Dis 11 ER + Inference

Wednesday, 7 April 2021 9:17 PM

Estimation methods (~~MAP, MLE~~)

Motivating example: Good doctor

A patient has symptom s. Being a good doctor, you want to infer the most probable cause c. (Suppose symptoms come from RV S and causes come from RV C. That is, patient has symptom S = s and cause C = c)

Mathematically, you'd want to find:

$\boxed{c^* = \arg\max_c \Pr(C=c \mid S=s)} = \arg\max_c \left(\frac{\Pr(S=s \mid C=c)\Pr(C=c)}{\cancel{\Pr(S=s)}} \right)$

$\boxed{MAP[C \mid S=s] = c^*}$

$= \arg\max_c \Pr(S=s \mid C=c)\Pr(C=c)$ 'prior'

from med. school books

past data

c = 'cold' up 90%

solve

$\Rightarrow \frac{d}{dc}[\quad] = 0$

sometimes, $\Pr(C=c \mid S=s)$ is PDF w/ form $e^{-\lambda x}$

to deal w/ exp, use $\ln(\cdot)$ ← useful trick

$= \arg\max_c \ln(\Pr(S=s \mid C=c)\Pr(C=c))$

"have more info"

MAP

situation, where we don't know anything about c

uniform

$c^* = \arg\max_c \Pr(C=c \mid S=s)$

$= \arg\max_c \frac{\Pr(S=s \mid C=c)\cancel{\Pr(C=c)}}{\cancel{\Pr(S=s)}}$

$= \arg\max_c \Pr(S=s \mid C=c)$

MLE

if we didn't know the $\Pr(C=c)$

$c^* = \arg\max_c \Pr(S=s \mid C=c)\cancel{\Pr(C=c)} = k$ (uniform)

cold, c2, c3,

$= \arg\max_c \Pr(S=s \mid C=c)$

3. MLE & MAP for Continuous Random Variables

Let X be an exponentially distributed random variable with mean 1. Given X, the random variable Y is exponentially distributed with rate X.

(a) Find MLE$[X \mid Y]$.

(b) Find MAP$[X \mid Y]$.

a) $MLE(X \mid Y=y) = \frac{1}{y}$

$x^* = \arg\max_x \cancel{\Pr(Y=y \mid X=x)}$

$= \arg\max_x x e^{-yx}$

$f_{Y|X}(y \mid x)$

$\sim Exp(x)$

$= \underset{x}{\arg\max} \; x e^{-yx}$

$= \underset{x}{\arg\max} \; \ln (x e^{-yx})$

$= \underset{x}{\arg\max} \; (\ln x) + (-yx)$

$\frac{d}{dx} \ln x - yx = 0 \Rightarrow \frac{1}{x} - y = 0$

$x = \frac{1}{y}$

$\text{MLE}[X|Y] = \frac{1}{Y}$

b) $x^* = \text{MAP}[X|Y=y]$

$= \underset{x}{\arg\max} \; f_{Y|X}(y|x) \, f_X(x) \sim \text{Exp}(1)$

$= \underset{x}{\arg\max} \; x e^{-yx} \cdot 1 e^{-1x}$

$= \underset{x}{\arg\max} \; (\ln x) + (-yx) - x$

$\frac{d}{dx}(\ln x - yx - x) = 0 \Rightarrow \frac{1}{x} - y - 1 = 0$

$\Rightarrow x^* = \frac{1}{y+1}$

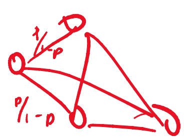
$\Rightarrow \text{MAP}[X|Y] = \frac{1}{Y+1}$

1. Generating Erdös-Renyi Random Graphs

True/False: Let G_1 and G_2 be independent Erdös-Renyi random graphs on n vertices with probabilities p_1 and p_2, respectively. Let $G = G_1 \cup G_2$, that is, G is generated by combining the edges from G_1 and G_2. Then, G is an Erdös-Renyi random graph on n vertices with probability $p_1 + p_2$. assume $p_1 + p_2 \leq 1$

ER graph $G(n,p)$

edges $e \in E$



each edge e exist w.p. p

doesn't exist w.p. $1-p$

$\binom{n}{2}$ possible edges

$\forall e' \in E(G)$ $\quad$ e' doesn't exist $\equiv$ e' not in $G_1 \wedge$ e' not in G_2

$\Rightarrow P(e' \text{ doesn't exist}) = (1-p_1)(1-p_2)$

if G were ER $G(n, p_1+p_2)$ $\quad \neq \quad 1-p_1-p_2$

$\forall e \in E(G), P(e \text{ not existing}) = 1-p_1-p_2$

$P(e' \text{ exist}) = \square \neq p_1+p_2 \Rightarrow G \neq \text{E.R}$

2. Random Graph

Consider a random undirected graph on n vertices, where each of the $\binom{n}{2}$ possible edges is present with probality p independently of all the other edges (i.e. an Erdös-Renyi random graph). If $p = 0$ we have a fully empty graph with n completely disconnected vertices; in contrast, if $p = 1$, every edge exists, the network is an n-clique, and every vertex is a distance one from every other vertex.

(a) Fix a particular vertex of the graph, and let D be a random variable which is equal to the degree of this vertex. What is the PMF of D? Calculate $\lambda \triangleq \mathbb{E}[D]$.

(b) Assume that $c = np$ is a constant, independent of n. For large values of n, how you would approximate the PMF of D?

a) $D \sim \text{Bin}(n-1, p) \Rightarrow \lambda = E(D) = (n-1)p$

b) $D \sim \text{Poi}(\underbrace{c}_{=(n-1)p})$ $\qquad \lim_{n\to\infty} \frac{pn}{n} = c$

$c^* = \text{most probable}_{c}\ \Pr(C=c \mid S=s)$

$= \arg\max_{c} \Pr(C=c \mid S=s)$

$= \arg\max_{c} \frac{\Pr(S=s \mid C=c)\Pr(C=c)}{\cancel{\Pr(S=s)}}$

Motivating example: Good doctor

$= \arg\max_{c} \Pr(S=s \mid C=c)\Pr(C=c)$

A patient has symptom s. Being a good doctor, you want to infer the most probable cause c. (Suppose symptoms come from RV S and causes come from RV C. That is, patient has symptom S = s, and cause C = c)

Mathematically, you'd want to find:

$\frac{d}{dc}(\quad) = 0$ solve for c

called MAP . . .

$c^* = \text{MAP}(C \mid S=s)$

called MAP . . .

$c^* = \underline{\mathrm{MAP}(C \mid S=s)}$

<u>new situation</u>

what if don't know $\underline{\Pr(C=c)}$

assume $\Pr(C=c)$ is <u>uniform</u>

$$c^* = \underset{c}{\operatorname{argmax}} \; \underline{\Pr(S=s \mid C=c)\,\cancel{\Pr(C=c)}}$$
$$\qquad\qquad\qquad\cancel{\Pr(S=s)}$$

$$= \underset{c}{\operatorname{argmax}} \; \Pr(S=s \mid C=c)$$

$\mathrm{MLE}(C \mid S=s)$