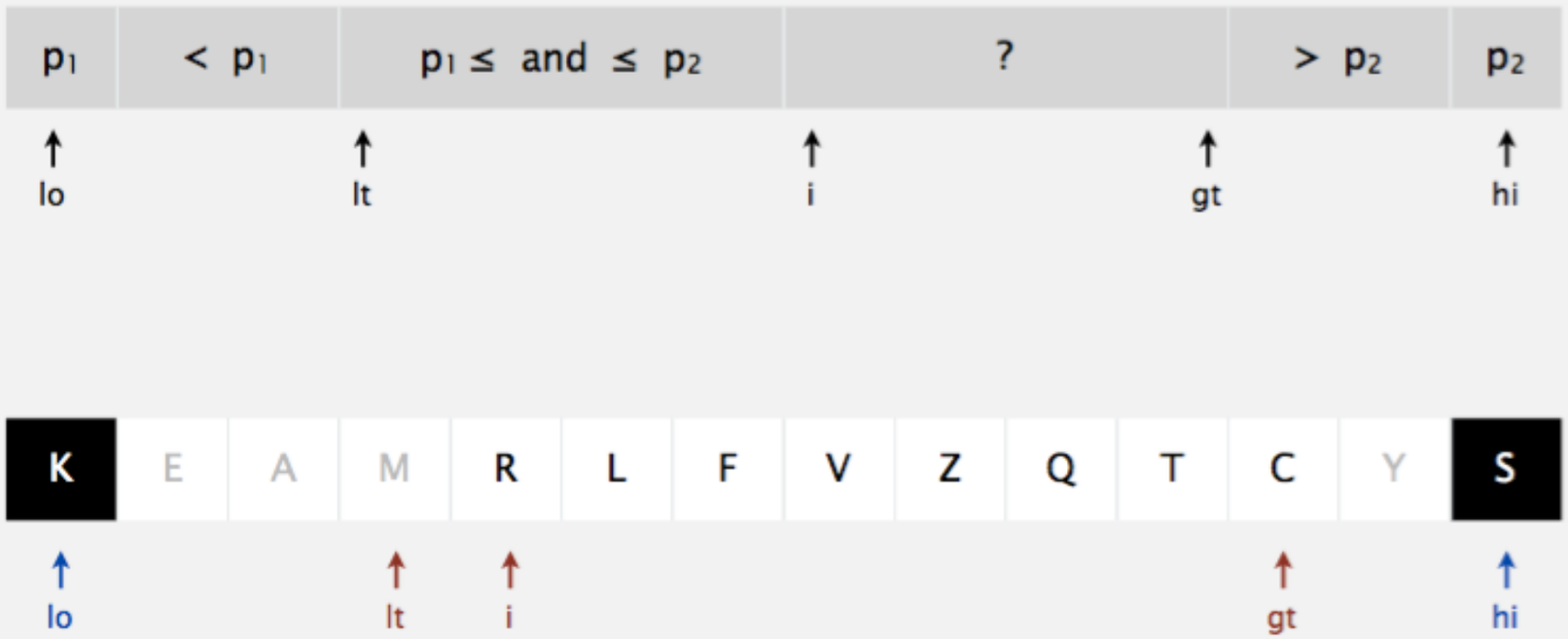

Partitioning: Dual-pivot partitioning

Dual-pivot partitioning demo

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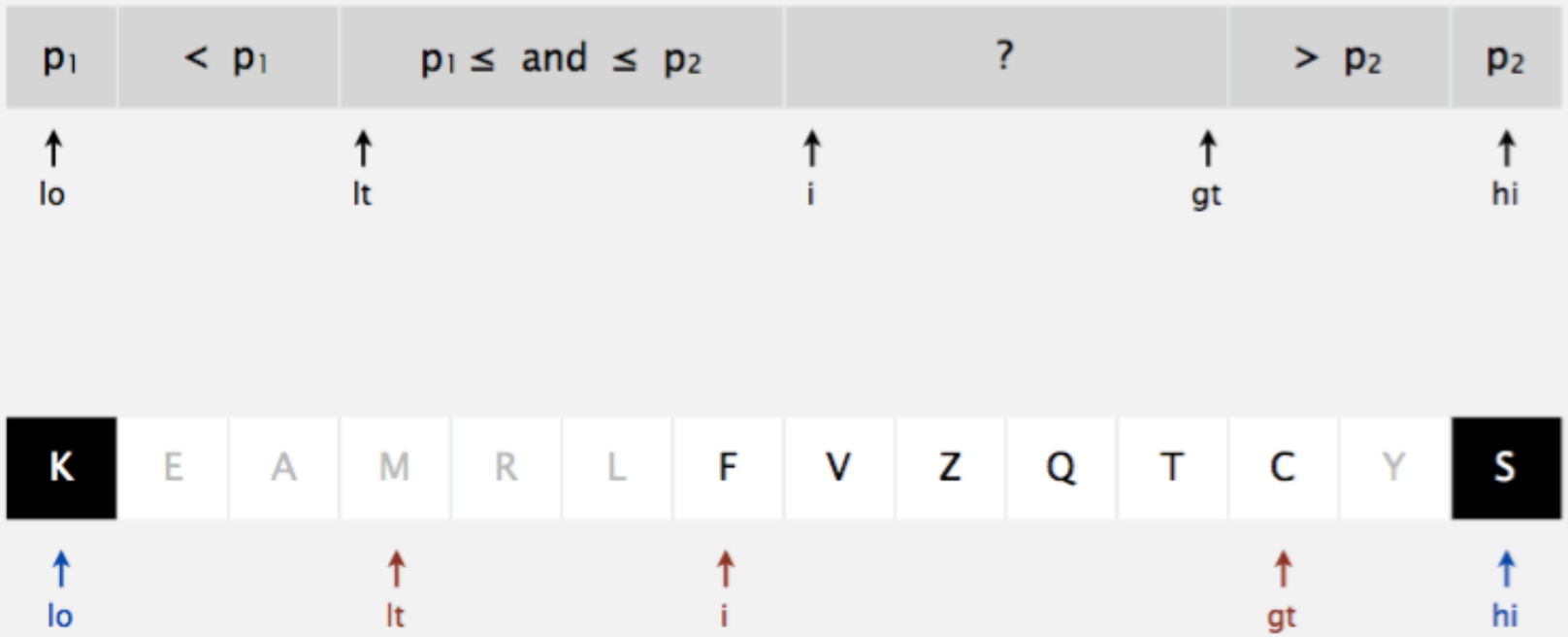

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Dual-pivot partitioning demo

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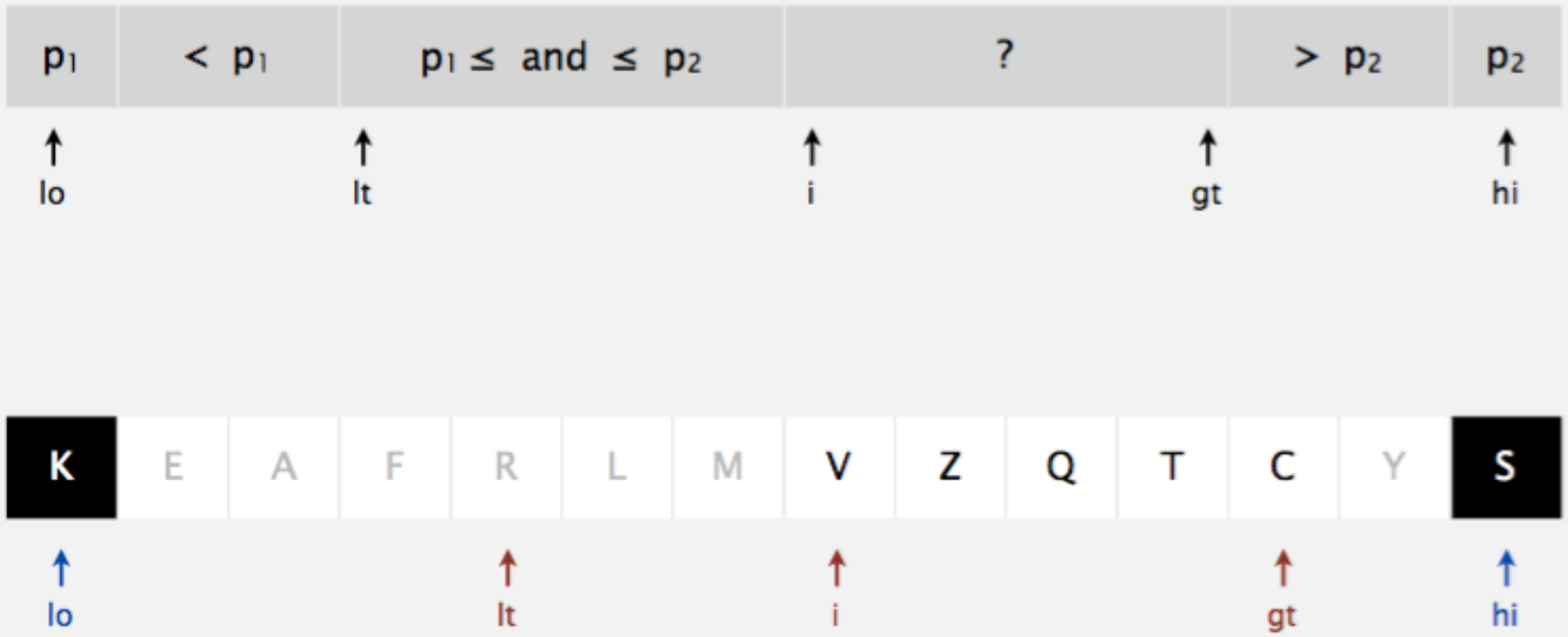

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Dual-pivot partitioning demo

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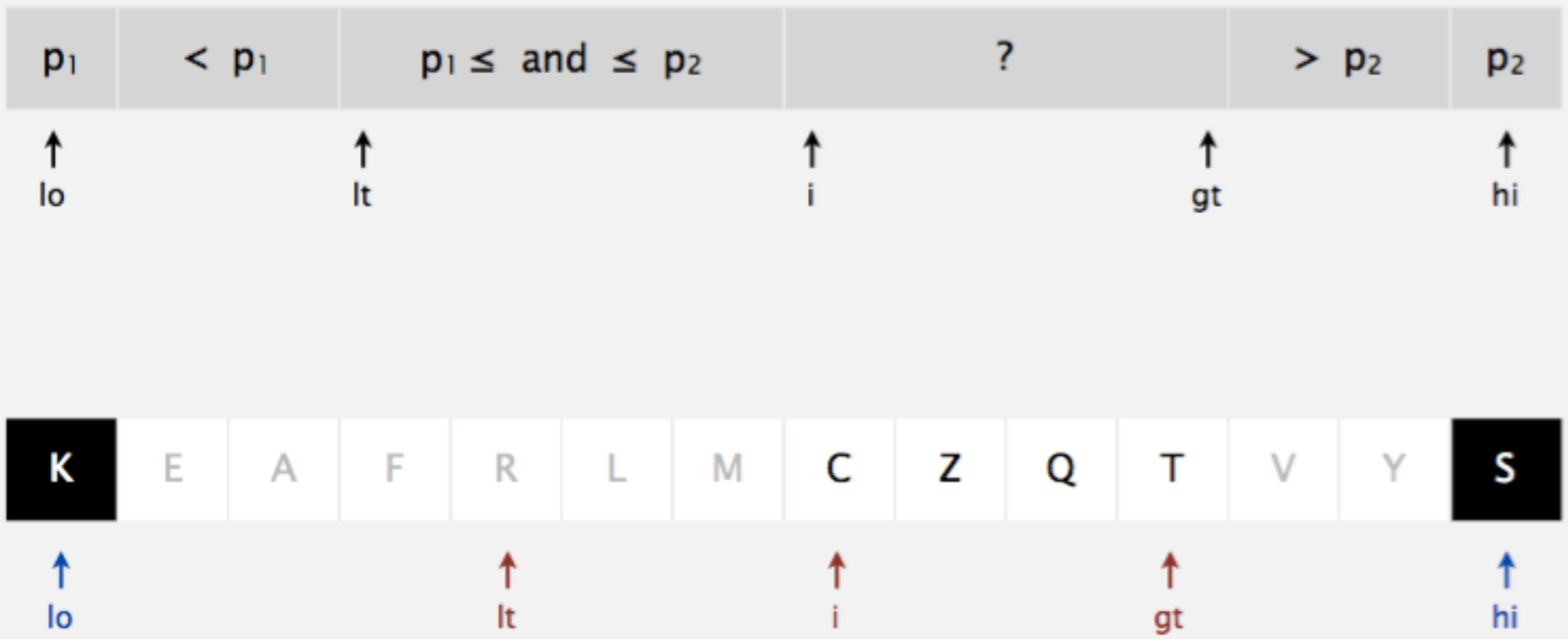

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Partitioning: Dual-pivot partitioning

Dual-pivot partitioning demo

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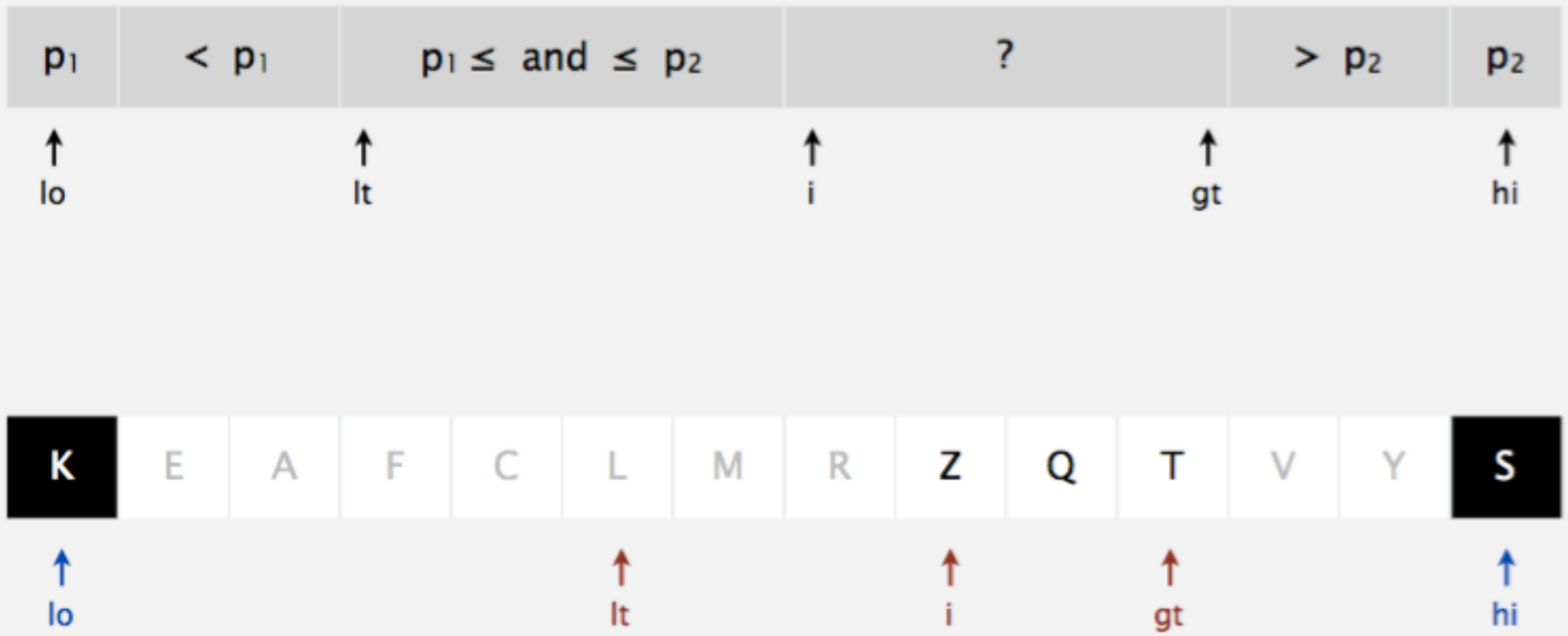

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Dual-pivot partitioning demo

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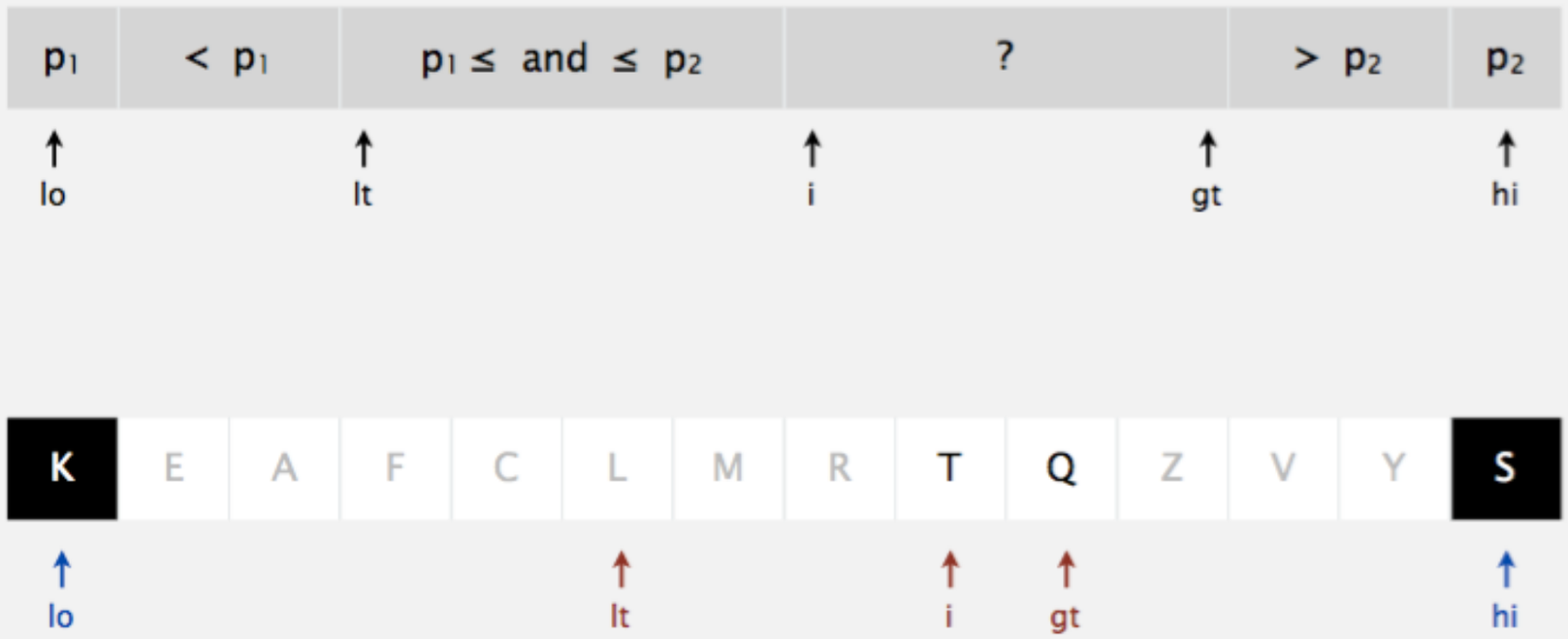

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Dual-pivot partitioning demo

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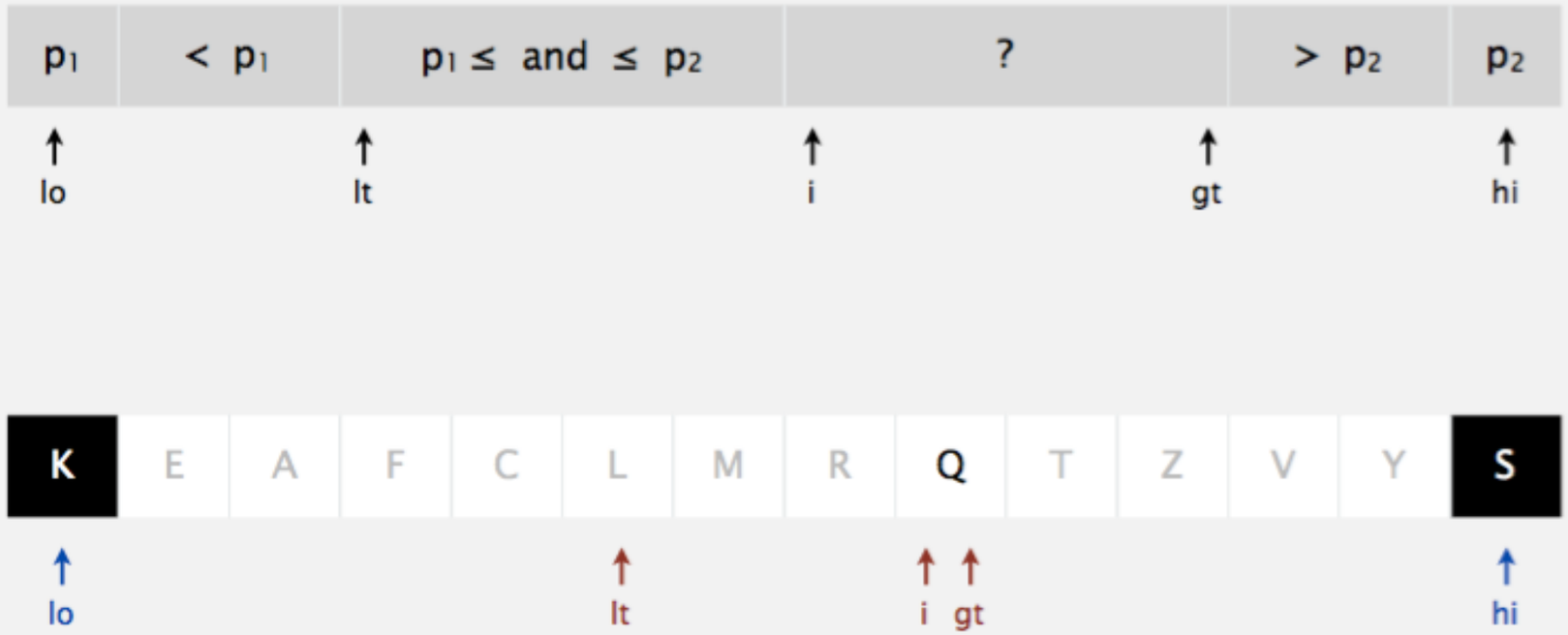

Partitioning: Dual-pivot partitioning

Dual-pivot partitioning demo

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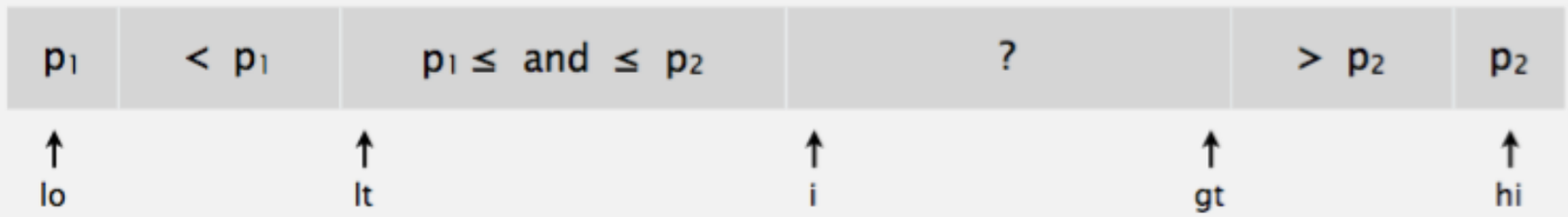
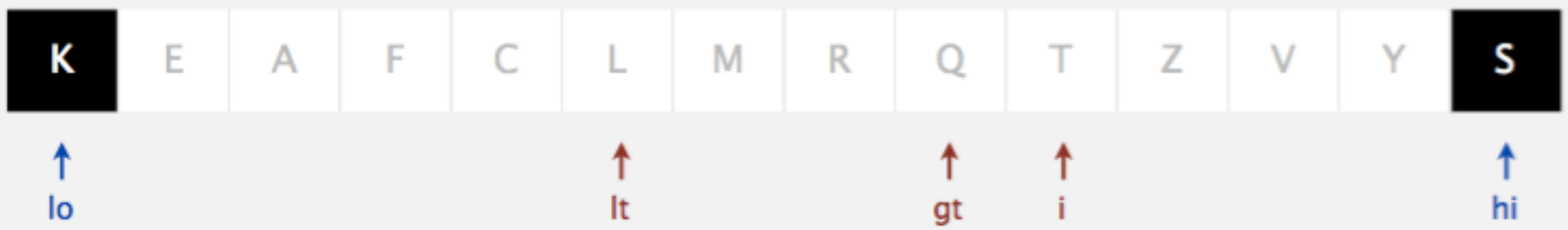




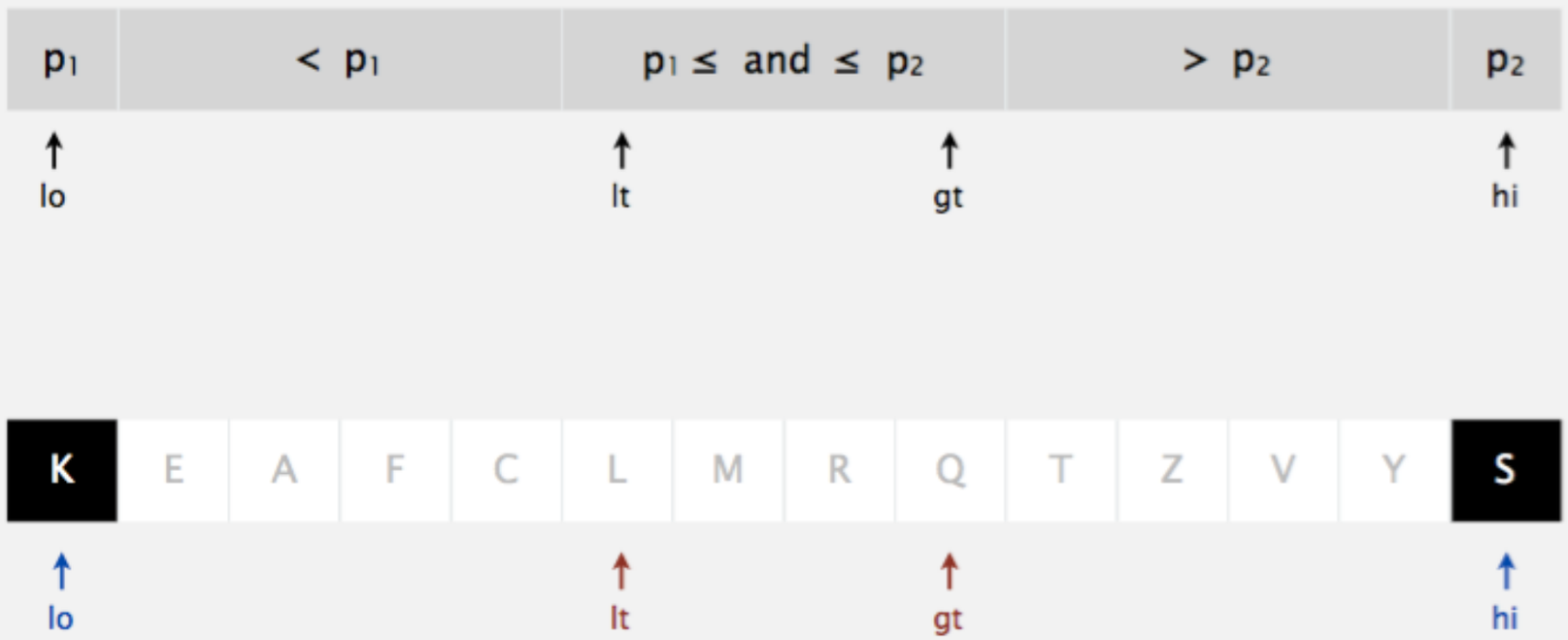
stop when pointers cross

Partitioning: Dual-pivot partitioning

Dual-pivot partitioning demo

Finalize.

- Exchange `a[lo]` with `a[--lt]`.
- Exchange `a[hi]` with `a[++gt]`.



Partitioning: Dual-pivot partitioning

Dual-pivot partitioning demo

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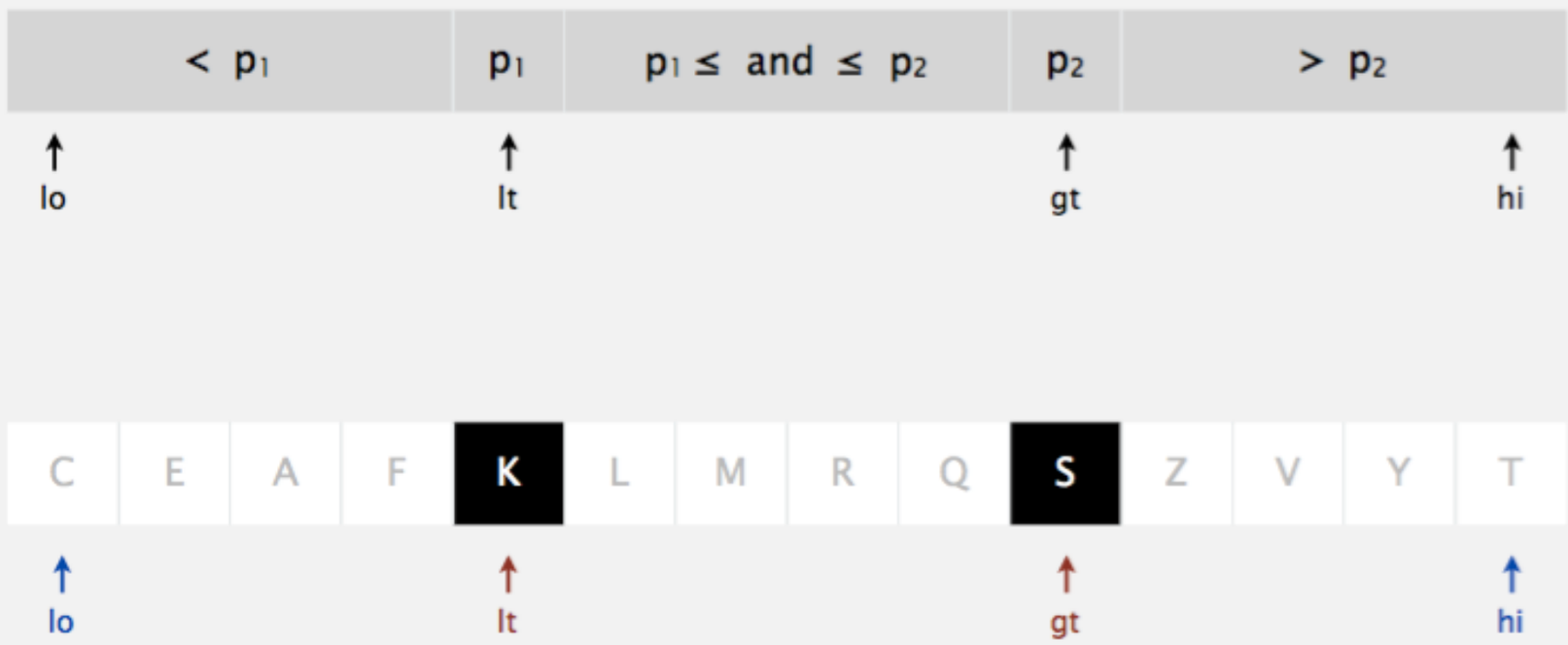


3-way partitioned

Quicksort

Quicksort: Java code for partitioning

```
private static int partition(Comparable[] a, int lo, int hi)
{
   int i = lo, j = hi+1;
   while (true)
   {
      while (less(a[++i], a[lo]))
         if (i == hi) break;

      while (less(a[lo], a[--j]))
         if (j == lo) break;

      if (i >= j) break;
      exch(a, i, j);
   }

   exch(a, lo, j);
   return j;
}
```

find item on left to swap

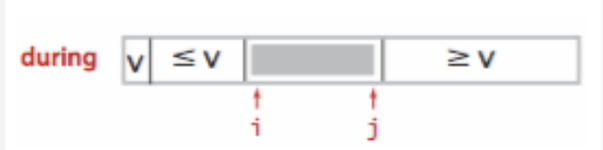
find item on right to swap

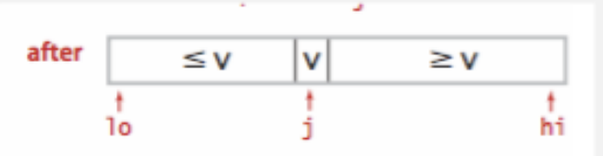
check if pointers cross

swap

swap with partitioning item

return index of item now known to be in place





Quicksort

Quicksort: Java implementation

```
public class Quick
{
   private static int partition(Comparable[] a, int lo, int hi)
   {  /* see previous slide */  }

   public static void sort(Comparable[] a)
   {
      StdRandom.shuffle(a);
      sort(a, 0, a.length - 1);
   }

   private static void sort(Comparable[] a, int lo, int hi)
   {
      if (hi <= lo) return;
      int j = partition(a, lo, hi);
      sort(a, lo, j-1);
      sort(a, j+1, hi);
  }
}
```

shuffle needed for performance guarantee (stay tuned)

Quicksort

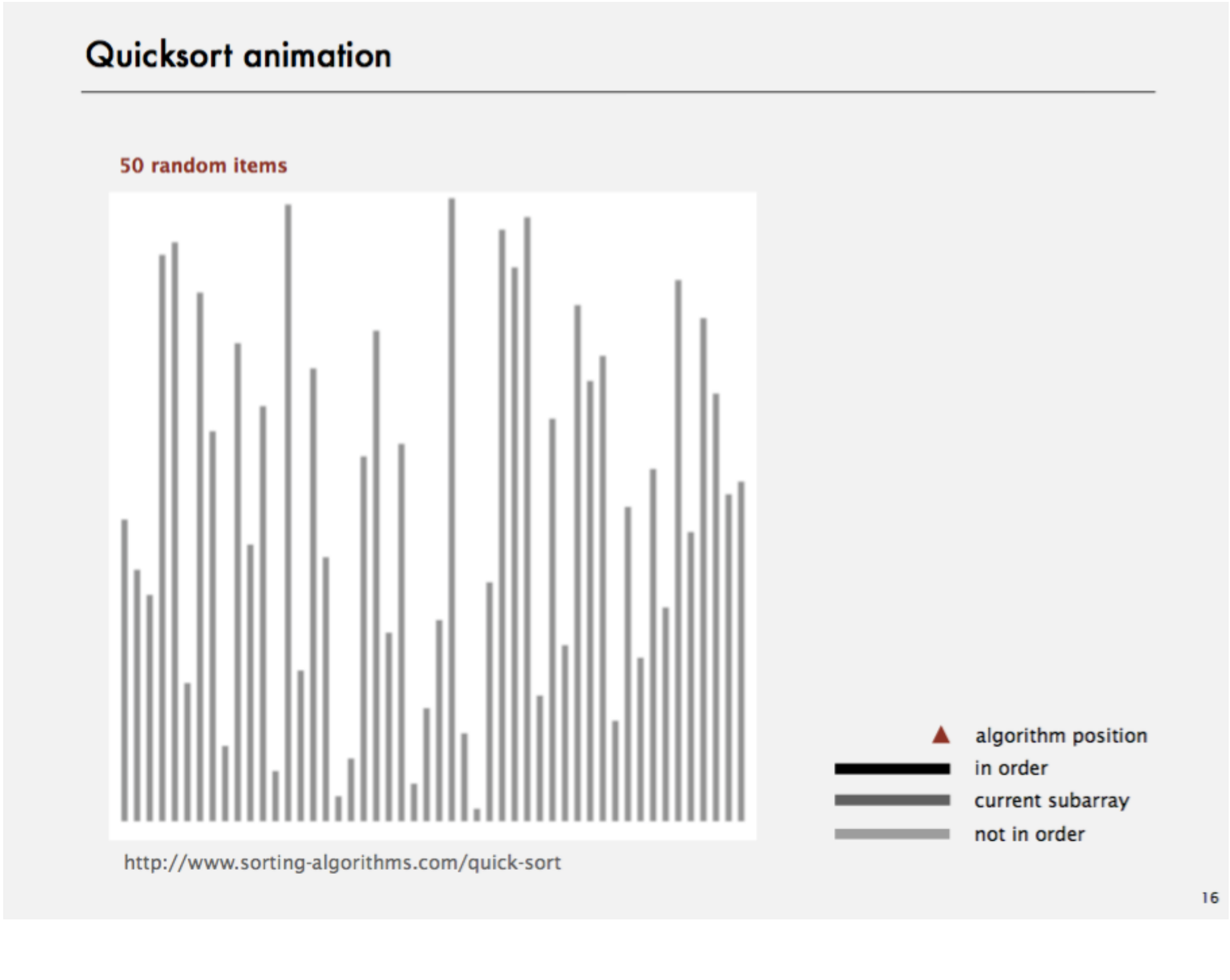
Quicksort trace

	lo	j	hi	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
initial values				Q	U	I	C	K	S	O	R	T	E	X	A	M	P	L	E
random shuffle				K	R	A	T	E	L	E	P	U	I	M	Q	C	X	O	S
	0	5	15	E	C	A	I	E	K	L	P	U	T	M	Q	R	X	O	S
	0	3	4	E	C	A	E	I	K	L	P	U	T	M	Q	R	X	O	S
	0	2	2	A	C	E	E	I	K	L	P	U	T	M	Q	R	X	O	S
	0	0	1	A	C	E	E	I	K	L	P	U	T	M	Q	R	X	O	S
	1		1	A	C	E	E	I	K	L	P	U	T	M	Q	R	X	O	S
	4		4	A	C	E	E	I	K	L	P	U	T	M	Q	R	X	O	S
	6	6	15	A	C	E	E	I	K	L	P	U	T	M	Q	R	X	O	S
	7	9	15	A	C	E	E	I	K	L	M	O	P	T	Q	R	X	U	S
	7	7	8	A	C	E	E	I	K	L	M	O	P	T	Q	R	X	U	S
	8		8	A	C	E	E	I	K	L	M	O	P	T	Q	R	X	U	S
	10	13	15	A	C	E	E	I	K	L	M	O	P	S	Q	R	T	U	X
	10	12	12	A	C	E	E	I	K	L	M	O	P	R	Q	S	T	U	X
	10	11	11	A	C	E	E	I	K	L	M	O	P	Q	R	S	T	U	X
	10		10	A	C	E	E	I	K	L	M	O	P	Q	R	S	T	U	X
	14	14	15	A	C	E	E	I	K	L	M	O	P	Q	R	S	T	U	X
	15		15	A	C	E	E	I	K	L	M	O	P	Q	R	S	T	U	X
result				A	C	E	E	I	K	L	M	O	P	Q	R	S	T	U	X

no partition for subarrays of size 1

Quicksort trace (array contents after each partition)

Quicksort


Quicksort animation
50 random items
algorithm position
in order
current subarray
not in order
http://www.sorting-algorithms.com/quick-sort
16

Quicksort

Quicksort: implementation details

Partitioning in-place. Using an extra array makes partitioning easier (and stable), but is not worth the cost.

Terminating the loop. Testing whether the pointers cross is trickier than it might seem.

Equal keys. When duplicates are present, it is (counter-intuitively) better to stop scans on keys equal to the partitioning item's key. ← stay tuned

Preserving randomness. Shuffling is needed for performance guarantee.

Equivalent alternative. Pick a random partitioning item in each subarray.

Quicksort: empirical analysis (1961)

Running time estimates:

- Algol 60 implementation.
- National-Elliott 405 computer.

Table 1

NUMBER OF ITEMS	MERGE SORT	QUICKSORT
500	2 min 8 sec	1 min 21 sec
1,000	4 min 48 sec	3 min 8 sec
1,500	8 min 15 sec*	5 min 6 sec
2,000	11 min 0 sec*	6 min 47 sec

* These figures were computed by formula, since they cannot be achieved on the 405 owing to limited store size.

sorting N 6-word items with 1-word keys

Elliott 405 magnetic disc (16K words)

Quicksort

Quicksort: empirical analysis

Running time estimates:

- Home PC executes 10^8 compares/second.
- Supercomputer executes 10^{12} compares/second.

	insertion sort (N^2)			mergesort (N log N)			quicksort (N log N)		
computer	thousand	million	billion	thousand	million	billion	thousand	million	billion
home	instant	2.8 hours	317 years	instant	1 second	18 min	instant	0.6 sec	12 min
super	instant	1 second	1 week	instant	instant	instant	instant	instant	instant

Lesson 1. Good algorithms are better than supercomputers.

Lesson 2. Great algorithms are better than good ones.

Quicksort

Quicksort: best-case analysis

Best case. Number of compares is $\sim N \lg N$.

			a[ ]														
lo	j	hi	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14
initial values			H	A	C	B	F	E	G	D	L	I	K	J	N	M	O
random shuffle			H	A	C	B	F	E	G	D	L	I	K	J	N	M	O
0	7	14	D	A	C	B	F	E	G	H	L	I	K	J	N	M	O
0	3	6	B	A	C	D	F	E	G	H	L	I	K	J	N	M	O
0	1	2	A	B	C	D	F	E	G	H	L	I	K	J	N	M	O
0		0	A	B	C	D	F	E	G	H	L	I	K	J	N	M	O
2		2	A	B	C	D	F	E	G	H	L	I	K	J	N	M	O
4	5	6	A	B	C	D	E	F	G	H	L	I	K	J	N	M	O
4		4	A	B	C	D	E	F	G	H	L	I	K	J	N	M	O
6		6	A	B	C	D	E	F	G	H	L	I	K	J	N	M	O
8	11	14	A	B	C	D	E	F	G	H	J	I	K	L	N	M	O
8	9	10	A	B	C	D	E	F	G	H	I	J	K	L	N	M	O
8		8	A	B	C	D	E	F	G	H	I	J	K	L	N	M	O
10		10	A	B	C	D	E	F	G	H	I	J	K	L	N	M	O
12	13	14	A	B	C	D	E	F	G	H	I	J	K	L	M	N	O
12		12	A	B	C	D	E	F	G	H	I	J	K	L	M	N	O
14		14	A	B	C	D	E	F	G	H	I	J	K	L	M	N	O
			A	B	C	D	E	F	G	H	I	J	K	L	M	N	O

Quicksort

Quicksort: worst-case analysis

Worst case. Number of compares is $\sim \frac{1}{2} N^2$.

			a[ ]														
lo	j	hi	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14
initial values			A	B	C	D	E	F	G	H	I	J	K	L	M	N	O
random shuffle			A	B	C	D	E	F	G	H	I	J	K	L	M	N	O
0	0	14	A	B	C	D	E	F	G	H	I	J	K	L	M	N	O
1	1	14	A	B	C	D	E	F	G	H	I	J	K	L	M	N	O
2	2	14	A	B	C	D	E	F	G	H	I	J	K	L	M	N	O
3	3	14	A	B	C	D	E	F	G	H	I	J	K	L	M	N	O
4	4	14	A	B	C	D	E	F	G	H	I	J	K	L	M	N	O
5	5	14	A	B	C	D	E	F	G	H	I	J	K	L	M	N	O
6	6	14	A	B	C	D	E	F	G	H	I	J	K	L	M	N	O
7	7	14	A	B	C	D	E	F	G	H	I	J	K	L	M	N	O
8	8	14	A	B	C	D	E	F	G	H	I	J	K	L	M	N	O
9	9	14	A	B	C	D	E	F	G	H	I	J	K	L	M	N	O
10	10	14	A	B	C	D	E	F	G	H	I	J	K	L	M	N	O
11	11	14	A	B	C	D	E	F	G	H	I	J	K	L	M	N	O
12	12	14	A	B	C	D	E	F	G	H	I	J	K	L	M	N	O
13	13	14	A	B	C	D	E	F	G	H	I	J	K	L	M	N	O
14		14	A	B	C	D	E	F	G	H	I	J	K	L	M	N	O
			A	B	C	D	E	F	G	H	I	J	K	L	M	N	O

Quicksort: average-case analysis

Proposition. The average number of compares C_N to quicksort an array of N distinct keys is $\sim 2N \ln N$ (and the number of exchanges is $\sim \frac{1}{3} N \ln N$).

Pf. C_N satisfies the recurrence $C_0 = C_1 = 0$ and for $N \geq 2$:

partitioning; left; right

$$C_N = (N+1) + \left(\frac{C_0 + C_{N-1}}{N}\right) + \left(\frac{C_1 + C_{N-2}}{N}\right) + \ldots + \left(\frac{C_{N-1} + C_0}{N}\right)$$

partitioning probability

- Multiply both sides by N and collect terms:

$$NC_N = N(N+1) + 2(C_0 + C_1 + \ldots + C_{N-1})$$

- Subtract from this equation the same equation for $N - 1$:

$$NC_N - (N-1)C_{N-1} = 2N + 2C_{N-1}$$

- Rearrange terms and divide by $N(N+1)$:

$$\frac{C_N}{N+1} = \frac{C_{N-1}}{N} + \frac{2}{N+1}$$

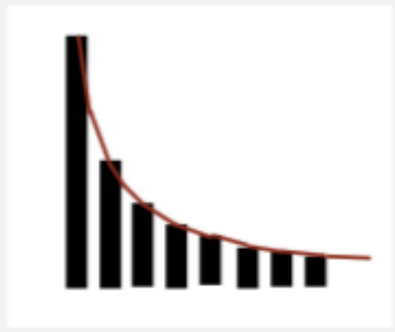
Quicksort: average-case analysis

- Repeatedly apply previous equation:

$$\begin{aligned}
\frac{C_N}{N+1} &= \frac{C_{N-1}}{N} + \frac{2}{N+1} \\
&= \frac{C_{N-2}}{N-1} + \frac{2}{N} + \frac{2}{N+1} \quad \longleftarrow \text{substitute previous equation} \\
&= \frac{C_{N-3}}{N-2} + \frac{2}{N-1} + \frac{2}{N} + \frac{2}{N+1} \\
&= \frac{2}{3} + \frac{2}{4} + \frac{2}{5} + \cdots + \frac{2}{N+1}
\end{aligned}$$

- Approximate sum by an integral:

$$\begin{aligned}
C_N &= 2(N+1)\left(\frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \cdots \frac{1}{N+1}\right) \\
&\sim 2(N+1)\int_3^{N+1} \frac{1}{x}\,dx
\end{aligned}$$



- Finally, the desired result:

$$C_N \sim 2(N+1)\ln N \approx 1.39 N \lg N$$

Quicksort

Quicksort: summary of performance characteristics

Quicksort is a (Las Vegas) randomized algorithm.

- Guaranteed to be correct.
- Running time depends on random shuffle.

Average case. Expected number of compares is $\sim 1.39\, N \lg N$.

- 39% more compares than mergesort.
- Faster than mergesort in practice because of less data movement.

Best case. Number of compares is $\sim N \lg N$.

Worst case. Number of compares is $\sim \frac{1}{2} N^2$.

[but more likely that lightning bolt strikes computer during execution]

Quicksort

Quicksort properties

Proposition. Quicksort is an in-place sorting algorithm.

Pf.

- Partitioning: constant extra space.
- Depth of recursion: logarithmic extra space (with high probability).

can guarantee logarithmic depth by recurring on smaller subarray before larger subarray (requires using an explicit stack)

Proposition. Quicksort is not stable.

Pf. [by counterexample]

i	j	0	1	2	3
		B_1	C_1	C_2	A_1
1	3	B_1	C_1	C_2	A_1
1	3	B_1	A_1	C_2	C_1
0	1	A_1	B_1	C_2	C_1

Quicksort: practical improvements

Insertion sort small subarrays.

- Even quicksort has too much overhead for tiny subarrays.
- Cutoff to insertion sort for ≈ 10 items.

```
private static void sort(Comparable[] a, int lo, int hi)
{
   if (hi <= lo + CUTOFF - 1)
   {
      Insertion.sort(a, lo, hi);
      return;
   }
   int j = partition(a, lo, hi);
   sort(a, lo, j-1);
   sort(a, j+1, hi);
}
```

Quicksort: practical improvements

Median of sample.

- Best choice of pivot item = median.
- Estimate true median by taking median of sample.
- Median-of-3 (random) items.

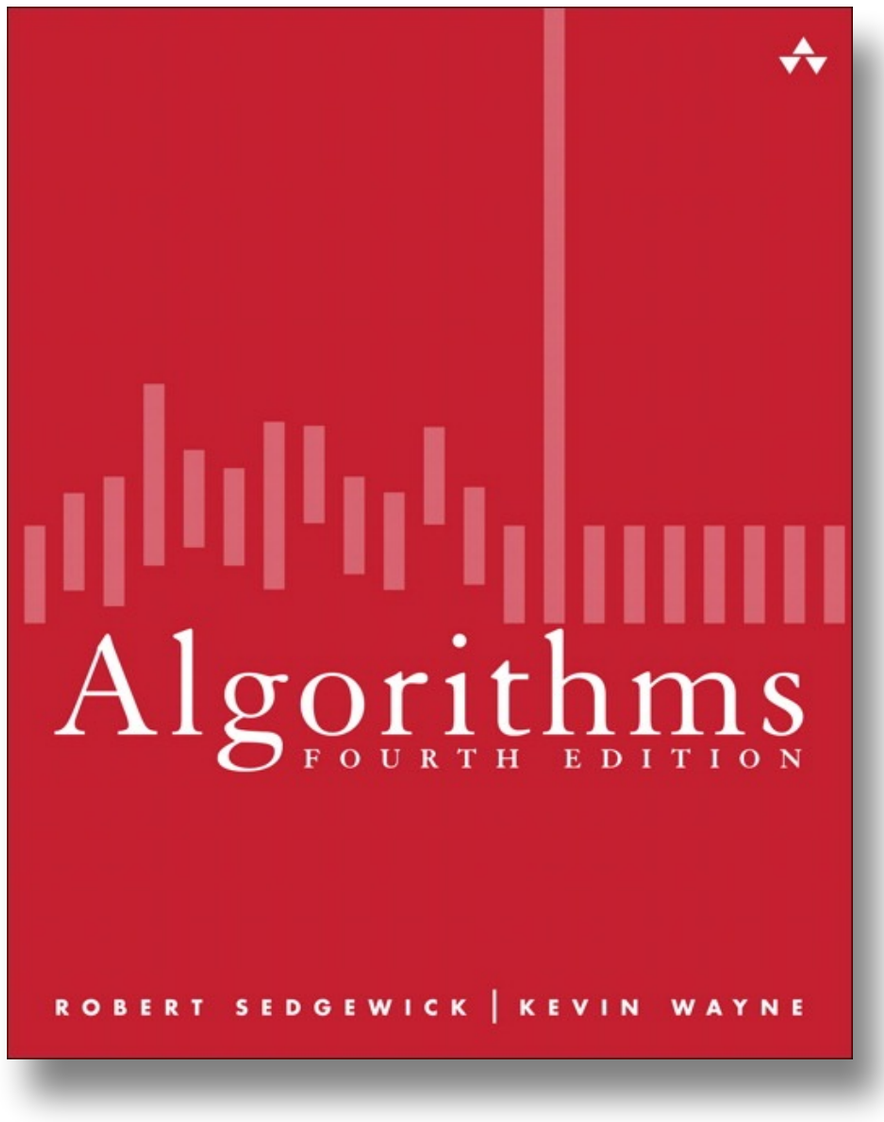
~ 12/7 N ln N compares (14% fewer)
~ 12/35 N ln N exchanges (3% more)

```
private static void sort(Comparable[] a, int lo, int hi)
{
   if (hi <= lo) return;

   int median = medianOf3(a, lo, lo + (hi - lo)/2, hi);
   swap(a, lo, median);

   int j = partition(a, lo, hi);
   sort(a, lo, j-1);
   sort(a, j+1, hi);
}
```

Lecture Overview



2.3 Quicksort

- Quicksort
- Selection
- Duplicate keys
- System sorts

Selection

Selection

Goal. Given an array of N items, find the k^{th} smallest item.

Ex. Min ($k = 0$), max ($k = N - 1$), median ($k = N/2$).

Applications.

- Order statistics.
- Find the "top k."

Use theory as a guide.

- Easy $N \log N$ upper bound. How?
- Easy N upper bound for $k = 1, 2, 3$. How?
- Easy N lower bound. Why?

Which is true?

- $N \log N$ lower bound? ← is selection as hard as sorting?
- N upper bound? ← is there a linear-time algorithm?

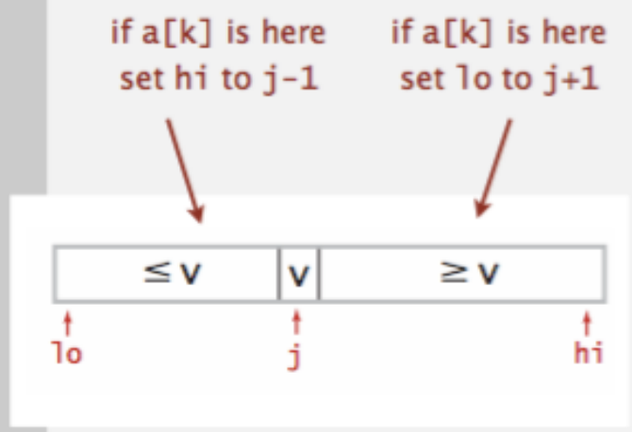
Quick-select

Partition array so that:

- Entry `a[j]` is in place.
- No larger entry to the left of `j`.
- No smaller entry to the right of `j`.

Repeat in one subarray, depending on `j`; finished when `j` equals `k`.

```
public static Comparable select(Comparable[] a, int k)
{
    StdRandom.shuffle(a);
    int lo = 0, hi = a.length - 1;
    while (hi > lo)
    {
       int j = partition(a, lo, hi);
       if      (j < k) lo = j + 1;
       else if (j > k) hi = j - 1;
       else            return a[k];
    }
    return a[k];
}
```



Selection

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- Easy N lower bound. Why?

Which is true?

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- N upper bound? ← is there a linear-time algorithm?

Quick-select: mathematical analysis

Proposition. Quick-select takes linear time on average.

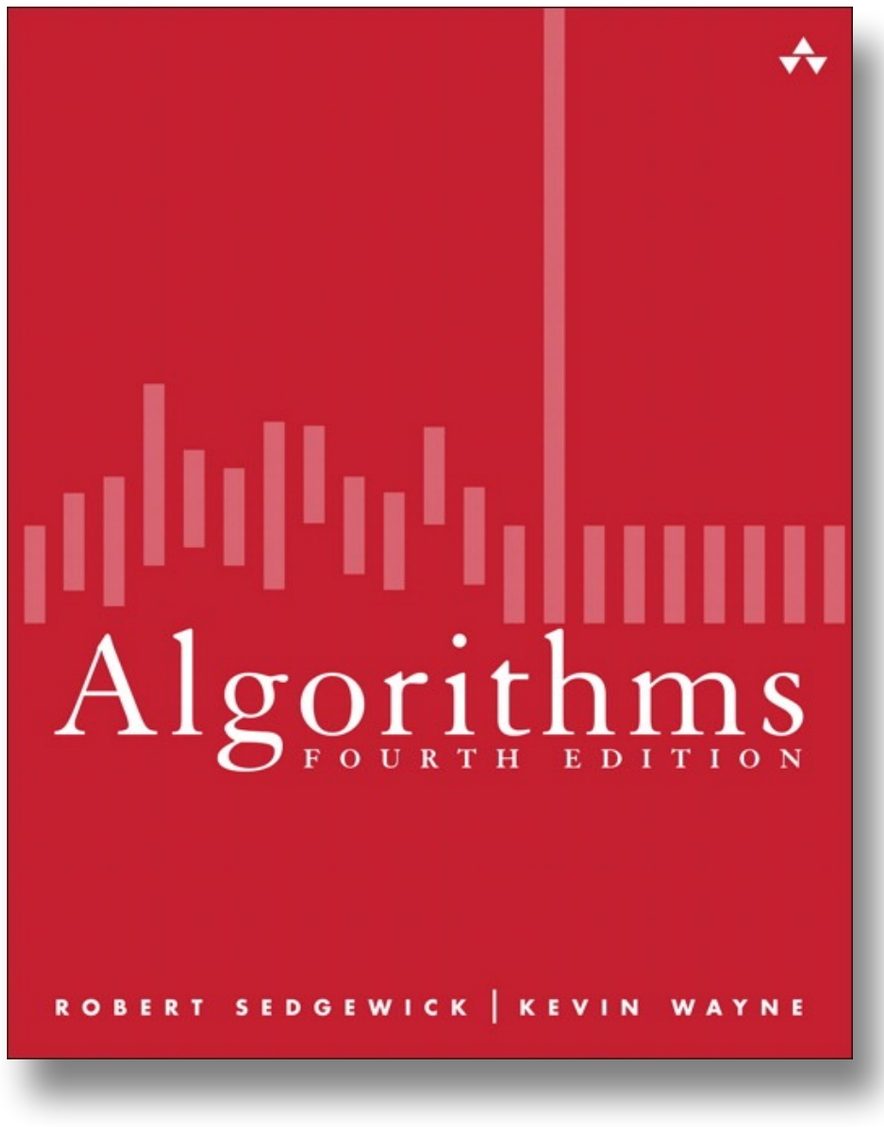
Pf sketch.

- Intuitively, each partitioning step splits array approximately in half: $N + N/2 + N/4 + \ldots + 1 \sim 2N$ compares.
- Formal analysis similar to quicksort analysis yields:

$$C_N = 2N + 2k \ln (N/k) + 2(N-k) \ln (N/(N-k))$$

- Ex: $(2 + 2 \ln 2) N \approx 3.38 N$ compares to find median $k = N/2$.

Lecture Overview



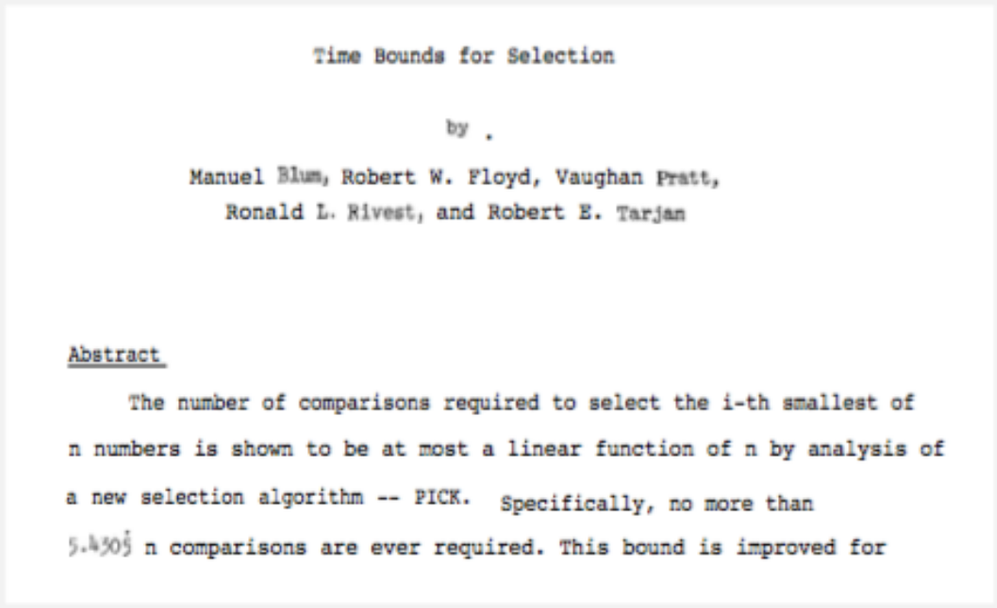
2.3 Quicksort

- Quicksort
- Selection
- Duplicate keys
- System sorts

Duplicate Keys

Theoretical context for selection

Proposition. [Blum, Floyd, Pratt, Rivest, Tarjan, 1973] Compare-based selection algorithm whose worst-case running time is linear.



Time Bounds for Selection

by

Manuel Blum, Robert W. Floyd, Vaughan Pratt,
Ronald L. Rivest, and Robert E. Tarjan

Abstract

The number of comparisons required to select the i-th smallest of n numbers is shown to be at most a linear function of n by analysis of a new selection algorithm -- PICK. Specifically, no more than 5.4305 n comparisons are ever required. This bound is improved for

Remark. Constants are high ⇒ not used in practice.

Use theory as a guide.

- Still worthwhile to seek practical linear-time (worst-case) algorithm.
- Until one is discovered, use quick-select if you don't need a full sort.

Duplicate Keys

Duplicate keys

Often, purpose of sort is to bring items with equal keys together.

- Sort population by age.
- Remove duplicates from mailing list.
- Sort job applicants by college attended.

Typical characteristics of such applications.

- Huge array.
- Small number of key values.

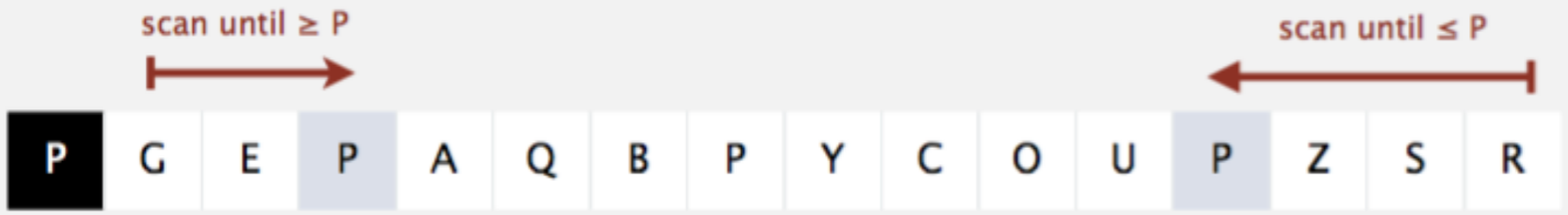
```
Chicago 09:25:52
Chicago 09:03:13
Chicago 09:21:05
Chicago 09:19:46
Chicago 09:19:32
Chicago 09:00:00
Chicago 09:35:21
Chicago 09:00:59
Houston 09:01:10
Houston 09:00:13
Phoenix 09:37:44
Phoenix 09:00:03
Phoenix 09:14:25
Seattle 09:10:25
Seattle 09:36:14
Seattle 09:22:43
Seattle 09:10:11
Seattle 09:22:54
```

key

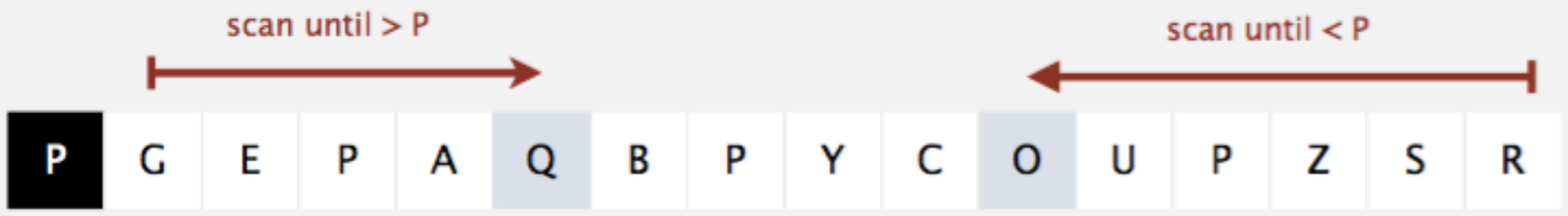
Duplicate Keys

Duplicate keys: stop on equal keys

Our partitioning subroutine stops both scans on equal keys.



Q. Why not continue scans on equal keys?



Duplicate Keys

Partitioning an array with all equal keys

		a[ ]															
i	j	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
		A	A	A	A	A	A	A	A	A	A	A	A	A	A	A	A
1	15	A	A	A	A	A	A	A	A	A	A	A	A	A	A	A	A
1	15	A	A	A	A	A	A	A	A	A	A	A	A	A	A	A	A
2	14	A	A	A	A	A	A	A	A	A	A	A	A	A	A	A	A
2	14	A	A	A	A	A	A	A	A	A	A	A	A	A	A	A	A
3	13	A	A	A	A	A	A	A	A	A	A	A	A	A	A	A	A
3	13	A	A	A	A	A	A	A	A	A	A	A	A	A	A	A	A
4	12	A	A	A	A	A	A	A	A	A	A	A	A	A	A	A	A
4	12	A	A	A	A	A	A	A	A	A	A	A	A	A	A	A	A
5	11	A	A	A	A	A	A	A	A	A	A	A	A	A	A	A	A
5	11	A	A	A	A	A	A	A	A	A	A	A	A	A	A	A	A
6	10	A	A	A	A	A	A	A	A	A	A	A	A	A	A	A	A
6	10	A	A	A	A	A	A	A	A	A	A	A	A	A	A	A	A
7	9	A	A	A	A	A	A	A	A	A	A	A	A	A	A	A	A
7	9	A	A	A	A	A	A	A	A	A	A	A	A	A	A	A	A
	8	A	A	A	A	A	A	A	A	A	A	A	A	A	A	A	A
	8	A	A	A	A	A	A	A	A	A	A	A	A	A	A	A	A

Duplicate Keys

Duplicate keys: partitioning strategies

Bad. Don't stop scans on equal keys.

[$\sim \frac{1}{2} N^2$ compares when all keys equal]

B A A B A B B B C C C A A A A A A A A A A A A

Good. Stop scans on equal keys.

[$\sim N \lg N$ compares when all keys equal]

B A A B A B C C B C B A A A A A A A A A A A A

Better. Put all equal keys in place. How?

[$\sim N$ compares when all keys equal]

A A A B B B B B C C C A A A A A A A A A A A A

Duplicate Keys

Duplicate keys: lower bound

Sorting lower bound. If there are n distinct keys and the i^{th} one occurs x_i times, any compare-based sorting algorithm must use at least

$$\lg \left(\frac{N!}{x_1!\, x_2! \cdots x_n!} \right) \sim -\sum_{i=1}^{n} x_i \lg \frac{x_i}{N}$$

$N \lg N$ when all distinct;
linear when only a constant number of distinct keys

compares in the worst case.

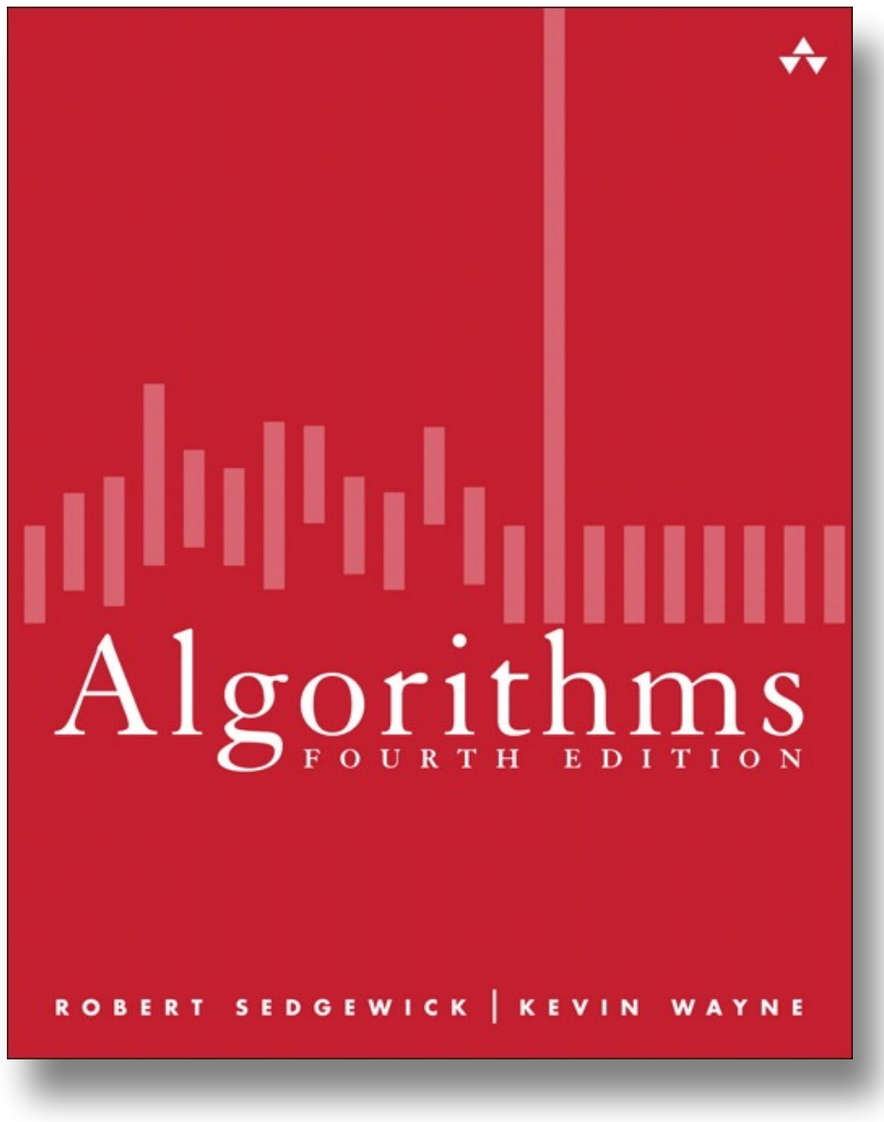
proportional to lower bound

Proposition. [Sedgewick-Bentley 1997]
Quicksort with 3-way partitioning is entropy-optimal.

Pf. [beyond scope of course]

Bottom line. Quicksort with 3-way partitioning reduces running time from linearithmic to linear in broad class of applications.

Lecture Overview



2.3 Quicksort

- Quicksort
- Selection
- Duplicate keys
- System sorts

Sorting applications

Sorting algorithms are essential in a broad variety of applications:

- Sort a list of names.
- Organize an MP3 library.
- Display Google PageRank results.
- List RSS feed in reverse chronological order.

obvious applications

- Find the median.
- Identify statistical outliers.
- Binary search in a database.
- Find duplicates in a mailing list.

problems become easy once items are in sorted order

- Data compression.
- Computer graphics.
- Computational biology.
- Load balancing on a parallel computer.

. . .

non-obvious applications

System Sorts

War story (system sort in C)

A beautiful bug report. [Allan Wilks and Rick Becker, 1991]

We found that qsort is unbearably slow on "organ-pipe" inputs like "01233210":

```
main (int argc, char**argv) {
   int n = atoi(argv[1]), i, x[100000];
   for (i = 0; i < n; i++)
      x[i] = i;
   for ( ; i < 2*n; i++)
      x[i] = 2*n-i-1;
   qsort(x, 2*n, sizeof(int), intcmp);
}
```

Here are the timings on our machine:

```
$ time a.out 2000
real    5.85s
$ time a.out 4000
real   21.64s
$time a.out 8000
real   85.11s
```

War story (system sort in C)

Bug. A `qsort()` call that should have taken seconds was taking minutes.

At the time, almost all `qsort()` implementations based on those in:

- Version 7 Unix (1979): quadratic time to sort organ-pipe arrays.
- BSD Unix (1983): quadratic time to sort random arrays of 0s and 1s.

System Sorts

Which sorting algorithm to use?

Many sorting algorithms to choose from:

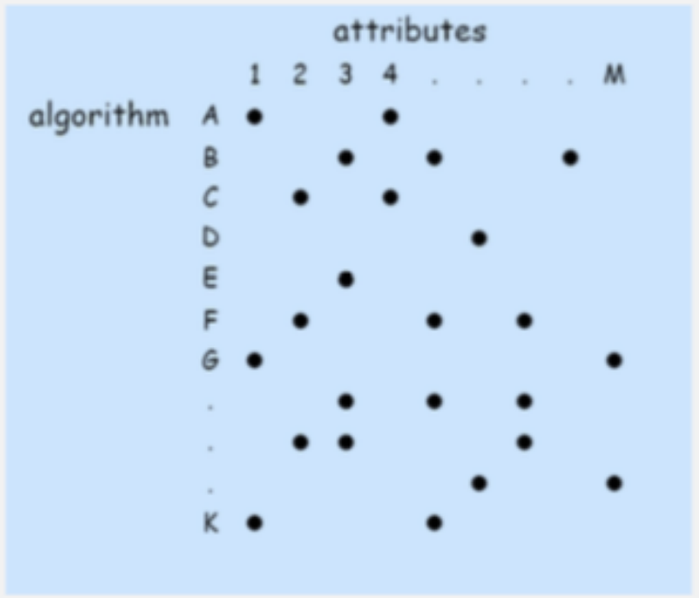
sorts	algorithms
elementary sorts	insertion sort, selection sort, bubblesort, shaker sort, ...
subquadratic sorts	quicksort, mergesort, heapsort, shellsort, samplesort, ...
system sorts	dual-pivot quicksort, timsort, introsort, ...
external sorts	Poly-phase mergesort, cascade-merge, psort, ...
radix sorts	MSD, LSD, 3-way radix quicksort, ...
parallel sorts	bitonic sort, odd-even sort, smooth sort, GPUsort, ...

System Sorts

Which sorting algorithm to use?

Applications have diverse attributes.

- Stable?
- Parallel?
- In-place?
- Deterministic?
- Duplicate keys?
- Multiple key types?
- Linked list or arrays?
- Large or small items?
- Randomly-ordered array?
- Guaranteed performance?



many more combinations of attributes than algorithms

Q. Is the system sort good enough?

A. Usually.

System Sorts

System sort in Java 7

`Arrays.sort()`.

- Has method for objects that are `Comparable`.
- Has overloaded method for each primitive type.
- Has overloaded method for use with a `Comparator`.
- Has overloaded methods for sorting subarrays.

Algorithms.

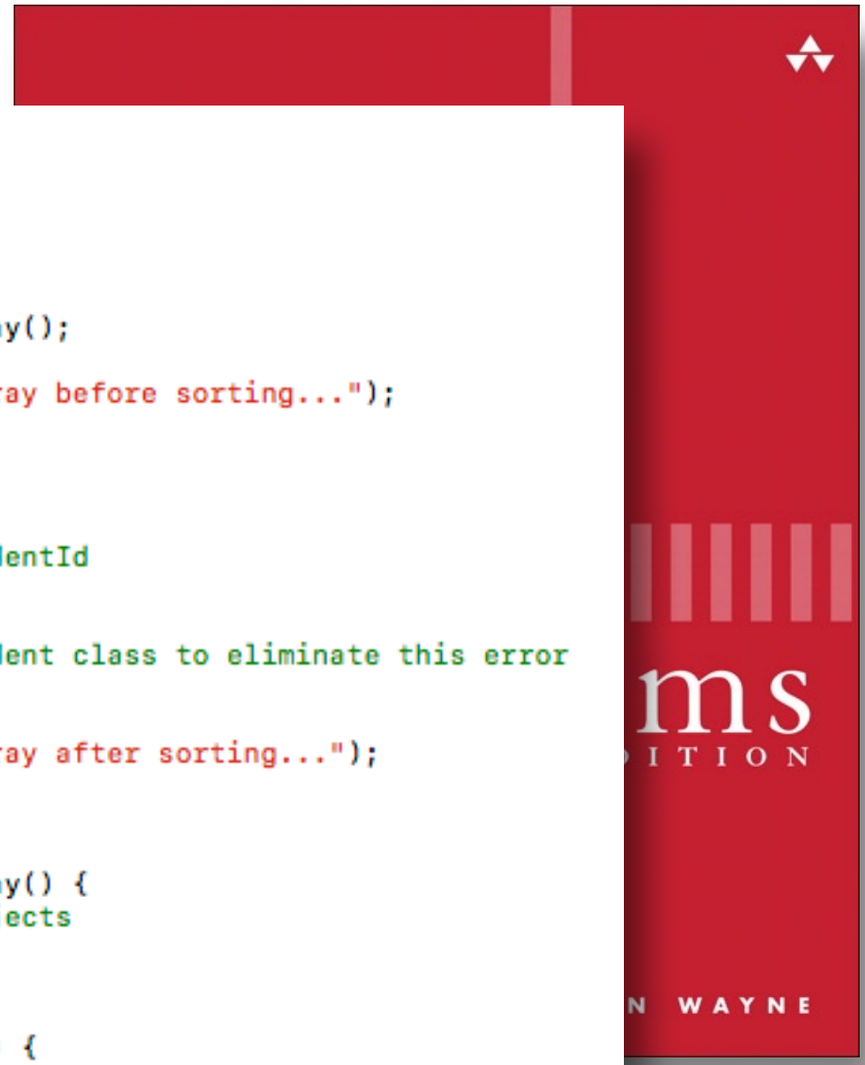
- Dual-pivot quicksort for primitive types.
- Timsort for reference types.

Q. Why use different algorithms for primitive and reference types?

Homework 4

Mergesort

```java
public class TestMergeSort {

    /**
     * @param args
     */
    public static void main(String[] args) {
        Student[] students = initializeStudentsArray();

        System.out.println("Displaying students array before sorting...");
        display(students);

        System.out.println("Being sorting...");
        /*
         * The Student array will be sorted by studentId
         * which is declared as a String
         */
        Merge.sort(students); // TODO: Fix the Student class to eliminate this error
        System.out.println("End sorting...");

        System.out.println("Displaying students array after sorting...");
        display(students);
    }

    private static Student[] initializeStudentsArray() {
        // TODO: Initialize an array of Student objects
        return null;
    }

    private static void display(Student[] students) {
        // TODO: Display the contents of the Students Array
    }

}
```



Assignment due next Monday at 11:59 PM

Homework 4

Mergesort

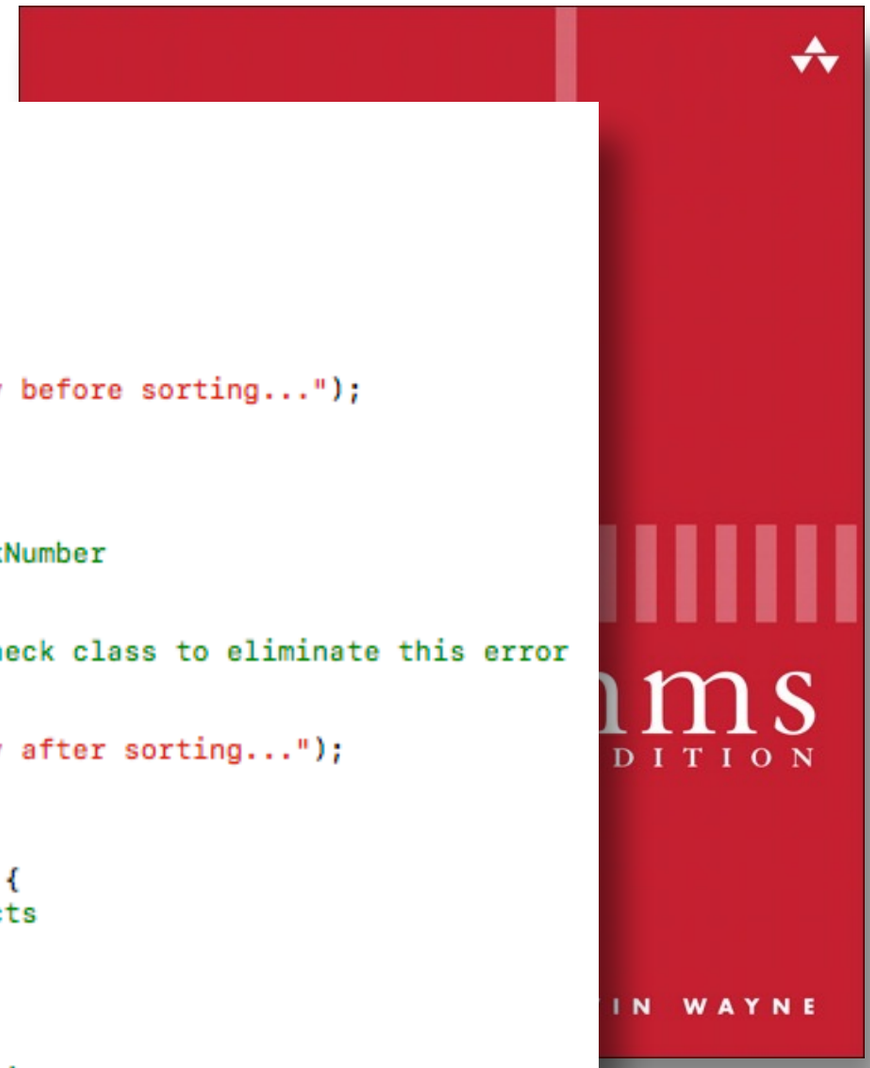
```
/*
 * This is an example of a successful run:
 *
    Displaying students array before sorting...
    Students [firstName=Joe, lastName=Jones, studentId=1001]
    Students [firstName=Adam, lastName=Ant, studentId=950]
    Students [firstName=Bill, lastName=Barnes, studentId=735]
    Students [firstName=Mark, lastName=Roth, studentId=1102]
    Students [firstName=Jerome, lastName=Howard, studentId=1050]
    Being sorting...
    End sorting...
    Displaying students array after sorting...
    Students [firstName=Adam, lastName=Ant, studentId=950]
    Students [firstName=Bill, lastName=Barnes, studentId=735]
    Students [firstName=Mark, lastName=Roth, studentId=1102]
    Students [firstName=Jerome, lastName=Howard, studentId=1050]
    Students [firstName=Joe, lastName=Jones, studentId=1001]
*
*/
```

Assignment due next Monday at 11:59 PM

Homework 4

Selectionsort

```java
public class TestSelectionSort {

    /**
     * @param args
     */
    public static void main(String[] args) {
        Check[] checks = initializeChecksArray();

        System.out.println("Displaying checks array before sorting...");
        display(checks);

        System.out.println("Being sorting...");
        /*
         * The Checks array will be sorted by checkNumber
         * which is declared as a Integer
         */
        Selection.sort(checks); // TODO: Fix the Check class to eliminate this error
        System.out.println("End sorting...");

        System.out.println("Displaying checks array after sorting...");
        display(checks);
    }

    private static Check[] initializeChecksArray() {
        // TODO: Initialize an array of Check objects
        return null;
    }

    private static void display(Check[] checks) {
        // TODO: Display the contents of the Check Array
    }

}
```



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Homework 4

Selectionsort

```
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 * This is an example of a successful run:
 *
    Displaying checks array before sorting...
    Check [checkNumber=1001, payTo=Joe Jones, date=Mon Feb 06 15:36:40 CST 2017]
    Check [checkNumber=950, payTo=Adam Ant, date=Mon Feb 06 15:36:40 CST 2017]
    Check [checkNumber=735, payTo=Bill Barnes, date=Mon Feb 06 15:36:40 CST 2017]
    Check [checkNumber=1102, payTo=Mark Roth, date=Mon Feb 06 15:36:40 CST 2017]
    Check [checkNumber=1050, payTo=Jerome Howard, date=Mon Feb 06 15:36:40 CST 2017]
    Being sorting...
    End sorting...
    Displaying checks array after sorting...
    Check [checkNumber=735, payTo=Bill Barnes, date=Mon Feb 06 15:36:40 CST 2017]
    Check [checkNumber=950, payTo=Adam Ant, date=Mon Feb 06 15:36:40 CST 2017]
    Check [checkNumber=1001, payTo=Joe Jones, date=Mon Feb 06 15:36:40 CST 2017]
    Check [checkNumber=1050, payTo=Jerome Howard, date=Mon Feb 06 15:36:40 CST 2017]
    Check [checkNumber=1102, payTo=Mark Roth, date=Mon Feb 06 15:36:40 CST 2017]
 *
 */
```

Assignment due next Monday at 11:59 PM