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The Boundary Value Problem

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1. INTRODUCTION

The purpose of this assignment is to evaluate the boundary value problem. Going off from the last project, this assignment focuses on the calculated initial values that will result in the boundary conditions that are set for the problem. The problem in question is projectile motion under different conditions, such as the inclusion of air resistance and the gravitational effect from a large moving body. To demonstrate this, the Runge-Kutta third order method and the iterative method are utilized.

2. ANALYTICAL METHOD

2.1. *Projectile Motion*

Before implementing the Runge-Kutta and iterative methods, the boundary value problem can be solved analytically. With projectile motion type of problems, the kinematic equations are utilized even when no initial values are provided. In this example, the height, distance, and time are initially provided. These are the boundary conditions that the initial values will have to meet and can be found in *Table 1*.

Table 1. Initial Boundary Conditions

Boundaries	**Boundary Values**
Height (h)	60 m
Distance (d)	60 m
Time (t)	5 s

Utilizing these boundary conditions, the velocity vector would be our solution. V_x would be found through Equation (1) while v_y can be determined through a re-worked version of one of the kinematic equations (Equations 2).

$$v_x = \frac{d}{t} \tag{1}$$

$$v_y = \frac{y}{t} + \frac{1}{2} * g * t \tag{2}$$

From these two equations, along with the boundary conditions that were set, the resulting velocity vector was calculated. These can be found in *Table 2*.

Table 2. Calculated Velocity Components

Velocity Component	**Value**
v_x	12 m
v_y	36.5 m

From the calculated velocity components, the position vector can be calculated and updated with respect to time. A plot of this motion can be found in *Figure 1*.

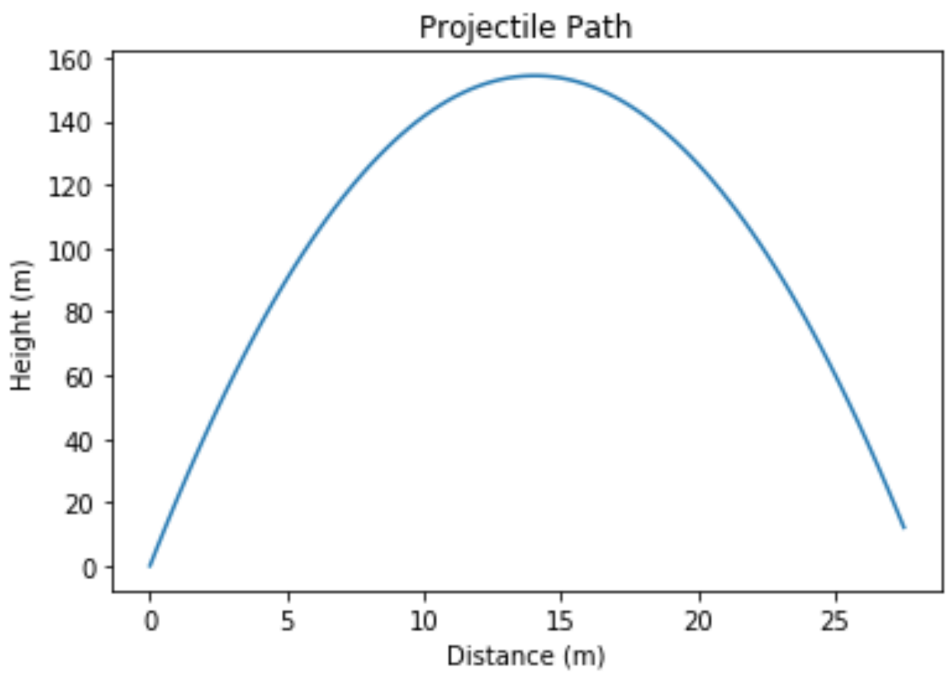

Figure 1. Plot depicting the projectile path that obeys the boundary conditions set.

2.1.1. *Runge-Kutta Version*

Besides the analytical solution, Runge-Kutta 3rd order method can also be utilized to calculate the initial values from the boundary value problem. As previously discussed, the Runge-Kutta method finds the solution by root-finding. With Runge-Kutta, the initial values can be determined by guessing and iteratively solving for them. Regardless of which method is utilized, the process is similar in that the roots are found and the plots will be the same. *Figure 2* depicts the projectile path as calculated iteratively by the Runge-Kutta method (the guessing method produces a plot exactly as this).

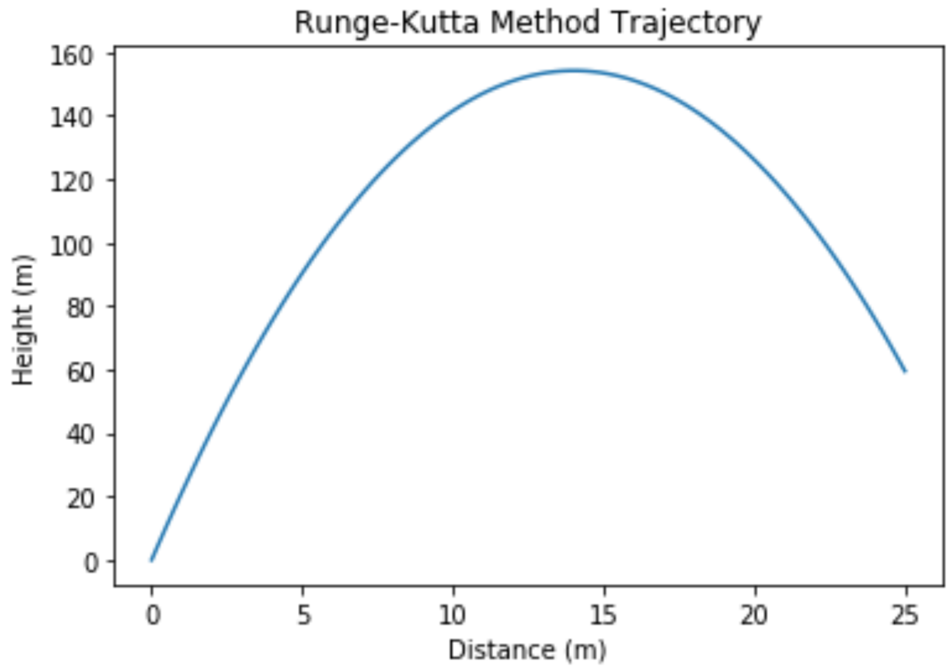

Figure 2. Plot depicting projectile motion by guessing the initial values.

2.1.2. *Air Resistance*

Until now, the projectile motion problem in question has been solved without taking air resistance into account. With the addition of air resistance, it is expected that the calculated velocity components will be larger in value now that there is a force that pushes against the object being launched. Equations (3) and (4) below can account for this increase in both the x and y direction of the velocity component. C is defined as my drag coefficient.

$$v_x = c\dot{x}^2 \tag{3}$$

$$v_y = -g - (c\dot{y}^2) \tag{4}$$

Table 3 shows the newly calculated velocity components once taking into account air resistance. *Figure 3* is a comparison plot of the projectile path as determined by Runge-Kutta both with and without air resistance.

Table 3. Calculated Velocity Components W/ Air Resistance

Velocity Vector	**Value**
v_x	19.46 m
v_y	76.15 m

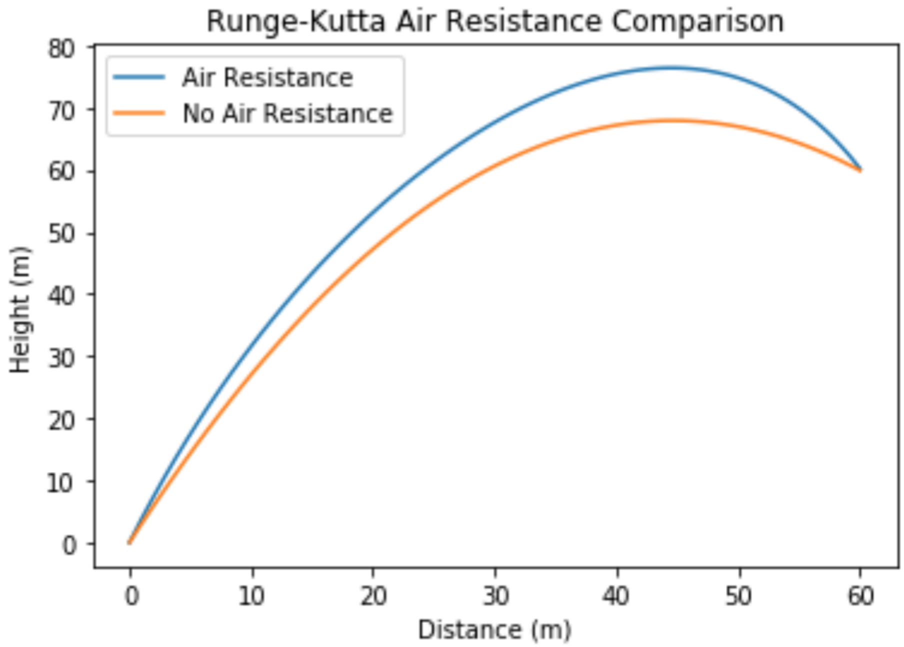

Figure 3. A visual comparison of the path the projectile takes with and without air resistance.

3. GRAVITY ASSIST

Within the confines of the projectile motion problem, gravity assist is a very important subset. Gravity assist is utilized to provide a satellite a 'boost' in velocity when the satellite is close in proximity to a large orbiting celestial body. As the name of the method describes, the boost in velocity is due to the gravitational effect from the large celestial body.NASA has utilized the maneuver with many different satellites, one of them being the *Cassini* spacecraft. *Figure 4* provides an illustration of *Cassini* undergoing the procedure.

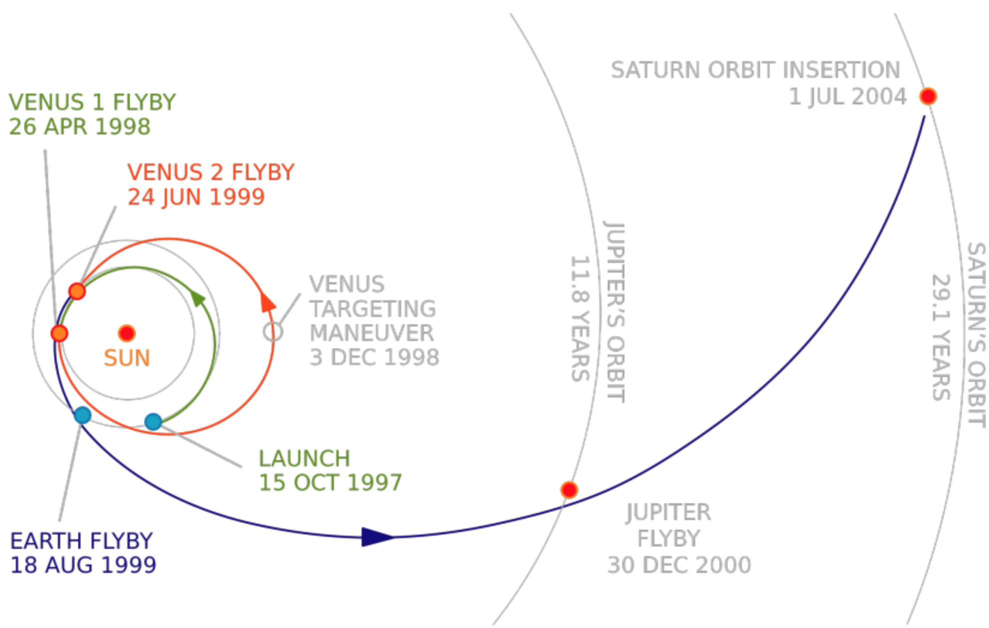

Figure 4. Timeline illustration of the Casssini spacecraft conducting the slingshot method due to gravity assist.

The Runge-Kutta method was utilized to model this procedure and the subsequent changes to the satellite's parameters. Many properties can be analyzed after a satellite utilizes gravitational assist to see what has changed. Besides velocity, kinetic energy, and potential energy can be analyzed to compare before and after the slingshot maneuver. Equations (5) and (6) were used to calculate the initial values.

$$\dot{v_x} = \frac{-GM(x_{craft} - x_{planet})}{r^3} \tag{5}$$

$$\dot{v_y} = \frac{-GM(y_{craft} - y_{planet})}{r^3} \tag{6}$$

Figure 5 displays the change in trajectory the spacecraft experiences when approaching a planet with similar parameters to those of Saturn.

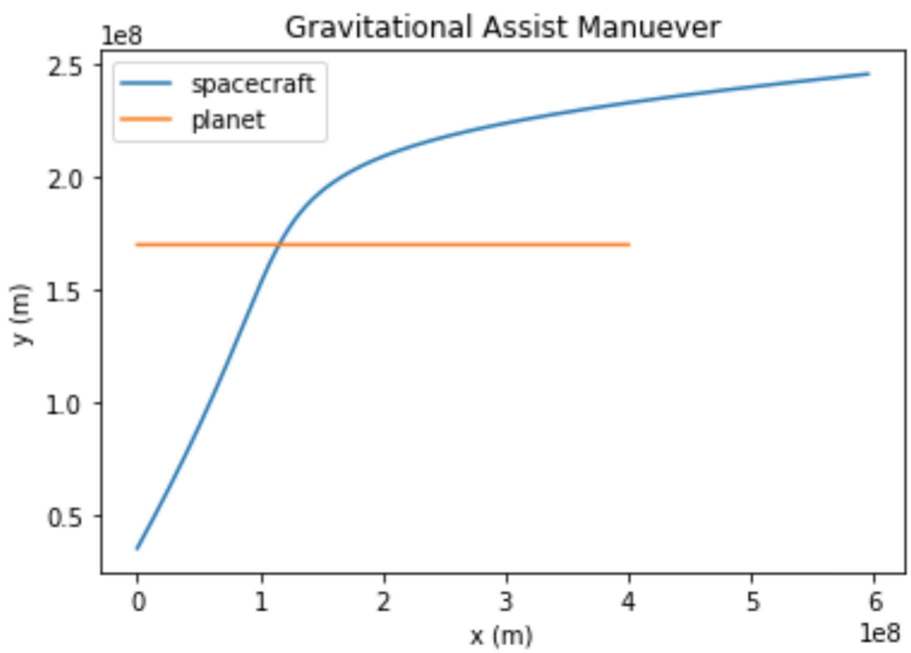

Figure 5. Plot of the slingshot method over a planet with parameters similar to Saturn.

As briefly mentioned before, trajectory of a satellite is not the only property that changes when undergoing the slingshot method. The velocity at which the satellite was moving also experiences a shift. Due to the gravitational assist, it is expected for the satellite's speed to sharply increase and decrease, yet maintain a faster speed than before the encounter. *Figure 6* demonstrates this change in speed.

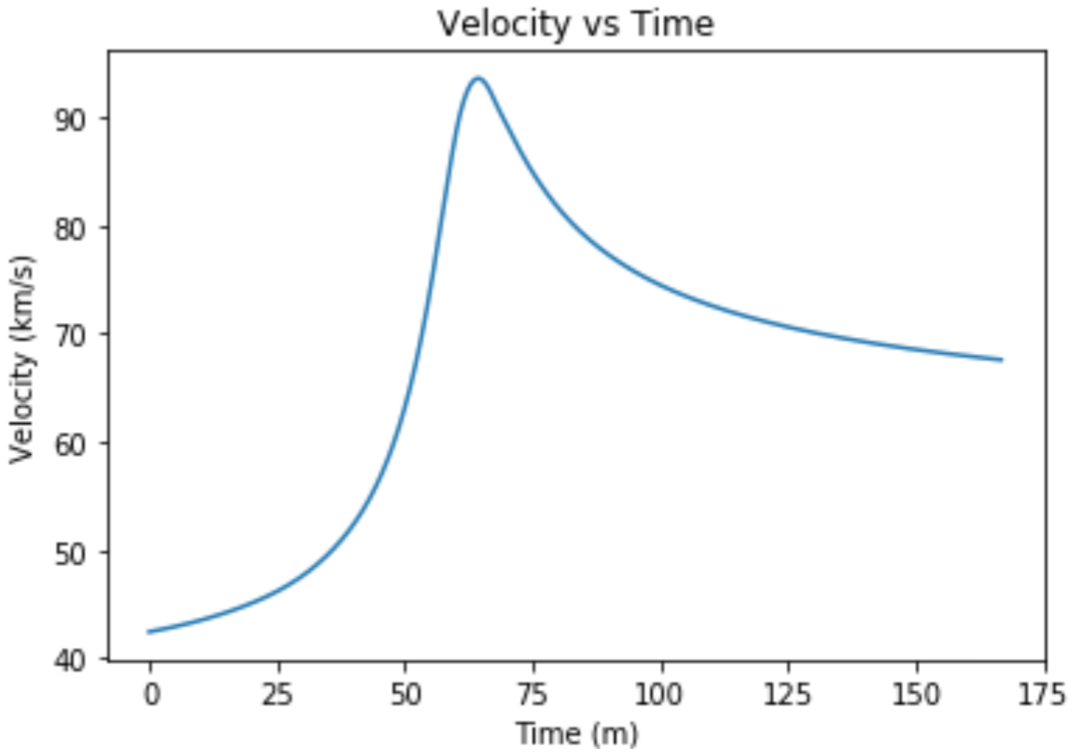

Figure 6. Plot that depicts the change is velocity over the change in time.

Besides trajectory and speed, energy is another property that experiences a shift, namely kinetic and potential energy. Due to the spacecraft and and the planet moving in the same direction, total energy change is positive and is therefore not conserved. *Figure 7* and *figure 8* showcase the shifts in the different types of energy present. The shift in kinetic energy is larger than the dip in potential energy and therefore, total energy is positive.

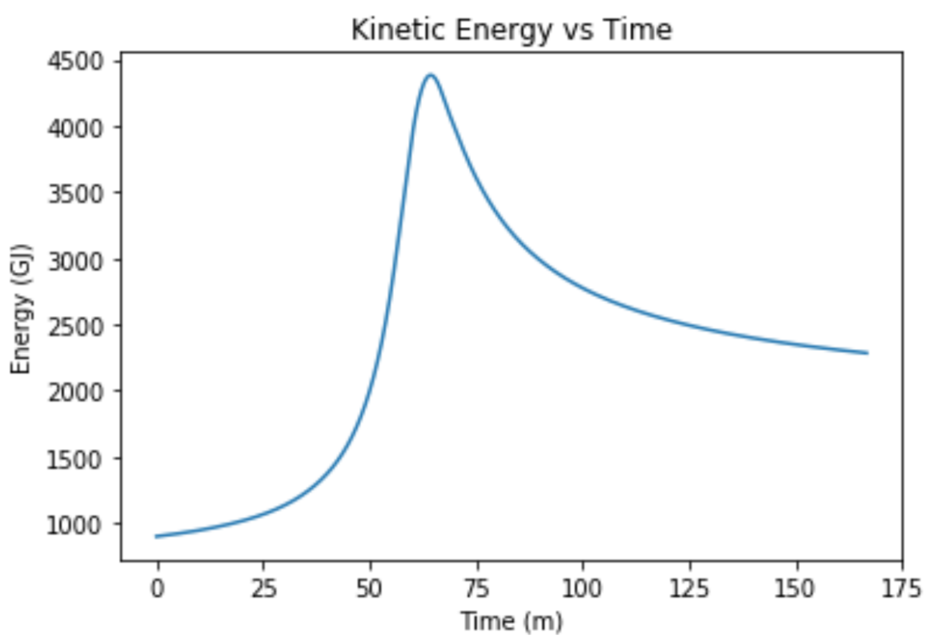

Figure 7. The change in energy can be observed as time continues.

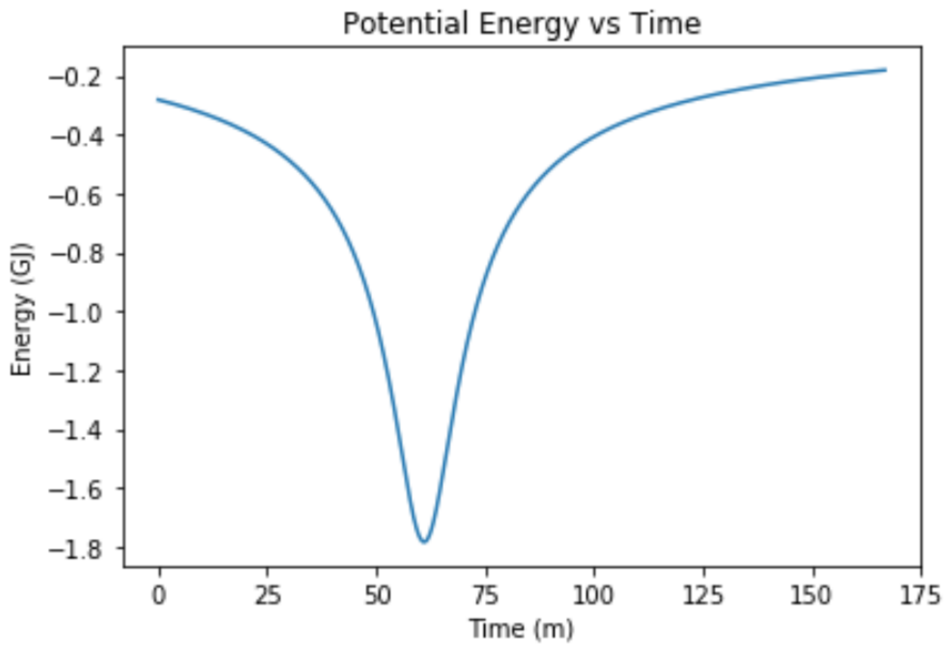

Figure 8. This plot indicates a dip in the potential energy around the time when the spacecraft is within close proximity of the planet.

4. IMPACT PARAMETERS AND VELOCITY RATIO

Another property that also gets affected is the impact parameter. The impact parameter (p) is the distance between the center of the planet and the approaching satellite. In this section, we explore how varying the impact parameter results in vastly different trajectories of the satellite. As the impact parameter becomes larger, so does the deflection the satellite experiences. *Figure 9* depicts these differing trajectories.

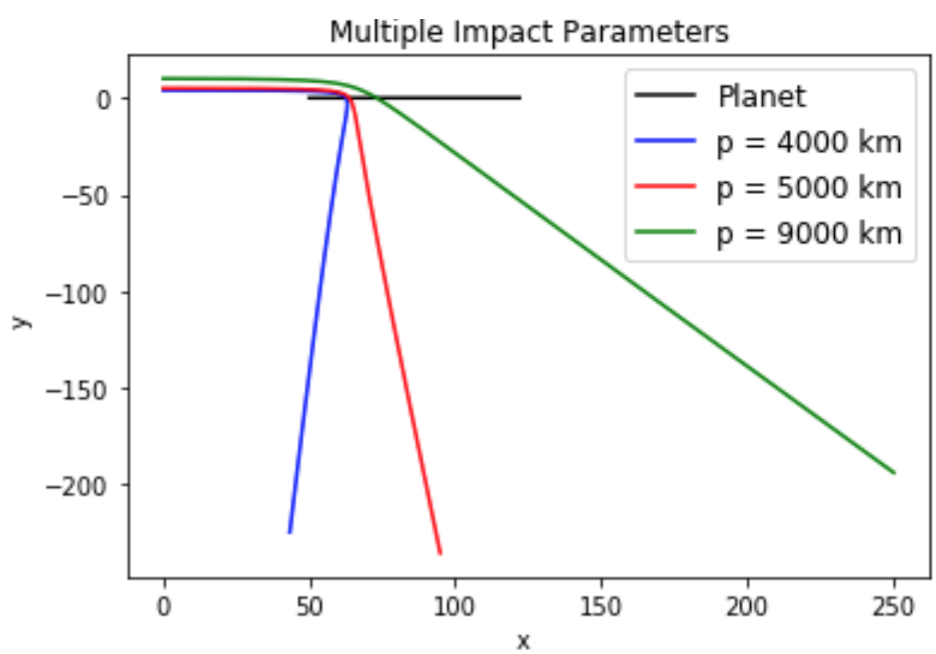

Figure 9. This plot explores the different projectile paths that can be observed, should the impact parameter vary.

Another parameter that experiences a change is the speed ratio. The speed ratio is the speed of the spacecraft divided by the planet's speed and is commonly denoted as chi. It should be noted that the speed ratio lives by specific limit and above this, the level of deflection becomes very apparent. Any value below this limit does not result in a strong deflection and may even result in the satellite being captured by the planet's orbit. *Figure 10* encapsulates this idea with the varying values of chi and their effect on the trajectory of the satellite.

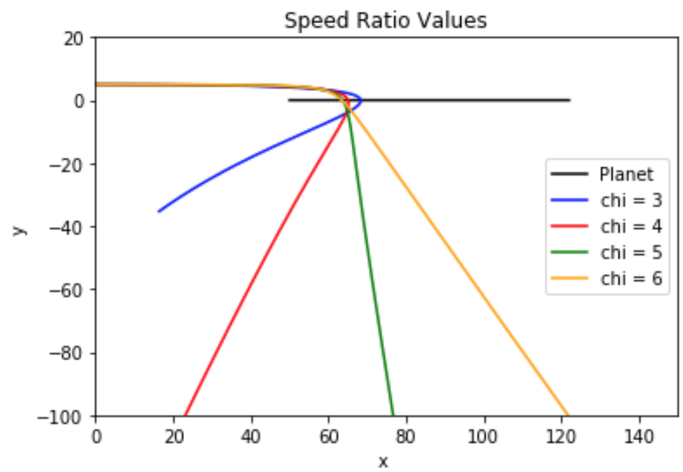

Figure 10. This plot depicts the projectile speed ration a spacecraft were experience according to the respective chi value.

5. MAXIMAL GRAVITY ASSIST

Another celestial body that is perfect for the slingshot maneuver is the Moon. In fact, it is ideal for giving satellites the maximal gravitational assist. As the name suggests, this is meant to result in the greatest gravitational assist available to the satellite. In this section, we attempt to find this optimal value for a satellite moving close to the Moon. *Figure 11* displays this optimal gravitational assist as evidenced by the wide deflection the satellite experiences when in close proximity to the Moon. The resulting value was determined to be 22.07. With this value, additional trajectories were plotted with values that neighbored this one.

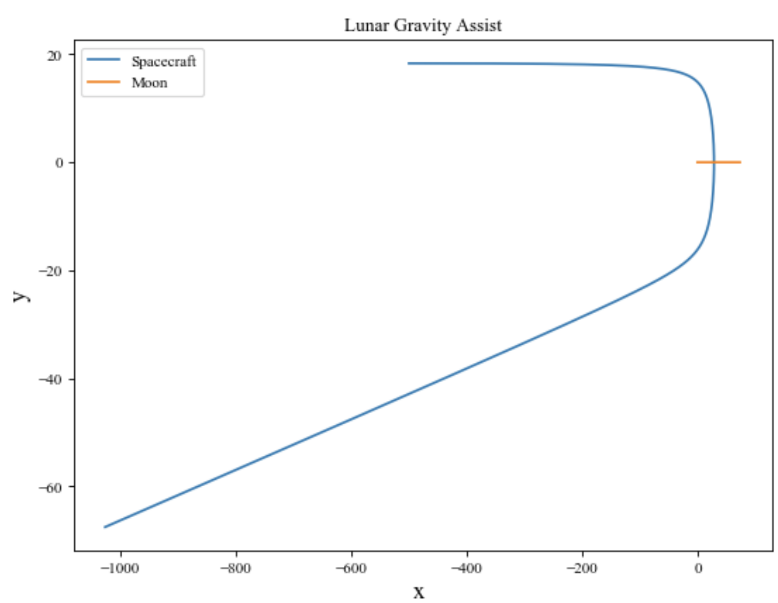

Figure 11. This plot displays the slingshot method on the Moon.

It can be observed that slight alterations to the chi value can result in dramatic changes to the trajectory. *Figure 12* depicts this wide range of trajectories along with their respective chi values.

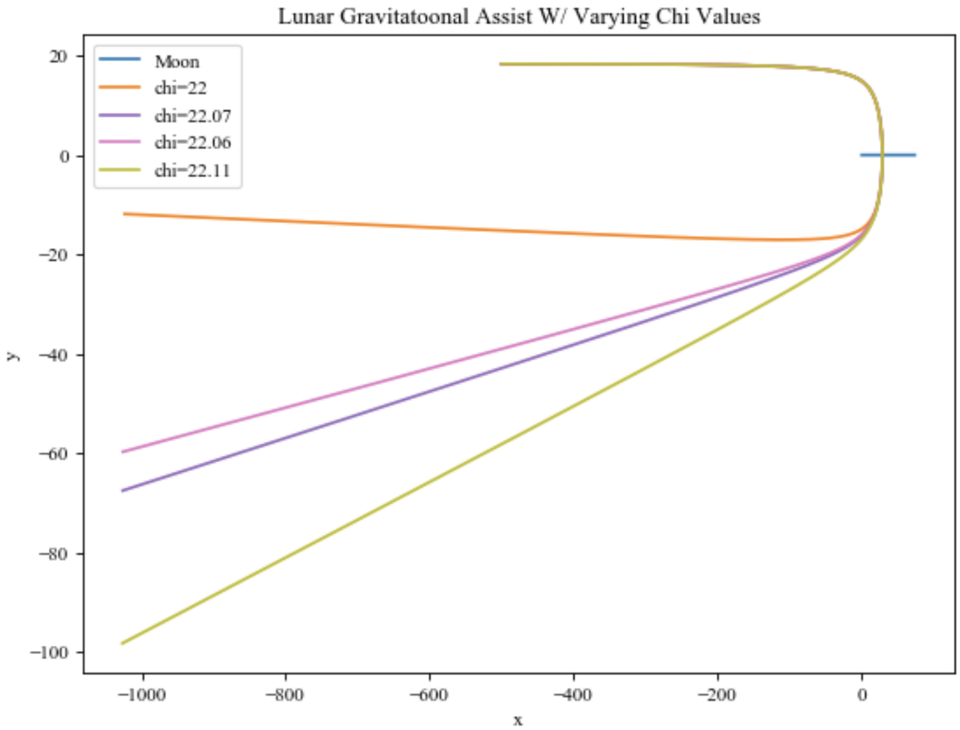

Figure 12. This plot indicates the varying gravitational trajectories that are near the optimal one.

6. CONCLUSION

As this paper proves, the Runge-Kutta method proves to be an accurate alternative in solving the boundary value problem, besides the kinematic equations. Utilizing the boundary conditions, the Runge-Kutta method can iteratively solve for the initial values or guess them. Both methods are so accurate that the plots produced are the same. Even when air resistance is taken into account, Runge-Kutta successfully calculates the roots of the problem.