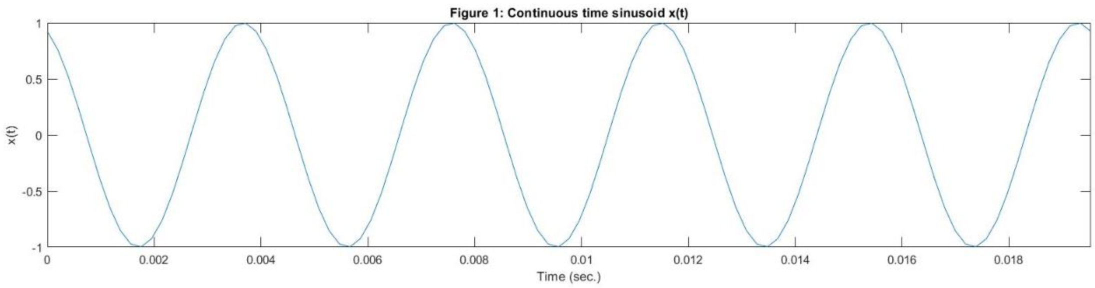
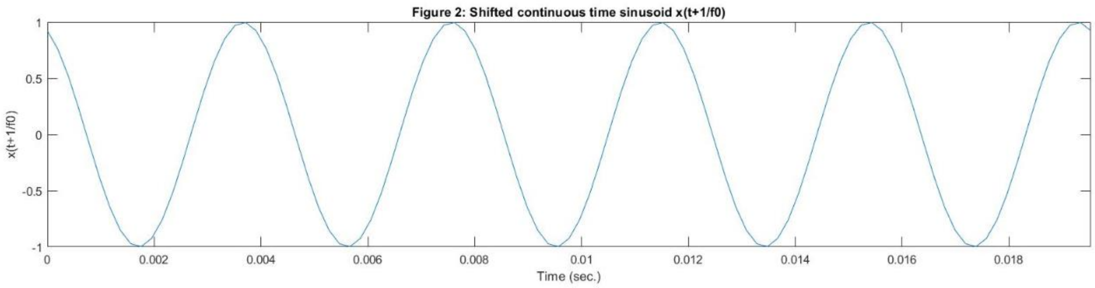
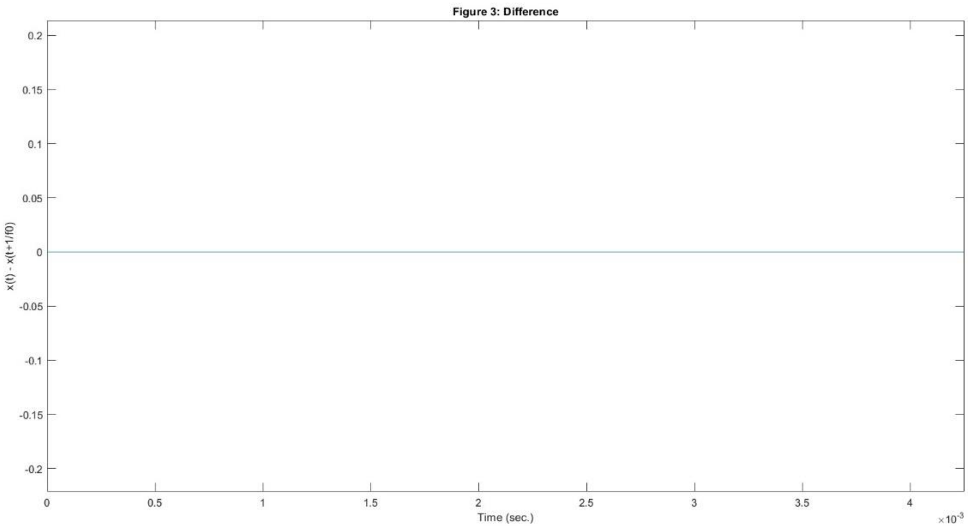
1. (40 pts) Connection between signal features and your ear/the sound you hear. Go through "Codes for problems 1 and 2" in ComputerHW1_SampleCodes.m, and try to

All the plots need to have the correct axis and label. (0.5 pts for each one)

(a) (10 pts) Demonstrate that x(t) is a periodic signal and f0 is truly the frequency via MATLAB graphic illustration as well. As for the MATLAB graphic demonstration, I mean that plot x(t) and x(t+1/f0) and show the two signals are exactly the same (had better include a plot of the difference of the two). (3 pts foreach plot)





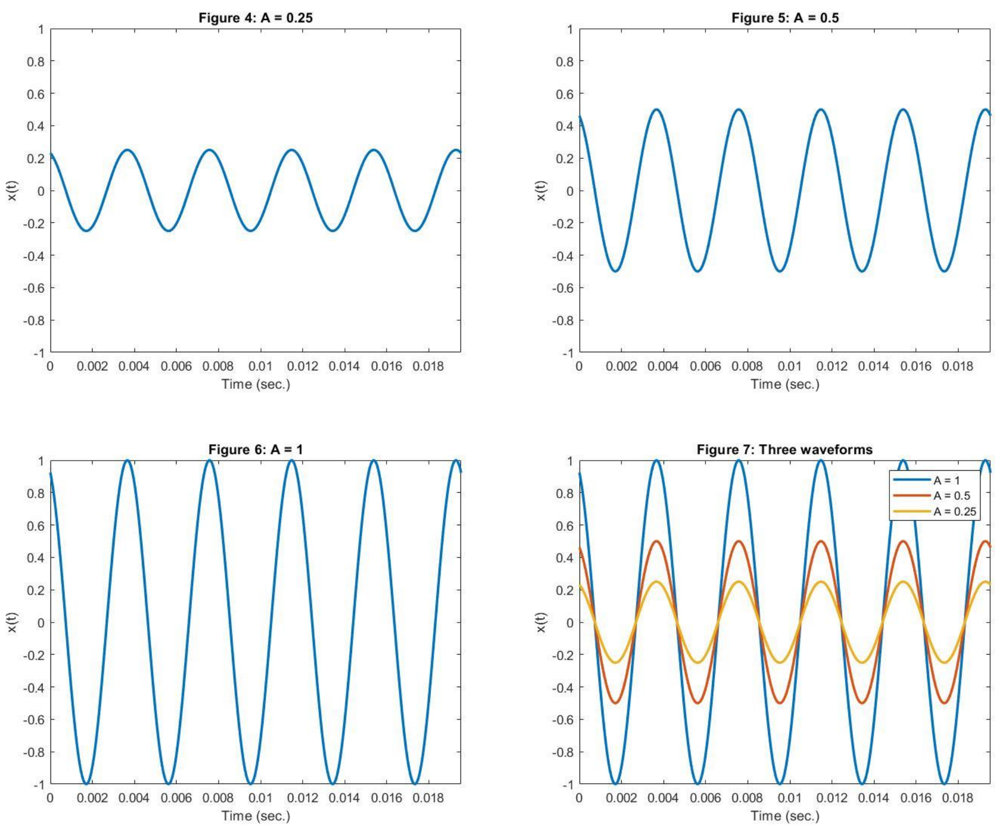


①. Figure 2 is the shifted version of figure 1 by $1/f_0$ in the time domain, also the shifted time $1/f_0$ is the period of $\mathrm{x(t)}$, and it can be found that $\mathrm{x}(\mathrm{t}+1/f_0) = \mathrm{x(t)}$ from the difference. Therefore, $\mathrm{x(t)}$ is a periodic signal and $1/f_0$ is the period.

②. For the mathematical interpretation, if we apply the time shift $1/f_0$ to the signal $x(t) = A\cos(2\pi f_0 t + \phi)$, the equation will become $x(t + 1/f_0) = A\cos(2\pi f_0(t + 1/f_0) + \phi) = A\cos(2\pi f_0 t + 2\pi + \phi) = A\cos(2\pi f_0 t + \phi) = x(t)$. The result will be the same as ①.

③. Figure 3 是將$x(t) - x(t + 1/f_0)$計算出來並 plot 出來的結果，點開來看可能會發現不為 0 但卻非常小的數字，這是因為 MATLAB 在計算的時候不可能計算到小數點下無窮多位而造成的誤差，程式本身並沒有錯誤。

(b) (10 pts) Assign the values of 0.25, 0.5, and 1, respectively, to the variable A while keeping all the other variables as default (i.e., don't modify the other codes). Please tell the changes in the signal by observing the plots and by listening to the sounds as well when the variable A goes from 0.25 to 1. Plot the signals with different values of A in one plot to illustrate your findings. (5 pts for plots, 5 pts for discussion)

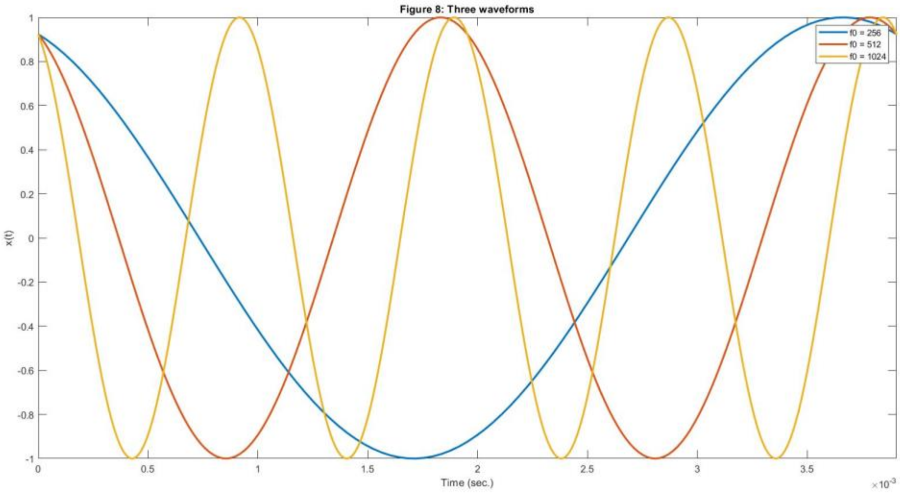


①. If we combine the three different waves into one image, the output is like figure 6.

②. From figure 4 to figure 7, it can be fount that giving different coefficient A changes the magnitude of the wave.

③. From the perspective of the sound, A means the volume of sound. The higher A, the louder of the sound.

(c) (10 pts) Assign the values of 256, 512, and 1024, respectively, to the variable **f0** while keeping all the other variables as default. Please tell the changes in the signal by observing the plots and by listening to the sounds as well when the variable **f0** goes from 256 to 1024. Again, plot the signals with different values of **f0** in one plot to illustrate your findings. (5 pts for plots, 5 pts for discussion)



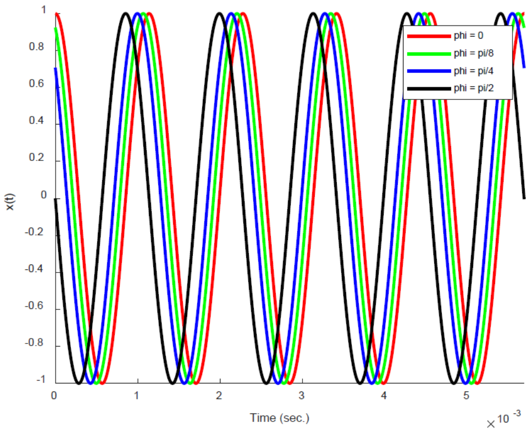
①. The blue line: f0 = 256, the red line: f0=512, the yellow line: f0 = 1024.

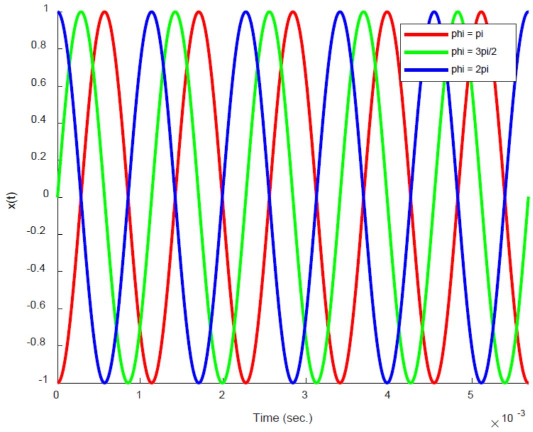
②. It can be found that giving different f_0 changes the frequency of the wave. In other words, f_0 changes the period of the cosine function (because the period is the reciprocal of the frequency).

③. Combine 3 waves into one image (figure 8). It can be found that with higher f_0, we will have the higher oscillation rate, which indicates the higher frequency.

④. From the perspective of the sound, f_0 indicates the pitch of sound.

(d) (10 pts) Assign the values of 0, π/8, π/4, π/2, π, 3π/2 and 2π, respectively, to the variable **phi** while keeping all the other variables as default. Please tell the changes in the signal by observing the plots and by listening to the sounds as well when the variable **phi** goes from 0 to 2π. In addition, tell what type of transformation of the independent variable (i.e., time shift, time reversal, and time scaling) the changes correspond to, and justify your answer via the plots and more specifically via a mathematical equation showing the relationship between the phase ϕ and the corresponding transformation of the independent variable. Is there any limitation for such a transformation to a sinusoidal signal by change of the phase ϕ? If your answer is yes, please describe the limitation. (4 pts for plots, 6 pts for discussion including the answer of “time shift” “time reversal” “time scaling”)





①. From the above figure, it can be found that the coefficient "phi (phase)" corresponds to a time shift of the signal relative to $\phi = 0$. Applying the mathematical equation, $x(t) = A\cos(2\pi f_0 t + \phi)$, if we give signal $x(t)$ a time shift t_0, the equation will become $x(t - t_0) = A\cos(2\pi f_0 (t - t_0) + \phi) = A\cos(2\pi f_0 t - 2\pi f_0 t_0 + \phi) = A\cos(2\pi f_0 t + \phi')$, where $\phi' = -2\pi f_0 t_0 + \phi$, so changing the phase term ϕ can be seen as changing the "time shift" of the signal.

②. From the perspective of the sound, theoretically, we should hear the sound delay for a while. However, most people's ears can't distinguish any phase difference in the sound. It can only be observed from the plots of the signals.

③. $x(t) = A\cos(2\pi f_0\, t + \phi) = A\cos(2\pi f_0\, t + \phi + 2k\pi)$, where k is an integer.

$$x(t + 1/f_0) = A\cos(2\pi f_0\, (t + 1/f_0) + \phi) = A\cos(2\pi f_0\, t + 2\pi f_0/f_0 + \phi)$$
$$= A\cos(2\pi f_0\, t + \phi + 2\pi) = A\cos(2\pi f_0\, t + \phi) = x(t).$$

For the limitation, the answer should be yes, and the limitation should be $0 \le \phi < 2\pi$ because $x(t)$ is periodic with a fundamental period of $1/f_0$ and the maximum observable time shift is $1/f_0$, which corresponds to 2π phase change.

2. (30 pts) Sampling a sinusoidal signal. Go through "Codes for problems 1 and 2" in ComputerHW1_SampleCodes.m, and answer the following questions.

(a) (10 pts) Change the value of the variable fsRatio from 20 down to 1.2 while keeping all the other variables as default. Note that you at least must try fsRatio = 20, 4, 2.5, 2.2, 2, 1.8, 1.4, and 1.2. Assume that the sound with fsRatio = 20 played by your computer sound card is the correct sound for x(t). Listen to the sounds when fsRatio goes from 20 down to 1.2 and tell your findings. More specifically, tell with what values of fsRatio (or with what sampling rate) you can hear the correct sound of x(t). You may justify this finding via the plots of x[n] with different fsRatio values. (You can google "how sound cards work" if you are interested).

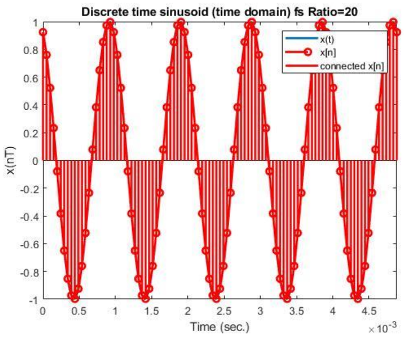
此題主要是讓同學們可以觀察取樣後類比訊號重建時的狀況，在未來課程(Ch4 Sampling)中會學到。

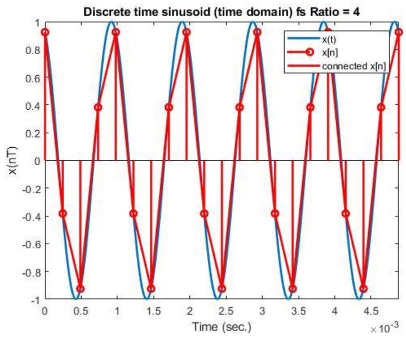
Ans : (5 pts for discussion)

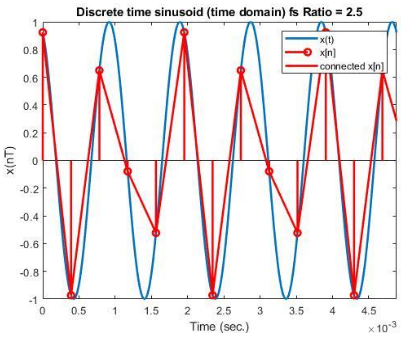
Reduce the fs ratio from 20 to 1.2 (try fs ratio = 20, 4, 2.5, 2.2, 2, 1.8, 1.4 and 1.2). In the beginning, the sound sounds the same with the fs Ratio from 20 to 2.2. After adjusting to 1.8, the pitch is obviously low and unclear which is different form the correct sound. With the sound card of TA , when fs Ratio = 2, chances are you will hear the same pitch as that of the correct sound but the volume of the sound may be different.

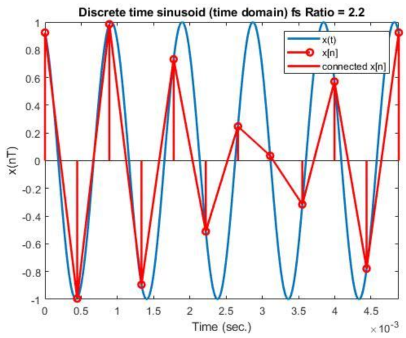
(5 pts for plots) (f0=1024)

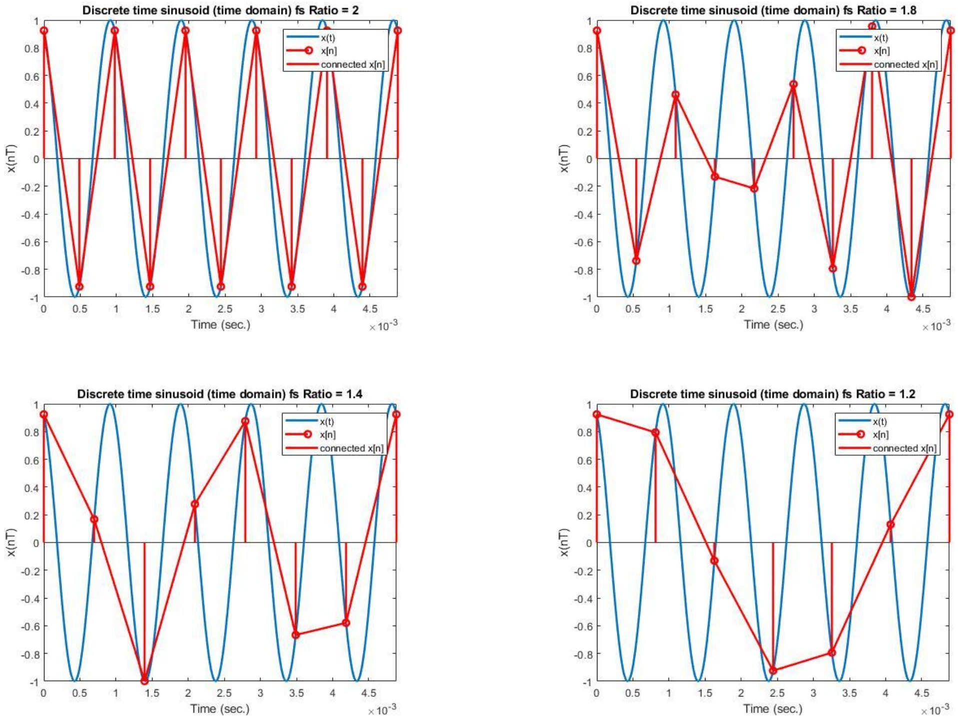
根據下列各圖可發現 fs ratio 大於 2.2 時皆可以看出週期與原訊號一樣 (音效卡實際在做時會有內插等，因此重建時會與原訊號一樣)，至於 2 與 2.2 可能會因音效卡的不同而有所失真，圖較難看出但實際聽會有所差異。











補充說明

經過取樣後，變成 $DT\ sinusoidal\ signal$ 後，等同已將 $absolute\ frequency$ 轉換成 $Normalized\ frequency = \frac{1}{fsRatio}$ ，當 $Normalized\ frequency$從 0 到 0.5 時 ($fsRatio$大於 2)， frequency 愈來愈大者，超過 0.5，基於 $DT\ periodic\ in\ frequency$ 的特性，相當是 normalized frequency 落在
-0.5 到 0 ($fsRatio$ 小於 2, $normalized\ frequency - 1$)，frequency 又會漸漸變低 (*)。所以按照上課教的 (參考 Topic 1 的投影片 第 26 頁)我們把$Normalized\ frequency$ 乘上 Fs 轉回 $absolute\ frequency$ 時，便會發現，當$fsRatio$小於 2 時，(sound card) 重建的 $CT\ absolute\ frequency$會比未取樣前原本 CT 的$absolute\ frequency$來得低 (*)。兩個(*)是呼應的。

(b) (10 pts) Following (2a), what feature of the sinusoidal signal x(t) (i.e., magnitude A, frequency f0, or phase ϕ) has been changed after the sampling so that you hear the incorrect sound? According to the computer experiments in (2a), comment on what is the smallest fsRatio (i.e., the smallest sampling rate) which allows you to hear the correct sound, that is, keeps the above recognized feature unchanged.

Ans : (8 pts for plots or discussion)

Note that sound() reconstructs continuous time signal x(t) from discrete time signal x[n] with the provided fs. From the above plots, after connecting the discrete dots of x[n] (or precisely x[n]), for fsRatio > 2, you still can find the same oscillation rate as that of the correct sound. Therefore, after reconstruction by sound(), you can hear the same pitch. The pitch has changed, so it is known that the frequency has changed, and it can be seen from the above figure.

(此題會依據音效卡不同而 fsRatio > 2 有差異，僅需描述實際聽到的情況即可)

Ans : (2 pts for discussion)

Supposedly, now you can answer the following questions which I raised in class. Under what sampling rate will the DT sinusoidal signals from sampling a CT sinusoidal signal preserve key properties of the CT signals so that the CT signals can be reconstructed from their DT counterparts?

fsRatio > 2. (1 pt) (會依據音效卡不同而有些微差異)

What do you think are the key properties?

frequency. (1 pt)

(c) (10 pts) If you carefully exam all the sampled sinusoidal signal, i.e., the DT sinusoidal signal x[n] via the plots, you will find out that all x[n] with the fsRatio values mentioned in (2a) are periodic signals. Does it mean that all the sampled sinusoidal signals x[n] are periodic signals? If your answer is no, please tell with what values of fsRatio, the DT sinusoidal signal x[n] is not periodic and verify your answer by plotting the aperiodic x[n].

Ans :

經過取樣後，變成 DT sinusoidal signal 後，等同已將 absolute frequency 轉換成 Normalized frequency:

$$fsRatio = \frac{F_s}{F_0} = \frac{1}{Normalized\ frequency}$$

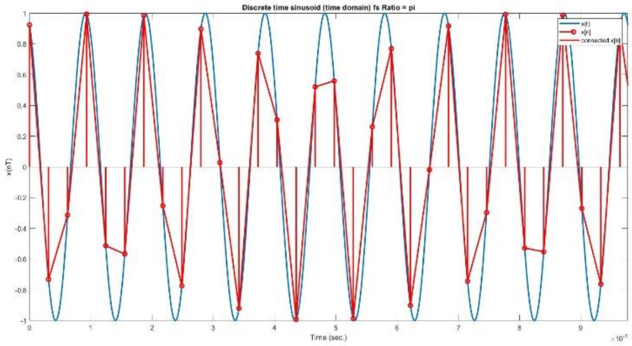
$$Normalized\ frequency = \frac{1}{fsRatio}$$

依據課堂提及

DT periodic signal 的 $Normal\ frequency$ 須為 rational number，故$fsRatio$

也必須為 rational number。

下圖是 fsRatio = pi(Not a peridoc number) 的 DT signal: aperiodic



3. Deciphering MATLAB codes into system models (i.e., mathematical operations).

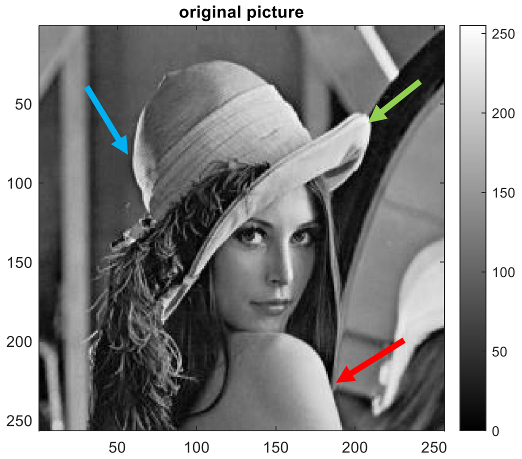
(a) (10%) Go through “3(a)” of “Codes for problem 3” in ComputerHW1_SampleCodes.m, and try to write down the 1D system model (i.e., difference equation converting input to output) used to process each row (i.e., input signal) of the provided Lena image (i.e., lena.jpg). Verify whether the 1D system is causal, linear and time invariant or not, tell what the system does to each row of Lena image and elaborate the reason why.

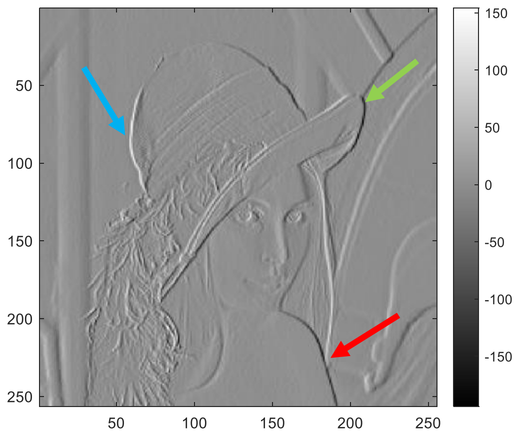
```
94   y = zeros(M, N-1); % system output
95   for m = 1:M,
96       y(m,:) = x(m, 2:end) - x(m,1:end-1); % What is the system model for the row input? Is this system causal, linear and time invariant?
97   end
98   figure
99   imagesc(y);
100  colormap(gray); % show gray scale image according to the provided gray-scale colormap
101  axis image
102  colorbar
```

首先我們會讀到一個 256 x 256 尺寸大小的圖片，這裡會存成二維矩陣 x。這題要我們去找出圖片所通過的系統，在 95~97 行的地方是整個系統做的事情，就是對圖片中的每一條 row，將兩個選取的陣列相減，得到新的陣列。x[m,2:end]相當於 x[m,2:256]就是選擇矩陣第 m 列(row)第 2~256 行(column)的範圍，而 x[m,1:end-1]就是選擇矩陣第 m row 和第 1~N-1 column 的範圍，因此會是兩個陣列。

這裡將陣列表示成 signal，x(m,2:end)對應成沒有 time delay 的訊號 x[n+1]，那麼 x(m,1:end-1)就會是 x[n]。因此整個系統就是$y[n] = x[n+1] - x[n]$。或者也可以把 x(m,2:end)視為 x[n]，這樣整個系統就會是$y[n] = x[n] - x[n-1]$，因為我們沒有特別指定 x[n]是要哪一個 signal。

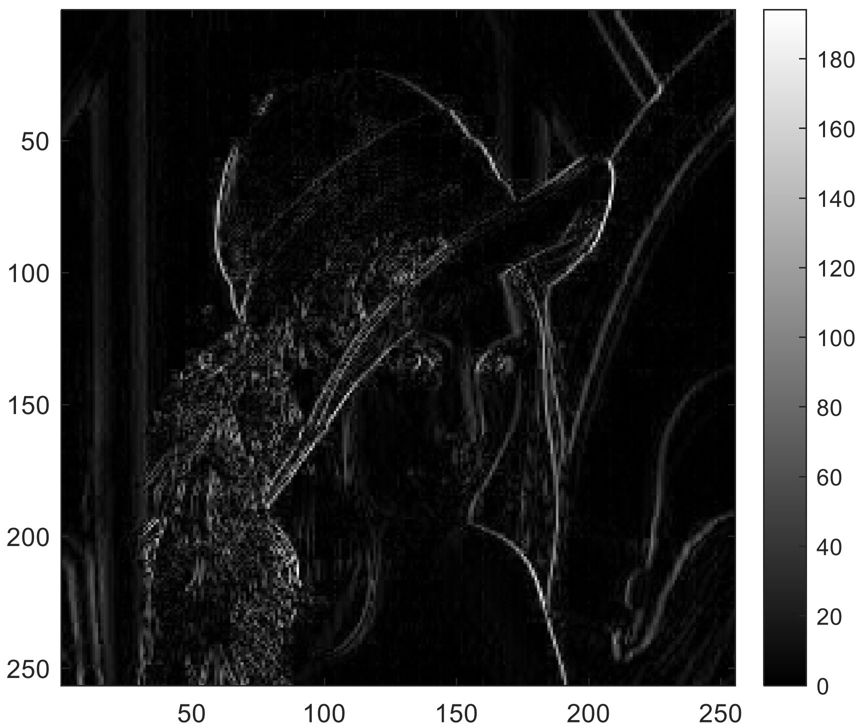
原本的圖片和經過系統後的圖片:





圖片經過處理之後，如果注意兩張圖箭頭指的地方的話，我們會看到圖片中的輪廓被顯現出來。原因是因為圖片中每個 row 相鄰兩個 pixel 的值差距很大的地方，會是物體和背景的交接處。而物體和背景的顏色因為相鄰 pixel 值差不多的關係，因此相減起來值的差異不大。

經過系統後取絕對值的圖片(這裡只是考慮圖片讀不到負值)。



如果觀察取了絕對值的圖片的話，會將相鄰之間差異很大的地方顯亮，因此只會看到亮亮的。

Check Properties:

Linearity:

$$y_1[n] = x_1[n] - x_1[n-1]$$
$$y_2[n] = x_2[n] - x_2[n-1]$$

令$x[n] = ax_1[n] + bx_2[n]$

則$y[n] = x[n] - x[n-1] = ax_1[n] + bx_2[n] - (ax_1[n-1] + bx_2[n-1])$

$= a(x_1[n] - x_1[n-1]) + b(x_2[n] - x_2[n-1]) = ay_1[n] + by_2[n]$

所以 system 符合 linearity

Causality:

如果是$y[n] = x[n+1] - x[n]$的話，就會是 Non-causal，因為 output 有未來的時間的 input 有關(x[n+1]這一項)。如果$y[n] = x[n] - x[n-1]$的話，就會是 Causal，因為所接收到的 output 僅和現在還有過去的時間的 input 有關。

Time-invariant:

$$x_1[n] \rightarrow y_1[n] = x_1[n] - x_1[n-1]$$
$$x_2[n] = x_1[n-n_0] \rightarrow y_2[n] = x_2[n] - x_2[n-1] = x_1[n-n_0] - x_1[n-n_0-1]$$
$$= y_1[n-n_0]$$

(b) (10%) Similar to (a), go through “3(b)” of “Codes for problem 3” in ComputerHW1_SampleCodes.m, and try to write down the 1D system model (i.e., difference equation converting input to output) used to process each row (i.e., input signal) of the provided Lena image (i.e., lena.jpg). Tell what the system does to each row of Lena image and elaborate the reason why. Again verify whether the 1D system is causal, linear and time invariant or not. If it is not causal, please remodel it into a causal form with the same function to each row and implement the causal system you model.

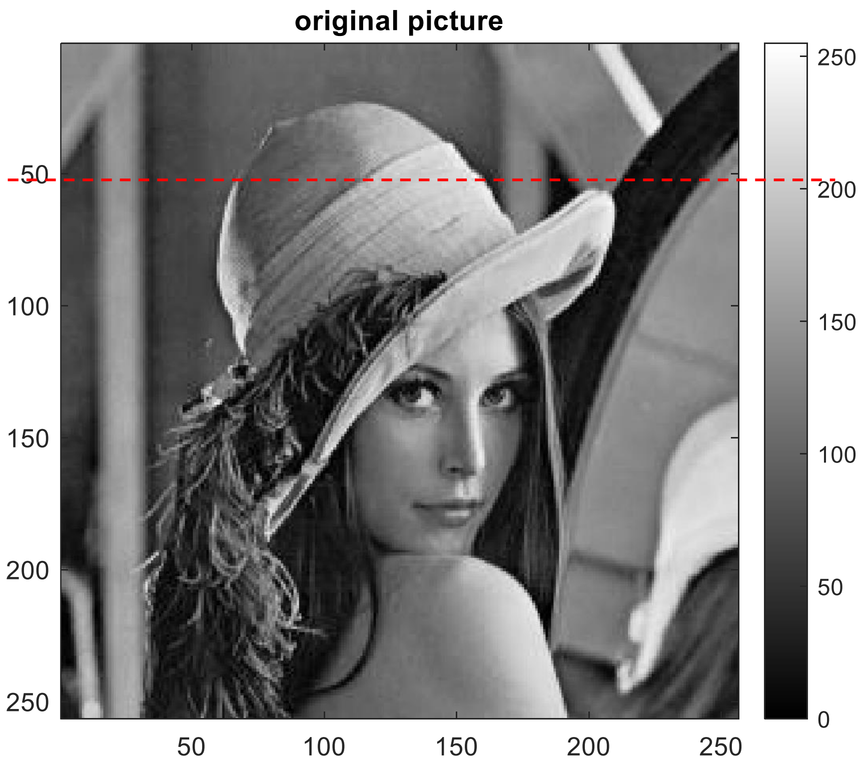
(write down model: 2pts
what system does to each row: 2pts
elaborate reason: 1pts
each property of system:1pts
remodel:2pts)

```
%% ---------- 3(b) ----------
y = zeros(M, N); % system output
for m = 1:M,
    for n = 3:N-2
        y(m,n) = sum(x(m, n-2:n+2))/5; % What is the system model for the row input? Is this system
    end
end
```

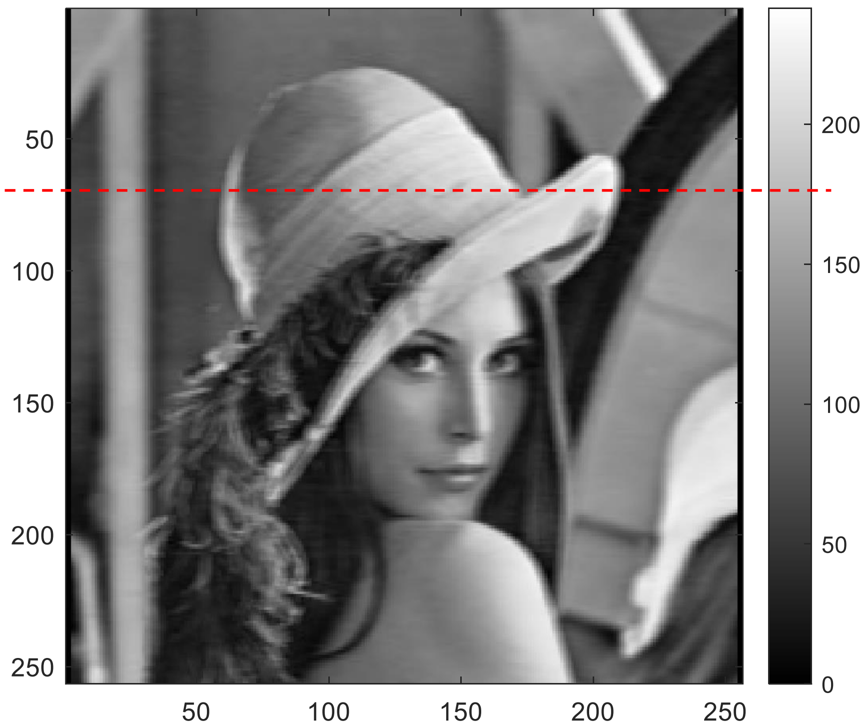
這題是將前面圖片的 pixel，取出 x(m,n-2:n+2)，將 5 個點加起來取平均，因此整個系統就是

$$y[n] = \frac{1}{5}\sum_{k=-2}^{k=2} x\,[n-k] = \frac{1}{5}(\mathrm{x}[\mathrm{n}-2] + \mathrm{x}[\mathrm{n}-1] + \mathrm{x}[\mathrm{n}] + \mathrm{x}[\mathrm{n}+1] + \mathrm{x}[\mathrm{n}+2])$$

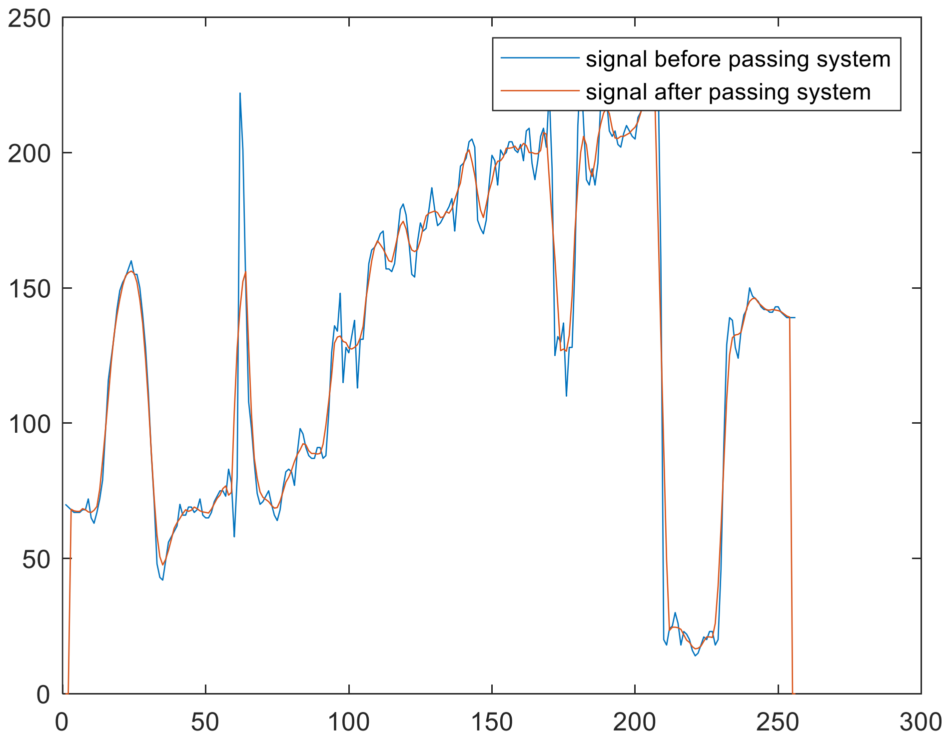
原本的圖片(右邊標示的是影像亮度所對應的數值):



經過系統後的圖片:



這裡我們取一條 row(紅線的部分)，來觀察經過系統前後的變化。



可以看到經過系統後，每條 row 的訊號變得比較平緩，代表每條 row 相鄰

兩個 pixel 的值的差距也變小，從上一小題可以知道物體的邊緣通常在相鄰兩個 pixel 上面會有很大的差距。而兩個 pixel 值的差距變小這件事情也代表物體的邊緣變得比較模糊，因此所出來的影像會比較模糊一些。

Check Properties:

Linearity:

$y_1[n] = \frac{1}{5}\sum_{k=-2}^{k=2} x_1[n-k]$，$y_2[n] = \frac{1}{5}\sum_{k=-2}^{k=2} x_1[n-k]$

令$\mathrm{x[n]} = \mathrm{a}x_1[n] + bx_2[n]$

則 $\mathrm{y[n]} = \frac{1}{5}\sum_{k=-2}^{k=2} x[n-k] = \frac{a}{5}\sum_{k=-2}^{k=2} x_1[n-k] + \frac{b}{5}\sum_{k=-2}^{k=2} x_1[n-k] = \mathrm{a}y_1[n] + by_2[n]$

Causality:

$y_1[n] = \frac{1}{5}\sum_{k=-2}^{k=2} x_1[n-k]$，當 k<0 的時候，$y_1[n]$與收到未來 input 有關，因此為 non causal。

Time-invariant:

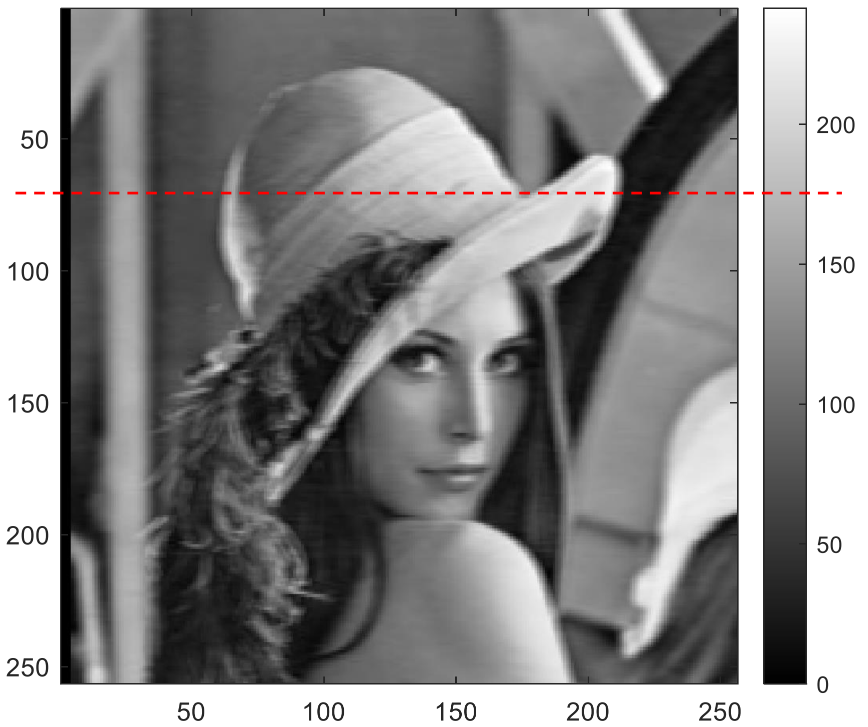
$x_1[n] \rightarrow \quad y_1[n] = \frac{1}{5}\sum_{k=-2}^{k=2} x_1[n-k]$

令$x_2[n] = x_1[n-n_0] \quad y_2[n] = \frac{1}{5}\sum_{k=-2}^{k=2} x_2[n-k] = \frac{1}{5}\sum_{k=-2}^{k=2} x_1[n-n_0-k] = y_1[n-n_0]$

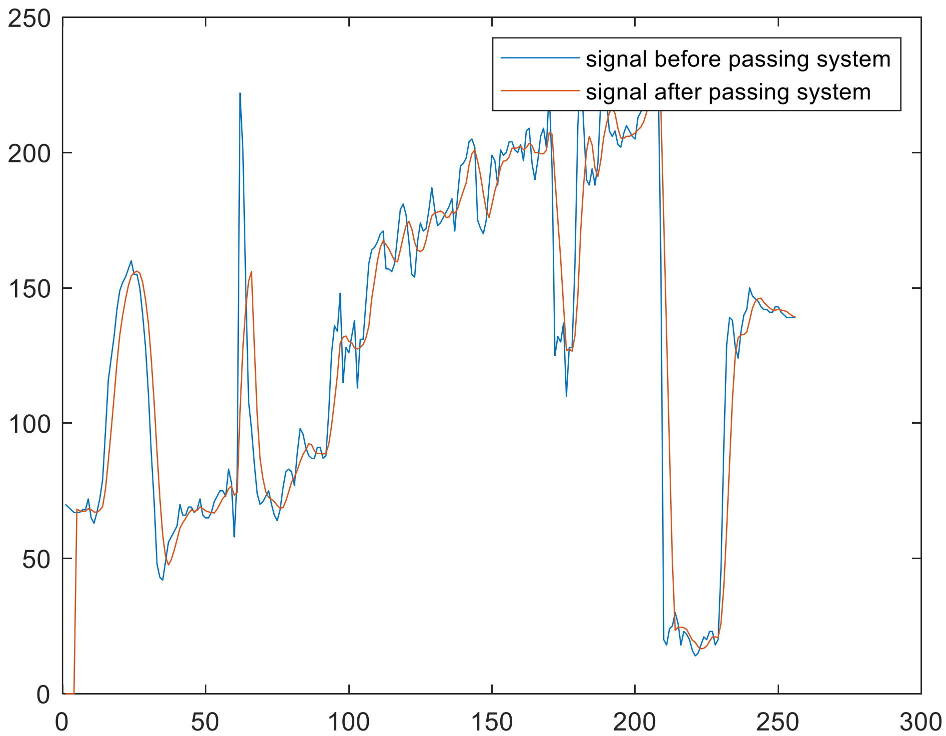
If the system is not causal, please re-model it into a causal system and implement the causal system you model

原本的系統是$\frac{1}{5}\sum_{k=-2}^{k=2} x[n-k]$，這裡的目的是在不改變整個系統功能(即作用)的狀況下，將系統改成 Causal 的形式，因此會變成$\frac{1}{5}\sum_{k=0}^{k=4} x[n-k]$，也就是輸入的地方改成 x[n-4]到 x[n]，以下是對應的 Matlab Code。

```
y = zeros(M, N); % system output
for m = 1:M,
    for n = 5:N
        y(m,n) = sum(x(m, n-4:n))/5;
    end
end
```



同樣從畫線的地方來看訊號通過系統的差異。



(c) (10%) go through the codes in “3(c)” of “Codes for problem 3” in ComputerHW1_SampleCodes.m, and try to write down the corresponding mathematical operations **(including equations and the name of these operations taught in the course lectures)** to each row of Lena image.

(寫出 y1 的公式:3 分　operation:1 分

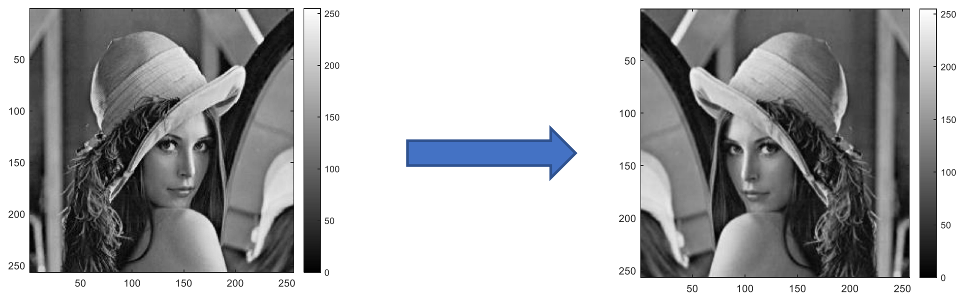
寫出 y2 的公式:3 分　operation:1 分

說明 y1+y2 會長怎樣 2 分)

```
y1 = (x + fliplr(x))/2;
figure
imagesc(y1);
colormap(gray); % show gray scale image according to the provided gray-scale colormap
axis image
colorbar

y2 = (x - fliplr(x))/2;
figure
imagesc(y2);
colormap(gray); % show gray scale image according to the provided gray-scale colormap
axis image
colorbar
```

CODE 裡面 fliplr 的用途是把圖片左右翻轉過來，如下圖所示。

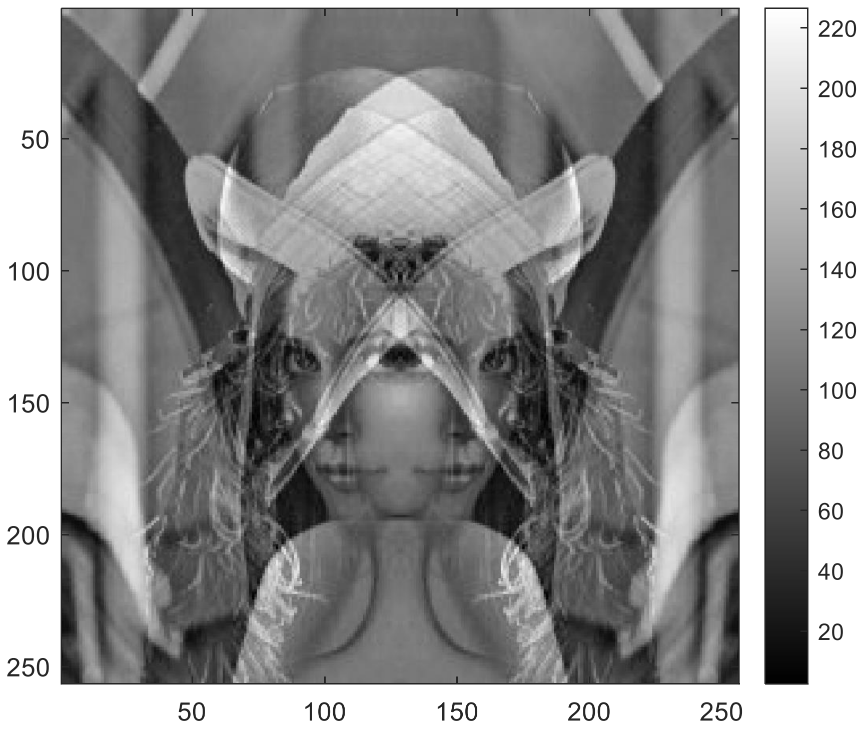


如果從訊號的角度來看，因為圖片的大小是 256*256，如果將影像左右翻轉，每條 row 的第 n 個點，會變成第 256-n 個點。令 N=256，因此翻轉後的訊號變成$x_{flip}[n] = x[N-n]$。

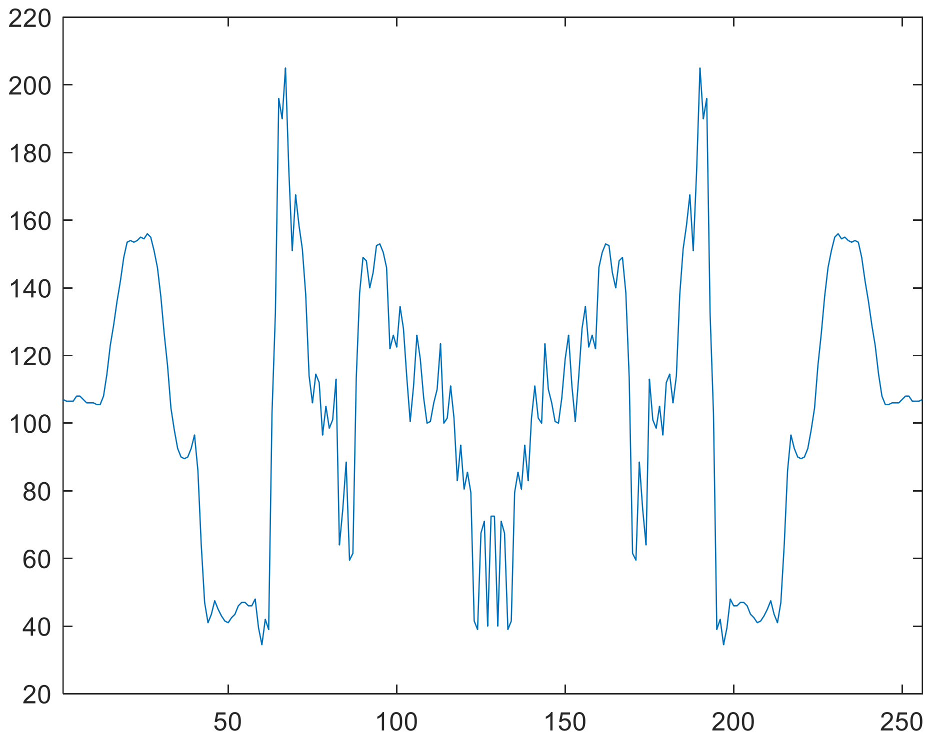
根據 code，y1 對應的式子如下:

$$y_1[n] = \frac{x[n] + x[N-n]}{2}$$

下面是 y1 的圖片



可以看到出來的圖片呈現左右對稱。取出一條 row 來看，結果如下。



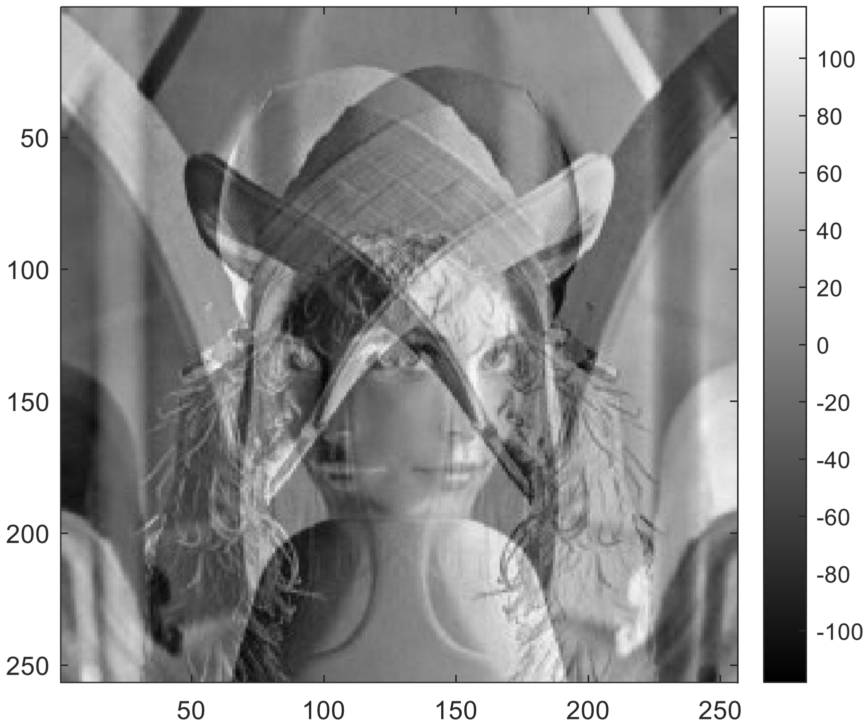
出來的訊號剛好會從中間左右兩邊對稱，如果 n=0 的位置剛好是在中間的

話，就會是偶函數。

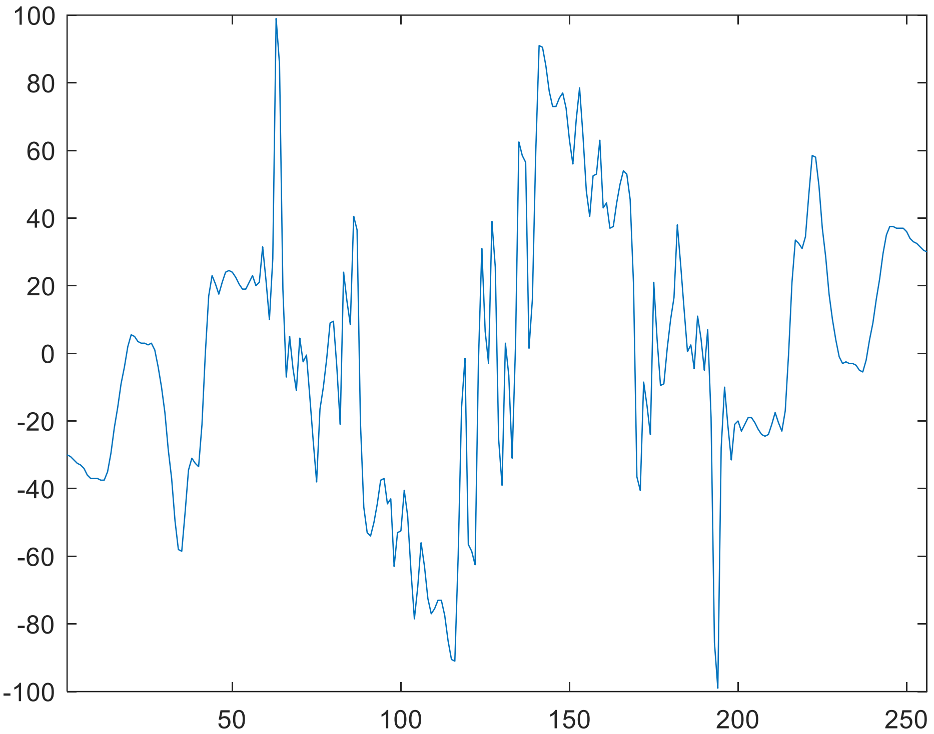
接著根據 code，y2 對應的式子如下:

$$y_2[n] = \frac{x[n] - x[N-n]}{2}$$

下面是 y2 的圖片



這裡出來的圖片好像可以看到左右對稱，只是亮暗不一樣。一樣取出一條 row，結果如下。



可以看到訊號從中間的地方呈現相反對稱，如果 n=0 的位置剛好是在中間的話，就會是奇函數。

```
x_new = y1 + y2; % is x_new equal to x? check it by x-x_new
figure
imagesc(x_new);
colormap(gray); % show gray scale image according to the provided gray-scale colormap
axis image
colorbar
```

這裡 x_new 就是把 y1 和 y2 這兩個訊號相加起來，也就是$y_1[n]+y_2[n]$，因此會得到原本的$x[n]$。

$$x_{new}[n]=y_1[n]+y_2[n]=\frac{x[n]+x[N-n]}{2}+\frac{x[n]-x[N-n]}{2}=\mathrm{x}[\mathrm{n}]$$

下面是 x_new 的圖片，會長得跟 x 一樣。

這題老師的目的是讓同學用圖片去觀察 even signal 和 odd signal，並且相加起來可以得到原本的影像，可以去看 even-odd decomposition。