Dis 6C: Continuous Probability Distributions

Wednesday, 29 July 2020 8:05 PM

Solve:

$Pr(X > 20 \mid X > 10) = Pr(X > 10)$

Given X ~ Geo(p) i.e. X is a geometric random variable, where the probability of success for each independent Bernoulli trial is p (think: # independent coin flips until first success)

Pr(X < 20 | X > 15) = $1 - Pr(X \geq 20 \mid X > 15)$

$= 1 - Pr(X > 19 \mid X > 15)$

$= 1 - Pr(X > 4)$

$= 1 - (1-p)^4$

Memoryless Property

Just like the geometric distribution, the Exponential distribution exhibits the memoryless property. Let $X \sim Exp(\lambda)$, then $P(X > x + t \mid X > t) = P(X > x)$.

Proof:

$$P(X > x + t \mid X > t) = \frac{P(X > x + t \cap X > t)}{P(X > t)} = \frac{P(X > x + t)}{P(X > t)} = \frac{e^{-\lambda(x+t)}}{e^{-\lambda t}} = e^{-\lambda x} = P(X > x)$$

Thanks, Khalil

Exponential RV's
= RV for "time (in days / hours / mins) until something happens"

(Geometric RV's continuous friend)

Anything confusing about this slide?

Continuous Analog of Geometric

Let $X \sim Exp(\lambda)$, where X is the number of seconds we have to wait.

Then $P(X > x) = e^{-\lambda x}$. This is the probability we have to wait at least x seconds.

We can consider a discrete time setting, in which we perform 1 trial every δ seconds (then we can make $\delta \to 0$ to get a continuous setting). Here we can say our success probability for a trial is $p = \lambda * \delta$. This makes sense since λ can be interpreted as a rate of success per unit time ($\lambda = \frac{p}{\delta}$). In the discrete case, the number of trials until first success is $Y \sim Geom(p)$.

Y is the time until 1st success

$$P(Y > k\delta) = (1-p)^k = (1 - \lambda\delta)^k$$

If we switch to time instead of trials via $t = k\delta$, we get:

$$P(Y > t) = P(Y > (\tfrac{t}{\delta})\delta) = (1 - \lambda\delta)^{\frac{t}{\delta}} \approx e^{-\lambda t}$$

as $\delta \to 0$.

How would you derive the PDF of the exponential RV Y, given P(Y > t)?

$= 1 - Pr(Y \leq t)$

$f_Y(x) = \frac{d}{dx} F_Y(x)$

Integrate e^(kx) dx: $\int e^{kx}\,dx$

$= \frac{e^{kx}}{k} + C$

$\Rightarrow F_Y(t) = 1 - e^{-\lambda t}$

$f_Y(t) = \lambda e^{-\lambda t}$

Normal RVs:

For any $\mu \in \mathbb{R}$ and $\sigma > 0$, a continuous random variable X with pdf

$$f(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$$

is called a normal random variable with mean parameter μ and variance σ^2, and we write $\mathcal{N}(\mu, \sigma^2)$

Thanks again

1 Exponential Distributions: Lightbulbs

A brand new lightbulb has just been installed in our classroom, and you know the life span of a lightbulb is exponentially distributed with a mean of 50 days.

(a) Suppose an electrician is scheduled to check on the lightbulb in 30 days and replace it if it is broken. What is the probability that the electrician will find the bulb broken?

Let X be the RV for the ______________________________

$X \sim \mathrm{Exp}(\frac{1}{50})$

$$P(X < 30) = \int_0^{30} f(x)\, dx$$

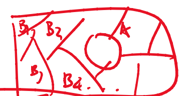
$$= \int_0^{30} \lambda e^{-\lambda x}\, dx = \int_0^{30} \frac{1}{50} e^{-\frac{30}{50}}\, dx$$

(b) Suppose the electrician finds the bulb broken and replaces it with a new one. What is the probability that the new bulb will last at least 30 days?

$1 - a)$

$=$

Pr



(c) Suppose the electrician finds the bulb in working condition and leaves. What is the probability that the bulb will last at least another 30 days?

$$\Pr(X \geq 60 \mid X \geq 30) = \Pr(X \geq 30)$$

$$\Pr(A) = \int_{x=-\infty}^{x=\infty} \Pr(A \cap X = x)$$

discrete

How would you apply the law of total probability to Pr(A) using a continuous random variable X? $X \in [-\infty, \infty]$

I.e. Pr(A) = ______________________________

$$\int_{-\infty}^{\infty} \Pr(A \mid X = x)\, f_X(x)\, dx \qquad x \in [x, x+\delta]$$

$$\Pr(A) = \sum_{i=1}^{n} \Pr(A \cap B_i) = \sum_{i=1}^{n} \Pr(A \mid B_i) \Pr(B_i)$$

$$= \int_{-\infty}^{\infty} \Pr(A \mid X = x) \Pr(X = x) = \int_{-\infty}^{\infty} f_X(x)\, dx$$

0

Law of Total Probability:

$$P(A) = \int_{-\infty}^{\infty} P(A|X = x) f_X(x)\, dx \qquad (5.16)$$

$$E[X] = E[E[X \mid Y]]$$

$$P(A) = E[\mathbb{1}(A)]$$

A / A^c

$$I(A) = \begin{cases} 1, & \text{if } A \\ 0, & \text{if } A^c \end{cases}$$

$$= 0 \times \Pr(I(A) = 0) + 1 \times \Pr(I(A) = 1)$$

$$= \Pr(\mathbb{1}(A) = 1) = \Pr(A)$$

$P(X<Y) =$

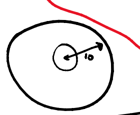
2 Darts Again

Edward and Khalil are playing darts.

Edward's throws are uniformily distributed over the entire dartboard, which has a radius of 10 inches. Khalil has good aim; the distance of his throws from the center of the dartboard follows an exponential distribution with parameter 1/2.

Say that Edward and Khalil both throw one dart at the dartboard. Let X be the distance of Edward's dart from the center, and Y be the distance of Khalil's dart from the center of the dartboard. What is $\mathbb{P}(X < Y)$, the probability that Edward's throw is closer to the center of the board than Khalil's? Leave your answer in terms of an unevaluated integral.

[*Hint:* X is not uniform over $[0, 10]$. Solve for the distribution of X by first computing the CDF of X, $\mathbb{P}(X < x)$.]



$X \sim$

$\Pr(X < x)$

$\boxed{Y \sim \mathrm{Exp}(\frac{1}{2})} \Rightarrow f_Y(x)\ \&\ F_Y(x)$

$$\Pr(X < x) = \boxed{F_X(x) = \frac{x^2}{100}}$$

$$\boxed{P(X < Y)} = \int_0^{10} \boxed{P(X < Y \mid X = x)} f_X(x)\, dx \qquad f_X(x) = \frac{x}{50}$$

$$= \int_0^{10} \boxed{P(Y > x)} f_X(x)\, dx$$

$$= \int_0^{10} (e^{-\lambda x}) \frac{x}{50}\, dx$$

$$P(A) = E[I(A)] \qquad \text{RV}$$

$$= E[E[I(A) \mid Y]]$$

$$= E[g(y)]$$

$$= \int_{y=-\infty}^{y=\infty} g(y)\, dy$$

$$P(X<Y) = \int_0^{\infty} \Pr(X < Y \mid Y = y) f_Y(y)\, dy$$

$$= \int_0^{\infty} \boxed{\Pr(X < y)} \lambda e^{-\lambda y}\, dy$$

$$= \int_0^{\infty} \frac{y^2}{100} \lambda e^{-\lambda y}\, dy$$

WTS

$$E[I(A) \mid Y] = \boxed{g(y) = P(A \mid Y = y) f_Y(y)}$$

3 Why Is It Gaussian?

Let X be a normally distributed random variable with mean μ and variance σ^2. Let $Y = aX + b$, where $a > 0$ and b are non-zero real numbers. Show explicitly that Y is normally distributed with

$$- \left(\int_0 \frac{y}{100} \lambda e \; dy \right)$$

$$E[I(A) \mid Y] =$$

3 Why Is It Gaussian?

Let X be a normally distributed random variable with mean μ and variance σ^2. Let $Y = aX + b$, where $a > 0$ and b are non-zero real numbers. Show explicitly that Y is normally distributed with mean $a\mu + b$ and variance $a^2\sigma^2$. The PDF for the Gaussian Distribution is $\frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$. One approach is to start with the cumulative distribution function of Y and use it to derive the probability density function of Y.

[1.You can use without proof that the pdf for any gaussian with mean and sd is given by the formula $\frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$ where μ is the mean value for X and σ^2 is the variance. 2. The drivative of CDF gives PDF.]

$$\boxed{\begin{aligned} E[Y] &= aE(X) + b \\ &= a\mu + b \\ Var(Y) &= a^2\sigma^2 \end{aligned}}$$

$$\lambda \quad = \quad \int_{-\infty} \quad dy$$

P $F_Y(x) = Pr(Y < x) = Pr(aX + b < x)$ $\qquad \frac{\mu}{\sigma^2}$

$= Pr\left(X < \frac{x-b}{a}\right)$ $\qquad \boxed{Z = \frac{X-\mu}{\sigma}}$ use

$= Pr\left(\sigma Z + \mu < \frac{x-b}{a}\right)$ $\qquad X = \sigma Z + \mu$

$= Pr(a\sigma Z + \mu a + b < x)$

$Y = \boxed{a\sigma Z + \mu a + b}$

$E(Y) = E(\cdots)$

$$f_Y(x) = \frac{d}{dx} \boxed{F_Y(x)}$$

$$\boxed{F_Y(x)} = Pr(Y < x) = Pr\left(X < \frac{x-b}{a}\right) = \boxed{F_X\left(\frac{x-b}{a}\right)}$$

$$\Rightarrow f_Y(x) = \frac{d}{dx} F_X\left(\frac{x-b}{a}\right)$$

$$= \frac{1}{a} f_X\left(\frac{x-b}{a}\right) \qquad X \sim N(\mu, \sigma^2)$$

$$= \frac{1}{a} \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{\left(\frac{x-b}{a} - \mu\right)^2}{2\sigma^2}}$$

$$= \frac{1}{\sqrt{2\pi a^2\sigma^2}} e^{-\frac{1}{2\sigma^2}\left(\frac{x-b-a\mu}{a}\right)^2}$$

$$= \frac{1}{\sqrt{2\pi a^2\sigma^2}} e^{-\frac{(x-(b+a\mu))^2}{2\sigma^2 a^2}} \qquad \mu$$

σ^2

$$\Rightarrow \boxed{Y \sim N(b + a\mu, \sigma^2 a^2)}$$