Problem Set 1

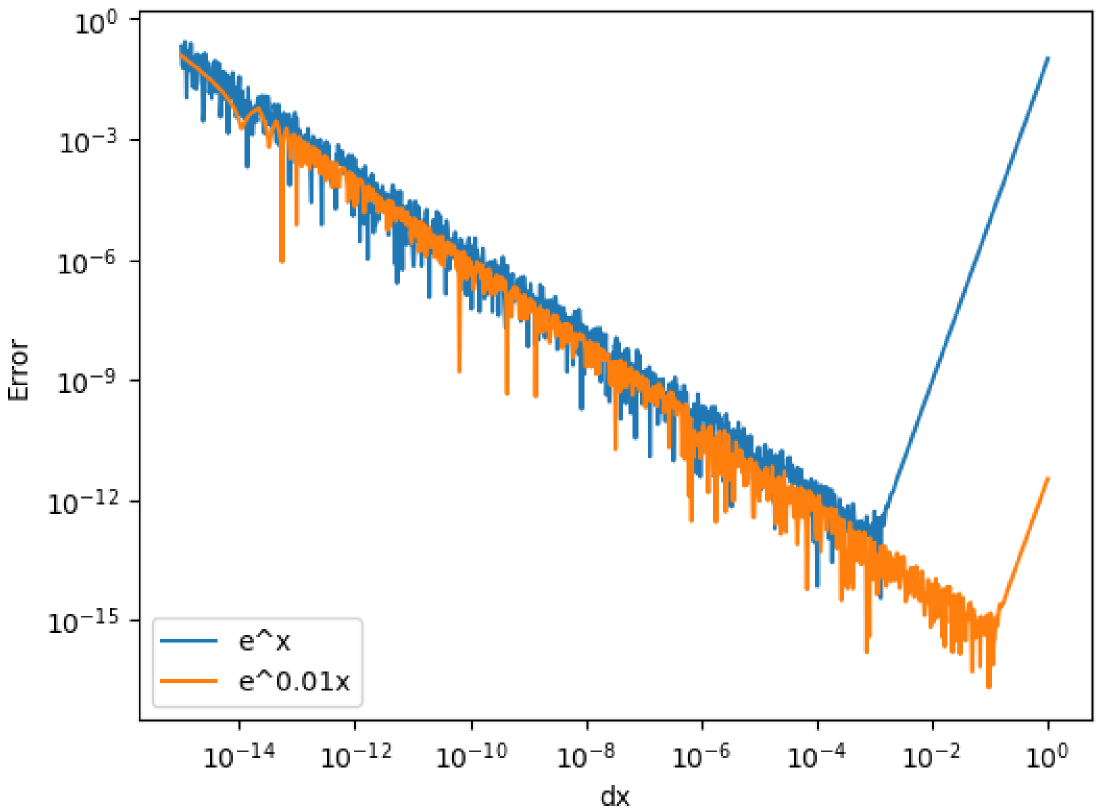
September 17, 2022

[1]:
```
import numpy as np
from matplotlib import pyplot as plt
```

0.0.1 Problem 1

[23]:
```
logdx = np.linspace(-15, 0, 1001)
dx = 10**logdx

def func(x):
    return np.exp(x)

def func1(x):
    return np.exp(x*0.01)

x0=1

def derivative(func, x, dx):
    return (func(x+2*dx) - func(x-2*dx) - 8*(func(x+dx) - func(x-dx)))/(-12*dx)

d = derivative(func, x0, dx)
d1 = derivative(func1, x0, dx)

plt.loglog(dx, np.abs(d-func(x0)), label='e^x')
plt.plot(dx, np.abs(d1-(func1(x0)*0.01)), label='e^0.01x') # derivative of e^0.
↪01x is 0.01*e^0.01x
plt.xlabel('dx')
plt.ylabel('Error')
plt.legend()
```

[23]: <matplotlib.legend.Legend at 0x10b9cadc0>



As we predicted, error is around 10^-1 and 10^-3 for double precision

0.0.2 Problem 2

[45]:
```
def third_der(fun, x, dx):
    return (fun(x+2*dx) - 2*fun(x+dx) + 2*fun(x-dx) - fun(x-2*dx))/(2*dx**3)

def ndiff(fun, x, full=False):
    err = 10**-15
    initialdx = 0.001
    newdx = (np.abs((3*fun(x)*err)/(2*third_der(fun, x, initialdx))))**(1/3)
    diff = np.abs((newdx - initialdx)/newdx)
    while(all(i >= 0.01 for i in diff)):
        temp = (np.abs((3*fun(x)*err)/(2*third_der(fun, x, newdx))))**(1/3)
        diff = np.abs((temp - newdx)/newdx)
        newdx = temp
    d = (fun(x+newdx) - fun(x-newdx))/(2*newdx)
    if(full):
        return d, newdx, np.abs((fun(x)*err)/(2*newdx) + (third_der(fun, x,
 ↪newdx)*newdx**2)/6)
    return d
```

[56]:

```
x = np.linspace(1,10,10)
d, dx, err = ndiff(np.exp, x, True)
print('derivative:', d, '\n')
print('dx:', dx, '\n')
print('error:', err, '\n')
```

```
derivative: [2.71828183e+00 7.38905610e+00 2.00855369e+01 5.45981500e+01
 1.48413159e+02 4.03428793e+02 1.09663316e+03 2.98095799e+03
 8.10308393e+03 2.20264658e+04]

dx: [1.09464279e-05 1.14407293e-05 1.22030728e-05 1.08595587e-05
 9.79561435e-06 9.79355099e-06 9.95490459e-06 1.05670221e-05
 8.84187971e-06 8.72644700e-06]

error: [1.85016952e-10 4.91132558e-10 1.18688635e-09 3.76790733e-09
 9.75159476e-09 2.68845047e-08 8.93405943e-08 1.94843274e-07
 5.78227626e-07 2.30428030e-06]
```

Problem Set 1

1.

a) $f(x \pm dx) \approx f(x) \pm f^1(x)dx + \frac{1}{2}f^2(x)dx^2 \pm \frac{1}{6}f^3(x)dx^3 + O(dx^4)$

$f(x \pm 2dx) \approx f(x) \pm 2f^1(x)dx + 2f^2(x)dx^2 \pm \frac{4}{3}f^3(x)dx^3 + O(dx^4)$

Let us find the derivatives

$f(x+dx) - f(x-dx) \approx 2f^1 dx + \frac{1}{3}f^3 dx^3 + O(dx^5)$

$f(x+2dx) - f(x-2dx) \approx 4f^1 dx + \frac{8}{3}f^3 dx^3 + O(dx^5)$

$$\begin{bmatrix} 4 & 2 \\ 8/3 & 1/3 \end{bmatrix}\begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \qquad \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} -1/12 \\ 2/3 \end{bmatrix}$$

$$\Rightarrow \frac{-\frac{1}{12}(f(x+2x) - f(x-2x)) + \frac{2}{3}(f(x+dx) - f(x-dx))}{dx} = f^1$$

Cancelled f^3 terms, leaving $f^1 dx$

b)

$$f^1 \sim \frac{(1+g_+\epsilon)(f(x+2x) + (-8)f(x+dx)) - (1+g_-\epsilon)(f(x-2x) \pm (-8)f(x-dx))}{-12dx}$$

Let us expand to order 5 in $f(x \pm dx)$, $f(x \pm 2dx)$ and take $|g_+ - g_-| = g \approx 1$, and discard ϵdx terms or higher

$$f^1 \sim \frac{9f\epsilon - 12f^1 dx + \frac{64}{120}f^5 dx^5 - 8\frac{2}{120}f^5 dx^5}{-12dx}$$

$$f' \sim -\frac{3}{4}\frac{f\epsilon}{dx} + f' - \frac{3}{4}f^5 dx^4$$

Then error in f': $\frac{3}{4}\left|\frac{f\epsilon}{dx} + f^5 dx^4\right|$

differentiate w.r.t dx: $\frac{3}{4}\left|\frac{-f\epsilon}{dx^2} + 4f^5 dx^3\right| = 0$

$$\Rightarrow dx = \left(\frac{f\epsilon}{4f^5}\right)^{1/5}$$

for $\epsilon = 10^{-7}$:

$f = e^x$ $\quad dx \approx 0.03$

$f = e^{0.01x}$ $\quad dx \approx 3$

for $\epsilon = 10^{-15}$:

$f = e^x$ $\quad dx \approx 5 \times 10^{-3}$

$f = e^{0.01x}$ $\quad dx \approx 5 \times 10^{-1}$

1. As before, $f(x+dx) - f(x-dx) \approx 2f'dx + \frac{1}{3}f^3 dx^3 + \frac{1}{60}f^5 dx^5$

$$f' \sim \frac{(1+g+\epsilon)f(x+dx) - (1+g+\epsilon)f(x-dx)}{2dx}$$

$$\sim \frac{f\epsilon + 2f'dx + \frac{1}{3}f^3 dx^3 + \frac{1}{60}f^5 dx^5}{2dx}$$

Error: $\left| \frac{f\varepsilon}{2dx} + \frac{1}{6} f^{3} dx^{2} + \cancel{\frac{1}{120} f^{5} dx^{4}} \right|$

differentiate: $\left| \frac{-f\varepsilon}{2dx^{2}} + \frac{1}{3} f^{3} dx \right| = 0$

$\Rightarrow dx = \left(\frac{3f\varepsilon}{2f^{3}} \right)^{1/3}$