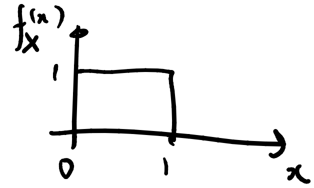
Dis 6B: Continuous Probability

Tuesday, 28 July 2020 7:24 PM

Suppose X is a **uniform continuous random variable** taking on values between **0 and 1** with uniform likelihood (think: the value the needle lands on when you spin a wheel with circumference of 1m)



Pr(X = 0.5) = 0

Note on language:

When dealing with **continuous random variables**, instead of probability mass functions [denoted P()], we use probability density functions [denoted $f_X(x)$].

Domain: $\mathbb{R}$

Range: $\mathbb{R}$

$$\boxed{\int_{-\infty}^{\infty} f(x)\, dx = 1}$$

These pdfs satisfy properties:

1. For all x, f(x) >= 0
2. Integrating f(x) from –inf to +inf (i.e across the entirety of its _________) dx = 1

a <= X <= b is an event, which is a subset of the sample space

Pr(a <= X <= b) = $\int_a^b f(x)\, dx$

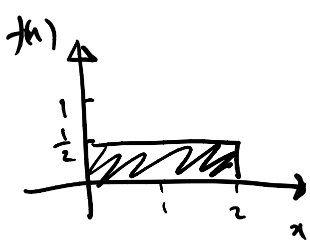
T/F: Pr (X=a)

Pr(a <= X <= b) = Pr(a < X < b)

Problem solving friend: Visualizing the integral (e.g simple continuous random variable X)

$$f(x) = \begin{cases} 0 & \forall x \in (-\infty, 0) \\ c & \forall x \in [0, 2] \\ 0 & \forall x \in (2, \infty) \end{cases}$$

(where c = ½)



Continuous case formula of E[X] = $\int_{-\infty}^{\infty} x f_X(x)\, dx$ =

$f_X(x) = f(x)$

Cumulative distribution function (cdf) of RV X, $\boxed{F(x) = P(X <= x)}$

In terms of CDFs: Pr(a <= X <= b) = F(b) - F(a)

In terms of CDFs: Pr(a <= X <= b) = $F(b) - F(a)$

Get PDF from CDF:	f(a) = $\frac{d}{dx} F(a)$
Get CDF from PDF	F(a) = $Pr(X \le a) = \int_{-\infty}^{a} f(x)\, dx$

Problem solving tip: when the **PDF is tricky to find, find the CDF** and convert the CDF to the PDF

Find the conditional PDF of continuous RV X given that event A has occurred:

$$P(A|B) = \frac{P(A \cap B)}{P(B)}$$

f_{X|A} (x) = $\begin{cases} \frac{f_X(x)}{P(A)}, & \forall x \in A \\ 0 & \forall x \notin A \end{cases}$

Consider a continuous random variable X, and an event A. Let $\mathcal{A}$ be the set of values $X(\omega)$ for all $\omega \in A$. So, $P(X \in \mathcal{A}) = P(A)$.

$$f_{X|A}(x)\delta = P(x \le X \le x + \delta | X \in \mathcal{A})$$
$$= \frac{P(x \le X \le x + \delta \cap X \in \mathcal{A})}{P(X \in \mathcal{A})} = \begin{cases} \frac{f_X(x)\delta}{P(A)} & \forall x \in \mathcal{A} \\ 0 & \text{otherwise} \end{cases}$$

(thanks Khalil)

1 Condition on an Event

The random variable X has the PDF

$$f_X(x) = \begin{cases} cx^{-2}, & \text{if } 1 \le x \le 2, \\ 0, & \text{otherwise.} \end{cases}$$

(a) Determine the value of c.

$$\int_{x=-\infty}^{x=\infty} f(x)\, dx = 1$$

$$\int_1^2 cx^{-2}\, dx = 1$$

$$c\left[-x^{-1}\right]_1^2 = 1$$

$$\Rightarrow \left(x^{-1}\right]_1^2 = -\frac{1}{c} \Rightarrow \left(\frac{1}{2} - 1\right) = -\frac{1}{c}$$

$$\Rightarrow c = 2$$

(b) Let A be the event $\{X > 1.5\}$. Calculate $\mathbb{P}(A)$ and the conditional PDF of X given that A has occurred.

$$P(A) = P(X > 1.5) = \int_{1.5}^{2} 2x^{-2}\, dx = \ldots$$

$P(A) = P(X > 1.5) = \int_{1.5} 2x^{-2}\,dx = \dots$

$f_{X|A}(x) = \dfrac{f_X(x)}{P(A)} = \dfrac{\quad}{\quad}$

2 Max of Uniforms

Let $X_1, \dots X_n$ be independent $U[0,1]$ random variables, and let $X = \max(X_1, \dots X_n)$. Compute each of the following in terms of n.

(a) What is the cdf of X?

(b) What is the pdf of X?

a) $F_X(k) = Pr(X \le k) = Pr(\max(X_1, \dots, X_n) \le k)$

$= Pr(X_1 \le k, \dots, X_n \le k) = \prod_{i=1}^{n} Pr(X_i \le k) \qquad 0 \le k \le 1$

$= k^n$

b) $f_X(x) = \frac{d}{dx} F_X(x) = \frac{d}{dx} x^n = \boxed{n x^{n-1}}$

(c) What is $\mathbb{E}[X]$?

(d) What is $Var[X]$?

c) $E(X) = \int_{-\infty}^{\infty} x f(x)\,dx = \int_0^1 n x^n = n\left[\frac{x^{n+1}}{n+1}\right]_0^1$

$= \frac{n}{n+1}(1-0) = \frac{n}{n+1}$

d) $Var(X) = E(X^2) - E^2(X) = \int_0^1 x^2\,dx - E^2(X)$

$= \int_0^1 n x^{n+1}\,dx - E^2(X) = \left(\frac{n x^{n+2}}{n+2}\right)\Big|_0^1 - E^2(X)$

$= \frac{n}{n+2} - \left(\frac{n}{n+1}\right)^2$

3 Darts but with ML

Suppose Alice and Bob are playing darts on a circular board with radius 1. When Alice throws a dart, the distance of the dart from the center is uniform $[0,1]$. When Bob throws the dart, the location of the dart is uniform over the whole board. Let X the a random variable corresponding to the distance of the player's dart from the board.

(a) What is the pdf of X if Alice throws

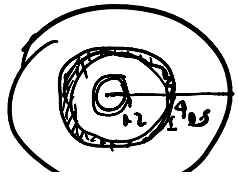
X: 0 —— 1 A

$X \sim U[a,b] \quad f_X(x) = c$

$Pr(X = 0.5)$

$\boxed{\int_a^b f_X(x)\,dx = 1}$

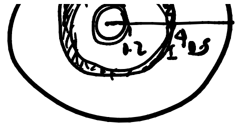
$Pr(0.5 \le X \le 0.5 + \delta x)$



B

$Pr(a) = 2\,Pr(b)$

$\Pr(0.5 \le X \le 0.5 + \delta x)$



$\Pr(a) = 2 \Pr(b)$

$f_X(x) = 1$

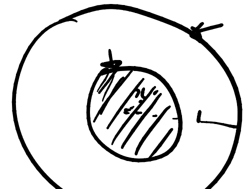
direct way on Thur lecture

(b) What is the pdf of X if Bob throws?

CDF way

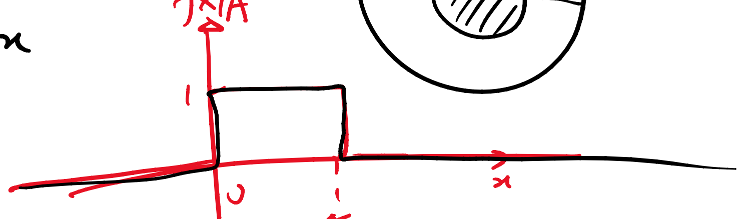
$$f_X(x) = \frac{d}{dx} F_X(x) = \frac{d}{dx} \Pr(X \le x)$$

$$= \frac{d}{dx}\left(\frac{\pi x^2}{\pi 1^2}\right) = 2x$$



$f_{X|A}$

$$f_{X|A}(x) = \begin{cases} 1 & x \in [0,1] \\ 0 & x \notin [0,1] \end{cases}$$



$f_X(x) = 1$

CDF of X is $F_X(k) = \Pr(X \le k) = \int_{-\infty}^{k} f_X(x)\, dx$

(c) Suppose we let Alice throw the dart with probability p, and let Bob throw otherwise. What is the pdf of X (your answer should be in terms of p)?

$$f_X(x) = \frac{d}{dx} F_X(x) = \frac{d}{dx} P(X \le x)$$

$f_{X|A}(x) = 1$

$f_{X|B}(x) = 2x$

$P(A) = \sum_{i=1}^{n} \Pr(A \cap B_i)$

law of total prob — General formula

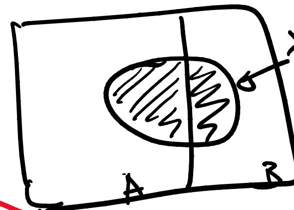
Event

$\Pr(X \le x) = \cancel{\Pr(X \le x | A \cup X | B)}$

Law of TP

$A \cup B = \Omega$

$A \cap B = \phi$



law of T.P

$$= \Pr(X \le x \wedge A) + \Pr(X \le x \wedge B)$$

$$= \Pr(X \le x | A) P(A) + \Pr(X \le x | B) \Pr(B)$$

$$= x p + \left(\int_0^x 2x\, dx\right)(1-p) = x p + (x^2)(1-p)$$

$$f_X(x) = \frac{d}{dx}$$

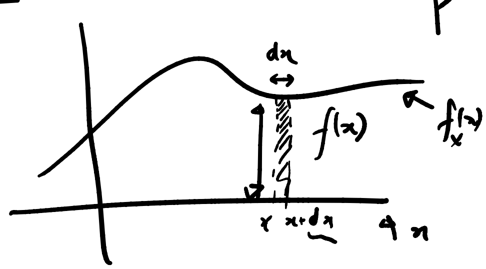
$$= p + 2x(1-p)$$

(d) Using the same premise as in part c, suppose you observe a dart on the board but don't know who threw it. Let x be the dart's distance from the center. We would like to come up with a decision rule to determine whether Alice or Bob is more likely to have thrown the dart given your observation, x. Specifically, if we let A be the event that Alice threw the dart and B be the event that Bob threw, we want to guess A if $\mathbb{P}[A|X \in [x, x+dx]] > \mathbb{P}[B|X \in [x, x+dx]]$ (what do these two probabilities have to sum up to?). For what values of p would we guess A? (your answer should be in terms of p)

$$P(X \in [x, x+dx]) \approx f_X(x)\, dx$$

$$= \Pr(x \le X \le x + dx)$$

$$= \int_x^{x+dx} f_X(x)\, dx$$



$\Pr(A|B) = \ldots$

$$\Pr(A | X \in [x, x+dx])$$

$$= \Pr(X \in [x, x+dx] | A) \Pr(A)$$

$$= \frac{\Pr(X \in (x, x+dx] \mid A)\,\Pr(A)}{\Pr(X \in (x, x+dx])}$$

rpl. A w/ B

Bayes' rule

LOTP

$$= \frac{1 \times p \,\cancel{(dx)}}{\cancel{(dx)}\,1 \times p + (2x)(1-p)\,\cancel{(dx)}} = \frac{p}{p + 2x(1-p)}$$

$$\Pr(B \mid X \in (\ldots]) = \frac{2x(1-p)}{p + 2x(1-p)}$$

Pick A if

$$\boxed{} > \boxed{}$$

i.e. $p > 2x(1-p)$

$$\Rightarrow \frac{p}{2(1-p)} > x$$