Dis 4B: Counting II (Combinatorial Proofs, Stars and Bars, Inclusion-Exclusion

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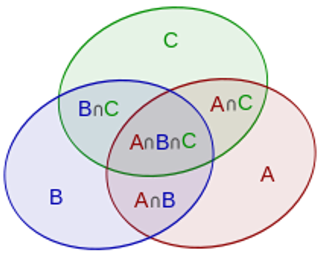
Principle of Inclusion - Exclusion

You have 3 finite sets A, B, C. You want to find the number of elements in A or B or C.
Visually, this is the total number of elements in the circles below:



$$|A \cap B \cap C| = |A| + |B| + |C| - |A \cap B| - |B \cap C| - |C \cap A| + |A \cap B \cap C|$$

Theorem 11.3 (Inclusion-Exclusion). *Let $A_1, \dots, A_n$ be arbitrary subsets of the same finite set A. Then,*

$$|A_1 \cup \cdots \cup A_n| = \sum_{k=1}^{n} (-1)^{k-1} \sum_{S \subseteq \{1,\dots,n\}:|S|=k} |\cap_{i \in S} A_i|. \tag{5}$$

Remarks:

1. The inner summation in (5) is over all size-k subsets of $\{1,2,\dots,n\}$. More explicitly, (5) can be written as

$$|A_1 \cup \cdots \cup A_n| = \sum_{i=1}^{n} |A_i| - \sum_{i<j} |A_i \cap A_j| + \sum_{i<j<k} |A_i \cap A_j \cap A_k| - \sum_{i<j<k<l} |A_i \cap A_j \cap A_k \cap A_l| + \cdots + (-1)^{n-1} |A_1 \cap A_2 \cap \cdots \cap A_n|, \tag{6}$$

where $\sum_{i<j}$ denotes summing over all $i, j \in \{1,\dots,n\}$ such that $i < j$, and so on.

1 The Count

(a) How many of the first 100 positive integers are divisible by 2, 3, or 5?

(Skip for now)

Visualization (?)

3 Bit String

How many bit strings of length 10 contain at least five consecutive 0's?

Hint:

$(A = B)$ $\left(\binom{n}{k} = \binom{n}{n-k}\right)$

Hint:

$A = B$ $\binom{n}{k} = \binom{n}{n-k}$

General Procedure:

1. **Make up a 'story' for the quantity we're counting.**
Sometimes the problem gives this to us, e.g. Kevin would like to select directors for his musical.
Usually, one side of the identity is obvious, e.g. $\binom{n}{2}$ means Kevin would like to select 2 directors for his

Common quantities:

- Number of ways to select k people from a group of n i.e. $\binom{n}{k}$
- Number of ways to assign n people to k groups i.e k^n

2. **Count the quantity using another method**

Strategies:

a) Can I interpret the algebra of the 'non-obvious' side to give me a hint?

b) Can I count this quantity from another **point of view**?

- e.g. rather than taking the perspective of the groups, take the perspective of the people (and vice

c) Can I consider an arbitrary "first" element, then consider the rest?

- Nice way to do this is to DRAW A LINE

d) Can I switch between equivalent algebraic forms to make the expression more interpretable?

e) Can I split the quantity into disjoint cases?

[NB these strategies are often related]

Interpreting algebra:

- Sum: breaking up quantity into disjoint (i.e. non-overlapping) **cases**

e.g.

(Let $Cyc(k,n)$ denote the number of distinct cycles of length k in a complete graph K_n)

Number of distinct cycles in complete graph K_n

A = $Cyc(3,n) + Cyc(4,n) + \ldots + Cyc(n,n)$ B

= $\sum_{k=3}^{n} Cyc(k,n)$

e.g.

When $n = 4$ in the previous setting, our sum has two elements. Same concept.

Number of distinct cycles in complete graph K_4 = $Cyc(3,4) + Cyc(4,4)$

- Nested sums: Fixing one quantity, then considering another e.g. $\sum_{i=0}^{n}\sum_{j=0}^{n} i + j$
- Subtraction in choosing: denoting that we're choosing over the elements that "remain" e.g. $\binom{n-i}{k}$
- Products: ways to make **sequential choices** $x_1 x_2 \ldots x_n$ $2 * \binom{n}{k} = \binom{n}{k} + \binom{n}{k}$
- Exponentials: same as a product, except the ways to make each choice remain constant e.g. x^5

2 CS70: The Musical

Edward, one of the previous head TA's, has been hard at work on his latest project, *CS70: The Musical*. It's now time for him to select a cast, crew, and directing team to help him make his dream a reality.

(a) First, Edward would like to select directors for his musical. He has received applications from $2n$ directors. Use this to provide a combinatorial argument that proves the following identity:

$$\binom{2n}{2} = 2\binom{n}{2} + n^2 = \binom{n}{2} + \binom{n}{2} + n^2$$

1. Make up a story for the quantity: . . . . words
2. Count Quantity using another method:
   - To help us do this, interpret algebra on 'non-obvious' side
   Hint: *DRAW A LINE* (of ______)

Divide quantity into 3 disjoint cases: ①, ②, ③ ... $= n^2 + 2\binom{n}{2}$

A | B

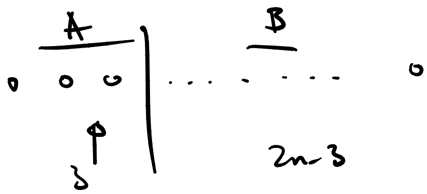
o o o o . . . . . | . . . . o o o

choose 2 from A + B:
① 1 from A, 1 from B $n * n$
② 2 from B $\binom{n}{2}$
③ 2 from A $\binom{n}{2}$

   - Alternative way to count quantity:

→ ③ 2 from A $\left(\binom{n}{2}\right)$

a') Prove the identity: $\binom{2n}{2} = \binom{3}{2} + \binom{2n-3}{2} + 3(2n-3)$



(b) Edward would now like to select a crew out of n people, Use this to provide a combinatorial argument that proves the following identity: $\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}$ (this is called pascal's identity)

1. Make up a story for the quantity:
2. Count Quantity using another method:
   - To help us do this, interpret algebra on 'non-obvious' side
     Hint: *DRAW A LINE* (of ______)

$\binom{n-1}{k}$

- Alternative way to count quantity:

(first person)

2 cases: i) don't choose person n for our group of k

ii) do " " " " "

$+$

$\binom{n-1}{k-1}$

(c) There are n actors lined up outside of Edward's office, and they would like a role in the musical (including a lead role). However, he is unsure of how many individuals he would like to cast. Use this to provide a combinatorial argument that proves the following identity: $\sum_{k=1}^{n} k\binom{n}{k} =$ → |ppl in the group|

$n2^{n-1}$ "yes" or "no"

What's the number of lead roles he'd like to cast? 1

What's the minimum number of individuals he'd like to cast? ≥ 1

1. Make up a story for the quantity: # of ways to choose a group of people where 1 of the group is a lead actor. (from a total of n ppl).
2. Count Quantity using another method:
   - To help us do this, interpret algebra on 'non-obvious' side
     Hint: *DRAW A LINE* (of ______)

→ ① Sum across all possible values for the

Hint: DRAW A LINE (or ________)

→ ① Sum across all possible values for the # of people in the group chosen.

- Alternative way to count quantity:

ⓚ $\binom{n}{k}$ · need to choose the 'lead'

$\binom{k}{1}$

$\sum_{k=1}^{n}\binom{n}{k}\cdot\binom{k}{1}$

② 1st choose lead role, then ~~determ~~ count the ways to pick the 'rest'

(d) Generalizing the previous part, provide a combinatorial argument that proves the following identity: $\sum_{k=j}^{n}\binom{n}{k}\binom{j}{k} = 2^{n-j}\binom{n}{j}$.

j leads

① find the group of actors first, then find leads

② Find the j leads, then find remaining

c) Abhi's choosing a 7 digit phone number and he wants the digits of his number to be strictly decreasing. How many choices for a new phone number does he have?

Hint: what does Abhi know about the digits of his number given they're strictly decreasing?

3, 7, 5, 2, 1, 6, 8 $\Leftrightarrow$ 8 7 6 5 3 2 1

bijection

a choice of 7 distinct numbes

digit satisfying strictly decreasing property

3 cases: start of 9, 8,

7: one way

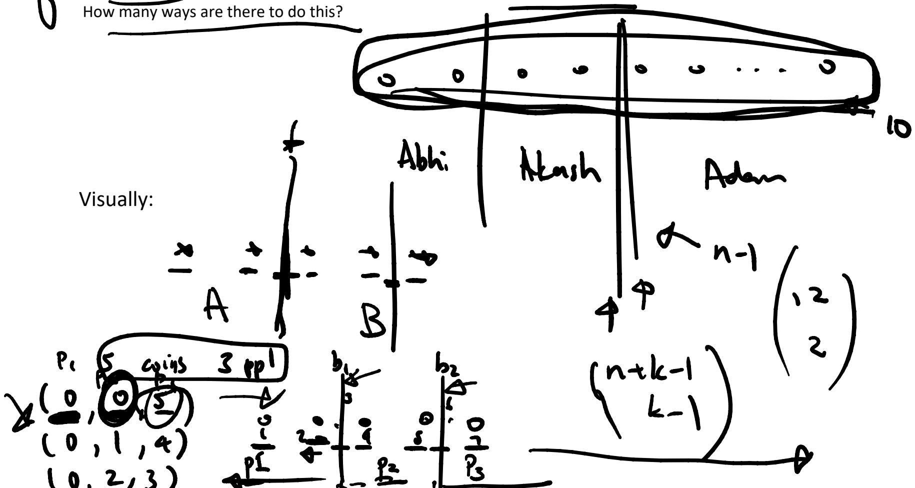
8: ways to 'drop' a digit (6 places)

9: 2 drops or one 'double' drop in $\boxed{9.65432}$

6 of these

b') Abhi, Akash, Adam want to share 10 Kevins (suppose the Kevins are identical) among themselves. How many ways are there to do this?

Visually:



(b) The Count is trying to choose his new 7-digit phone number. Since he is picky about his numbers, he wants it to have the property that the digits are non-increasing when read from left to right. For example, 9973220 is a valid phone number, but 9876545 is not. How many choices for a new phone number does he have?

Hint: for any choice of 7 numbers, how many permutations satisfy the property that the digits are non-increasing?

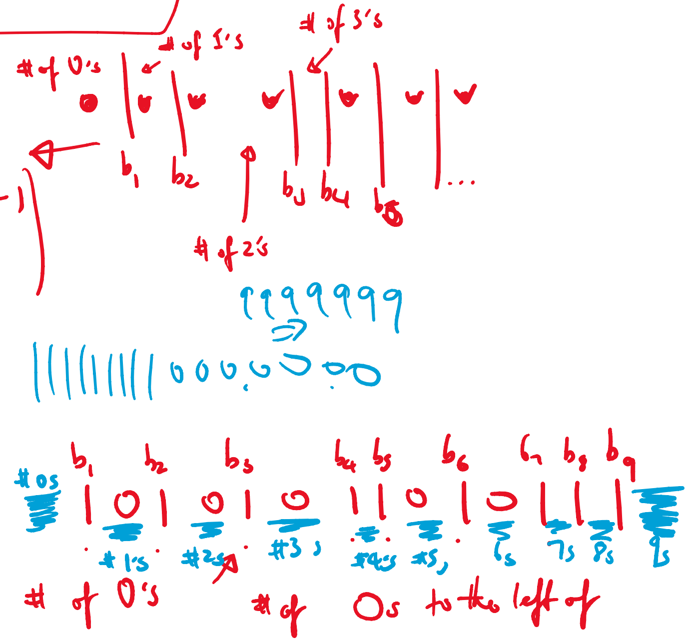
1 1 1 2 3 4 3

specifying the ... ⟷ a 7-digit non-↓ing

Hint: for any choice of 7 numbers, how many permutations satisfy the property that the digits are non-increasing?

specifying the # of times each digit (0-9) appears $\Leftrightarrow$ a 7-digit non-↑ing number

	0	1	2	3	....	9
# of times						



$$\binom{10+7-1}{7}$$

☆ & bars

5 coins & 3 ppl. Find # of ways to split.

○ ○ ○ ○ ○

this diagram $\longleftrightarrow$ one way of splitting the coins

O | O O | O O $\Leftrightarrow$

A	B	C
1	2	2

O | | O O O O $\Leftrightarrow$

A	B	C
1	0	4

Q: 2 coins, 51 people. Find # of ways to split the coins.