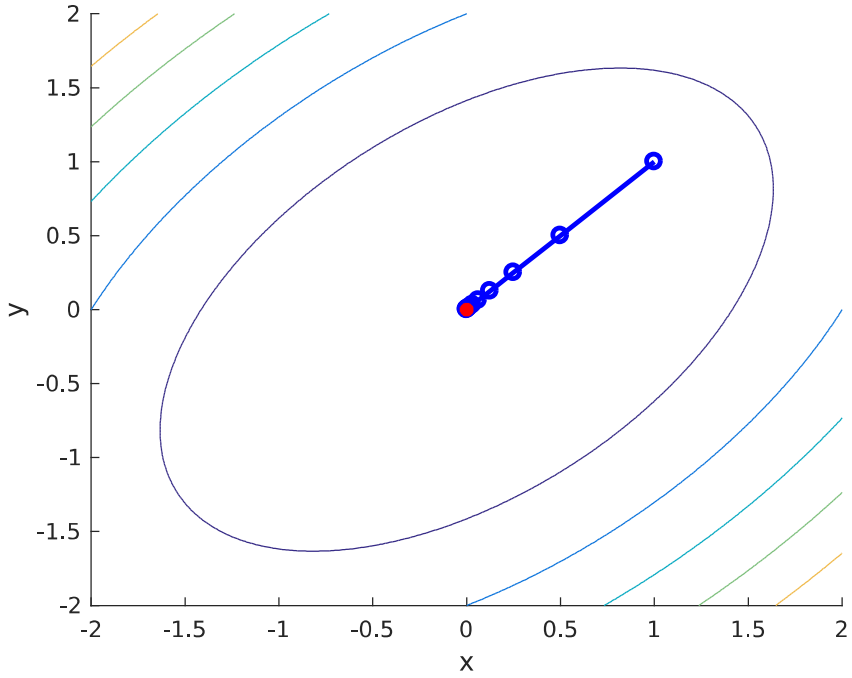
# EECE 5550: Mobile Robotics




Lecture 14: Optimization

# Recap

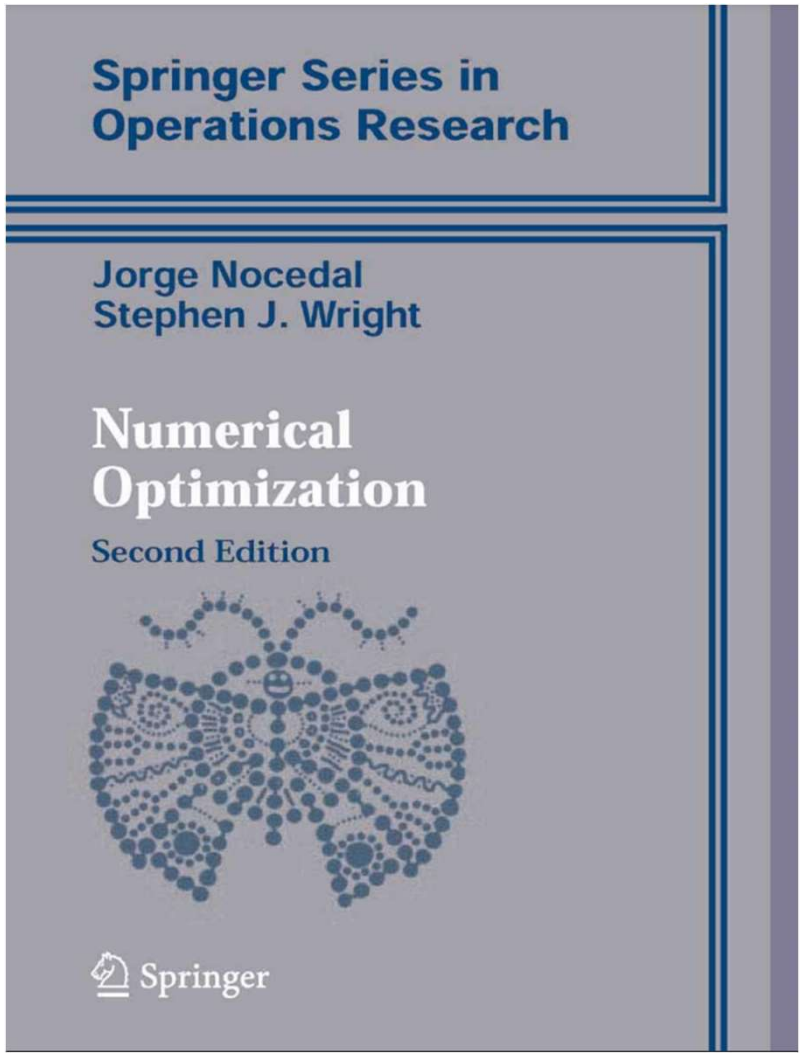
**Last time:**

- Factor graphs
- Simultaneous localization and mapping (SLAM)
- Maximum likelihood estimation

**This time:**

- Basic theory of optimization
  (i.e. how to *actually do* MLE)

# References

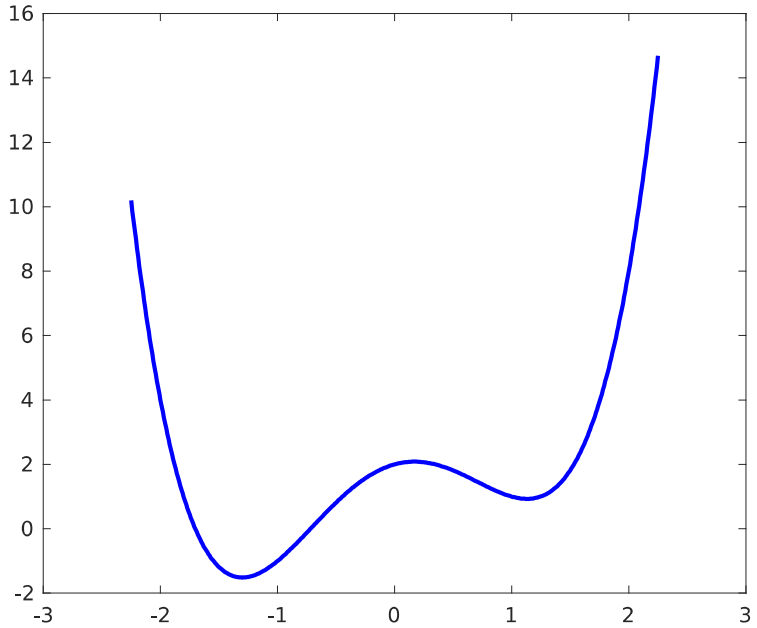

Springer Series in
Operations Research
Jorge Nocedal
Stephen J. Wright
Numerical
Optimization
Second Edition
Springer

# The Main Idea

**Given:** $f: \mathbb{R}^n \to \mathbb{R}$, we want to

$$\min_{x \in \mathbb{R}^n} f(x)$$

**Problem:** We have *no idea* how to actually do this …




**Main idea:** Let's *approximate* $f$ with a simple *model function* $m$, and use that to search for a minimizer of $f$.

# Optimization Meta-Algorithm

**Given:** A function $f: \mathbb{R}^n \to \mathbb{R}$ and an initial guess $x_0 \in \mathbb{R}^n$ for a minimizer

**Iterate:**

- Construct a *model* $m_i(h) \approx f(x_i + h)$ of $f$ near $x_i$.
- Use $m_i$ to search for a *descent direction h* ($f(x + h) < f(x)$)
- Update $x_{i+1} \leftarrow x_i + h$

**until convergence**

# A first example

Let's consider applying the Basic Algorithm to minimize

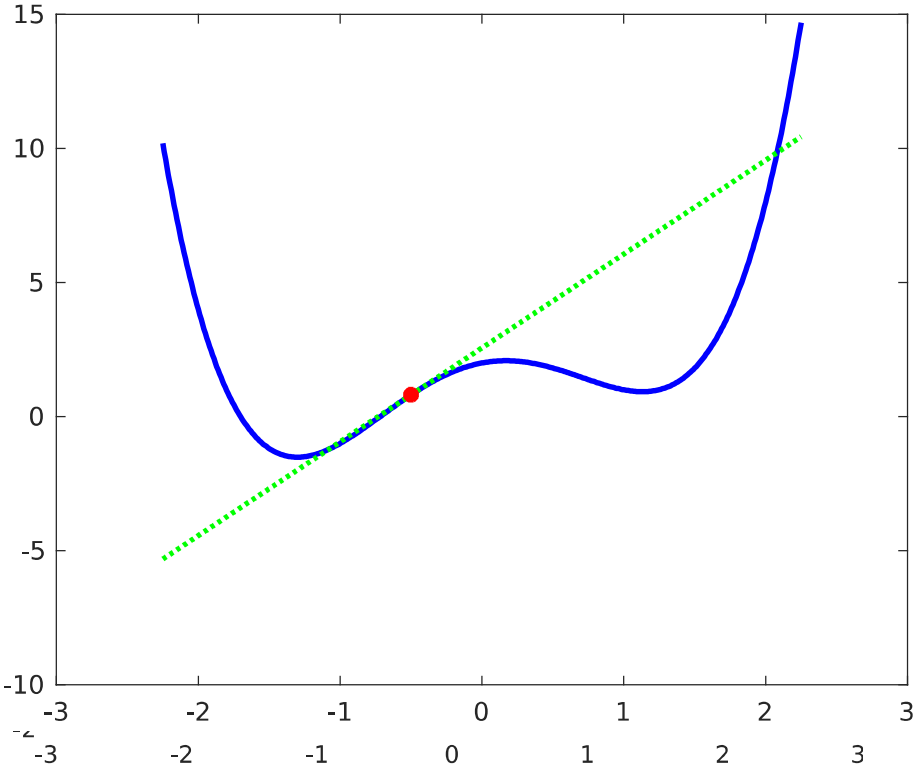
$$f(x) = x^4 - 3x^2 + x + 2$$

starting at $x_0 = -\frac{1}{2}$.

**Q:** How can we *approximate* (*model*) $f$ near $x_0$?

**A:** Let's try linearizing! Take

$$m_0(h) \triangleq f(x_0) + f'(x_0)h$$



# Gradient descent

**Given:**

- A function $f: \mathbb{R}^n \to \mathbb{R}$
- An initial guess $x_0 \in \mathbb{R}^n$ for a minimizer
- Sufficient decrease parameter $c \in (0,1)$, stepsize shrinkage parameter $\tau \in (0,1)$
- Gradient tolerance $\epsilon > 0$

**Iterate:**

- Compute search direction $p = -\nabla f(x_i)$ at $x_i$
- Set initial stepsize $\alpha = 1$
- *Backtracking line search*: Update $\alpha \leftarrow \tau\alpha$ until the *Armijo-Goldstein sufficient decrease condition*:

$$f(x_i + \alpha p) < f(x_i) - c\alpha\|p\|^2$$

  is satisfied
- Update $x_{i+1} \leftarrow x_i + \alpha p$

**until** $\|\nabla f(x_i)\| < \epsilon$

# Exercise: Minimizing a quadratic

Try minimizing the quadratic:

$$f(x, y) = x^2 - xy + \kappa y^2$$

using gradient descent, starting at $x_0 = (1,1)$ and using $c, \tau = \frac{1}{2}$ and $\epsilon = 10^{-3}$, for a few different values of $\kappa$, say

$$\kappa \in \{1, 10, 100, 1000\}$$

**Q:** If you plot function value $f(x_i)$ vs. iteration number *i*, what do you notice?

**Gradient Descent**

**Given:**

- A function $f: \mathbb{R}^n \to \mathbb{R}$
- An initial guess $x_0 \in \mathbb{R}^n$ for a minimizer
- Sufficient decrease parameter $c \in (0,1)$, stepsize shrinkage parameter $\tau \in (0,1)$
- Gradient tolerance $\epsilon > 0$

**Iterate:**

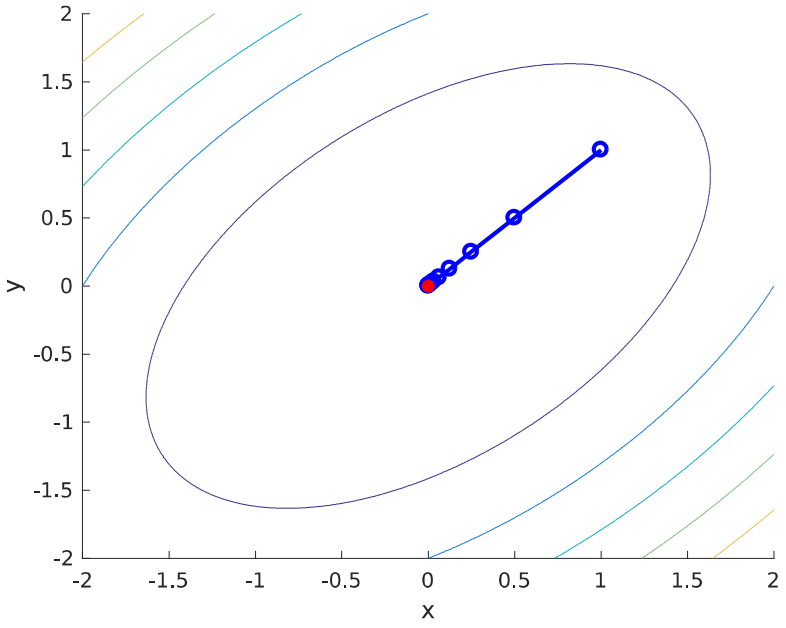
- Compute search direction $p = -\nabla f(x_i)$ at $x_i$
- Set initial stepsize $\alpha = 1$
- Line search: update $\alpha \leftarrow \tau\alpha$ until

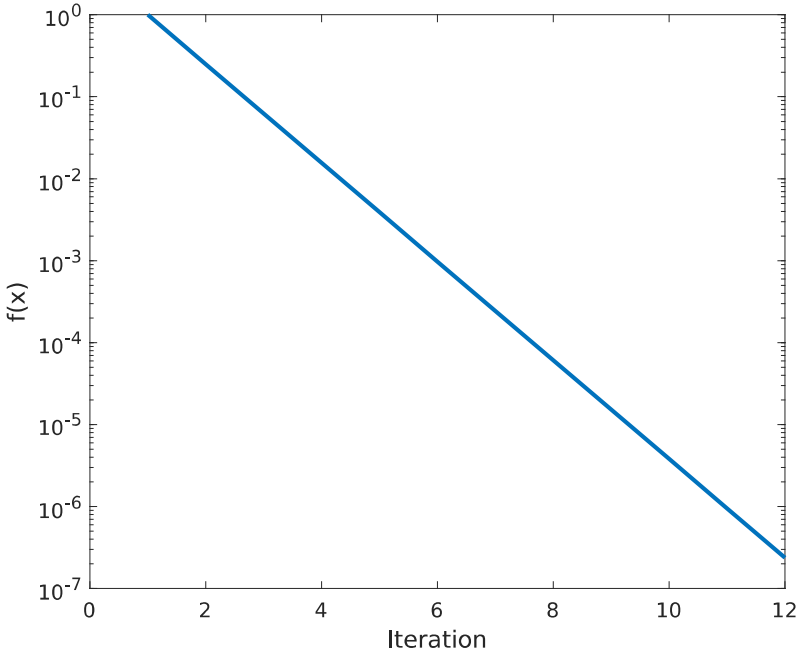
$$f(x_i + \alpha p) < f(x_i) - c\alpha\|p\|^2$$

- Update $x_{i+1} \leftarrow x_i + \alpha p$

**until** $\|\nabla f(x_i)\| < \epsilon$

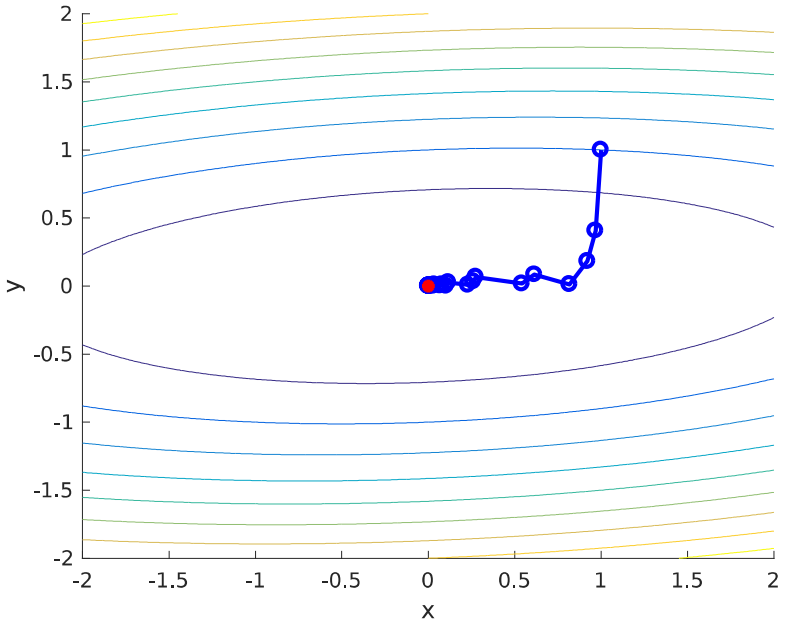
# Exercise: Minimizing a quadratic

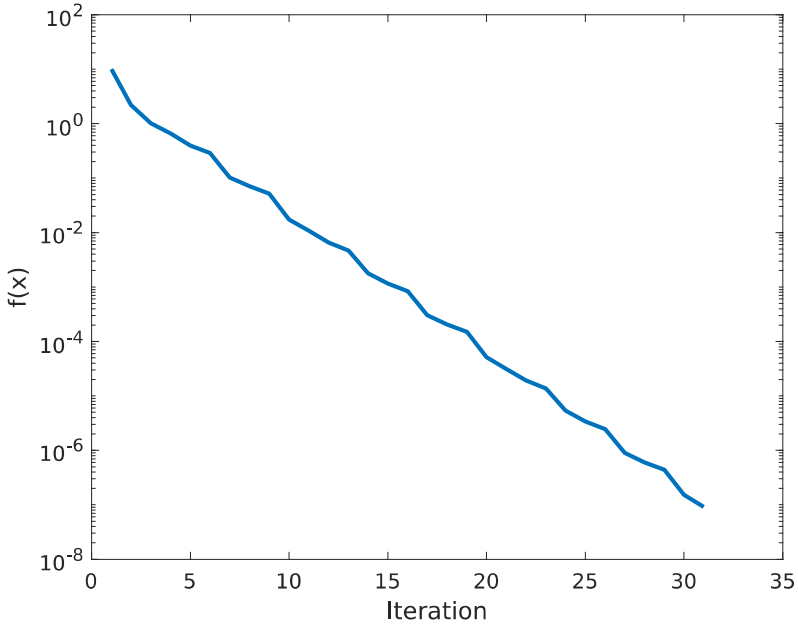



$\kappa = 1$

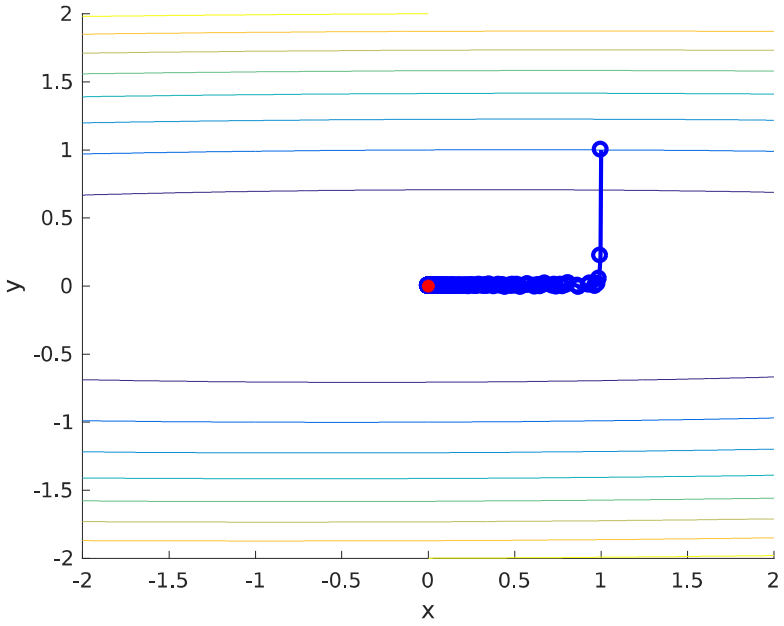
# Exercise: Minimizing a quadratic

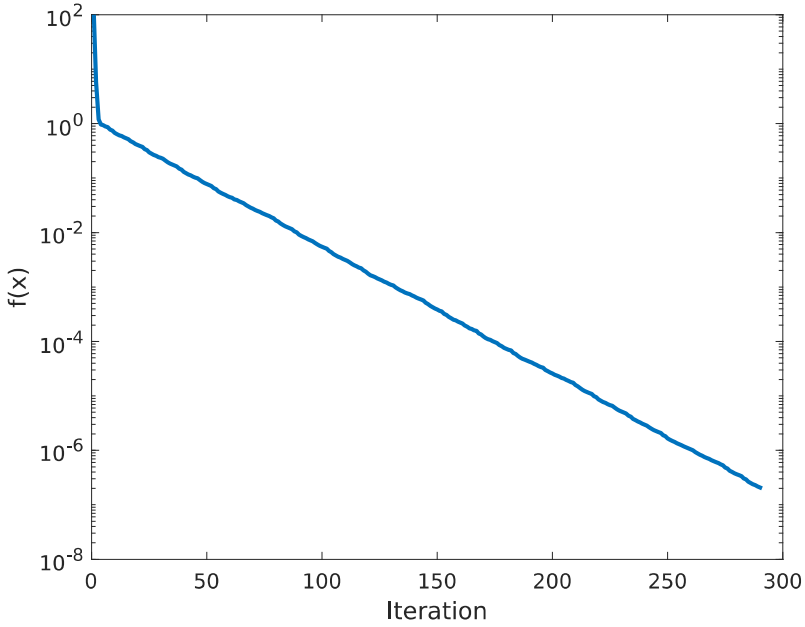



$\kappa = 10$

# Exercise: Minimizing a quadratic




$\kappa = 100$

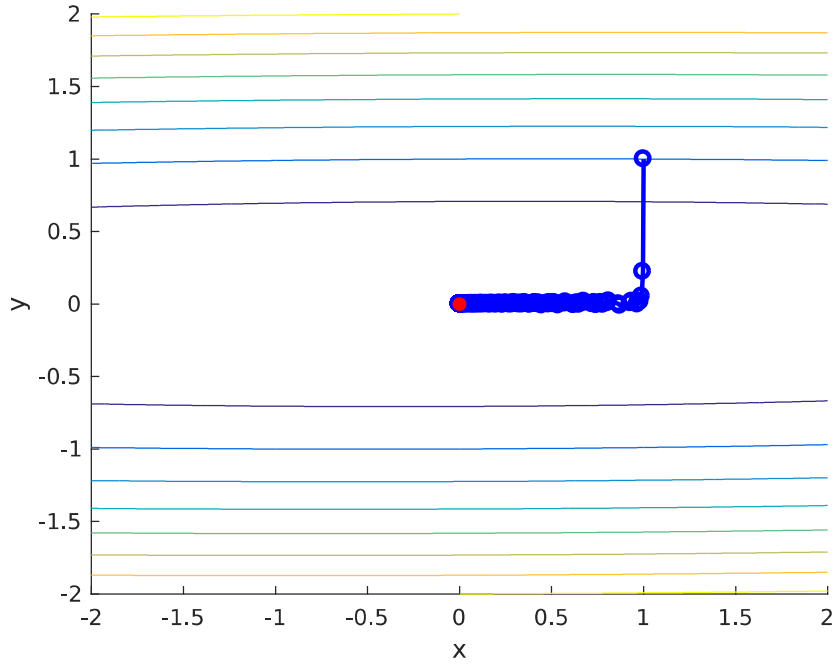
# The problem of conditioning

Gradient descent doesn't perform well when $f$ is *poorly conditioned* (has "stretched" contours).

**Q:** How can we improve our local model:

$$m_i(h) = f(x_i) + \nabla f(x_i)^T h$$

so that it handles curvature better?




WE NEED TO GO
DEEPER

# Second-order methods

Let's try adding in curvature information using a *second-order* model for $f$:

$$m_i(h) = f(x_i) + \nabla f(x_i)^T h + \frac{1}{2} h^T \nabla^2 f(x) h$$

**NB:** If $\nabla^2 f(x) \succ 0$, then $m_i(h)$ has a *unique minimizer*:

$$h_N = -\left(\nabla^2 f(x_0)\right)^{-1} \nabla f(x_0)$$

In that case**,** using the update rule:

$$x_{i+1} \leftarrow x_i + h_N$$

gives ***Newton's method***

# Exercise: Minimizing a quadratic

Let's try minimizing the same quadratic:

$$f(x, y) = x^2 - xy + \kappa y^2$$

this time using Newton's method, starting at $x_0 = (1,1)$ and using $\epsilon = 10^{-3}$, for

$$\kappa \in \{1, 10, 100, 1000\}$$

If you plot function value $f(x_i)$ vs. iteration number *i*, what do you notice?

## Newton's method

**Given:**

- A function $f: \mathbb{R}^n \to \mathbb{R}$
- An initial guess $x_0 \in \mathbb{R}^n$ for a minimizer
- Gradient tolerance $\epsilon > 0$

**Iterate:**

- Compute gradient $\nabla f(x_i)$ and Hessian $\nabla^2 f(x_i)$
- Compute Newton step:
$$h_N = -\left(\nabla^2 f(x_0)\right)^{-1} \nabla f(x_0)$$
- Update $x_{i+1} \leftarrow x_i + h_N$
- **until** $\|\nabla f(x_i)\| < \epsilon$

# Quasi-Newton methods

Newton's method is ***fast***! (It has a *quadratic* convergence rate)

**But:**

- $h_N$ is only guaranteed to be a descent direction if $\nabla^2 f(x_i) \succ 0$
- Computing exact Hessians can be expensive!

**Quasi-Newton methods:** Use a *positive-definite approximate Hessian* $B_i$ in the model function:

$$m_i(h) = f(x_i) + \nabla f(x_i)^T h + \frac{1}{2} h^T B_i h$$

$\Rightarrow m_i(h)$ *always* has a unique minimizer:

$$h_{QN} = -B_i^{-1} \nabla f(x_i)$$

$\Rightarrow h_{QN}$ is *always* a descent direction!

# Quasi-Newton method with line search

**Given:**

- A function $f: \mathbb{R}^n \to \mathbb{R}$
- An initial guess $x_0 \in \mathbb{R}^n$ for a minimizer
- Sufficient decrease parameter $c \in (0,1)$, stepsize shrinkage parameter $\tau \in (0,1)$
- Gradient tolerance $\epsilon > 0$

**Iterate:**

- Compute gradient $g_i = \nabla f(x_i)$ and positive-definite Hessian approximation $B_i$ at $x_i$
- Compute quasi-Newton step:

$$h_{QN} = -B_i^{-1} g_i$$

- Set initial stepsize $\alpha = 1$
- *Backtracking line search*: Update $\alpha \leftarrow \tau\alpha$ until the *Armijo-Goldstein sufficient decrease condition*:

$$f\left(x_i + \alpha h_{QN}\right) < f(x_i) + c\alpha g_i^T h_{QN}$$

  is satisfied
- Update $x_{i+1} \leftarrow x_i + \alpha h_{QN}$

**until** $\|g_i\| < \epsilon$

# Quasi-Newton methods (cont'd)

Different choices of $B_i$ give different QN algorithms
$\Rightarrow$ Can trade off *accuracy* of $B_i$ with *computational cost*

**LOTS** of possibilities here!

- Gauss-Newton
- Levenberg-Marquardt
- (L-)BFGS
- Broyden
- etc …

$\Rightarrow$ Don't be afraid to experiment ☺!

# Special case: The Gauss-Newton method

A quasi-Newton algorithm for minimizing a *nonlinear least-squares objective*:

$$f(x) = \|r(x)\|^2$$

where $r$: $\mathbb{R}^m \to \mathbb{R}^n$ is a *vector-valued residual function*.

Uses the local quadratic model obtained by *linearizing r*:

$$m_i(h) = \|r(x_i) + J(x_i)h\|^2$$

where $J(x_i) \triangleq \frac{\partial r}{\partial x}(x_i)$ is the *Jacobian* of *r*.

Equivalently:

$$g_i = 2J(x_i)^T r(x_i), \qquad B_i = 2J(x_i)^T J(x_i)$$

**NB:** In this case, the update *h* is the solution of a linear least-squares problem

# A word on linear algebra

The dominant cost (memory + time) in a QN method is *linear algebra*:

- Constructing the Hessian approximation $B_i$
- Solving the linear system:

$$B_i h_{QN} = -h_{QN}$$

$\Rightarrow$ Fast/robust linear algebra is ***essential*** for efficient QN methods

- Take advantage of sparsity in $B_i$!

- **NEVER, NEVER, NEVER INVERT $B_i$ directly!!!**
  - It's incredibly expensive, and unnecessary
  - **Use instead** [cf. Golub & Van Loan's *Matrix Computations*]:
    - *Matrix factorizations:* QR, Cholesky, $LDL^T$
    - *Iterative linear methods:* conjugate gradient

A word on linear algebra

***NEVER INVERT $B_i$!!!***

# Optimization methods: Cheat sheet

## First-order methods

Use only gradient information

- **Pro:** Local model is inexpensive
- **Con:** Slow (linear) convergence rate

**Canonical example:** Gradient descent

**Best for:**

- Moderate accuracy
- Very large problems

## Second-order methods

Use (some) 2nd-order information

- **Pro:** Fast (superlinear) convergence
- **Con:** Local model can be expensive

**Canonical example:** Newton's method

**Best for:**

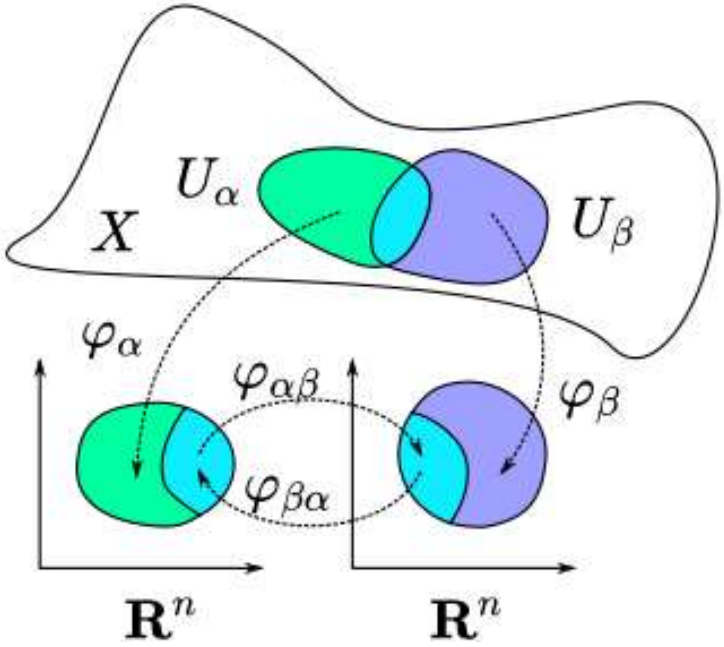
- High accuracy
- Small to moderately large problems

# Optimization on Manifolds

**Main idea (recap):** Search for a descent direction *h* using a *local model* *m* of the objective *f*:

$$m_i(h) = f(x_i) + \nabla f(x_i)^T h + \frac{1}{2} h^T B_i h$$

**NB:** model *m* is built using (approximate) derivative information ($B_i \approx \nabla^2 f(x_i)$)




**Recall:**

- **Key point:** Derivatives are *local*: $df_x$ only depends upon *f*'s behavior in an infinitesimally small open set around *x*.
- Smooth manifolds are spaces in which every point *x* has an open set U that is diffeomorphic to an open set in $\mathbb{R}^n$

⇒We can apply exactly the same approach to optimize functions on smooth manifolds!

## Quasi-Newton optimization on $\mathbb{R}^n$

**Iterate:**

1. Compute gradient $g_i = \nabla f(x_i)$ and positive-definite Hessian approximation $B_i \approx \nabla^2 f(x_i)$ at $x_i$
2. Compute quasi-Newton step:

$$h_{QN} = -B_i^{-1} g_i$$

3. Set initial stepsize $\alpha = 1$
4. *Backtracking line search*: Update $\alpha \leftarrow \tau\alpha$ until the *Armijo-Goldstein sufficient decrease condition*:

$$f(x_i + \alpha h_{QN}) < f(x_i) + c\alpha g_i^T h_{QN}$$

   is satisfied
5. Update $x_{i+1} \leftarrow x_i + \alpha h_{QN}$

**until** $\|g_i\| < \epsilon$

## Quasi-Newton optimization on a manifold *M*

**Iterate:**

1. Compute gradient $g_i = \nabla f(x_i)$ and positive-definite Hessian approximation $B_i \approx \nabla^2 f(x_i)$ at $x_i$
2. Compute quasi-Newton step:

$$h_{QN} = -B_i^{-1} g_i$$

3. Set initial stepsize $\alpha = 1$
4. *Backtracking line search*: Update $\alpha \leftarrow \tau\alpha$ until the *Armijo-Goldstein sufficient decrease condition*:

$$f(Retr_{x_i}(\alpha h_{QN})) < f(x_i) + c\alpha g_i^T h_{QN}$$

   is satisfied
5. Update $x_{i+1} \leftarrow Retr_{x_i}(\alpha h_{QN})$

**until** $\|g_i\| < \epsilon$

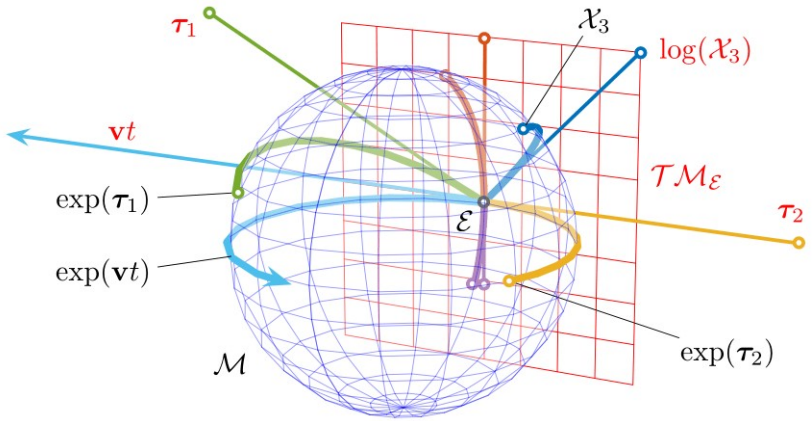
⇒ The only difference is that on a general manifold, in steps 4 & 5 we need a *retraction operator* $Retr_x: T_x(M) \rightarrow M$ that describes how to move along the manifold *M* from *x* in the direction of a tangent vector $\dot{v} \in T_x(M)$

# Special case: Optimization on Lie groups

**Recall:** Exponential map $\exp: Lie(G) \to G$ is a map from Lie algebra Lie(G) (tangent space at the identity $e \in G$) to $G$

Given some function $f: G \to \mathbb{R}$ and some point $x_i \in G$, we can construct the following *pullback* $\hat{f}: Lie(G) \to \mathbb{R}$ of $f$ to Lie(G) at $x_i$ :

$$\hat{f}_{x_i}(\xi) \triangleq f(x_i \cdot \exp(\xi))$$




**NB:**

- $y = x_i \cdot \exp(\xi)$ maps a neighborhood of $\xi = 0$ in Lie(G) to a neighborhood of $x_i$ in $G$.
- $\hat{f}_{x_i}$ is a function between Euclidean spaces
  ⇒We know exactly how to differentiate this!
- $\hat{f}_{x_i}(0) = f(x_i)$

**Therefore:** We can build a local model of $\hat{f}_{x_i}$ at $\xi = 0$, just as before:

$$\hat{m}_i(h) = f(x_i) + \nabla \hat{f}(0)^T h + \frac{1}{2} h^T B h,$$

then find a descent direction $h \in Lie(G)$ for $\hat{m}_i(h)$, then apply the retraction (update):

$$x_{i+1} \leftarrow x_i \exp(h)$$

# Quasi-Newton method on a Lie group G

**Given:**

- A function $f: G \to \mathbb{R}$
- An initial guess $x_0 \in \mathbb{R}^n$ for a minimizer
- Sufficient decrease parameter $c \in (0,1)$, stepsize shrinkage parameter $\tau \in (0,1)$
- Gradient tolerance $\epsilon > 0$

**Iterate:**

- Construct pullback function $\hat{f}_{x_i}(\xi) \triangleq f(x_i \cdot \exp(\xi))$ at $x_i$
- Compute gradient $g_i = \nabla \hat{f}_{x_i}(0)$ and positive-definite Hessian approximation $B_i \approx \nabla^2 \hat{f}_{x_i}(0)$ at $\xi = 0$
- Compute quasi-Newton step:

$$h_{QN} = -B_i^{-1} g_i$$

- Set initial stepsize $\alpha = 1$
- *Backtracking line search*: Update $\alpha \leftarrow \tau\alpha$ until the *Armijo-Goldstein sufficient decrease condition*:

$$f(x_i \cdot \exp(\alpha h)) < f(x_i) + c\alpha g_i^T h_{QN}$$

  is satisfied
- Update $x_{i+1} \leftarrow x_i \cdot \exp(\alpha h_{QN})$

**until** $\|g_i\| < \epsilon$