CS 4610/5335 – Lecture 2
Representations and Transformations

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1/24/22

Material adapted from:
1. Robert Platt, CS 4610/5335
2. Peter Corke, Robotics, Vision and Control
3. Oussama Khatib, Stanford CS 223A

One-stop shop

Piazza @16:

Instructor masking poll (@14, closed)

Onboarding questionnaire (Google form)

Ex0 (on Canvas)

Responses to feedback (@6)

Materials from Spring 2020 (Piazza > Resources)

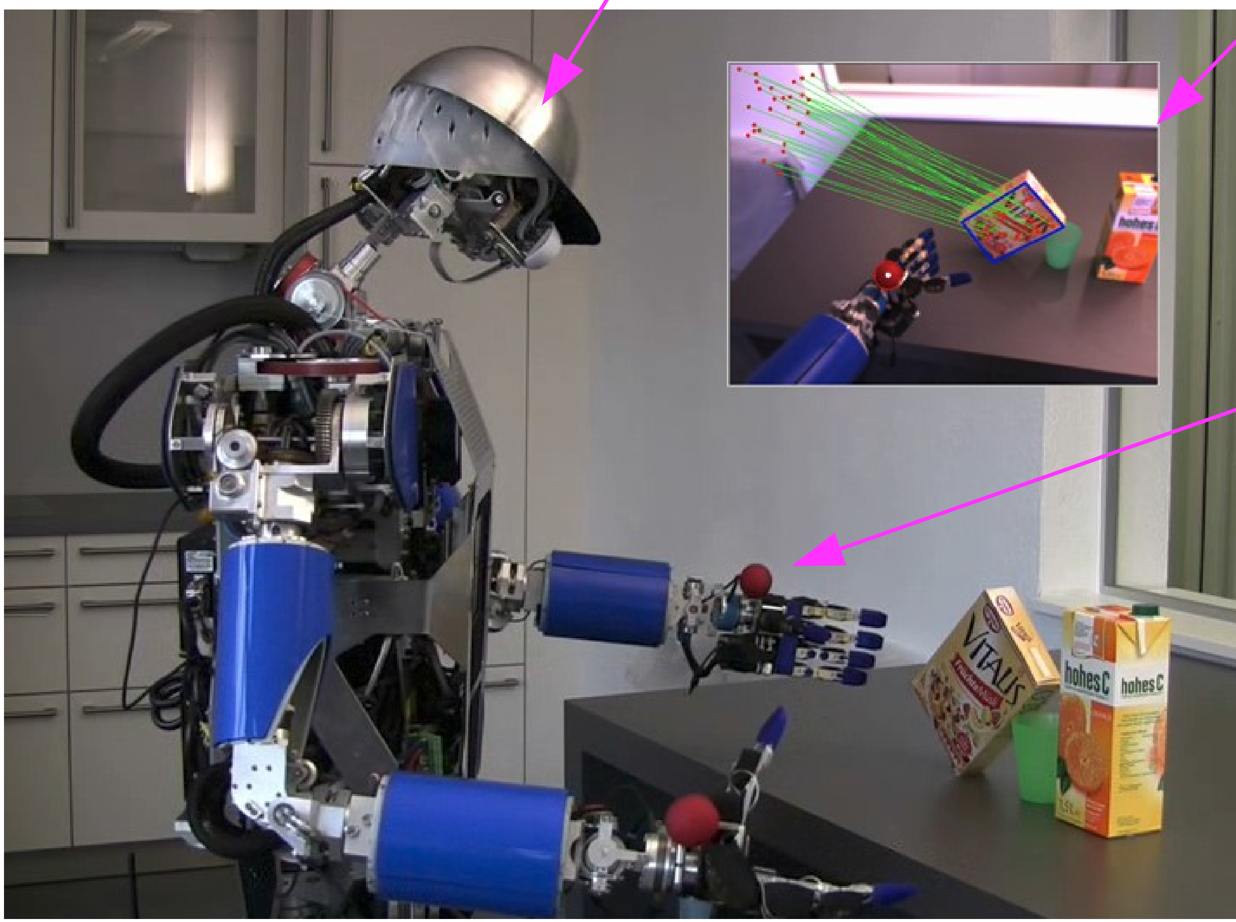
Instructions for MATLAB and Robotics Toolbox (@7)

Thread on “leveling up” (@15)

Transformations

Joint encoders tell us head angle

Visual perception tells us object position and orientation (pose)

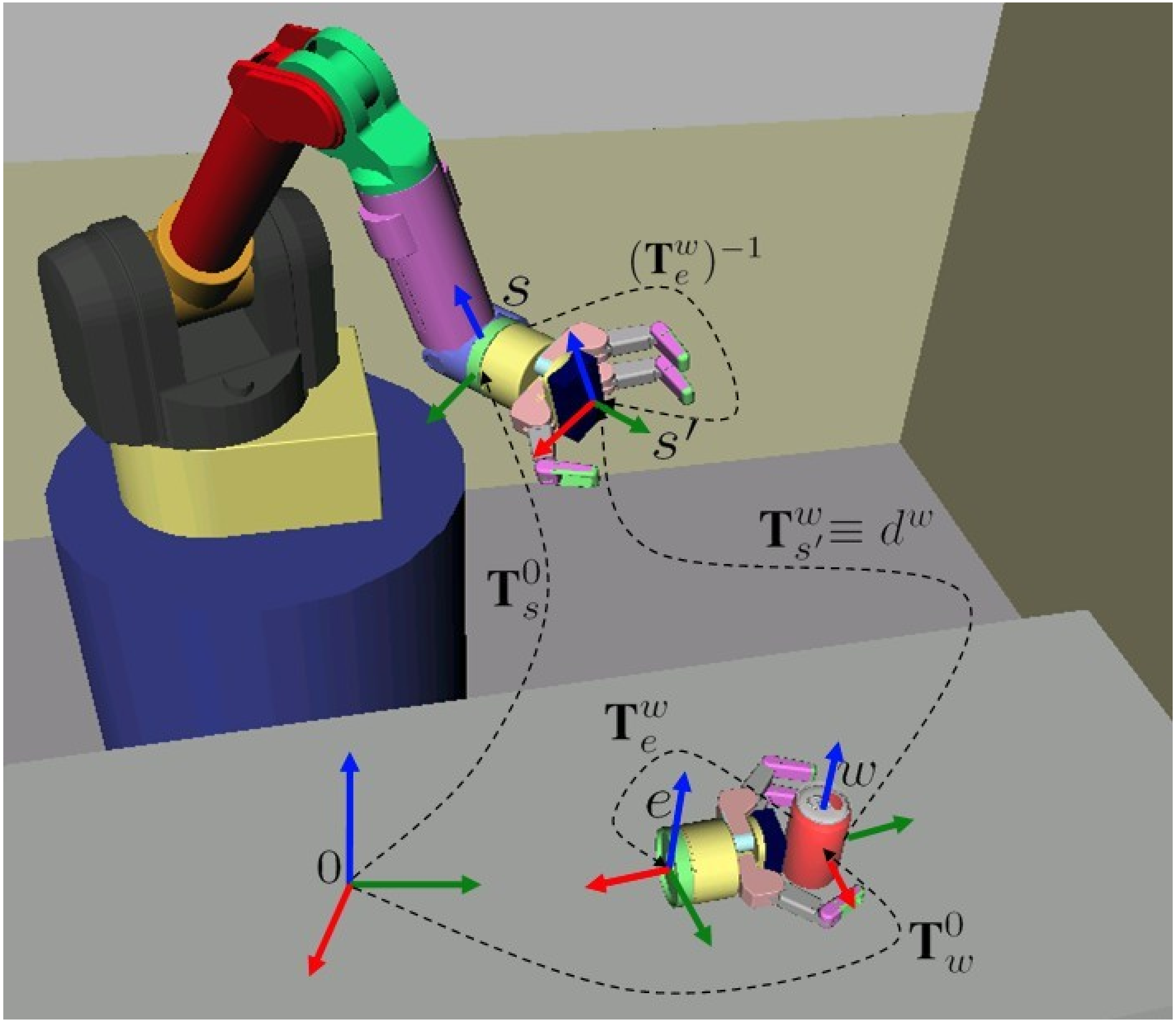


KIT Humanoid

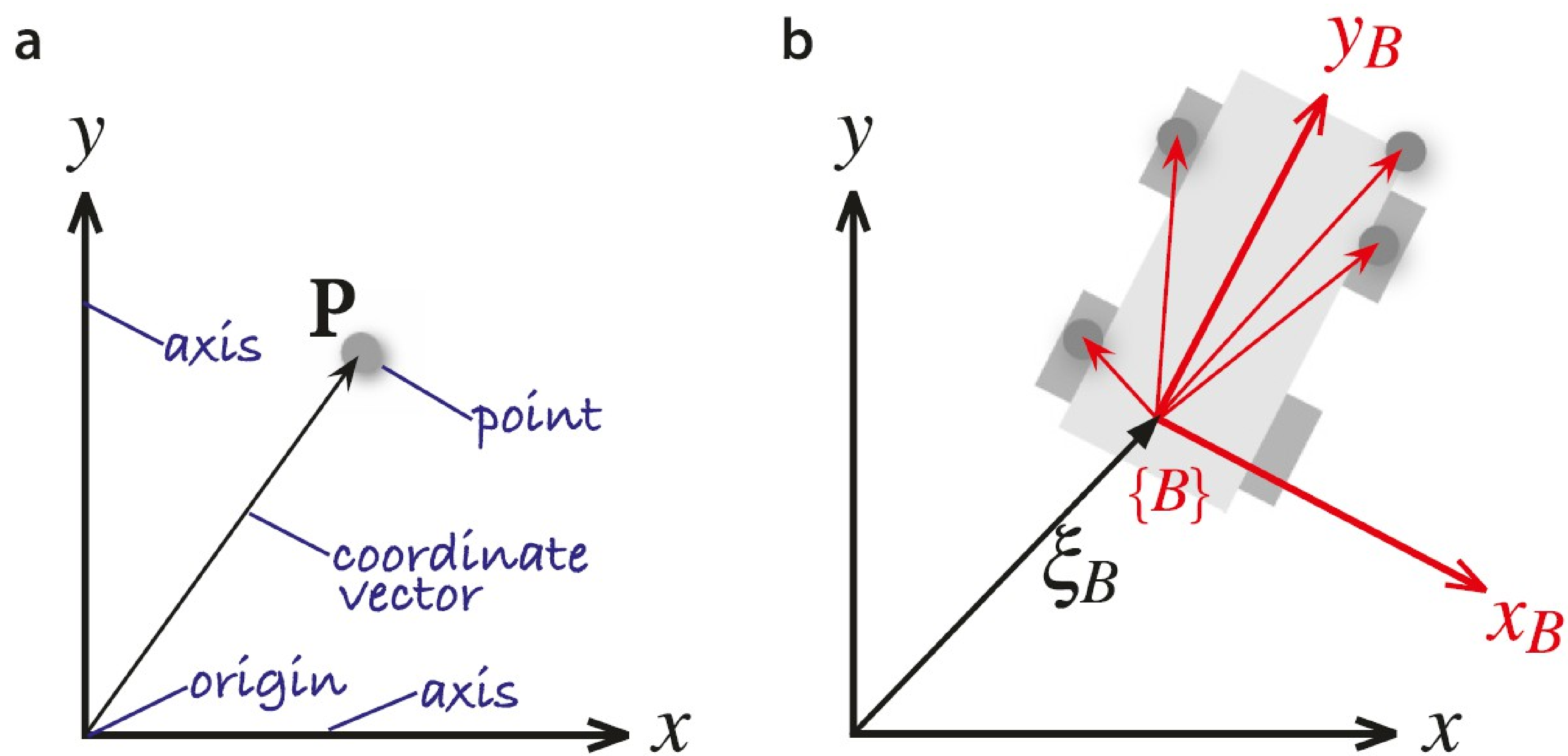
Need to know where hand is...

Need to tell the hand where to move!

Transformations

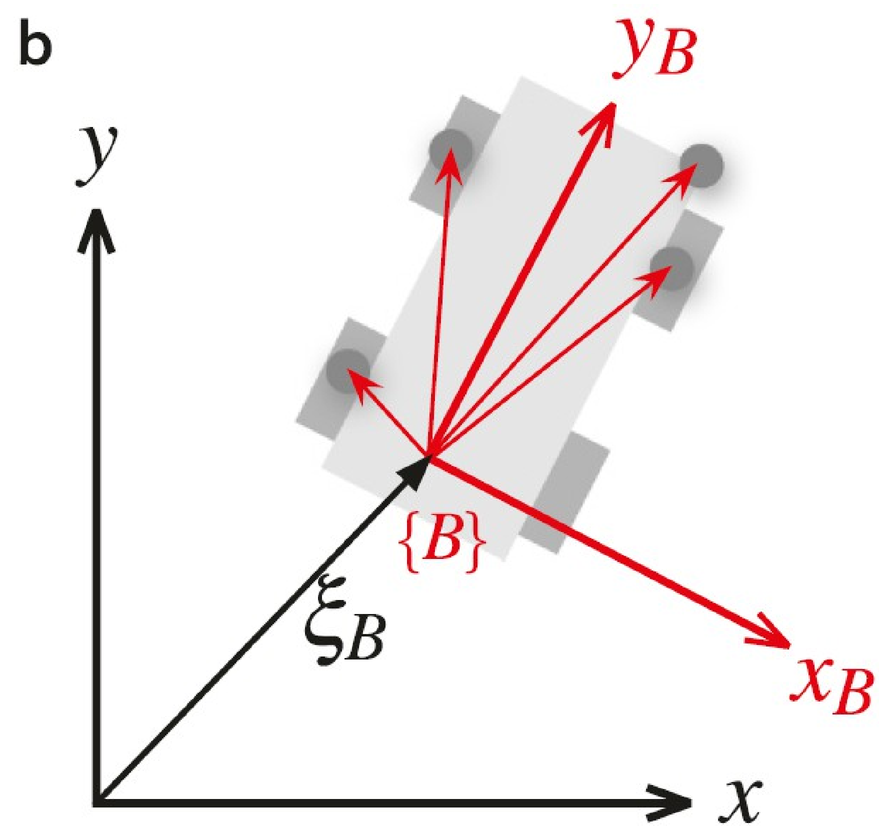


Representations



Representations

Fig. 2.1.
a The point **P** is described by a coordinate vector with respect to an absolute coordinate frame.
b The points are described with respect to the object's coordinate frame $\{B\}$ which in turn is described by a relative pose ξ_B. Axes are denoted by thick lines with an open arrow, vectors by thin lines with a swept arrow head and a pose by a thick line with a solid head



Transformations

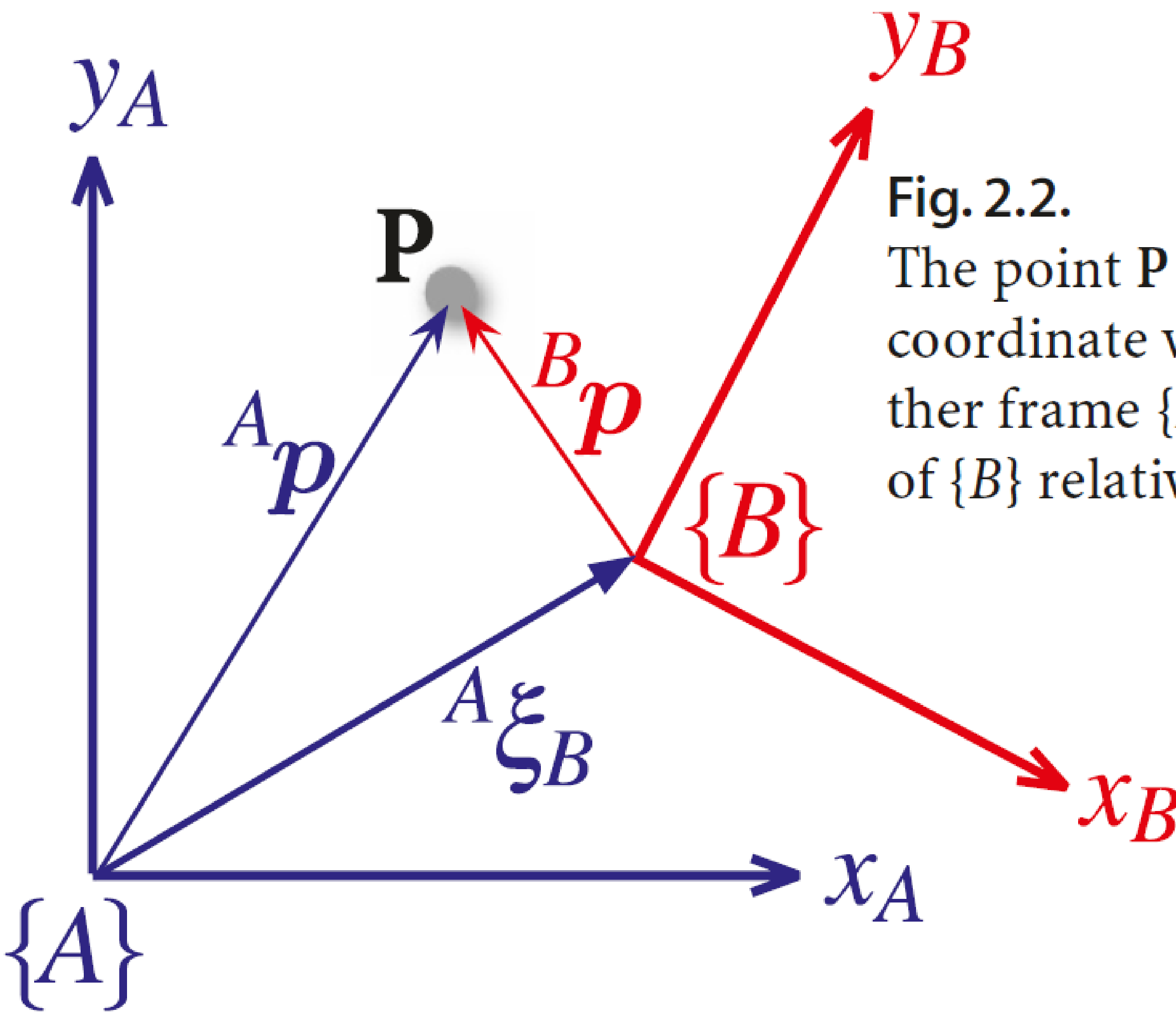


Fig. 2.2.
The point **P** can be described by coordinate vectors relative to either frame $\{A\}$ or $\{B\}$. The pose of $\{B\}$ relative to $\{A\}$ is ${}^{A}\xi_B$

Transformations

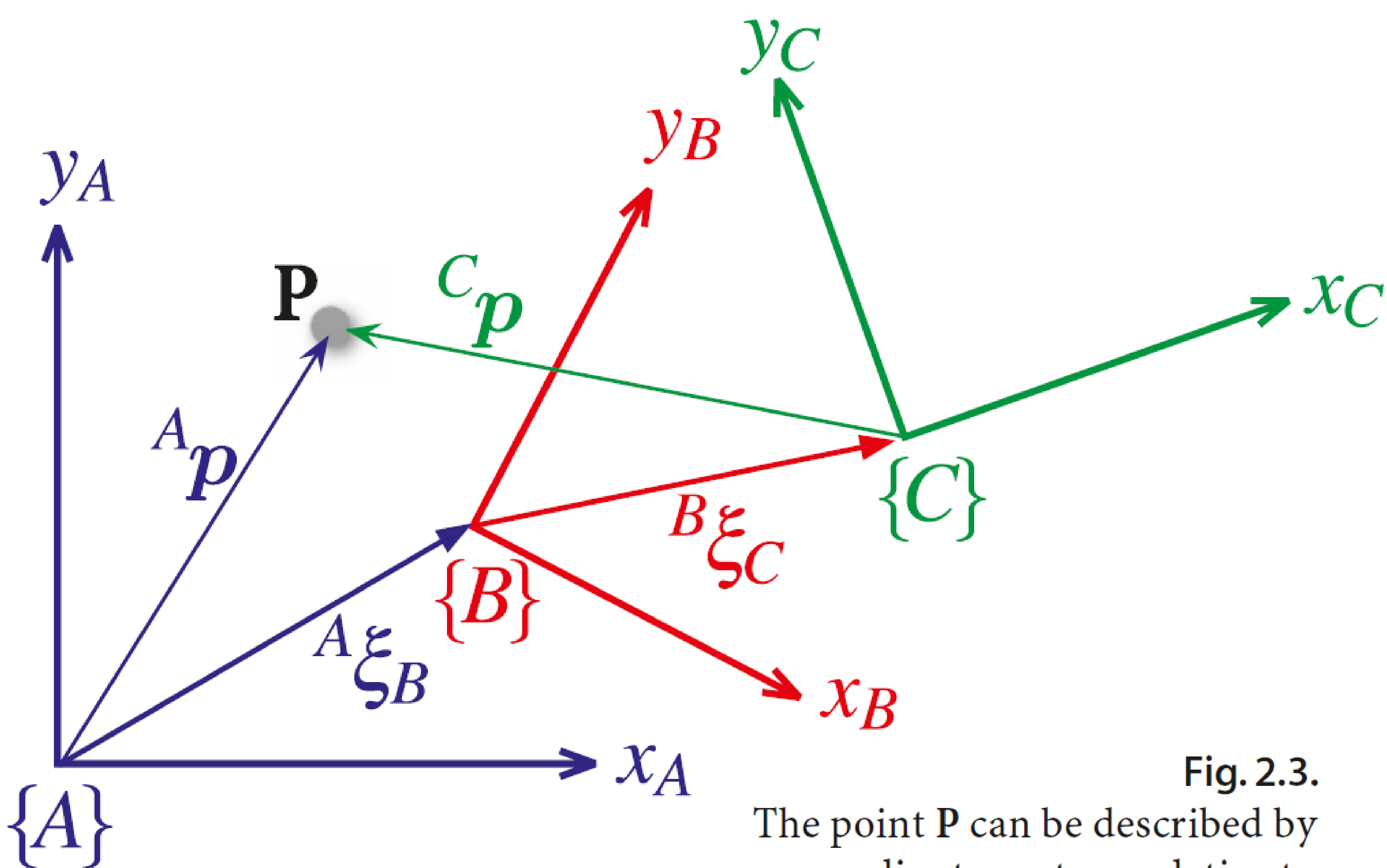


Fig. 2.3.
The point $\mathbf{P}$ can be described by coordinate vectors relative to either frame $\{A\}$, $\{B\}$ or $\{C\}$. The frames are described by relative poses

Transformations

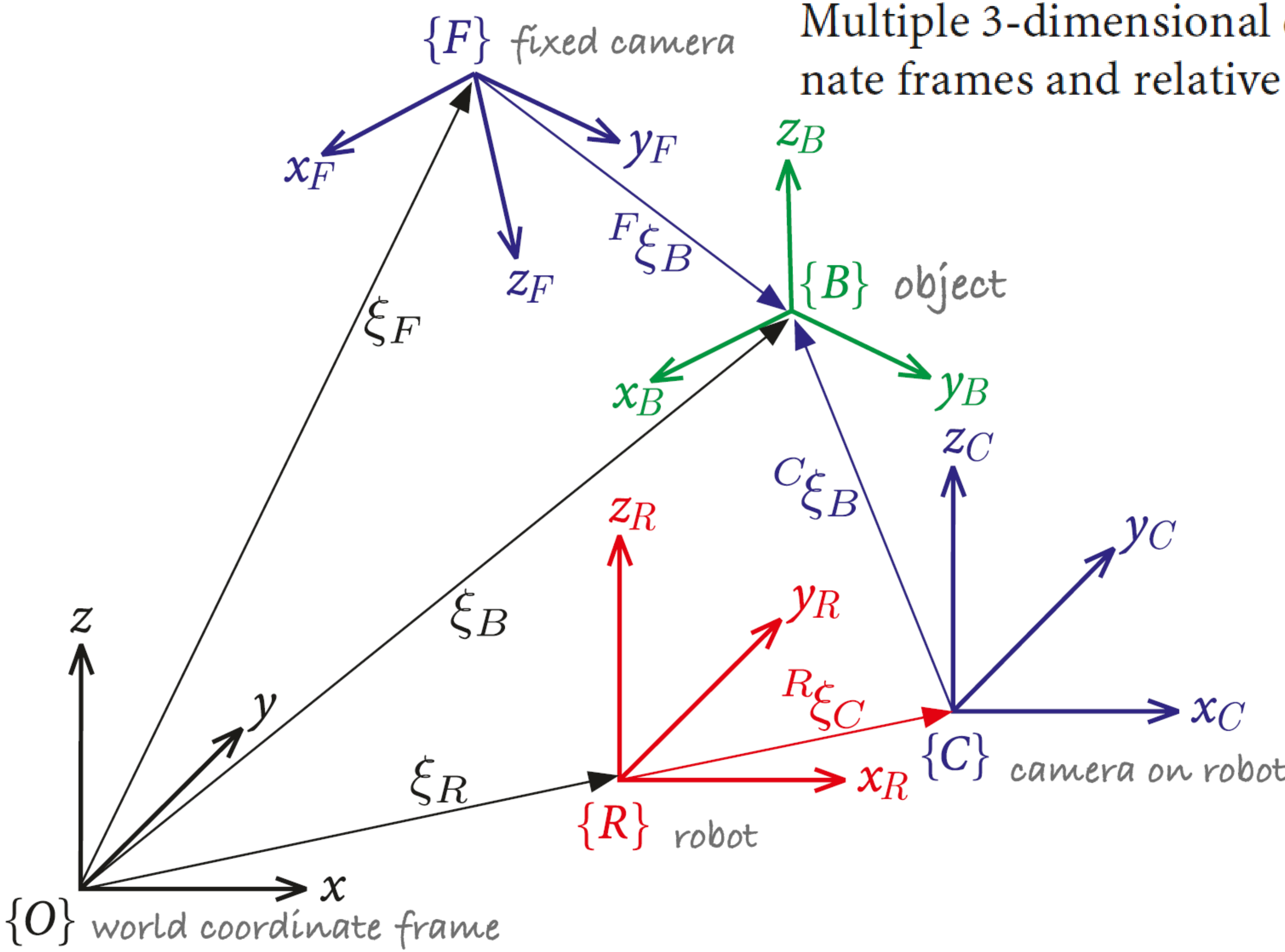


Fig. 2.4.
Multiple 3-dimensional coordinate frames and relative poses

Transformations

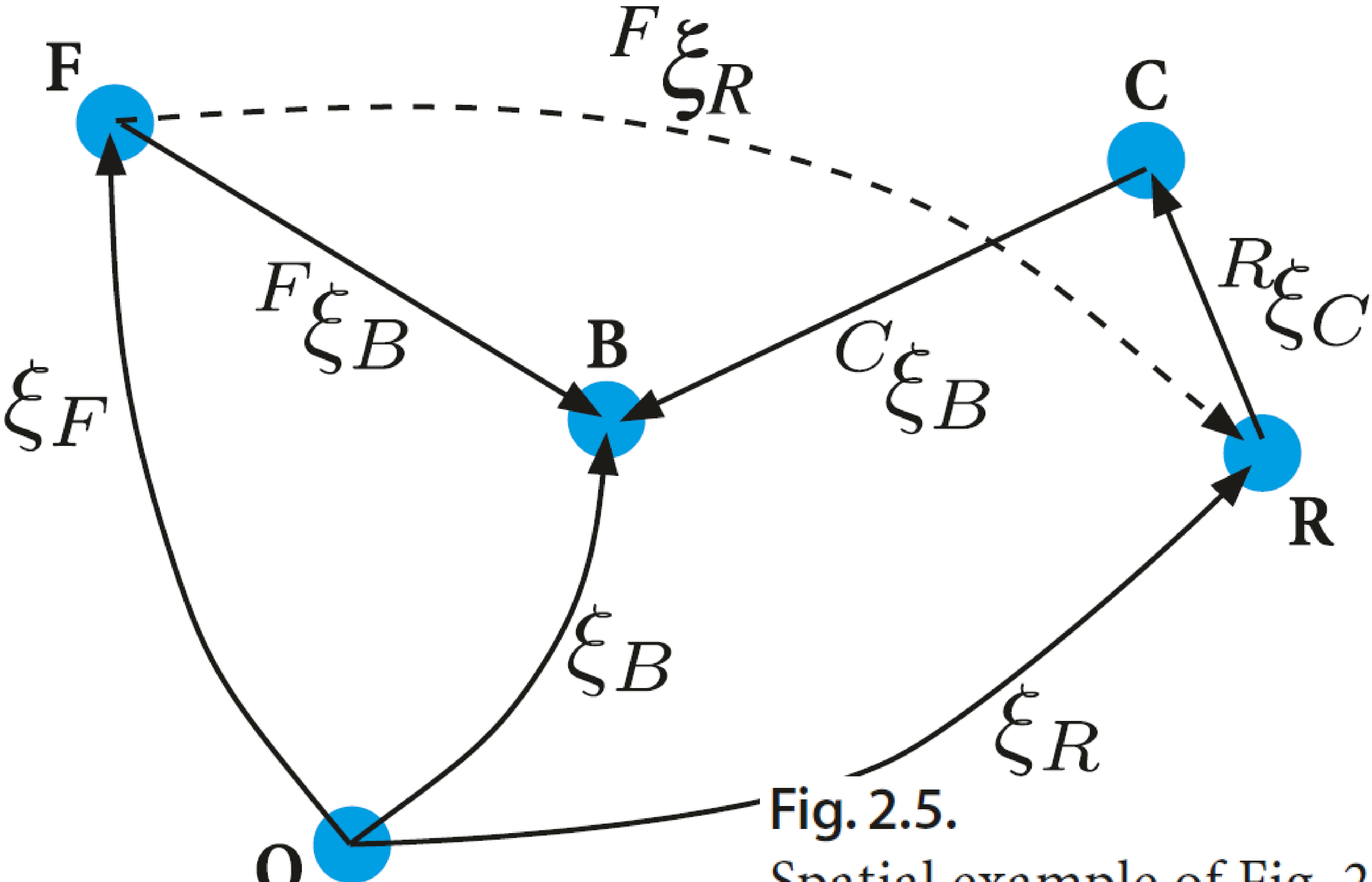


Fig. 2.5.
Spatial example of Fig. 2.4 expressed as a directed graph

Outline

Vector / matrix refresher

SO(2): 2-D rotation / orientation

SE(2): 2-D transformations

SO(3): 3-D rotation / orientation

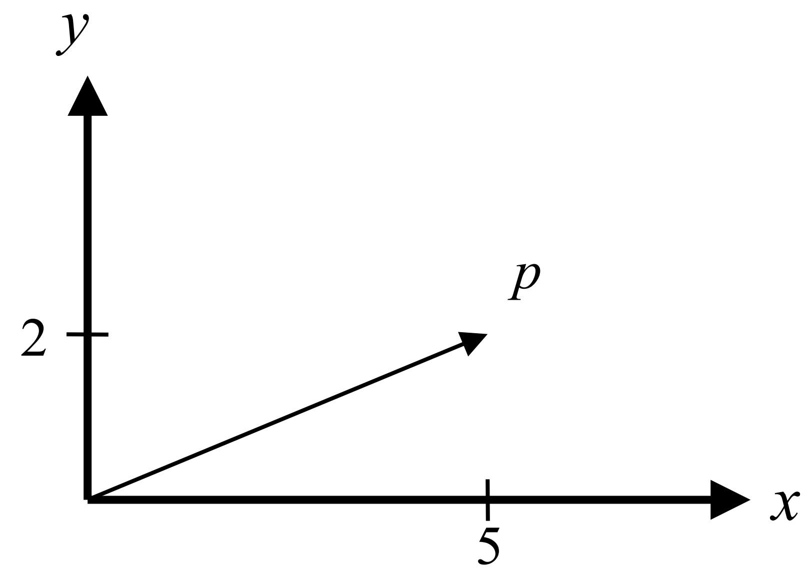
SE(3): 3-D transformations

More representations for 3-D rotations (time permitting)
- Euler angles
- Axis-angle representations
- Quaternions

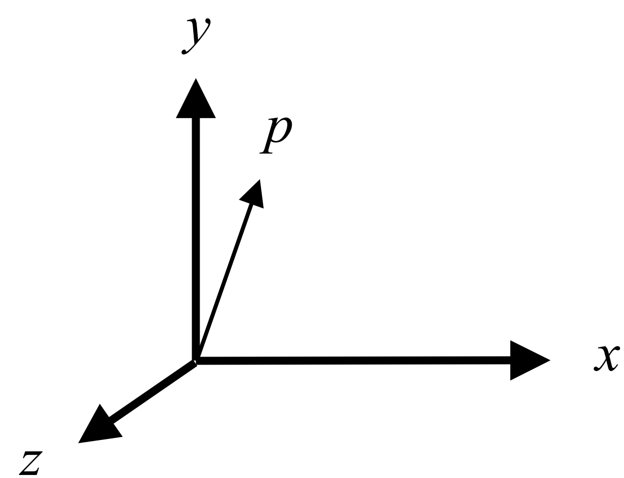
Representing position: Vectors

$$p = \begin{pmatrix} 5 \\ 2 \end{pmatrix}$$ ("column" vector)

$$p = (5, 2)$$ ("row" vector)



$$p = \begin{pmatrix} 2 \\ 5 \\ 2 \end{pmatrix}$$

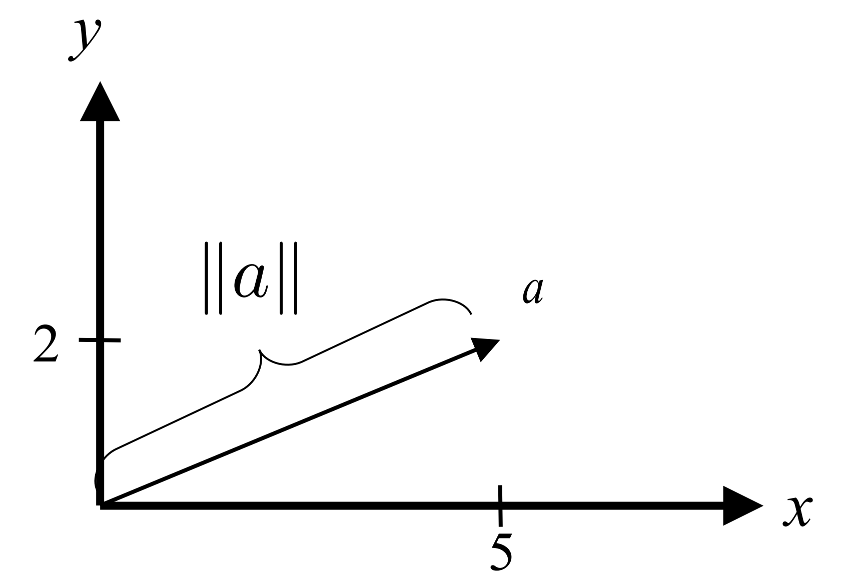


Unit vectors

These are the elements of *a*: $a = \begin{pmatrix} a_x \\ a_y \end{pmatrix}$

Vector length/magnitude:

$$\|a\| = \sqrt{a_x^2 + a_y^2}$$



Definition of unit vector: $\|\hat{a}\| = 1$

How convert a non-zero vector *a* into a unit vector pointing in the same direction?

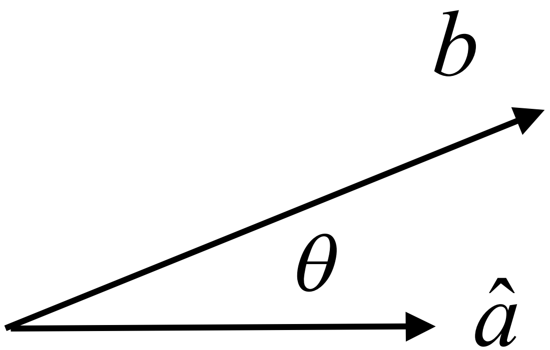
$$\hat{a} = \frac{a}{\sqrt{a_x^2 + a_y^2}}$$

Orthogonal vectors

First, define the dot product: $$a \cdot b = a_x b_x + a_y b_y$$

$$= |a||b|\cos(\theta)$$

Under what conditions is the dot product zero?



Orthogonal vectors

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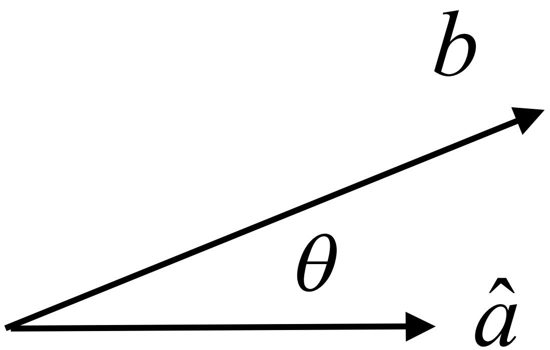

$$a \cdot b = 0$$ when: $$a = 0$$

or, $$b = 0$$

or, $$\cos(\theta) = 0$$

Orthogonal vectors

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$a \cdot b = 0$ when: $a = 0$

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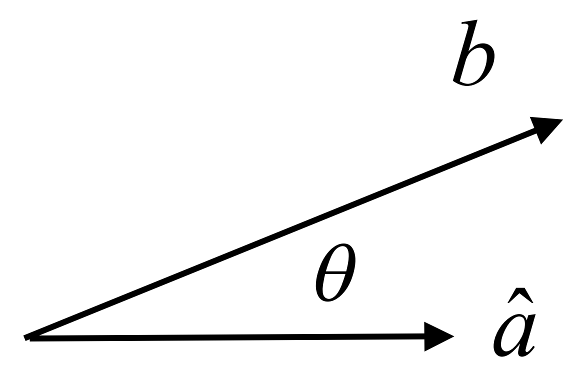
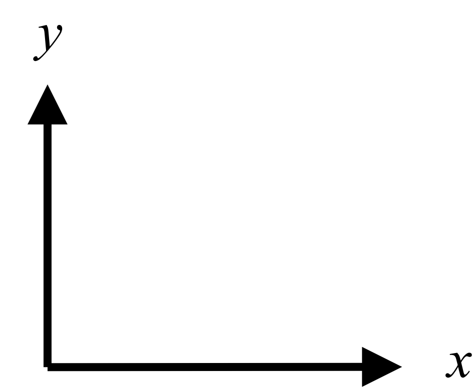


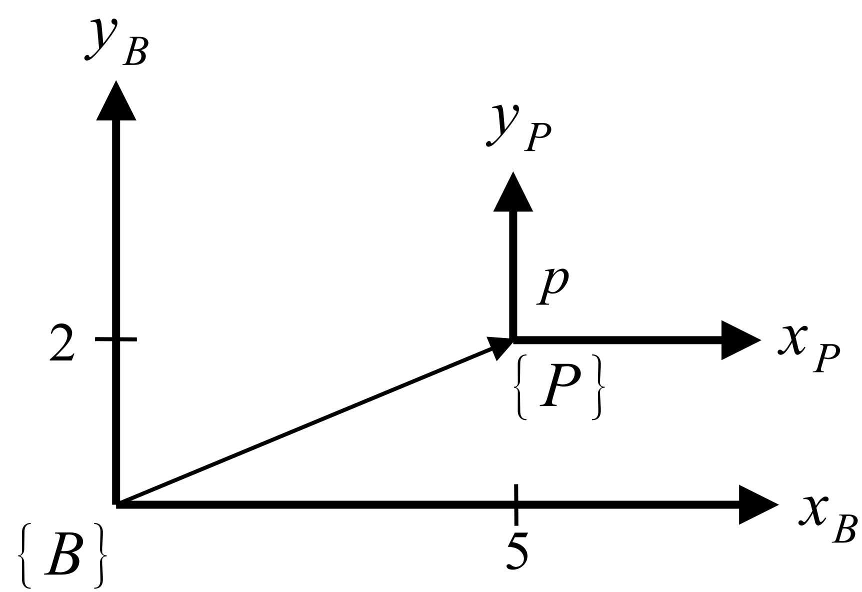
Unit vectors are orthogonal iff
the dot product is zero:

x is orthogonal to y iff $x \cdot y = 0$



Representing translation: Vectors

- Vectors can represent translation between two different reference frames with the same orientation
- The prefix superscript denotes the reference frame in which the vector is represented



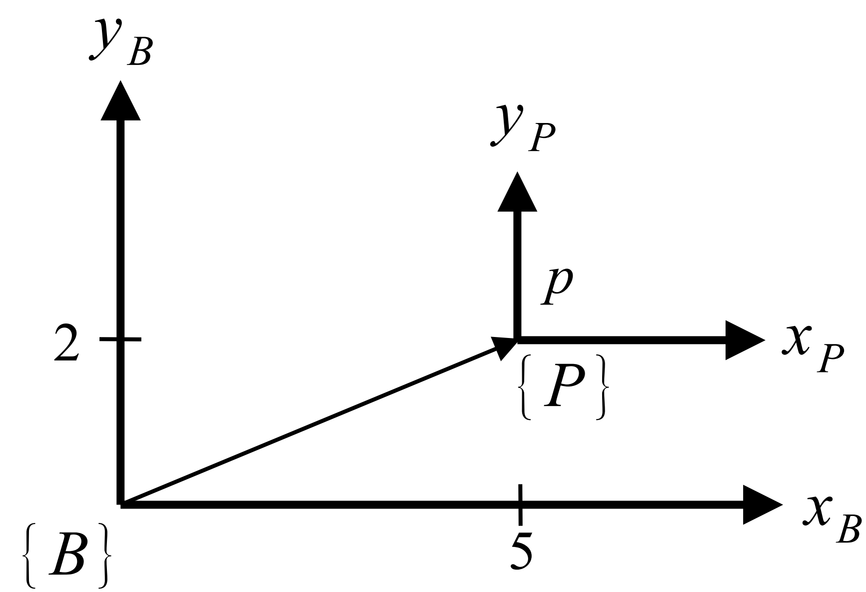
$$^{B}p = \begin{pmatrix} 5 \\ 2 \end{pmatrix} \qquad ^{P}p = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

Same point, two different reference frames

Representing translation: Vectors

- Axes are *orthogonal* unit basis vectors

This means "perpendicular"

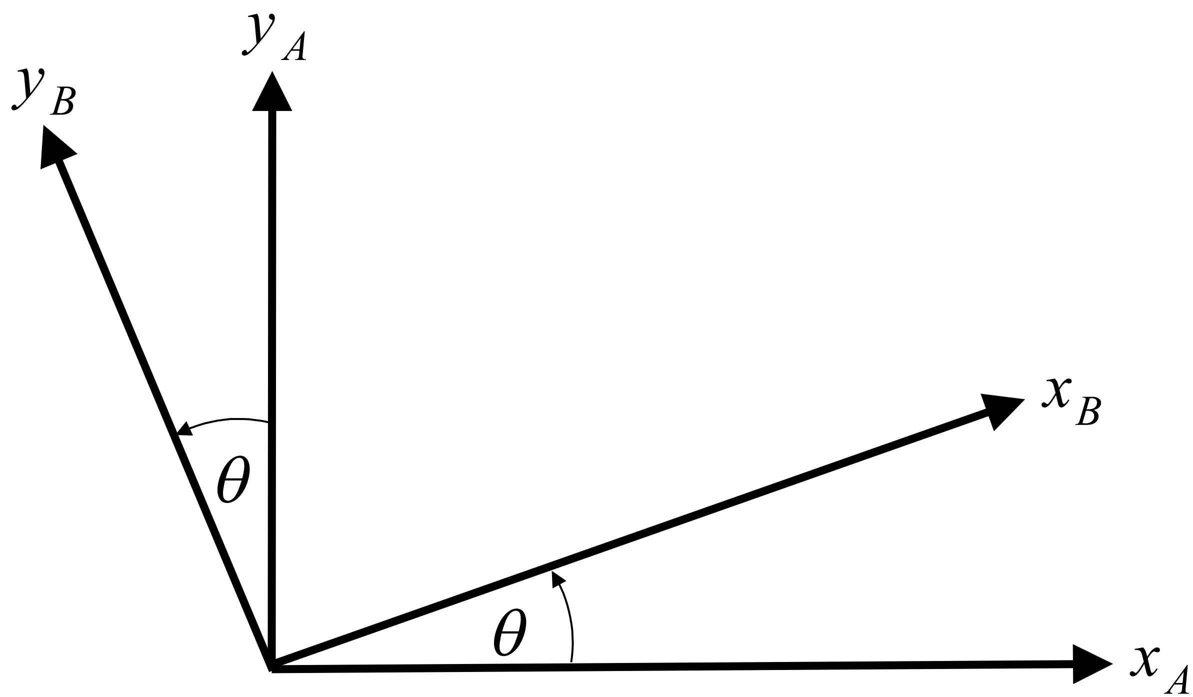


$\hat{x}_B$ ← A vector of length one pointing in the direction of the base frame *x* axis

y_B ← Base *{ B }* frame *y* axis

y_P ← *{ P }* frame *y* axis

Representing rotation: Matrices



$$ {}^A\hat{x}_B = \begin{pmatrix} \cos(\theta) \\ \sin(\theta) \end{pmatrix} \qquad {}^A R_B = \begin{pmatrix} {}^A\hat{x}_B & {}^A\hat{y}_B \end{pmatrix} = \begin{pmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{pmatrix} $$

$$ {}^A\hat{y}_B = \begin{pmatrix} -\sin(\theta) \\ \cos(\theta) \end{pmatrix} \qquad {}^B R_A = \begin{pmatrix} {}^B\hat{x}_A & {}^B\hat{y}_A \end{pmatrix} = \begin{pmatrix} \cos(\theta) & \sin(\theta) \\ -\sin(\theta) & \cos(\theta) \end{pmatrix} = \left({}^A R_B \right)^{-1} = {}^A R_B^T $$

Matrix multiplication

$$A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \qquad B = \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix}$$

$$AB = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix} = \begin{pmatrix} a_{11}b_{11} + a_{12}b_{21} & a_{11}b_{12} + a_{12}b_{22} \\ a_{21}b_{11} + a_{22}b_{21} & a_{21}b_{12} + a_{22}b_{22} \end{pmatrix}$$

Can represent dot product as a matrix multiply:

$$a \cdot b = a^T b = a_x b_x + a_y b_y$$

Matrix transpose

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} \qquad A^T = \begin{pmatrix} a_{11} & a_{21} & a_{31} \\ a_{12} & a_{22} & a_{32} \\ a_{13} & a_{23} & a_{33} \end{pmatrix}$$

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$$

$$p = \begin{pmatrix} 5 \\ 2 \end{pmatrix} \longrightarrow p^T = (5, 2)$$

Important property: $\mathbf{A}^T\mathbf{B}^T = (\mathbf{BA})^T$

Outline

✔ Vector / matrix refresher

SO(2): 2-D rotation / orientation

SE(2): 2-D transformations

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SE(3): 3-D transformations

More representations for 3-D rotations (time permitting)
- Euler angles
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2-D transformation

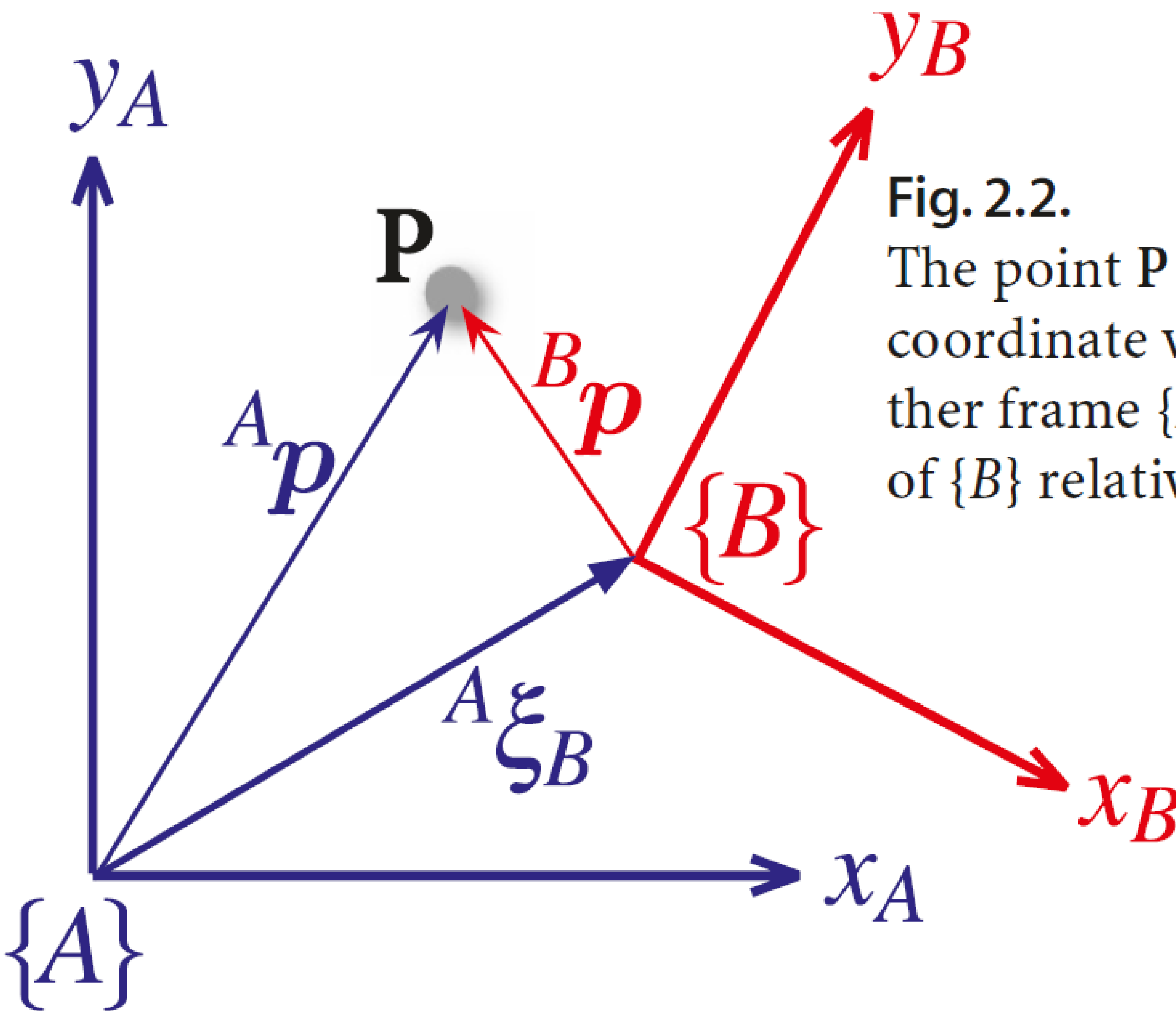

Fig. 2.2.
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2-D transformation

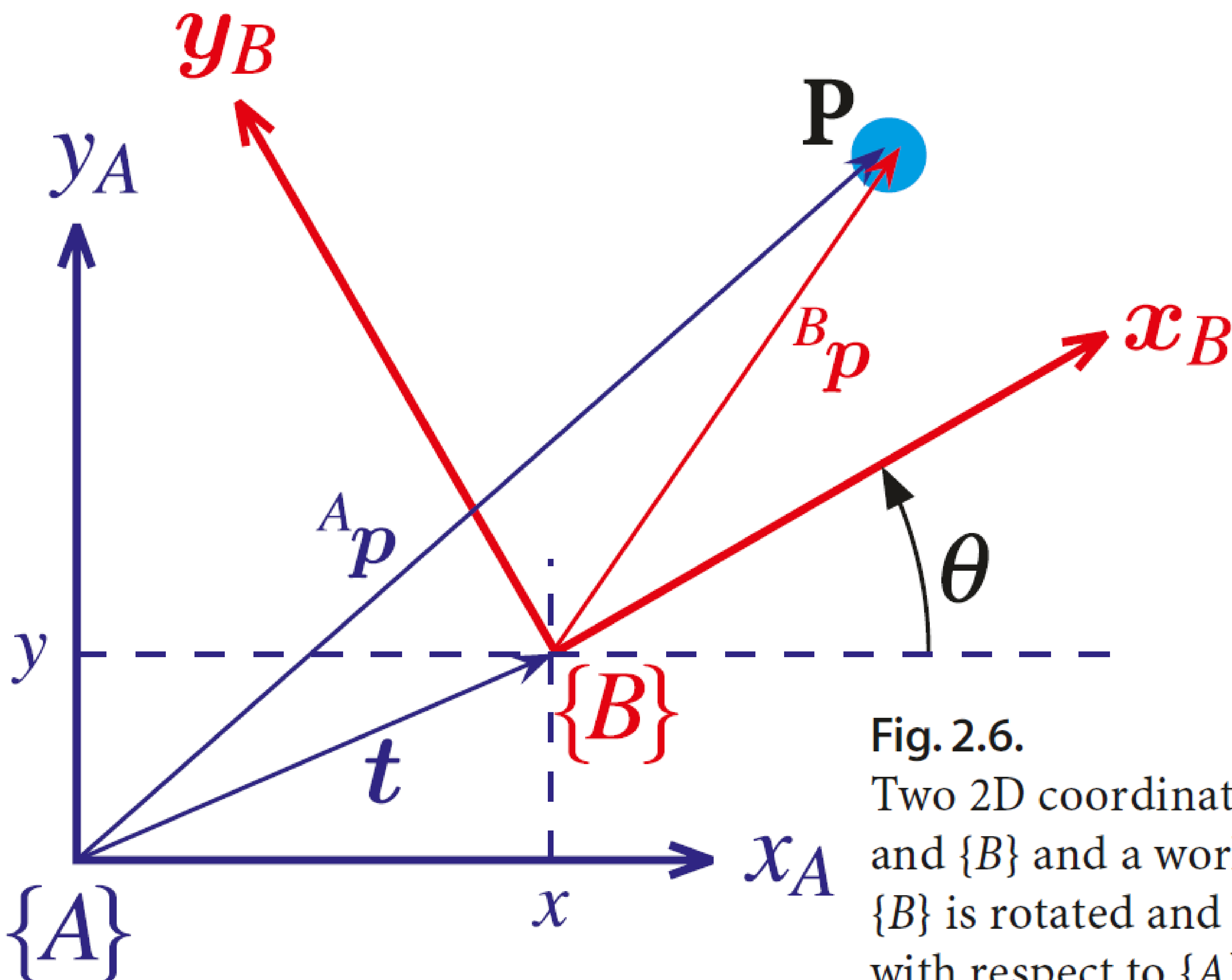


Fig. 2.6.
Two 2D coordinate frames $\{A\}$ and $\{B\}$ and a world point **P**. $\{B\}$ is rotated and translated with respect to $\{A\}$

2-D rotation

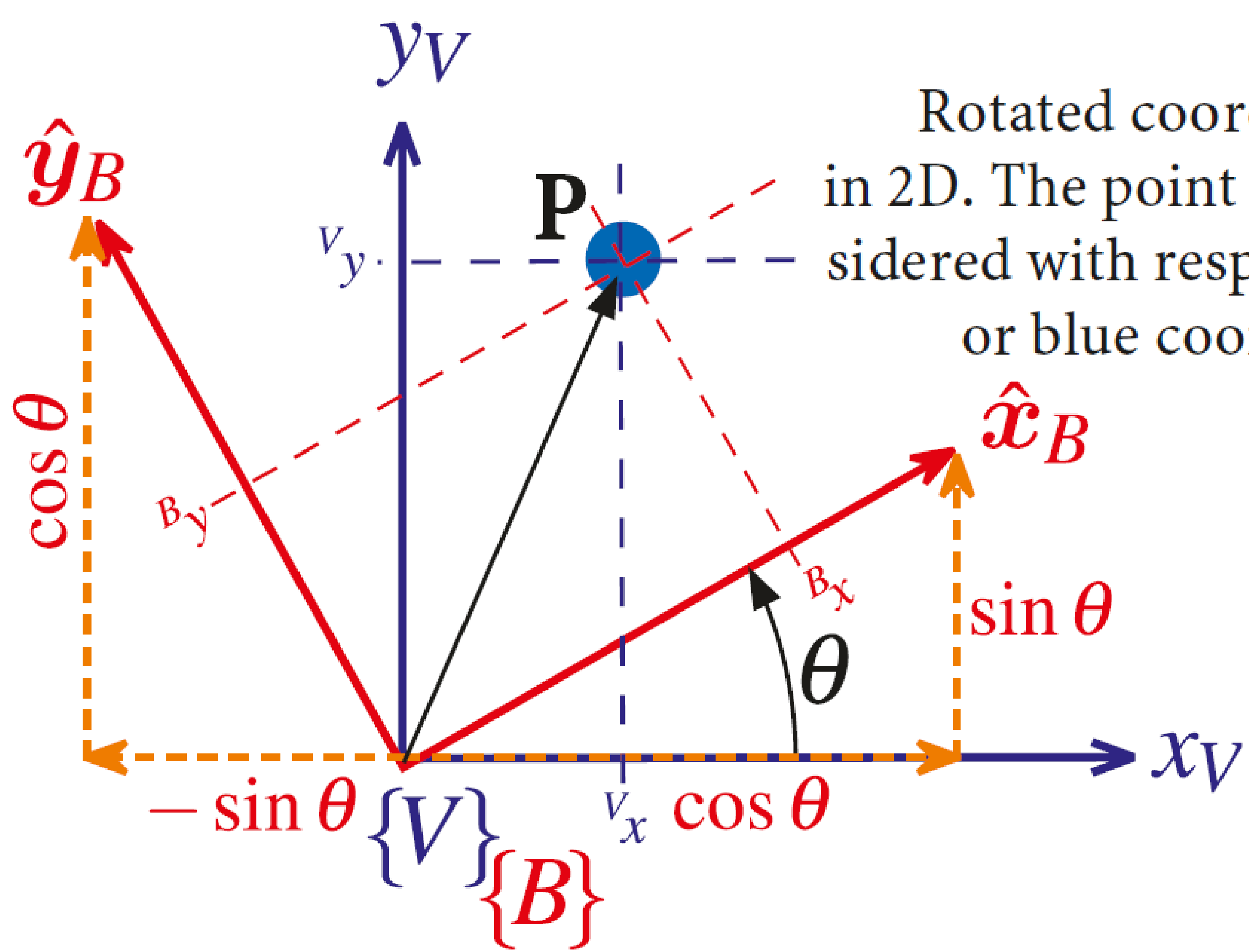


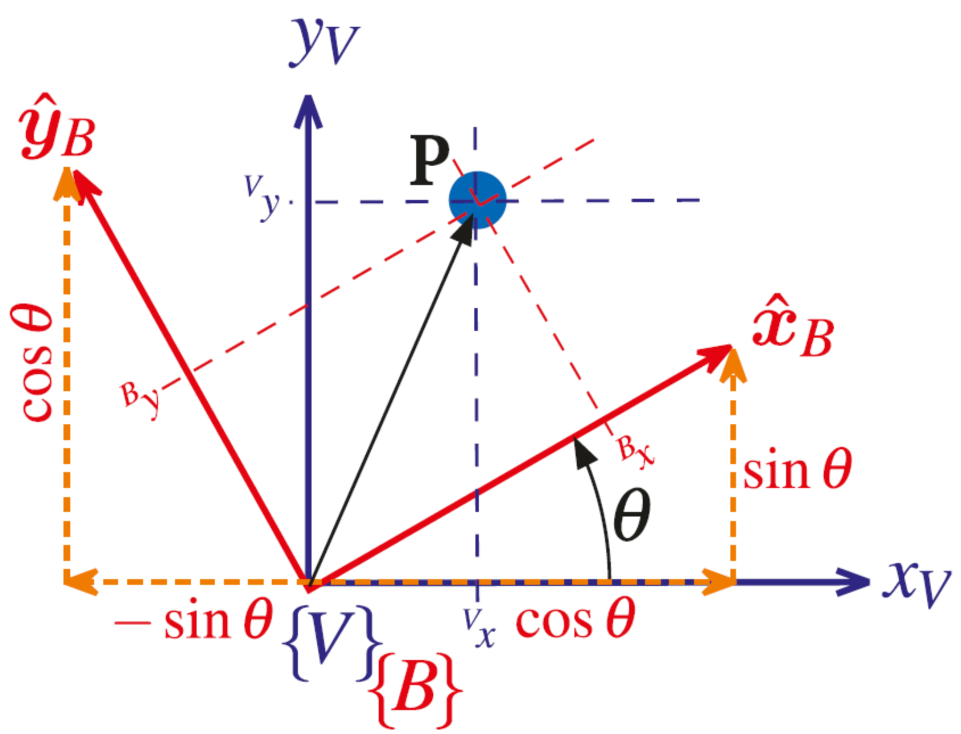
Fig. 2.7. Rotated coordinate frames in 2D. The point **P** can be considered with respect to the red or blue coordinate frame

2-D rotation

Relationship between bases:

$$\hat{\boldsymbol{x}}_B = \cos\theta\hat{\boldsymbol{x}}_V + \sin\theta\hat{\boldsymbol{y}}_V$$

$$\hat{\boldsymbol{y}}_B = -\sin\theta\hat{\boldsymbol{x}}_V + \cos\theta\hat{\boldsymbol{y}}_V$$



2-D rotation

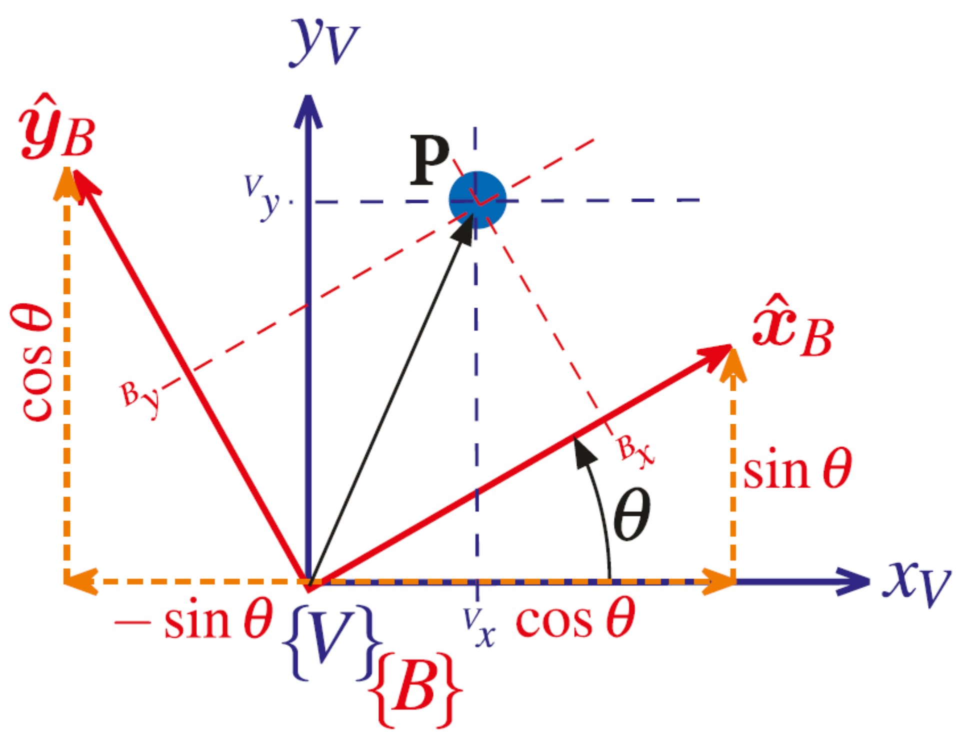
Relationship between bases:

$$\hat{\boldsymbol{x}}_B = \cos\theta\hat{\boldsymbol{x}}_V + \sin\theta\hat{\boldsymbol{y}}_V$$

$$\hat{\boldsymbol{y}}_B = -\sin\theta\hat{\boldsymbol{x}}_V + \cos\theta\hat{\boldsymbol{y}}_V$$

Coordinate transformation:

$$\begin{pmatrix} {}^{V}x \\ {}^{V}y \end{pmatrix} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} {}^{B}x \\ {}^{B}y \end{pmatrix}$$

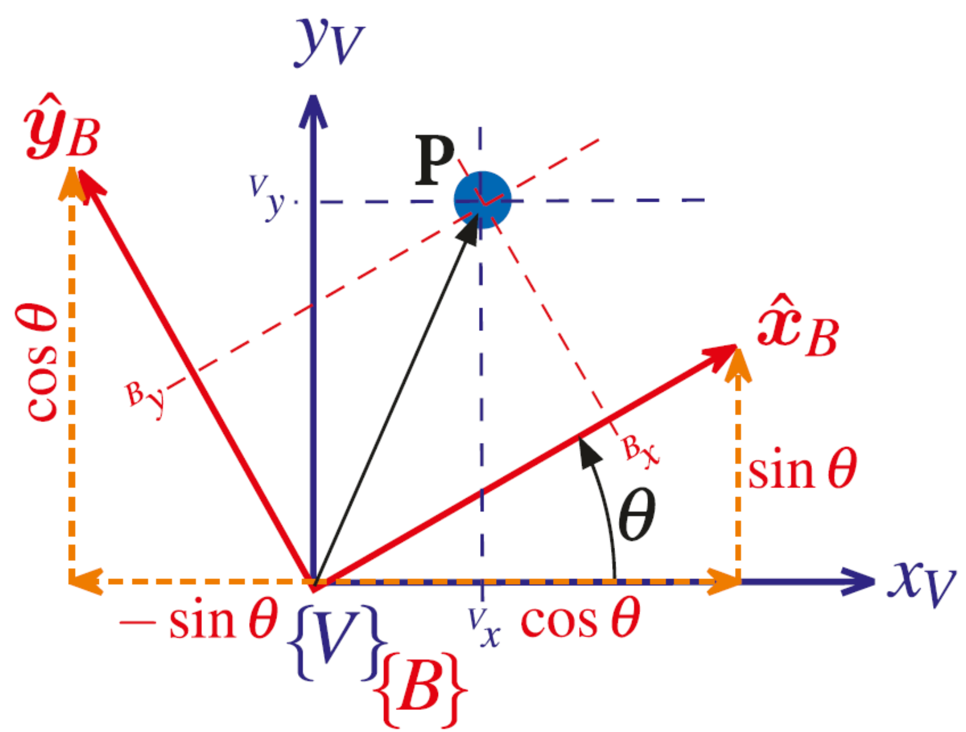


2-D rotation

Relationship between bases:

$$\hat{\boldsymbol{x}}_B = \cos\theta\,\hat{\boldsymbol{x}}_V + \sin\theta\,\hat{\boldsymbol{y}}_V$$

$$\hat{\boldsymbol{y}}_B = -\sin\theta\,\hat{\boldsymbol{x}}_V + \cos\theta\,\hat{\boldsymbol{y}}_V$$



Coordinate transformation:

$$\begin{pmatrix} {}^V x \\ {}^V y \end{pmatrix} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} {}^B x \\ {}^B y \end{pmatrix}$$

To obtain rotation matrix from V to B:
- Express each B axis unit vector in V coordinates
- Insert each as a <u>column</u> vector

$${}^V R_B = \begin{pmatrix} {}^V\hat{x}_B & {}^V\hat{y}_B \end{pmatrix}$$

2-D rotation

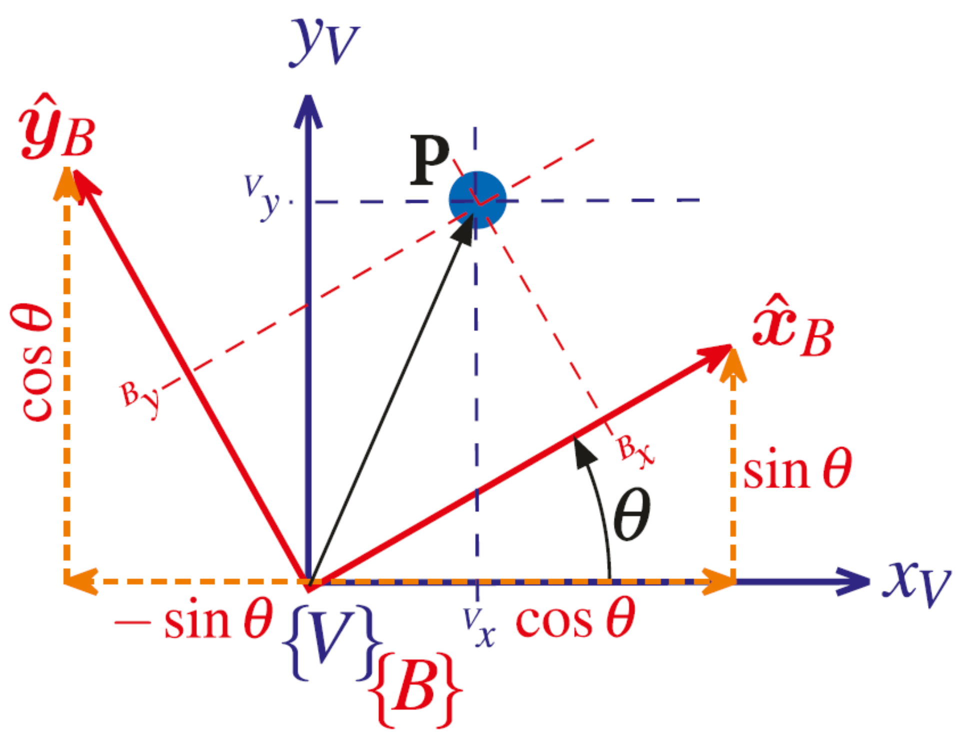
Relationship between bases:

$$\hat{\boldsymbol{x}}_B = \boxed{\cos\theta\hat{\boldsymbol{x}}_V + \sin\theta\hat{\boldsymbol{y}}_V}$$

$$\hat{\boldsymbol{y}}_B = -\sin\theta\hat{\boldsymbol{x}}_V + \cos\theta\hat{\boldsymbol{y}}_V$$

Coordinate transformation:

$$\begin{pmatrix} {}^{V}x \\ {}^{V}y \end{pmatrix} = \begin{pmatrix} \boxed{\begin{matrix}\cos\theta \\ \sin\theta\end{matrix}} & \begin{matrix}-\sin\theta \\ \cos\theta\end{matrix} \end{pmatrix} \begin{pmatrix} {}^{B}x \\ {}^{B}y \end{pmatrix}$$



To obtain rotation matrix from V to B:
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$${}^{V}R_B = \begin{pmatrix} \boxed{{}^{V}\hat{x}_B} & {}^{V}\hat{y}_B \end{pmatrix}$$

2-D rotation

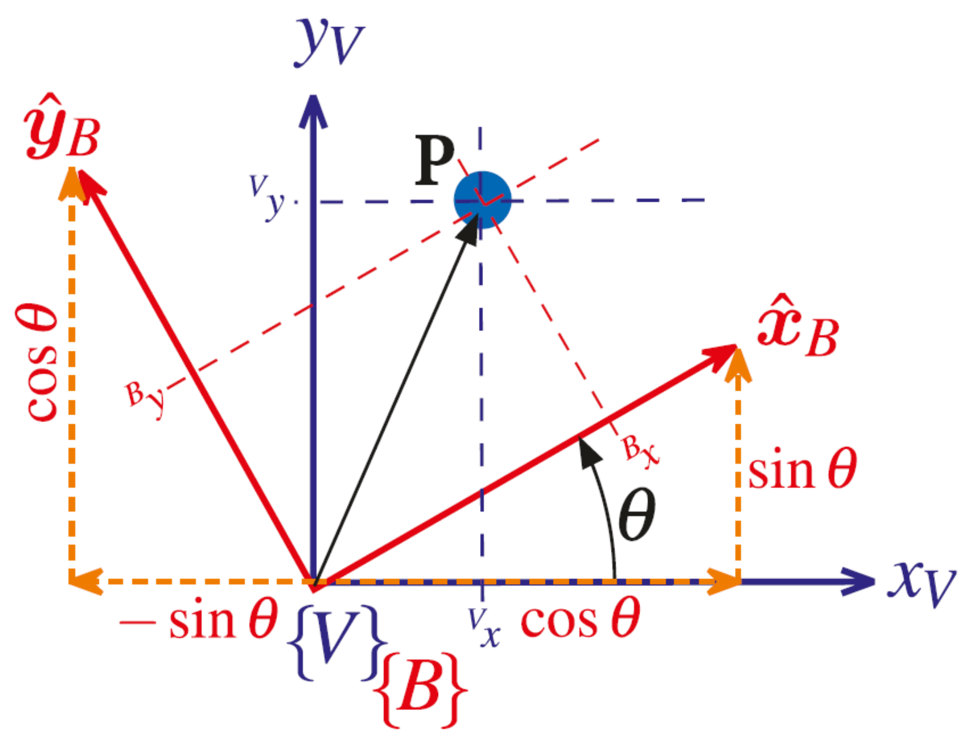
Relationship between bases:

$$\hat{\boldsymbol{x}}_B = \cos\theta\,\hat{\boldsymbol{x}}_V + \sin\theta\,\hat{\boldsymbol{y}}_V$$

$$\hat{\boldsymbol{y}}_B = \boxed{-\sin\theta\,\hat{\boldsymbol{x}}_V + \cos\theta\,\hat{\boldsymbol{y}}_V}$$

Coordinate transformation:

$$\begin{pmatrix} {}^V x \\ {}^V y \end{pmatrix} = \begin{pmatrix} \cos\theta & \boxed{\begin{matrix} -\sin\theta \\ \cos\theta \end{matrix}} \\ \sin\theta & \end{pmatrix} \begin{pmatrix} {}^B x \\ {}^B y \end{pmatrix}$$



To obtain rotation matrix from V to B:
- Express each B axis unit vector in V coordinates
- Insert each as a <u>column</u> vector

$${}^V R_B = \begin{pmatrix} {}^V\hat{x}_B & \boxed{{}^V\hat{y}_B} \end{pmatrix}$$

2-D rotation

Relationship between bases:

$$\hat{\boldsymbol{x}}_B = \cos\theta\,\hat{\boldsymbol{x}}_V + \sin\theta\,\hat{\boldsymbol{y}}_V$$

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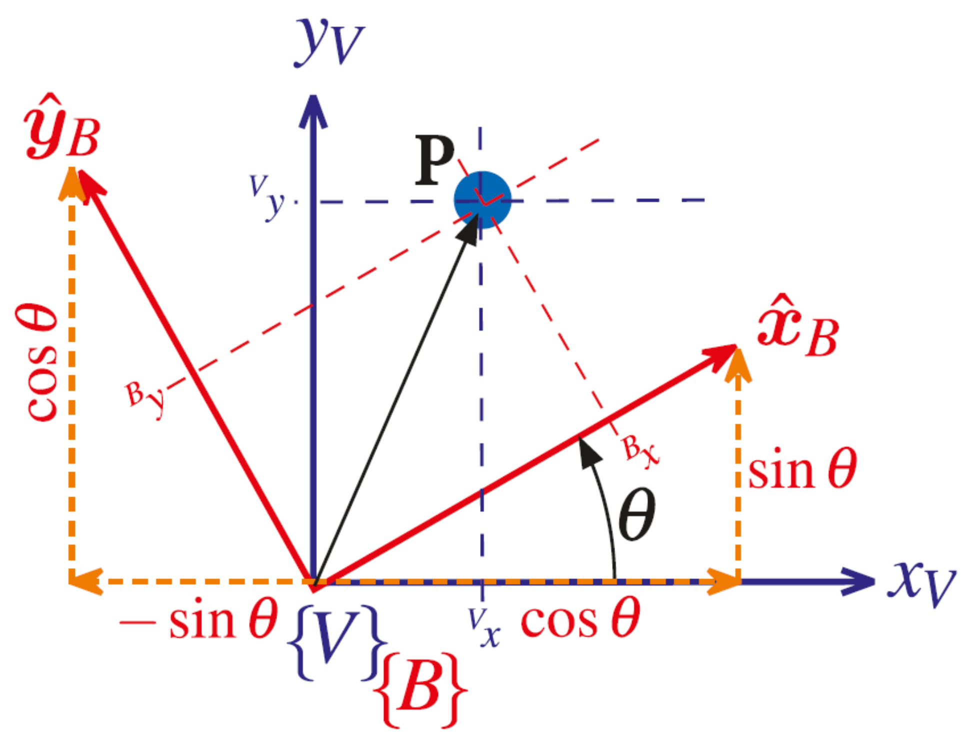

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- Express each B axis unit vector in V coordinates
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$${}^V R_B = \begin{pmatrix} {}^V\hat{x}_B & {}^V\hat{y}_B \end{pmatrix}$$

$$\begin{pmatrix} {}^V x \\ {}^V y \end{pmatrix} = {}^V\boldsymbol{R}_B \begin{pmatrix} {}^B x \\ {}^B y \end{pmatrix}$$

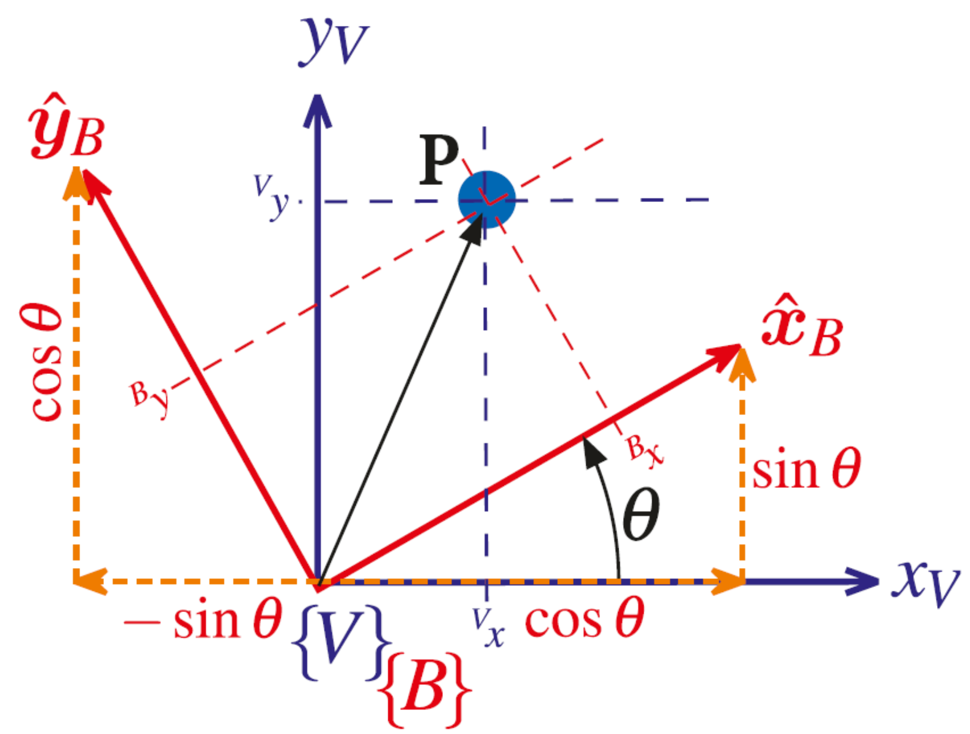
Notation

The following lines equations the same thing:
Relating coordinates of *p* in *B* frame
to coordinates of *p* in *V* frame

$$\begin{pmatrix} {}^{V}x \\ {}^{V}y \end{pmatrix} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} {}^{B}x \\ {}^{B}y \end{pmatrix}$$

$$\begin{pmatrix} {}^{V}x \\ {}^{V}y \end{pmatrix} = {}^{V}\boldsymbol{R}_{B} \begin{pmatrix} {}^{B}x \\ {}^{B}y \end{pmatrix}$$

$${}^{V}p = {}^{V}R_{B}\, {}^{B}p$$



Rotation matrices: Properties

A rotation matrix is a 2x2 (or 3x3) matrix *R* such that:

1. $R^T R = I$
2. $det(R) = +1$

Rows and columns are unit length and orthogonal

Right handed coordinate frame

Rotation matrix inverse equals transpose:

$$R^T R = I$$
$$R = (R^T)^{-1}$$
$$R = (R^{-1})^T$$
$$R^T = R^{-1}$$

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Rows and columns are unit length and orthogonal

Right handed coordinate frame

$$R = \begin{pmatrix} r_{11} & r_{12} \\ r_{21} & r_{22} \end{pmatrix}$$

Unit vectors and orthogonal to each other

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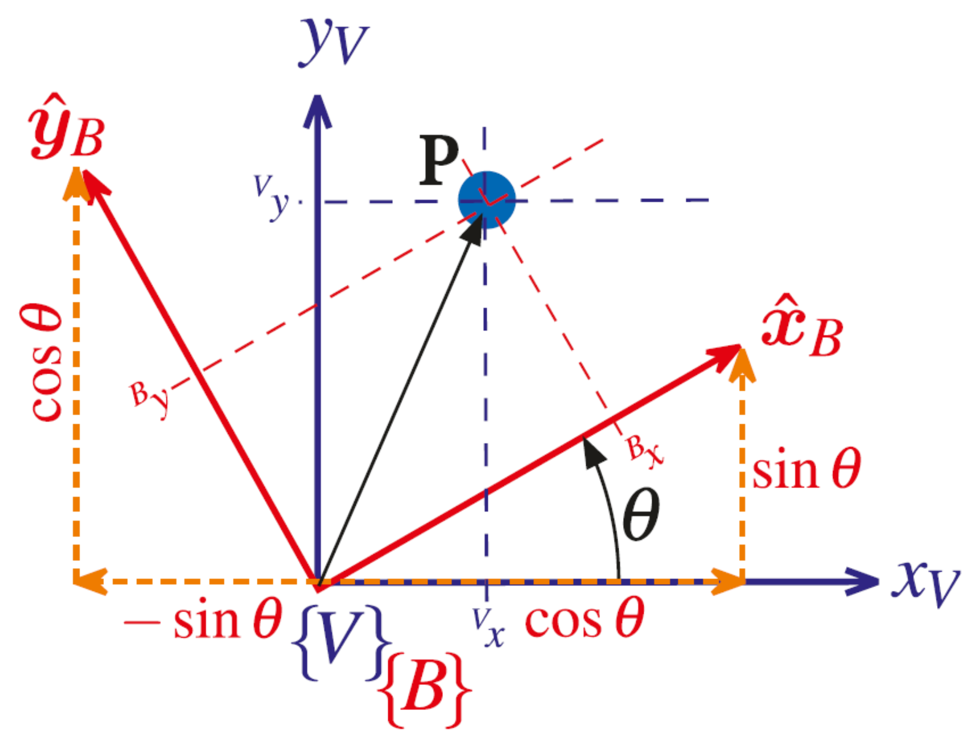
Unit vectors and orthogonal to each other

Rotation matrices: Inverse

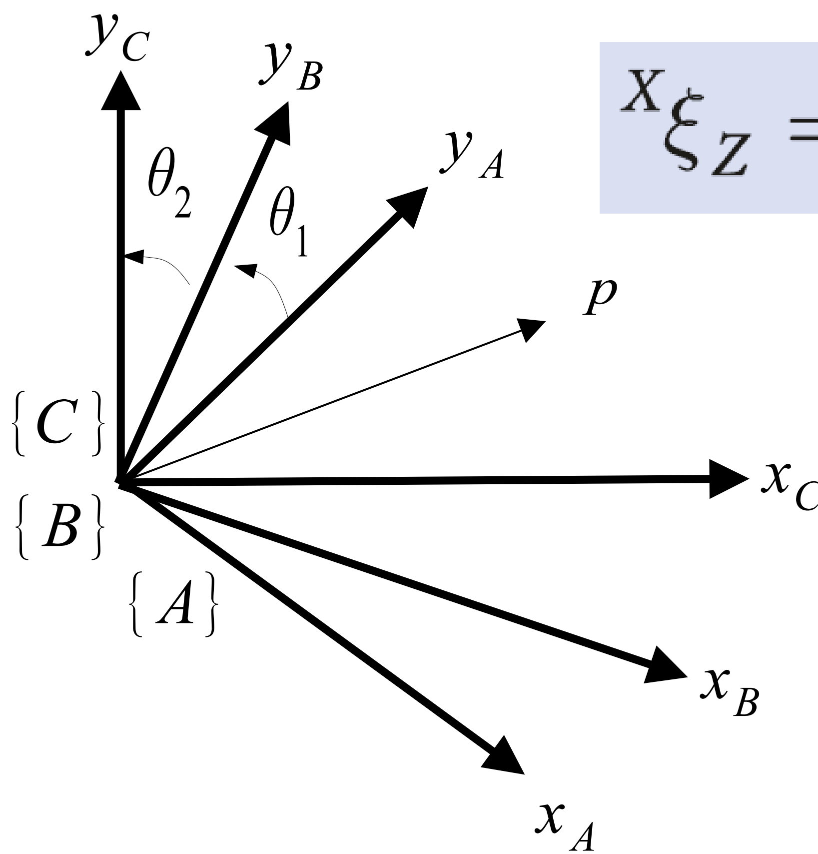
$$\begin{pmatrix} {}^{V}x \\ {}^{V}y \end{pmatrix} = {}^{V}\boldsymbol{R}_{B} \begin{pmatrix} {}^{B}x \\ {}^{B}y \end{pmatrix}$$



$$\begin{pmatrix} {}^{B}x \\ {}^{B}y \end{pmatrix} = \left({}^{V}\boldsymbol{R}_{B}\right)^{-1} \begin{pmatrix} {}^{V}x \\ {}^{V}y \end{pmatrix} = \left({}^{V}\boldsymbol{R}_{B}\right)^{T} \begin{pmatrix} {}^{V}x \\ {}^{V}y \end{pmatrix} = {}^{B}\boldsymbol{R}_{V} \begin{pmatrix} {}^{V}x \\ {}^{V}y \end{pmatrix}$$

Note that inverting the matrix is the same as swapping the superscript and subscript, which leads to the identity $\boldsymbol{R}(-\theta) = \boldsymbol{R}(\theta)^{T}$.

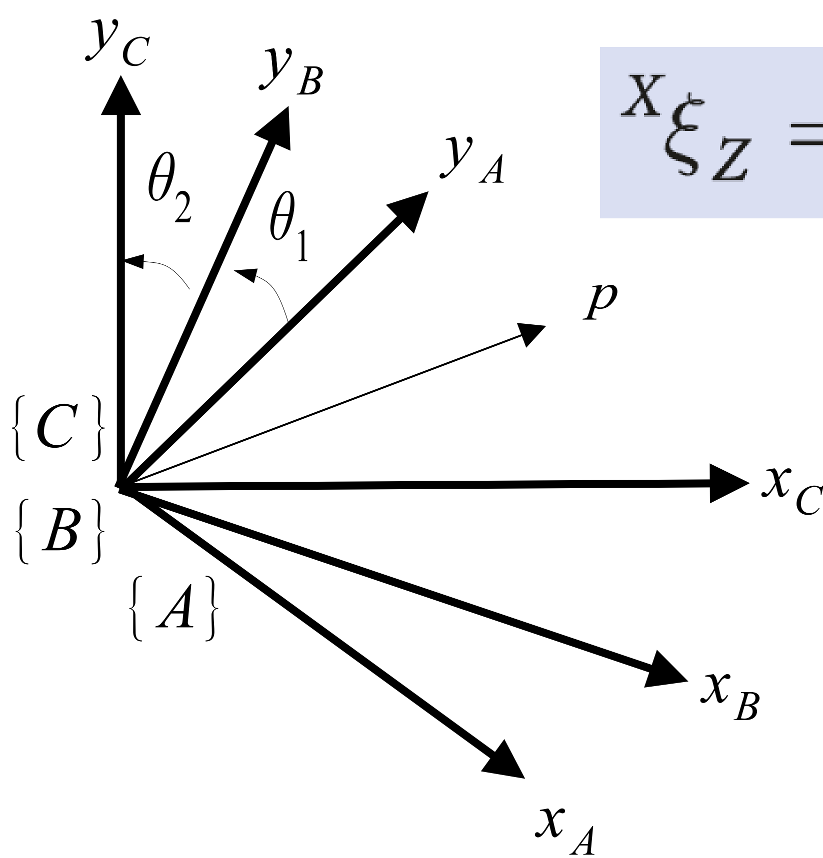
Rotation matrices: Composition



$$ {}^{X}\xi_{Z} = {}^{X}\xi_{\cancel{Y}} \oplus {}^{\cancel{Y}}\xi_{Z} $$

$$ {}^{A}R_{C} = {}^{A}R_{B}\,{}^{B}R_{C} $$

Rotation matrices: Composition



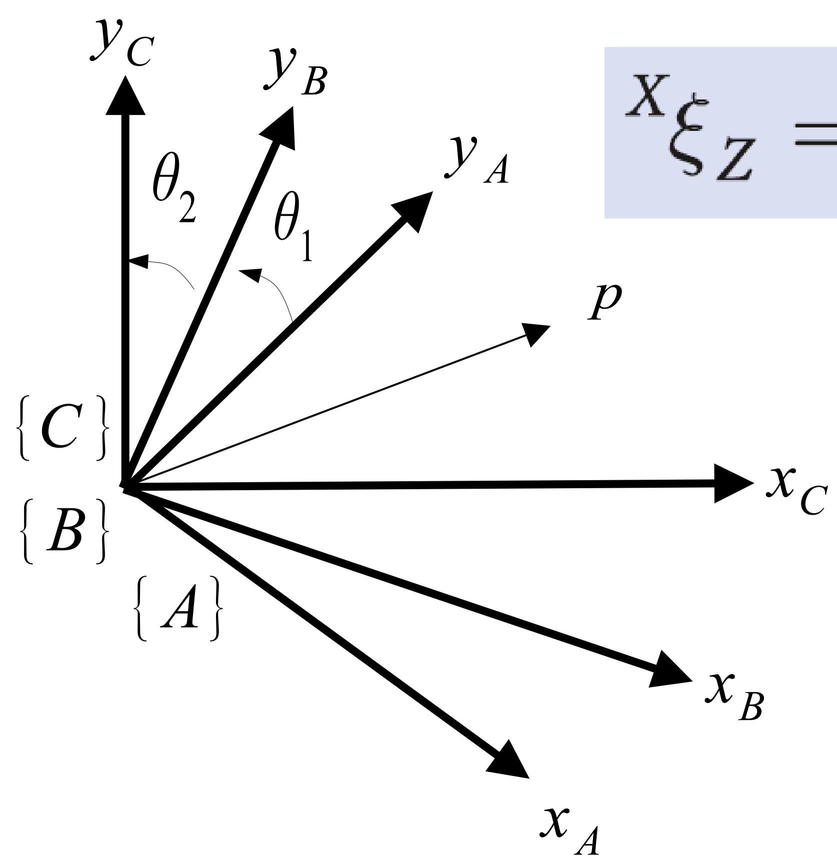
$$^{X}\xi_{Z} = {}^{X}\xi_{\cancel{Y}} \oplus {}^{\cancel{Y}}\xi_{Z}$$

$$^{A}R_{C} = {}^{A}R_{B}\,{}^{B}R_{C}$$

$$^{A}R_{C} = \begin{pmatrix} \cos(\theta_1) & -\sin(\theta_1) \\ \sin(\theta_1) & \cos(\theta_1) \end{pmatrix} \begin{pmatrix} \cos(\theta_2) & -\sin(\theta_2) \\ \sin(\theta_2) & \cos(\theta_2) \end{pmatrix} = \begin{pmatrix} c_1 c_2 - s_1 s_2 & -c_1 s_2 - s_1 c_2 \\ s_1 c_2 + c_1 s_2 & c_1 c_2 - s_1 s_2 \end{pmatrix}$$

$$= \begin{pmatrix} c_{12} & -s_{12} \\ s_{12} & c_{12} \end{pmatrix}$$

Rotation matrices: Composition



$$^{X}\xi_{Z} = {}^{X}\xi_{\cancel{Y}} \oplus {}^{\cancel{Y}}\xi_{Z}$$

$$^{A}R_{C} = {}^{A}R_{B}\,{}^{B}R_{C}$$

Shorthand for $\cos(\theta_1)$

$$^{A}R_{C} = \begin{pmatrix} \cos(\theta_1) & -\sin(\theta_1) \\ \sin(\theta_1) & \cos(\theta_1) \end{pmatrix} \begin{pmatrix} \cos(\theta_2) & -\sin(\theta_2) \\ \sin(\theta_2) & \cos(\theta_2) \end{pmatrix} = \begin{pmatrix} c_1 c_2 - s_1 s_2 & -c_1 s_2 - s_1 c_2 \\ s_1 c_2 + c_1 s_2 & c_1 c_2 - s_1 s_2 \end{pmatrix}$$

Shorthand for $\cos(\theta_1 + \theta_2)$

$$\equiv \begin{pmatrix} c_{12} & -s_{12} \\ s_{12} & c_{12} \end{pmatrix}$$

There are just a few algebraic rules:

$$\xi \oplus 0 = \xi, \quad \xi \ominus 0 = \xi$$
$$\xi \ominus \xi = 0, \quad \ominus \xi \oplus \xi = 0$$

where 0 represents a zero relative pose. A pose has an inverse

$$\ominus {}^X\xi_Y = {}^Y\xi_X$$

which is represented graphically by an arrow from $\{Y\}$ to $\{X\}$. Relative poses can also be composed or compounded

$${}^X\xi_Y \oplus {}^Y\xi_Z = {}^X\xi_Z$$

It is important to note that the algebraic rules for poses are different to normal algebra and that composition is *not* commutative

$$\xi_1 \oplus \xi_2 \neq \xi_2 \oplus \xi_1$$

with the exception being the case where $\xi_1 \oplus \xi_2 = 0$. A relative pose can transform a point expressed as a vector relative to one frame to a vector relative to another

$${}^X\boldsymbol{p} = {}^X\xi_Y \cdot {}^Y\boldsymbol{p}$$

Summary: 2-D rotation

$$ {}^{X}\boldsymbol{R}_{Y}(\theta) = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} $$

is a 2-dimensional rotation matrix with some special properties:

- it is *orthonormal* (also called *orthogonal*) since each of its columns is a unit vector and the columns are orthogonal.
- the columns are the unit vectors that define the axes of the rotated frame Y with respect to X and are by definition both unit-length and orthogonal.
- it belongs to the special orthogonal group of dimension 2 or $\boldsymbol{R} \in \mathbf{SO}(2) \subset \mathbb{R}^{2\times 2}$. This means that the product of any two matrices belongs to the group, as does its inverse.
- its *determinant* is $+1$, which means that the length of a vector is unchanged after transformation, that is, $\|{}^{Y}\boldsymbol{p}\| = \|{}^{X}\boldsymbol{p}\|, \forall\theta$.
- the inverse is the same as the transpose, that is, $\boldsymbol{R}^{-1} = \boldsymbol{R}^{T}$.

Outline

✔ Vector / matrix refresher

✔ SO(2): 2-D rotation / orientation

SE(2): 2-D transformations

SO(3): 3-D rotation / orientation

SE(3): 3-D transformations

More representations for 3-D rotations (time permitting)
- Euler angles
- Axis-angle representations
- Quaternions

2-D transformation: Rotation + translation

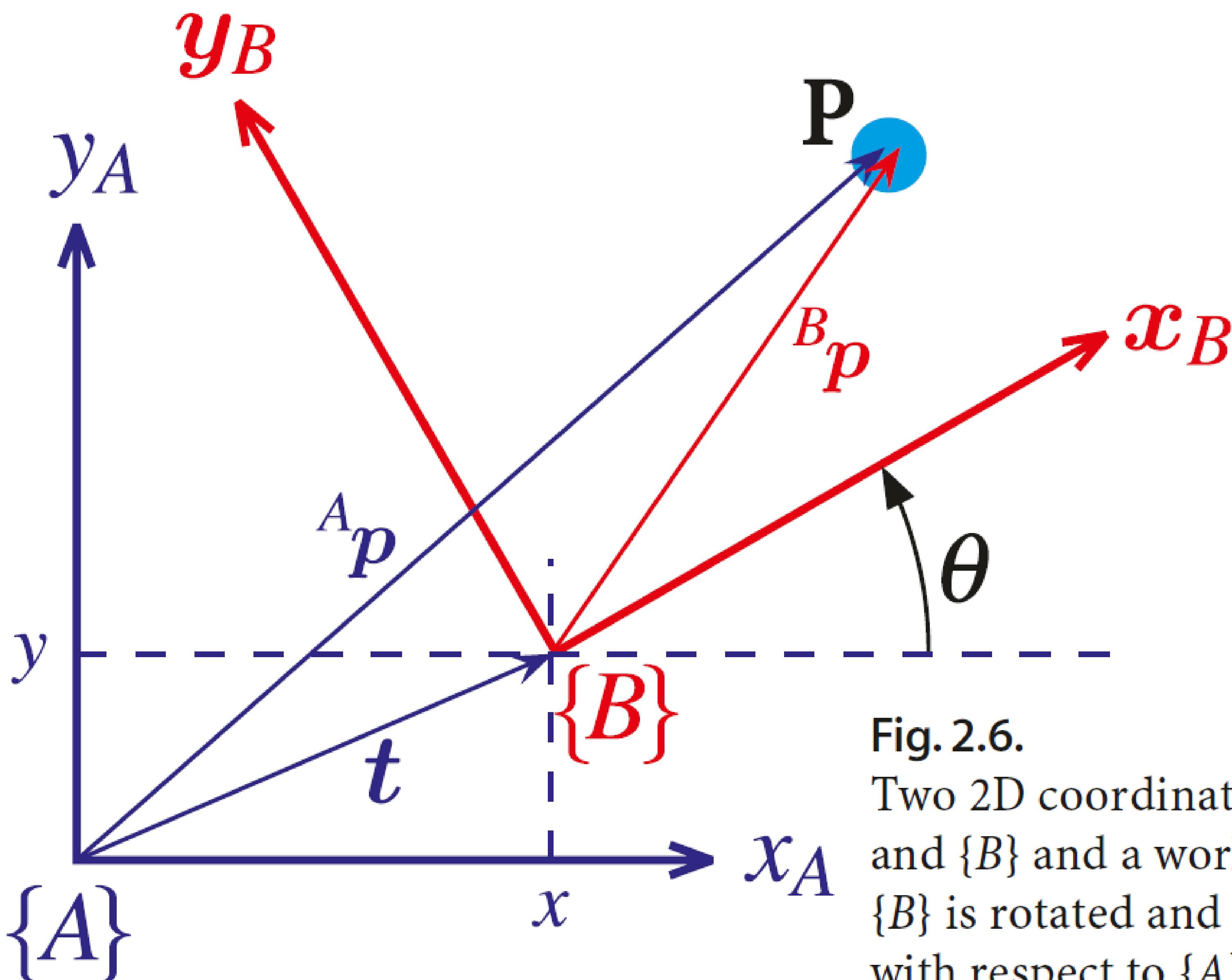
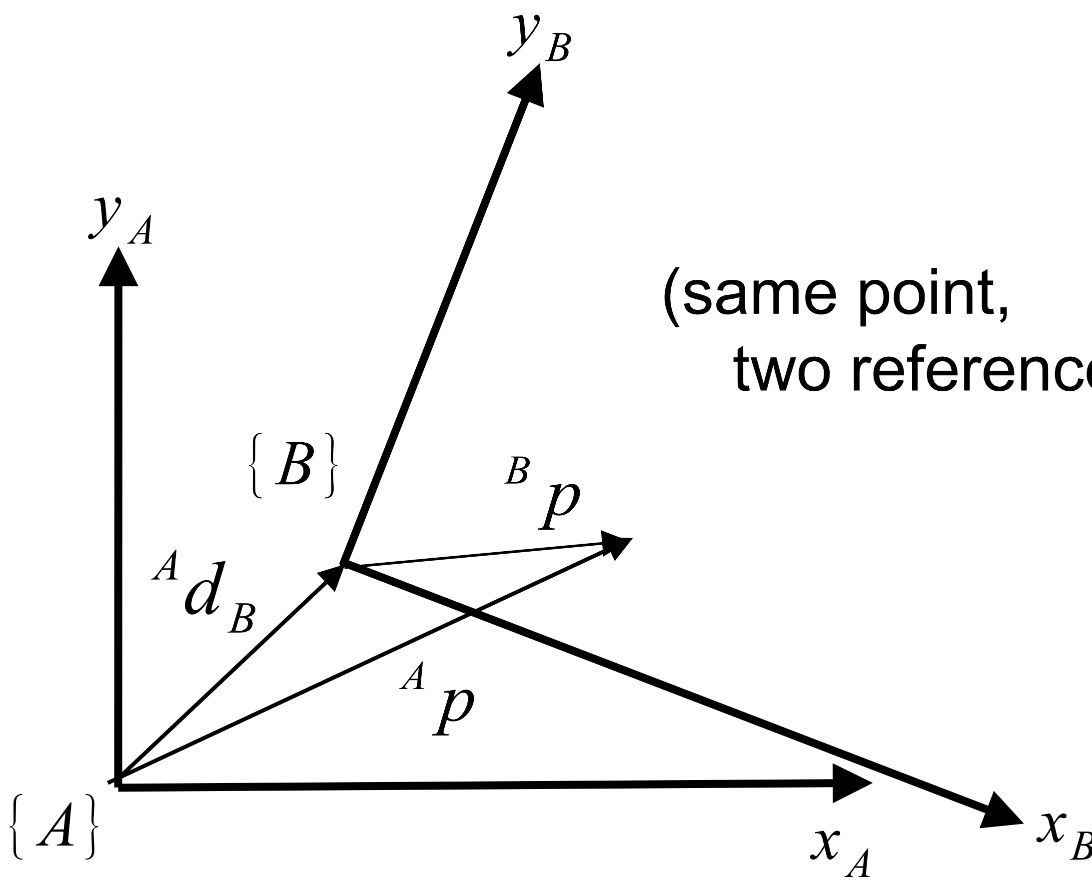


Fig. 2.6.
Two 2D coordinate frames $\{A\}$ and $\{B\}$ and a world point **P**. $\{B\}$ is rotated and translated with respect to $\{A\}$

Homogeneous transformation



(same point,
two reference frames)

$$ {}^{A}p = {}^{A}R_B \, {}^{B}p + {}^{A}d_B $$

Homogeneous transformation

$$\begin{pmatrix} {}^{A}x \\ {}^{A}y \end{pmatrix} = \begin{pmatrix} {}^{V}x \\ {}^{V}y \end{pmatrix} + \begin{pmatrix} x \\ y \end{pmatrix}$$

$$= \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} {}^{B}x \\ {}^{B}y \end{pmatrix} + \begin{pmatrix} x \\ y \end{pmatrix}$$

$$= \begin{pmatrix} \cos\theta & -\sin\theta & x \\ \sin\theta & \cos\theta & y \end{pmatrix} \begin{pmatrix} {}^{B}x \\ {}^{B}y \\ 1 \end{pmatrix}$$

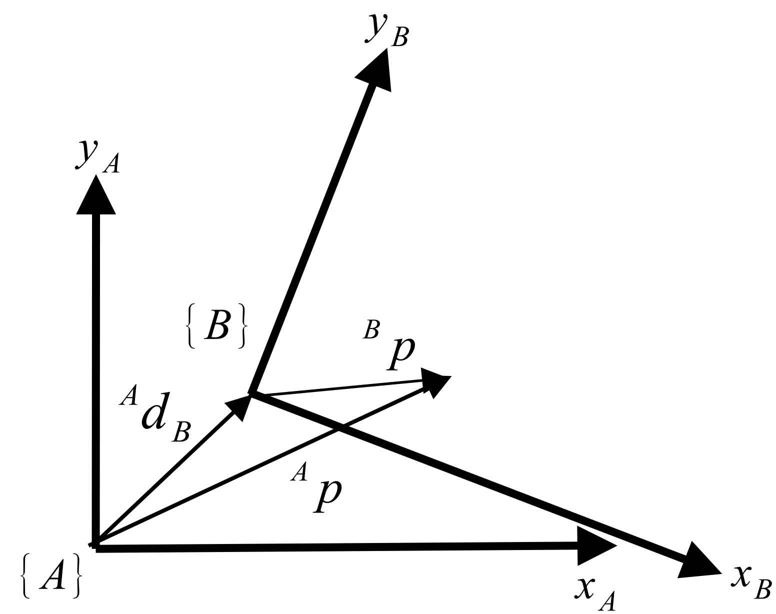
Homogeneous transformation

$$^{A}p = {}^{A}R_{B}\,{}^{B}p + {}^{A}d_{B}$$

$$= \begin{pmatrix} ^{A}R_{B} & ^{A}d_{B} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} ^{B}p \\ 1 \end{pmatrix}$$

always zeros

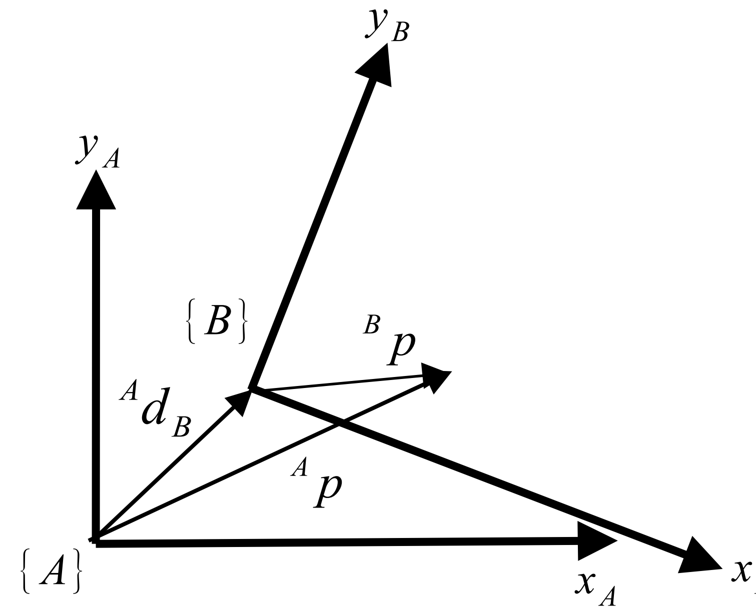
always one



Homogeneous transformation

$$^{A}p = {}^{A}R_{B}\, {}^{B}p + {}^{A}d_{B}$$

$$= \begin{pmatrix} {}^{A}R_{B} & {}^{A}d_{B} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} {}^{B}p \\ 1 \end{pmatrix}$$



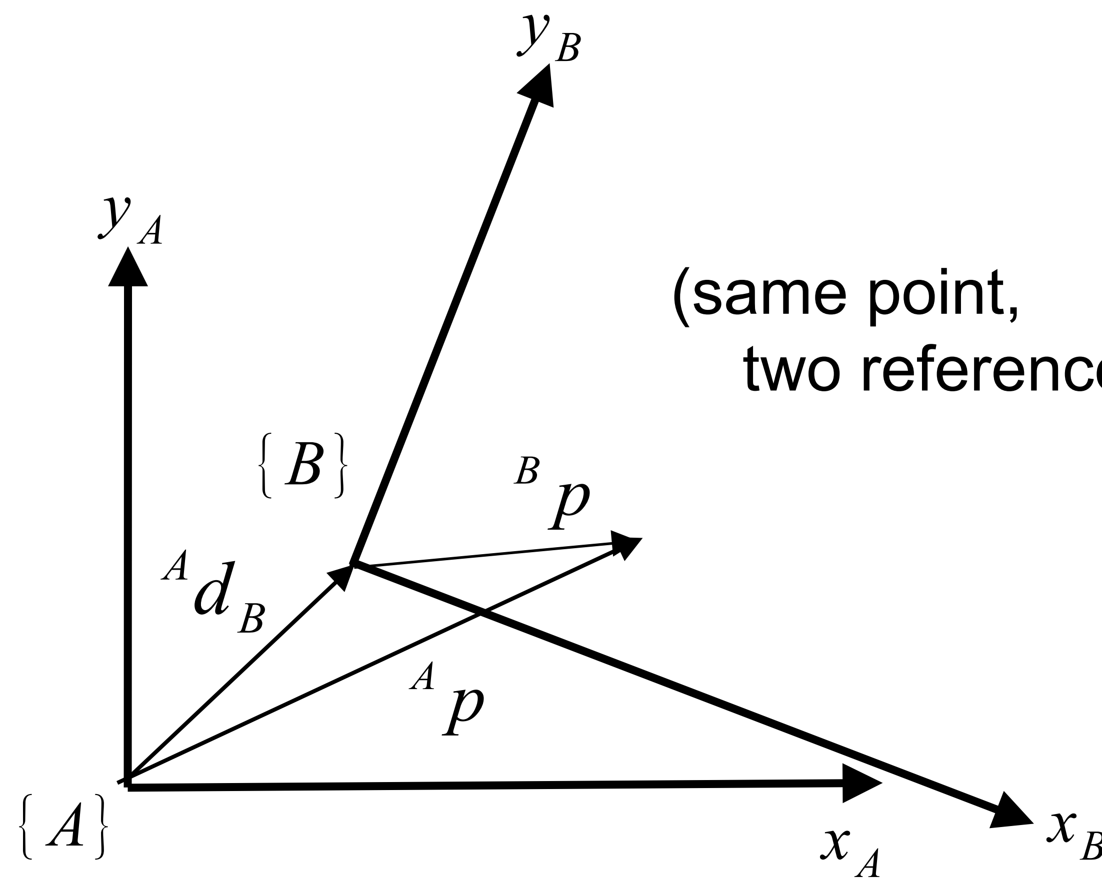
always zeros

always one

$$\begin{pmatrix} {}^{A}x \\ {}^{A}y \\ 1 \end{pmatrix} = \begin{pmatrix} {}^{A}\boldsymbol{R}_{B} & \boldsymbol{t} \\ \boldsymbol{0}_{1\times 2} & 1 \end{pmatrix} \begin{pmatrix} {}^{B}x \\ {}^{B}y \\ 1 \end{pmatrix}$$

$$\begin{aligned} {}^{A}\tilde{\boldsymbol{p}} &= \begin{pmatrix} {}^{A}\boldsymbol{R}_{B} & \boldsymbol{t} \\ \boldsymbol{0}_{1\times 2} & 1 \end{pmatrix} {}^{B}\tilde{\boldsymbol{p}} \\ &= {}^{A}\boldsymbol{T}_{B}\, {}^{B}\tilde{\boldsymbol{p}} \end{aligned}$$

Homogeneous transformation: Inverse



$$ {}^{B}p = {}^{B}R_A\, {}^{A}p - {}^{B}R_A\, {}^{A}d_B = {}^{B}R_A \left({}^{A}p - {}^{A}d_B \right) $$

Homogeneous transformation: Inverse

Can also derive it from the forward Homogeneous transform:

$$^{B}p = {}^{B}R_{A}\,{}^{A}p + {}^{B}d_{A}$$

$$^{A}p = {}^{B}R_{A}^{T}\left({}^{B}p - {}^{B}d_{A}\right) = \left({}^{B}T_{A}\right)^{-1}\begin{pmatrix} {}^{B}p \\ 1 \end{pmatrix}$$

where $\left({}^{B}T_{A}\right)^{-1} = \begin{pmatrix} {}^{B}R_{A}^{T} & -{}^{B}R_{A}^{T}\,{}^{B}d_{A} \\ 0 & 1 \end{pmatrix}$

Summary: 2-D transformation

$$T = \begin{pmatrix} \cos\theta & -\sin\theta & x \\ \sin\theta & \cos\theta & y \\ 0 & 0 & 1 \end{pmatrix}$$

A concrete representation of relative pose ξ is $\xi \sim \boldsymbol{T} \in \mathbf{SE}(2)$ and $\boldsymbol{T}_1 \oplus \boldsymbol{T}_2 \mapsto \boldsymbol{T}_1 \boldsymbol{T}_2$ which is standard matrix multiplication

$$\boldsymbol{T}_1 \boldsymbol{T}_2 = \begin{pmatrix} \boldsymbol{R}_1 & \boldsymbol{t}_1 \\ \boldsymbol{0}_{1\times 2} & 1 \end{pmatrix} \begin{pmatrix} \boldsymbol{R}_2 & \boldsymbol{t}_2 \\ \boldsymbol{0}_{1\times 2} & 1 \end{pmatrix} = \begin{pmatrix} \boldsymbol{R}_1 \boldsymbol{R}_2 & \boldsymbol{t}_1 + \boldsymbol{R}_1 \boldsymbol{t}_2 \\ \boldsymbol{0}_{1\times 2} & 1 \end{pmatrix}$$

One of the algebraic rules from page 21 is $\xi \oplus 0 = \xi$. For matrices we know that $\boldsymbol{T}\boldsymbol{I} = \boldsymbol{T}$, where $\boldsymbol{I}$ is the identify matrix, so for pose $0 \mapsto \boldsymbol{I}$ the identity matrix. Another rule was that $\xi \ominus \xi = 0$. We know for matrices that $\boldsymbol{T}\boldsymbol{T}^{-1} = \boldsymbol{I}$ which implies that $\ominus \boldsymbol{T} \mapsto \boldsymbol{T}^{-1}$

$$\boldsymbol{T}^{-1} = \begin{pmatrix} \boldsymbol{R} & \boldsymbol{t} \\ \boldsymbol{0}_{1\times 2} & 1 \end{pmatrix}^{-1} = \begin{pmatrix} \boldsymbol{R}^T & -\boldsymbol{R}^T \boldsymbol{t} \\ \boldsymbol{0}_{1\times 2} & 1 \end{pmatrix}$$

For a point described by $\tilde{\boldsymbol{p}} \in \mathbb{P}^2$ then $\boldsymbol{T} \bullet \tilde{\boldsymbol{p}} \mapsto \boldsymbol{T}\tilde{\boldsymbol{p}}$ which is a standard matrix-vector product.

Feedback

Piazza thread: 1/24 Lec 02 Feedback

Please post your answers to the following anonymously.

1. What did you like so far?

2. What was unclear?

3. Any additional feedback / comments?