Dis 5A: Bayes' Rule

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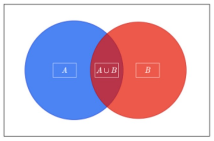
A Guide to Bayes' Rule

Where does $P(A|B) = \frac{Pr(A \cap B)}{Pr(B)}$ come from?



Back to basics

Let's think about what $Pr(X)$ means.

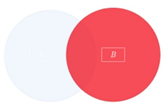
First, we're notice that X is an event in an experiment. For example, $[X = \text{roll an even number} = (\text{roll } 2, \text{roll } 4, \text{roll } 6)]$ is an event in the experiment of rolling a fair 6-sided die.

Then, $Pr(X)$ is the **probability that a random outcome** o of our experiment **satisfies the property**: $o \in X$. Specifically, o satisfies that property that o is an even number.

Conditional Probability

What does $P(A|B)$ mean?

You can interpret the "$|$ event" operator as one of 'reducing' the sample space to the elements in event.



(The "$|B$" operator reduces the sample space from Ω to B)

i.e. if you considered $P(A)$, you're implicitly considering $P(A|\Omega)$. In terms of the diagram, Ω is everything in the white rectangle. When you consider $P(A|B)$, Ω effectively becomes B.

Bayes' Rule: Intuitive derivation

Now, let's look at $Pr(A \cap B)$.

$Pr(A \cap B)$ is the **probability that a random outcome** o of our experiment **satisfies the property**: $o \in A \cap B$. Specifically, o satisfies that property that $o \in A$ and $o \in B$.

3 steps:

1. For $o \in A$ and $o \in B$, we need $o \in B$. This happens with probability $Pr(B)$, by definition.
2. Once we have $o \in B$, we also need $o \in A$. This happens with probability $Pr(A|B)$ also by definition
3. Altogether, we have $Pr(A \cap B) = Pr(B)Pr(A|B)$

2 Poisoned Smarties

Supposed there are 3 men who are all owners of their own Smarties factories. Burr Kelly, being the brightest and most innovative of the men, produces considerably more Smarties than his competitors and has a commanding 45% of the market share. Yousef See, who inherited his riches, lags behind Burr and produces 35% of the world's Smarties. Finally Stan Furd, brings up the rear with a measly 20%. However, a recent string of Smarties related food poisoning has forced the FDA investigate these factories to find the root of the problem. Through his investigations, the inspector found that one Smarty out of every 100 at Kelly's factory was poisonous. At See's factory, 1.5% of Smarties produced were poisonous. And at Furd's factory, the probability a Smarty was poisonous was 0.02.

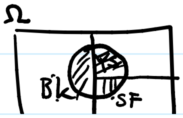
(a) What is the probability that a randomly selected Smarty will be safe to eat?

1) Let Bk be .. from . . .
YS
SF
S

$Pr(Bk) = 0.45$ $Pr(S^c | Bk) = 0.01$
$Pr(YS) = 0.35$ $Pr(S^c | YF) = 0.015$
$Pr(SF) = 0.20$ $Pr(S^c | SF) = 0.02$

2) $Pr(S) =$
$= Pr($

Ω



$Pr(S) = Pr(S \cap Bk) + Pr(S \cap YS) + Pr(S \cap SF)$
$= Pr(S | Bk) Pr(Bk) + \ldots$

$Pr(S) = Pr(S \cap Bk) + Pr(S \cap YS) + Pr(S \cap SF)$

$= Pr(S|Bk) Pr(Bk) + \cdots$

(b) If we know that a certain Smarty didn't come from Burr Kelly's factory, what is the probability that this Smarty is poisonous?

$P(A) = Pr(A \cap B_1) + \cdots$

1) $Pr(S) - Pr(S \cap Bk)$

$Bk^c = YS \cup SF$

Find: $Pr(S^c | Bk^c) = Pr(S^c | YS \cup SF)$

$= \dfrac{Pr(S^c | YS) Pr(YS) + Pr(S^c | SF) Pr(SF)}{Pr(YS \cup SF)}$

$= \dfrac{Pr(\boxed{S^c \cap YS \cup SF})}{Pr(YS \cup SF)} = \dfrac{\boxed{Pr(S^c \cap YS) + Pr(S^c \cap SF)}}{Pr(YS) + Pr(SF)}$

(c) Given this information, if a randomly selected Smarty is poisonous, what is the probability it came from Stan Furd's Smarties Factory?

$Pr(SF | S^c) = \dfrac{Pr(SF \cap S^c)}{Pr(S^c)} = \dfrac{Pr(S^c | SF) Pr(SF)}{1 - a)}$

arg. by symmetry

Cindy has a bag of n blue balls and m red balls. She randomly selects a ball to play tennis with her acquaintance Yasaman.
What's the probability that the 4th ball she selects is blue?
(Assume n, m >= 4)

$(\frac{}{n})(\frac{}{n-1})(\frac{}{n-2})(\frac{}{n-3})$

cases: # blue balls before

$\frac{n}{n+m}$

$b\ r\ r\ r$

$\Updownarrow$

$r\ b\ r\ r$

each is

$\_ \_ \_ \_ \_ \_ \_$

$n+m$

$Pr(1^{st} \text{ ball is blue}) = \dfrac{n}{n+m} = \dfrac{|\text{outcomes where 1st is blue}|}{|\Omega|}$

$Pr(2^{nd} \text{ ball is blue})\ ? = \dfrac{|\text{outcomes where 2nd is blue}|}{|\Omega|}$

n-1 blues left to "place" in a slot

"randomly fixed 1st blue slot" $\binom{n+m-1}{n-1}$

b b r r r b ... r b r b

n+m

"$\binom{n+m-1}{n-1}$

3 Bag of Coins

Your friend Forrest has a bag of n coins. You know that k are biased with probability p (i.e. these coins have probability p of being heads). Let F be the event that Forrest picks a fair coin, and let B be the event that Forrest picks a biased coin. Forrest draws three coins from the bag, but he does not know which are biased and which are fair.

(a) What is the probability of three coins being pulled in the order FFB?

$Pr(FFB) \cdot \left(\frac{n-k}{n}\right)\left(\frac{n-k-1}{n-1}\right)\left(\frac{k}{n-2}\right)$

$Pr(C_1 = F, C_2 = F, C_3 = B)$

$P(A \cap B \cap C) = P(A | (B \cap C)) P(B \cap C)$

$P(A \cap B \cap C) = P(A \mid (B \cap C))\, P(B \cap C)$

$\Pr(C_2 = F, C_3 = B \mid C_1 = F)\, \Pr(C_1 = F)$

$= \Pr(C_3 = B \mid C_2 = F, C_1 = F)\, \Pr(C_2 = F \mid C_1 = F)\, \Pr(C_1 = F)$

(b) What is the probability that the third coin he draws is biased?

$k, \; n-k$

$\frac{k}{n}$

$\frac{k}{n}$ $\Pr(1^{st} \text{ is biased}) = \Pr(2^{nd} \text{ is biased}) = \Pr(3^{rd} \text{ is biased})$ ... n^{th}

$= \frac{|1^{st} = \text{biased}|}{|\Omega|}$

$\frac{|2^{nd} = \text{biased}|}{|\Omega|}$

b b r r b b r r b . b b "one outcome"

1^{st}

1^{st} = blue

(c) What is the probability of picking at least two fair coins?

b

3^{rd}

3^{rd} = blue

$1 - \frac{n-k}{n}$

$1 - \Pr(0 \text{ or } 1 \text{ fair coins})$

(d) Given that Forrest flips the second coin and sees heads, what is the probability that this coin is biased?

$\Pr(H \mid B) = p$

$\Pr(H \mid F) = \frac{1}{2}$

$\Pr(B \mid H) =$