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T.E.R. : INTEREST-RATE DERIVATIVES VALUATION USING MULTINOMIAL TREES (ARBRES RECOMBINANTS POUR L'ÉVALUATION DE PRODUITS DE TAUX.)

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Abstract

"An interest-rate derivative is a financial instrument based on an underlying financial security whose value is affected by changes in interest rates."[investopedia]. This is how investopedia site web describe an interest rate derivative. All interest-rate derivative has a price that show his real state in the market where it is negotiate, a state juged by his current rate. Nowadays, pricing that financials instruments is really far from a simple equation solving. It takes since 1973, that complexe short-rate models has been developped, using real strong mathematics implementations to give to the products his real price according to his rate and all the risks involved. Theses models, with differents assumptions and differents parameters, try to determinate (with approximations of course) the future evolution of the rates in time by drawing his most probable rate curve. Here is the utility of lattice model, it is used by some or all short-rate models do modelise all the possible way of the rate evolution , trough the time.

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Introduction

According to the 2016, Bank For International Settlements (BIS) estimations [BIS2016], the notional amounts outstanding is about US$434.740 trillion for OTC interest rate contracts, this make the interest rate derivatives market the largest one in the world. The size of this market, make it very important and very complex in the finance world.

So, all these derivatives, need to be priced according to his underlying interest rate evolution, and this is why complex and modern short-rate model are made for. Short-rate models describes the future evolution of interest rates by the short rate evolution, and to do it, they use binomial or trinomial trees accroding to the model.

In our redaction, we explore the general definitions and ideas, and the differents assumptions of the models. Then we study four differents most used short rate model through the time, wich are, Vasicek model, Cox-Ingersoll-Ross model, Ho-lee model and Hull & white model. And then we practice the implementation of trinomial tree with the Hull & White model for valuation of a zero-coupon in a risk neutral situation.

Chapter 1

INTEREST RATE DERIVATIVES AND MULTINOMIAL TREES.

A sound understanding of some notions is essential to a good explanation of the models developed later

1.0.1 Some definitions[investopedia]

1. **hort rate:**

   The interest rate at short term (1 year).

2. **Derivative**

   "A derivative is a security with a price that is dependent upon or derived from one or more underlying assets."

3. **Futures**

   "A futures contract (or simply futures, colloquially) is an agreement between two parties for the sale of an asset at an agreed upon price."

4. **Forwards**

   Forward contracts are another important kind of derivative similar to futures contracts, the key difference being that unlike futures, forward contracts (or "forwards") are not traded on exchange, but rather are only traded over-the-counter.

5. **LIBOR : London Interbank Offered Rate**

   "LIBOR or ICE LIBOR (previously BBA LIBOR) is a benchmark rate that some of the world's leading banks charge each other for short-term loans. It stands for IntercontinentalExchange London Interbank Offered

Rate and serves as the first step to calculating interest rates on various loans throughout the world. LIBOR is administered by the ICE Benchmark Administration (IBA)."

6. **Swaps**

   "A swap is a derivative contract through which two parties exchange financial instruments. "

7. **Interest rate swap**

   "In an interest rate swap, the parties exchange cash flows based on a notional principal amount (this amount is not actually exchanged) "

8. **Swaptions**

   A swaption is an option on swap.

9. **Volatility**

   "Volatility is a statistical measure of the dispersion of returns for a given security or market index. Volatility can either be measured by using the standard deviation or variance between returns from that same security or market index. Commonly, the higher the volatility, the riskier the security."

10. **Risks**

    "The chance that an investment's actual return will be different than expected. Risk includes the possibility of losing some or all of the original investment."

11. **Zero-coupon bond**

    "A zero-coupon bond is a debt security that doesn't pay interest (a coupon) but is traded at a deep discount, rendering profit at maturity when the bond is redeemed for its full face value."

12. **Risk neutral**

    "Risk neutral is indifference to risk. The risk-neutral investor would be in the middle of the continuum represented by risk-seeking investors at one end, and risk-averse investors at the other extreme. Risk-neutral measures find extensive application in the pricing of derivatives.

We developed now two notions more in detail .In fact, Zero coupon rates and forward rates are used in the following models.

Zero-coupon rates: A zero-coupon interest rate is the rate of interest earned on an investment with no intermediate payment, there is just one payment at the maturity T of the zero-coupon rates. Therefore, there are just two cash flows, one at the beginning, the initial flow (the purchase price) and one at the end for the reimbursement of the nominal N with the interests. It's internal interest rate of return.

The zero-coupon rates are very useful to find other interest rates. The zero coupon rate$r(t,T)$ in discrete-time at time t, with a maturity T, is defined, like this:

$$B(t,T) = \frac{1}{(1+r(t,T))^{T-t}}$$

the zero-coupon curve is the evolution of $r(t,T)$ in function of T,

$$r(t,T) = -\frac{ln(B(t,T))}{T-t}$$

With B(t,T) the zero coupon bond which is just a bond at zero rate,

$$B(t,T) = e^{-r(t,T)(T-t)}$$

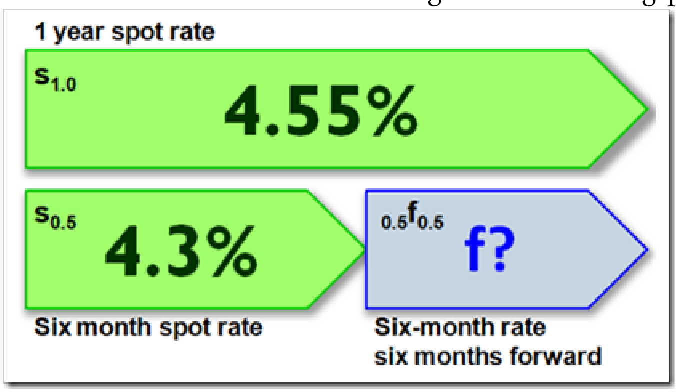
The limit of term structure of the interest rates $r(t,T)$ give the zero rate $r(t)$ which is a free risk rate.

$$r(t) = lim_{T \to t}\, r(t,T)$$

The forward rates

"Forward interest rates are the future rates of interest implied by current zero rates for periods of time in the future."

Figure 1.0.1: Forward rates.dlittlethingsthatmatter.blogspot.ca



If we supposed we want to issue debt for 1 year. There are different possibilities, we can directly issue debt at the zero rate for 1-years investment but we can also fractionate the period in two. In fact, we can issue debt at the zero rate for 6-months investment and for the next 6-months at the forward rate on the period which actually start in six months and finish in one years.
Subsequently, in the Hull and White model, to determinate the zero-coupon bond price, we need the instantaneous forward rate. which is defined with the following formula,

$$F(t,T) = lim_{T_2 \to T_1} \frac{ln(B(t,T_1)) - ln(B(t,T_2))}{T_2\text{-}T_1} = lim_{T_2 \to T_1} \frac{1}{T_2\text{-}T_1} ln(\frac{B(t,T_1)}{B(t,T_2)})$$

$$F(t,T) = lim_{T_2 \to T_1} \frac{1}{T_2\text{-}T_1} ln(\frac{e^{-r(t,T_1)(T_1-t)}}{e^{-r(t,T_2)(T_2-t)}})$$

In particular,

$$F(0,T) = lim_{T_2 \to T_1} \frac{r(0,T2)T_2 - r(0,T_1)T_1}{T_2\text{-}T_1}$$

With $T_2 > T_1$and $T = T_2 - T_1$.

If we suppose that $T_1 = T$ and $T_2 = T + \Delta t$, with Δt infinitesimal growth rate and if $R_1$and R_2 are the zero rates for maturities $T_1$and T_2, we have R_F, the forward interest rate for the period of time between $T_1$and T_2 ,with the following formula,

$$R_F = \frac{R_2T_2 - R_1T_1}{T_2 - T_1}$$

1.0.2 Short-rate model and interest rate derivative.

As we define, an interest-rate derivative is a financial instrument based on an underlying financial security whose value is affected by changes in interest rates. Interest-rate derivatives are hedges used by institutional investors such as banks to combat the changes in market interest rates. Individual investors are more likely to use interest-rate derivatives as a speculative tool - they hope to profit from their guesses about which direction market interest rates will move. That mean that, before to invest in an interest rate derivative, the investor need to know, what will be the rates movements during the agreement. He need to know what is his real risk in this investment, will he gain or loose money, how and when. So he need to know the future, and to do it, he will use short-rate models. According to his situation, his market, the type of derivative, he will chose the model that is the mos t representative of his current situation.

Short-rate model for interest rate derivatives, is a relative complex mathematical model that describe the future evolution of interest rate by modelising his very short time interval rate curve, the short rate curve. Short-rate model appear arround the 1970's years, and the firs notable one, the Vasicek model, came out in 1977. Since, models became more and more complex, strong, and reflect the real market.

Short rate model, modelise the short rate, by using mainly a mathematical solution, stochastics processes. For assumption, shor rate variations depend on time and is the main factor of the global evolution of rates. This is why, short rate model use relative complex stochastics processes to reach his goal, to determine the futures evolution of interest rate. But there are others important parameters that come to complete the equations, like current rates, rates's volatility using the past rate curve, economics and market random effects and some parameters for statistical fit.

1.0.3 Interest rate derivative valuation and multinomial trees.

In his implementation, the models use multinomial trees according to his method. In fact in his implementation, from a time t to t+1, the rate can go up or down for binomial trees and for trinomial trees the rate can go up, stay stable or go down. And once the trees implemented, the interest rate curve can be approximated using the differents probabilities give by the models. And then an approximated most probable interest rate curve can be draw and it will give the real value of the derivative according to his type. For a swaption it will permit to price it, and for an interest rate swap it will permit to an investor to evaluate his risk of gain or loose his money.

Chapter 2

SHORT-RATE WITH ONE-FACTOR MODELS.

The risk-free short rate, r, at time t is the rate that applies to an infinitesimally short period of time at time t. It is sometimes referred to as the $instantaneous\ short\ rate$. Bond prices, option prices, and other derivative prices depend only on the process followed by r in a risk-neutral world. The process for r in the real world is not used. the traditional risk-neutral world is a world where, in a very short time period between t and $t + \Delta t$, investors earn on average $r(t)\Delta t$.

2.1 Eliquilibrium models

Equilibrium models usually start with assumptions about economic variables and derive a process for the short rate, r. They then explore what the process for r implies about bond prices and option prices.

In a one-factor equilibrium model, the process for r involves only one source of uncertainty. Usually the risk-neutral process for the short rate is described by an Itô process of the form

$$dr = m(r)dt + s(r)dz$$

The instantaneous drift, m, and instantaneous standard deviation, s, are assumed to be functions of r, but are independent of time. The assumption of a single factor is not as restrictive as it might appear. A one-factor model implies that all rates move in the same direction over any short time interval, but not that they all move by the same amount. The shape of the zero curve can therefore change with the passage of time.

2.1.1 Vasicek's model (1977).

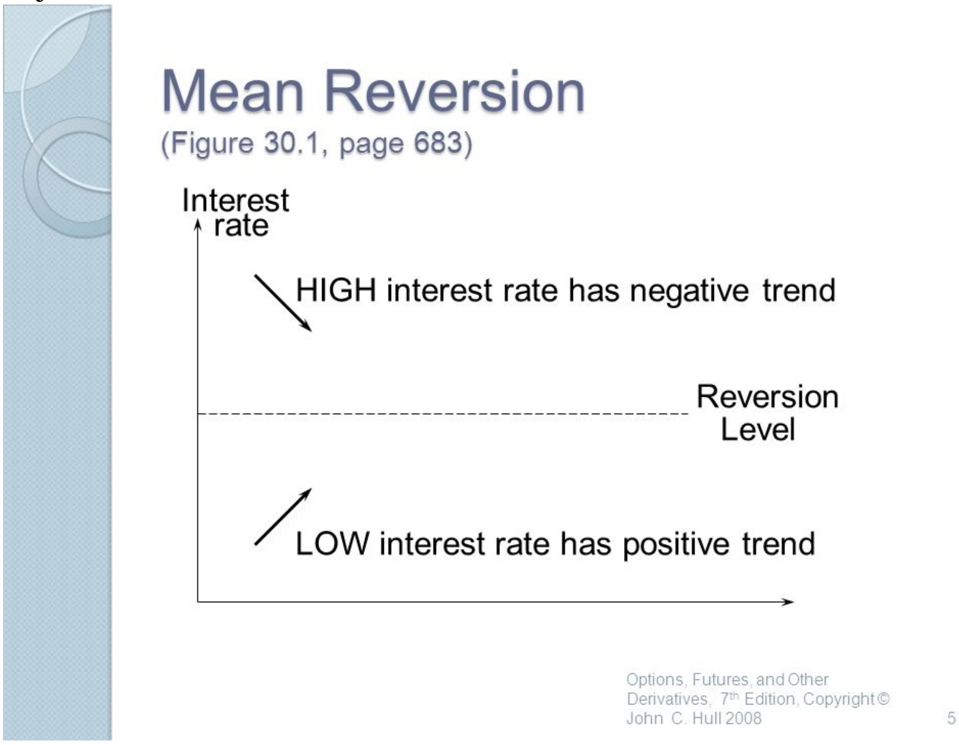
2.1.1.1 Model presentation

The Vasicek's model is from a paper by Oldrich Vasicek and titled "An Equilibrium Characterization of the Term Structure" published in 1977. Definition. In Vasicek mdel, the risk-neutral process for r is

$$dr(t) = a(\theta\text{-}r(t))dt + \sigma dW(t)$$

It's a Ornstein-Uhlenbeck's process and a, θ and σ are nonnegative constants. W is Wiener process. σ is a volatility of the short rate. θ is the equilibrium level , the mean of r and θ is the rate (speed) of the mean-reverting. In fact, it's a process which has the characteristic to be mean-reverting, i.e. the interest rate fluctuating around his mean.
Let's take an example to illustrate the trend of the interest rate to return toward his mean. If r the interest rate is bigger than ϑ his mean value, then the term (the drift) will be negative. Which involve a downward trend in the interest rate and vice versa. We can notice that more K will be important more the rate will be reach the reversion level rapidly.
Finally, we can notice that the process '"noise" are more or less amplified by the standard deviation

Figure 2.1.1: Main-reversion.options futures and other derivatives 7th edition.J.C.HULL 2008



The solution of the Vasicek's model is

$$r(t) = r(s)e^{-a(t-s)} + \theta(1 - e^{-a(t-s)}) + \sigma \int_s^t e^{-a(t-u)} dW(u)$$

For the proof, we have to use the lemme of Itô.
Proposal. In Vasicek's model, The zero-coupon bond price at t with a maturity at T is given by,

$$P(t,T) = exp[(\theta - \frac{\sigma^2}{2a^2})(B(t,T) - (T-t)) - \frac{\sigma^2}{4a}B(t,T)^2]$$

with,

$$B(t,T) = \frac{1-e^{-a(T-t)}}{a}$$

And the zero-coupon curve at t with a maturity at T is,

$$R(t,T) = -\frac{1}{T-t} lnB(t,T) = -R_\infty + (R_\infty - r)\frac{1-e^{-a(T-t)}}{a(T-t)} - \frac{\sigma^2}{4a^3(T-t)}(1 - e^{-k(T-t)})^2$$

2.1.1.2 Advantages et limits

- An advantage of Vasicek's model is that we can have an explicit formula of the stochastic differential equation (SDE), i.e. we can determinate the Bond and zero-coupon rates.
- Another advantage is that we have a Gaussian distribution which is known and simple to use.
- However, Gaussian distribution is also a limit of the model, because we can have interest rates with negative values.
- We also are limited to represent all shapes of today's term strcuture of interest due to simplicity of the model. Specially, it is due to drift is not function of time, as mentioned above.

2.1.2 Cox–Ingersoll–Ross's (CIR) model (1985)[12]

2.1.2.1 Model presentation[12]

Cox, Ingersoll, and Ross model is also an equilibrium model. In the Vasicek's model, short term interest rate,r, can become negative. Cox, Ingersoll, and Ross have proposed an alternative model where rates are always non-negative. The risk-neutral process for r that they propose as alternative model is :

$$dr = a(b - r)dt + \sigma\sqrt{r}dz$$

where a, b, and σ are nonnegative constants. This has the same mean-reverting drift as Vasicek, but the standard deviation of the change in the short rate in a short period of time is proportional to $\sqrt{r}$. This means that, as the short-term interest rate increases, the standard deviation increases. Bond prices in the CIR model have the same general form as those in Vasicek's model,

$$P(t, T) = A(t, T)\exp(-B(t, T)r(t))$$

but the functions $B(t, T)$ and $A(t, T)$ are different :

$$B(t, T) = \frac{2(\exp(\gamma(T - t)}{(\gamma + a)(\exp(\gamma(T - 1)) - 1) + 2\gamma}$$

and

$$A(t, T) = \left[\frac{2\gamma\exp((a + \gamma)(T - t)/2)}{(\gamma + a)(\exp(\gamma(T - 1)) - 1) + 2\gamma}\right]^{2ab/\sigma^2}$$

with $\gamma = \sqrt{a^2 + 2\sigma^2}$

To see this result, we substitute $m = a(b-r)$ and $s = \sigma\sqrt{r}$ into differential equation (equation à determiner plus tard) to get :

$$\frac{\partial f}{\partial t} + a(b-r)\frac{\partial f}{\partial r} + \frac{1}{2}\sigma^2 r\frac{\partial f}{\partial r^2} = rf$$

As in the case of Vasicek's model, we can prove the bond-pricing result by substituting $f = A(t,T)\exp(-B(t,T)r)$ into the differential equation. In this case, $A(t,T)$ and $B(t,T)$ are solutions of

$$B_t - aB - \frac{1}{2}\sigma^2 B^2 + 1 = 0, \quad A_t - abAB = 0$$

Furthermore, the boundry condition $P(T,T) = 1$ is satisfied.

The $A(t,T)$ and $B(t,T)$ functions are different for Vasicek and CIR, but for both models

$$P(t,T) = A(t,T)\exp(-B(t,T)r(t))$$

so that

$$\frac{\partial P(t,T)}{\partial r(t)} = -B(t,T)P(t,T)$$

Then the zero rate at time t for a period of $T-t$ is

$$R(t,T) = -\frac{1}{T-t}\ln A(t,T) + \frac{1}{T-t}B(t,T)r(t)$$

2.1.2.2 Advantages and limits

The Vasicek and CIR models are examples of $time - homogeneous$, equilibrium models. A disadvantage of such models is that they give a set of theoretical prices for bonds which will not normally match precisely the actual prices that we observe in the market. This led to the development of some $time - inhomogeneous$ Markov models for r(t), most notably those due to Ho & Lee (1986) and Hull & White (1990). (Interest-Rate Models Andrew J.G. Cairns)

2.2 No-Arbitrage models

Definition : The No-Arbitrage principle[7]**:** The market model is arbitrage free or viable if and only if no arbitrage possibility can appear on a financial market, which means, there does not exist any self-financing portfolio strategy h with initial value $V_0(h) = 0$ such that the final value $V_T(h) > 0$ with strictly positive probability ($P(V_T(h) > 0) > 0$) or $P - almost$ surely. More precisely, no investor can start with nothing and make a profit without risk at the end.

2.2.1 Ho-Lee model (1986).

2.2.1.1 Model presentation

Ho-Lee model is a short-rate model very used in bond pricing, options, swaptions and other derivatives. It is also used to modelise the evolution of future interest rate. Developped in 1986 by Thomas Ho and Sang Bin Lee, it was the first arbitrgae free model of interest rate.

The model born from the fact that "The Cox-Ingersoll-Ross, Vasicek and Brennan- Schwartz models all focused on a theoretical construct called the short rate of interest as a starting point for their analysis. Like the Black-Scholes model used to price derivatives based on stocks, these interest-rate models used one or, at most, two so-called state variables to describe future states of the world. Though this simplistic approach made analysis of future uncertainty feasible, the interest-rate models had the drawback of treating the discount curve as something spit out by the model rather than as an input. The discount curve is the set of zero-coupon-bond prices of all possible maturities. Unfortunately, this model output didn't necessarily match market bond prices."[4]

The basics assumptions of the Ho-Lee model are[19] :

- (A1) The market is frictionless. there are no taxes and no transaction costs, and all securities are perfectly divisible.

- (A2) The market clears at discrete points in time, wich are separated in regular intervals. For simplicity, we use each period as a unit of time. we define a discount bond of maturity T to be a bond that pays \$1 at the end of the Tth period, with no other payments to its holder.

- (A3) The bond market is complete. There exists a discount bond for each maturity $n(n = 0, 1, 2 \cdots)$.

- (A4) At each time$n$, there are a finite number of states of the world. For state i, we denote the equilibrium price of the discount bond of maturity T by $P_i^{(n)}(T)$. Note that $P_i^{(n)}(\cdot)$ is a function that relates the price of a discount bond to its maturity. This function is called the discount function. Within the context of the model, the discount function completely describes the term structure of interest rates of the i^{th} state at time$n$.

In real market, from a period to another one in the time, it is very difficult, or even impossible to predict the interest rate of a zero-coupon, so his bond price.

To considere that fact, the Ho and Lee model assume that, the bond price of a non-zero coupon is a stochastic price process wich follows the binomial model. That is to say, if t designs the present time, at the time $t+1$ the price can go up or $down$ relative to an additional factors related to the economics problems, which is uncertain and no previsible. And that factor is modelised by the $perturbation\ factor$ which is a non deterministic function of the expiration time $T-t$, denoted by $\eta(T-t)$, and defines a stochastic behaviour according to the way the price evolute.

2.2.1.2 Equations and parameters[12]

Ho and Lee proposed the first no-arbitrage model of the term structure in a paper in 1986.10 They presented the model in the form of a binomial tree of bond prices with two parameters: the short-rate standard deviation and the market price of risk of the short rate. It has since been shown that the continuous-time limit of the model in the traditional risk-neutral world is

$$dr = \theta(t)dt + \sigma dz$$

where σ, the instantaneous standard deviation of the short rate, is constant and $\theta(t)$ is a function of time chosen to ensure that the model fits the initial term structure. The variable $\theta(t)$ defines the average direction that r moves at time t. This is independent of the level of r. Ho and Lee's parameter that concerns the market price of risk is irrelevant when the model is used to price interest rate derivatives.

Technical Note 31 at www.rotman.utoronto.ca/ ~ hull/TechnicalNotes shows that :

$$\theta(t) = F_t(0,t) + \sigma^2 t$$

where $F(0,t)$ is the instantaneous forward rate for a maturity t as seen at time zero and the subscript t denotes a partial derivative with respect to t. As an approximation, $\theta(t)$ equals $F_t(0,t)$. This means that the average direction that the short rate will be moving in the future is approximately equal to the slope of the instantaneous forward curve.

Technical Note 31 also shows that

$$P(t,T) = A(t,T)\exp(-r(t)(T-t))$$

where

$$\ln\frac{P(0,T)}{P(0,t)} = (T-t)F(0,t) - \frac{1}{2}\sigma^2 t(T-t)^2$$

with $F(0,t) = -\partial \ln P(0,t)/\partial t$. The zero-coupon bond prices, $P(0,t)$, are know for all t from today's term structure of interest rates. Equation of $P(t,T)$ therefore gives the price of a zero-coupon bond at a future time t in terms of the short rate at time t and the prices of bonds today.

2.2.1.3 Advantages and limits[7]

Ho-lee model, surely bring something new and consistent to the valuation of derivative, it begin where . However, the model presents some disadvantages:

- Negative rates can appear in the model for some cases.
- The model considere that zero-coupon rates have the same volatility so it can only simulate the interest with constant volatility and that does not reflect the complexity of interest rate modeling.

2.2.2 Hull & White model (1990).

2.2.2.1 Model presentation

The original Hull-White's model is from a paper by J.Hull and A.White titled "Pricing Interest Rate Derivative Securities" published in 1990.
This model is an extention of the Vasicek model. Hull-White was intended to fite the intial term structure. It's a risk neutral model.

$$dr(t) = (\theta(t) - ar(t))dt + dW(t)$$

σ is constant and positive.σ is the standard deviation of the shorte rate. ϑ(t) is "a function of time chosen to ensure that the model fits the initial term structure".
The Hull-White model is main-reverting as the Vasicek model, "a" is the rate of the reverting. It's also a positive constant. dr is the increase of the short rate r during the time interval dt and dw as in the Vasicek model is a Wiener process.

The short rate is,

$$r(t) = r(s)e^{-a(T-s)} + \int_s^t \theta(u)e^{-a(t-u)}du + \sigma \int_s^t e^{-a(T-u)}dW(u)$$

The zero-coupon bond prices at time t with a maturity at T are given by

$$P(t,T) = A(t,T)e^{-B(t,Y)r(t)}$$

Where

$$B(t,T) = \frac{1}{a}(1 - e^{-a(T-t)})$$

and

$$A(t,T) = \frac{P(0,T)}{P(0,t)}exp(B(t,T)F(0,T) - \sigma^2 B(t,T)^2 \frac{(1 - e^{-2at})}{4a})$$

$F(0,T)$ is the instantaneous forward rate determinate at time 0 for a forward beginning at t .$P(0,t)$ and $P(0,T)$ are the zero-coupon bond prices at time 0 respectively with a maturity at t and at T.

2.2.2.2 Advantages and limits

- The main advantage is that ϑ calibrate perfectly with today's term structure
- A limit of this model is the possibility of negative rate due to a Gaussian distribution.
- Another limit is that we are in risk neutral world, but in the reality a financial institution doesn't place it in such position.

Chapter 3

HULL AND WHITE MODEL AND TRINOMIAL TREES

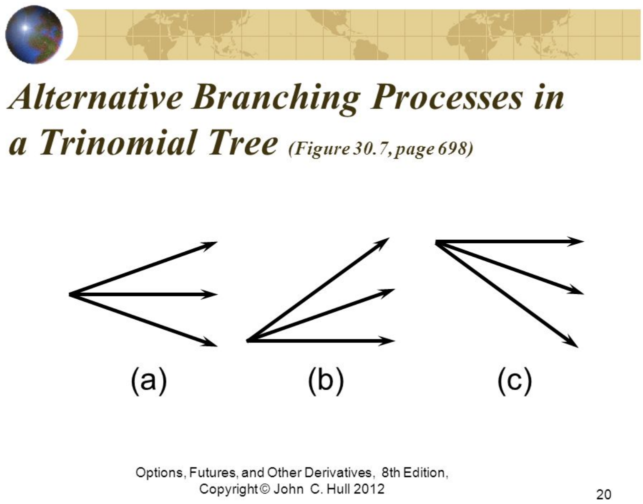
This part intends to propose a representation of the evolution of the short rate though time with an interest rate trees. An interest rate tree is a discrete-time representation of the stochastic process for the short rate. The procedure is therefore appropriate for no-arbitrage models and not for equilibrium model. We choose the Hull and White model for our application.
Hull and White have proposed a two stage procedure which is presented in a paper titled *"Numerical procedures for implementing term structure models I: Single-Factor Models,in Journal of Derivatives"*, published in 1994. Hull and White published another paper titled *"USING HULL-WHITE INTEREST-RATE TREES"* published in 1996, it's from this paper that our application is based in large part.
The first part explains what a trinomial tree for yield curve models is, or more precisely, this part describes the operating of a tree. The second part put in place the Hull and White procedure and the third place is a numerical application.
To realize our application, we use the language and environment R.

3.1 Trinomial interest rate tree

A trinomial interest rate tree is a very natural and simple representation. Each node represents an interest rate at a specific time and the nodes are linked by branches in particular way. A branch will determinate the evolution of the interest rate between two time step.
There are three possibilities of evolution for the interest rate, with a certain probability, therefore a node has tree branch. The time step is the same between each step (delta t).
The vertical distance between the nodes on the tree is same between each step,

it is deltla R.
Each node has coordinates (i, j) where i represent the time (column) and j the vertical position in the tree (row). The variable i is a positive integer and j is a positive or negative integer.
Tree alternative branching possibilities exist. The first one, the standard trinomial branching is “up one, straight along and down one”. The second one is “up, two up, straight along” and the third one is c”. The following figure illustrates these different branching more clearly:

Figure 3.1.1: Branching types



3.2 A Hull and White model implementation with trinomial interest rate tree

It should be recalled that in Hull and White model for the short rate ‘r’ is

$$dr(t) = (\theta(t) - ar(t))dt + dW(t)$$

The model is risk neutral.

And, Δt, the time step, is constant. In the case where the limit of Δttends to zero, we can postulate that the delta t rate R, follows the same process as r.

We can write that,

$$dR(t) = (\theta(t) - aR(t))dt + dW(t)$$

The first stage of the procedure

The first stage will consist of building a tree for a variable $R1$. $R1$ is R with $\theta(t)$ and the initial value of R equal to zero.

Therefore the process is,

$$dR_1(t) = (-aR_1(t))dt + \sigma dW(t)$$

ΔR_1 equal to $\sqrt{3V}$. V is $var(\Delta R_1)$. This choice is justified from 'the viewpoint of minimization'. It's the vertical distance between the nodes on the tree.

The expected value and variance of ΔR_1 are:

$$E[\Delta R_1] = (e^{-a\Delta t} - 1)R = MR_1,$$

with, $M = (e^{-a\Delta t} - 1)$

$$Var(\Delta R_1) = V = \sigma^2 \frac{(1 - e^{-a\Delta t})}{2a}$$

"the spacing between interest rates on the tree,ΔR , is set as" $\sigma\sqrt{3\Delta t}$.

The coordinates (i, j) refer to the node where $t = i\Delta t$ and $R_1 = j\Delta R$. As mentioned above, we have three branching. The tree use the standard branching "up one, straight along and down one" until thresholds of the index j, the vertical position in the tree. In fact, Hull and white integrate two limits value $jmin$ and $jmax$ which are respectively the minimum and the maximum value where the transition probabilities are always positive (when $a > 0$). We can always justify this choice with the concept of mean reversion.

The mechanism is the following: when the index j reach $jmin$ we switch the branching from the original branching to the "up, two up, straight along" branching. And when the index j reach $jmax$ we switch the branching from the original branching to the " down , two down, straight along" branching. The values are :

$$jmax = Ent(\frac{0.184}{a\Delta \mathrm{t}}) + 1 \; and \; jmin = -jmax$$

The probabilities are chosen to match with the expected value and variance of ΔR_1, mentioned above. Moreover, the sum of probabilities of the three different evolution of the short rate (pu, pm, pd) must be one. Therefore, we have a system of three equation with three unknown, pu, pm, pd.

For the standard branching we have,

$$\begin{cases} (E[\Delta R_1] =) & MR_1 = (p_u - p_d)\Delta R \\ (Var(\Delta R_1) =) & \sigma^2 \frac{(1-e^{-a2\Delta t})}{2a} = (p_u - p_d)\Delta R^2 - (p_u + p_d)^2 \Delta R^2 \\ p_u + p_d + pm = 1 \end{cases}$$

by solving the system of equations, we have,

$$\begin{cases} p_u = \frac{1}{6} + \frac{j^2M^2+jM}{2} \\ pm = \frac{2}{3} - j^2M^2 \\ p_d = \frac{1}{6} + \frac{jM^2-jM}{2} \end{cases}$$

To obtain the probabilities for the other branching, we solve in the same way, the difference is that R_1doesn't take the same values due to difference value taking by the index j.

For the branching when j reach $jmax$, we have ,

$$\begin{cases} p_u = \frac{7}{6} + \frac{j^2M^2-3jM}{2} \\ pm = -\frac{1}{3} - j^2M^2 - 2jM \\ p_d = \frac{1}{6} + \frac{jM^2+jM}{2} \end{cases}$$

For the branching when j reach $jmin$, we have,

$$\begin{cases} p_u = \frac{1}{6} + \frac{j^2M^2-jM}{2} \\ pm = -\frac{1}{3} - j^2M^2 - 2jM \\ p_d = \frac{7}{6} + \frac{jM^2-3jM}{2} \end{cases}$$

The second stage of the procedure

In this stage , we construct the final tree. From the tree for R1 we construct the tree for R. The purpose of this stage is to match the tree with the initial term structure. To do that, we define $\alpha(t)$ which is function of time.

$R(t) = R_1(t) + \alpha(t)$

Hull and White use an iterative procedure to determinate $\alpha(t)$. According to this perspective, Hull-White use Arrow Debreu asset noted Q. $Q(i,j)$ have a value of 1$ if the node (i,j) is reached otherwise $Q(i,j)$ pays 0$.

The value of $Q(0,0)$ is 1 and $\alpha(0)$ is equal to the initial maturity of the zero coupon rate.

We note $P(0, R(i)i\Delta t)$ the price of the zero coupon bond maturing at time $i\Delta t$.

$$P(0, R(i)i\Delta t) = \sum_{j=-n_i}^{n_i} Q_{i,j} exp[-(\alpha_i + j\Delta R)\Delta t]$$

Where n_i is the number of nodes on each side of the central node at time $i\Delta$t

With,

$$\alpha_i = \frac{ln(\sum_{j=-n_i}^{n_i} Q_{i,j} e^{-j\Delta R\Delta t}) - ln(P(0, R(i+1)(i+1)\Delta t))}{\Delta t}$$

And,

$$Q_{i,j} = \sum_l Q_{i-1,l} p(l,j) exp[-(\alpha_{i-1} + l\Delta R)\Delta t]$$

Where $p(l,j)$ is the probability of moving from node $(i-1,l)$ to node (i,j).

The operating is to calculate $Q(i,j)$ and α_i iteratively. In fact, to calculate α_i we need to have the value of $Q(i,j)$. To calculate $Q(i,j)$, we need to know all $Q(k,j)$ with k=1,...,i-1. And to calculate $Q(i,j)$ we need to have the value of α_{i-1} etc.

Chapter 4

APPLICATION : HULL & WHITE AND TRINOMIAL TREE.

In this part, we present an application of the trinomial tree with the software R. In the application, inputs are the parameter a and th volatility σ of the HULL-WHITE model, the delta t-period Δt , the maturity T which is the end of the tree and the initial zero rates.

First example

To begin, we choose a volatility σ= 0.01, a delta t period Δt= 1 and maturity $T = 3$, both expressed in years. The zero rate at maturity $T = 1$ is 3.824% , at $T = 2$ is 4.512% and at$T = 3$ is 5.086%. The parameter $a = 0.1$. The first stage The first tree $R1$ refer to the tree without $\theta(t)$ (see part 3). So we obtain,

Figure 4.0.1: Tree of the first stage. Exemple 1

	1	2	3
2	0	0.00000000	0.03464102
1	0	0.01732051	0.01732051
0	0	0.00000000	0.00000000
-1	0	-0.01732051	-0.01732051
-2	0	0.00000000	-0.03464102

To calculate ΔR, we use an approximation. In fact, the exact formulation of ΔR is $\sqrt{3\sigma^2 \frac{(1-e^{-a\Delta t})}{2a}}$ but we use limited development, so we have $\Delta R = \sqrt{3\sigma^2 \Delta t}$.

In our context, ΔR is equal to$0.01\sqrt{3 * 1} = 0.01732$

We have also have to specify $jmax = -jmin$ according to the formula $jmax = Ent(\frac{0.184}{a\Delta t}) + 1$, here we find $jmax = 2$
Obviously, The short rates in the first stage tree are the same for each node with the same index j due to absence of $\theta(t)$. Now, let's see the building of the second stage
The second stage
As see in part 3, we have to calculate $\alpha(t)$. $\alpha(t)$ translate the nodes to match with the initial short rates. We obtain,

Figure 4.0.2: Alpha.Example 1

0.03824000	0.05205000	0.07819359

Which are $\alpha(0)$, $\alpha(1)$. and $\alpha(2)$.To finish, we just have to add alpha with values found in the tree of the first stage Finally, the final tree is ,

Figure 4.0.3: Tree of the second stage . Example 1

0.00000	0.00000000	0.11283460
0.00000	0.06937051	0.09551409
0.03824	0.05205000	0.07819359
0.00000	0.03472949	0.06087308
0.00000	0.00000000	0.04355257

Unlike the tree of the first stage, the values of the rates are determined by the branching probabilities. We stock the probabilities in a matrix,

Figure 4.0.4: Branching probabilities. Example 1

	2	1	0	-1	-2
2	0.8992908	0.01109333	0.08961592	0.00000000	0.0000000
1	0.1236133	0.65761075	0.21877592	0.00000000	0.0000000
0	0.0000000	0.16666667	0.66666667	0.16666667	0.0000000
-1	0.0000000	0.00000000	0.21877592	0.65761075	0.1236133
-2	0.0000000	0.00000000	0.08961592	0.01109333	0.8992908

For example, the probability to move to a node with $j = 1$ to $j = 0$ is 21,877592%

Second example

To understand the importance of the volatility, we make another example with a volatility equal to 0.05

The first stage

With the new volatility of 5%, we obtain a significant impact on the values of the rates ,

Figure 4.0.5: Tree of the first stage.Example 2

0	0.00000000	0.17320508
0	0.08660254	0.08660254
0	0.00000000	0.00000000
0	-0.08660254	-0.08660254
0	0.00000000	-0.17320508

The extremes rates of the first stage tree move up +/-3.5% to +/-17.3%.

The second stage

The new Alpha values are,

Figure 4.0.6: Alpha.Example 2

0.0382400	0.0532500	0.1452258

We note that $\alpha(1)$ move up 5.2% to 5.3%. Cleary, the increase is reasonable. Furthermore, $\alpha(2)$ move up 7.8% to 14.5% which is an important increase.

The final tree is,

Figure 4.0.7: Tree of the second stage . Example 2

0.00000	0.00000000	0.31843090
0.00000	0.13985254	0.23182836
0.03824	0.05325000	0.14522582
0.00000	-0.03335254	0.05862328
0.00000	0.00000000	-0.02797926

Minimum rate at maturity 3 years with a volatility of 1% is 4.3% while with a volatility of 5% we have a rate of -2.7%. Maximum rates move up 11,3% to 31,8%.

Third example

In the example, we keep a volatility of 5% but we modified "a" which is the parameter of reverting. We change the value of a=0.1 to a=0.05. The first tree doesn't change. The value jmax move up 2 to 4. The branching probabilities are,

Figure 4.0.8: Branching probabilities

	4	3	2	1	0	-1	-2	-3	-4
4	0.8930718	0.01877417	0.08815407	0.0000000	0.0000000	0.0000000	0.00000000	0.00000000	0.0000000
3	0.1042144	0.64525955	0.25052609	0.0000000	0.0000000	0.0000000	0.00000000	0.00000000	0.0000000
2	0.0000000	0.12265323	0.65715239	0.2201944	0.0000000	0.0000000	0.00000000	0.00000000	0.0000000
1	0.0000000	0.00000000	0.14347066	0.6642881	0.1922412	0.0000000	0.00000000	0.00000000	0.0000000
0	0.0000000	0.00000000	0.00000000	0.1666667	0.6666667	0.1666667	0.00000000	0.00000000	0.0000000
-1	0.0000000	0.00000000	0.00000000	0.0000000	0.1922412	0.6642881	0.14347066	0.00000000	0.0000000
-2	0.0000000	0.00000000	0.00000000	0.0000000	0.0000000	0.2201944	0.65715239	0.12265323	0.0000000
-3	0.0000000	0.00000000	0.00000000	0.0000000	0.0000000	0.0000000	0.25052609	0.64525955	0.1042144
-4	0.0000000	0.00000000	0.00000000	0.0000000	0.0000000	0.0000000	0.08815407	0.01877417	0.8930718

The new Alpha values are,

Figure 4.0.9: Alpha.Example 3

0.03824000	0.06169719	0.28929857

the final tree is,

Figure 4.0.10: Tree of the second stage . Example 3

0.00000	0.00000000	0.0000000
0.00000	0.00000000	0.0000000
0.00000	0.00000000	0.4625036
0.00000	0.14829973	0.3759011
0.03824	0.06169719	0.2892986
0.00000	-0.02490535	0.2026960
0.00000	0.00000000	0.1160935
0.00000	0.00000000	0.0000000
0.00000	0.00000000	0.0000000

We notice that all values move up. In part 2, we explain that a is the rate (speed) of the mean-reverting. We notice that more a will be important more quickly the rate will be reaching the reversion level rapidly. We can interpret in our example that the drift $a(\theta(t) - r)$ are positive and when we increase a we decrease the interest rate toward the reversion level. Therefore ϑ(t) is higher than the rate r.

Fourth example

If we make another example with $a = 0.001$, the final tree is,

Figure 4.0.11: Tree of the second stage . Example 4

0.00000	0.00000000	16.11197
0.00000	0.15569215	16.02537
0.03824	0.06908961	15.93877
0.00000	-0.01751293	15.85217
0.00000	0.00000000	15.76556

We can see that the values the rates are very high. It's coherent with the theory. In fact, a is 100 times as slow as old value so it's normal that we are very far of the reversion value due to the slow reversion.
Calibration In reality with true values, we have to calibrate the model. To calibrate the model, we have to estimate a and the volatility σ. The objective is to choose a and the volatilityσ to have "goodness of fit" measure minimized. "The goodness of fit" measure the most commonly used is , $Min \sum (U_i - V_i)^2$. Where U_i is the market price of the ith calibrating instrument and V_i is the price given by the model for this instrument.

Conclusion

To begin, we can say that interest's rates tree is simple to use with a simple mechanism. The characteristic of mean reversion is very interesting because observable in the reality. Another quality of the model is the compatibility with the zero coupon curve . In our application, we could observe the importance of the volatility. In fact, higher volatility give interest rates more variable. In the application, the volatility is constant , in other model more sophisticate, volatilities are function of time. We notice the limits of the model mentioned in the second part. In fact, we have negatives rates. As regards calibrations difficulties, it's seemed to be true referring to some applications of the calibration, we have consulted. There are alternative model to compensate the problem of negatives rates. For instance, the Black-Karasinski model. In this model we doesn't have $f(r) = r$ like in the Hull and White model but $f(r) = ln(r)$, therefore negative rate are not possible. However, BM model is less friendly to use than HW. Amelioration could be also done with model with 2 or more factors. However, the calibration is very difficult.

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Annexes

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