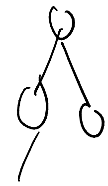
Dis2

Thursday, 28 January 2021 1:20 PM

1. Suspicious Game

You are playing a card game with your friend: you take turns picking a card from a deck (you may assume that you never run out of cards). If you draw one of the special "bullet" cards, then you lose the game. Unfortunately, you do not know the contents of the deck. Your friend claims that 1/3 of the deck is filled with "bullet" cards. You don't trust your friend fully, however: you believe that he is lying with probability 1/4. You assume that if your friend is lying, then the opposite is true: 2/3 of the deck is filled with "bullet" cards!

What is the probability that you win the game if you go first?



$$P(W) = P(W \cap ly) + P(W \cap ly^c)$$

$$P(W \cap B_2) + P(W \cap B_4) + P(W \cap B_6) + \dots \dots$$

$$= \boxed{P(W \mid ly)}\, P(ly)^{=\frac{1}{4}} + \boxed{P(W \mid ly^c)}\, P(ly^c)^{=\frac{3}{4}}$$

$$= P(W \mid P(\text{bullet}) = \tfrac{2}{3})\, \tfrac{1}{4} + P(W \mid P(\text{bullet}) = \tfrac{1}{3})\, \tfrac{3}{4}$$

want $\boxed{P(W \mid p(\text{bullet}) = q)}$

geometric series

$$\sum_{i=0}^{\infty} ar^i = \frac{a}{1-r}$$

$$= P(W \cap \text{bullet is drawn on an even turn})$$

$$= (1-q)q + (1-q)^3 q + (1-q)^5 q + (1-q)^7 q + \dots$$

P(bullet is 2^{nd} turn) bullet @ 4^{th} turn 6^{th} 8^{th}

$$= q \sum_{i=0}^{\infty} (1-q)^{2i+1} = q(1-q) \sum_{i=0}^{n} (1-q)^{2i} = q(1-q) \times \frac{1}{1-(1-q)^2}$$

$$= q(1-q) \times \frac{1}{1-(1-2q+q^2)}$$

$$= q(1-q) \times \frac{1}{2q-q^2} = \frac{q(1-q)}{q(2-q)} = \frac{1-q}{2-q}$$

$$\left(\frac{1-\frac{2}{3}}{2-\frac{2}{3}}\right)\frac{1}{4} + \left(\frac{1-\frac{1}{3}}{2-\frac{1}{3}}\right)\frac{3}{4} = \dots$$

2. Packet Routing

Consider a system with n inputs and n outputs. At each input, a packet appears independently with probability p. If a packet appears, it is destined for one of the n outputs uniformly randomly, independently of the other packets.

(a) Let X_1 denote the number of packets destined for the first output. What is the distribution of X_1?

(b) A collision happens when two or more packets are destined for the same output. What is the expected number of total collisions C?

a) $X_1 \sim \text{Bin}\left(n, \frac{p}{n}\right)$

\# of trials pr of "success"

\# of trials   pr of "success"

a)

$P($ i^{th} input goes into 1^{st} output$)$ = $P($ packet appears $\cap$ packet goes to 1^{st} output$)$

$= P($goes to 1^{st} output $|$ pkt appears$)$ $P($pkt appears$)$

$= \frac{1}{n} \times p = \frac{p}{n}$

b) $I_i :=$ indicator for if there's a collision on the i^{th} output

$C = I_1 + \cdots + I_n$

$E[C] = \cdots = n E[I_1] = n \times p(I_1 = 1)$ there is a collision on i^{th} output

$= n(1 - p(\text{no collisions on } 1^{st})) = n(1 - p($one pkt on 1^{st} output or zero pkts on 1^{st} output$))$

$= n(1 - p(X_1 = 0) - p(X_1 = 1)) = n\left(1 - \binom{n}{0}\left(\frac{p}{n}\right)^0\left(1-\frac{p}{n}\right)^n - \binom{n}{1}\left(\frac{p}{n}\right)^1\left(1-\frac{p}{n}\right)^{n-1}\right)$

$= n\left(1 - \left(1-\frac{p}{n}\right)^n - p\left(1-\frac{p}{n}\right)^{n-1}\right)$

3. Sampling without Replacement

Suppose you have N items, G of which are good and B of which are bad (B, G, and N are positive integers, $B + G = N$). You start to draw items without replacement, and suppose that the first good item appears on draw X. Compute $\mathbb{E}[X]$ and $\mathbb{E}[(X-1)^2]$

$E[X] = \sum_x x P(X = x) = \sum_{x=0}^{\infty} P(X \geq x)$

$E[X] = E[X_1 + X_2 + \cdots + X_b]$

$E(X) = \sum_{i=0}^{x-1} \frac{B-i}{N-i}$

label each bad item from $1, \ldots, B$

$X_i :=$ the i^{th} bad item goes before the 1^{st} drawn good

$\equiv$ i^{th} bad item goes before all good items

$\boxed{X = 1 + X_1 + X_2 + \cdots + X_B}$

$E(X) = 1 + B\, E[X_1] = 1 + B\, p(X_1 = 1)$

$= 1 + B\, p(1^{st}$ bad before all good items$)$

$= 1 + \frac{B}{G+1}$

B G G G . . . . G

G of these

G of these

$$\frac{1}{G+1}$$

$$1 + \frac{B}{G+1}$$

$$E((X-1)^2) = E\left((X_1 + \cdots + X_B)^2\right) = \boxed{B\,E[X_1^2] + B(B-1)\,E(X_1X_2)}$$

$$= B\,P(X_1 = 1) + B(B-1)\,P(\text{1st \& 2nd bad before all good})$$

$$\binom{G+2}{2}$$

$$= \frac{B}{G+1} + B(B-1) \times \frac{1}{\binom{G+2}{2}}$$

$$= \frac{B}{G+1} + B(B-1) \times \frac{1}{\frac{(G+2)!}{G!\,2!}} = \frac{B}{G+1} + \frac{B(B-1)\,2}{(G+1)(G+2)}$$