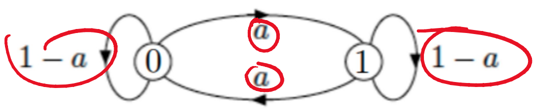
Dis 7C: Markov Chains

Wednesday, 5 August 2020 9:07 PM

How'd you explain what a Markov Chain is to a friend who's never heard of it before?

Picture:



In terms of thinking, a Markov Chain has a bunch of 'states'. "You" are the particle, and you move around the state with certain 'transition probabilities".

You also obey the Markov property: Future is independent of the past; the future depends only on the present.

Formally a Markov Chain is a sequence of random variables X_n, n = 0, 1, 2, … satisfying the Markov Property. A Markov Chain contains a sample space, a transition probability matrix and an initial distribution.

1 Markov Chain Basics

A Markov chain is a sequence of random variables X_n, $n = 0, 1, 2, \ldots$. Here is one interpretation of a Markov chain: X_n is the state of a particle at time n. At each time step, the particle can jump to another state. Formally, a Markov chain satisfies the Markov property:

$$\boxed{\mathbb{P}(X_{n+1} = j \mid X_n = i, X_{n-1} = i_{n-1}, \ldots, X_0 = i_0) = \mathbb{P}(X_{n+1} = j \mid X_n = i),} \tag{1}$$

for all n, and for all sequences of states $i_0, \ldots, i_{n-1}, i, j$. In other words, the Markov chain does not have any memory; the transition probability only depends on the current state, and not the history of states that have been visited in the past.

(a) In lecture, we learned that we can specify Markov chains by providing three ingredients: $\mathscr{X}$, P and π_0. What do these represent, and what properties must they satisfy?

(For Q1, we implicitly assume that X is a set of consecutive natural numbers starting from 0, e.g.
X = {0, 1, 2, 3, 4, …, m})

Btw, do you know what pie_n represents?

Andrew Lin 3 days ago π_i is a row vector that contains the probabilities of being in each state at time i, so $\pi_i(k)$ is the probability that we are in state k at time i. π_0 is the initial distribution of the system, giving the probabilities of being in each of the states of the Markov chain at time $t = 0$.
In terms of the random variables X_i, π_i represents the probability distribution of X_i, so $P(X_i = k) = \pi_i(k)$.
~ An instructor (Kevin Lin) thinks this is a good comment ~

undo good comment | 2

(b) If we specify $\mathscr{X}$, P, and π_0, we are implicitly defining a sequence of random variables X_n, $n = 0, 1, 2, \ldots$, that satisfies (1). Explain why this is true.

$$\pi_0 = (P(X_0=0), P(X_0=1), \ldots P_r(X_0=k))$$

$$\pi_0 = [\pi_0(0), \pi_0(1), \ldots, \pi_0(k)]$$

(c) Calculate $\mathbb{P}(X_1 = j)$ in terms of π_0 and P. Then, express your answer in matrix notation. What is the formula for $\mathbb{P}(X_n = j)$ in matrix form?

$$P(X_1=j) = (\pi_0 P)[j]$$

$$\pi_1 = \pi_0 P$$

$$P(X_n = j) = (\pi_0 P^n)[j]$$

What's one way to show that a process (that contains a sample space X and transition probability matrix P) is NOT a Markov Chain?

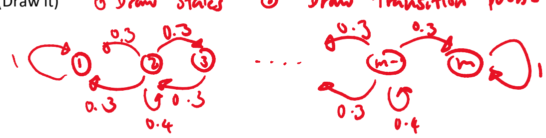
1. ~~Markov Prop~~ s

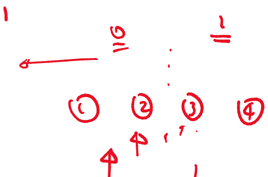
2. $\sum \neq 1$

2 Can it be a Markov Chain?

(a) A fly flies in a straight line in unit-length increments. Each second it moves to the left with probability 0.3, right with probability 0.3, and stays put with probability 0.4. There are two spiders at positions 1 and m and if the fly lands in either of those positions it is captured. Given that ~~hte~~ the fly starts between positions 1 and m, model this process as a Markov Chain.

(Draw it) ① Draw states ② Draw transition probs





(Same scenario: fly starts between 1 and m, but not on either 1 or m)

(b) Take the same scenario as in the previous part with $m = 4$. Let $Y_n = 0$ if at time n the fly is in position 1 or 2 and let $Y_n = 1$ if at time n the fly is in position 3 or 4. Is the process Y_n a Markov chain?

Prove your claim

$$\neq P(Y_{n+1} = 1 \mid Y_n = 0, Y_{n-1} = 1)$$

i.e.

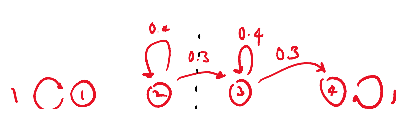
if $\Pr(Y_2 = 1 \mid Y_1 = 0, Y_0 = 1) \neq \Pr(Y_2 = 1 \mid Y_1 = 0, Y_0 = 0)$

then Markov Prop. doesn't hold

$2 \to 1$

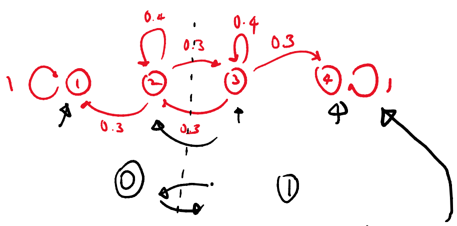
Hint: choose any π_0, & show

X_0 X_1 X_2 ...



Hint: choose any π_0, & show

$X_0\ X_1\ X_2$ = 3, 2, 3



$$LHS = \frac{Pr(Y_2=1, Y_1=0, Y_0=1)}{Pr(Y_1=0, Y_0=1)} = \frac{\frac{1}{2} \times 0.3 \times 0.3}{\frac{1}{2} \times 0.3} = 0.3$$

let $\pi_0 = [0, \frac{1}{2}, \frac{1}{2}, 0]$ if $A = B$, $Pr(A) = Pr(B)$

$X_2 = \_,\ X_1 = \_,\ X_0 = \_$

$$RHS = \frac{Pr(Y_2 = 1, Y_1 = 0, Y_0 = 0)}{Pr(Y_1=0, Y_0=0)} = \frac{Pr(X_2=3, X_1=2, X_0=2)}{Pr(X_1=2, X_0=2) + Pr(X_1=\cancel{1}, X_0=2)}$$

ϕ

$$= \frac{\cancel{\frac{1}{2}} \times 0.4 \times 0.3}{\cancel{\frac{1}{2}} \times 0.4 + \cancel{\frac{1}{2}} \times 0.3} \neq 0.3$$

$\Rightarrow$ M Prop doesn't hold...

3 Allen's Umbrella Setup

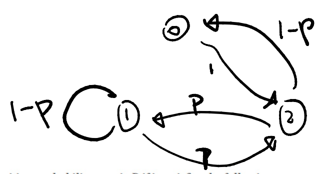
Every morning, Allen walks from his home to Soda, and every evening, Allen walks from Soda to his home. Suppose that Allen has two umbrellas in his possession, but he sometimes leaves his umbrellas behind. Specifically, before leaving from his home or Soda, he checks the weather. If it is raining outside, he will bring his umbrella (that is, if there is an umbrella where he currently is). If it is not raining outside, he will forget to bring his umbrella. Assume that the probability of rain is p.

(a) Model this as a Markov chain. What is $\mathscr{X}$? Write down the transition matrix.

Assume Allen brings at most one umbrella if it's raining and there's at least one umbrella where he currently is.

What makes sense to put as the states of the Markov chain?

# of umbrellas @ current location i.e. $\mathscr{X} = \{0, 1, 2\}$



$$[\pi(0), \pi(1), \pi(2)] \begin{bmatrix} P(0,0) & P(0,1) & P(0,2) \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \\ & & P(2,2) \end{bmatrix} = (\pi(0), \pi(1), \pi(2))$$

ϕ

3

1

$\pi(0) + \pi(1) + \pi(2) = 1$

Page 5 – 7 of notes

Definition 24.1. *A distribution π is* invariant *for the transition probability matrix P if it satisfies the following balance equations:*

$$\pi = \pi P. \qquad (10)$$

$P \times P = P^2$ P^n

(b) What is the transition matrix after 2 trips? n trips? Determine if the distribution of X_n converges to the invariant distribution, and compute the invariant distribution. ~~Determine the long-term fraction of time that Allen will walk through rain with no umbrella.~~

① Irred.

② Aperiodic $\rightarrow$ Inv.

$$P = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1-p & p \\ 1-p & p & 0 \end{bmatrix}$$

Determine if the distribution of X_n converges ... and compute the invariant distribution.

Definition 24.1. *A distribution π is* invariant *for the transition probability matrix P if it satisfies the following balance equations:*

$$\pi = \pi P. \qquad (10)$$

let $\pi = [a\ b\ c]$

$$\pi = \pi P \Rightarrow [a\ b\ c] \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1-p & p \\ 1-p & p & 0 \end{bmatrix} = [a\ b\ c]$$

(For some reason, the solutions don't answer everything after "n trips?")

Fraction of Time in States

How much time does a Markov chain spend in state i, in the long term? That is, what is the long term fraction of time that $X_n = i$? We can write this long term fraction of time as

$$\lim_{n\to\infty} \frac{1}{n} \sum_{m=0}^{n-1} 1\{X_m = i\}.$$

To understand this expression, note that $\sum_{m=0}^{n-1} 1\{X_m = i\}$ counts the number of steps among steps $\{0, 1, \ldots, n-1\}$ that $X_m = i$. Thus, $\frac{1}{n}\sum_{m=0}^{n-1} 1\{X_m = i\}$ is the fraction of time among the first n steps that $X_m = i$. By taking the limit, we obtain the long term fraction of time that $X_m = i$.

To study this long term fraction of time, we need one property: irreducibility.

Definition 24.2 (Irreducible)**.** *A Markov chain is* irreducible *if it can go from every state i to every other state j, possibly in multiple steps.*

Theorem 24.3. *Consider a finite irreducible Markov chain with state space $\mathscr{X}$ and transition probability matrix P. Then, for any initial distribution π_0,*

$$\lim_{n\to\infty} \frac{1}{n} \sum_{m=0}^{n-1} 1\{X_m = i\} = \pi(i), \forall i \in \mathscr{X}. \tag{15}$$

In (15), $\pi = \{\pi(i), i \in \mathscr{X}\}$ is an invariant distribution. Consequently, the invariant distribution exists and is unique.