Derivation of fisher information of a multivariate Gaussian density

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We are interested to find the fisher information of this density with respect to parameter s upon which μ and Σ depends. This notes assumes fluency in matrix calculus (denominator layout) in the matrix cookbook or wikipedia.

For scalar x, scalar function f, vectors a, b, c, and matrices A, B, C:

$$d\frac{a^\intercal Ab}{da} = Ab$$

$$d\frac{a^\intercal Ab}{dA} = d\frac{Tr(a^\intercal Ab)}{dA} = d\frac{Tr(Aa^\intercal b)}{dA} = ab^\intercal$$

$$d\frac{f(b(x))}{dx} = \left[\frac{df(b)}{db}\right]^\intercal \nabla_x b$$

$$d\frac{f(A(x))}{dx} = Tr\left[\left(\frac{dx}{dA}\right)^\intercal \nabla_x A\right]$$

$$d\frac{a(c)^\intercal b}{dc} = \frac{da^\intercal}{dc}b$$

$$d\frac{\log A}{dx} = A^{-T}\nabla_x A$$

$$dA^{-1} = -A^{-1}dAA^{-1}$$

$$d\frac{A^{-1}}{dx} = -A^{-1}\nabla A A^{-1}$$

$$d\frac{Tr(AB^{-1}C)}{dB} = -B^{-T}CAB^{-T}$$

The multivariate Gaussian density is written as:

$$p(x) = \frac{1}{\sqrt{|2\pi\Sigma|}} \exp\left(-\frac{1}{2}(x-\mu)^\intercal \Sigma^{-1}(x-\mu)\right) = \frac{|\Lambda|^{1/2}}{(2\pi)^{D/2}} \exp\left(-\frac{1}{2}(x-\mu)^\intercal \Lambda(x-\mu)\right)$$

where Λ is the precision matrix and is symmetric. Define

$$\nabla\mu = \frac{d\mu}{ds}$$

$$\nabla\Sigma = \frac{d\Sigma}{s}$$

$$\nabla\Lambda = \frac{d\Lambda}{ds} = \frac{d\Sigma^{-1}}{ds} = -\Sigma^{-1}\nabla\Sigma\Sigma^{-1}$$

In the following, we will use the fact that $\Lambda = \Lambda^\intercal$

And write the log density as:

$$\log p(x) = \frac{1}{2}\log|\Lambda| - \frac{1}{2}(x-\mu)^\mathsf{T}\Lambda(x-\mu) + constant$$

There two ways ahead. One can write out the second derivative of the log density using the chain rule. It will be composed of four terms in which two of them will involve a score term $\frac{d\log p(x)}{d\mu}$ and $\frac{d\log p(x)}{d\Lambda}$. The expectation of these two terms will be zeros, so we ended up having only two terms. However, $\frac{d^2\log p(x)}{d\mu^2}$ is a matrix and $\frac{d^2\log p(x)}{d\Lambda^2}$ is a four-dimentional tensor which we should avoid... It is still possible to work, but less clear.

The second way is to compute the first derivative first in terms of the parameters, and then take derivative again.

$$\frac{d\log p(x)}{ds} = \left[\frac{d\log p(s)}{d\mu}\right]^\mathsf{T}\nabla\mu + Tr\left[\frac{d\log p(s)}{d\Lambda}^\mathsf{T}\nabla\Lambda\right]$$

$$= \underbrace{(x-\mu)^\mathsf{T}\Lambda\nabla\mu}_{M_1} + \frac{1}{2}\underbrace{Tr\left[\left[\Lambda^{-1} - (x-\mu)(x-\mu)^\mathsf{T}\right]\nabla\Lambda\right]}_{M_2}$$

One can notice that if we use the expectation of squared score function, the cross terms should be zero because they contain odd moments of Gaussian variables. However, it is also less clear how to write the third moment with vectors, especially with the trace. Let's continue with the dumb way.

Take the derivative of M_1 and then evaluate the negative expectation

$$\frac{dM_1}{ds} = \left[\frac{dM_1}{d\mu}\right]^\mathsf{T}\nabla\mu + Tr\left[\left[\frac{dM_1}{d\Lambda}\right]^\mathsf{T}\nabla\Lambda\right]$$

$$= \left[-\Lambda\nabla\mu + \nabla^2\mu\Lambda(x-\mu)\right]^\mathsf{T}\nabla\mu + Tr\left[[(x-\mu)\nabla\mu^\mathsf{T}]^\mathsf{T}\nabla\Lambda\right]$$

$$-\left\langle\frac{dM_1}{ds}\right\rangle = \nabla\mu^\mathsf{T}\Lambda\nabla\mu = \nabla\mu^\mathsf{T}\Sigma^{-1}\nabla\mu$$

Do the same for M_2

$$\frac{dM_2}{ds} = \left[\frac{dM_2}{d\mu}\right]^\mathsf{T}\nabla\mu + Tr\left[\left[\frac{dM_2}{d\Lambda}\right]^\mathsf{T}\nabla\Lambda\right]$$

$$= -\left[\frac{d}{d\mu}Tr\left[(x-\mu)(x-\mu)^\mathsf{T}\nabla\Lambda\right]\right]^\mathsf{T}\nabla\mu + Tr\left[\left[\frac{d}{d\Lambda}Tr\left[\Lambda^{-1}\nabla\Lambda\right] - \frac{d}{d\Lambda}Tr\left[(x-\mu)(x-\mu)^\mathsf{T}\nabla\Lambda\right]\right]^\mathsf{T}\nabla\Lambda\right]$$

$$= -2(x-\mu)^\mathsf{T}\nabla\Lambda\nabla\mu + Tr\left[\left[-\Lambda^{-1}\nabla\Lambda\Lambda^{-1} + \Lambda^{-1}\nabla^2\Lambda - (x-\mu)(x-\mu)^\mathsf{T}\nabla^2\Lambda\right]^\mathsf{T}\nabla\Lambda\right]$$

$$= -2(x-\mu)^\mathsf{T}\nabla\Lambda\nabla\mu + Tr\left[-\Lambda^{-1}\nabla\Lambda\Lambda^{-1}\nabla\Lambda\right] + Tr\left[\left[\Lambda^{-1} - (x-\mu)(x-\mu)^\mathsf{T}\right]\nabla^2\Lambda\nabla\Lambda\right]$$

$$-\left\langle\frac{dM_2}{ds}\right\rangle = 0 + Tr\left[-\Lambda^{-1}\nabla\Lambda\Lambda^{-1}\nabla\Lambda\right] + Tr\left[\left[\Lambda^{-1}\nabla^2\Lambda - \Sigma\nabla^2\Lambda\right]\nabla\Lambda\right]$$

$$-\left\langle\frac{dM_2}{ds}\right\rangle = Tr\left[\Lambda^{-1}\nabla\Lambda\Lambda^{-1}\nabla\Lambda\right] = Tr\left[\Lambda^{-1}\Sigma^{-1}\nabla\Sigma\Sigma^{-1}\Lambda^{-1}\Sigma^{-1}\nabla\Sigma\Sigma^{-1}\right] = Tr\left[\nabla\Sigma\Sigma^{-1}\nabla\Sigma\Sigma^{-1}\right]$$

The last equality is because the first contains first moment, and the third term is zero even though we are not sure what $\nabla^2\Lambda$ looks like.

So the fisher information is then

$$J(s) = \nabla\mu^\mathsf{T}\Sigma^{-1}\nabla\mu + \frac{1}{2}Tr\left[\nabla\Sigma\Sigma^{-1}\nabla\Sigma\Sigma^{-1}\right]$$