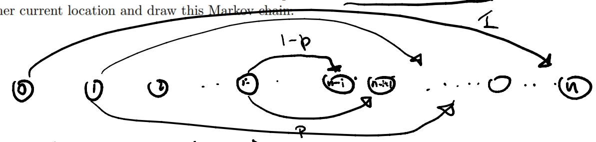
Dis 7 Markov Chains

Wednesday, March 3, 2021   5:04 PM

1. Umbrellas

A professor has n umbrellas, for some positive integer n. Every morning, she commutes from her home to her office, and every night she commutes from her office back home. On every commute, if it is raining outside, she takes an umbrella (if there is at least one umbrella at her starting location); otherwise she does not take any umbrellas. Assume that on each commute, it rains with probability $p \in (0,1)$ independently of all other times. Give the state space and transition probabilities for the Markov chain which corresponds to the number of umbrellas she has at her current location and draw this Markov chain.



$$P(X_{n+1} = n-i \mid X_n = i) = 1-p$$

if we don't take umbrella

$$P(X_{n+1} = n-i+1 \mid X_n = i) = p$$

$$P(X_{n+1} = n \mid X_n = 0) = 1$$

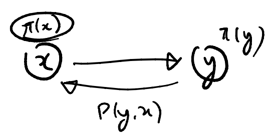
2. Detailed Balance Equations

DBE ⇒ BE

(a) Suppose we have a finite irreducible Markov chain on a state space $\mathcal{X}$ with transition matrix P. Show that if there exists a distribution π on $\mathcal{X}$ that satisfies the **detailed balance equations**:

$$\pi(x)P(x,y) = \pi(y)P(y,x) \text{ for all } x, y \in \mathcal{X},$$

then π is a stationary distribution of the chain (i.e. $\pi P = \pi$).



(b) Suppose π is a stationary distribution for a finite irreducible Markov chain. Is this sufficient for π to satisfy the detailed balance equations?

$$P \in \mathbb{R}^{|X| \times |X|}$$

$$[P]_{ij} = P(i,j)$$

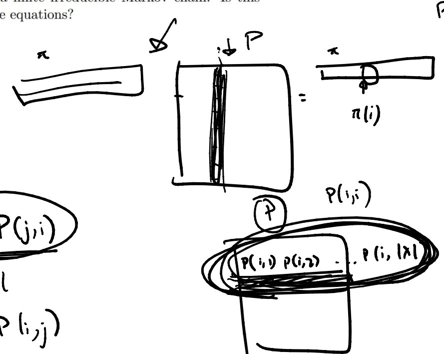
$$\pi(i) = (\pi P)(i)$$

$$(\pi P)(i) = \sum_{j=1}^{|X|} \pi(j) P(j,i)$$

$$= \sum_{j=1}^{|X|} \pi(i) P(i,j)$$

$$= \pi(i) \sum_{j=1}^{|X|} P(i,j) \to 1$$

$$= \pi(i)$$



Σ of rows $= 1$

(balance)

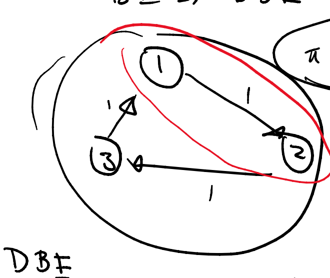
stat. dist $\equiv \pi P = \pi$

b) if it true BE ⇒ DBE ?

if π is unique, then

BE ⇒ DBE

$\pi = [\frac{1}{3} \ \frac{1}{3} \ \frac{1}{3}]$



$\pi = (a \ b \ c)$

$a + b + c = 1$

DBE $\forall x, y \in X$, say $x = 1$, $y = 2$

$$\pi(x) P(x,y) = \pi(y) P(y,x)$$

3. Random Walk on an Undirected Graph

Consider a random walk on an undirected connected finite graph (that is, define a Markov chain where the state space is the set of vertices of the graph, and at each time step, transition to a vertex chosen uniformly at random out of the neighborhood of the current vertex). What is the stationary distribution?

Hint: Instead of trying to work it out algebraically, think about what the stationary distribution is. Visualize doing the random walk on the graph (in your mind or on paper) and see which vertex you visit more and which you visit less. Then make a conjecture based on the properties of the graph and test to see if it is a stationary distribution.

deg(u) to quantify P(u,v)

$$\forall v \in V \quad \pi(v) = \frac{\deg(v)}{\sum_{x \in V} \deg(x)}$$

$\forall x, u \in V$

$\sum_{x \in V} \deg(x)$

$\forall x, y \in V$

$$\pi(x) P(x,y) \overset{?}{=} \pi(y) P(y,x)$$

LHS $\frac{\deg(x)}{\sum_{u \in V} \deg(u)} \cdot \frac{1}{\deg(x)} = \frac{1}{\sum \deg(u)}$

RHS $\frac{\deg(y)}{\sum \deg(u)} \cdot \frac{1}{\deg(y)} = \frac{1}{\sum \deg(u)}$

if x, y aren't connected, DBE still holds

DBE $\Rightarrow$ BE $\Leftrightarrow$ stationary

$$\pi = \pi P$$

$$\pi(x) P(x,y) = \pi(y) P(y,x)$$

$\frac{1}{3} \times 1 \neq \frac{1}{3} \times 0$

$\Rightarrow$ DBE doesn't hold

$E[\frac{1}{X}]$, $X \in \{0, 1, 2, 3\}$

$E[\frac{1}{X}]$ & $U(-1, 1)$

$X \in [-1, \infty]$

$$E[f(X)] = \int_0^\infty \frac{1}{x} p_X(x)\, dx$$

$P_{X,Y}(x,y)$ $= \frac{P_{X,Y}(x,y)}{P_X(x) P_Y(y)} \leftarrow Z$

Z is distributed as $P_{X,Y}$

$Z =$

$$I(X;Y) = E\left[\log \frac{P(X,Y)}{P(X)P(Y)}\right]$$

let $Z = (X, Y) \leftarrow$

$$\Rightarrow I(X;Y) = E(f(X,Y))$$ where f is log

$$f(X,Y) = \log \frac{P(X,Y)}{P(X)P(Y)}$$

$$f(E[Z]) = \log(P(E[X], E(Y])$$

$E(X]; E(Y]$

$$H(X|Y) = E\Big[\log \underbrace{\frac{1}{P_{X|Y}(X|Y)}}_{Z}\Big]$$

$$= \sum_x \sum_y P_{X,Y}(x,y) \log \frac{1}{P_{X|Y}(x|y)}$$

$$E(g(Z))$$

$$= \sum_{z \in \mathcal{Z}} f(z) \, p_Z$$

$$= \sum_x \sum_y f_{X,Y}(x,y)$$