Draft version February 15, 2022
Typeset using LaTeX default style in AASTeX631

Population Models

Kevin B. Moposita[1]

[1] *Villanova University*
800 Lancaster Avenue
Villanova, PA 19085, USA

1. INTRODUCTION

The topic of this assignment is a segue from the last assignment. The logistical equation that was utilized to determine the bifurcation points and Feigenbaum's constant is more commonly encountered when analyzing populations. With the logistic equation, the rate of population growth is proportional to the population size. However, the logistical equation only takes the population cap into account and not any other circumstance such as loss of life or other events that would otherwise impact the population amount. A great example of this, unfortunately, is the current pandemic that has changed everyone's life. Within these circumstances, there are multiple constraints that can be placed due to the constant fluctuation of a person's health.

2. VERHULST MODEL

As previously mentioned, the logistical equation utilized to approximate Feigenbaum's constant is also utilized to model population growth. Equation (1) presents this equation.

$$x_{n+1} = rx_n(1 - x_n) \tag{1}$$

With increasing values of r, the logistic model would behave asymptotically and eventually in an oscillatory manner. From this equation, we can modify it to take into account a population constraint that is time-dependent. Equation (2) refers to the Verhulst equation.

$$\dot{P} = kP(1 - \frac{P}{N}) \tag{2}$$

In this equation, P is the population size, k is the growth rate, and N is the size constraint placed on the population. The population that will be looked at is Ecuador's population from the 1960's until 2020. *Figure 1* models Ecuador's population during this time period. As seen in the plot, the normal population growth seems to be exponential.

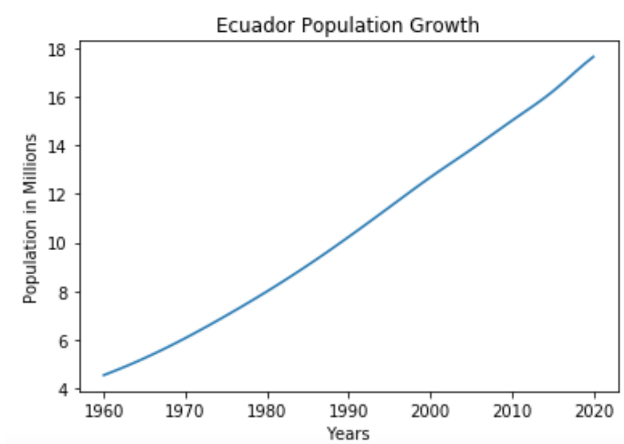

Figure 1. Modeling of Ecuador's population from 1960 to 2020.

Additionally, it is also possible to model the growth rate of the population during this period of time. As can be seen, the general trend of the growth rate steeply declines and quickly rises and falls right after the beginning of the 2010's. This is expected as it would be unrealistic for the growth rate to continuously increase year after year. This is best explained by the inhabitants beginning to utilize the resources around them for the next couple of decades to come. And with no influx of additional resources, the growth rate continues to drop until just a little after 2010, when the country really begins to look to trade.

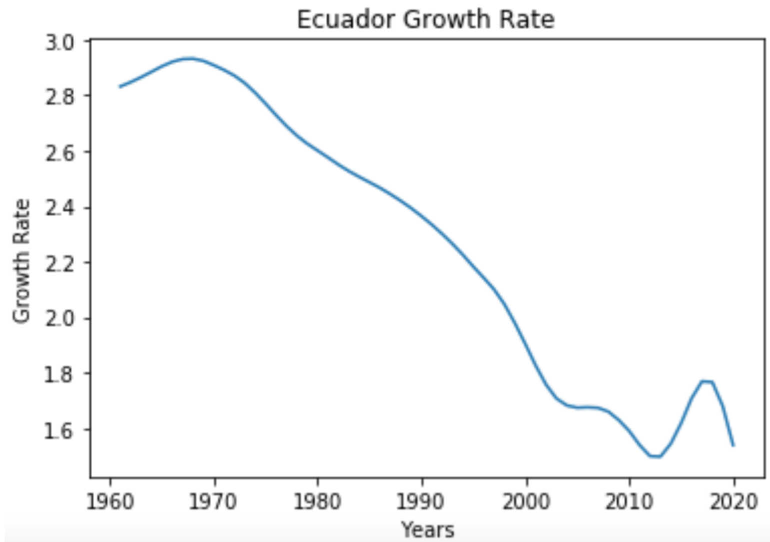

Figure 2. Ecuador's growth rate from 1960 to 2020.

We can also attempt to plot our own fit to the population model in *Figure 1.* With the Verhulst equation, we can incorporate constraints that would allow for an accurate model of the population. N, representing a size constraint, is an effective example of this. At some point, the rate of population growth has to decrease, just as *Figure 2* is shown as doing. Population growth is not forever, and barring an infinite source of resources, population will be capped off.

Utilizing Ecuador's national population data from the 1800's until 2020, an analytical fit can be added to the model. This can be accomplished with Equation (3) below:

$$P(t) = \frac{K}{1 + (\frac{K - P_0}{P_0})e^{-rt}} \tag{3}$$

In this equation, K is defined as the carrying capacity, which is the highest amount the population is able to grow to, assuming infinite time. P_0 represents the initial population, while r is the growth rate. t signifies time. *Figure 3* displays the Verhulst fit overlaying on the population model of Ecuador. As it can be observed, the Verhulst fit actually does not fully encapsulate the entire population. Instead, the fit begins to cap off much earlier than expected and this is due to the low carrying capacity that was utilized for this fit. The calculated values of both the carrying capacity and the rate of growth can be found in *Table 1* below.

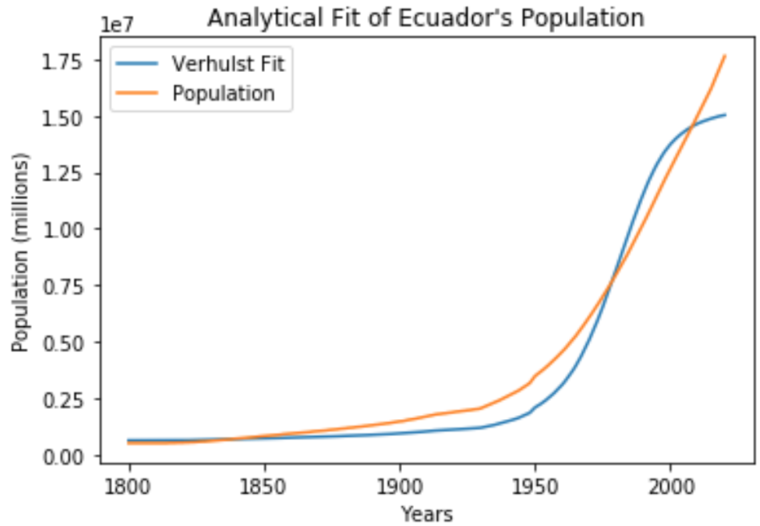

Figure 3. Verhulst fit overlaying on the population model of Ecuador.

Figure 3 shows the Verhulst fit capping off the population just a bit after 2020, while the population model clearly surpasses this constraint.

Table 1. Parameter Values of the Verhulst Fit

Parameter	**Value**
K	1.5203 x 10^7
r	4.4334 x 10^{-1}

In addition to the Verhuls fit, an exponential fit can also be added of the population model. Unlike the Verhulst model, the exponential model is not dependent on a carrying capacity parameter, just the growth rate. This modification of the equation can be seen in Equation (4).

$$N = P_0 e^{rt} \tag{4}$$

For this equation, N becomes the new population. The rest of the variables stay the same. *Figure 4* is the exponential model that was created utilizing a new calculated rate of growth. *Table 2* contains the value of this growth rate.

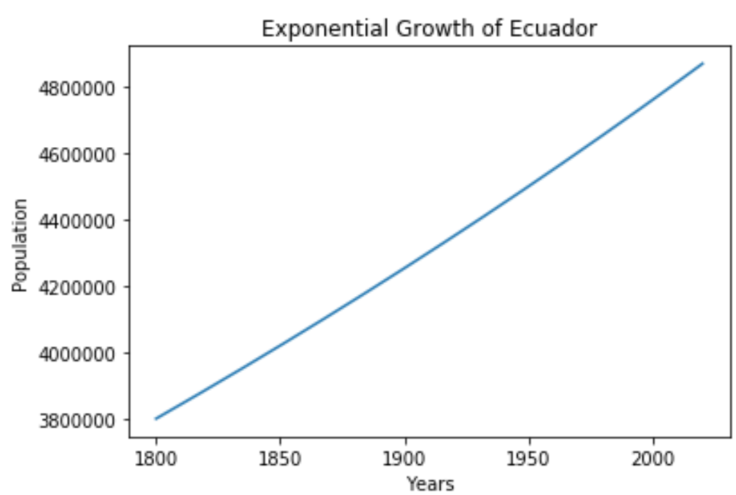

Figure 4. The calculated exponential model of Ecuador's population with r = 0.00112.

Table 2. Growth Rate Parameter of Exponential Model

Parameter	**Value**
r	0.00112

While *Figure 4* is supposed to behave as an exponential model, it does not. Instead, the model appears to be increasing more linearly than exponentially. We have no current idea as to why this is happening, though suspicions are placed upon the very low calculated r value.

3. RABBITS AND FOXES MODEL

Just as the Verhulst equation is useful with modeling constrained populations, the Lotka-Volterra equation is useful in modeling interactions that also utilize differing growth rates. However, there is more than one rate of growth present in these differential equations, unlike before. Equations (5) and (6) represent these differential equations which are meant to represent the relationship between rabbits and fox (predator and prey). While the equations utilized carry four variables, they can be further simplified so that both equations are now only dependent on a single parameter.

$$\dot{R} = \alpha R - \beta RF \tag{5}$$

$$\dot{F} = -\gamma F + \delta RF \tag{6}$$

$$\dot{R} = \alpha R(1 - F) \tag{7}$$

$$\dot{F} = -\gamma F(1 - R) \tag{8}$$

$$R' = pR(1 - F) \tag{9}$$

$$F' = -\frac{F}{p}(1 - r) \tag{10}$$

The first set of differential equations for rabbits and foxes were utilized for simplicity. The parameters and their respective values can be found below, in *Table 3*.

Table 3. Growth Parameters of Rabbits and Foxes

Parameter	**Value**
α	1
β	0.2
γ	0.15
δ	1.2

In this model, α is representative of the rate of birth of the rabbits, β is the rate of interaction between rabbits and foxes, γ is the rate at which the foxes eat the rabbits during said interaction, and δ is the rate at which foxes die, should they not encounter any rabbits. *Figure 5* is a plot that models both the population of rabbits and foxes against time. *Figure 6* is a zoomed-in version of this model where the peaks are more pronounced. Aside from these parameters, both the initial populations of rabbits and foxes were set to 10.

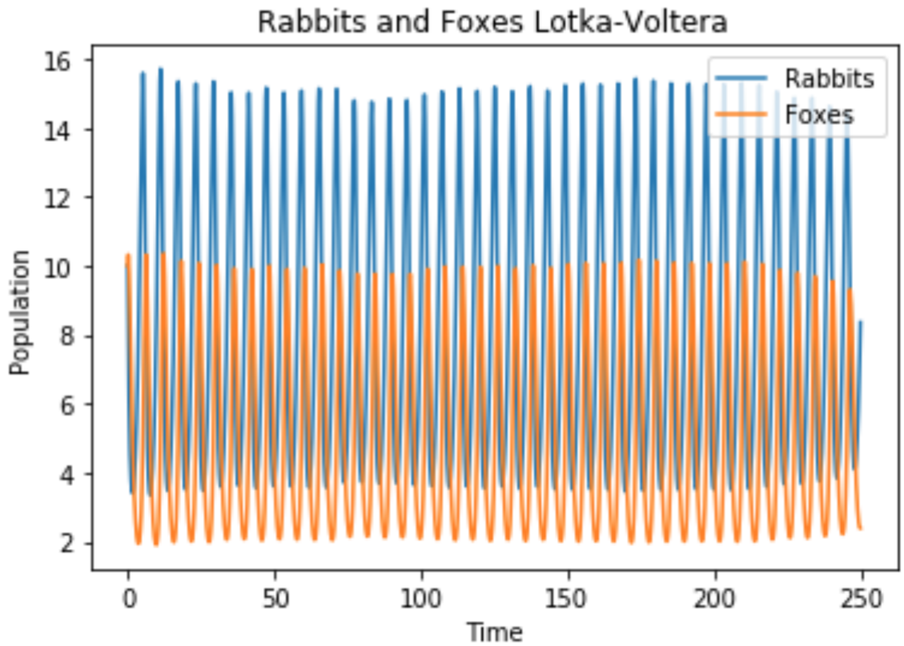

Figure 5. The population model of both rabbits and foxes against time.

As noticed in *Figure 5*, both populations more in an oscillatory manner, which is expected. However, the foxes appear to have much larger oscillations than the rabbits and this can be attributed to the initial conditions that were previously shown. Additionally, it is worth noting how the cap of the population of rabbits appears to be around the same region and the same is true for the foxes. *Figure 6* provides a closer look at the set of oscillations for both animals. The rounded tip is due to the high number of iterations that was utilized.

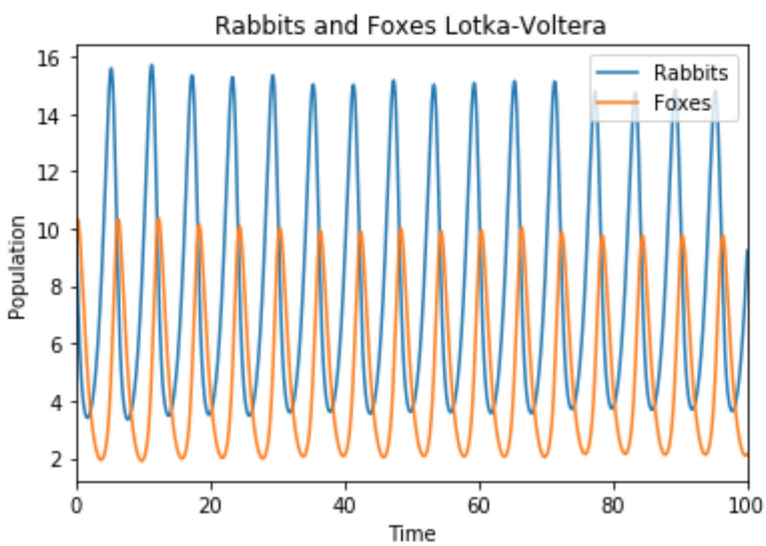

Figure 6. A zoomed-in version of Figure 5.

Figure 7 is a plot of the phase diagram between rabbits and foxes. In this diagram, the appearance of the plot tells us two things: it is a periodic function and the direction of the phase diagram is counter-clockwise. The plot is determined to be periodic as it eventually returns to the same path and cycles through, periodically. The direction of the plot is determined to be counter-clockwise as it follows what is happening in the actual scenario. While the number of foxes is low, the population of rabbits will begin to increase. As the population of rabbits begins to increase, so does the population of the foxes and will begin to eat more rabbits. As the number of rabbits decrease, so does the population of foxes. The cycle then continuously repeats.

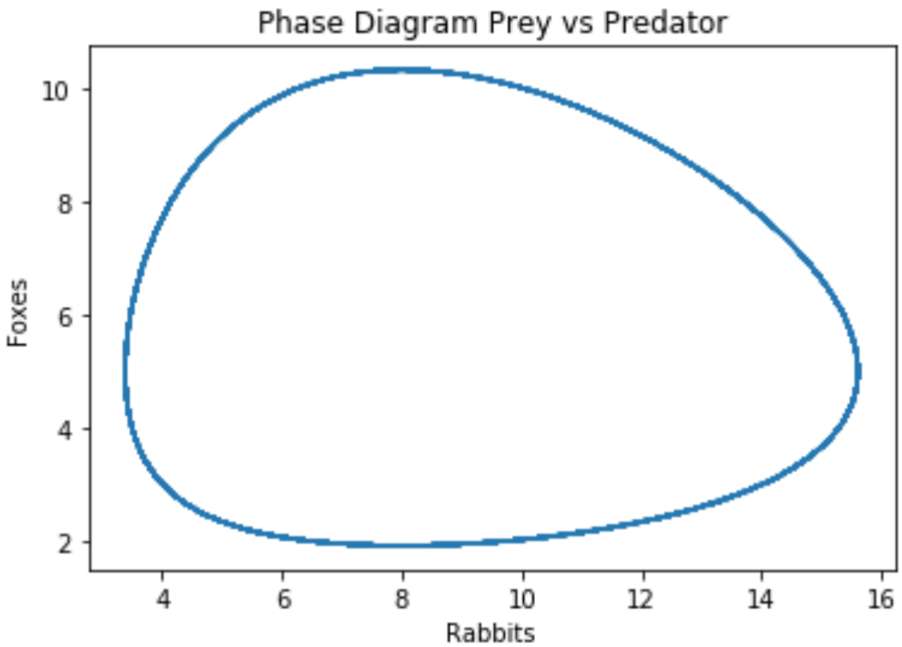

Figure 7. The phase diagram of rabbits and foxes.

4. PANDEMIC MODEL

Another way to model populations is through a simulated (not really) scenario in which the population was going through a pandemic. In this pandemic, the population would have to be categorized into three groups, those are are healthy, those who have gotten sick, and those who have recovered from the virus and gained immunity. Within the model, a person who is sick can spread the virus to the rest of the population (according to the determined contact rate). From there, those that are sick have a pre-determined chance of recovering in a certain amount of days and can now be considered immune. The differential equations (Equations 11, 12, and 13) that will be used for this model.

$$\dot{H} = -\alpha HS \tag{11}$$

$$\dot{S} = \alpha HS - \beta S \tag{12}$$

$$\dot{I} = \beta S \tag{13}$$

Within these differential equations, α is representative of the rate that healthy people get sick and β is representative of the rate at which people will become immune. These parameters, along with the rest, can be found below in *Table 4*.

Table 4. The parameters of the Pandemic.

Parameter	**Value**
α	0.5
β	0.29
H	0.97
S	0.03
I	0

Utilizing the three differential equations above, along with the pre-determined parameters, a model of the pandemic population can be created. *Figure 7* models this scenario and the three categories of the population would look after 100 days. As evidenced in the plot, the amount of healthy people takes a tumble approximately 20 days after it begins. While this is going on, the population of those that are sick begins to increase, which is the relationship that we were expecting. Simultaneously, the population of those that are immune begins to increase. After this period of time, the population of those that are healthy plateaus just below 0.4. The population of those that are sick drops down and never increases again. The population that are now immune continues to increase for about 10 more days and plateaus at approximately 0.7. Analyzing the population of those that were sick, we can determine a few more properties. The maximum number of sick people at any given time was about 12% of the population. Overall, approximately 0.73% of the entire population was sick. The time at which the maximum number of people who were sick occurred after 12 days.

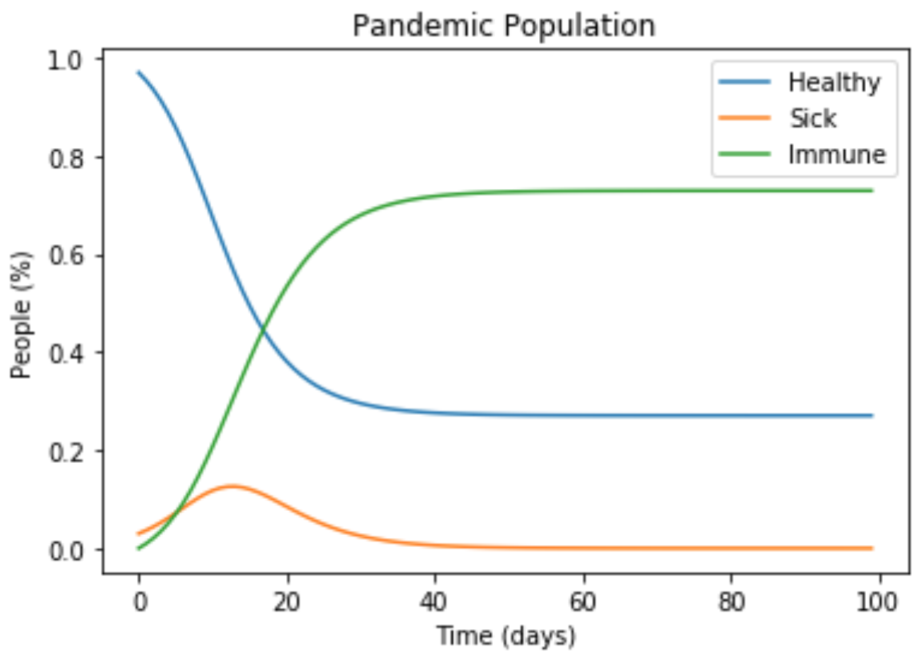

Figure 8. The population model for the pandemic scenario.

Now, what if the population developed a vaccine and common sense dictated they should take it (unfortunately not the case in today's world)? To account for this, the starting number of the immune population is no longer zero. For this scenario, 10% of the population is now immune. How would this change our previous model and the properties that we've determined? *Figure 9* displays the model with the introduction of inoculation and compared to the previous model, there is a bigger discrepancy between sick and immune. With this new model, the maximum number of sick is now 8%, the overall number of sick is 59% of the population, and the time at which the maximum number of sick people took place was now 13 days.

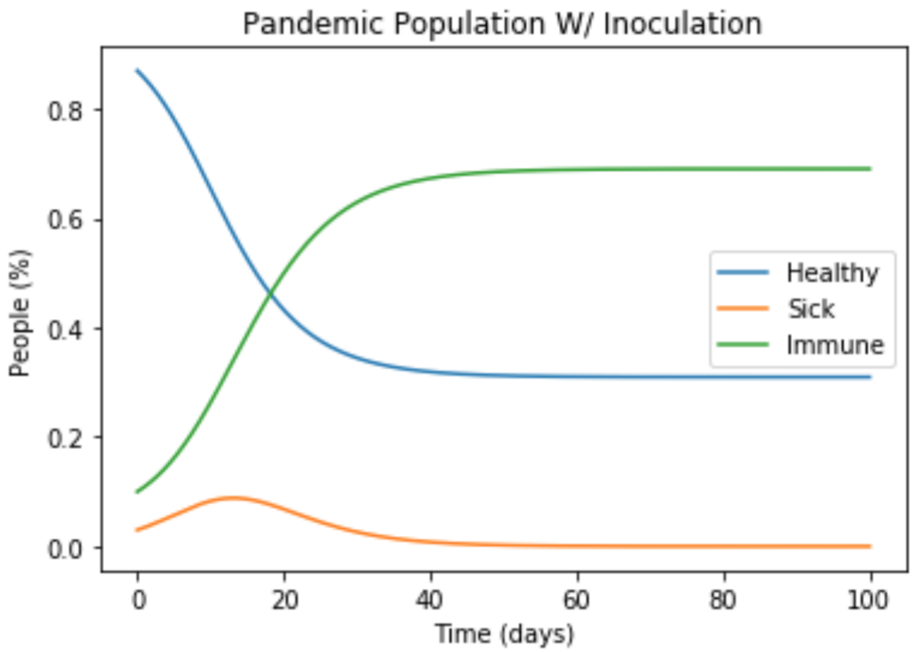

Figure 9. Pandemic population with the introduction of modern science.

Table 5 displays the new parameters utilized in the new scenario.

Table 5. New Parameters for Inoculation

Parameter	**Value**
α	0.5
β	0.29
H	0.87
S	0.03
I	0.10

5. CONCLUSION

The purpose of this assignment was to branch off of the last assignment and analyze different types populations with different methods. While the logistical equation (Equation 1) is useful for modeling population growth, there is no constraint present that would cap off the population. This is where the Verhulst equation (Equation 2) shines as it allows for a calculated population constraint. Aside from this, an exponential model is also an interesting equation that models a population without the carrying capacity. Another method of mapping populations is through the modeling of the interactions between rabbits and foxes (prey and predator). Utilizing the differential equations above, their relationship can be modeled. We find that in this model, a plot of the respective phase diagram would show that it is periodic and moves in the counter-clockwise direction. One last example involves modeling the Covid-19 pandemic, which has unfortunately ruined many people's lives (though it helped me reach my weight goal). In this scenario, rates were utilized to model the relationship between parts of the population that were healthy, sick, and immune. With the rates and the differential equations, a model was created to observe the behavior of the three groups against a function of time.