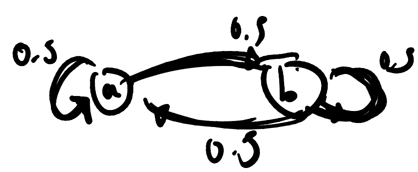
Dis 7D: Markov Chains

Thursday, 6 August 2020 4:33 PM



$d(a) = \gcd\{1, 2, 4, 3, \ldots$
$= 1$

1 Markov Chain Terminology

In this question, we will walk you through terms related to Markov chains.

1. (Irreducibility) A Markov chain is irreducible if, starting from any state i, the chain can transition to any other state j, possibly in multiple steps.

2. (Periodicity) $d(i) := \gcd\{n > 0 \mid P^n(i,i) = \mathbb{P}[X_n = i \mid X_0 = i] > 0\}$, $i \in \mathscr{X}$. If $d(i) = 1 \; \forall i \in \mathscr{X}$, then the Markov chain is aperiodic; otherwise it is periodic.

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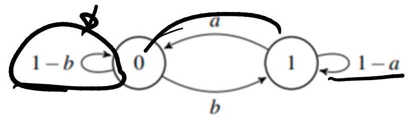
3. (Matrix Representation) Define the transition probability matrix P by filling entry (i, j) with probability $P(i, j)$.

4. (Invariance) A distribution π is invariant for the transition probability matrix P if it satisfies the following balance equations: $\pi = \pi P$.



(a) For what values of a and b is the above Markov chain irreducible? Reducible?

$\left(\text{not } a = 0 \text{ and not } b = 0 \right)$ Red

(b) For ~~$a = 1, b = 1$~~, prove that the above Markov chain is periodic.

$d(0) = \gcd(2, 4, 6, \ldots) = 2 \neq 1$

(c) For $0 < a < 1$, $0 < b < 1$, prove that the above Markov chain is aperiodic.

$d(0) = \gcd(1, \ldots) = 1$

(d) Construct a transition probability matrix using the above Markov chain.

$$P = \begin{bmatrix} 1-b & b \\ a & 1-a \end{bmatrix}$$

(e) Write down the balance equations for this Markov chain and solve them. Assume that the Markov chain is irreducible.

$\pi = \pi P$, $\pi(0) + \pi(1) = 1 \iff \pi(1) = 1 - \pi(0)$

$$[\pi(0)\ \ \pi(1)] \begin{bmatrix} 1-b & b \\ a & 1-a \end{bmatrix} = [\pi(0)\ \ \pi(1)]$$

$\Rightarrow (1-b)\pi(0) + a\pi(1) = \pi(0)$

sub. $\pi(1)$ in

$(1-b)\pi(0) + a(1 - \pi(0)) = \pi(0)$

$a = (a+b)\pi(0)$

$\Rightarrow \pi(0) = \frac{a}{a+b}$

sub. $\pi(1)$ in

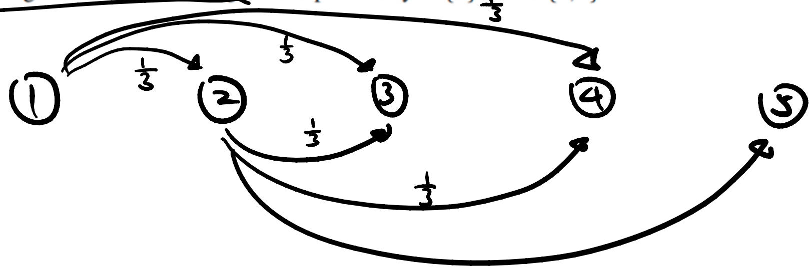
$(1-b)\pi(0) + a(1-\pi(0)) = \pi(0)$

$\Rightarrow \cancel{\pi(0)} - b\pi(0) + a - a\pi(0) = \cancel{\pi(0)}$

$\Rightarrow \boxed{\pi(0) = \frac{a}{a+b} \quad \pi(1) = \frac{b}{a+b}}$

2 Skipping Stones

We consider a simple Markov chain model for skipping stones on a river, but with a twist: instead of trying to make the stone travel as far as possible, you want the stone to hit a target. Let the set of states be $\mathscr{X} = \{1,2,3,4,5\}$. State 3 represents the target, while states 4 and 5 indicate that you have overshot your target. Assume that from states 1 and 2, the stone is equally likely to skip forward one, two, or three steps forward. If the stone starts from state 1, compute the probability of reaching our target before overshooting, i.e. the probability of $\{3\}$ before $\{4,5\}$.



$p(i)$ = Pr of reaching 3 from i before overshooting

$p(5) = 0 = p(4)$ $\quad p(3) = 1$ $\quad p(2) = \frac{1}{3}$

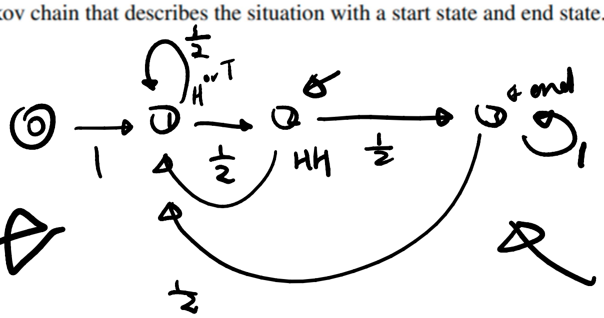
$p(1) = \frac{1}{3}p(2) + \frac{1}{3}p(3)$

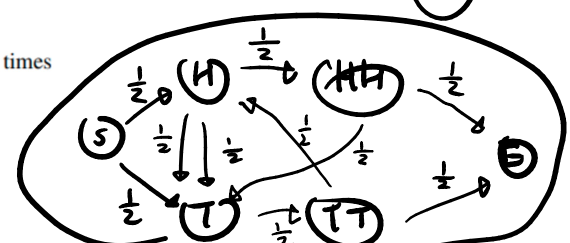
$= \frac{1}{3}\cdot\frac{1}{3} + \frac{1}{3} = \frac{4}{9}$

3 Consecutive Flips

Suppose you are flipping a fair coin (one Head and one Tail) until you get the same side 3 times (Heads, Heads, Heads) or (Tails, Tails, Tails) in a row.

(a) Construct an Markov chain that describes the situation with a start state and end state.





$Pr(2 \to 3) =$

$Pr(HHH \cup TTT \mid 2 \text{ cons. throws})$

HH or TT

$= Pr(HHH \mid 2 \text{ cons. throws}) + Pr(TTT \mid 2 \text{ cons. throws})$

$= \frac{1}{4} + \frac{1}{4} = \frac{1}{2}$

(b) Given that you have flipped a (Tails, Heads) so far, what is the expected number of flips to see the same side three times?

$p(i)$ = ex. # flip to get to 3 from i

$p(3) = 0$

$\beta(3) = 0$

$\beta(2) = 1 + \frac{1}{2}\cancel{\beta(3)}^{0} + \frac{1}{2} \times \beta(1) \Rightarrow \beta(2) = 1 + \frac{1}{2}\beta(1)$

$\beta(1) = 1 + \frac{1}{2}\beta(1) + \frac{1}{2}\beta(2) \Rightarrow \frac{1}{2}\beta(1) = 1 + \frac{1}{2}\beta(2)$

$= \frac{1}{4} + \frac{1}{4} = \frac{1}{2}$

(c) What is the expected number of flips to see the same side three times, beginning at the start state?

$\Rightarrow \beta(2) = 1 + 1 + \frac{1}{2}\beta(2)$

$\frac{1}{2}\beta(2) = 2$

$\beta(2) = 4$

$3 = \frac{1}{2}\beta(1)$

$\Rightarrow 6 = \beta(1)$

$\beta(0) = 1 + 1 \times \beta(1) = 1 + 6$

$= 7$