

# CS 4610/5335 – Lecture 10

## Kalman Filtering (n-D)

Lawson L.S. Wong  
Northeastern University  
2/23/22

Material adapted from:

1. Robert Platt, CS 4610/5335
2. Sebastian Thrun, Wolfram Burgard, & Dieter Fox,  
Probabilistic Robotics

# Announcements

Friday OH slightly shifted this week: 3 – 5 PM

Ex2 due Friday (2/25)

Ex3 will be out ~Fri/Sat

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Project: Continue adding details to your threads

- We will review them in the next week  
and possibly leave comments / questions
  
- Craft a two-month plan  
(lay out a weekly/biweekly plan; 10 weeks left)
- Starting work in simulation is highly recommended  
(plan for hardware to arrive mid-/late March)

# Robotics talk of interest!

Tesca Fitzgerald  
Postdoctoral Fellow  
Robotics Institute, CMU

Learning to address novel situations  
through human-robot collaboration

Friday, February 25  
10:30 – 11:30 AM  
366 West Village H  
also on Zoom: 950 4773 0161 (423306)



“My research is centered around interactive robot learning.  
... I develop algorithms that allow a robot to structure and  
interpret its interactions with a human teacher in order to  
adapt its task knowledge to novel situations.  
... My work contributes toward a future  
of adaptive, collaborative robots.”

# Outline

Kalman filter (1-D, simplified) + example

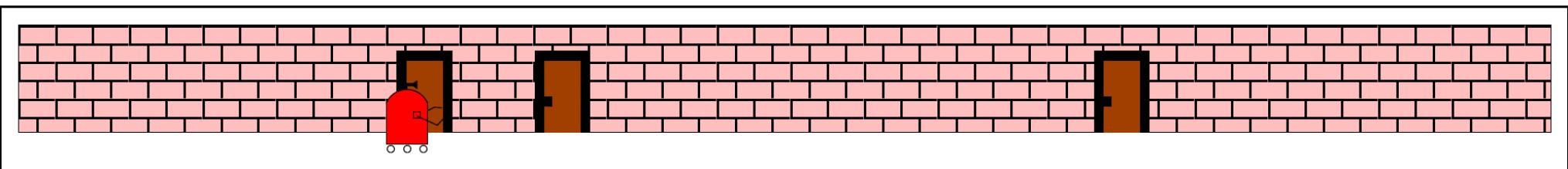
Derivation of 1-D Kalman filter

Kalman filter (1-D, general) + example

Kalman filter (n-D) + example (time-permitting)

# Recap: Localization example (1-D)

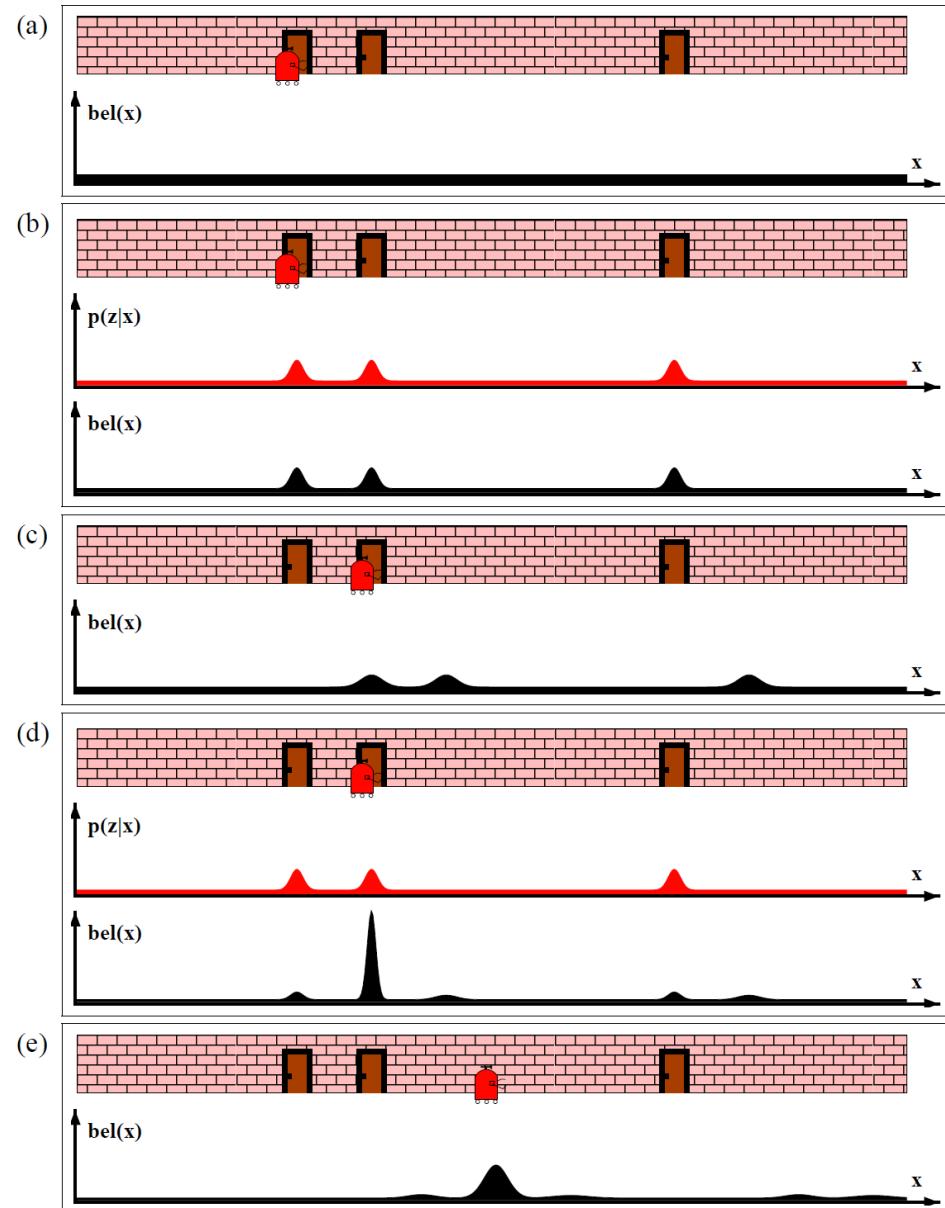
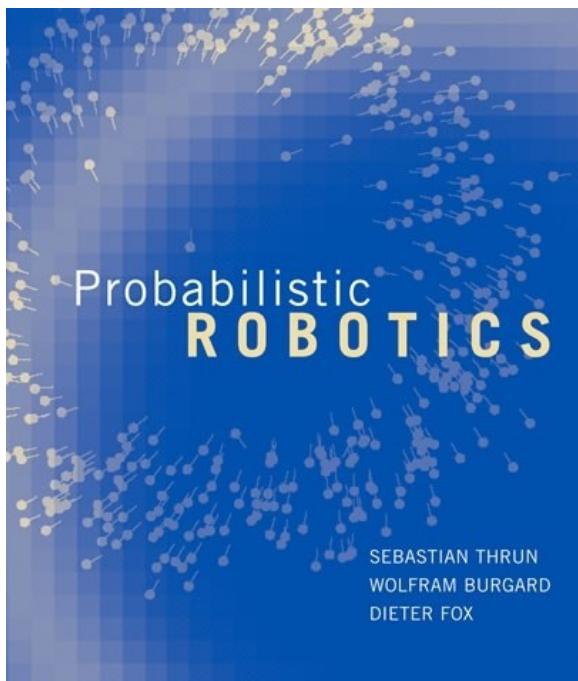
From “Probabilistic Robotics” (Ch. 7-8)  
by Sebastian Thrun, Wolfram Burgard, & Dieter Fox



**Figure 7.4** Example environment used to illustrate mobile robot localization: One-dimensional hallway environment with three indistinguishable doors. Initially the robot does not know its location except for its heading direction. Its goal is to find out where it is.

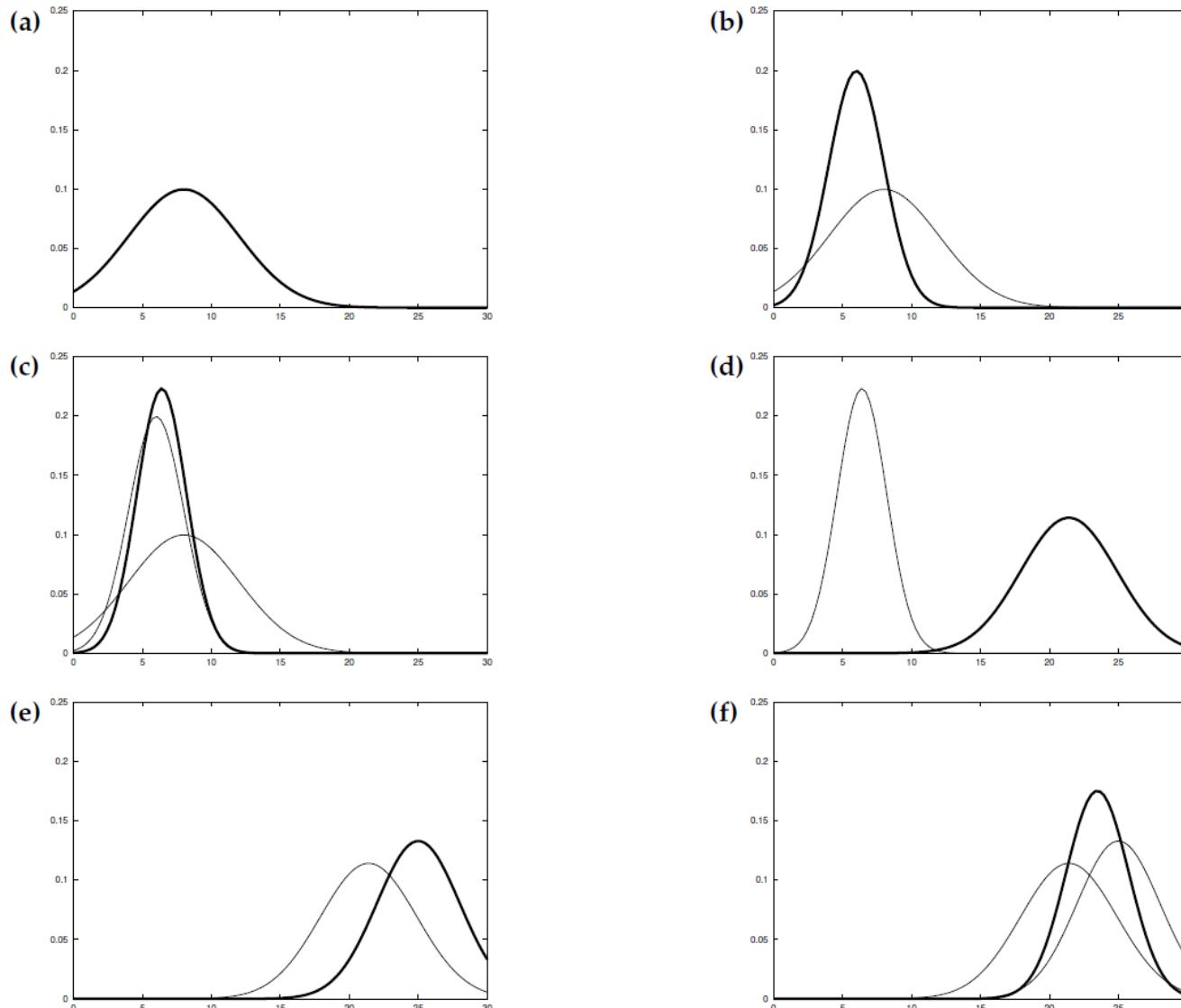
# Recap: Localization example (1-D)

From  
“Probabilistic Robotics”  
(Ch. 7-8)  
by  
Sebastian Thrun,  
Wolfram Burgard,  
& Dieter Fox



**Figure 7.4** Illustration of the Markov localization algorithm. Each picture depicts the position of the robot in the hallway and its current belief  $bel(x)$ . (b) and (d) additionally depict the observation model  $p(z_t | x_t)$ , which describes the probability of observing a door at the different locations in the hallway.

# Recap: Kalman Filter (1-D)



**Figure 3.2** Illustration of Kalman filters: (a) initial belief, (b) a measurement (in bold) with the associated uncertainty, (c) belief after integrating the measurement into the belief using the Kalman filter algorithm, (d) belief after motion to the right (which introduces uncertainty), (e) a new measurement with associated uncertainty, and (f) the resulting belief.

# On the board: 1-D Kalman Filter (simplified)

Input at  $t$ :  $\hat{x}_t, \hat{\sigma}_t^2$

Kalman filter (1-D, simplified)

Prediction step	$\hat{x}_{t+1}^+ = \hat{x}_t + u_t$	Know: $u_t, \sigma_v^2$
"Innovation" - (error)	$v_{t+1}^+ = z_{t+1} - \hat{x}_{t+1}^+$	$z_{t+1}, \sigma_w^2$
Kalman gain	$K_{t+1} = \frac{\hat{\sigma}_{t+1}^2}{\hat{\sigma}_{t+1}^2 + \sigma_w^2}$	

Assumptions / model:

- $x_t$  = location at time  $t$
- $z_t$  = observation at  $t$
- $x_{t+1} = x_t + u_t + v_t$
- $z_{t+1} = x_{t+1} + w_{t+1}$
- $v_t \sim N(0, \sigma_v^2)$  obs noise
- $w_t \sim N(0, \sigma_w^2)$  noise

$x_t \sim N(\hat{x}_t, \hat{\sigma}_t^2)$

estimated loc.      estimated variance

$$\begin{aligned}\hat{x}_{t+1} &= \hat{x}_{t+1}^+ + K_{t+1} v_{t+1} \\ \hat{\sigma}_{t+1}^2 &= (1 - K_{t+1}) \hat{\sigma}_{t+1}^2\end{aligned}$$

Posterior: Estimated loc. at  $t+1 \sim N(\hat{x}_{t+1}, \hat{\sigma}_{t+1}^2)$

# Outline

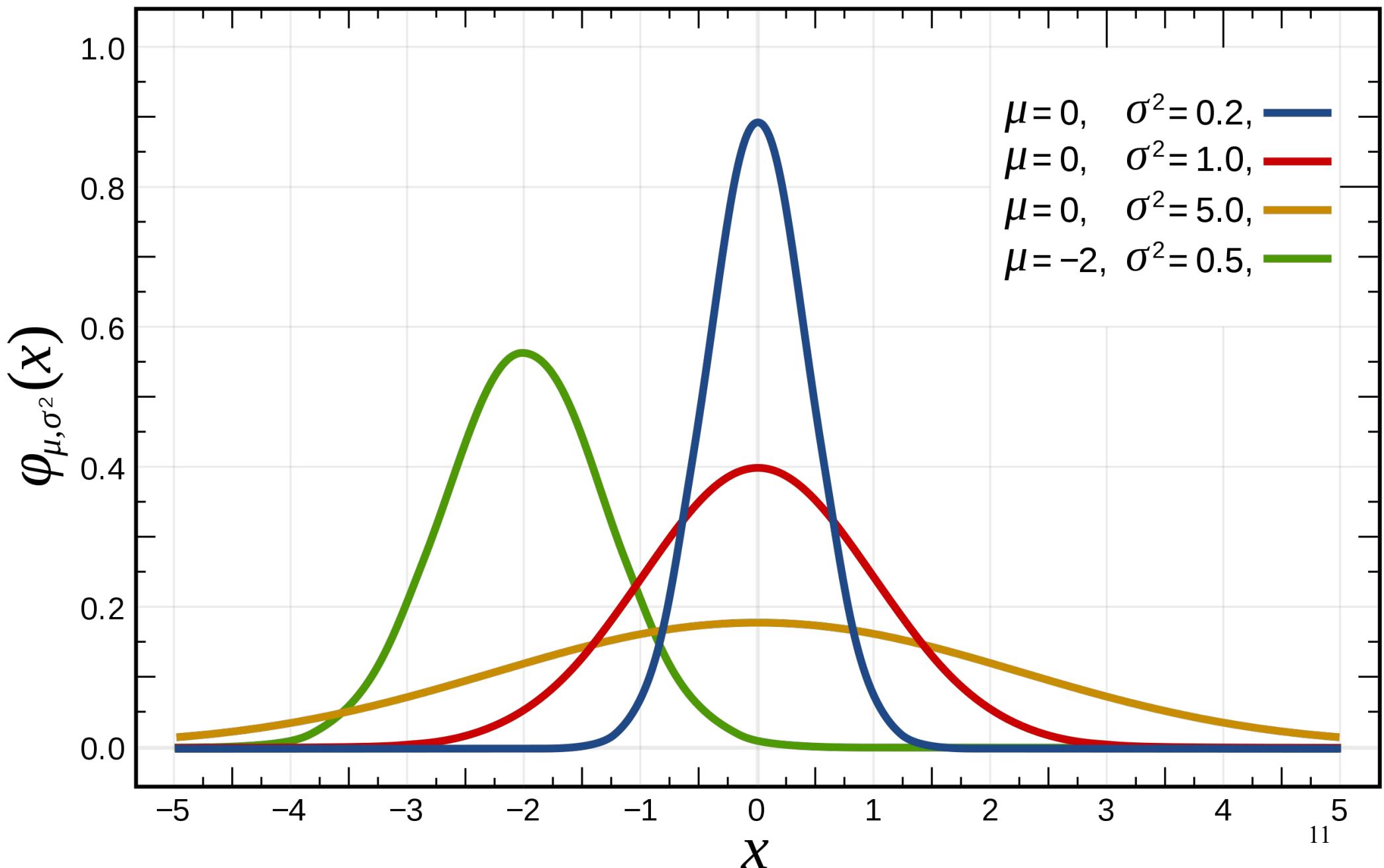
- ✓ Kalman filter (1-D, simplified) + example

Derivation of 1-D Kalman filter

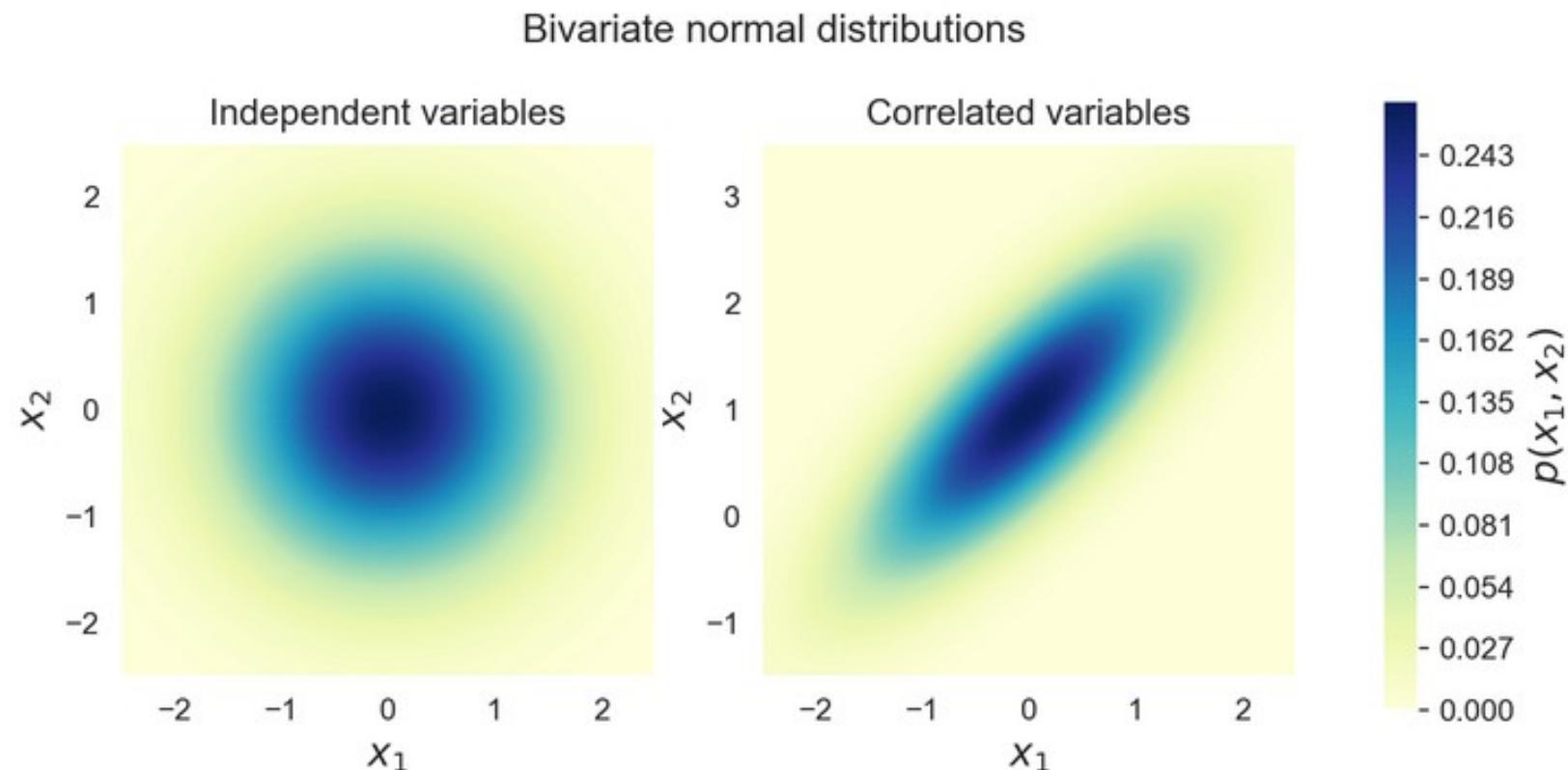
Kalman filter (1-D, general) + example

Kalman filter (n-D) + example (time-permitting)

# Gaussian distribution / Normal distribution (1-D)



# Gaussian distribution / Normal distribution (2-D)



# On the board: Properties of Gaussian distributions

$X \sim N(\mu_x, \sigma_x^2)$ ,  $Y \sim N(\mu_y, \sigma_y^2)$  then  $X+Y \sim N(\mu_x + \mu_y, \sigma_x^2 + \sigma_y^2)$

$X \sim N(\mu, \sigma^2)$  then  $aX+b \sim N(a\mu+b, a^2\sigma^2)$

$X \sim N(\mu, \sigma^2)$ ,  $Z \sim N(hx+b, \sigma_z^2)$ , then

$$\begin{bmatrix} X \\ Z \end{bmatrix} \sim N\left(\begin{bmatrix} \mu \\ hx+b \end{bmatrix}, \begin{bmatrix} \sigma^2 & h\sigma^2 \\ h\sigma^2 & h^2\sigma^2 + \sigma_z^2 \end{bmatrix}\right)$$

$$\begin{bmatrix} X \\ Y \end{bmatrix} \sim N\left(\begin{bmatrix} \mu_x \\ \mu_y \end{bmatrix}, \begin{bmatrix} \sigma_x^2 & \sigma_{xy} \\ \sigma_{yx} & \sigma_y^2 \end{bmatrix}\right)$$

$$\text{then } X|Y \sim N\left(\mu_x + \frac{\sigma_{xy}}{\sigma_y^2}(y - \mu_y), \sigma_x^2 - \frac{\sigma_{xy}^2}{\sigma_y^2}\right)$$

Suppose  $x_t \sim N(\hat{\mu}_t, \hat{P}_t)$

Prediction:

$$\begin{aligned} \hat{\mu}_{t+1}^+ &= F\hat{\mu}_t + Gv_t \\ \hat{P}_{t+1}^+ &= F\hat{P}_t F^\top + V \end{aligned}$$

Update:

$$v_{t+1} = z_{t+1} - H\hat{\mu}_{t+1}^+ \quad (\text{innovation})$$

$$K_{t+1} = \hat{P}_{t+1}^+ H^\top (H\hat{P}_{t+1}^+ H^\top + W)^{-1}$$

$$\hat{\mu}_{t+1} = \hat{\mu}_{t+1}^+ + K_{t+1} v_{t+1}$$

$$\hat{P}_{t+1} = (I - K_{t+1} H) \hat{P}_{t+1}^+$$

# Outline

✓ Kalman filter (1-D, simplified) + example

✓ Derivation of 1-D Kalman filter

Kalman filter (1-D, general) + example

Kalman filter (n-D) + example (time-permitting)

# On the board: 1-D Kalman Filter (general)

$X_{t+1} = fX_t + gU_t + V_t$	$V_t \sim N(0, \sigma_v^2)$ known	Current belief: $\hat{X}_t, \hat{\sigma}_t^2$	$\frac{\hat{X}_t}{X_t} \rightarrow$
$Z_{t+1} = hX_{t+1} + W_{t+1}$	$W_t \sim N(0, \sigma_w^2)$	Predicted mean: $\hat{X}_{t+1}^+ = f\hat{X}_t + gU_t$	$U_t = \text{Change in } x$
$X_t$ : State at time t	$V_t$ : Transition / process noise	Predicted variance: $\hat{\sigma}_{t+1}^2 = f^2 \hat{\sigma}_t^2 + \sigma_v^2$	$X_{t+1} = X_t + U_t + V_t$
$Z_t$ : Measurement / observation at time t	$W_t$ : Meas. / observation noise	Innovation: $\gamma_{t+1} = Z_{t+1} - h\hat{X}_{t+1}^+$	$Z_{t+1}$ : Measured location
$U_t$ : "Control input" at time t	Initial belief: $X_0 \sim N(\hat{X}_0, \hat{\sigma}_0^2)$	Kalman gain: $K_{t+1} = h\hat{\sigma}_{t+1}^2 / (h^2 \hat{\sigma}_{t+1}^2 + \sigma_w^2)$	$\hat{X}_2 = 5 \quad U_2 = 3 \quad \sigma_v^2 = .6$
$f, g, h$ : Known scalars ( $\mathbb{R}$ )	Update	Posterior Mean: $\hat{X}_{t+1} = \hat{X}_{t+1}^+ + K_{t+1} \gamma_{t+1}$	$\hat{\sigma}_2^2 = 2 \quad Z_3 = 7.5 \quad \sigma_w^2 = .1$
		Posterior variance: $\hat{\sigma}_{t+1}^2 = (1 - K_{t+1}) \hat{\sigma}_{t+1}^2$	

$X_t = \# \text{ infected at day } t \sim N(0, 1000^2)$   
 $X_{t+1} = 1.1 X_t + (-1000) + V_t$   
 $Z_t = \# \text{ positive tests at day } t$   
 $Z_{t+1} = .1 X_{t+1} + W_t \sim N(0, 100^2)$

68-95-99.7 rule

$$\begin{aligned}
 \hat{X}_3^+ &= \hat{X}_2 + U_2 & \hat{\sigma}_3^2 &= \hat{\sigma}_2^2 + \sigma_v^2 \\
 8 &= 5 + 3 & 2.6 &= 2 + .6 \\
 \gamma_3 &= Z_3 - \hat{X}_3^+ & K_3 &= \frac{2.6}{2.6 + .4} = \frac{2.6}{3} \\
 &= 7.5 - 8 = -.5 & \hat{\sigma}_3^2 &= (1 - K_3) \hat{\sigma}_2^2 \\
 \hat{X}_3 &= \hat{X}_3^+ + K_3 \gamma_3 & &= 8 + \frac{2.6}{3} \cdot (-.5) \\
 &= 8 - \frac{1.3}{3} \approx 7.6 & &= 8 - \frac{1.3}{3} \approx 7.6
 \end{aligned}$$

# On the board: 1-D Kalman Filter (general)

1-D.  $X_{t+1} = f X_t + g u_t + v_t$ ,  $v_t \sim N(0, \sigma_v^2)$

$Z_{t+1} = h X_{t+1} + w_t$ ,  $w_t \sim N(0, \sigma_w^2)$

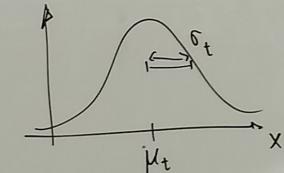
obs.

Prediction  $\left\{ \begin{array}{l} \hat{\mu}_{t+1}^+ = f \hat{\mu}_t + g u_t \\ \hat{\sigma}_{t+1}^2 = f^2 \hat{\sigma}_t^2 + \sigma_v^2 \end{array} \right.$

Innovation —  $v_{t+1} = Z_{t+1} - h \hat{\mu}_{t+1}^+$

Kalman gain —  $K_{t+1} = \frac{h \hat{\sigma}_{t+1}^2}{h \hat{\sigma}_{t+1}^2 + \sigma_w^2}$

Time  $t$ :  $P(X_t) = N(\hat{\mu}_t, \hat{\sigma}_t^2)$



Update step:

$$\begin{aligned} \hat{\mu}_{t+1} &= \hat{\mu}_{t+1}^+ + K_{t+1} v_{t+1} \\ \hat{\sigma}_{t+1}^2 &= \hat{\sigma}_{t+1}^{2+} - K_{t+1} h \hat{\sigma}_{t+1}^{2+} \\ &= (1 - K_{t+1} h) \hat{\sigma}_{t+1}^{2+} \end{aligned}$$

# On the board: 1-D Kalman Filter (general)

System:  $X_{t+1} = fX_t + gU_t + V_t$ ,  $V_t \sim N(0, \sigma_v^2)$

$$Z_{t+1} = hX_{t+1} + W_{t+1}, \quad W_t \sim N(0, \sigma_w^2)$$

Kalman filter:

$$\hat{M}_{t+1}^+ = f\hat{X}_t + gU_t$$

$$\hat{\sigma}_{t+1}^2 = f^2 \hat{\sigma}_t^2 + \sigma_v^2$$

$$Y_{t+1} = Z_{t+1} - h\hat{M}_{t+1}^+$$

$$K_{t+1} = \frac{h\hat{\sigma}_{t+1}^2}{h^2 \hat{\sigma}_{t+1}^2 + \sigma_w^2}$$

$$\hat{M}_{t+1} = \hat{M}_{t+1}^+ + K_{t+1} Y_{t+1}$$

$$\hat{\sigma}_{t+1}^2 = \left(1 - \frac{h^2 \hat{\sigma}_{t+1}^2}{h^2 \hat{\sigma}_{t+1}^2 + \sigma_w^2}\right) \hat{\sigma}_{t+1}^2 = \frac{\sigma_w^2 \hat{\sigma}_{t+1}^2}{h^2 \hat{\sigma}_{t+1}^2 + \sigma_w^2}$$

Belief:

$$X_t \sim N(\hat{M}_t, \hat{\sigma}_t^2)$$

$$fX_t + gU_t \sim N\left(\frac{f\hat{M}_t + gU_t}{\hat{M}_t}, \frac{f^2 \hat{\sigma}_t^2}{\hat{M}_t}\right)$$

# Outline

- ✓ Kalman filter (1-D, simplified) + example
- ✓ Derivation of 1-D Kalman filter
- ✓ Kalman filter (1-D, general) + example

Kalman filter (n-D) + example (time-permitting)

# On the board (bottom): n-D Kalman Filter

$X_{t+1} = fX_t + gU_t + V_t$        $V_t \sim N(0, \sigma_v^2)$  known  
 $Z_{t+1} = hX_{t+1} + W_{t+1}$        $W_{t+1} \sim N(0, \sigma_w^2)$

$X_t$ : State at time  $t$   
 $Z_t$ : Measurement / observation at time  $t$   
 $U_t$ : "Control input" at time  $t$   
 $f, g, h$ : Known scalars ( $\mathbb{R}$ )

$V_t$ : Transition / process noise  
 $W_t$ : Meas. / observation noise

Initial belief:  $X_0 \sim N(\hat{X}_0, \hat{\sigma}_0^2)$

Current belief:  $\hat{X}_t, \hat{\sigma}_t^2$   
 Predicted mean:  $\hat{X}_{t+1}^+ = f\hat{X}_t + gU_t$   
 Predicted variance:  $\hat{\sigma}_{t+1}^2 = f^2\hat{\sigma}_t^2 + \sigma_v^2$   
 Innovation:  $\hat{V}_{t+1} = Z_{t+1} - h\hat{X}_{t+1}^+$   
 Kalman gain:  $K_{t+1} = h\hat{\sigma}_{t+1}^2 / (h^2\hat{\sigma}_{t+1}^2 + \sigma_w^2)$   
 Posterior mean:  $\hat{X}_{t+1} = \hat{X}_{t+1}^+ + K_{t+1}\hat{V}_{t+1}$   
 Posterior variance:  $\hat{\sigma}_{t+1}^2 = (1 - K_{t+1}h)\hat{\sigma}_{t+1}^2 + \sigma_w^2$

$\vec{x}$   $\rightarrow$   
 $\vec{X}_t$   
 $U_t$  = Change in  $x$   $\downarrow$  Slippage  
 $Z_{t+1} = X_t + U_t + V_t$   
 $Z_{t+1}$  = Measured location  
 $= X_{t+1} + W_{t+1} + \text{noise}$   
 $\hat{X}_2 = 5 \quad U_2 = 3 \quad \sigma_v^2 = .6$   
 $\hat{\sigma}_2^2 = 2 \quad Z_3 = 7.5 \quad \sigma_w^2 = .4$

$\vec{X}_{t+1} = F\vec{X}_t + G\vec{U}_t + \vec{V}_t$   
 $\vec{Z}_{t+1} = H\vec{X}_{t+1} + \vec{W}_t$   
 $\vec{X}_t \in \mathbb{R}^n \quad U_t \in \mathbb{R}^m \quad \vec{Z}_t \in \mathbb{R}^p$

$F: n \times n \quad \vec{V}_t \in \mathbb{R}^n \sim N(0, \Sigma_v)$   
 $G: n \times m \quad \vec{W}_t \in \mathbb{R}^p \sim N(0, \Sigma_w)$   
 $H: p \times n$

Belief:  $N(\hat{X}_t, \hat{\Sigma}_t)$

$\begin{bmatrix} X \\ Y \\ Z \end{bmatrix}_{t+1} = \begin{bmatrix} X \\ Y \\ Z \end{bmatrix}_t + \begin{bmatrix} \Delta x \\ \Delta y \\ \Delta z \end{bmatrix}_t + \vec{V}_t$   
 Control: Change in pos.  
 $\begin{bmatrix} X \\ Y \\ Z \end{bmatrix}_{t+1} = \begin{bmatrix} X \\ Y \\ Z \end{bmatrix}_t + \begin{bmatrix} \sqrt{\Delta t} \\ \Delta x \\ \Delta z \end{bmatrix}_t + \vec{V}_t = \begin{bmatrix} 1 & \Delta t \\ 0 & 1 \end{bmatrix} \begin{bmatrix} X \\ Y \\ Z \end{bmatrix}_t + \begin{bmatrix} 0 \\ 1 \end{bmatrix} \Delta V_x + \vec{V}_t$   
 $\vec{X}_{t+1} = F\vec{X}_t + G\vec{U}_t + \vec{V}_t$

# On the board: n-D Kalman Filter

Linear system:

$$\vec{X}_{t+1} = F\vec{X}_t + G\vec{U}_t + \vec{V}_t$$

$$\vec{Z}_{t+1} = H\vec{X}_{t+1} + \vec{W}_{t+1} + \vec{b}$$

$$\vec{V}_t \sim N(\vec{0}, V)$$

$$\vec{W}_t \sim N(\vec{0}, W)$$

n State dims  
m Control dims.  
p Obs. dims.

$$F: \mathbb{R}^{n \times n}$$

$$G: \mathbb{R}^{n \times m}$$

$$H: \mathbb{R}^{p \times n}$$

Prediction

$$\begin{cases} \hat{\mu}_{t+1}^+ = F\hat{\mu}_t + GU_t \\ \hat{\Sigma}_{t+1}^+ = F\hat{\Sigma}_t F^T + V \end{cases}$$

Innovation

$$\vec{v}_{t+1} = \vec{Z}_{t+1} - (I - \hat{\mu}_{t+1}^+ + \vec{b})$$

Kalman gain

$$K_{t+1} = \hat{\Sigma}_{t+1}^+ H^T (H\hat{\Sigma}_{t+1}^+ H^T + W)^{-1}$$

Update

$$\begin{cases} \hat{\mu}_{t+1} = \hat{\mu}_{t+1}^+ + K_{t+1} v_{t+1} \\ \hat{\Sigma}_{t+1} = (I_n - K_{t+1} H) \hat{\Sigma}_{t+1}^+ \end{cases}$$

Belief:  $N(\hat{\mu}, \hat{\Sigma})$

# Feedback

## Piazza thread: 2/23 Lec 10 Feedback

Please post your answers to the following anonymously.

1. What did you like so far?
2. What was unclear?
3. Did you have a good long weekend?
4. Any additional feedback / comments?

# On the board: Bayesian filtering

## Bayes Filters

$$P(x_0)$$

$$P(x_{t+1} | x_t)$$

$$P(y_t | x_t)$$

Initial

$$\mathcal{N}(\mu_0, \Sigma_0)$$

Transition

$$f, g, \sigma_v^2$$

Observation

$$h, \sigma_w^2$$

Discrete

Particle filter

Kalman filter

$D = \# \text{ dimensions}$

## Belief representation

Vector of  $N$  #'s,  $0 \leq p(x_i) \leq 1$   
 $N$  states

Set of  $M$  particles

Each particle is a state

Gaussian distribution

Parameters:  $\hat{\mu}_t, \hat{\Sigma}_t$   
 $D \times 1$  vector       $D \times D$  matrix

## Prediction



Rif

Matrix-vector  
multiplication

Simulate one step  
forward

$$\hat{\mu}_{t+1}^+, \hat{\Sigma}_{t+1}^+$$

## Update

Element-wise mult  
+ normalization       $R$   
 $N = R^D$

Weighting & particles  
+ resampling       $S_R$ :  
 $O(MD)$

$$\frac{\hat{\mu}_{t+1}, \hat{\Sigma}_{t+1}}{D^2 + D}$$