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Dynamical Chaos

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1. INTRODUCTION

The purpose of this assignment is to understand, plot, and analyze dynamical systems, and being watchful of potential chaos. Dynamical systems are where functions describe a time dependence of a point in a geometrical space. Within these systems, they are described by a state that is further described by a vector of real numbers, at any given time t. Dynamical systems always follows the *evolutionary rule*, a function that describes which future state will develop as a result of the current state, though the future states are generally predictable. Different types of sets and maps were assigned to prove this. This is best observed when plotting the logistic map of the Feigenbaum set and analyzing the bifurcation points and its stability. The Mandelbrot set, a classical fractal, demonstrates how some initial values are bound within the set and the geometric shape it forms. Arnold's cat map demonstrates the chaos that's included when an image is encrypted and later decrypted. The Chirikov standard map is an area-preserving map that describes dynamical variables.

2. THE FEIGENBAUM SET

2.1. *Introduction*

One way to study and analyze dynamical chaos is to look and plot the Feigenbaum set. The Feigenbaum set is a set of numbers that are bound according to the logistical equation (Equation 1) below:

$$x_{t+1} = Rx_t(1 - x_t) \tag{1}$$

Within this equation, chaos can ensue according to the respective value of R through multiple iterations. Developed by Mitchell Feigenbaum, there are various values of R in which bifurcation occurs. With these R values, a ratio is taken utilizing Equation (2). This ratio is the limit between the distances of the bifurcation points. As additional bifurcation points are taken into account, the ratio eventually converges to approximately 4.6639, Feigenbraum's constant. Generally, the logistic map for Feigenbaum's set is depicted as so in *Figure 1*.

$$ratio = \frac{a_{n-1} - a_{n-2}}{a_n - a_{n-1}} \tag{2}$$

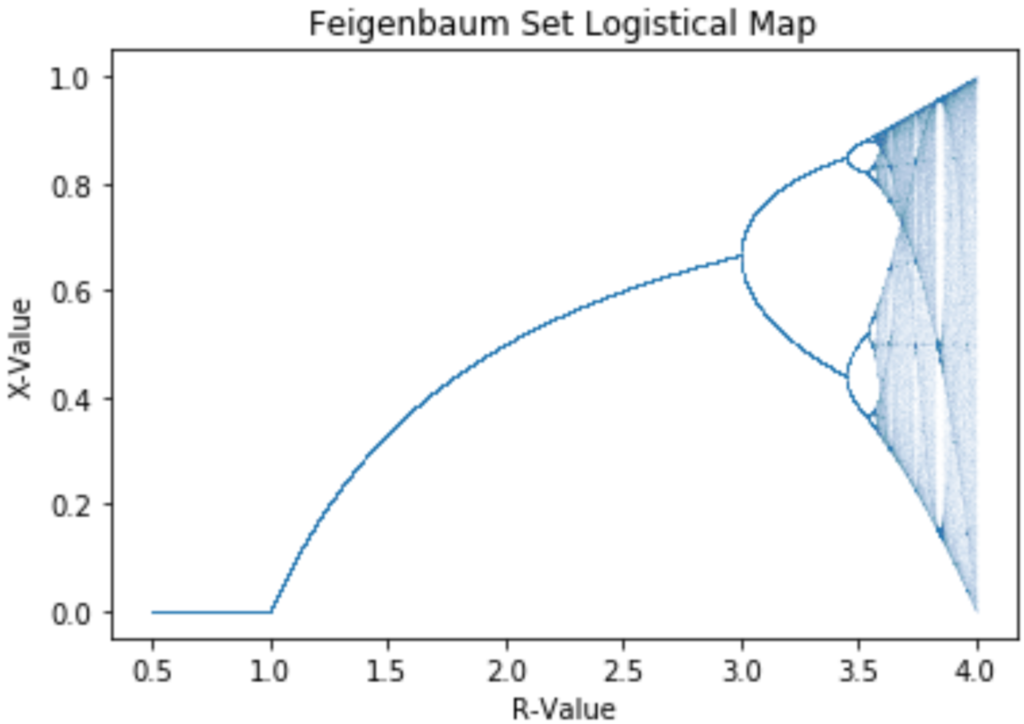

Figure 1. Depiction of the logistic map of Feigenbaum's set with R values 0.5 to 4.

As it can be observed in *Figure 1*, a couple of bifurcation points can be easily identified before the map begins to lean to chaos, as evidenced by the apparent random plot movement on the right side. However, upon a closer inspection, it reveals smaller bifurcation points. *Figure 2* and *Figure 3* provide a closer look at the first two bifurcation points from the logistic map.

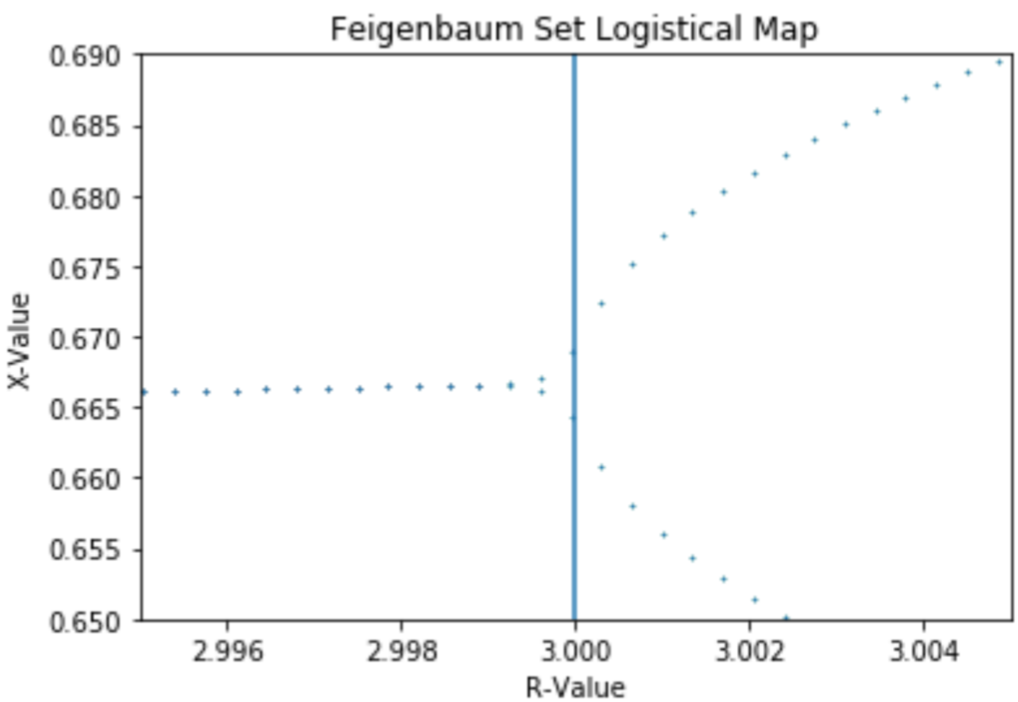

Figure 2. A close up of the first bifurcation point. The vertical line pinpoints the bifurcation point.

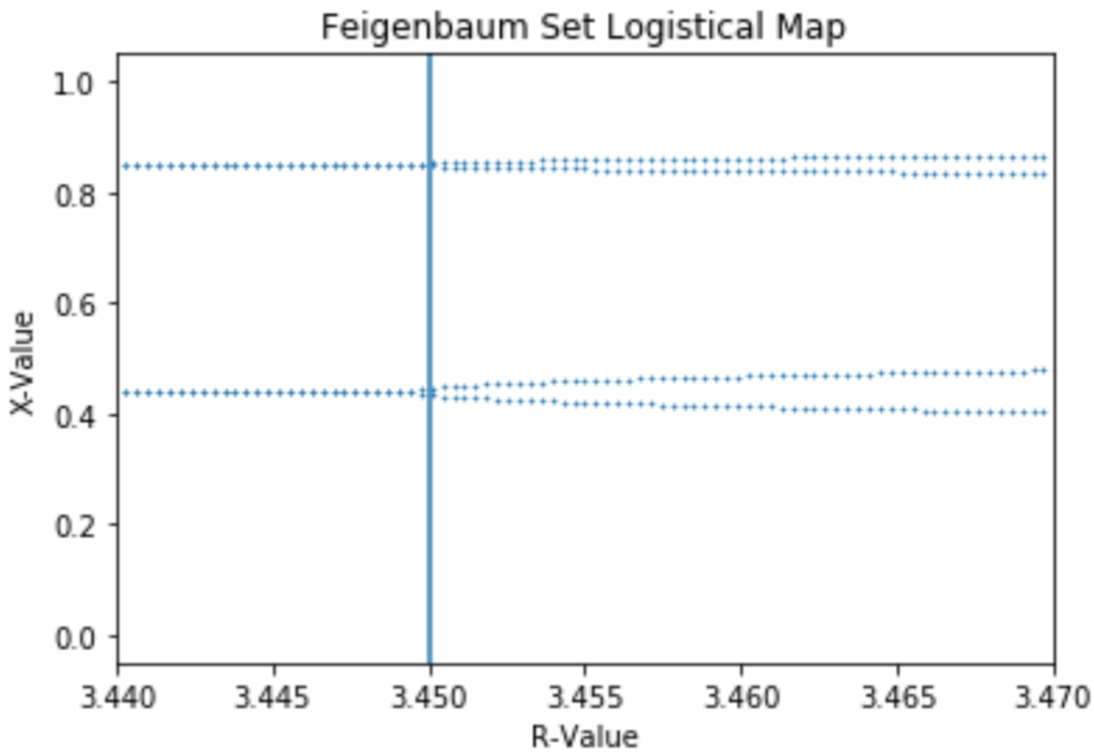

Figure 3. A zoom-in on the second bifurcation point. The vertical line pinpoints the second bifurcation points.

Analyzing the logistic map, the first six bifurcation points were determined and can be found in *Table 1* below.

Table 1. Parameters Obtained from the First Six Bifurcation Points

N	Period	Bifurcation Value
1	2	3.00
2	4	3.4494897
3	8	3.5440903
4	16	3.5644073
5	32	3.5687594
6	64	3.5696916

With these bifurcation points, Feigenbaum's constant was approximated to 4.668 utilizing Equation (2).

2.1.1. *Descending to Chaos*

As previously mentioned, as the value of R increases, the logistic map tends to favor disorder. More specifically, as R gets closer to 4, chaos begins to ensue. *Figure 4* demonstrates the plot of the logistic equation when R is 2.75. After a small number of iterations, the plot stabilizes into a pattern. *Figure 5* is the plot when R is 3.15 and *Figure 6* is the plot when R is 3.75. As observed in *Figure 6*, when R gets closer to 4, random points are generated, also known as chaos.

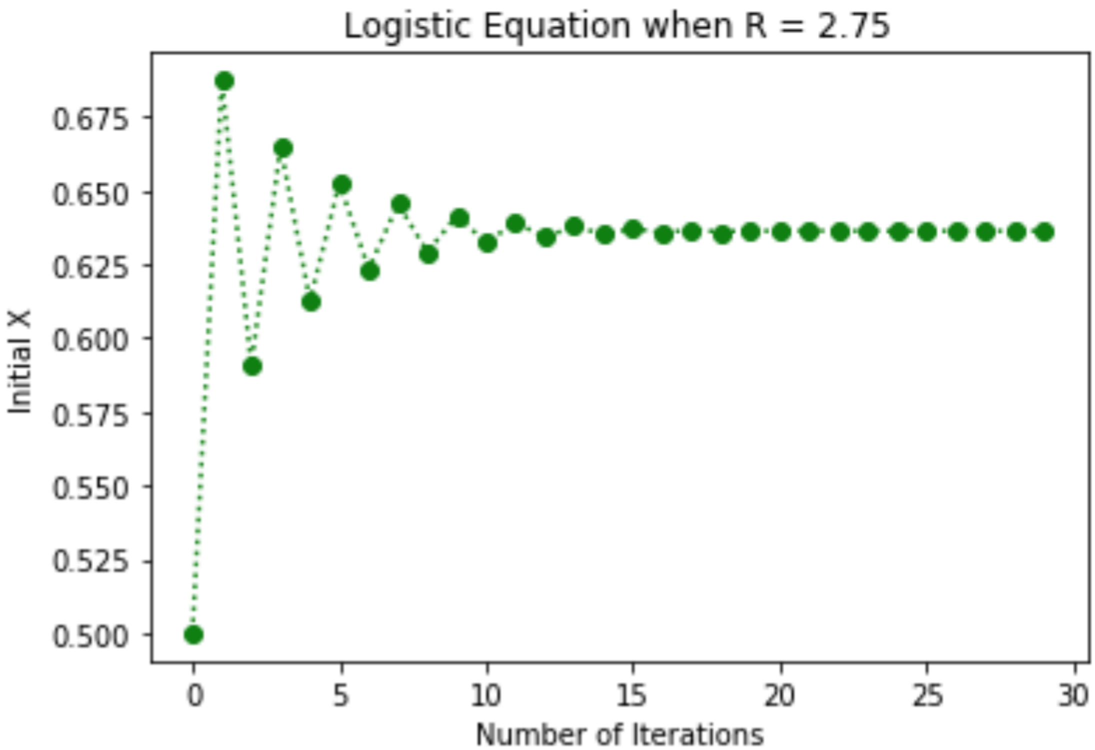

Figure 4. The logistic map when R is equal to 2.75.

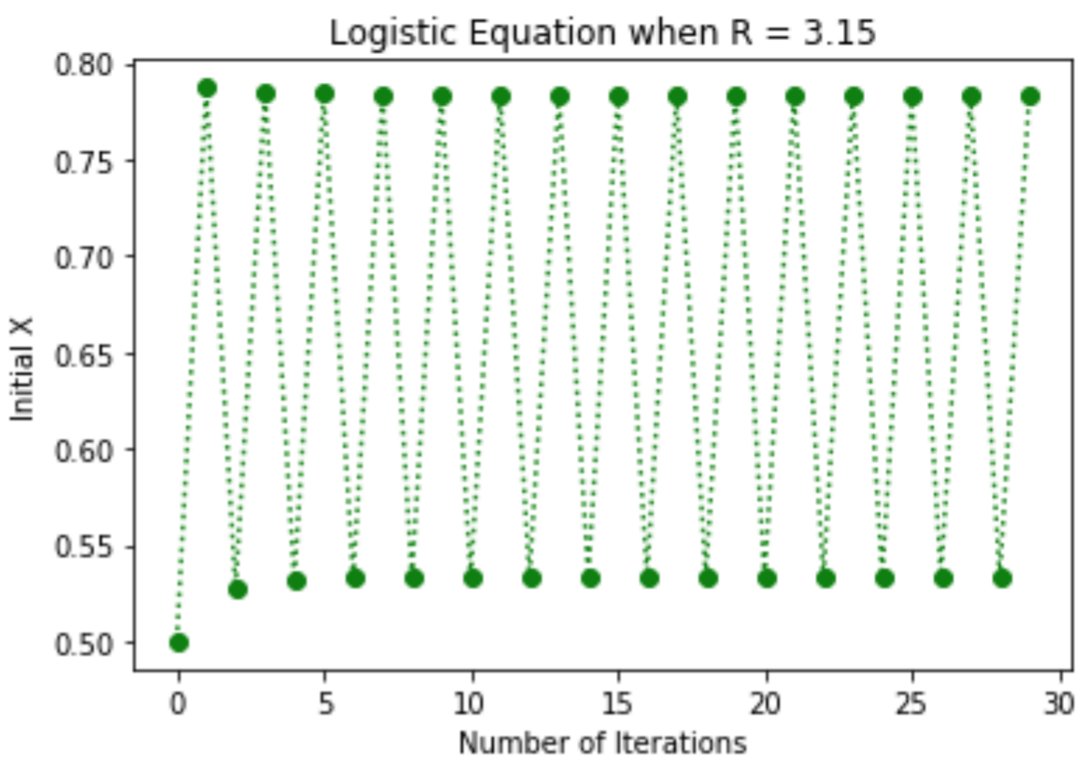

Figure 5. The logistic map when R is equal to 3.15.

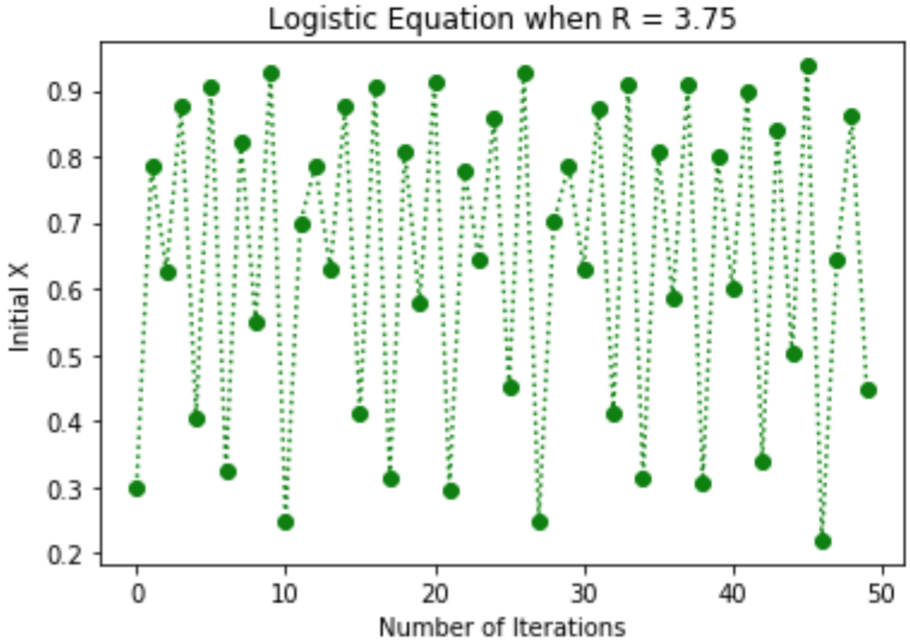

Figure 6. The logistic map when R is equal to 3.75. Unlike the two previous plots, this plot is very chaotic.

One interesting thing to note about these plots is analyzing where they stabilize. *Figure 1* can be inspected when looking at the R range 0 to 1. Within this range, the x values stabilize at zero, making it the attractor for this range. When looking at the range from 1 to 2, the x values tend to stabilize asymptotically, as depicted in *figure 4*, making the region between 0.625 and 0.650 the attractor for this range. *Figure 5* depicts the plot behaving in a oscillatory manner. As a result of this, there is no stability from this point onward and the plot continues periodically. There is no attractor within this range.

Additionally, when zooming in into one of the pockets of the logistic map, the same initial structure is observed. If a pocket of one of these were to be zoomed in, the same thing would occur. This is due to the fact that the Feigenbaum structure is self-similar. *Figure 7* displays the zoomed-in view of one of the pockets.

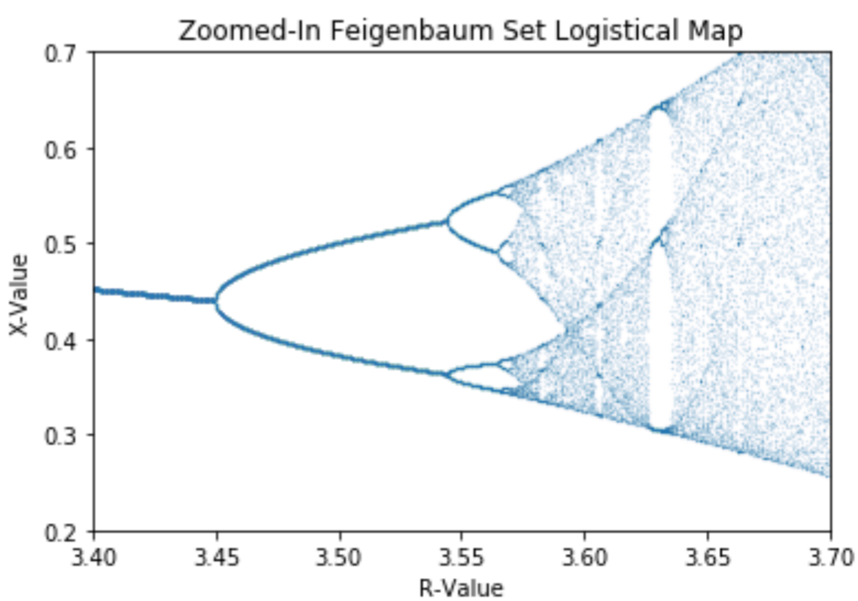

Figure 7. Zooming into one of the pockets of the logistic map reveals that the Feigenbaum set is self-similar.

3. MANDELBROT SET

Aside from the Feigenbaum set, the Mandelbrot set is also utilized to study chaos. The Mandelbrot set is a complex quadratic polynomial. Additionally, it is also one of the most well-known fractals that when plotted, produces a figure that depicts the complex numbers that are bounded by the set. The complex numbers that form a part of the set are determined by Equation (3) and (4) below. C is meant to represent the complex numbers.

$$z_0 = 0 \tag{3}$$

$$z_{n+1} = z_n^2 + c \tag{4}$$

There is a threshold in which the magnitude of the complex number will not exceed. Those numbers that do not exceed it will be a part of the set and those that do are not a part of it. *Figure 8* demonstrates a color-coded plot of the Mandelbrot set. The difference in color is meant to represent the number of iterations necessary to cross certain thresholds. The dark purple color that surrounds the pattern signifies the complex numbers that are not bounded by the set. The yellow color is indicative of the complex numbers that are bounded by the limit and therefore a part of the set.

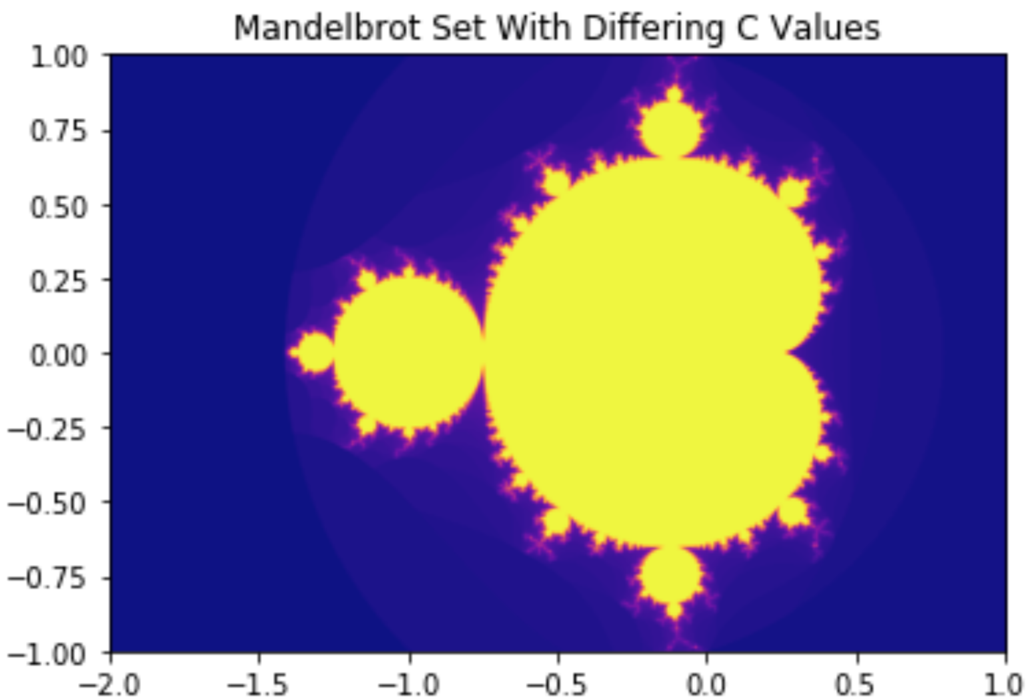

Figure 8. A color-coded plot of the Mandelbrot set.

Additionally, the plot of the Mandelbrot set can also prove that it is a fractal. Being that a fractal is an infinitely repeating pattern, a zoomed-in version of the plot reveals this characteristic. The closer that a certain region of the plot is observed, the patter would still remain, regardless of the scale. *Figure 9* demonstrates this pattern when the plot is zoomed in near the bottom region of where the big and little circles connect.

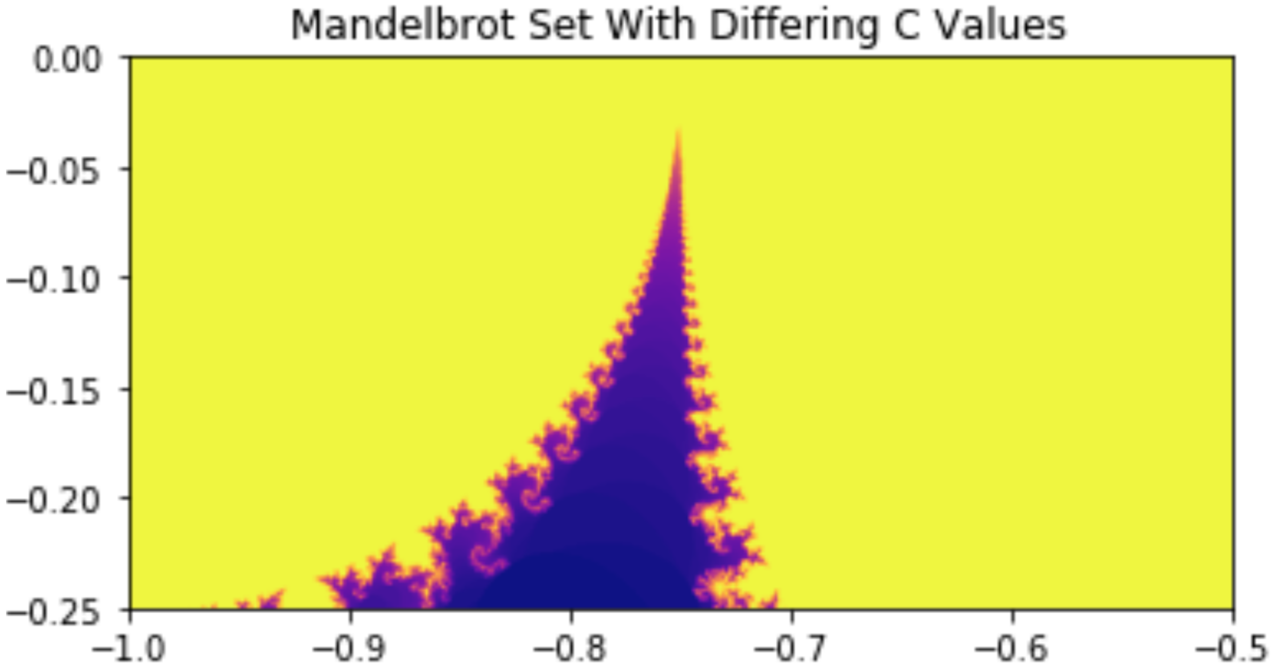

Figure 9. A zoomed-in version of the Mandelbrot set that supports the idea of the set being a fractal.

4. ARNOLD'S CAT

As with the Feigenbaum and Mandelbrot sets, another iterative function tha can be analyzed is Arnold's cat map. Arnold's cat map is described as a chaotic map through which an image of n pixels is stretched and whose positions are randomized until the original dimensions are restored, from encryption to decryption. This is a good way to analyze chaos as the number of iterations necessary to decrypt the image follows a random order. Therefore it is difficult to approximate how many iterations are necessary depending on the number of pixels. For individual Arnold's cat maps however, it is periodic with the set number of iterations, meaning there is a set number of iterations in which the image will be restored to its original form.

This method of encryption is done through transformation matrices that is then applied to every single pixel within the image. Equation (5) is what gets applied to all pixels:

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x + y \\ x + 2y \end{bmatrix} \tag{5}$$

According to the iteration number that is being sampled, the images' pixels will be scrambled accordingly. The only way to determine the correct iteration number that will restore the image back to normal is through sampling. *Figure 10* depicts the resulting image when the incorrect iteration number is sampled. The pixels look very randomized and is just generally chaotic. *Figure 11* is the resulting image when the correct iteration number is sampled. The original image is restored.

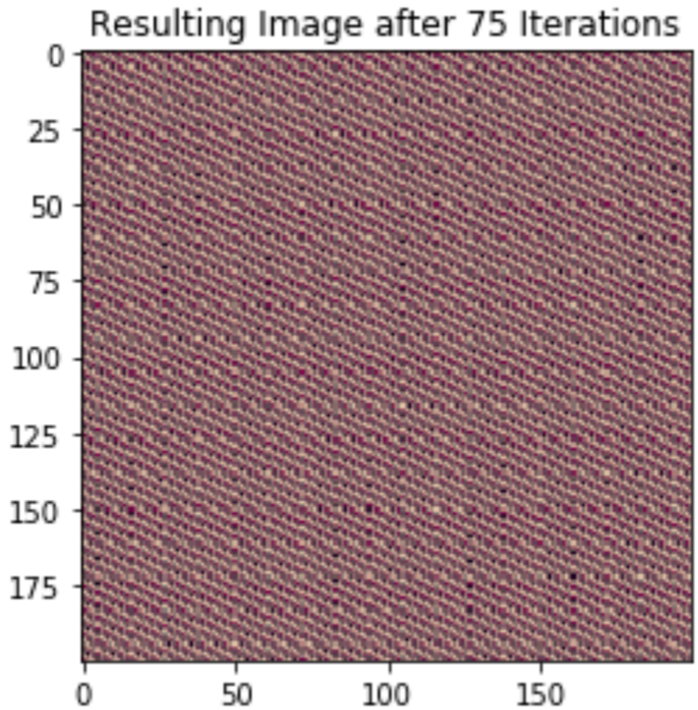

Figure 10. Resulting image when the iteration count is incorrect.

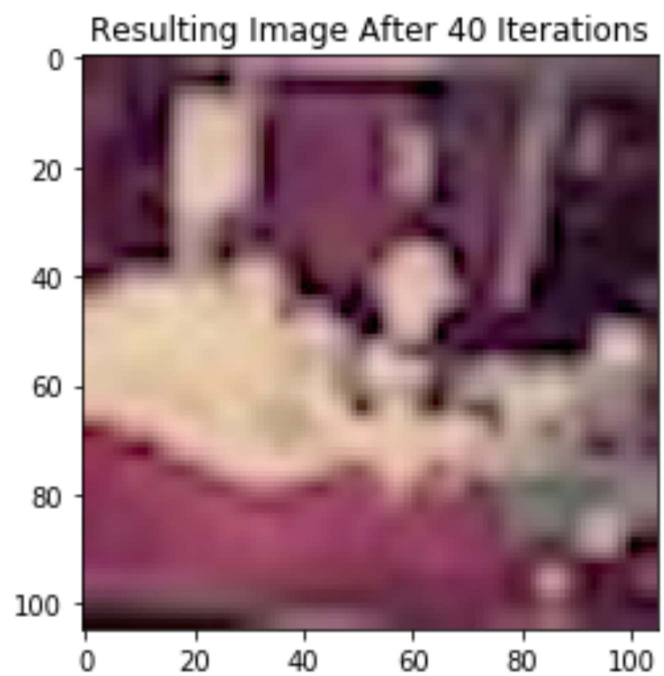

Figure 11. Resulting image when the iteration count is correct.

To see a more general outlook on how n pixels dictate the number of iterations needed to correctly restore the image, *figure 12* was plotted. As it can be observed, there is no identifiable pattern. The plot appears to be chaotic and is so, as no period is found. The correct iteration seems randomized, regardless of the number of pixels.

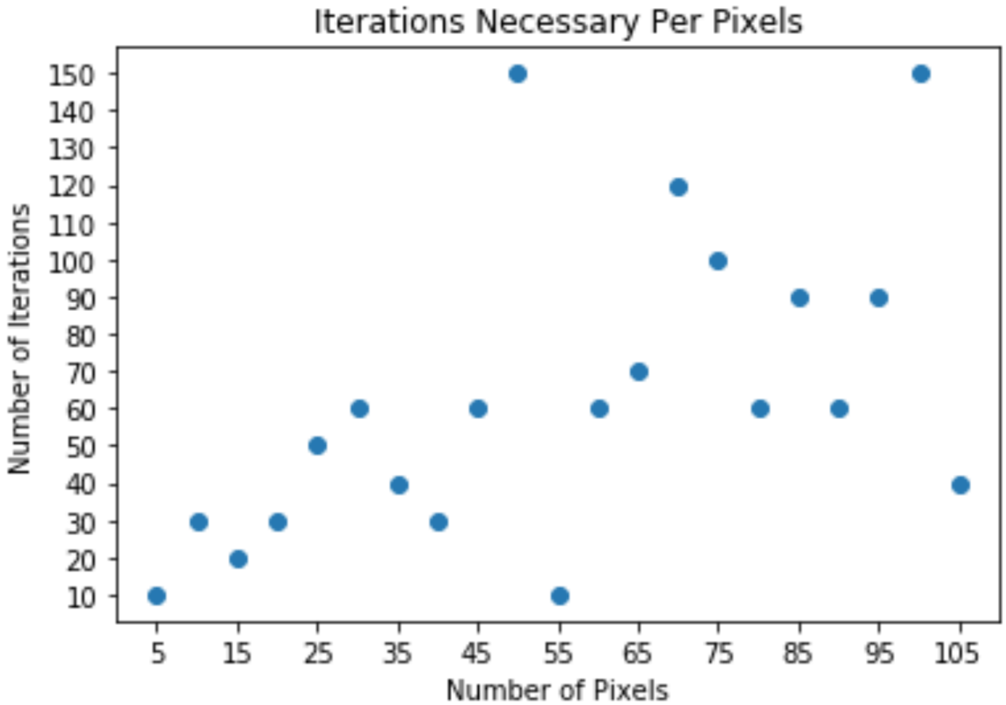

Figure 12. Depiction of the iterations necessary to restore an image, according to the number of pixels.

5. CHIRIKOV'S STANDARD MAP

As with the previous sets, the standard map can also depict the potential path to chaos. The map itself is an iterative function that deals with variables momentum and coordinates (p,x). The map is plotted with the help of Equation (6) and (7). K is representative of the gamma parameter, which dictates the level of chaos on the map.

$$p_{n+1} = p + Ksin(x) \tag{6}$$

$$x_{n+1} = x + p_{n+1} \tag{7}$$

As the equations display, the momentum parameter is constantly updated and takes into account the gamma parameter. This updated p parameter is then placed into the coordinate parameter, x and is continuously updated, therefore producing the standard map. *Figure 13* displays the structure of the standard map when K is 0.

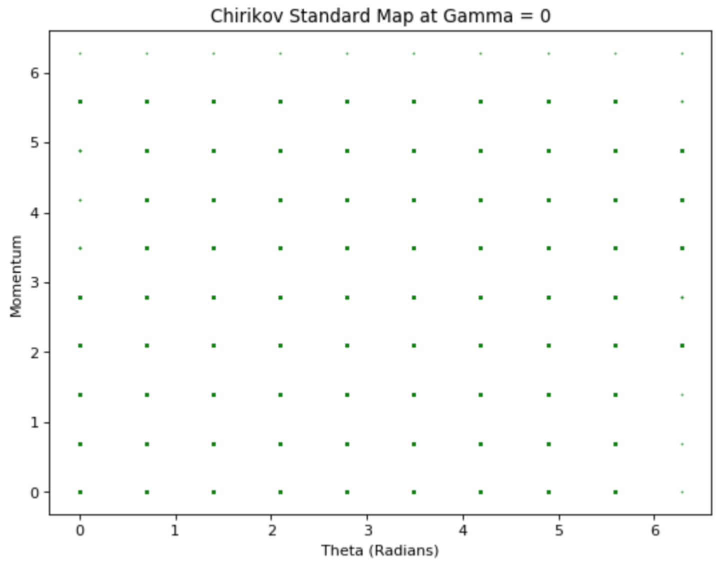

Figure 13. Depiction of Chirikov's standard map when gamma is set to zero.

Because gamma is zero, we expect no chaos to be present, in fact far from it. As the figure depicts, it is a structured array of data points. There appears to be no degree of chaos. However, as gamma inceases, we start to notice that the standard map is more prone to chaotic structure. *Figure 14* shows the resulting standard map when gamma is set to 0.5. Though more chaotic than the previous plot, there is still evidence of order present.

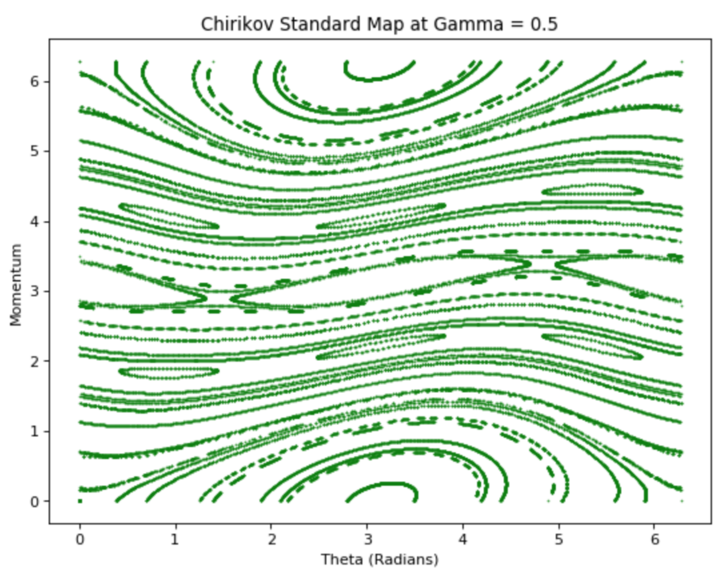

Figure 14. Depiction of the standard mp when gamma is set to 0.5.

However, once gamma is set to higher values such as 7 and above, chaos immediately manifests itself. As observed on *figure 15*, everything looks to be randomized. There is no patterned structure present at all.

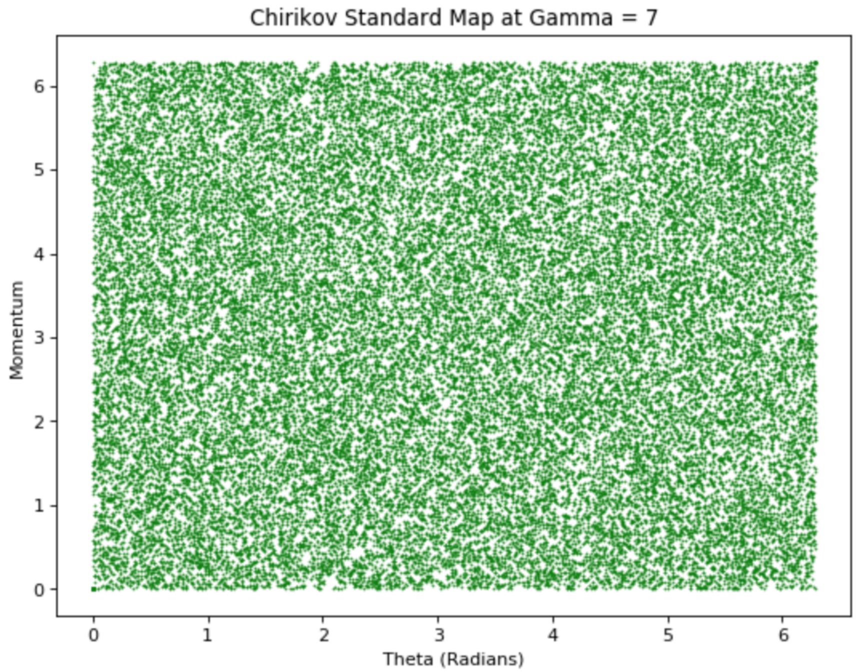

Figure 15. Depiction of the standard map when gamma is set to 7.

6. CONCLUSION

As this paper demonstrates, there are various sets of iterative functions that can aid with the understanding of dynamical systems and chaos. The Feigenbaum set's bifurcation points show a level of stability before leaning towards chaos, as R increases. Mandelbrot's set of complex numbers creates a color-coded pattern comprised of member numbers that creates a pattern. Upon closer inspection, it can be observed that the Mandelbrot set is a fractal, which is also chaotic. Arnold's cat map displays levels of chaos due to the randomization of the correct number of iterations required to restore the image back to its original state. The standard map utilizes initial values to predict the areas on the phase space map that will lead to stability and towards chaos.