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Discrete Population Models

Kevin B. Moposita[1]

[1] *Villanova University*
800 Lancaster Avenue
Villanova, PA 19085, USA

1. INTRODUCTION

In this assignment, we were once again looking at population models, but with a twist. Unlike the previous assignment where deterministic models were analyzed, the population models we're observing are stochastic, which take into account a level of unpredictability and randomness. Within these models, the extinction of a group of species are also depicted, whereas the deterministic models depict a constant cycle of population rebound.

Within this problem set, different scenarios involving stochastic populations are modeled with different results each time. More specifically, a Markov chain is a type of stochastic model that within a system, describes the possible events in a sequence. Of these possibilities, the sequence should sum up to 1. Equation (1) depicts this rule. The probability of the system is dependent on the previous state of the system. This probability can be determined with Equation (2) below.

$$\sum_{j=1}^{S} P_{i,j} = 1 \tag{1}$$

$$S_N = M^N S_0 \tag{2}$$

In Equation (2), N is representative of the number of iterations the model undergoes, S is the probability of the next state, S_0 is the initial probability, and M is the matrix that holds the rest of the states. In order for the matrix to be as accurate as can be, a high number of iterations must be met to ensure the probabilities within the matrix converge and multiplied by another matrix that contain the initial probabilities. Equation (3) displays the resulting matrix that carries the resulting probabilities.

$$M = \begin{bmatrix} P_{1,1} & P_{1,2} & ... & P_{1,j} & ... & P_{1,S} \\ P_{2,1} & P_{2,2} & ... & P_{2,j} & ... & P_{2,S} \\ P_{i,1} & P_{i,2} & ... & P_{i,j} & ... & P_{i,S} \\ P_{S,1} & P_{S,2} & ... & P_{S,j} & ... & P_{S,S} \end{bmatrix} \tag{3}$$

This calculates the final probabilities of each state of the system.

2. RESTAURANT DASH

In this model, the objective is to determine and analyze the probability that a table within a restaurant is available if the initial condition is set to no guests to begin with. P will be representative of the probability of a person arriving and Q will be representative of the departure probability. The equations below were assigned to specific places within the matrix so to account for possible arrivals and departures, with respect to the number of tables available.

$$M_{0,0} = 1 - P \tag{4}$$

$$M_{0,1} = P \tag{5}$$

$$M_{1,0} = Q(1 - P) \tag{6}$$

$$M_{1,1} = (1-P)(1-Q) + PQ \tag{7}$$

$$M_{2,1} = Q(1-P) \tag{8}$$

$$M_{n,n} = (1-Q) \tag{9}$$

$$M_{n,n-1} = Q \tag{10}$$

$$M_{n-2,n-1} = P(1-Q) \tag{11}$$

$$M_{n-1,n-1} = (1-P)(1-Q) + PQ \tag{12}$$

$$M_{n-1,n} = P(1-Q) \tag{13}$$

For these next three equations, they take into account areas in the matrix that are not covered by the previous ones.

$$M_{i,i} = (1-P)(1-Q) + PQ \tag{14}$$

$$M_{i+1,i} = Q(1-P) \tag{15}$$

$$M_{i-1,i} = P(1-Q) \tag{16}$$

To utilize these equations, initial conditions must be introduced. *Table 1* provides an organized list of the parameters that will be introduced along with their values.

Table 1. Restaurant Parameters

Parameter	**Value**
# of Tables	10
N	100000
P	0.3
Q	0.4

With these initial conditions, the Markovian matrix was calculated. The Markovian matrix dictates all the different possible probabilities from all the states. Equation (17) contains all of said probabilities. With the Markovian matrix, it is set to a high value of N iterations to ensure that the matrix values converge to set values and are multiplied by the initial conditions of the states within the system. The product are the final availability probabilities of all 10 seats (or all the states within the system).To ensure this was done correctly, the final probabilities are summed up and should equal to 1. Equation (18) contains the final probabilities of all 10 states.

$$
M = \begin{bmatrix}
0.7 & 0.28 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0.3 & 0.54 & 0.28 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0.18 & 0.54 & 0.28 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0.18 & 0.54 & 0.28 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0.18 & 0.54 & 0.28 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0.18 & 0.54 & 0.28 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0.18 & 0.54 & 0.28 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0.18 & 0.54 & 0.28 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0.18 & 0.54 & 0.28 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0.18 & 0.54 & 0.4 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0.18 & 0.6
\end{bmatrix} \tag{17}
$$

$$
P = \begin{bmatrix}
0.252665 \\
0.2707125 \\
0.17402947 \\
0.11187609 \\
0.07192034 \\
0.0462345 \\
0.02972218 \\
0.01910712 \\
0.01228315 \\
0.00789631 \\
0.00355334
\end{bmatrix} \tag{18}
$$

As expected, when all the values of the final probabilities matrix are summed up, it equals to 1, indicating accuracy. Another way of looking at these values is through a bar graph, such as in *Figure 1*. In this plot, it is easier to distinguish which seat (or state) has the highest probability of being occupied, which would be seat 1. The tenth seat has the lowest probability of being occupied.

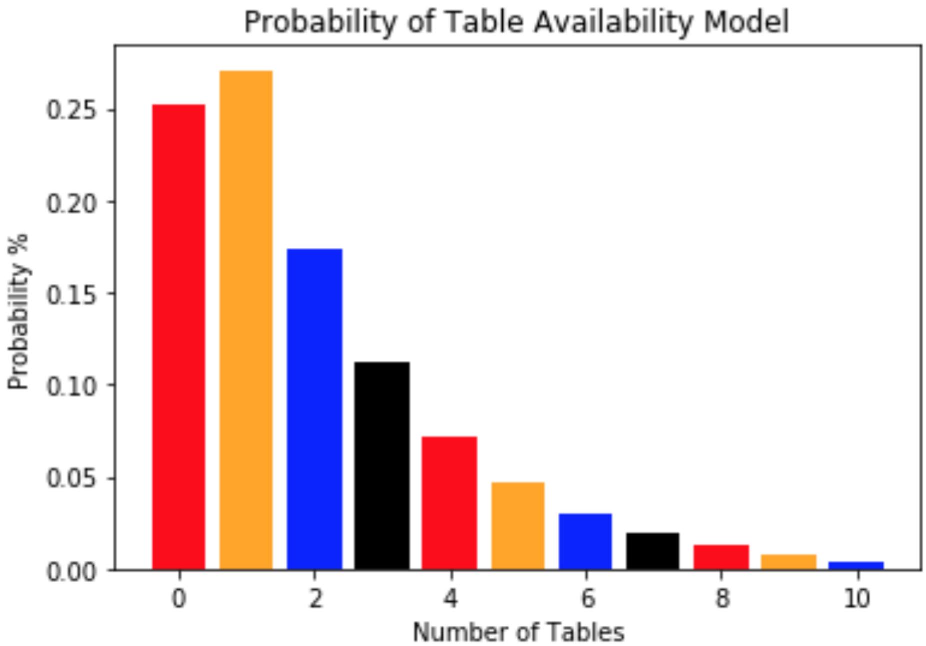

Figure 1. Bar graph of the final probabilities of each state within the system.

3. REAL-ER EPIDEMIC POPULATION MODEL

Once again, the not-so-imaginary pandemic scenario is being revisited. Previously, this same situation was modeled through a deterministic population model, as can be seen in *Figure 2*. However, the model being created will now be stochastic in nature, which would also make the simulation more realistic.

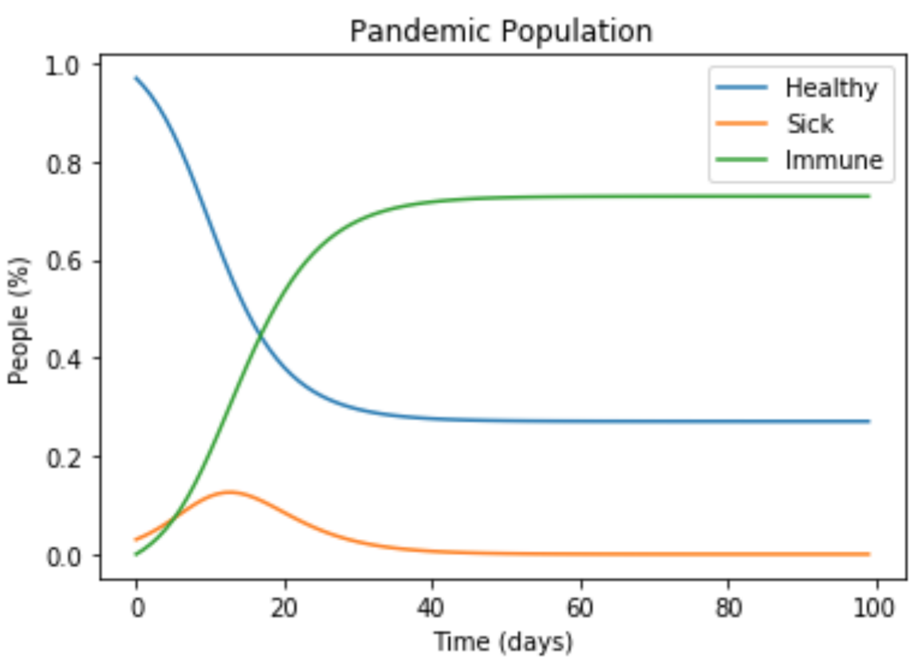

Figure 2. Previous deterministic model of the pandemic without accounting for inoculation.

To take stochasticity into account, the improved model will now take into account death. In this scenario, death can come in different forms. Should a healthy person become sick, there is now a possibility of either recovering and perishing from it, and not just the former as was previously modeled. Regardless, death is now an inevitable outcome that can be modeled.

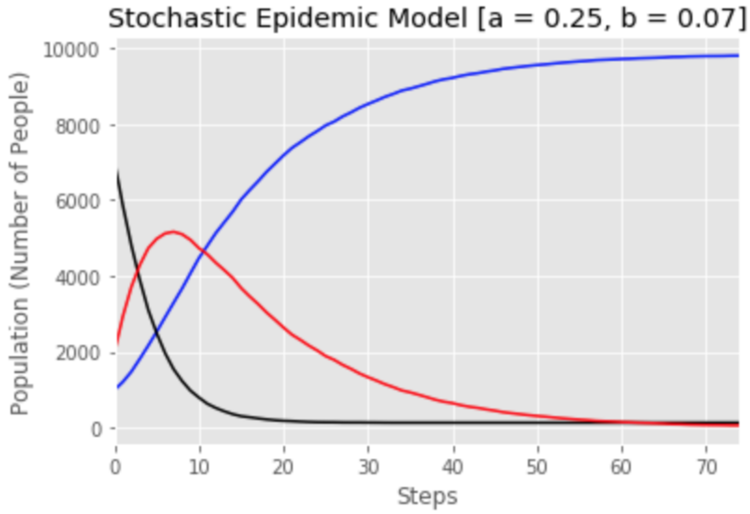

Figure 3. Stochastic model of the Epidemic

As evidenced in *Figure 3*, stochasticity once again rules as times elapses. While the immune (blue plot) population increases, the healthy population (black) and sick population (red) all eventually move to zero.

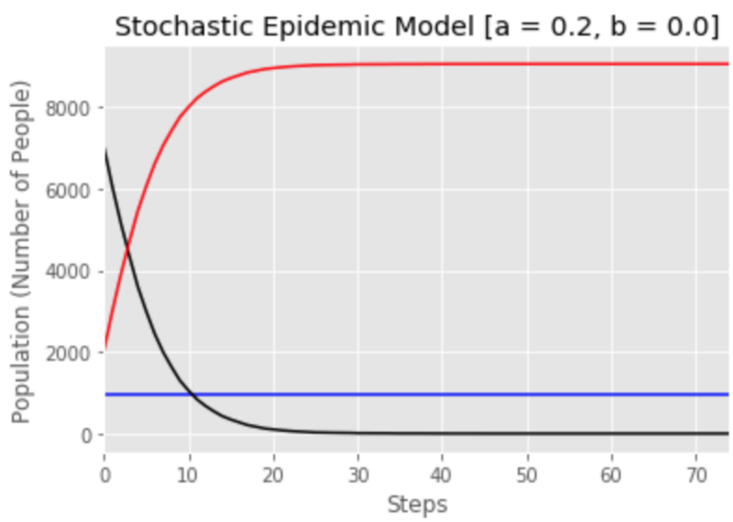

Figure 4. Model with no immunity given.

Figure 4 demonstrates what happens if the probability of immunity turns to 0. In this scenario, the healthy population very quickly dies off.

4. REAL-ER RABBITS AND FOXES

In another homage to the previous assignment, the rabbits and fox relationship can once again be revisited. As previously mentioned, this initial model was deterministic in nature, meaning that either population always had the supernatural ability to bounce back from extinction. As a result, that relationship was periodic as the cycle would infinitely repeat itself. However, with this stochastic model, death now now another parameter that can be taken into account. With this latest addition, we expect the relationship to no longer be periodic, as realistically no population can come back once it has become extinct. *Figure 5* shows the previous Lotka-Voltera plot of the deterministic version of the model.

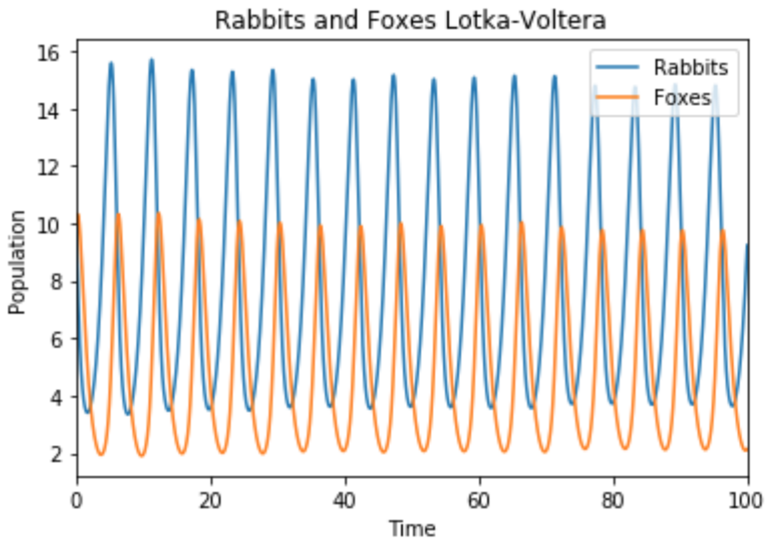

Figure 5. The previous model was periodic in nature due to the oscillatory behavior displayed.

To better model the interaction between the two species, *Figure 5* displays the location of both groups on a grid. It is through this grid where both groups of animals will interact and depending on the initial probabilities, can be fatal.

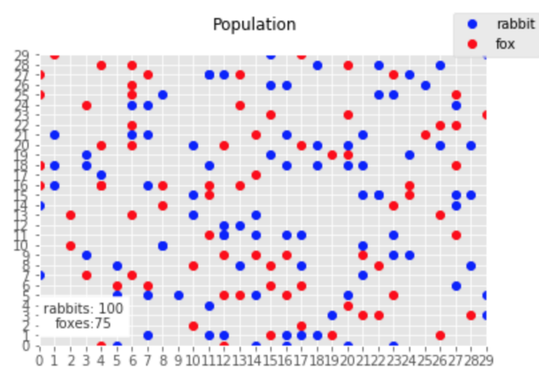

Figure 6. Grid that displays the locations of the rabbits and foxes.

It is also important to take into account that in this model, as time goes on, each species has a random opportunity to 'spawn' into the grid. However, as mentioned before, once the population of either one has reached zero, there is no going back.

Figure 7 demonstrates the plot of both rabbit and fox populations as a function of time. As it can observed, as times goes on, the ultimate fate of both populations is total extinction. Due to the high initial population of rabbits, the foxes experience a sharp increase in their respective population. However, as the number of rabbits begins to dwindle, the fox population experiences a decline. The brief moments of population increase is attributed to the 'spawn' factor previously noted. However, even this cannot save the fox population as after some time, they experience extinction as well.

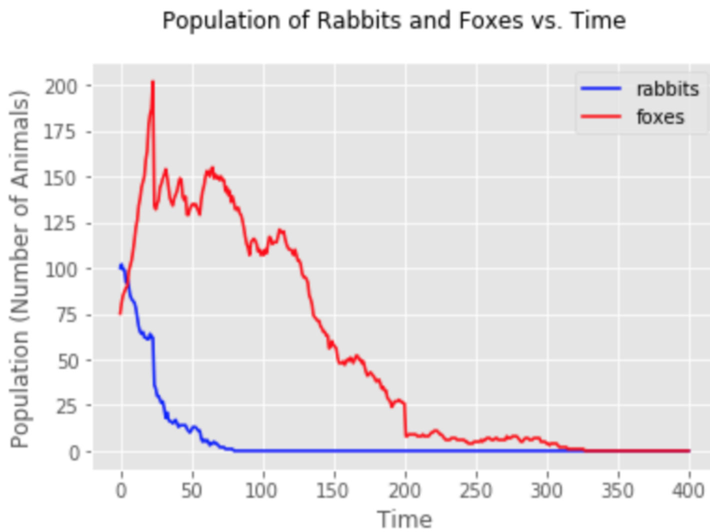

Figure 7. Rabbit and fox population over time

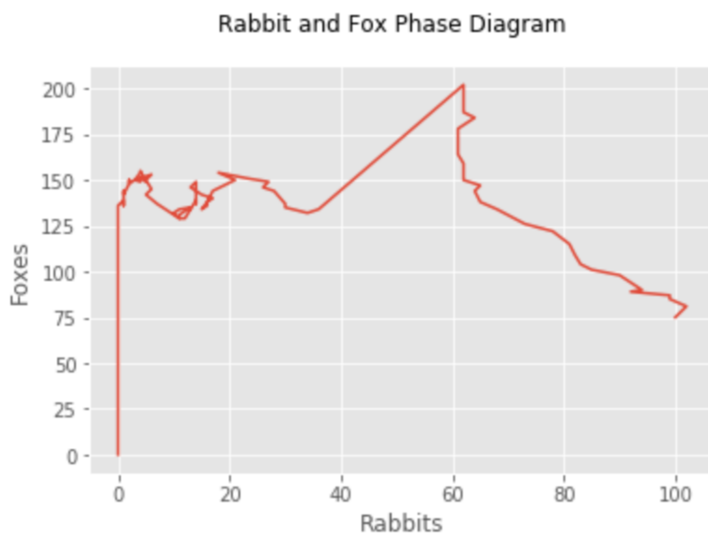

Figure 8. Phase diagram of foxes and rabbits

Taking a look at the phase diagram, *Figure 8*, it looks very chaotic. More interestingly, the phase diagram never displays an orbit nor a period. This is due to the fact that both species perished too early, before the plot made a single orbit. As expected, there was no period present within the phase diagram.

5. CONCLUSION

The purpose of this assignment was to re-imagine how population models actually work and to make it as realistic as possible. While we know that deterministic models are helpful to a certain extent, it is actually the stochastic model that better simulates real-life scenarios. The first scenario does exactly that in providing probabilities of which tables at a restaurant would most likely be occupied at any given moment. The pandemic scenario and the rabbits v.s foxes have also been revamped so to be more realistic as well. Previously, these population models were able to rebound, given enough time, and as a result would be periodic. This was due to the absence of death within the models, but the new stochastic models very well take that into account. Gone are the days where both rabbits and foxes are able to overcome extinction and re-populate the land. In our new models, once the species is dead, there is nothing that can be done to bring them back to the land of the living.