Certifiably Correct SLAM

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NEU EECE 5550 Mobile Robotics
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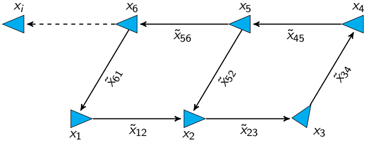
Recap: Pose-graph SLAM

Given:

- Unknown poses $x_1, \ldots, x_n \in \mathrm{SE}(d)$
- Noisy relative measurements $\tilde{x}_{ij} = (\tilde{t}_{ij}, \tilde{R}_{ij}) \approx x_i^{-1} x_j$



Find: An estimate $\hat{x} = (\hat{x}_1, \ldots, \hat{x}_n)$ for the hidden states:

$$\hat{x} \in \operatorname*{argmin}_{\substack{t_i \in \mathbb{R}^d \\ R_i \in \mathrm{SO}(d)}} \sum_{(i,j) \in \vec{\mathcal{E}}} \kappa_{ij} \| R_j - R_i \tilde{R}_{ij} \|_F^2 + \tau_{ij} \| t_j - t_i - R_i \tilde{t}_{ij} \|_2^2$$

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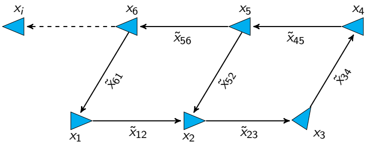

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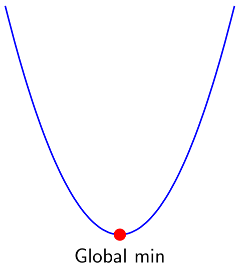
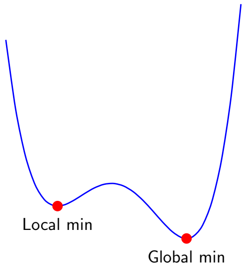
The Big Problem: This is a *high-dimensional*, *nonconvex* problem

Recap: Pose-graph SLAM

Convex



Nonconvex



- Local min $\Rightarrow$ *global* min
- *Easy* to optimize

- Can have *many* local minima
- *Hard* to optimize (generally)

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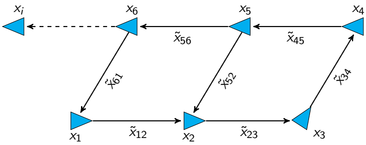

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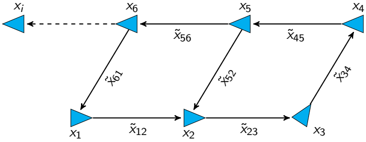

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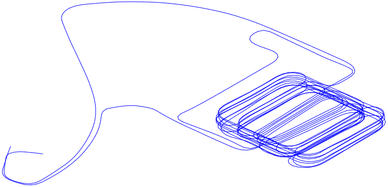
The Big Problem: This is a *high-dimensional*, *nonconvex* problem
$\Rightarrow$ **LOTS** of *local minima*
$\Rightarrow$ **No guarantees** on estimates recovered via *local search*!

The problem of nonconvexity

The problem of nonconvexity



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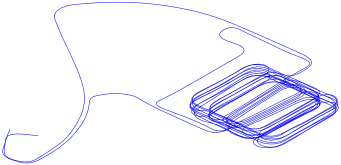
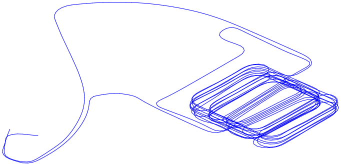


Optimal estimate

The problem of nonconvexity



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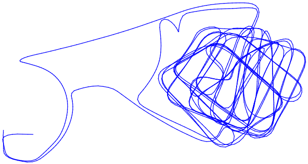
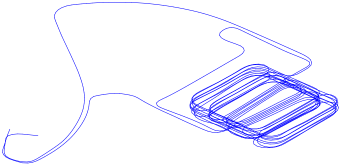


Suboptimal critical point

The problem of nonconvexity



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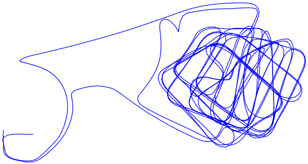

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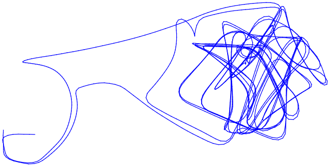

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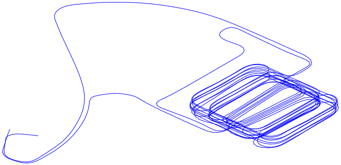

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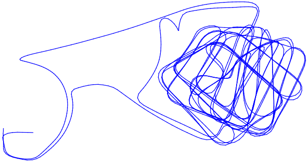

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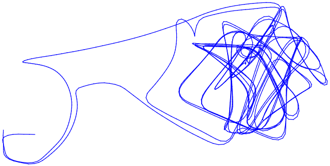

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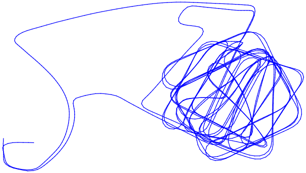

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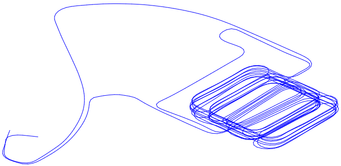
The problem of nonconvexity

Main Question: Is it possible to *ensure* good SLAM performance?



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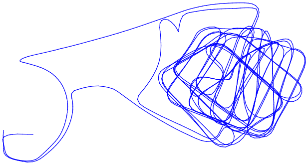

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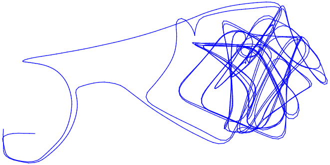

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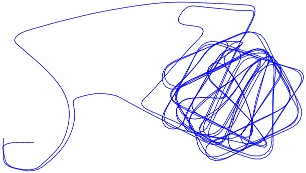

Suboptimal critical point

The Main Idea: Convex relaxation for the win!

We can't efficiently solve (NP-hard) pose-graph SLAM in general

[1] Bandeira 2016; Chandrasekaran et al. 2012; Chandrasekaran et al. 2013.

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We can't efficiently solve (NP-hard) pose-graph SLAM *in general*

But: Maybe we can *well-approximate* "*reasonable*" instances?

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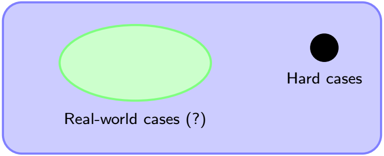
Intuition: Nature is not "out to get you"



All pose-graph SLAM instances

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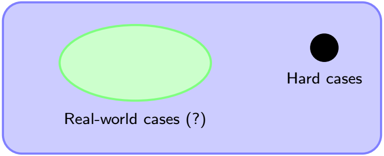

All pose-graph SLAM instances

Question: How to *generate* these approximations?

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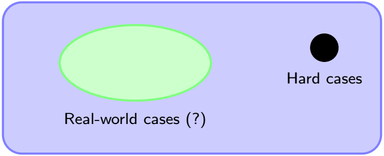

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One general approach: *Convex relaxation*
$\Rightarrow$ Solve a *convex approximation* of original problem

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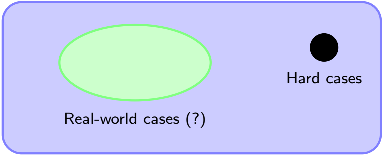

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- Standard machinery for constructing these
- Admit (some) formal guarantees on solution quality.[1]

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In this talk

1. **Lagrangian duality**: lower bounds for optimization problems via *convex relaxation*
2. **Solution verification**: *certificates of optimality* for pose-graph SLAM
3. **SE-Sync**: Fast *global* pose-graph optimization

A primer on Lagrangian duality

Main idea: Compute *lower bounds* for the *optimal value* of an optimization problem:

$$p^* = \min_{x \in \mathbb{R}^n} f(x) \quad \text{s.t.} \quad \left\{ \begin{array}{ll} g_i(x) = 0, & i = 1, \ldots, l \\ h_j(x) \leq 0, & j = 1, \ldots, m \end{array} \right\}$$

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Define the *Lagrangian*:

$$\mathcal{L} \colon \mathbb{R}^n \times \mathbb{R}^l \times \mathbb{R}^m_+ \to \mathbb{R}$$

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Key observation: For any $\nu \geq 0$, λ, and feasible x:

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The dual function

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Key fact: The dual function is *always concave*.

The Lagrangian dual problem

For all $\nu \geq 0$, λ, and feasible x, we have the *lower bound*:

$$d(\lambda, \nu) \leq p^* \leq f(x).$$

How to get the *tightest* possible bound?

The Lagrangian dual problem

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How to get the *tightest* possible bound? Maximize dual function!

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$$d^* = \max_{\nu \geq 0,\, \lambda} d(\lambda, \nu)$$

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$\Rightarrow$ Lagrangian dual problem is *always convex*

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Recall: The dual function is *always concave*
$\Rightarrow$ Lagrangian dual problem is *always convex*
$\Rightarrow$ We can *always* compute the *optimal lower-bound d^**

Strong duality

Primal problem

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Lagrangian duality: $d^* \leq p^*$

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Dual problem

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Lagrangian duality: $d^* \leq p^*$

Strong duality: $d^* = p^*$

Observe: If strong duality holds and x^* and (λ^*, ν^*) are primal and dual solutions, then:

$$p^* = f(x^*) = \inf_{x \in \mathbb{R}^n} \mathcal{L}(x, \lambda^*, \nu^*) = d(\lambda^*, \nu^*) = d^*.$$

Strong duality

Primal problem

$$p^* = \min_{x \in \mathbb{R}^n} f(x) \quad \text{s.t.} \quad \left\{ \begin{array}{l} g(x) = 0 \\ h(x) \leq 0 \end{array} \right\}$$

Dual problem

$$d^* = \max_{\nu \geq 0,\, \lambda} d(\lambda, \nu)$$

Lagrangian duality: $d^* \leq p^*$

Strong duality: $d^* = p^*$

Observe: If strong duality holds and x^* and (λ^*, ν^*) are primal and dual solutions, then:

$$p^* = f(x^*) = \inf_{x \in \mathbb{R}^n} \mathcal{L}(x, \lambda^*, \nu^*) = d(\lambda^*, \nu^*) = d^*.$$

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$\Rightarrow x^*$ is an *unconstrained minimizer* of $\mathcal{L}(x, \lambda^*, \nu^*)$.

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$$p^* = f(x^*) = \inf_{x \in \mathbb{R}^n} \mathcal{L}(x, \lambda^*, \nu^*) = d(\lambda^*, \nu^*) = d^*.$$

$\Rightarrow$ x^* is an *unconstrained minimizer* of $\mathcal{L}(x, \lambda^*, \nu^*)$.

$\Rightarrow$ Can recover *primal* solution x^* from (convex) dual solution (λ^*, ν^*) if minimizing $\mathcal{L}(x, \lambda^*, \nu^*)$ is easy.

Lagrangian duality: Summary

Primal problem

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Lagrangian: $\mathcal{L}(x, \lambda, \nu) \triangleq f(x) + \lambda^{\mathsf{T}} g(x) + \nu^{\mathsf{T}} h(x).$

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$\Rightarrow$ Dual function provides *lower bounds* on p^*

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Lagrangian duality: $d(\lambda, \nu) \leq f(x)$ for all $\nu \geq 0$, λ, feasible x
$\Rightarrow$ Dual function provides *lower bounds* on p^*

Dual problem:

- Provides *tightest* achievable lower bound $d^* \leq p^*$
- $d(\lambda, \nu)$ *concave* $\Rightarrow$ dual problem *convex*

In this talk

1. **Lagrangian duality**: lower bounds for optimization problems via *convex relaxation*
2. **Solution verification**: *certificates of optimality* for pose-graph SLAM
3. **SE-Sync**: Fast *global* pose-graph optimization

Assessing the quality of a pose-graph SLAM solution

Main idea: Use Lagrangian duality to *bound the suboptimality* of a candidate pose-graph SLAM solution.

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Let $x \in \mathrm{SE}(d)^n$ be a candidate solution of pose-graph MLE:

$$p^* = \min_{\substack{t_i \in \mathbb{R}^d \\ R_i \in \mathrm{SO}(d)}} \sum_{(i,j) \in \vec{\mathcal{E}}} \kappa_{ij} \| R_j - R_i \tilde{R}_{ij} \|_F^2 + \tau_{ij} \left\| t_j - t_i - R_i \tilde{t}_{ij} \right\|_2^2 .$$

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Lagrangian duality: $d^* \leq p^* \leq f(x)$. This implies:

$$f(x) - p^* \leq f(x) - d^*.$$

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$$f(x) - p^* \leq f(x) - d^*.$$

- Left-hand side is *suboptimality* of x.
- Right-hand side can be computed from (convex) *dual problem*.

$\Rightarrow$ We can *bound global suboptimality* of x using d^*!

Deriving the dual problem I: Simplifying the MLE

Pose-graph MLE

$$p^*_{\mathrm{MLE}} = \min_{\substack{t_i \in \mathbb{R}^d \\ R_i \in \mathrm{SO}(d)}} \sum_{(i,j) \in \vec{\mathcal{E}}} \kappa_{ij} \| R_j - R_i \tilde{R}_{ij} \|_F^2 + \tau_{ij} \left\| t_j - t_i - R_i \tilde{t}_{ij} \right\|_2^2$$

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Solving for $t \triangleq (t_1, \ldots, t_n)$ in terms of $R \triangleq (R_1, \ldots, R_n)$ using a generalized Schur complement...

Deriving the dual problem I: Simplifying the MLE

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[Lots of algebra...]

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Solving for $t \triangleq (t_1, \ldots, t_n)$ in terms of $R \triangleq (R_1, \ldots, R_n)$ using a generalized Schur complement:

Simplified pose-graph MLE

$$p^*_{\text{MLE}} = \min_{R \in \text{SO}(d)^n} \operatorname{tr}(\tilde{Q} R^\mathsf{T} R)$$

Deriving the dual problem II: Orthogonal relaxation

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Deriving the dual problem II: Orthogonal relaxation

Simplified pose-graph MLE

$$p^*_{\text{MLE}} = \min_{R \in \text{SO}(d)^n} \operatorname{tr}(\tilde{Q} R^\mathsf{T} R)$$

$$\text{SO}(d) = \left\{ R \in \mathbb{R}^{d \times d} \mid R^\mathsf{T} R = I_d,\ \det(R) = +1 \right\}$$

Deriving the dual problem II: Orthogonal relaxation

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Relaxing the determinantal constraint...

Deriving the dual problem II: Orthogonal relaxation

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Relaxing the determinantal constraint:

Orthogonally-relaxed MLE

$$p_{\mathrm{O}^*=\min_{R \in \mathrm{O}(d)^n}} \operatorname{tr}(\tilde{Q} R^\mathsf{T} R)$$

NB: $\mathrm{SO}(d)^n \subset \mathrm{O}(d)^n \Rightarrow p_{\mathrm{O}^* \le p^*_{\mathrm{MLE}}}$.

Deriving the dual problem III: Constructing the dual

Relaxed MLE (intrinsic)

$$p_{O^*} = \min_{R \in O(d)^n} \operatorname{tr}(\tilde{Q} R^\mathsf{T} R)$$

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$$p_{\mathrm{O}^*} = \min_{R \in \mathrm{O}(d)^n} \operatorname{tr}(\tilde{Q} R^\mathsf{T} R)$$

$\Longrightarrow$

Relaxed MLE (extrinsic)

$$p_{\mathrm{O}^*} = \min_{R \in \mathbb{R}^{d \times dn}} \operatorname{tr}(\tilde{Q} R^\mathsf{T} R)$$
$$\text{s.t.} \quad R_i^\mathsf{T} R_i = I_d \quad i = 1, \dots, n$$

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$$\text{s.t.} \quad R_i^\mathsf{T} R_i = I_d \quad i = 1, \ldots, n$$

Dual function:

$$d(\Lambda_1, \ldots, \Lambda_n) \triangleq \inf_{R \in \mathbb{R}^{d \times dn}} \operatorname{tr}(\tilde{Q} R^\mathsf{T} R) + \sum_{i=1}^{n} \operatorname{tr}\left(\Lambda_i (I_d - R_i^\mathsf{T} R_i)\right)$$

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Matricized form: For $\Lambda = \operatorname{Diag}(\Lambda_1, \ldots, \Lambda_n)$:

$$d(\Lambda) = \inf_{R \in \mathbb{R}^{d \times dn}} \operatorname{tr}\left((\tilde{Q} - \Lambda) R^\mathsf{T} R\right) + \operatorname{tr}(\Lambda)$$

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$\implies$

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Dual problem:

$$d^* = \max_{\Lambda \in \mathrm{SBD}(d,n)} \operatorname{tr}(\Lambda) \quad \text{s.t.} \quad \tilde{Q} - \Lambda \succeq 0$$

Deriving the dual problem IV: Putting it all together

Pose-graph MLE

$$p^*_{\text{MLE}} = \min_{X \in \text{SE}(d)^n} \operatorname{tr}(\tilde{M} X^\mathsf{T} X)$$

$$\Downarrow$$

Orthogonally-relaxed MLE

$$p_{O^*=\min_{R \in O(d)^n} \operatorname{tr}(\tilde{Q} R^\mathsf{T} R)}$$

$$\Downarrow$$

Lagrangian dual

$$d^* = \max_{\Lambda \in \text{SBD}(d,n)} \operatorname{tr}(\Lambda)$$
$$\text{s.t.} \quad \tilde{Q} - \Lambda \succeq 0$$

Lower bounds: $d^* \leq p_{O^* \leq p^*_{\text{MLE}}}$

Suboptimality upper bound:

$$f(x) - p^*_{\text{MLE}} \leq f(x) - d^* \quad \forall x \in \text{SE}(d)^n$$

Deriving the dual problem IV: Putting it all together

Pose-graph MLE

$$p^*_{\text{MLE}} = \min_{X \in \text{SE}(d)^n} \operatorname{tr}(\tilde{M} X^\mathsf{T} X)$$

$\Downarrow$

Orthogonally-relaxed MLE

$$p_{\text{O}^*} = \min_{R \in \text{O}(d)^n} \operatorname{tr}(\tilde{Q} R^\mathsf{T} R)$$

$\Downarrow$

Lagrangian dual

$$d^* = \max_{\Lambda \in \text{SBD}(d,n)} \operatorname{tr}(\Lambda) \quad \text{s.t.} \quad \tilde{Q} - \Lambda \succeq 0$$

Lower bounds: $d^* \leq p_{\text{O}^*} \leq p^*_{\text{MLE}}$

Suboptimality upper bound:

$$f(x) - p^*_{\text{MLE}} \leq f(x) - d^* \quad \forall x \in \text{SE}(d)^n$$

Key Question: How *tight* are these?

Deriving the dual problem IV: Putting it all together

Pose-graph MLE

$$p^*_{\text{MLE}} = \min_{X \in \text{SE}(d)^n} \operatorname{tr}(\tilde{M} X^{\mathsf{T}} X)$$

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Lower bounds: $d^* \leq p_{\text{O}^* \leq p^*_{\text{MLE}}}$

Suboptimality upper bound:

$$f(x) - p^*_{\text{MLE}} \leq f(x) - d^* \quad \forall x \in \text{SE}(d)^n$$

Theorem (Rosen et al. 2016)

Let $\underline{Q}$ be the data matrix constructed from noiseless measurements. Then $\exists \beta \triangleq \beta(\underline{Q}) > 0$ s.t. if $\|\underline{Q} - \tilde{Q}\| < \beta$, then $d^* = p_{\text{O}^* = p^*_{\text{MLE}}}$.

Deriving the dual problem IV: Putting it all together

Pose-graph MLE

$$p^*_{\text{MLE}} = \min_{X \in \text{SE}(d)^n} \operatorname{tr}(\tilde{M} X^\mathsf{T} X)$$

$\Downarrow$

Orthogonally-relaxed MLE

$$p_{\text{O}^*=\min_{R \in \text{O}(d)^n} \operatorname{tr}(\tilde{Q} R^\mathsf{T} R)}$$

$\Downarrow$

Lagrangian dual

$$d^* = \max_{\Lambda \in \text{SBD}(d,n)} \operatorname{tr}(\Lambda) \quad \text{s.t.} \quad \tilde{Q} - \Lambda \succeq 0$$

Lower bounds: $d^* \leq p_{\text{O}^* \leq p^*_{\text{MLE}}}$

Suboptimality upper bound:

$$f(x) - p^*_{\text{MLE}} \leq f(x) - d^* \quad \forall x \in \text{SE}(d)^n$$

Theorem (Rosen et al. 2016)

Let $\underline{Q}$ be the data matrix constructed from noiseless measurements. Then $\exists \beta \triangleq \beta(\underline{Q}) > 0$ s.t. if $\|\underline{Q} - \tilde{Q}\| < \beta$, then $d^* = p_{\text{O}^*=p^*_{\text{MLE}}}$.

- *Guarantees strong duality* for sufficiently small noise
- $f(x) = d^*$ *certifies* x's optimality

Pose-graph solution verification (version 1.0)[2]

Input: Candidate pose-graph SLAM solution $x = (t, R) \in \mathrm{SE}(d)^n$
Output: Upper-bound on x's suboptimality

- Solve dual problem:

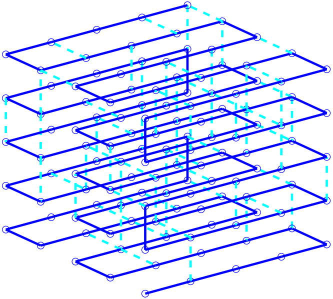
$$d^* = \max_{\Lambda \in \mathrm{SBD}(d,n)} \operatorname{tr}(\Lambda)$$
$$\text{s.t.} \quad \tilde{Q} - \Lambda \succeq 0.$$

- **Return** $f(x) - d^*$

[2] L. Carlone et al. "Lagrangian Duality in 3D SLAM: Verification Techniques and Optimal Solutions". In: *IEEE/RSJ Intl. Conf. on Intelligent Robots and Systems (IROS)*. Hamburg, Germany, Sept. 2015.

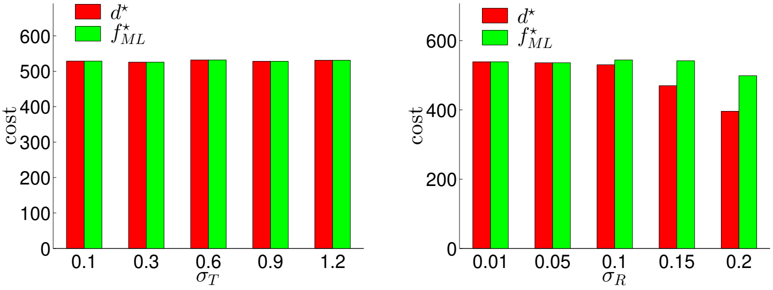
Experiments: When does strong duality hold?

We simulate a small 3D grid map, varying measurement noise...



Experiments: When does strong duality hold?

Tightness of the relaxation exhibits a *phase transition* as a function of the rotational measurement noise:

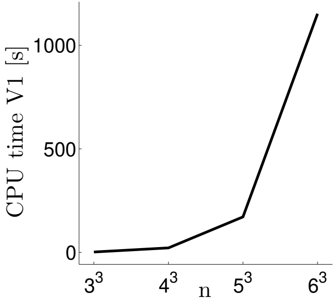


(**NB:** .1 rad $\approx$ 6 degrees)

Technical challenge: Scalability

General-purpose semidefinite programming solvers don't scale well.

$$d^* = \max_{\Lambda \in \mathrm{SBD}(d,n)} \operatorname{tr}(\Lambda) \quad \text{s.t.} \quad \tilde{Q} - \Lambda \succeq 0$$



Typical pose-graphs are 1–2 orders of magnitude larger than this.

Exploiting strong duality

Assume:

1. Strong duality holds
2. $R^* \in \mathrm{SO}(d)^n$ and $\Lambda^* \in \mathrm{SBD}(d, n)$ are primal-dual optimal pair

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Then: R^* is an *unconstrained minimizer* of $\mathcal{L}(R, \Lambda^*)$, where:

$$\mathcal{L}(R, \Lambda) \triangleq \mathrm{tr}(\tilde{Q}R^\mathsf{T}R) + \sum_{i=1}^{n} \mathrm{tr}\left(\Lambda_i(I_d - R_i^\mathsf{T}R_i)\right).$$

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Therefore:

$$\nabla_R \mathcal{L}(R^*, \Lambda^*) = 0.$$

Since Lagrangian is *affine* in Λ, this is *linear* equation in Λ^*.

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Since Lagrangian is *affine* in Λ, this is *linear* equation in Λ^*.

Therefore: *If* $R \in \mathrm{SO}(d)^n$ is optimal, *and* strong duality holds, we can *construct* the corresponding dual certificate Λ from R:

$$\Lambda(R) = \frac{1}{2} \operatorname{BlockDiag}_d \left(\tilde{Q} R^\mathsf{T} R + R^\mathsf{T} R \tilde{Q}\right).$$

Pose-graph solution verification (version 2.0)[3]

Input: Candidate pose-graph SLAM solution $x = (t, R) \in \mathrm{SE}(d)^n$
Output: Upper-bound on x's suboptimality

- Construct candidate dual certificate:

$$\Lambda(R) = \frac{1}{2}\,\mathrm{BlockDiag}_d\left(\tilde{Q}R^\mathsf{T}R + R^\mathsf{T}R\tilde{Q}\right).$$

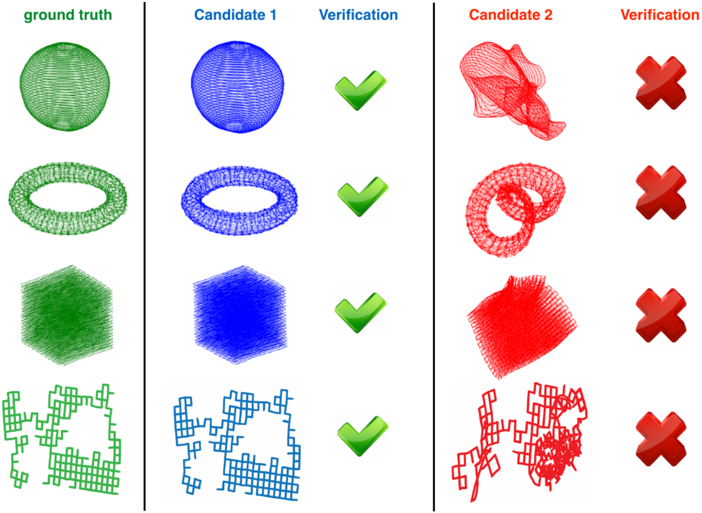
- Compute minimum eigenvalue $\lambda_{\min}$ of $\tilde{Q} - \Lambda(R)$.

- If $\lambda_{\min} \geq 0$ ($\Leftrightarrow \Lambda(R)$ is dual feasible):
  Return $f(x) - \mathrm{tr}(\Lambda(R))$

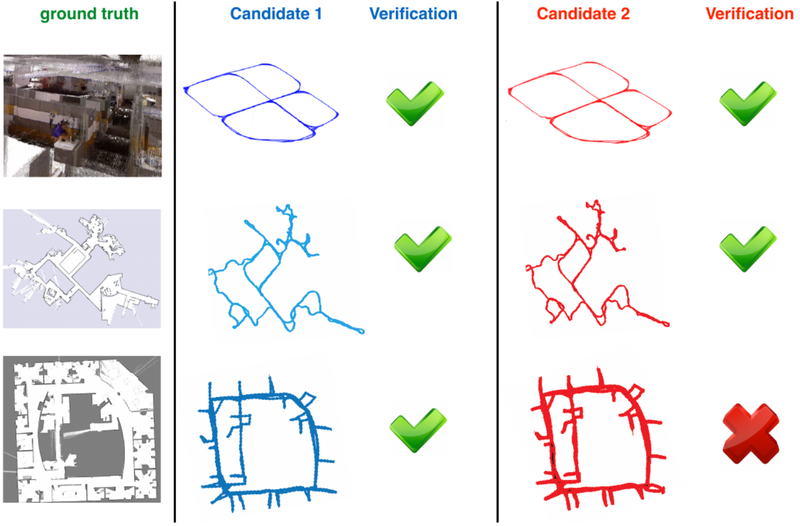
- Else:
  Return $+\infty$

[3] L. Carlone et al. "Lagrangian Duality in 3D SLAM: Verification Techniques and Optimal Solutions". In: *IEEE/RSJ Intl. Conf. on Intelligent Robots and Systems (IROS)*. Hamburg, Germany, Sept. 2015.

Experiments: Simulated data



Experiments: Real data



In this talk

1. **Lagrangian duality**: lower bounds for optimization problems via *convex relaxation*
2. **Solution verification**: *certificates of optimality* for pose-graph SLAM
3. **SE-Sync**: Fast *global* pose-graph optimization

From verification to global optimization

The story so far: We can *certify* the optimality of $\hat{x} \in \mathrm{SE}(d)^n$ whenever *strong duality* holds.

But: How can we *obtain* such an $\hat{x}$?

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Observe: Any $\hat{x}$ that could be *certified* (using strong duality) could also be *directly computed* (from a solution Λ of the dual).

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Key question: Can we do this *efficiently*?

SDP redux: What makes SDPs hard?

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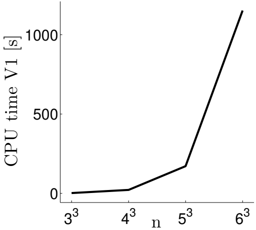
$$\text{s.t.} \quad \tilde{Q} - \Lambda \succeq 0$$



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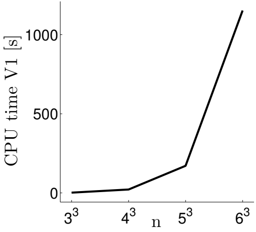

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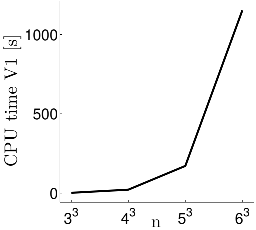

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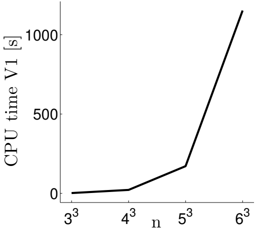

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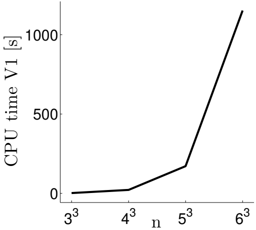

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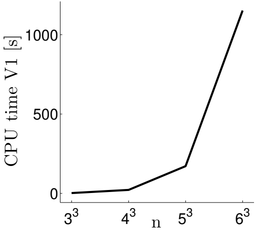

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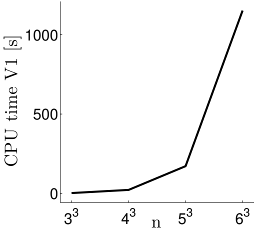

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But: Maybe we can build a *specialized* solver for PGO?

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SDPception

Simplified pose-graph MLE

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Question: How does the dual of the dual relate to the (primal) MLE?

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$\Rightarrow$ Dual of dual is a *convex relaxation* by *expanding MLE's feasible set*

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$\Rightarrow$ We can compute *global minima* of MLE by solving SDP

$\Rightarrow$ Solutions of SDP are *low-rank*

Exploiting low-rank structure

We expect a *low-rank* solution $Z^* = {Y^*}^\mathsf{T} Y^*$ for:

$$p^*_{\text{SDP}} = \min_{Z \in \mathbb{S}_+^{dn}} \operatorname{tr}(\tilde{Q}Z) \quad \text{s.t.} \quad \operatorname{Diag}_d(Z) = (I_d, \ldots, I_d).$$

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Main idea: Replace Z with its *low-rank factorization* $Y^\mathsf{T} Y$:[5]

$$p^*_{\mathrm{SDPLR}} = \min_{Y \in \mathbb{R}^{r \times dn}} \operatorname{tr}(\tilde{Q} Y^\mathsf{T} Y) \quad \text{s.t.} \quad \operatorname{Diag}_d(Y^\mathsf{T} Y) = (I_d, \ldots, I_d)$$

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Payoffs:

- $Y^\mathsf{T} Y \succeq 0$ for *all* Y $\Rightarrow$ PSD constraint is *redundant*
  $\Rightarrow$ Rank-restricted factorization is an *NLP* (vs. SDP)

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- $r \geq \operatorname{rank}(Z^*)$ for some optimal Z^* $\Rightarrow$ $(Y^*)^\mathsf{T} Y^*$ solves SDP.

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Exploiting geometric structure[6]

Rank-restricted SDP, NLP form

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[6] N. Boumal. "A Riemannian Low-Rank Method for Optimization Over Semidefinite Matrices with Block-Diagonal Constraints". arXiv preprint: arXiv:1506.00575v2. 2015

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$$p^*_{\text{SDPLR}} = \min_{Y \in \mathbb{R}^{r \times dn}} \operatorname{tr}(\tilde{Q} Y^\mathsf{T} Y)$$

$$\text{s.t.} \quad \operatorname{BlockDiag}_d(Y^\mathsf{T} Y) = (I_d, \ldots, I_d)$$

The constraints are equivalent to:

$$Y_i^\mathsf{T} Y_i = I_d, \quad Y_i \in \mathbb{R}^{r \times d}.$$

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Ensuring global optimality

Simplified pose-graph MLE

$$p^*_{\mathrm{MLE}} = \min_{R \in \mathrm{SO}(d)^n} \mathrm{tr}(\tilde{Q} R^\mathsf{T} R)$$

$$\Downarrow$$

Semidefinite relaxation

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Question:
What have we actually gained?

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If $Y \in \text{St}(d, r)^n$ is a rank deficient 2nd-order critical point for rank-restricted NLP, then:

- *Y is a global minimizer of Riemannian NLP*
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$\Rightarrow$ We can use (fast) *local* search to find *globally optimal* solutions!

SE-Sync

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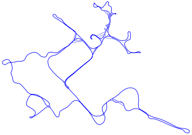
3. Round $Y^* \to \hat{R} \in \text{SO}(d)^n$ using truncated SVD.
4. Return $\{\hat{R}, p^*_{\text{SDP}}\}$.

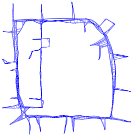
Payoff: If strong duality holds:

- $\hat{R}$ is *globally optimal*
- $\text{tr}(\tilde{Q} \hat{R}^\mathsf{T} \hat{R}) = p^*_{\text{SDP}}$ *certifies* it

Experimental results: Large-scale SLAM benchmarks

	# Poses	PDL-GN (GTSAM)		SE-Sync		
		Objective value	Time [s]	Objective value	Time [s]	Rel. suboptimality
csail	1045	3.170×10^{1}	0.029	3.170×10^{1}	0.010	7.844×10^{-16}
intel	1728	5.235×10^{1}	0.120	5.235×10^{1}	0.071	1.357×10^{-16}
ais2klinik	15115	1.885×10^{2}	12.472	1.885×10^{2}	1.981	2.412×10^{-15}
garage	1661	1.263×10^{0}	0.415	1.263×10^{0}	0.468	1.618×10^{-14}
cubicle	5750	7.171×10^{2}	2.456	7.171×10^{2}	0.754	2.061×10^{-15}
rim	10195	5.461×10^{3}	6.803	5.461×10^{3}	2.256	5.663×10^{-15}


csail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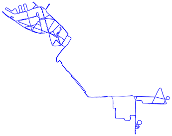
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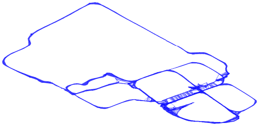
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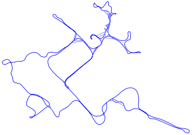
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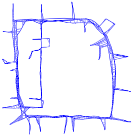
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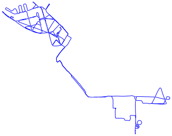
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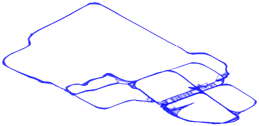
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Experimental results: Large-scale SLAM benchmarks

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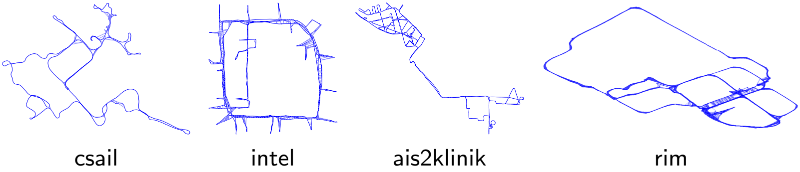

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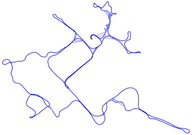

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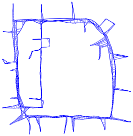

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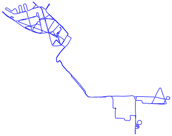

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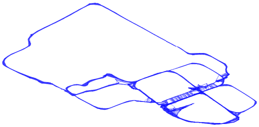
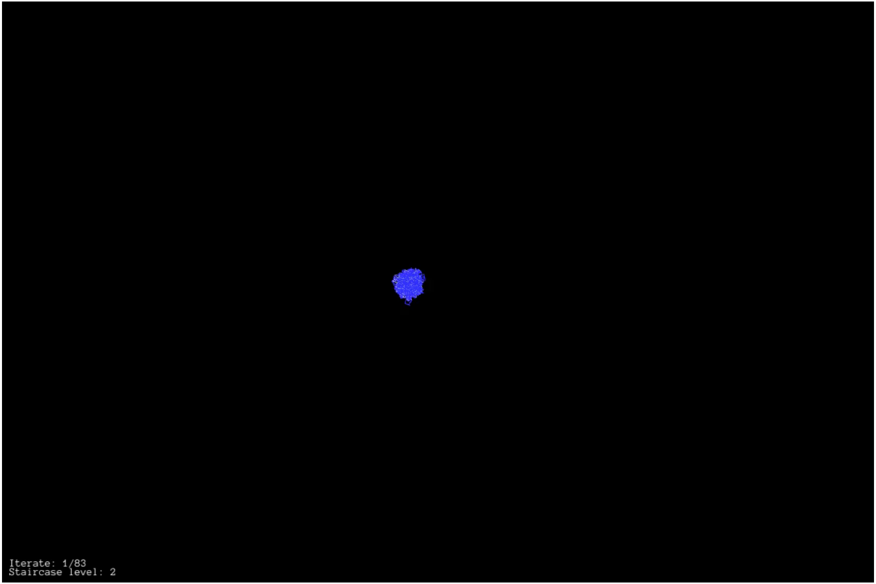


rim

Visualization: Grid world



Certifiably Correct SLAM

Main Takeaway: *Convex relaxation* is a powerful tool!

This talk:

- Lagrangian duality & Lagrangian relaxation
- Pose-graph solution verification
- **SE-Sync:** first *practical*, *provably correct* SLAM algorithm[7]

[7] D.M. Rosen et al. "SE-Sync: A Certifiably Correct Algorithm for Synchronization over the Special Euclidean Group". In: *Intl. J. of Robotics Research* 38.2–3 (Mar. 2019), pp. 95–125.

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New survey on certifiably correct perception (and other stuff!)[8]

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