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Wednesday, January 27, 2021 8:32 PM

Recall

Iterated expectations : $E[X] = E[E[X|Y]]$

Conditional expectation : $E[X|A] = \int_{x=-\infty}^{x=\infty} x \, f_{X|A}(x) \, dx$

[X is a R.V, A is an event]

Law of the unconscious statistician : $E[g(X)] = \int_{-\infty}^{\infty} g(x) \, f_X(x) \, dx$

(LOTUS)

3. Triangle Density

Consider random variables X and Y which have a joint PDF uniform on the triangle with vertices at $(0,0), (1,0), (0,1)$.

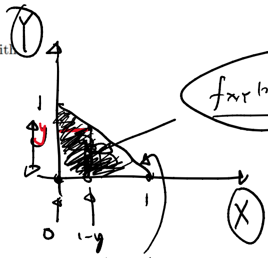
(a) Find the joint PDF of X and Y.

(b) Find the marginal PDF of Y.

(c) Find the conditional PDF of X given Y.

(d) Find $\mathbb{E}[X]$ in terms of $\mathbb{E}[Y]$.

(e) Find $\mathbb{E}[X]$.

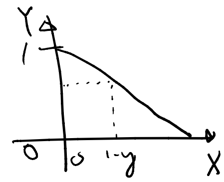


$f_{XY}(x,y) = c$

when (x,y) is in triangle

a) $\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f_{X,Y}(x,y) \, dx \, dy = 1$

$\int_0^1 \int_0^{1-y} c \, dx \, dy = 1 \Rightarrow c=2$



b) $f_Y(y) = \int_0^{1-y} f_{X,Y}(x,y) \, dx$ $\quad$ 2

$= 2(1-y)$

c) $f_{X|Y}(x|y) = \dfrac{f_{X,Y}(x,y)}{f_Y(y)} = \dfrac{2}{2(1-y)} = \dfrac{1}{1-y}$

d) Find $E(X)$ in terms of $E(Y)$

$a = 2b+3$

$E[X] = E[E[X|Y]]$

$E[X|Y=y]$

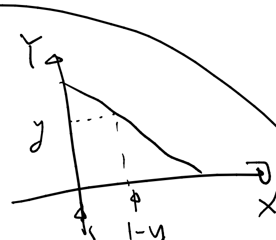
$g(Y)$

$E(Y) = \int_{-\infty}^{\infty} y \, f_Y(y) \, dy$

$E(X) = E[g(Y)] = \int_{y=-\infty}^{y=\infty} g(Y=y) f_Y(y) \, dy$

E

$E[X|Y=y] = \int_0^{1-y} x \, f_{X|Y}(x|y) \, dx$



$E(X|Y)$

$= \int_0^{1-y} \dfrac{x}{1-y} \, dx = \left[\dfrac{x^2}{2(1-y)}\right]_0^{1-y} = \dfrac{(1-y)^2}{2(1-y)} = \dfrac{1-y}{2}$

$\int_{y=-\infty}^{y=\infty} f_Y(y) \, dy$

$E[X] = \int_0^1 \dfrac{1-y}{2} f_Y(y) \, dy = \dfrac{1}{2}\int_0^1 (1-y) f_Y(y) \, dy = \dfrac{1}{2}\left(\int_0^1 f_Y(y) \, dy - E[Y]\right)$

$= \dfrac{1}{2}(1 - E(Y))$

2. Poisson Merging

(a) Let $X \sim \text{Poisson}(\lambda)$ and $Y \sim \text{Poisson}(\mu)$ be independent Poisson random variables. Prove that $X + Y \sim \text{Poisson}(\lambda + \mu)$. (This property known as **Poisson merging** will be extensively used when we discuss Poisson processes.)

Note that it is **not** sufficient to use linearity of expectation to say that $X + Y$ has mean $\lambda + \mu$. You are asked to prove that the *distribution* of $X + Y$ is Poisson.

Hint: You may find the binomial theorm helpful, which states $(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}$.

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(b) Suppose that you and your friend are folding paper cranes. However, you both are very slow at it. The number of paper cranes you and your friend can fold in an hour follows a Poisson distribution with mean 3 and mean 2 respectively (A Poisson is appropriate here since finishing a paper crane is a rare event).
What is the distributions of the total number of paper cranes you and your friend can fold together in an hour?

$= \frac{1}{2}(1 - E(Y))$

$E(X) = E(Y)$

$\Rightarrow \frac{1}{2}(1-E(Y)) = E(Y)$

$1 - E(Y) = 2E(Y)$

$E(Y) = \frac{1}{3} = E(X)$

a) WTS $\forall k \in \mathbb{Z}^+$ $P(X+Y=k) = \frac{(\mu+\lambda)^k}{k!} e^{-(\mu+\lambda)}$

$= \sum_{i=0}^{k} P(X=i)P(Y=k-i) = \sum_{i=0}^{k} \frac{\lambda^i}{i!} e^{-\lambda} \frac{\mu^{k-i}}{(k-i)!} e^{-\mu}$

$= e^{-(\lambda+\mu)} \sum_{i=0}^{k} \frac{\lambda^i \mu^{k-i}}{i!(k-i)!} = e^{-(\lambda+\mu)} \sum_{i=0}^{k} \frac{k!}{i!(k-i)!} \left(\frac{1}{k!}\right) \lambda^i \mu^{k-i}$

$\frac{k!}{i!(k-i)!} = \binom{k}{i}$ $\quad = \frac{e^{-(\lambda+\mu)}}{k!} \sum_{i=0}^{k} \binom{k}{i} \lambda^i \mu^{k-i}$ $\quad (\lambda = x,\ \mu = y)$

$= \frac{e^{-(\lambda+\mu)}}{k!} (\lambda+\mu)^k \Rightarrow X+Y \sim Poi(\lambda+\mu)$

b) $Poi(5)$

1. **Covariance Matrix and Independence**

For random variables $X_1, X_2, \ldots, X_n$, define the covariance matrix as a matrix Σ with entries $\Sigma_{ij} = \text{cov}(X_i, X_j)$ for all $i, j \in \{1, \ldots, n\}$. For this question, let $\mathbf{X} = [X_1, X_2, \ldots, X_n]^T$ and assume $\mathbb{E}[\mathbf{X}] = \mathbf{0}$. $\quad X \in \mathbb{R}^{n\times 1}$

(a) Show that Σ is symmetric and positive semi-definite (PSD). Recall that a square matrix $A \in \mathbb{R}^{n\times n}$ is symmetric iff $A = A^T$ and PSD iff $\mathbf{u}^T A \mathbf{u} \geq 0$ for all $\mathbf{u} \in \mathbb{R}^n$.
Hint: For any vector of random variables, $\Sigma = \mathbb{E}[(\mathbf{X} - \mathbb{E}[\mathbf{X}])(\mathbf{X} - \mathbb{E}[\mathbf{X}])^T]$, can you see for yourself why this is true?

(b) Show that if the X_i's are pairwise independent, then Σ is a diagonal matrix.

(c) Give an example of two random variables X_1 and X_2 with a diagonal covariance matrix, but X_1 and X_2 are not independent.

$\forall i,j$ WTS $\Sigma_{ij} = (\Sigma^T)_{ij} = \Sigma_{ji} = Cov(X_j, X_i)$

$\quad \| $

$Cov(X_i, X_j)$

from hint $\Sigma = E[XX^T]$

a) $\forall u \in \mathbb{R}^n$, $\quad u^T \Sigma u = u^T E[XX^T] u$

$= E[u^T X X^T u]$

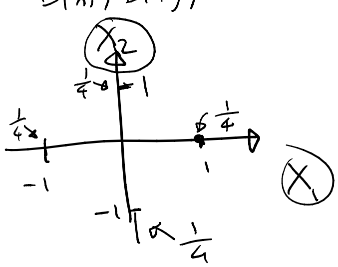
$= E[\underbrace{(X^T u)^T}_{\text{scalar}} \underbrace{(X^T u)}_{\text{scalar}}]$

$= E(S^2) \geq 0$, where $S = X^T u$

b) $\forall i \neq j$ $\quad \Sigma_{ij} = Cov(X_i, X_j)$

$= E(X_i X_j) - E(X_i)E(X_j)$

$= E(X_i)E(X_j) - \quad '' \quad '' \quad = 0$



$Cov(X_1, X_2) = 0$