Dis4

Wednesday, February 10, 2021 10:16 AM

Taylor expansion $e^{sX} = f(x) = \sum_{i=0}^{\infty} \frac{f^i(a)}{i!}(x-a)^i$

MGFs

$MGF_X(s) = E[e^{sX}]$

$\frac{d^k}{ds^k} MGF_X(s)\Big|_{s=0} = E[X^k]$

Tells us the 'moments' of a RV

$MGF_X(0) = E[e^{0X}] = 1$

Derived distributions

Given a RV as a function of a known continuous RV, find pdf of the continuous RV

$X \sim N(0,1)$

X^3

$P(X^3 = k)$

$P(X^3 = k) = P(X = \sqrt[3]{k})$

instead, for continuous RV, find CDF & differentiate for pdf

MGF of $X \sim N(0,1)$

$E[e^{sX}] = \int_{-\infty}^{\infty} e^{sx} f_X(x)\, dx$

$= \ldots$

3. Transform Practice

Consider a random variable Z with transform

$$M_Z(s) = \frac{a - 3s}{s^2 - 6s + 8}, \quad \text{for } |s| < 2.$$

Calculate the following quantities:

(a) The numerical value of the parameter a.

(b) $\mathbb{E}[Z]$.

(c) $\text{var}(Z)$.

$M_Z(0) = 1 = \frac{a - \cancel{3s}}{8} \Rightarrow a = 8$

b) kth moment of r.v. X is $E[X^k] = \frac{d^k}{ds^k} MGF_X(s)\Big|_{s=0}$

$E[Z^1] = \frac{d^1}{ds^1} MGF(s)\Big|_{s=0} = \frac{d}{ds}\left(\frac{a-3s}{s^2-6s+8}\right) = \frac{g'h - h'g}{h^2}\Big|_{s=0}$

$= \frac{(-3)(s-4)(s-2) - (2s-6)(a-3s)}{(s-4)^2(s-2)^2}\Big|_{s=0} = \frac{-3(-4)(-2) - (-6)(8)}{(-4)^2(-2)^2}$

$= \frac{-24+48}{64} = \frac{24}{64} =$

2. Revisiting Facts Using Transforms

(a) The MGF of a Poisson X with mean λ is $M_X(s) = \mathbb{E}[\exp(sX)] = \exp(\lambda(\exp(s) - 1))$. Let $X \sim \text{Poisson}(\lambda)$, $Y \sim \text{Poisson}(\mu)$ be independent. Calculate the MGF of $X + Y$ and use this to show that $X + Y \sim \text{Poisson}(\lambda + \mu)$.

(b) The MGF of $X \sim \text{Exponential}(\lambda)$ is $M_X(s) = \mathbb{E}[\exp(sX)] = \lambda/(\lambda - s)$. Use this to find all the moments of X.

Hint: You could take the kth derivatives. However, think about writing the MGF in two different ways as infinite series and match terms.

(c) The MGF of $X \sim \mathcal{N}(0, 1)$ is $\exp(\frac{s^2}{2})$. Similar as the above part, use this to find all the moments of X.

Hint: Derivatives are not required here either.

$MGF_X(s) = e^{\frac{s^2}{2}} = E[e^{sX}]$

$g(X)$ $g(Y)$

$E[e^{sZ}] = E[e^{s(X+Y)}] = E[e^{sX}e^{sY}] = E[e^{sX}]E[e^{sY}]$

$\varphi_X(s) = E[e^{sX}]$

$= e^{\lambda(e^s-1)} e^{\mu(e^s-1)} = e^{(\lambda+\mu)(e^s-1)}$

$\Rightarrow X+Y \sim \text{Poi}(\lambda+\mu)$

b) $e^{sX} = 1 + sX + \frac{(sX)^2}{2!} + \frac{(sX)^3}{3!} + \ldots$

$= 1 + a_1 s + a_2 s^2 + a_3 s^3 + \ldots$

$1 + b_1 s + b_2 s^2 + \ldots$

i) $E[e^{sX}] = \frac{1}{1!} + \frac{s\,E[X]}{1!} + \frac{s^2 E[X^2]}{2!} + \ldots = \sum_{i=0}^{\infty} \frac{s^i}{i!} E[X^i]$

$\frac{a}{1-r} = \sum_{i=0}^{\infty} a r^i$

$s^k \quad E[X^k] = (s\ldots)^k$

$$\boxed{\frac{a}{1-r} = \sum_{i=0}^{\infty} a r^i}$$

ii) $E(e^{sX}) = \dfrac{\lambda}{\lambda - s} = \dfrac{1}{1-(s/\lambda)} = \displaystyle\sum_{i=0}^{\infty} \left(\frac{s}{\lambda}\right)^i$

$\frac{s^k}{k!} E[X^k] = \left(\frac{s}{\lambda}\right)^k$

$\Rightarrow E[X^k] = \dfrac{k!}{\lambda^k}$

(a) Suppose that X has the **standard normal distribution**, that is, X is a continuous random variable with density function

$$f(x) = \frac{1}{\sqrt{2\pi}} \exp\Big(-\frac{x^2}{2}\Big).$$

What is the density function of $\exp X$? (The answer is called the **lognormal distribution**.)

$e^x \notin [-ves]$

$f_{e^x}(\cdot) > 0$

(b) Suppose that X is a continuous random variable with density f. What is the density of X^2?

$e^x \geq 0$

(c) What is the answer to the previous question when X has the standard normal distribution? (This is known as the **chi-squared distribution**.)

a) Let $Y = e^X$, $k>0$ $\quad f_Y(k) = \frac{d}{dk} F_Y(k) = \frac{d}{dk} \Pr(Y \leq k)$ $\quad Y = e^X$

ln both sides

$f_X(x) = \frac{d}{dx} F_X(x)$

$= \frac{d}{dk} \Pr(X \leq \ln k)$

$= \frac{d}{dk} F_X(\ln k)$

$= \frac{d}{dk}\ln k \; F_X'(\ln k)$

$= \frac{1}{k} f_X(\ln k)$

b) $Y = X^2$, $k>0$

$f_Y(k) = \frac{d}{dk} F_Y(k) = \frac{d}{dk} \Pr(X^2 \leq k)$

$= \frac{d}{dk} \Pr(-\sqrt{k} \leq X \leq \sqrt{k}) = \frac{d}{dk} \displaystyle\int_{-\sqrt{k}}^{\sqrt{k}} f_X(x)\, dx$

f $\quad g$ $\quad h$

$$\boxed{\frac{d}{dx} \int_{g(x)}^{f(x)} h(t)\, dt = h(f(x))\, f'(x) - h(g(x)) \cdot g'(x)}$$

$= f_X(\sqrt{k}) \frac{d}{dk}\sqrt{k} - f_X(-\sqrt{k}) \frac{d}{dk} -\sqrt{k}$

$\frac{d}{dk} k^{\frac{1}{2}}$

$= \dfrac{f_X(\sqrt{k})}{2\sqrt{k}} + \dfrac{f_X(-\sqrt{k})}{2\sqrt{k}}$

k^{th} mmt of RV X

$MGF_X(s) \rightarrow E[X^k]$

$\frac{d^k}{ds^k} MGF_X(s) \Big|_{s=0} = E[X^k]$

3. First Time to Decrease

Let $X_1, X_2, \ldots, X_n, \ldots$ be a sequence of independent and identically distributed (i.i.d.) continuous random variables with common PDF f.

(a) Argue that $\Pr(X_i = X_j) = 0$ for $i \neq j$.

(b) Calculate $\Pr(X_1 \leq X_2 \leq \cdots \leq X_{n-1})$.

(c) Let N be a random variable which is equal to the first time that the sequence of the random variables will decrease, i.e.

$$N = \min\{n \in \mathbb{Z}_{\geq 2} \mid X_{n-1} > X_n\}.$$

Calculate $\mathbb{E}[N]$.

Chunk $X_1, \ldots, X_n$

Packets $Y_1, \ldots, Y_n$

can contain
multiple chunks

$D_i \sim p(\cdot)$

$Y_1 = X_1 \oplus X_2 \oplus X_5$

decode

$Y_1 = X_2$

A

start

B

finish

after peeling k chunks

$n-k$

randomly choose Y_i ← deg. d

$E[X^k]$ k^{th} moment of X

MGF

$MGF_X(s) = E[e^{sX}]$ always

$\frac{d^k}{d^k s} MGF_X(s) \Big|_{s=0} = E[X^k]$

$X \sim N(0,1)$

$MGF_X(s) =$