Dis 4A: Counting

Tuesday, 14 July 2020   9:13 PM

"If people do not believe that mathematics is simple, it is only because they do not realize how complicated life is."

John von Neumann

1 Clothing Argument

(a) There are four categories of clothings (shoes, trousers, shirts, hats) and we have ten distinct items in each category. How many distinct outfits are there if we wear one item of each category?

(b) How many outfits are there if we wanted to wear exactly two categories?

$$\binom{4}{2} 10 \times 10 = \binom{4}{2} \times 100 = \frac{4!}{2!\,2!} \times 100$$

$$= 3 \times 2 \times 100$$

(c) How many ways do we have of hanging four of our ten hats in a row on the wall? (Order matters.)

$$\underline{10} \times \underline{9} \times \underline{8} \times \underline{7} = \binom{10}{4} 4 \times 3 \times 2 \times 1$$

(d) We can pack four hats for travels (order doesn't matter). How many different possibilities for packing four hats are there? Can you express this number in terms of your answer to part (c)?

$$\frac{c)}{4!} \qquad \binom{10}{4} = \frac{10!}{4!\,(10-4)!}$$

$$\boxed{\binom{n}{k} = \frac{n!}{(n-k)!\,k!}}$$

$p_1, p_2, p_3, \ldots, p_n$

repr. a choice of k ppl by an ordered sequence.

$$\frac{(p_1, p_2, p_3)}{(p_1, p_3, p_2)}$$

$$\underbrace{\_\ \_\ \_\ \_\ \cdots\ \_}_{n \times (n-1) \times (n-2) \cdots (n-k+1)} = \frac{n!}{(n-k)!\,k!}$$

$v_1, v_2, \ldots, v_{10}$

$e_1 = (v_1, v_2)$

$|E| = |\text{unique vertex pairs}|$

2 Counting on Graphs + Symmetry

(a) How many distinct undirected graphs are there with n labeled vertices? Assume that there can be at most one edge between any two vertices, and there are no edges from a vertex to itself. The graphs do not have to be connected.

$$\boxed{\text{possible \# of edges}} \quad \binom{n}{2} = \frac{n!}{2!\,(n-2)!} = \frac{n(n-1)}{2}$$

$$\boxed{e_1, \ldots, e_{\binom{n}{2}}}$$

$$2 \times 2 \times 2 \times \cdots \times 2 = 2^{\binom{n}{2}}$$

Dealing with rotating (and inverting) objects

e.g cycles in a graph, bracelet colorings, rubik's cube colorings

Draw a LINE

Draw a LINE

b_1, b_2, b_3, b_4

1. Label 'beads': b_1, b_2, b_3, …, b_n
2. Represent one object via **ordered sequence** of the 'beads'
3. Divide by *repetitions (*# sequences representing same object)

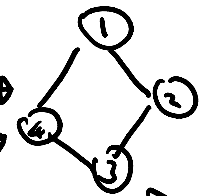
(c) How many ways are there to color a bracelet with n beads using n colors, such that each bead has a different color? Note: two colorings are considered the same if one of them can be obtained by rotating the other.

1, 2, 3, 4

2, 3, 4, 1

3, 4, 1, 2

4, 1, 2, 3



$$\frac{n!}{n}$$

Visual = Mapping from M sequences to one object

(b) How many distinct cycles are there in a complete graph K_n with n vertices? Assume that cycles cannot have duplicated edges. Two cycles are considered the same if they are rotations or inversions of each other (e.g. (v_1, v_2, v_3, v_1), (v_2, v_3, v_1, v_2) and (v_1, v_3, v_2, v_1) all count as the same cycle).

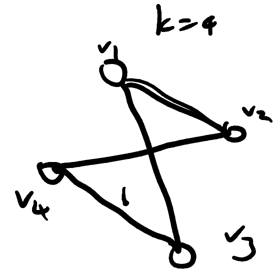
Hint: how many distinct cycles are there of length k in such a K_n?

$$\boxed{\binom{n}{k} \frac{k!}{2k}}$$

reps = rotations & inversion

$$\sum_{k=3}^{n} \binom{n}{k} \frac{k!}{2k}$$

k=4



v_1, v_2, v_3, v_4

2, 3, 4, 1

|||

3, 4, 1, 2

&

1, 3, 4, 2 + 1, 2, 3, 4

4, 3, 2, 1

(d) How many ways are there to color the faces of a cube using exactly 6 colors, such that each face has a different color? Note: two colorings are considered the same if one can be obtained from the other by rotating the cube in any way.

$f_1, f_2, \ldots f_6$

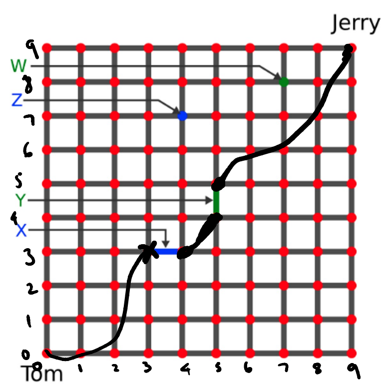
$$\# \text{ colorings} = \frac{① \text{ total \# of ordered sequences}}{② \text{ 'repetitions'}}$$

(each ordered sequence represents an object)

$$= \frac{6!}{24}$$

3 Maze

Let's assume that Tom is located at the bottom left corner of the 9×9 maze below, and Jerry is located at the top right corner. Tom of course wants to get to Jerry by the shortest path possible.



ways $(0,0) \to (3,3)$ × # ways $(4,3) \to (9,9)$

$\binom{6}{3} + \binom{11}{6}$

9U, 9R

(a) How many such shortest paths exist?

Hint: How many 'up's & 'right's are there in a shortest path?

Hint: draw _ _ _ _ _ ...... _ _ _ _ _, where each _ represents Tom's 'move' up or 'move' right

U _ _ U . . . . . . U . . . R R

1 2 3 17 18

$$\binom{18}{9} = \frac{18!}{9!\,9!}$$

U U R R R U

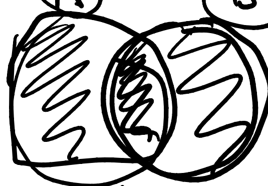
(b) How many shortest paths pass through the edge labeled X? The edge labeled Y? Both the edges X and Y? Neither edge X nor edge Y?

Hint: split the journey up and find number of paths for each section of the journey
Hint: (Neither edge X nor edge Y) Inclusion-Exclusion

Counting cardinality of sets

$|T| - |X \cup Y|$

A B



$A \cap B$

$|A \cup B| = |A| + |B| - |A \cap B|$