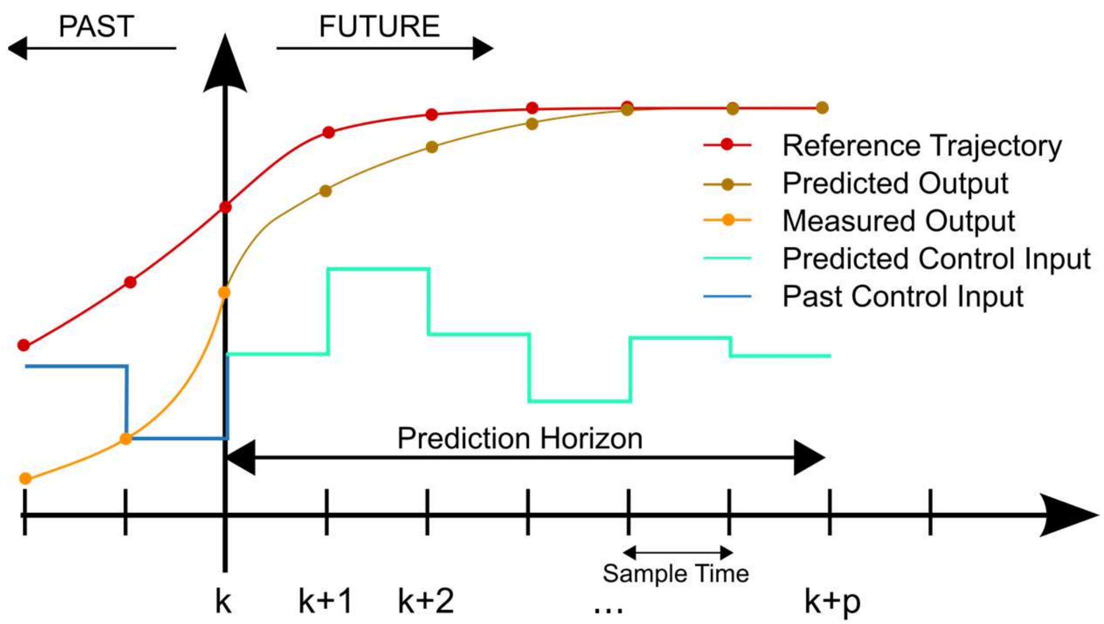
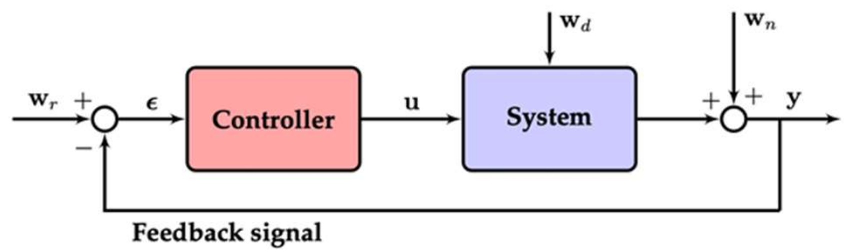
EECE 5550: Mobile Robotics



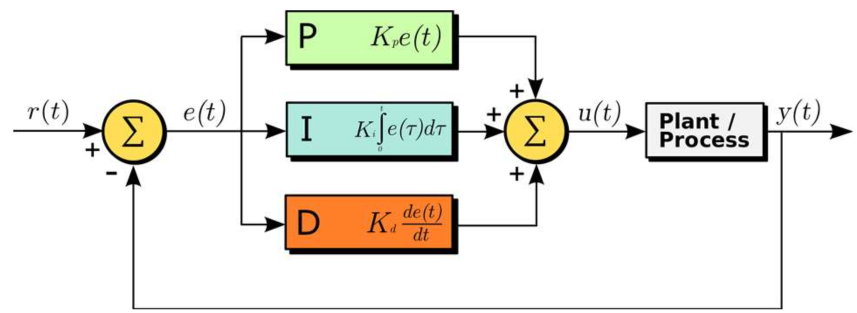
Lecture 18: Optimal and Model Predictive Control

Recap



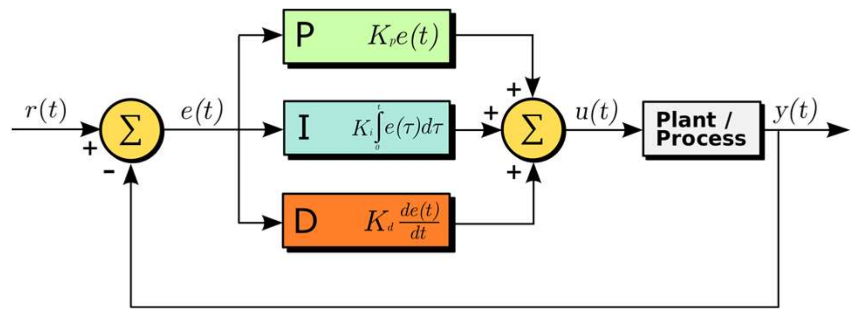
Last time: Basic feedback control

- Control system formulation
- State feedback control
- Proportional-Integral-Derivative (PID) control



PID Control Recap

- **Benefits of PID control**
- Conceptually simple
- Relatively straightforward tuning procedures



But:

- No natural way of including state or control constraints
  - State constraint: Should have altitude > 0 for a drone
  - Control constraint: Turtlebots can't do warp 9

- PID formulation doesn't support *costs* or *preferences* for controls or trajectories.
  - Trajectory preference: "faster is better"
  - Control cost: More aggressive controls typically require more energy, fuel, etc.

Punchline: We might like to have a more expressive framework for modeling these constraints / preferences, and designing controllers that account for them

https://youtu.be/meSItatXQ7M

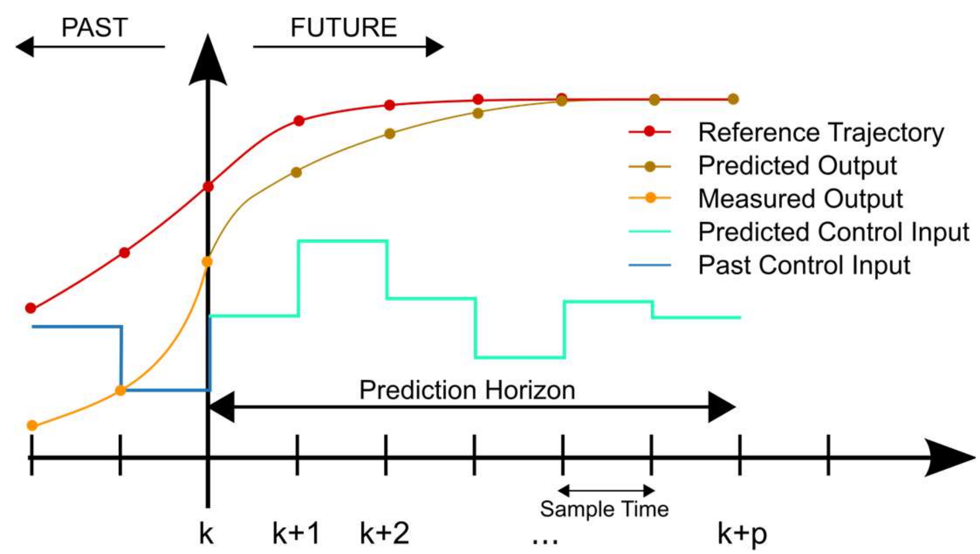
Plan of the day

Optimal control

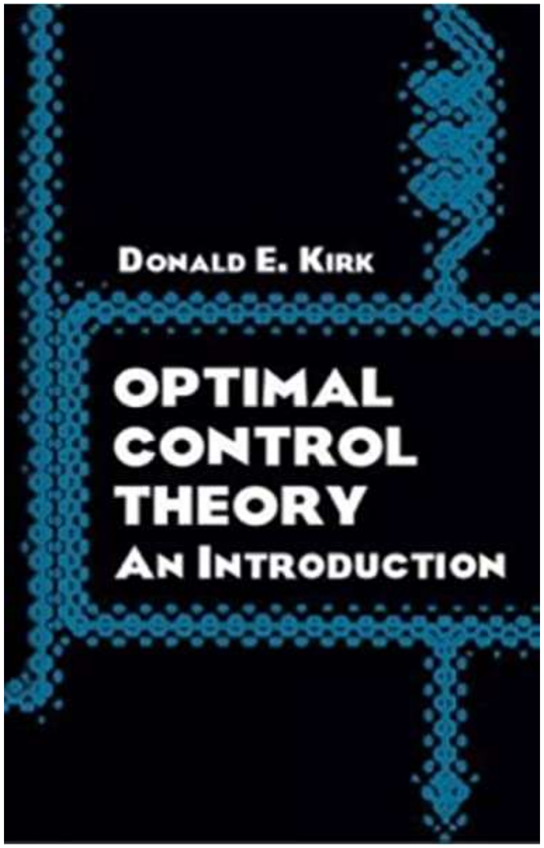
- Problem formulation
- Bellman's optimality principle

Solving the optimal control problem

- Indirect methods: Solving the Bellman-Jacobi-Hamilton equation
- Direct methods: Model predictive control



References



Optimal Control Theory: An Introduction

Survey paper: "A Survey of Numerical Methods for Optimal Control", by A.V. Rao

Optimal control: Problem formulation

Find: The final time $t_f \in \mathbb{R}$, control input $u\colon \left[t_0, t_f\right] \to \mathbb{R}^m$, and trajectory (or path) $x\colon \left[t_0, t_f\right] \to \mathbb{R}^n$ to

Performance index — Terminal cost — Running cost

Minimize:

$$J[x, u, t_f] \triangleq E\left(x\left(t_f\right), t_f\right) + \int_{t_0}^{t_f} L[x(t), u(t), t]\, dt$$

subject to:

$$\dot{x}(t) = f[x(t), u(t), t]$$ state equation (system dynamics)

$$h(x(t), u(t), t) \leq 0$$ path (state) constraints

$$e\left(x\left(t_f\right), t_f\right) = 0$$ endpoint (boundary) conditions

Optimal control: Problem formulation

Very general formulation! Supports:

- Expressing *preferences* for trajectories & controls (via performance index *J*)
- Complex (nonlinear) system dynamics f
- General state and control constraints (via path constraints *h*)
- Partially-constrained boundary condition (via endpoint condition *e*)

But: Decision variables include *functions* $x(t)$ and $u(t)$

- These are *infinite-dimensional objects*
- OCP is ***very*** hard to solve in general …

Optimal control problem (OCP)

Minimize:

$$J[x, u, t_f] \triangleq E\big(x(t_f), t_f\big) + \int_{t_0}^{t_f} L[x(t), u(t), t]\, dt$$

subject to:

$$\dot{x}(t) = f[x(t), u(t), t]$$

$$h(x(t), u(t), t) \leq 0$$

$$e\big(\, x(t_f), t_f\big) = 0$$

Bellman's optimality principle

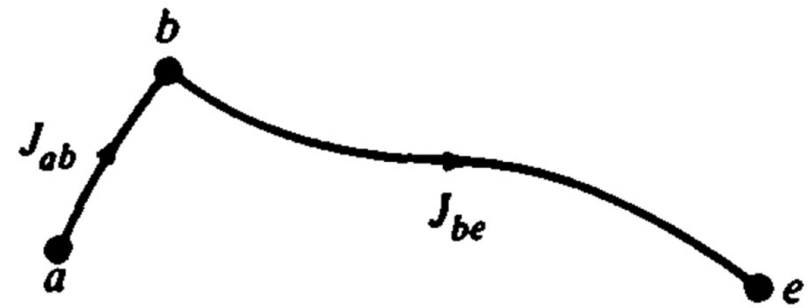
In order to *search* for an optimal solution $u^*(t), x^*(t)$ of the control problem, we at least need some way of characterizing what optimal solutions look like.

Bellman's *principle of optimality* describes a simple yet very useful property of optimal solutions.

Let $u: [a, e] \to \mathbb{R}^m$ be an optimal control for OCP with state $x: [a, e] \to \mathbb{R}^n$.
Suppose that we follow u over the interval [a,b] for some $b \in (a, e)$, arriving at state *x*(*b*).

What can we say about the control $u: [b, e] \to \mathbb{R}^m$ that sends the state from *x*(*b*) to *x*(*e*)?

Key observation: $u: [b, e] \to \mathbb{R}^m$ must be an optimal control for moving from *x*(*b*) to *x*(*e*)!



Proof (by contradiction): Suppose that there were some other control $v: [b, e] \to \mathbb{R}^m$ such that $J_{be}(v) < J_{be}(u)$
Define $w: [a, e] \to \mathbb{R}^m$ to be the control that applies u on [a,b] and *v* on [b,e].
Then:

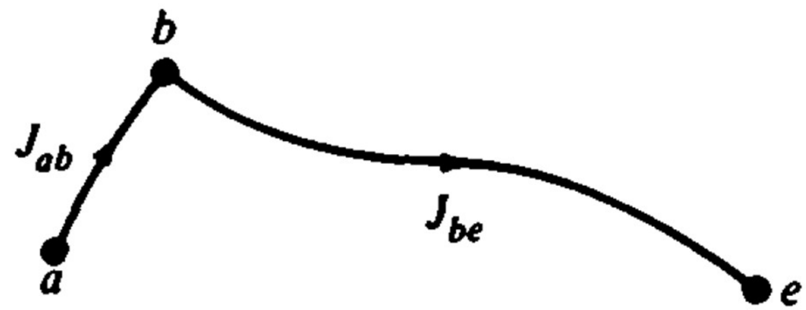
$$J_{ae}(w) = J_{ab}(u) + J_{be}(v) < J_{ab}(u) + J_{be}(u) = J_{ae}(u)$$

But this contradicts optimality of *u*. **QED**

Bellman's optimality principle

"An optimal policy has the property that whatever the initial state and initial decision are, the remaining decisions must constitute an optimal policy with regard to the state resulting from the first decision"

$$J_{ae}^{*} = J_{ab}^{*} + J_{be}^{*}$$



The optimal value function

Define the *value function* (or *cost-to-go function*) $V: \mathbb{R}^n \times [0, T] \to \mathbb{R}$ by:

$$V(x, t) \triangleq \min_{u \in U(x)} E(x(T), T) + \int_{t_0}^{t_f} L[x(t), u(t), t] \, dt$$

where here the set $U(x)$ denotes the set of admissible controls starting at state *x* at time *t*.
NB: $V(x, t)$ reports the cost of the optimal control policy *u*, *starting at state x at time t*.

What can we say about *V*?
Let's consider an optimal control $u(t)$ and trajectory $x(t)$ that start at state x_0 at time t_0.
By Bellman's principle, given any $\Delta t > 0$, we have:

$$V(x_0, t_0) = \underbrace{\min_{u \in U(x_0)} \int_{t_0}^{t_0 + \Delta t} L[x(t), u(t), t] \, dt}_{\text{(Optimal) cost to move from } x(t) \text{ to } x(t+\Delta t)} + \underbrace{V(x(t + \Delta t), t + \Delta t)}_{\text{Cost-to-go from } x(t+\Delta t)}$$

The Hamilton-Jacobi-Bellman (HJB) equation

$$V(x_0, t_0) = \min_{u \in U(x_0)} \int_{t_0}^{t_0+\Delta t} L[x(t), u(t), t]\, dt + V(x(t + \Delta t), t + \Delta t)$$

Taylor expansion of the right-hand side for small Δt:

$$\min_{u \in U(x_0)} \underbrace{L(x_0, u, t_0)\Delta \mathrm{t}}_{\text{Differential of integral term}} + V(x_0, t_0) + \underbrace{\frac{\partial V(x_0, t_0)}{\partial x} \cdot \dot{x}(t)\Delta t + \frac{\partial V(x_0, t_0)}{\partial t}\Delta t}_{\text{Total differential of optimal value function}} + O(\Delta t)$$

$$= \min_{u \in U(x_0)} L(x_0, u, t_0)\Delta \mathrm{t} + V(x_0, t_0) + \frac{\partial V(x_0,t_0)}{\partial x} \cdot \underbrace{f[x_0, u, t_0]}_{\text{System dynamics}}\Delta t + \frac{\partial V(x_0,t_0)}{\partial t}\Delta t + O(\Delta t^2)$$

Now subtract $V(x_0, t_0)$ from both sides, and take limit as $\Delta t \to 0$ to obtain the Hamilton-Jacobi-Bellman eq:

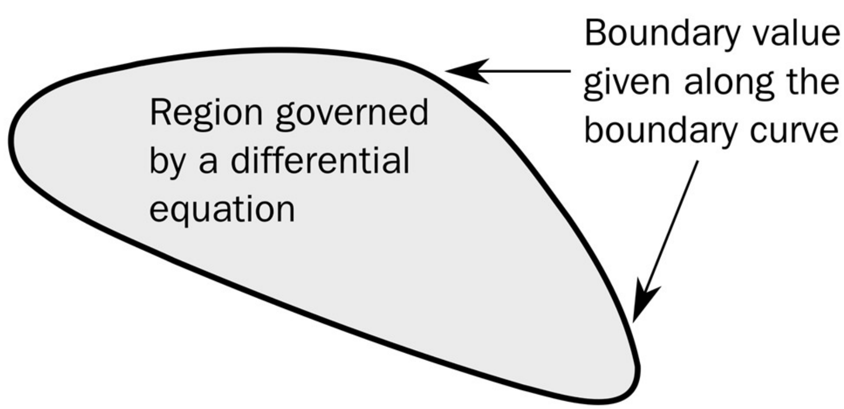
$$\boxed{\frac{\partial V(x, t)}{\partial x} + \min_{u \in U(x)} \left\{ \frac{\partial V(x, t)}{\partial x} \cdot f(x, u, t) + L(x, u, t) \right\} = 0}$$

The Hamilton-Jacobi-Bellman Boundary Value Problem

Main Takeaway: The value function $V(x,t)$ is the solution of the following boundary value problem:

$$\frac{\partial V(x,t)}{\partial x} + \min_{u \in U(x)} \left\{ \frac{\partial V(x,t)}{\partial x} \cdot f(x,u,t) + L(x,u,t) \right\} = 0$$

subject to: $V(x,T) = E(x,T)$ (endpoint cost)



Optimal control via the value function

Suppose that we have the value function $V(x, t)$ in hand. How can we use this to solve our control problem?

Recall: The value function $V(x, t)$ gives the optimal *cost-to-go* from state *x* at time *t*.
⇒If we are state *x* at time *t*, we should pick the control *u* towards the *lowest-cost reachable state*

From previous slides, the time derivative of *V* along trajectory $x(t)$ determined by *u* is:

$$\frac{d}{dt}[V(x(t), t)] = \frac{\partial V(x,t)}{\partial t} + \frac{\partial V(x,t)}{\partial x} \cdot f(x, u, t_0) + L(x, u, t)$$

⇒We should pick the control *u* to minimize the right-hand side above:

$$\boxed{u^*(x, t) = argmin_{u \in U(x)} \left\{ \frac{\partial V(x, t)}{\partial x} \cdot f(x, u, t_0) + L(x, u, t) \right\}}$$

Key point: This shows that knowing the value function gives us an *optimal control policy*: a *pointwise mapping* from the state *x* and time *t* to the optimal control action *u.*

The Discrete Case

The same approach applies to discrete-time control problems. Consider:

Minimize:

$$J[x,u] \triangleq E\left(x_{t_f}\right) + \sum_{t=0}^{t_f} L(x_t, a_t)$$

subject to:

$$x_{t+1} = f(x_t, a_t)$$

$$a_t \in \Gamma(x_t)$$

$$e\left(x(t_f)\right) = 0$$

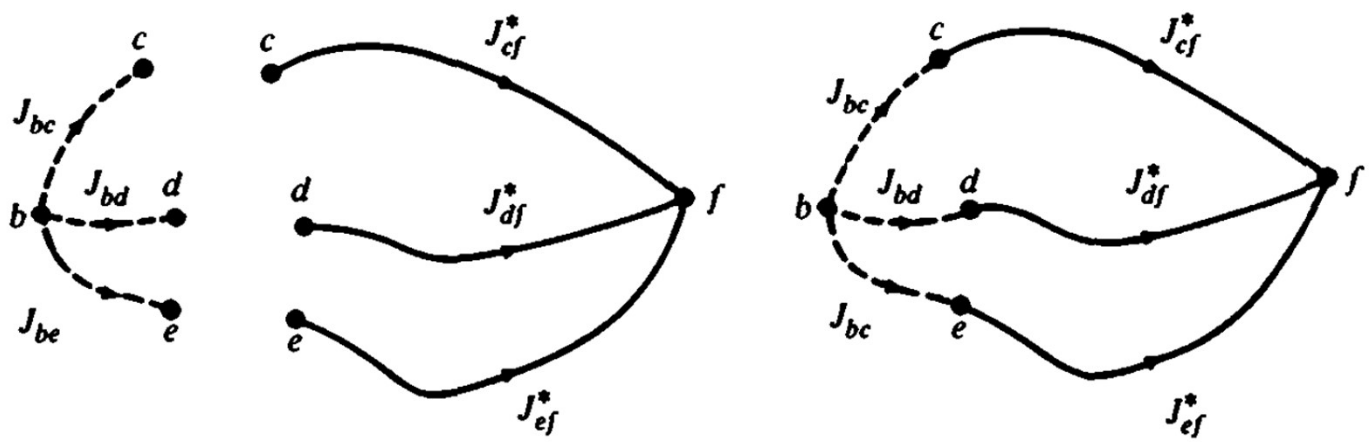
where:

- *E* is the terminal cost function
- *L* is running cost function
- *f* is the state transition function
- $\Gamma(x)$ is the set of *admissible actions* in state *x*
- *e* is the endpoint constraint

The Discrete Case: Bellman's Equation

Applying Bellman's principle, we find that the value function $V(x)$ satisfies Bellman's equation:

$$V(x) = \underbrace{\min_{a \in \Gamma(x)} L(x,a)}_{\text{Cost to move from } x \text{ to next state } y = f(x,a)} + \underbrace{V(f(x,a))}_{\text{Cost-to-go from } y}$$



Constructing the discrete-time value function

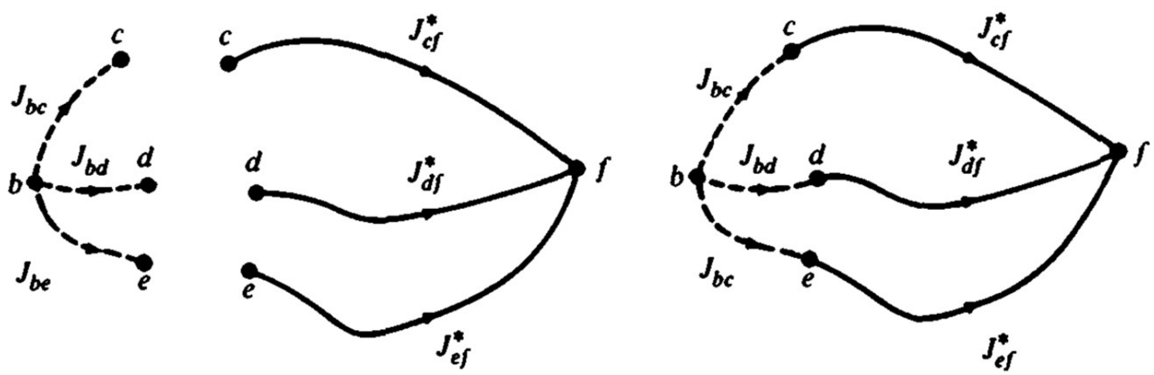
The discrete-time Bellman equation suggests a natural algorithm for constructing the value function via *dynamic programming* (specifically: *backwards induction*)

Base case: For each *terminal state* x, the cost-to-go is simply the terminal cost:

$$V(x) = E(x)$$

Recursive step: Working *backwards* (from the terminal states), calculate the value of *previous* states using the Bellman equation:

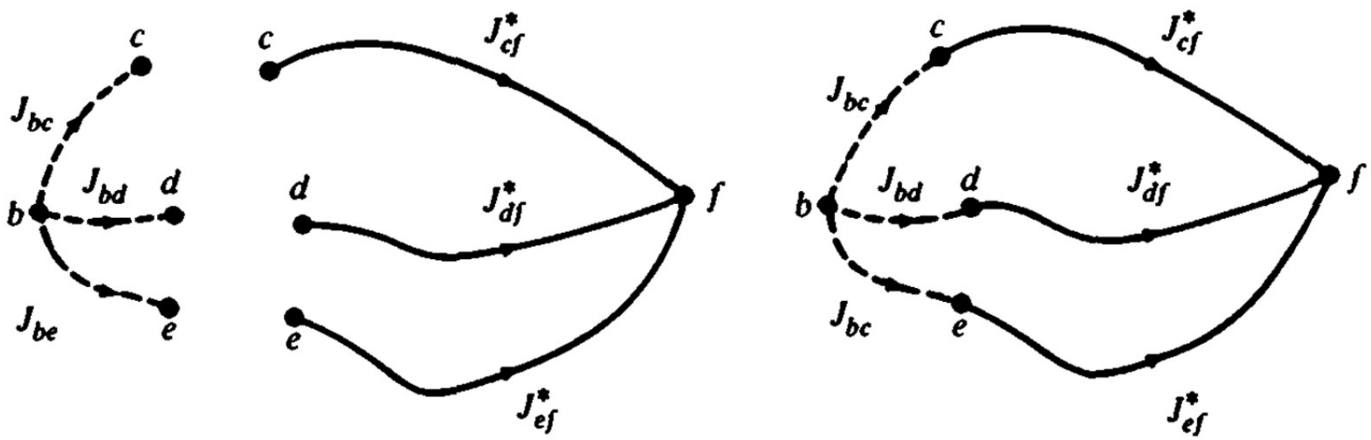
$$V(x) = \min_{a \in \Gamma(x)} L(x, a) + V(f(x, a))$$



Discrete-time optimal control via the value function

As in the continuous-time case, we can recover an *optimal control policy* *u* directly from the value function $V(x)$:

$$u(x) = argmin_{a \in \Gamma(x)} L(x, a) + V(f(x, a))$$



Exercise

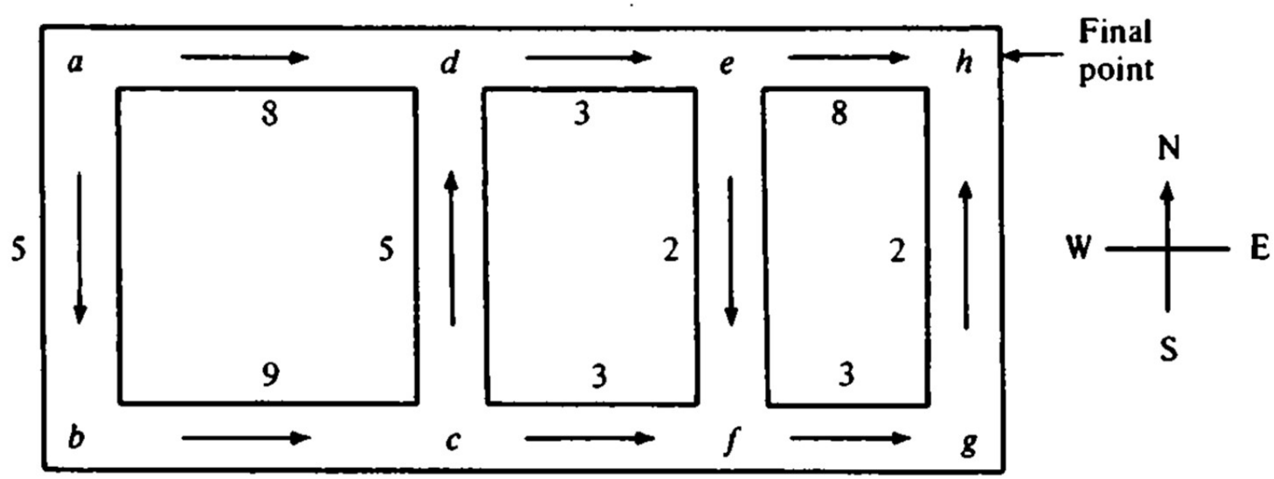
Given: The following road network, with travel costs as indicated on the edges and goal state *h*

Find:

- Optimal value function *V*(*x*)
- Optimal control policy *u* that assigns to each state *x* its optimal *heading* (N,S,E,W)

$$V(x) = \min_{a \in \Gamma(x)} L(x,a) + V(f(x,a))$$

$$u(x) = argmin_{a \in \Gamma(x)} L(x,a) + V(f(x,a))$$



Solution

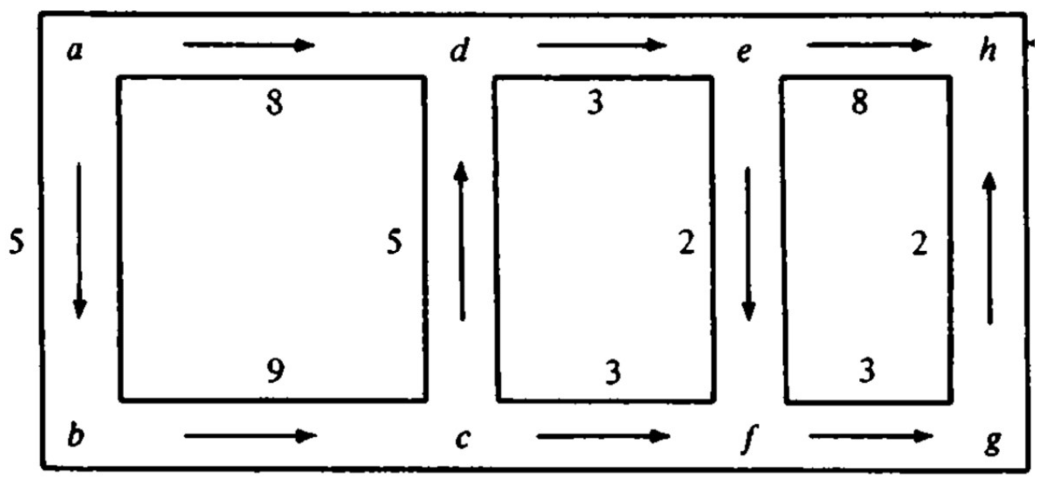


Table 3-1 CALCULATION OF OPTIMAL HEADINGS BY DYNAMIC PROGRAMMING

Current intersection α	*Heading* u_i	*Next intersection* x_i	*Minimum cost from* α *to h via* x_i $J_{\alpha x_i} + J^*_{x_i h} = C^*_{\alpha x_i h}$	*Minimum cost to reach h from* α $J^*_{\alpha h}$	*Optimal heading at* α $u^*(\alpha)$
g	N	*h*	2 + 0 = 2	2	N
f	E	*g*	3 + 2 = 5	5	E
e	E S	*h* *f*	8 + 0 = 8 2 + 5 = 7	7	S
d	E	*e*	3 + 7 = 10	10	E
c	N E	*d* *f*	5 + 10 = 15 3 + 5 = 8	8	E
b	E	*c*	9 + 8 = 17	17	E
a	E S	*d* *b*	8 + 10 = 18 5 + 17 = 22	18	E

Optimal control: Recap

The optimal control formulation provides a very expressive framework for modeling problems with nonlinear dynamics and state and control constraints

BUT: Optimal control problems are *very* hard to solve exactly

- Continuous case: HJB boundary-value problem
- Discrete-time case: Dynamic programming

⇒Both of these suffer from the curse of dimensionality

Key question: Can we do something more tractable?

Optimal control problem (OCP)

Minimize:

$$J\left[x, u, t_f\right] \triangleq E\left(x\left(t_f\right), t_f\right) + \int_{t_0}^{t_f} L[x(t), u(t), t]\, dt$$

subject to:

$$\dot{x}(t) = f[x(t), u(t), t]$$

$$h(x(t), u(t), t) \leq 0$$

$$e\left(x\left(t_f\right), t_f\right) = 0$$

Model Predictive Control (MPC)

Motivation: (Global) optimality is an *extremely strong* requirement. We can probably make due with a "pretty good" (but cheaper) control solution that satisfies the problem constraints.

Main idea: We will make two simplifying assumptions:

- We will only plan out to a (fixed) *planning horizon T*
- We will consider a *parametric family* of controls $\hat{u}(\cdot;\alpha)\colon [0,T] \to \mathbb{R}^m$ with *finite-dimensional parameter* $\alpha \in \mathbb{R}^p$.
  [Ex: polynomials up to order *k*, piecewise constant functions with *k* segments, etc.]

Now observe:

- Control $\hat{u}(t;\alpha)$ is determined by choice of α.
- Having fixed α, system dynamics for *x* then reduce to:

$$\dot{x} = f(x, \hat{u}(t;\alpha), t) = f(x, t; \alpha), \qquad x(0) = x_0$$

This is a first-order ODE for *x*(*t*). We can easily solve this using e.g. Runge-Kutta.
⇒The *trajectory* $x(t;\alpha)$ is likewise parameterized by α.

Payoff: Under this modeling assumption for the controls, the *entire control problem* reduces to a *nonlinear program over* α.

MPC: Problem transcription

Optimal control problem (OCP)

$$\min_{u,x} E\big(x(t_f), t_f\big) + \int_{t_0}^{t_f} L[x(t), u(t), t]\, dt$$

subject to: $\dot{x}(t) = f[x(t), u(t), t]$

$$h(x(t), u(t), t) \leq 0$$

$$e\big(\, x(t_f), t_f\big) = 0$$

MPC subproblem (NLP)

$$\min_{\alpha \in \mathbb{R}^p} E(x(t), T) + \int_0^T L[x(t), \hat{u}(t; \alpha), t]\, dt$$

subject to: $\dot{x}(t) = f[x(t), \hat{u}(t; \alpha), t]$

$$h(x(t), \hat{u}(t; \alpha), t) \leq 0$$

$$e(x(T)) = 0$$

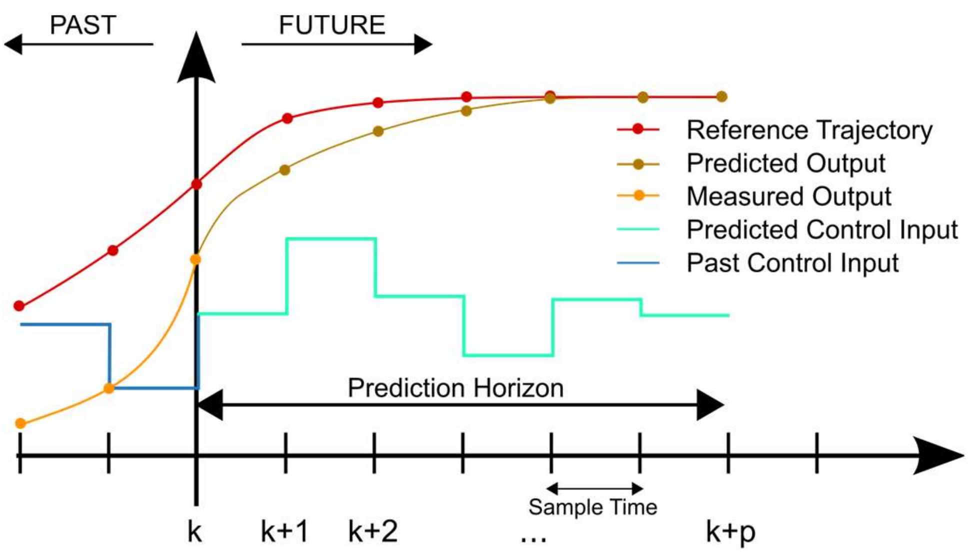
Key point: Assuming a finite horizon *T* and finite-dimensional parameterization $u(\cdot; \alpha): [0, T] \to \mathbb{R}^m$ for controls reduces the optimal control problem to a standard (finite-dimensional) nonlinear program

Payoff: We can (approximately) solve sparse NLPs *very* fast

Model Predictive Control

Repeat:

1. Given current state $x(0) = x_0$, solve MPC subproblem to generate (locally) optimal control $\hat{u}(\cdot;\alpha^*): [0,T] \to \mathbb{R}^m$ out to planning horizon
2. Apply first stage of control $\hat{u}$
3. Go to step 1



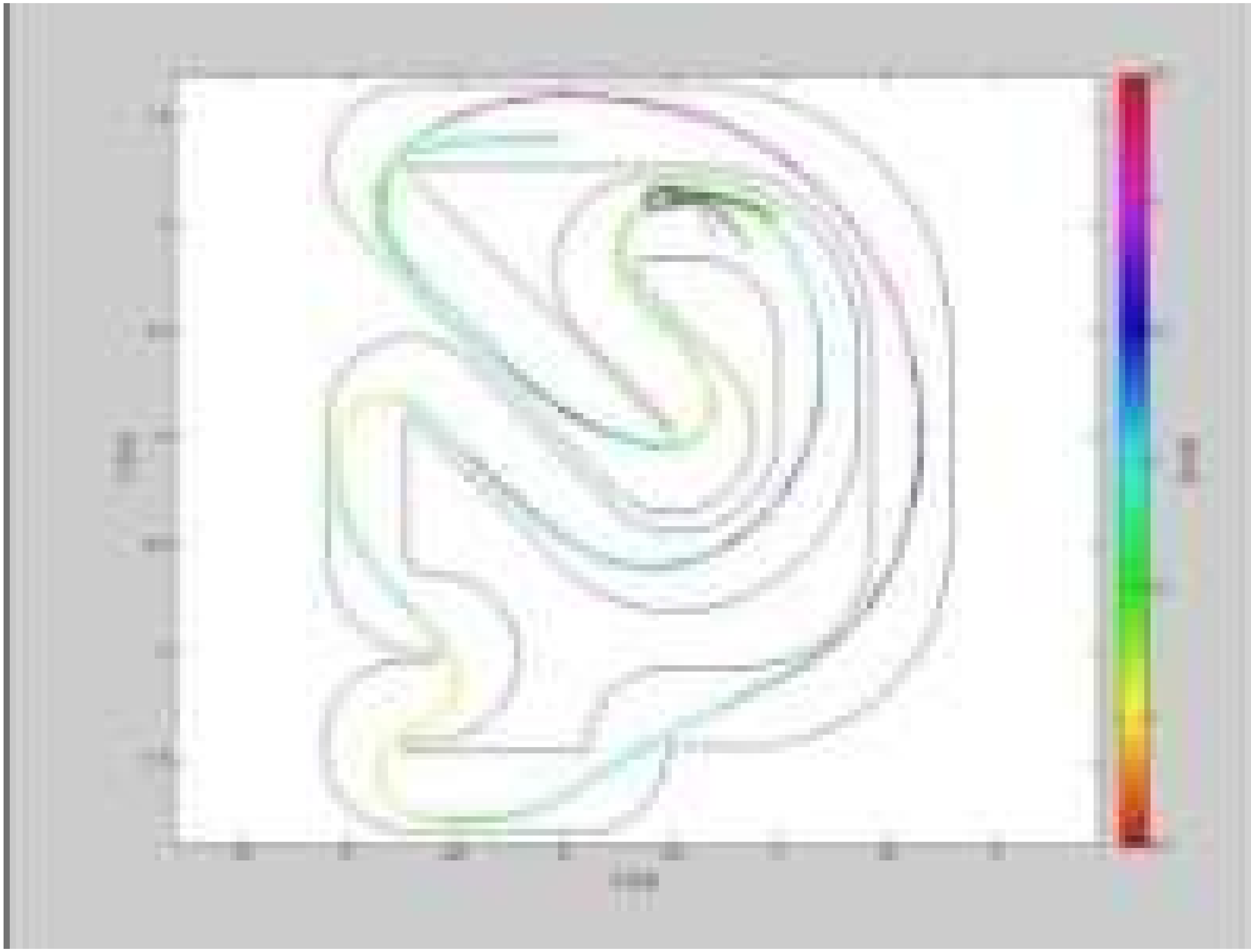
MPC subproblem (NLP)

$$\min_{\alpha \in \mathbb{R}^p} E(x(t),T) + \int_0^T L[x(t), \hat{u}(t;\alpha), t]\, dt$$

subject to: $\dot{x}(t) = f[x(t), \hat{u}(t;\alpha), t]$

$$h(x(t), \hat{u}(t;\alpha), t) \leq 0$$

$$e(x(T)) = 0$$



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Optimal and model Predictive control : Summary

Very general formulation! Supports:

- Expressing *preferences* for trajectories & controls (via performance index *J*)
- Complex (nonlinear) system dynamics *f*
- General state and control constraints
- Partially-constrained boundary conditions

But: Decision variables include *functions* $x(t)$ and $u(t) \Rightarrow$ VERY hard to solve in general

Model Predictive Control (MPC):

Approximate optimal control strategy:

- Uses (fixed) finite planning horizon *T*
- Assume parametric family of controls $\hat{u}(\cdot;\alpha^*): [0,T] \to \mathbb{R}^m$ w/ parameter α

Optimal control problem (OCP)

Minimize:

$$J[x,u,t_f] \triangleq E\big(x(t_f),t_f\big) + \int_{t_0}^{t_f} L[x(t),u(t),t]\,dt$$

subject to:

$$\dot{x}(t) = f[x(t),u(t),t]$$

$$h(x(t),u(t),t) \le 0$$

$$e\big(\,x(t_f),t_f\big) = 0$$

Payoff: Under these assumptions, MPC problem is a finite-dimensional nonlinear program

⇒We can solve these *very fast*