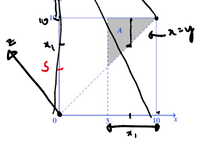
Dis 6D: More Continuous Probability

Thursday, 30 July 2020 10:55 PM

$$
\begin{aligned}
Pr(A) &= E[1_A] \\
&= E[E[1_A|X]] \\
&= E[Pr(A|X)] \\
&= \int_{-\infty}^{\infty} f_X(x)Pr(A|X=x)dx
\end{aligned}
$$

$Pr(A|X)$ is a random variable and we apply the formula for the expected value of a RV.

Here's a good test to see if you need to review: Suppose z = yx^2.
Compute the volume with A (on the x-y plane) as its base and z as its 'ceiling'.



$$\text{vol} = \int_{x=5}^{x=10}\int_{y=x}^{y=10} z \; dy\,dx$$

Properties of joint PDFs

$\int_{-\infty}^{\infty}\int$ f_{X, Y} (x, y) dx dy = 1

Conditional PDFs in terms of joint and simple PDFs

f_{X|Y} (x|y) = $\dfrac{f_{X,Y}(x,y)}{f_Y(x)}$

Marginal PDFs from Joint PDFs [say you have f_{X, Y} (x, y)]

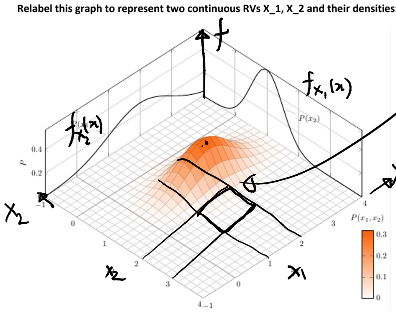
f_X (x) = $\int_{y=-\infty}^{y=\infty} f_{X,Y}(x,y)\,dy$

Probability to integral of a PDF

$P(a \le X \le b | Y = y)$

= $\int_{x=a}^{x=b} f_{X|Y}(x|y)\,dx$

Relabel this graph to represent two continuous RVs X_1, X_2 and their densities:



$f(x,y)$

$\boxed{f_{X_1,X_2}(x_1,x_2)}$

$P(1 \le X_1 \le 2,\ 2 \le X_2 \le 3)$

$f(x) = x^2$

$\int_{x=0}^{x=1} x^2\,dx$

1 Continuous Joint Densities

The joint probability density function of two random variables X and Y is given by $f(x,y) = Cxy$ for $0 \le x \le 1, 0 \le y \le 2$, and 0 otherwise (for a constant C).

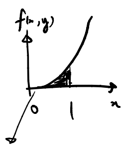
(a) Find the constant C that ensures that $f(x,y)$ is indeed a probability density function.

$$\int_0^1\int_0^2 Cxy\,dy\,dx = 1 \iff C\int_0^1\int_0^2 xy\,dy\,dx = 1$$

$$\iff C\int_0^1 x\left[\frac{y^2}{2}\right]_0^2 dx = 1 \iff C\int_0^1 2x\,dx = 1$$

$$\iff C\left[x^2\right]_0^1 = 1$$

$$C = 1$$



(b) Find $f_X(x)$, the marginal distribution of X.

$$f_X(x) = \int_{-\infty}^{\infty} xy\,dy = \int_0^2 xy\,dy = x\left[\frac{y^2}{2}\right]_0^2 = 2x$$

(c) Find the conditional distribution of Y given $X = x$.

① write algebra ② use formula

k is an arbitrary input.

$$f_{Y|X}(k|x) = \frac{f_{Y,X}(k,x)}{f_X(x)} = \frac{xk}{2x} = \frac{k}{2}$$

$f_Y(y) \approx Pr(Y=y)$ $\forall k$, $Pr(Y=k) = \binom{n}{k}p^k(1-p)^{n-k}$

Probability distr

function

cont → PDFS

discrete → PMFs

(d) Are X and Y independent?

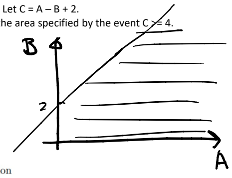
i) $f_{X,Y}(x,y) = f_X(x)\,f_Y(y)$

or

ii) $f_{Y|X}(k|x) = f_Y(k)$

Algebra to Graph practice:

Let A ~ Exp(a), B ~ Exp(b). Let C = A – B + 2.
Draw a graph and shade the area specified by the event C >= 4.



$A - B + 2 \ge 4$

$A - B \ge 2$

$A \ge 2 + B$

2 Uniform Distribution

You have two spinning wheels, each having a circumference of 10 cm with values in the range $[0, 10)$ marked on the circumference. If you spin both (independently) and let X be the position of the first spinning wheel's mark and Y be the position of the second spinning wheel's mark, what is the probability that $X \ge 5$, given that $Y \ge X$?

① Hint: write out the algebra first

② Hint: Are X, Y random variables? If so, what are they?

③ Hint: draw out the graph and shade in the area we're integrating over to get the 'volume' that corresponds to the

$f()$

$f : \mathbb{R} \to \mathbb{R}$

$\forall x_i$ define/specify $f(x_i) = y_i$

$f(x_1) = y_1$

$f(k)$

$f_Y(k)$

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① $\boxed{Pr(X \geq 5 \mid Y \geq X)}$ → A

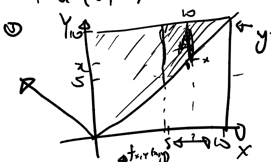
$= \dfrac{Pr(X \geq 5, Y \geq X)}{Pr(Y \geq X)}$ → areas, → area

$= \dfrac{\frac{1}{8}}{\frac{1}{2}} = \frac{1}{4}$

② $X \sim U(0,10)$

$Y \sim U(0,10)$

③



$Pr(A) = \int_5^{10}\int_x^{10} f_{X,Y}(x,y)\,dy\,dx$

$= \int_5^{10}\int_x^{10} \frac{1}{100}\,dy\,dx$

$= \int_5^{10} \left[\frac{1}{100}y\right]_x^{10} dx = \int_5^{10} \left(\frac{1}{100}(10-x)\right) dx$

$= \frac{1}{100}\int_5^{10} 10 - x\,dx$

$= \frac{1}{100}\left[10x - \frac{x^2}{2}\right]_5^{10}$

$= \frac{1}{100}\left(\left(100 - \frac{100}{2}\right) - \left(50 - \frac{25}{2}\right)\right)$

$= \frac{1}{100}\left(50 - \frac{75}{2}\right)$

$= \frac{1}{100}\left(\frac{100-75}{2}\right) = \frac{25}{2 \times 100} = \frac{1}{8}$

$f_Y(k)$

Follow up T/F:

f_X

X and Y aren't uniform RVs but are still continuous RVs bounded between 0 and 10, the area argument for finding Pr(X >= 5 | Y >= X) applies.

$\int_a^b\int^c f_{X,Y}(x,y)\,dx\,dy$

$f_{X,Y}(x,y) = f_X(x) f_Y(y)$ ← X, Y indep from ① statement.

$= \left(\frac{1}{10}\right)\left(\frac{1}{10}\right) = \frac{1}{100}$

$f_{X,Y}(x,y) = f_{X|Y}(x|y)\, f_Y(y)$

$= f_X(x)\, f_Y(y)$

Kevin Lin 4 hours ago

Re "bayes is just Pr[A|B] = Pr[A\cap B]/Pr[B] I e. area of the intersection over area of one the event being conditioned on"

We can only translate the num & denom probabilities into finding areas because finding areas is equivalent to doing a double integral on the joint densities. If A, B weren't both U(0,10) we wouldn't be able to use the area trick

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3 Exponential Practice

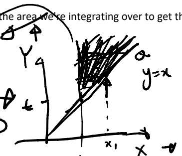
Let $X \sim \text{Exponential}(\lambda_X)$ and $Y \sim \text{Exponential}(\lambda_Y)$ be independent, where $\lambda_X, \lambda_Y > 0$. Let $U = \min\{X,Y\}$, $V = \max\{X,Y\}$, and $W = V - U$.

$\int_t^{\infty}$

(a) Compute $\mathbb{P}(U > t, X \leq Y)$ for $t \geq 0$.

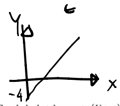
② Hint: draw out the graph and shade in the area we're integrating over to get the 'volume' that corresponds to the probability

① $= Pr(t < X \leq Y)$



$X \leq Y + 4$

=



③ $\int_{x=t}^{x=\infty}\int_{y=x}^{y=\infty} f_{X,Y}(x,y)\,dy\,dx$

$= \lambda_X\lambda_Y \int_t^{\infty}\int_x^{\infty} e^{-\lambda_X x} e^{-\lambda_Y y}\,dy\,dx$

$X, Y \geq 0$

(b) Use the previous part to compute $\mathbb{P}(X \leq Y)$. Conclude that the events $\{U > t\}$ and $\{X \leq Y\}$ are independent.

$P(X \leq Y) = Pr(U > 0, X \leq Y)$

(c) Compute $\mathbb{P}(W > t \mid X \leq Y)$.

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(d) Use the previous part to compute $\mathbb{P}(W > t)$.

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(a) Compute $\mathbb{P}(U > t, X \le Y)$, for $t \ge 0$.

(e) Calculate $\mathbb{P}(U > u, W > w)$, for $w > u > 0$. Conclude that U and W are independent. [*Hint*: Think about the approach you used for the previous parts.]