CS 4610/5335 – Lecture 11
Extended Kalman filtering

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2/28/22

Material adapted from:
1. Robert Platt, CS 4610/5335
2. Sebastian Thrun, Wolfram Burgard, & Dieter Fox, Probabilistic Robotics

Announcements

Ex3 out (due 3/11)

Ex1 grading almost done

Keep working on project:
Continue adding details to your threads
- We will review them this week
  and possibly leave comments / questions

- Craft a two-month plan
  (lay out a weekly/biweekly plan; 10 weeks left)
- Starting work in simulation is highly recommended
  (plan for hardware to arrive mid-/late March)

Outline

Today:

Kalman filter (n-D)

Extended Kalman filter (EKF)

Next time:

Localization with a known map (landmarks)

Mapping (landmarks) with known location

Simultaneous Localization and Mapping (EKF-SLAM)

Recap: Kalman Filter (1-D)

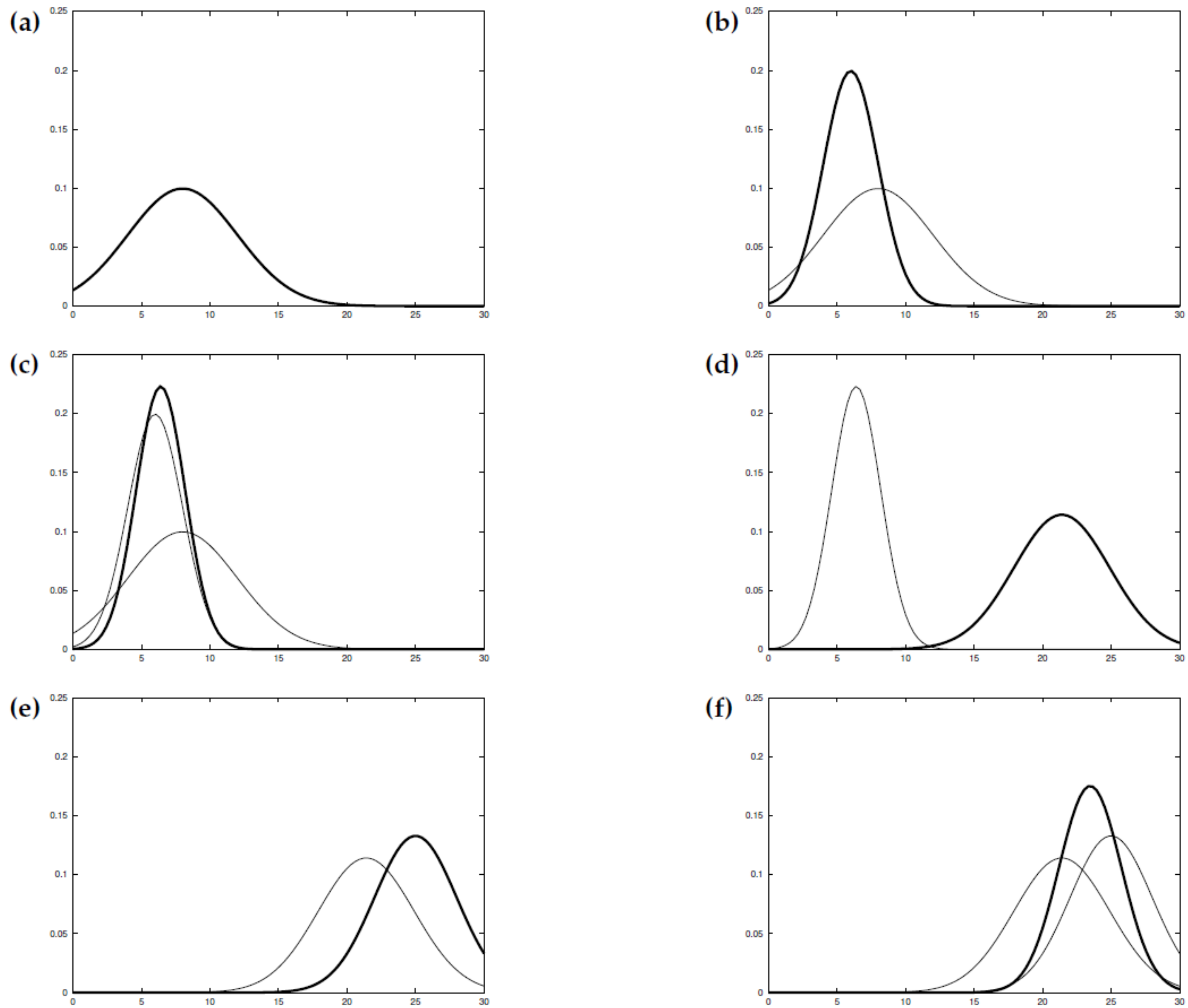
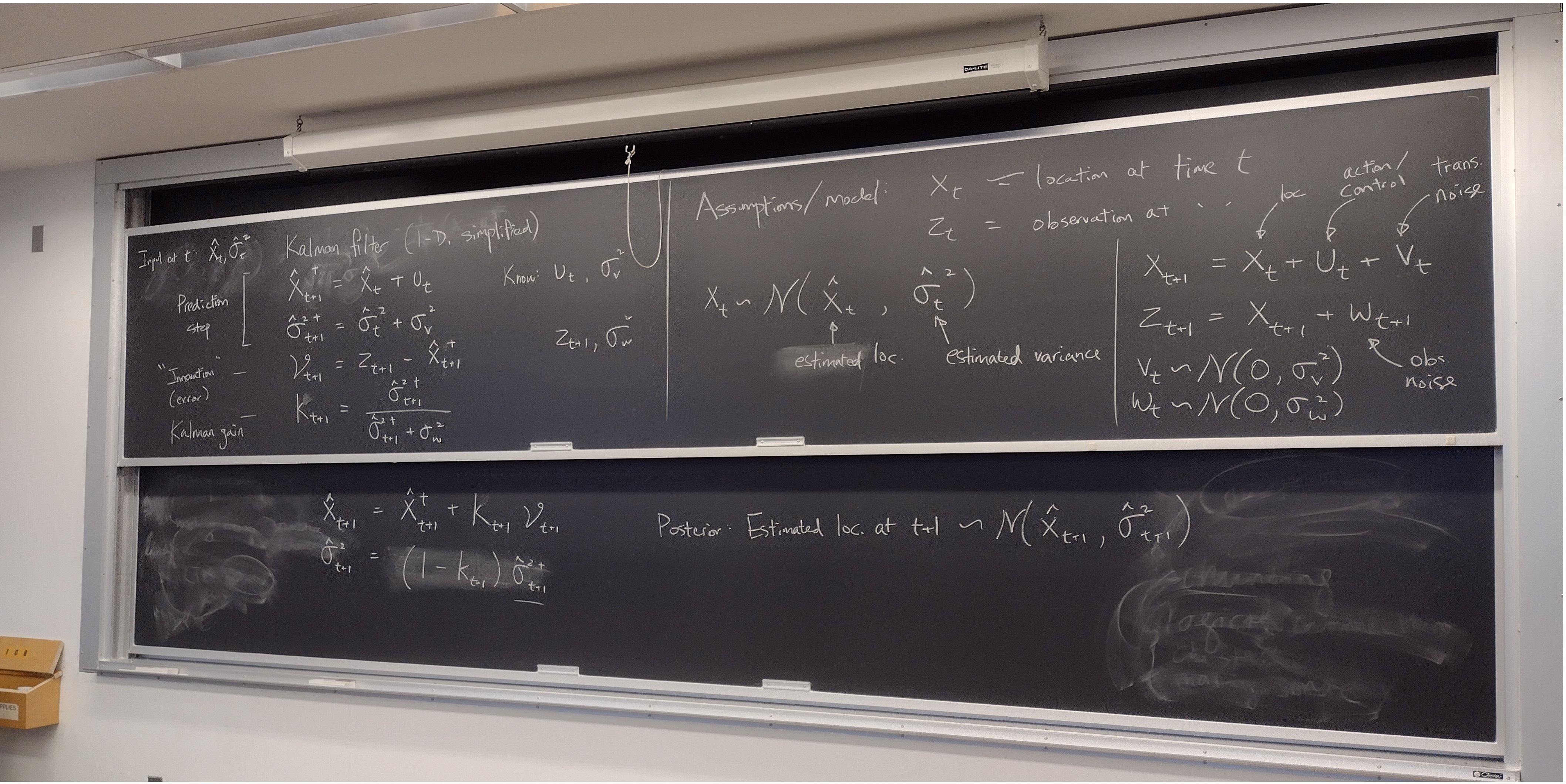
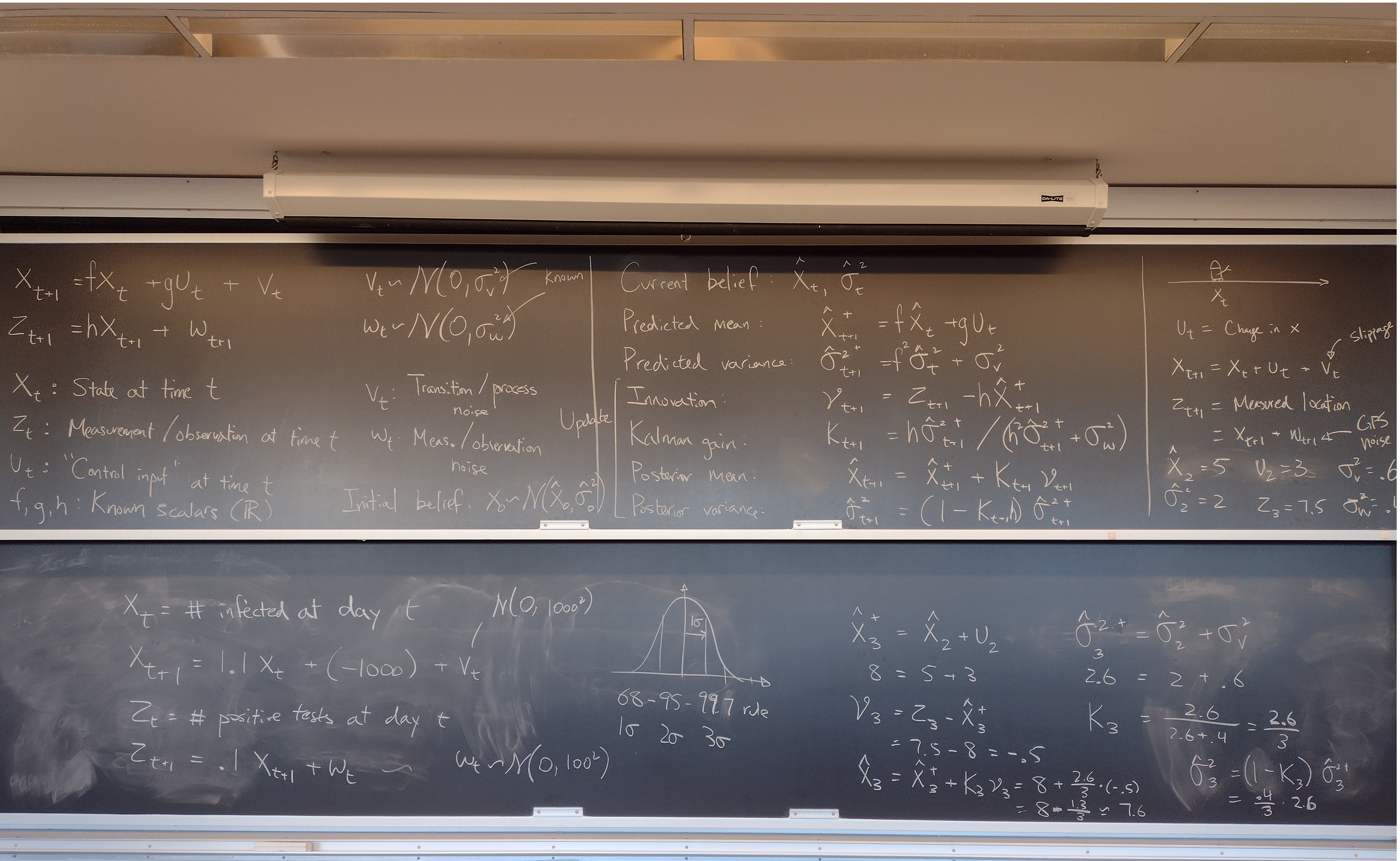


Figure 3.2 Illustration of Kalman filters: (a) initial belief, (b) a measurement (in bold) with the associated uncertainty, (c) belief after integrating the measurement into the belief using the Kalman filter algorithm, (d) belief after motion to the right (which introduces uncertainty), (e) a new measurement with associated uncertainty, and (f) the resulting belief.

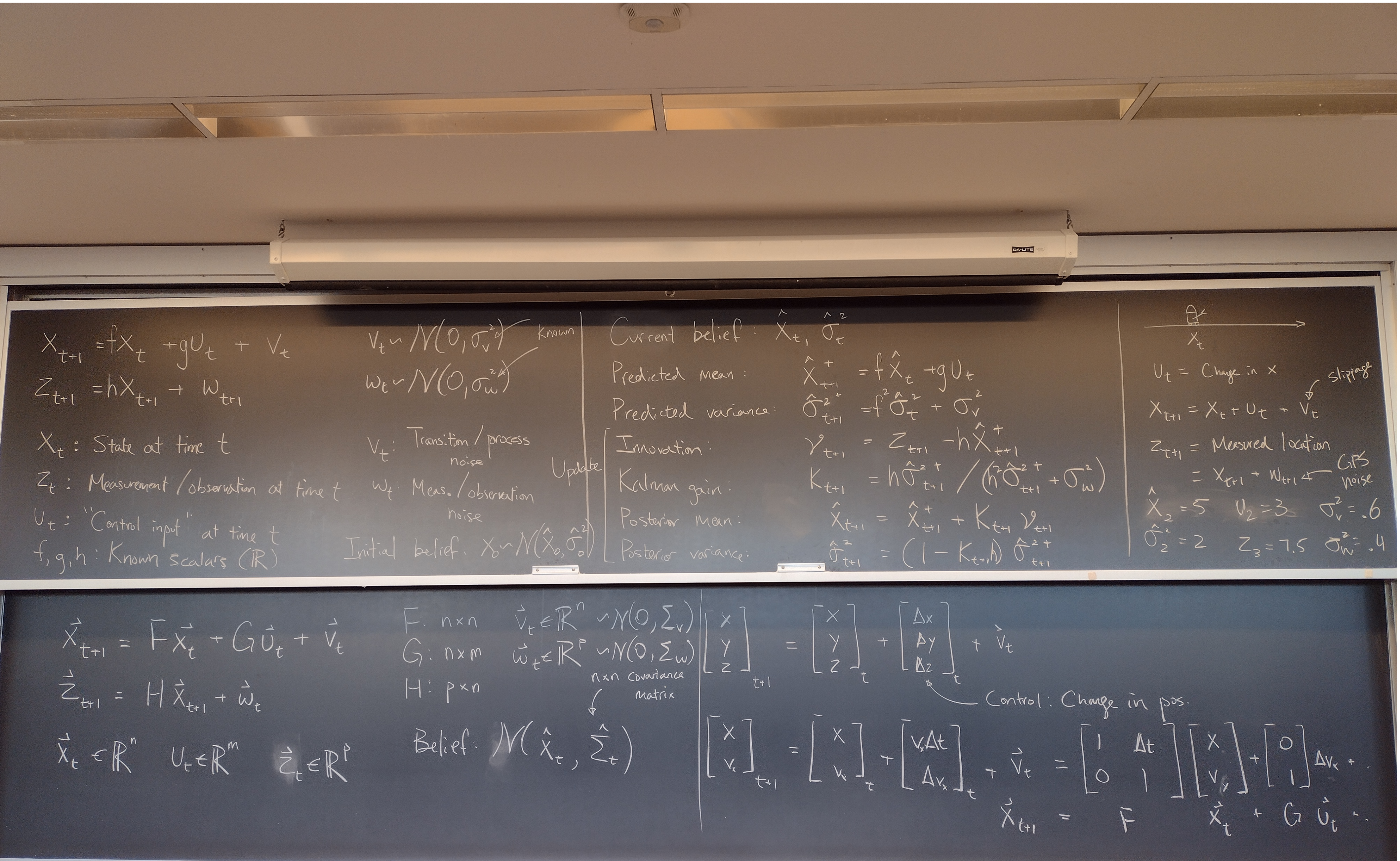
On the board: 1-D Kalman Filter (simplified)



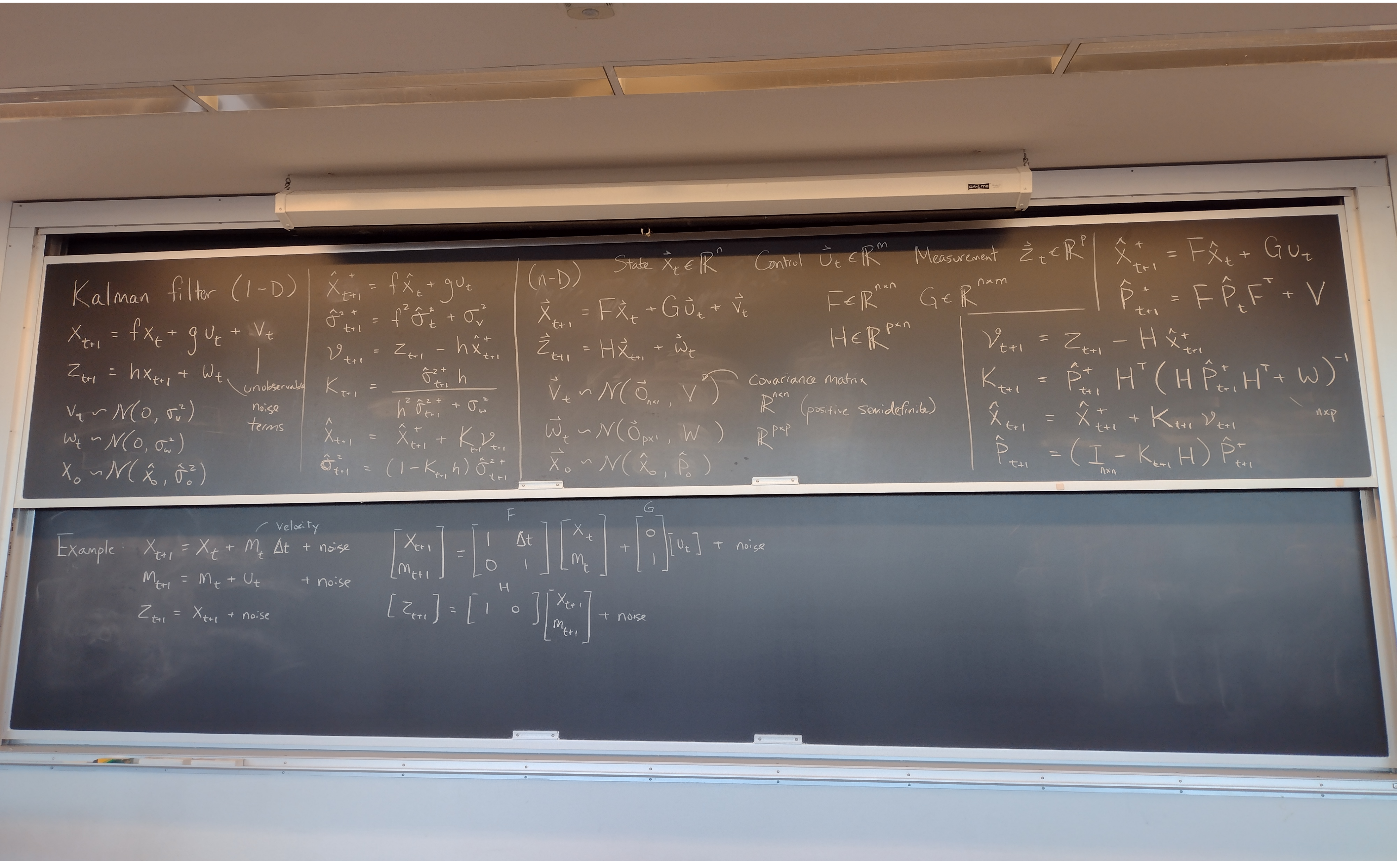
On the board: 1-D Kalman Filter (general)



On the board (bottom): n-D Kalman Filter



On the board: n-D Kalman Filter



Outline

✔ Kalman filter (n-D)

Extended Kalman filter (EKF)

Linearity in Kalman filters

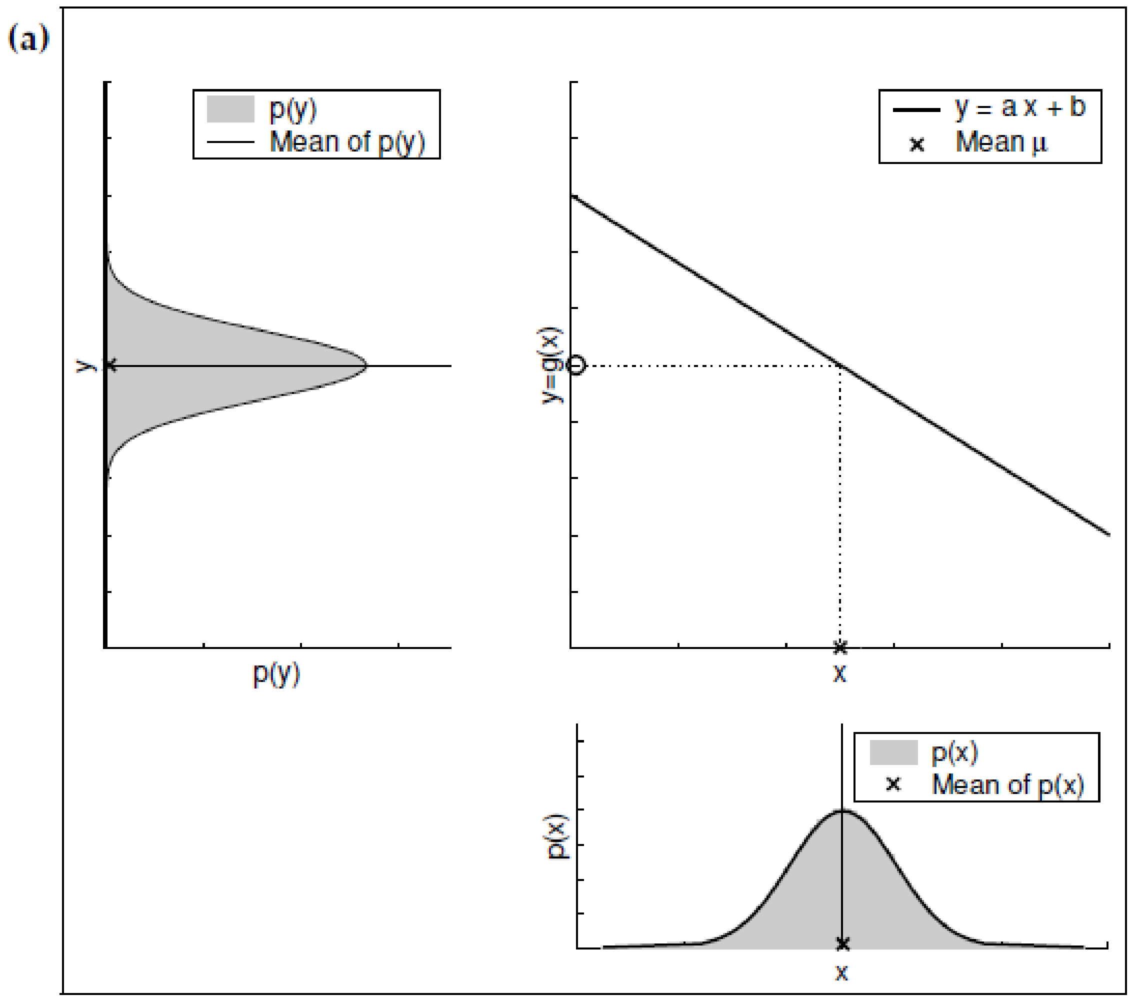


Figure 3.3 (a) Linear transformation of a Gaussian random variable. The lower right plots show the density of the original random variable, X. This random variable is passed through the function displayed in the upper right graphs (the transformation of the mean is indicated by the dotted line). The density of the resulting random variable Y is plotted in the upper left graphs.

Many systems are nonlinear!

$$
\begin{aligned}
x_{t+1} &= f(x_t, u_t) \\
&\neq Ax_t + Bu_t
\end{aligned}
$$

Nonlinear systems

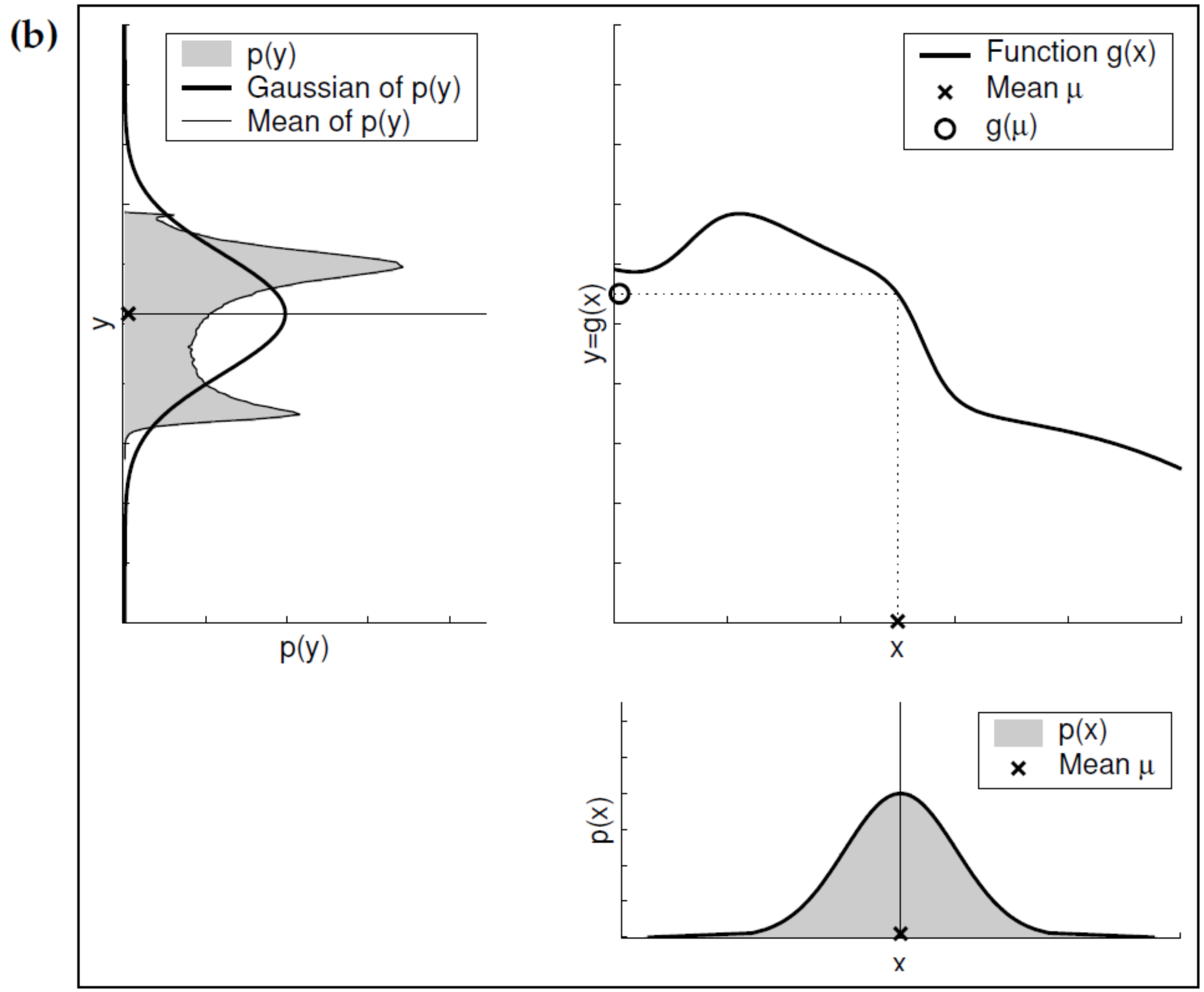


Figure 3.3 (a) Linear and (b) nonlinear transformation of a Gaussian random variable. The lower right plots show the density of the original random variable, X. This random variable is passed through the function displayed in the upper right graphs (the transformation of the mean is indicated by the dotted line). The density of the resulting random variable Y is plotted in the upper left graphs.

Extended Kalman filter

$$
\begin{aligned}
x_{t+1} &= f(x_t, u_t) \\
&\neq Ax_t + Bu_t
\end{aligned}
$$

Linearize around current mean:

$$
\begin{aligned}
x_{t+1} &= f(x_t, u_t) \\
&\approx f(\mu_t, u_t) + A_t(x_t - \mu_t)
\end{aligned}
$$

$$A_t = \frac{\partial f}{\partial x}(\mu_t, u_t)$$

$$
\begin{aligned}
z_{t+1} &= h(x_t) \\
&\approx h(\mu_t) + C_t(x_t - \mu_t)
\end{aligned}
$$

$$C_t = \frac{\partial h}{\partial x}(\mu_t)$$

Extended Kalman filter: Idea

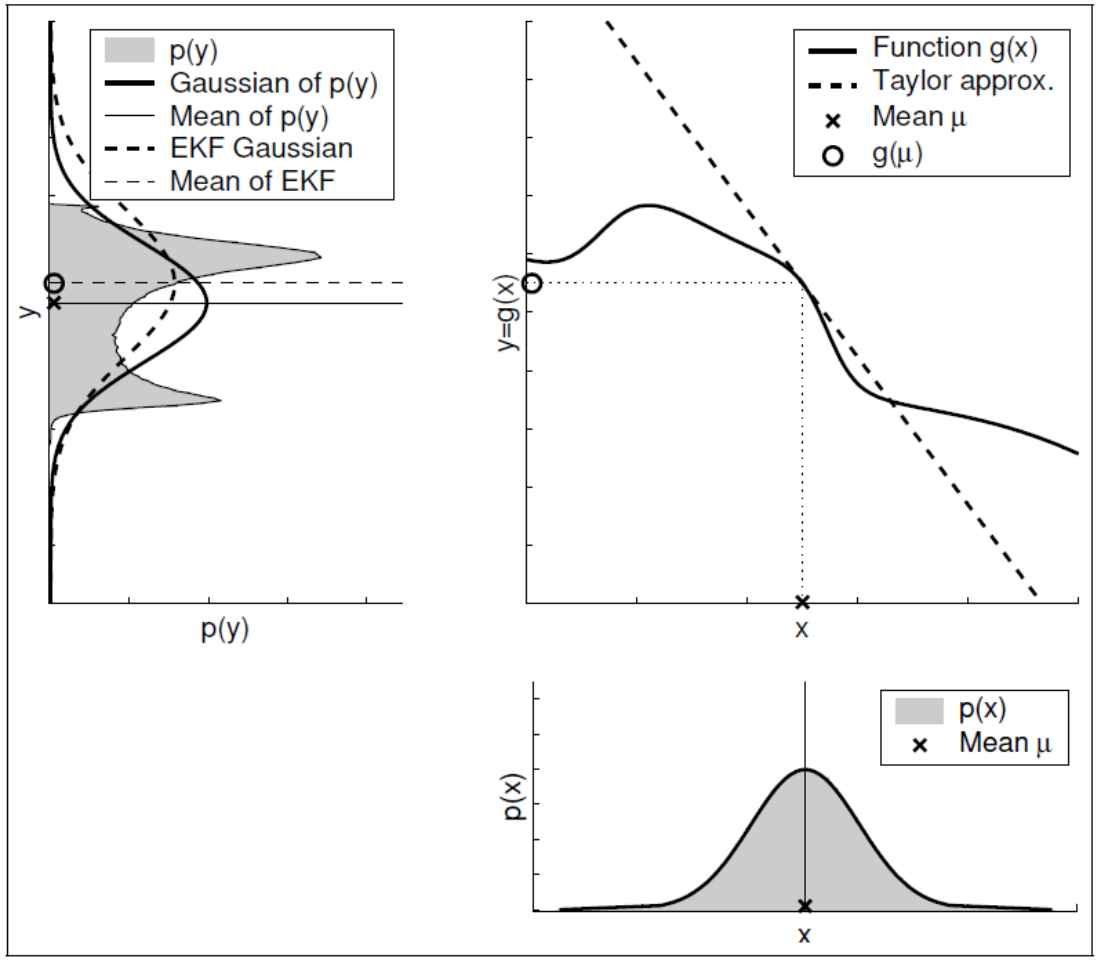


Figure 3.4 Illustration of linearization applied by the EKF. Instead of passing the Gaussian through the nonlinear function g, it is passed through a linear approximation of g. The linear function is tangent to g at the mean of the original Gaussian. The resulting Gaussian is shown as the dashed line in the upper left graph. The linearization incurs an approximation error, as indicated by the mismatch between the linearized Gaussian (dashed) and the Gaussian computed from the highly accurate Monte-Carlo estimate (solid).

Extended Kalman filter: Idea

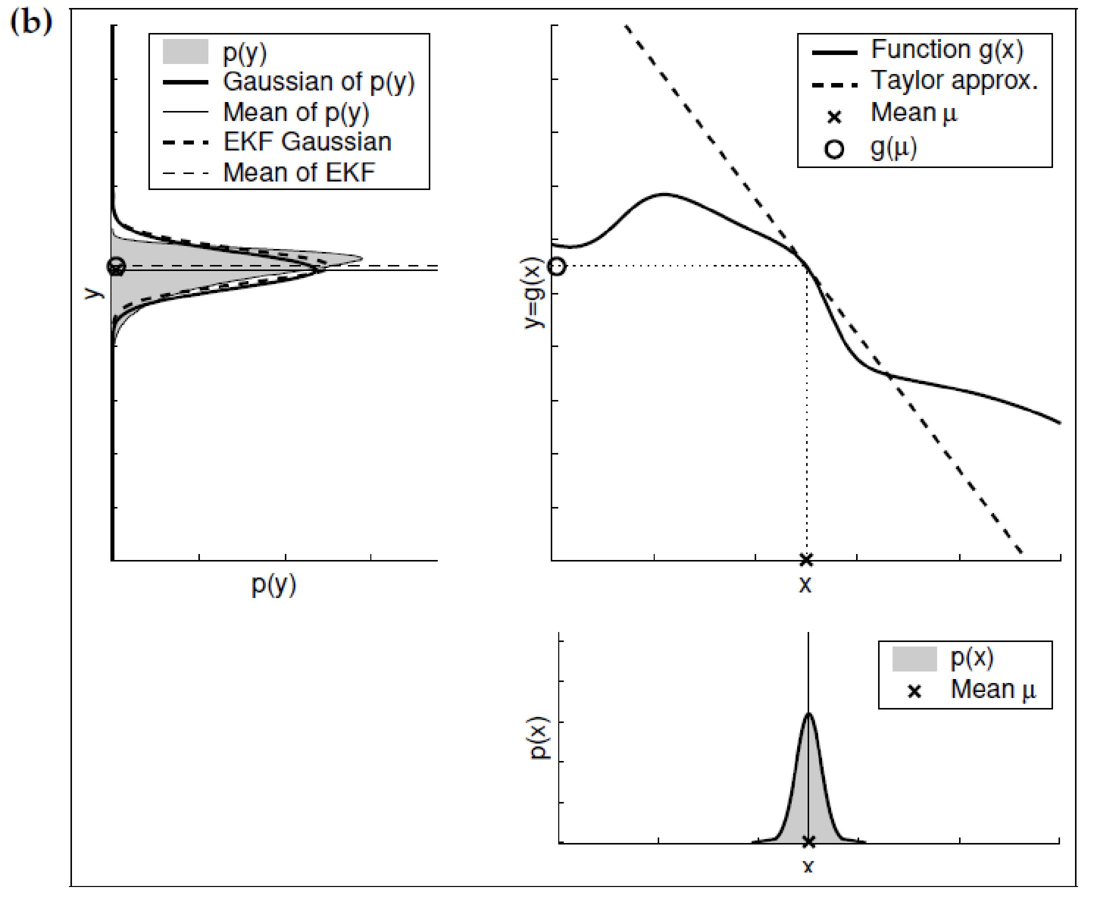


Figure 3.5 Dependency of approximation quality on uncertainty. Both Gaussians (lower right) have the same mean and are passed through the same nonlinear function (upper right). The higher uncertainty of the left Gaussian produces a more distorted density of the resulting random variable (gray area in upper left graph). The solid lines in the upper left graphs show the Gaussians extracted from these densities. The dashed lines represent the Gaussians generated by the linearization applied by the EKF.

Extended Kalman filter

$$\begin{aligned} x_{t+1} &= f(x_t, u_t) \\ &\neq Ax_t + Bu_t \end{aligned}$$

Linearize around current mean:

$$\begin{aligned} x_{t+1} &= f(x_t, u_t) \\ &\approx f(\mu_t, u_t) + A_t(x_t - \mu_t) \end{aligned}$$

$$A_t = \frac{\partial f}{\partial x}(\mu_t, u_t)$$

$$\begin{aligned} z_{t+1} &= h(x_t) \\ &\approx h(\mu_t) + C_t(x_t - \mu_t) \end{aligned}$$

$$C_t = \frac{\partial h}{\partial x}(\mu_t)$$

Then continue filtering assuming linearity

Extended Kalman filter

$$\begin{aligned} x_{t+1} &= f(x_t, u_t) \\ &\neq Ax_t + Bu_t \end{aligned}$$

Linearize around current mean:

$$\begin{aligned} x_{t+1} &= f(x_t, u_t) \\ &\approx f(\mu_t, u_t) + A_t(x_t - \mu_t) \end{aligned}$$

$$A_t = \frac{\partial f}{\partial x}(\mu_t, u_t)$$

$$\begin{aligned} z_{t+1} &= h(x_t) \\ &\approx h(\mu_t) + C_t(x_t - \mu_t) \end{aligned}$$

$$C_t = \frac{\partial h}{\partial x}(\mu_t)$$

Then continue filtering assuming linearity
In *n*-D case, A_t and C_t above are <u>Jacobian</u> matrices

Extended Kalman filter

H.2 Nonlinear Systems – Extended Kalman Filter

For the case where the system is not linear it can be described generally by two functions: the state transition (the motion model in robotics) and the sensor model

$$\boldsymbol{x}\langle k{+}1\rangle = \boldsymbol{f}\big(\boldsymbol{x}\langle k\rangle, \boldsymbol{u}\langle k\rangle, \boldsymbol{v}\langle k\rangle\big) \tag{H.9}$$

$$\boldsymbol{z}\langle k\rangle = \boldsymbol{h}\big(\boldsymbol{x}\langle k\rangle, \boldsymbol{w}\langle k\rangle\big) \tag{H.10}$$

and as before we represent model uncertainty, external disturbances and sensor noise by Gaussian random variables $\boldsymbol{v}$ and $\boldsymbol{w}$.

We linearize the state transition function about the current state estimate $\hat{\boldsymbol{x}}_k$ as shown in Fig. H.2 resulting in

$$\boldsymbol{x}'\langle k{+}1\rangle \approx \boldsymbol{F}_x \boldsymbol{x}'\langle k\rangle + \boldsymbol{F}_u \boldsymbol{u}\langle k\rangle + \boldsymbol{F}_v \boldsymbol{v}\langle k\rangle \tag{H.11}$$

$$\boldsymbol{z}'\langle k\rangle \approx \boldsymbol{H}_x \boldsymbol{x}'\langle k\rangle + \boldsymbol{H}_w \boldsymbol{w}\langle k\rangle \tag{H.12}$$

where $\boldsymbol{F}_x = \partial \boldsymbol{f} / \partial \boldsymbol{x} \in \mathbb{R}^{n\times n}$, $\boldsymbol{F}_u = \partial \boldsymbol{f} / \partial \boldsymbol{u} \in \mathbb{R}^{n\times m}$, $\boldsymbol{F}_v = \partial \boldsymbol{f} / \partial \boldsymbol{v} \in \mathbb{R}^{n\times n}$, $\boldsymbol{H}_x = \partial \boldsymbol{h} / \partial \boldsymbol{x} \in \mathbb{R}^{p\times n}$ and $\boldsymbol{H}_w = \partial \boldsymbol{h} / \partial \boldsymbol{w} \in \mathbb{R}^{p\times p}$ are Jacobians of the functions $\boldsymbol{f}(\cdot)$ and $\boldsymbol{h}(\cdot)$.

Extended Kalman filter

$$\hat{\boldsymbol{x}}^{+}\langle k+1\rangle = \boldsymbol{f}(\hat{\boldsymbol{x}}\langle k\rangle, \hat{\boldsymbol{u}}\langle k\rangle)$$ predict state one step ahead

$$\hat{P}^{+}\langle k+1\rangle = F_x \hat{P}\langle k\rangle F_x^T + F_v V F_v^T$$ project covariance one step ahead

prediction phase

$$\boldsymbol{\nu} = \boldsymbol{z}^{\#}\langle k+1\rangle - h(\hat{\boldsymbol{x}}^{+}\langle k+1\rangle, \boldsymbol{p}_i)$$ new information – innovation

$$K = P^{+}\langle k+1\rangle H_x^T \left(H_x P^{+}\langle k+1\rangle H_x^T + H_w W H_w^T\right)^{-1}$$ how to distribute the innovation to states – Kalman gain

$$\hat{\boldsymbol{x}}\langle k+1\rangle = \hat{\boldsymbol{x}}^{+}\langle k+1\rangle + K\boldsymbol{\nu}$$ state updated with innovation

$$\hat{P}\langle k+1\rangle = \hat{P}^{+}\langle k+1\rangle - K H_x \hat{P}^{+}\langle k+1\rangle$$ updated covariance

update phase

Fig. 6.6.
Summary of extended Kalman filter algorithm showing the prediction and update phases

F_x , F_v , H_x , H_w are Jacobian matrices

Extended Kalman filter

Procedure EKF

Input : $\hat{\boldsymbol{x}}\langle k\rangle \in \mathbb{R}^n$, $\hat{\boldsymbol{P}}\langle k\rangle \in \mathbb{R}^{n\times n}$, $\boldsymbol{u}\langle k\rangle \in \mathbb{R}^m$, $\boldsymbol{z}\langle k+1\rangle \in \mathbb{R}^p$, $\hat{\boldsymbol{V}} \in \mathbb{R}^{n\times n}$, $\hat{\boldsymbol{W}} \in \mathbb{R}^{p\times p}$

Output: $\hat{\boldsymbol{x}}\langle k+1\rangle \in \mathbb{R}^n$, $\hat{\boldsymbol{P}}\langle k+1\rangle \in \mathbb{R}^{n\times n}$

– *linearize about* $\boldsymbol{x} = \hat{\boldsymbol{x}}\langle k\rangle$

compute Jacobians: $\boldsymbol{F}_x \in \mathbb{R}^{n\times n}$, $\boldsymbol{F}_v \in \mathbb{R}^{n\times n}$, $\boldsymbol{H}_x \in \mathbb{R}^{p\times n}$, $\boldsymbol{H}_w \in \mathbb{R}^{p\times p}$

– *the prediction step*

$$\hat{\boldsymbol{x}}^+\langle k+1\rangle = \boldsymbol{f}\left(\hat{\boldsymbol{x}}\langle k\rangle, \boldsymbol{u}\langle k\rangle\right) \quad \textit{// predict state at next time step}$$

$$\hat{\boldsymbol{P}}^+\langle k+1\rangle = \boldsymbol{F}_x\hat{\boldsymbol{P}}\langle k\rangle\boldsymbol{F}_x^T + \boldsymbol{F}_v\hat{\boldsymbol{V}}\boldsymbol{F}_v^T \quad \textit{// predict covariance at next time step}$$

– *the update step*

$$\nu = \boldsymbol{z}\langle k+1\rangle - \boldsymbol{h}\left(\hat{\boldsymbol{x}}^+\langle k+1\rangle\right) \quad \textit{// innovation : measured – predicted sensor value}$$

$$\boldsymbol{K} = \boldsymbol{P}^+\langle k+1\rangle\boldsymbol{H}_x^T\left[\boldsymbol{H}_x\boldsymbol{P}^+\langle k+1\rangle\boldsymbol{H}_x^T + \boldsymbol{H}_w\hat{\boldsymbol{W}}\boldsymbol{H}_w^T\right]^{-1} \quad \textit{// Kalman gain}$$

$$\hat{\boldsymbol{x}}\langle k+1\rangle = \hat{\boldsymbol{x}}^+\langle k+1\rangle + \boldsymbol{K}\nu \quad \textit{// update state estimate}$$

$$\hat{\boldsymbol{P}}\langle k+1\rangle = \hat{\boldsymbol{P}}^+\langle k+1\rangle - \boldsymbol{K}\boldsymbol{H}_x\hat{\boldsymbol{P}}^+\langle k+1\rangle \quad \textit{// update covariance estimate}$$

Algorithm H.1.
Procedure EKF

Feedback

Piazza thread: 2/28 Lec 11 Feedback

Please post your answers to the following anonymously.

1. What did you like so far?

2. What was unclear?

3. How many hours did you spend on Ex2?

4. What was your favorite question on Ex2?

5. What was your least favorite question on Ex2?

6. Any additional feedback / comments?