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Fourier Transform and Signal Processing

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1. INTRODUCTION

One of the most commonly used signal processing tools are Fourier transforms. Some of these functions are described in the time domain in which h changes along with time, t. A similar layout can be said for the frequency domain, in which H (amplitude) is a function of frequency, f, through all real frequencies. Functions h(t) and H (f) are representative of the same function but in different ways. Though this may be the case, Equation (1) and (2) allows for a constant switch between the two.

$$H(f) = \int_{-\infty}^{\infty} h(t)e^{i2\pi ft}dt \tag{1}$$

$$h(t) = \int_{-\infty}^{\infty} H(f)e^{-i2\pi ft}df \tag{2}$$

For the discrete Fourier transform that includes N points, it is given by Equation (3).

$$H(f_n)_n \equiv \sum_{k=1}^{N-1} h_k e^{i2\pi kn/N} \tag{3}$$

Equation (3)'s inverse is then demonstrated through Equation (4):

$$h(t_k) \equiv h_k = \frac{1}{N}\sum_{n=1}^{N-1} H_n e^{-i2\pi kn/N} \tag{4}$$

An important aspect to cover is the Fast Fourier Transform (FFT). FFT is an algorithm that determines both the discrete Fourier transform and the inverse of the signal. This determination allows for faster operations to take place. Equation (5) and (6) are of the discrete versions of the FFT for convolution and correlation.

$$(r * s)_j = \sum_{k=-N/2+1}^{N/2} s_{j-k} r_k \tag{5}$$

$$(r * s)_j = \sum_{k=0}^{N/2} r_{j+k} s_k \tag{6}$$

S_j which is the signal function, has period, N, meaning the response function, r_k is within the interval -$\frac{N}{2}$, $\frac{N}{2}$. Equations (7) and (8) are utilized to then determine the convolution and correlation.

$$F(r * s) = F(r)F(s) \tag{7}$$

$$F(r * s) = F(r)F^*(s) \tag{8}$$

Within this assignment, the Fourier transform will be utilized to analyze different sorts of signals.

2. FOURIER TRANSFORM

As previously mentioned, Fourier transforms are useful in analyzing signals so to observe their respective frequencies. In this section, the Fourier transform, inverse Fourier transform, and the power spectral. density of sine, cosine, square pulse, triangle, and delta will be taken. *Figure 1* displays the sine function for reference of the signal being analyzed. *Figure 2* is the Fourier transform of sine, *Figure 3* is the inverse Fourier transform, and *Figure 4* is the power spectral density of sine.

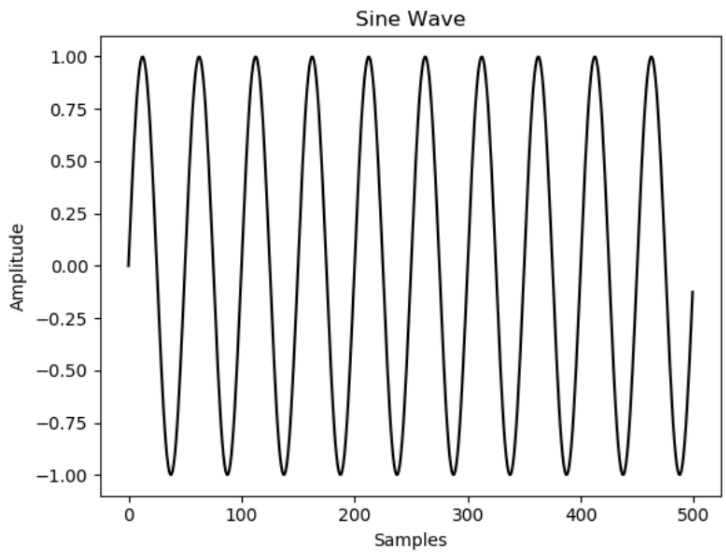

Figure 1. Original sine function.

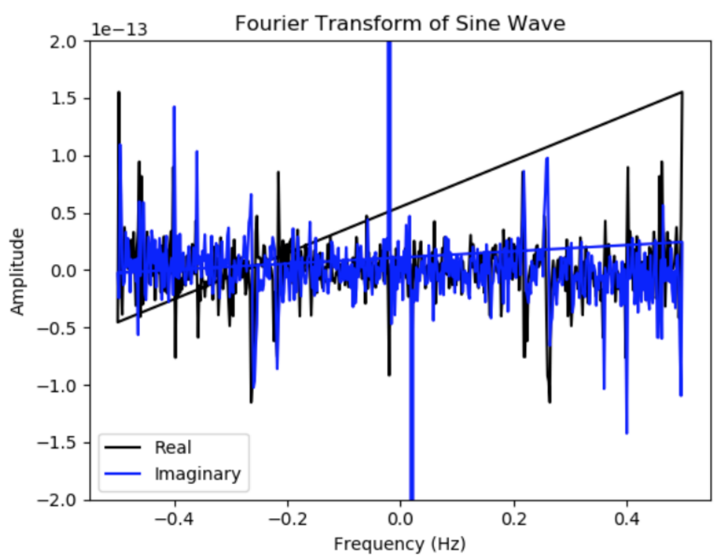

Figure 2. Fourier transform of the sine function.

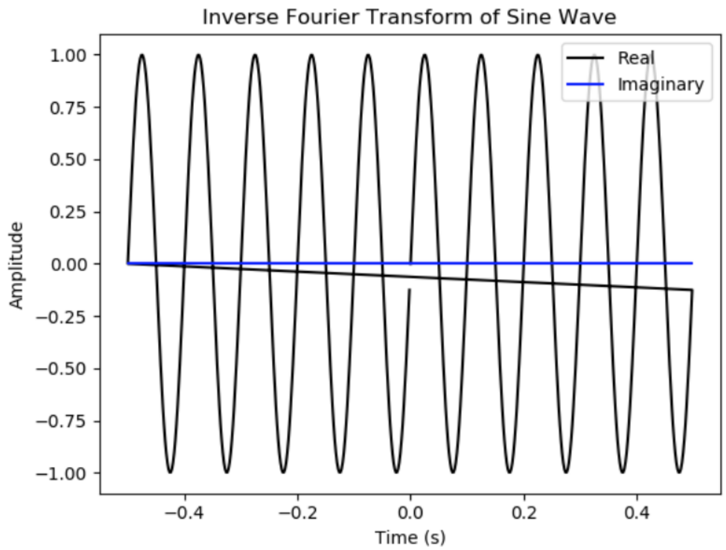

Figure 3. Inverse Fourier transform of sine.

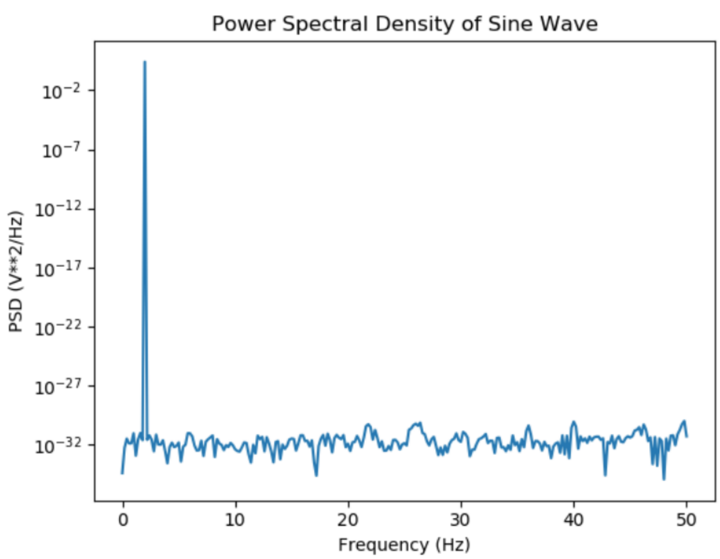

Figure 4. Power spectral density of sine.

As it can be observed in *Figure 2*, smaller amplitudes fluctuate around zero. The inverse Fourier transform, *Figure 3* is most similar to the original sine signal.

The process is then repeated with a cosine signal this time. *Figure 5* is the original cosine signal, again utilized as a reference plot. *Figure 6* is the Fourier transform of the signal, *Figure 7* is of the inverse Fourier transform, and *Figure 8* is of the power spectral density.

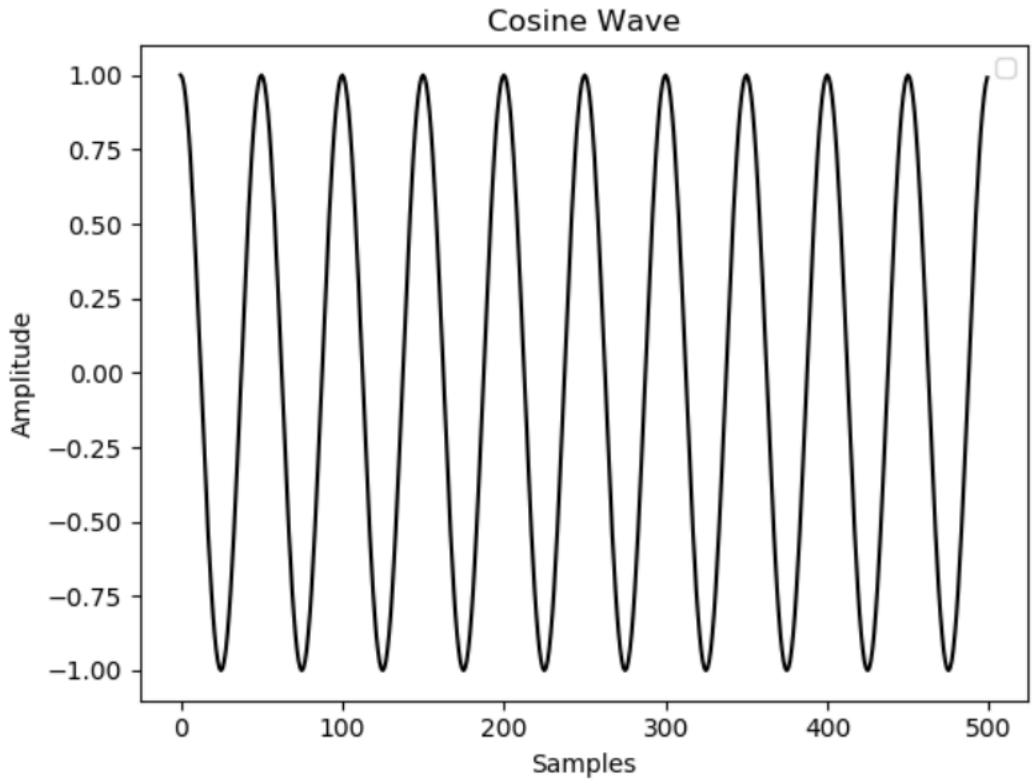

Figure 5. Original cosine signal.

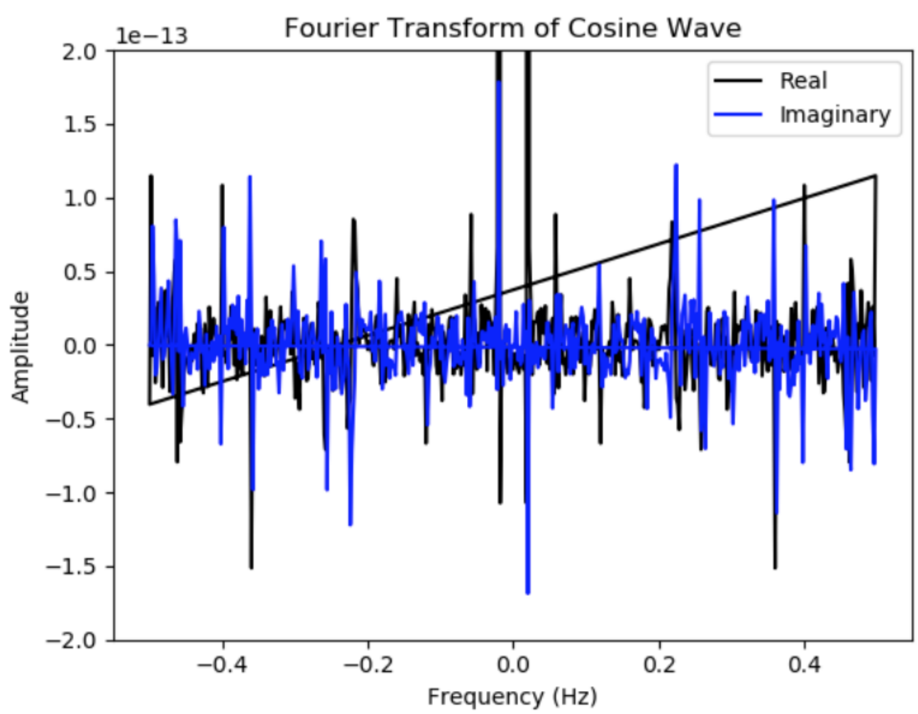

Figure 6. Fourier transform of cosine.

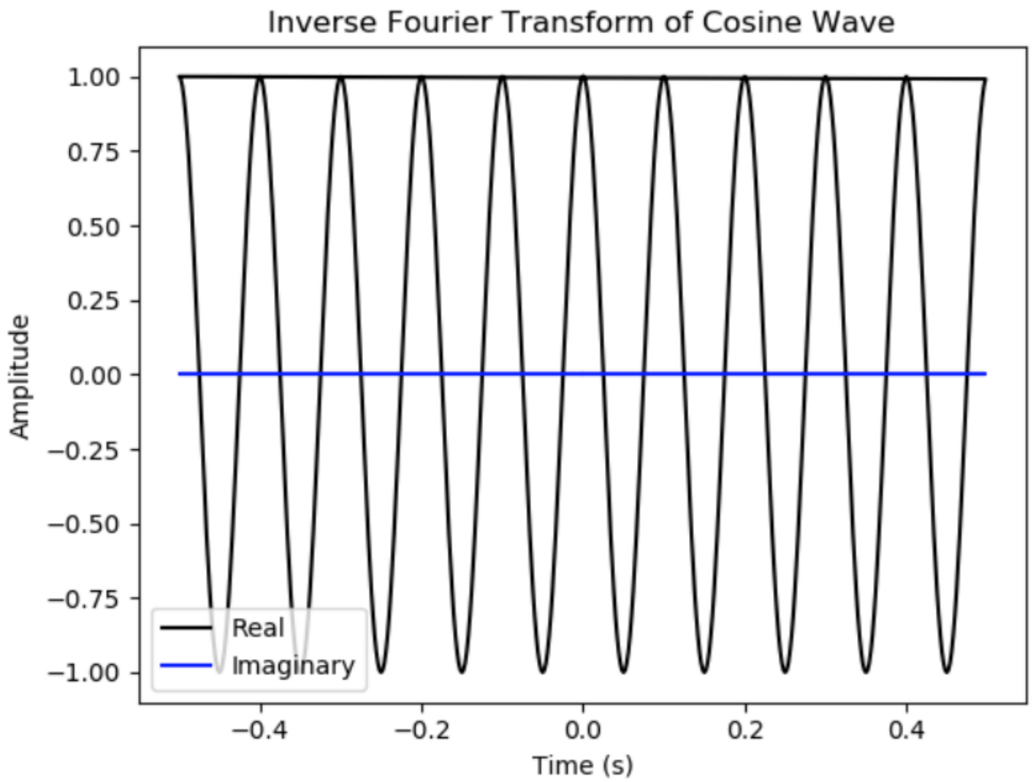

Figure 7. Inverse transform of cosine

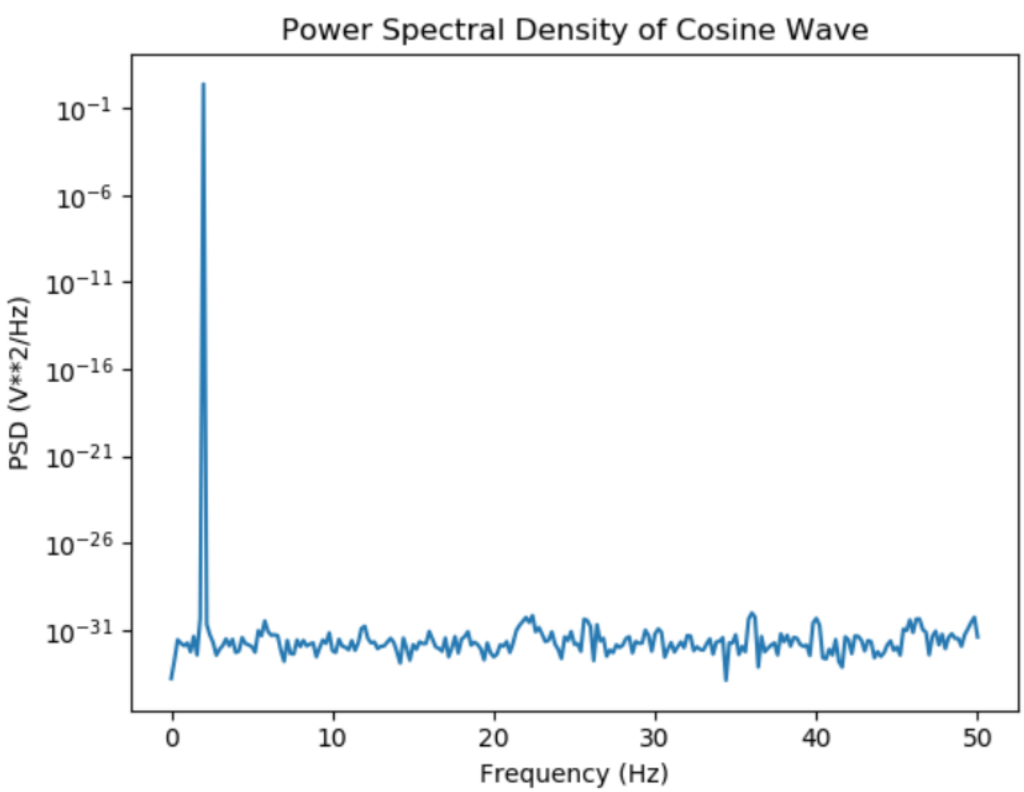

Figure 8. Power spectral density of cosine.

For the most part, the resulting plots for cosine are similar to that of the sine signal.

The square pulse is the next to be analyzed. *Figure 9* is the reference square signal, *Figure 10* is the Fourier transform of the signal, *Figure 11* is the inverse Fourier transform, and *Figure 12* is the power spectral density.

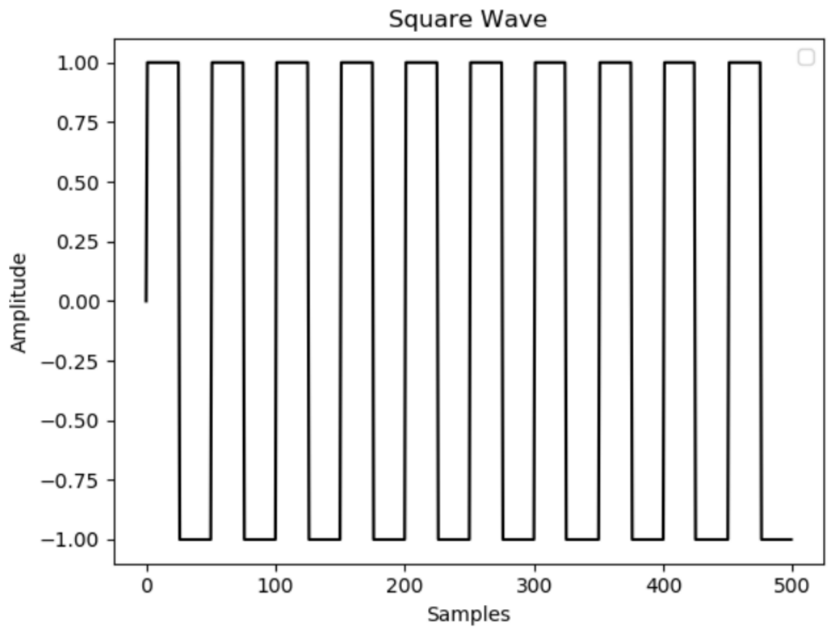

Figure 9. Original square pulse.

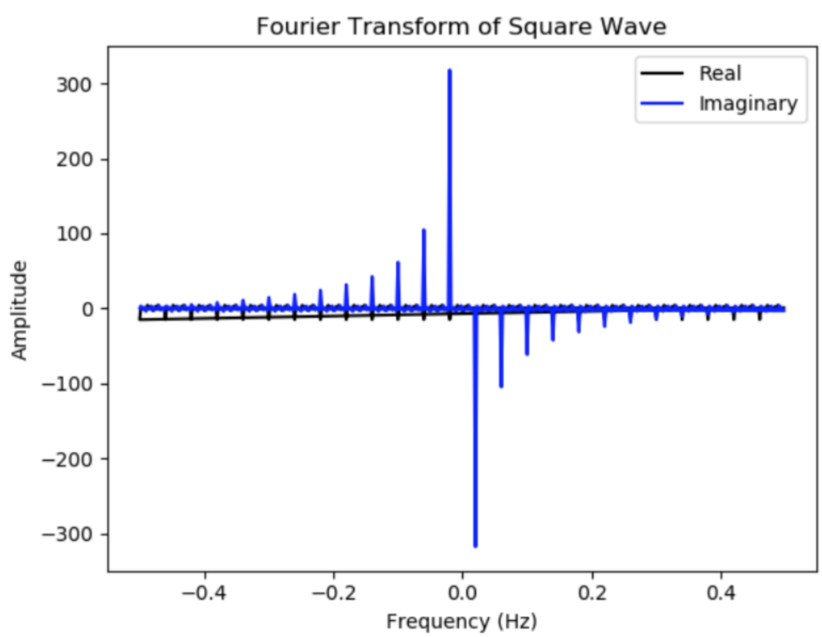

Figure 10. Fourier transform of square pulse.

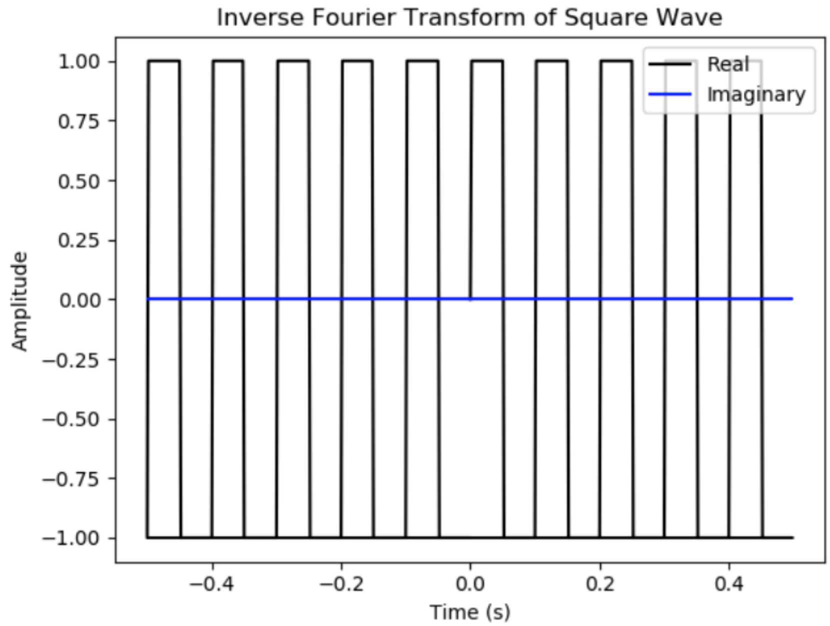

Figure 11. Inverse Fourier transform.

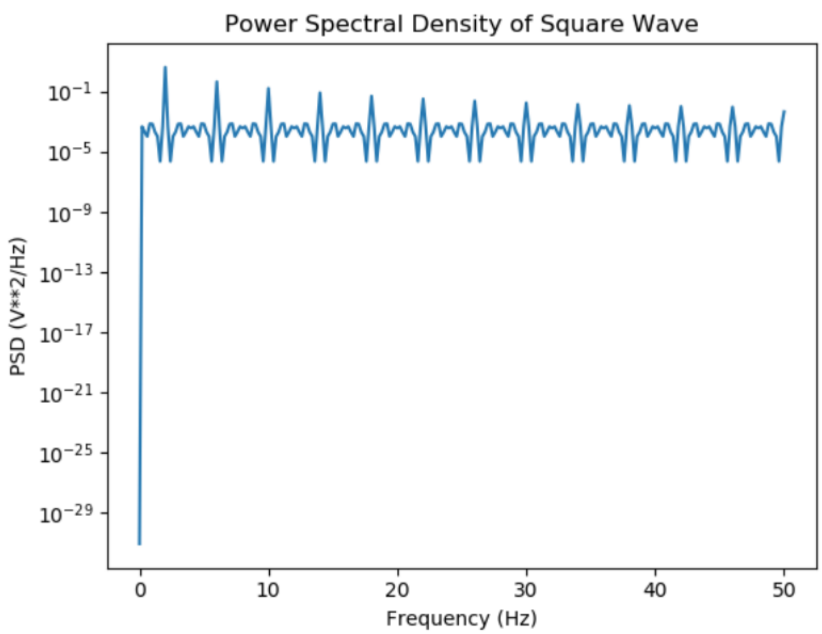

Figure 12. Power spectral density.

Something interesting to point out is that in *Figure 10*, the imaginary part of the plot stands out much more than the real part.

The next type of signal being analyzed is a triangle signal. *Figure 13* is the original triangle signal, *Figure 14* is the Fourier transform of the function, *Figure 15*, is the inverse Fourier transform, and *Figure 16* is the power spectral density.

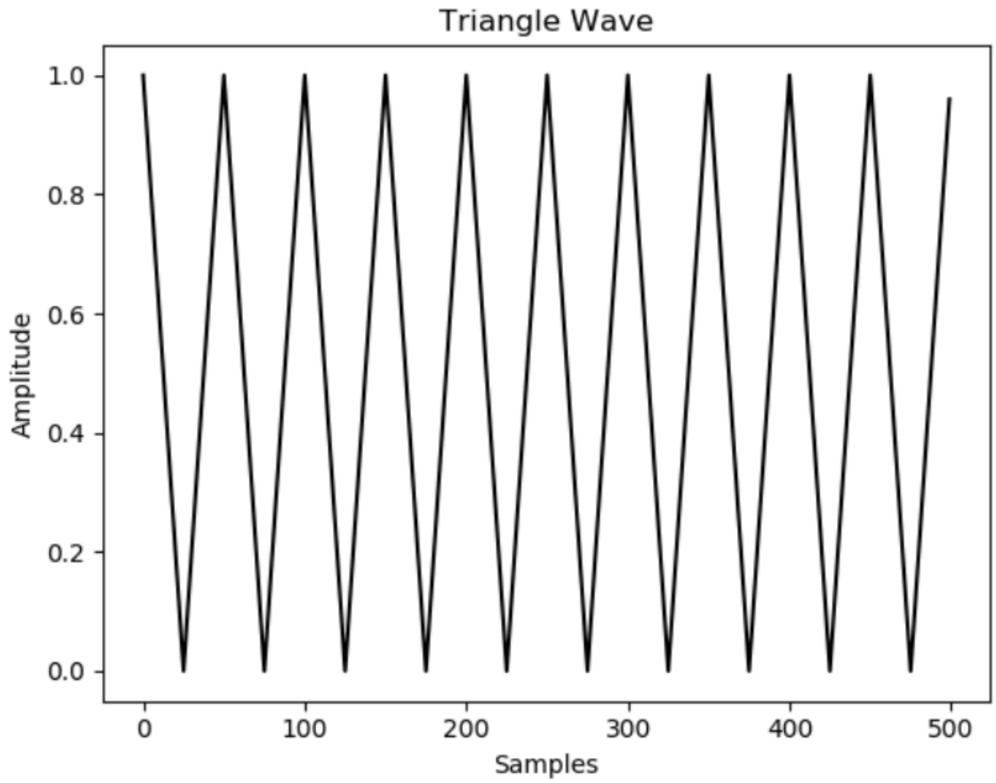

Figure 13. Original triangle signal.

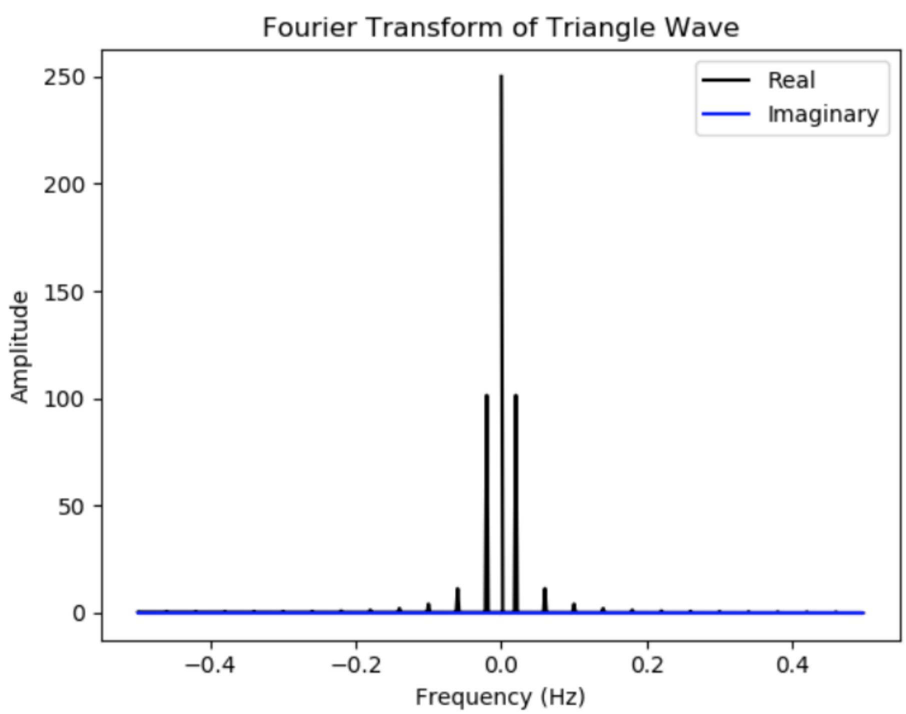

Figure 14. Fourier transform of triangle signal.

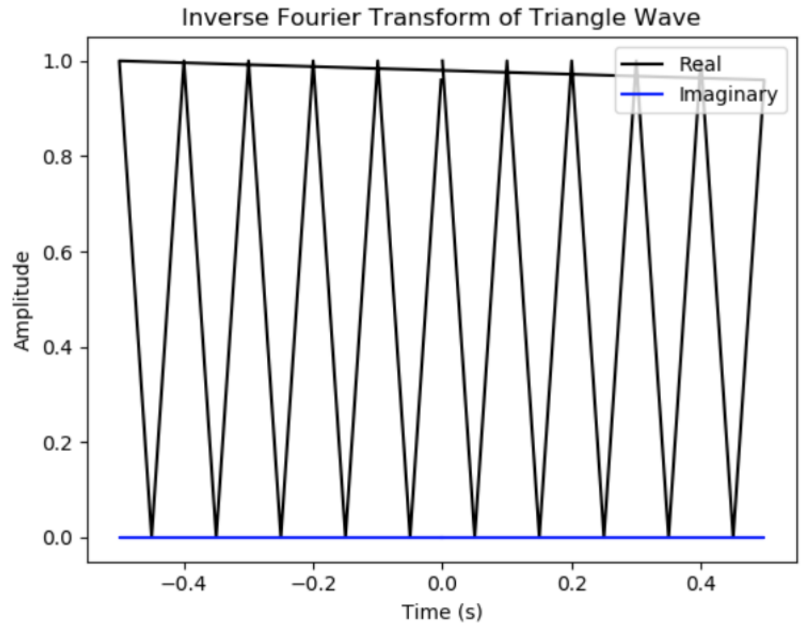

Figure 15. Inverse Fourier transform of triangle signal.

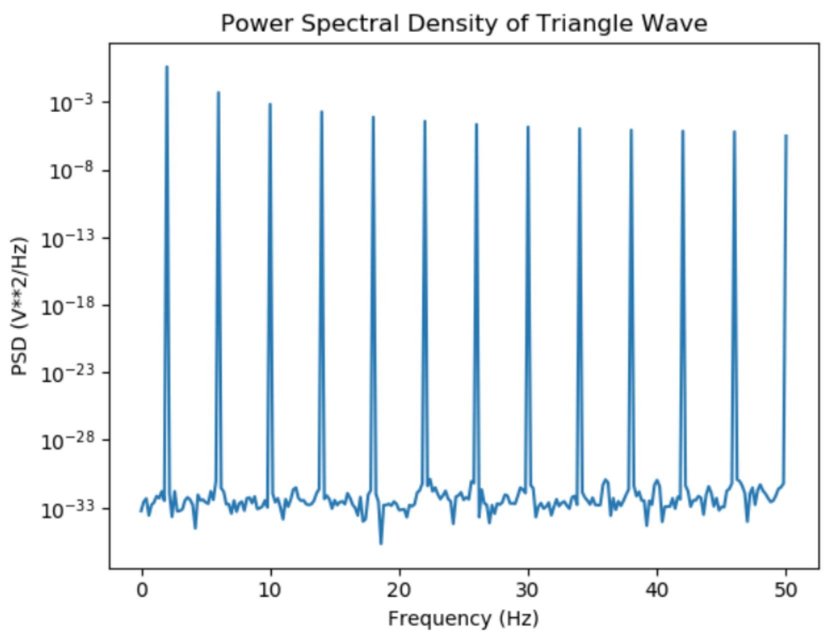

Figure 16. Power spectral density of triangle signal.

For the triangle signal, the Fourier transform indicates that the real component was more prominent at around 0 Hz in comparison to the imaginary aspect.

The final signal type being analyzed is a delta wave. *Figure 17* is the original delta signal, *Figure 18* is the Fourier transform, *Figure 19* is the inverse Fourier transform, and *Figure 20* is the power spectral density.

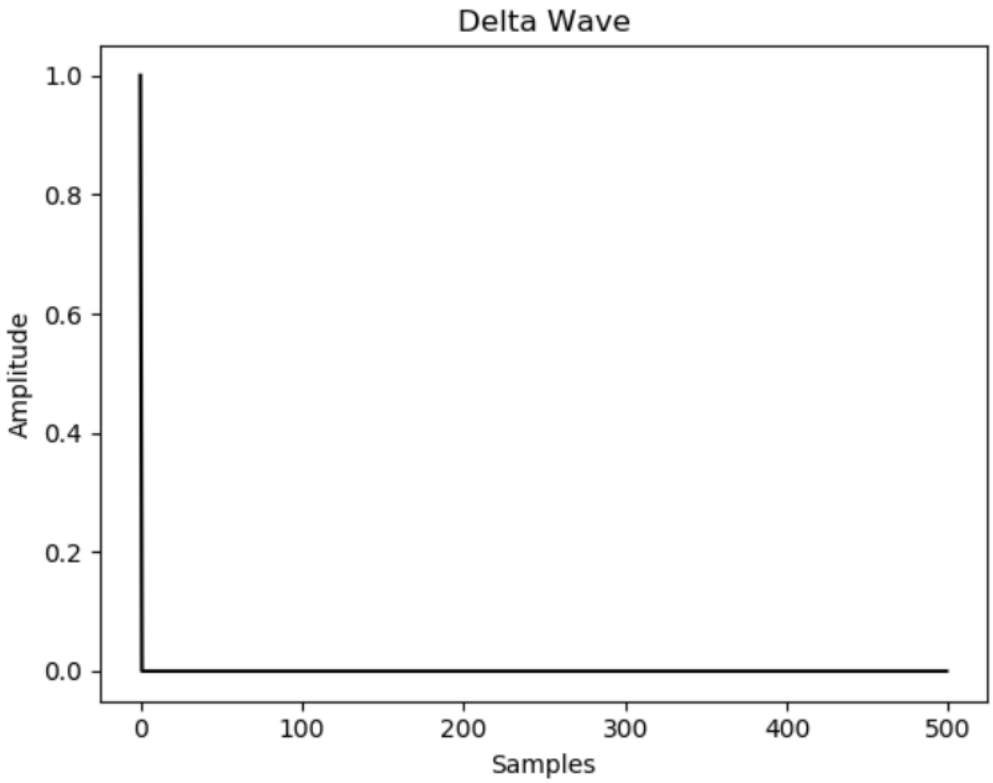

Figure 17. Original delta signal.

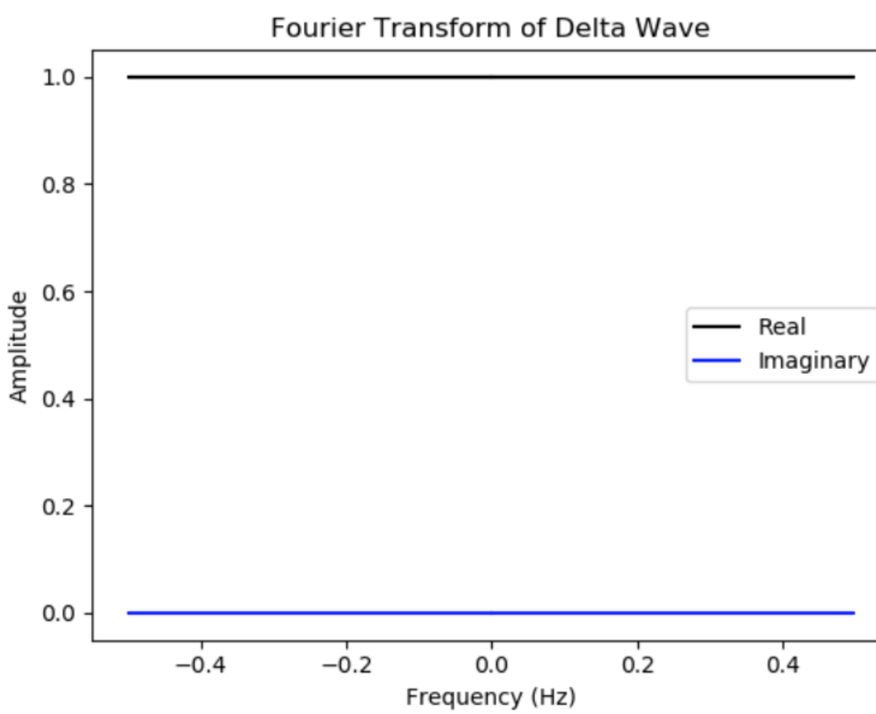

Figure 18. Fourier transform of delta signal.

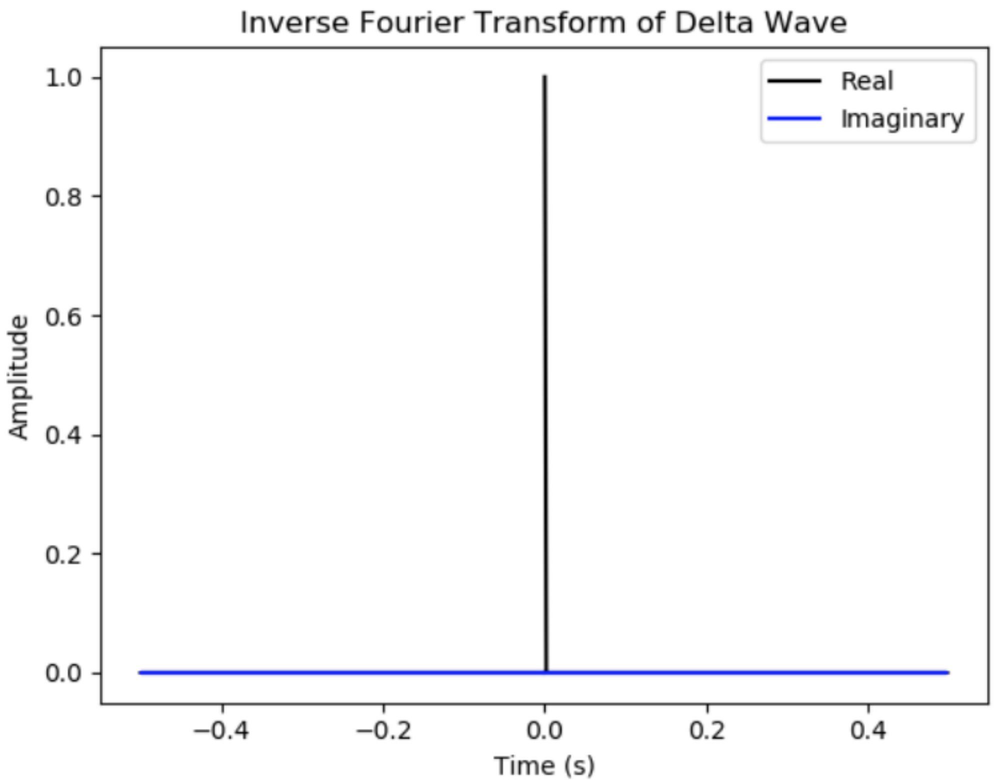

Figure 19. Inverse Fourier transform of delta signal.

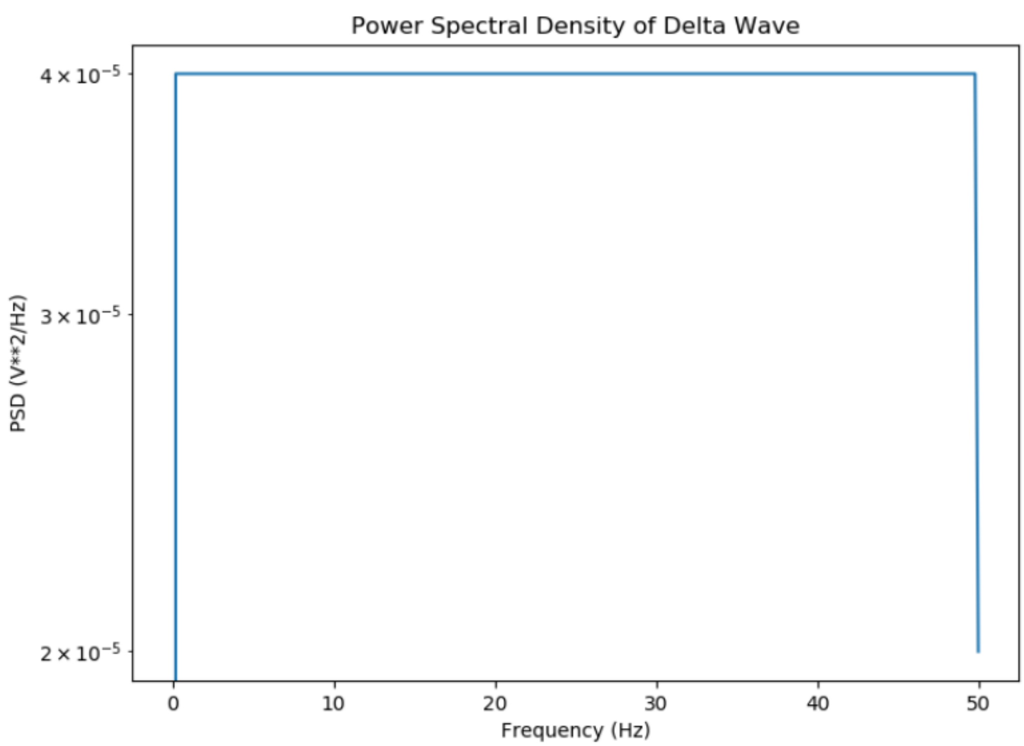

Figure 20. Power spectral density of delta signal.

3. ALIASING

Another part of signal processing is aliasing. Aliasing is when a frequency is incorrectly carried back as the 'correct signal' due to issues relating to sampling too coarsely. *Figure 21* provides a depiction of how the correct signal gets distorted due to the sampling chosen.

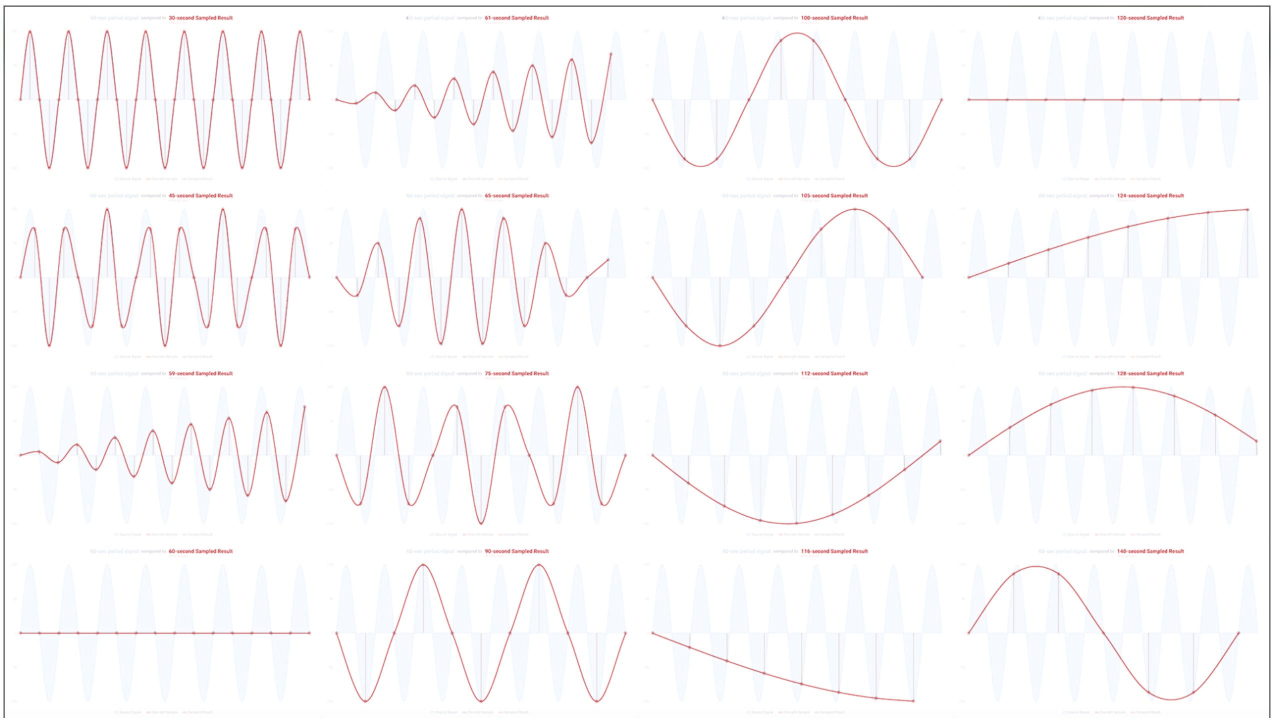

Figure 21. Different types of aliasing depending on the number of reference points utilized.

As *Figure 21* demonstrates, the lower the number of points being used to match the correct signal, the higher the probability it is incorrect. *Figure 22* demonstrates this for a sine wave. As the plot demonstrates, only utilizing 4 reference points is not enough to accurately recreate the original sine wave.

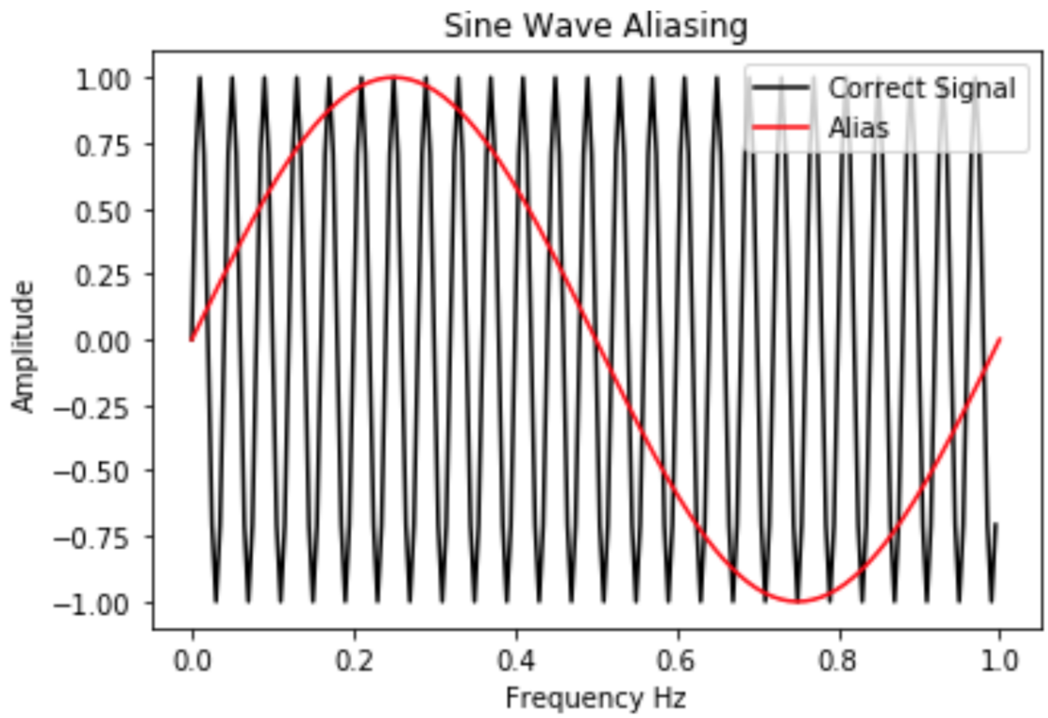

Figure 22. Aliasing on a sine wave.

To further refine the resulting signal to better match the correct one, it is recommended to sample at a higher rate so as to include as many reference points as possible. *Figure 23* depicts the same plot as before, but with a larger sample size which increases the accuracy of the signal.

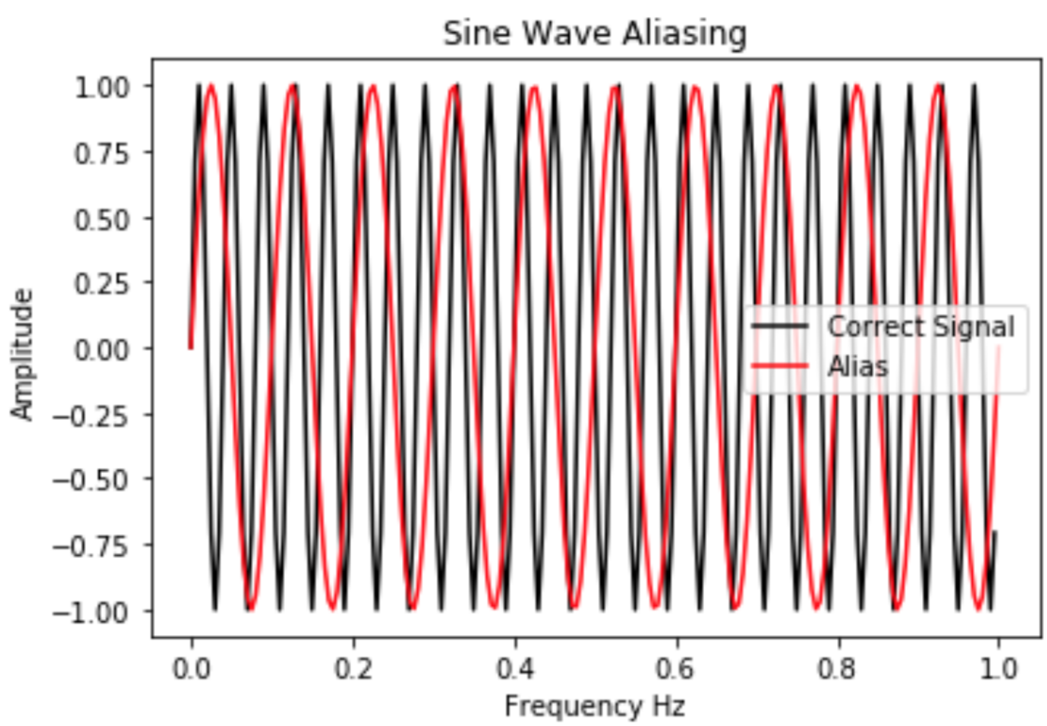

Figure 23. Increasing the sample number increases the accuracy of the signal and lowers the distortion.

4. GUITAR ANALYSIS

In addition to the signals that we have been analyzing, even frequencies from the strum of a guitar can be analyzed utilizing Fourier transforms. The sound wave goes through different ranges of frequencies. *Figure 23* is a plot of the sound wave from a guitar. Frequency is measured as a function of time.

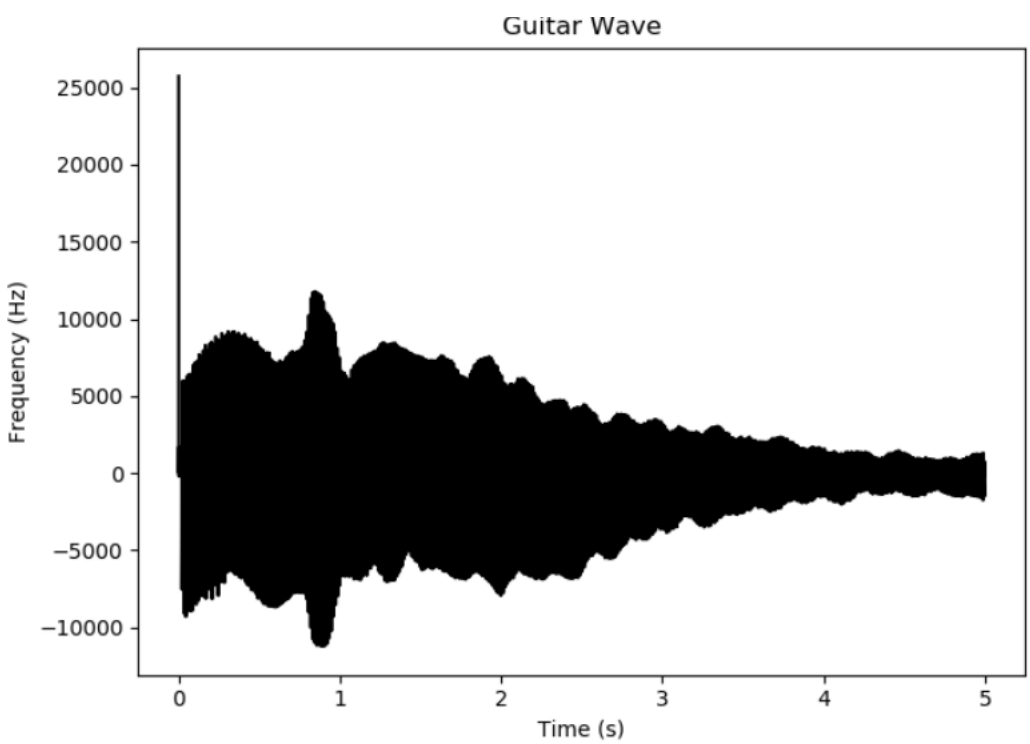

Figure 24. Sound wave from a guitar.

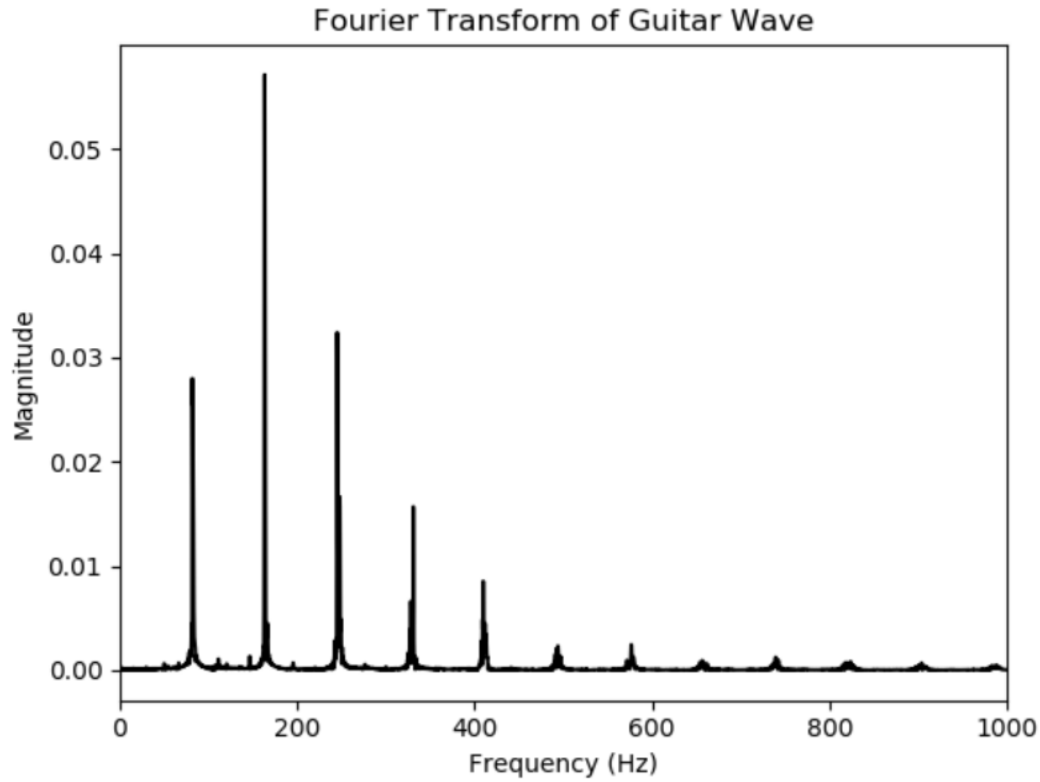

Figure 25. The Fourier transform of the sound wave.

As observed in *Figure 24*, the highest peak clocked in at 180 Hz and each subsequent peak decreased as frequency increased.

5. BACH'S PIECE

Another piece of music worth analyzing is Johann Sebastian Bach's Partita. From a brief snippet, it has been sampled at the rates of 882, 1378, 2756, 5512, 11025, and 44100 Hz. In order to best visualize how under sampling looks like, the plots of the mentioned frequencies will be shown in a descending order. *Figure 25* is of the snippet at 44100 Hz, *Figure 26* is of the 11025 Hz, *Figure 27* is at 5512 Hz, *Figure 28* is at 2756 Hz, *Figure 29* is at 1378 Hz, and *Figure 30* is at 882 Hz.

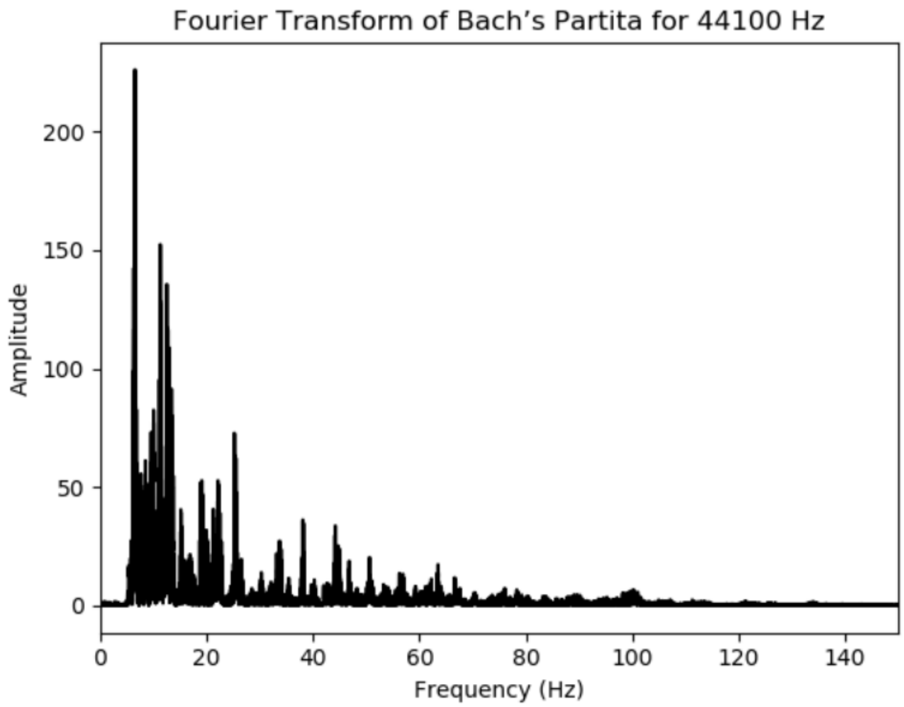

Figure 26. Snippet at 44100 Hz.

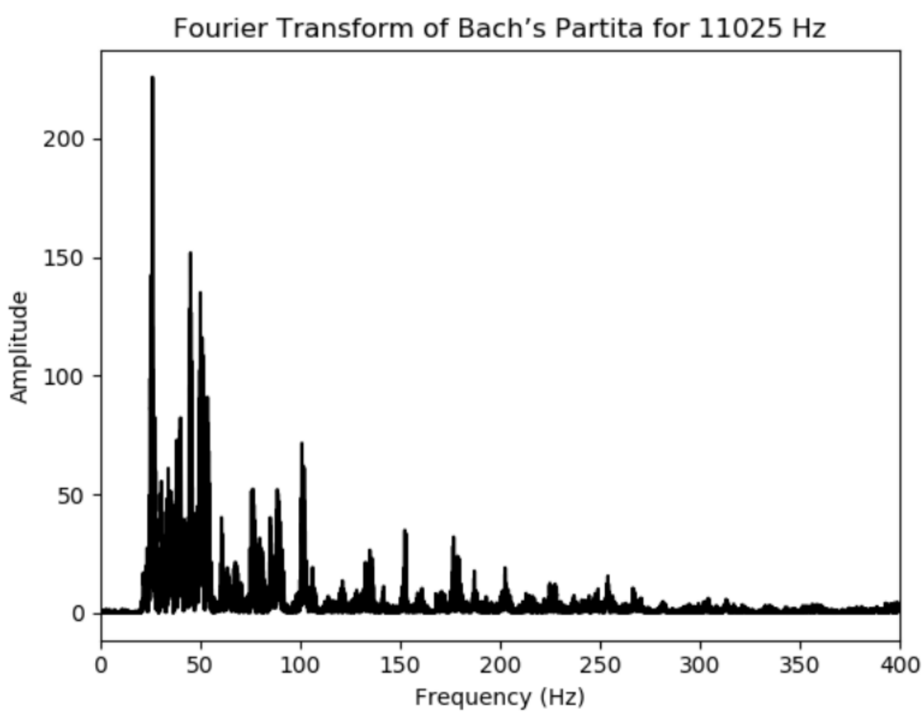

Figure 27. Snippet at 11025 Hz.

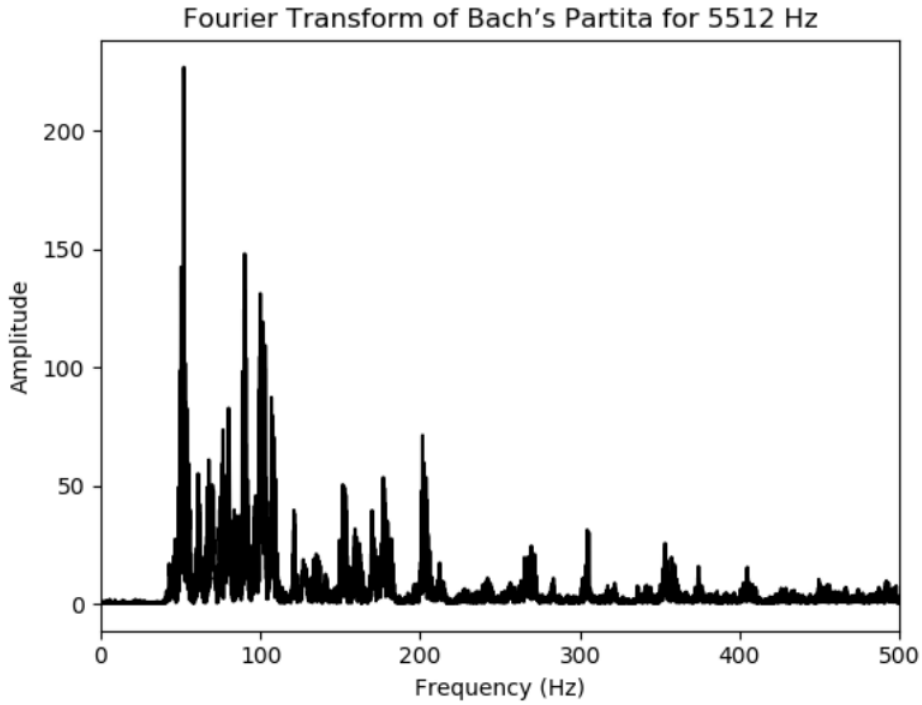

Figure 28. Snippet at 5512 Hz.

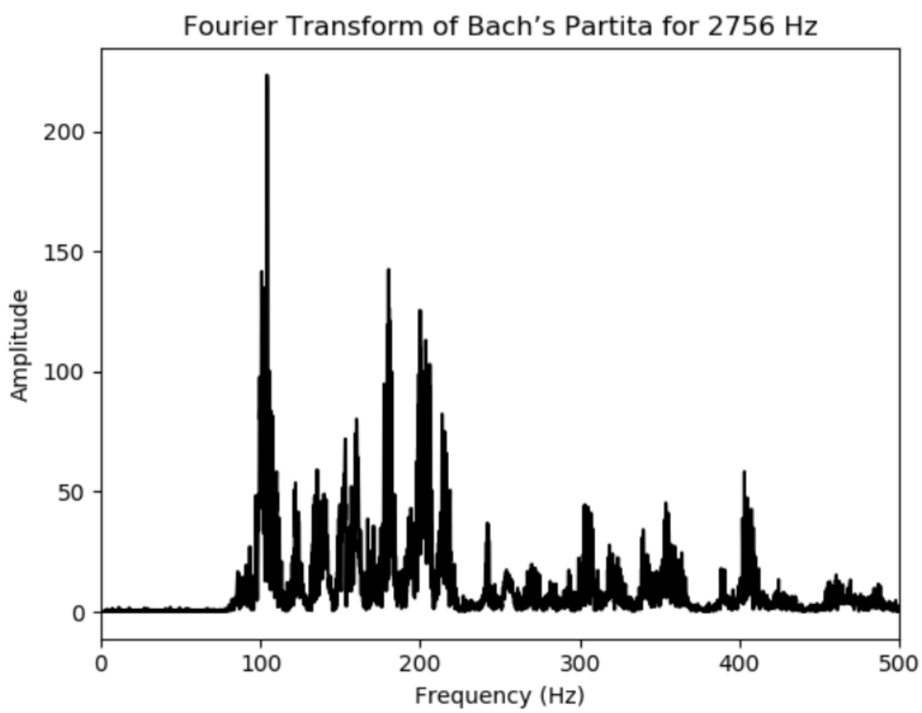

Figure 29. Snippet at 2756 Hz.

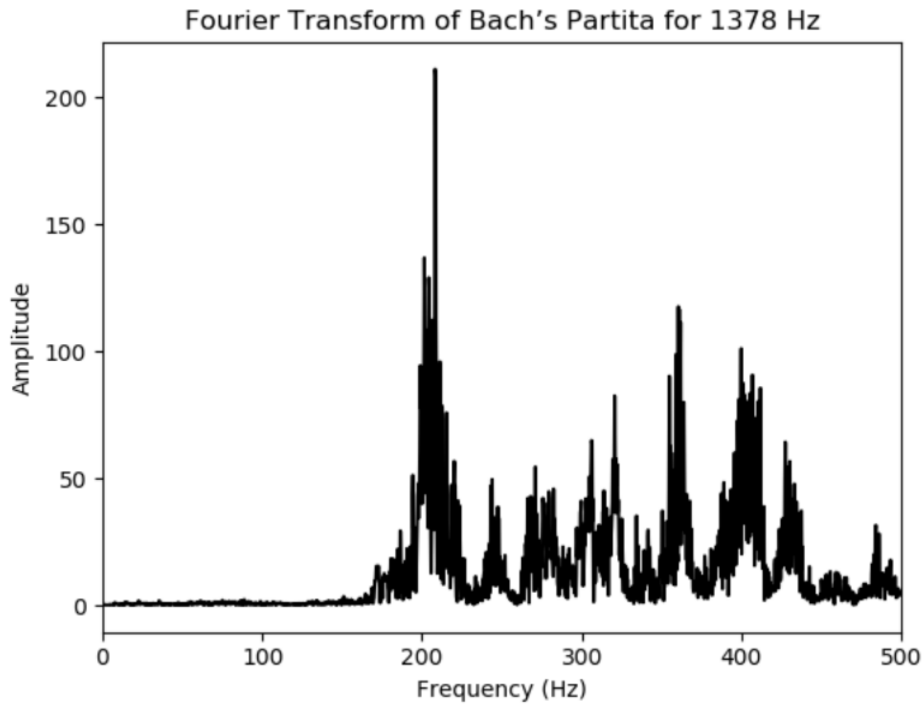

Figure 30. Snippet at 1378 Hz.

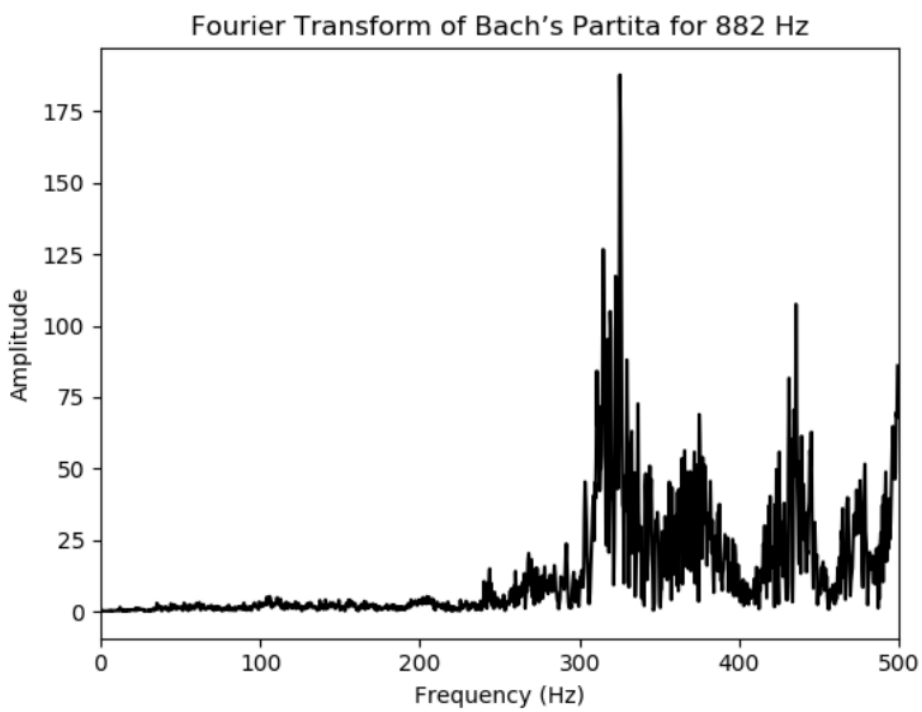

Figure 31. Snippet at 882 Hz.

6. SWEET SOUND OF BOILING WATER

Another sound that can be placed under the scope of autocorrelation is that of boiling water. Doing this allows for us to observe the time lag of a signal over time..*Figure 31* is a plot of the sound wave of boiling water, *Figure 32* is the Fourier transform of the signal, and *Figure 33* is the autocorrelation plot.

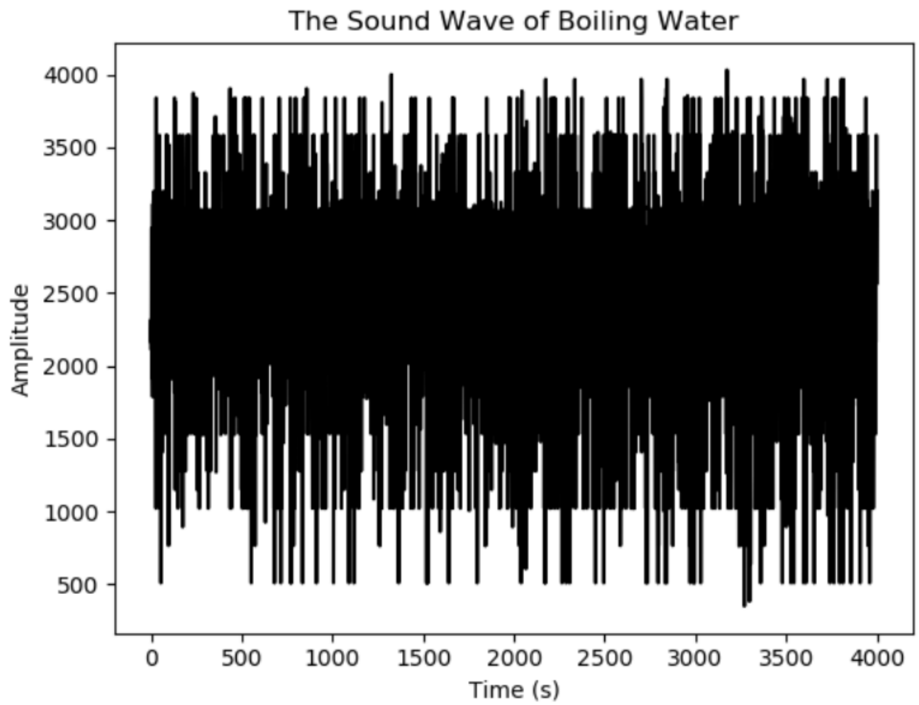

Figure 32. Boiling water sound wave.

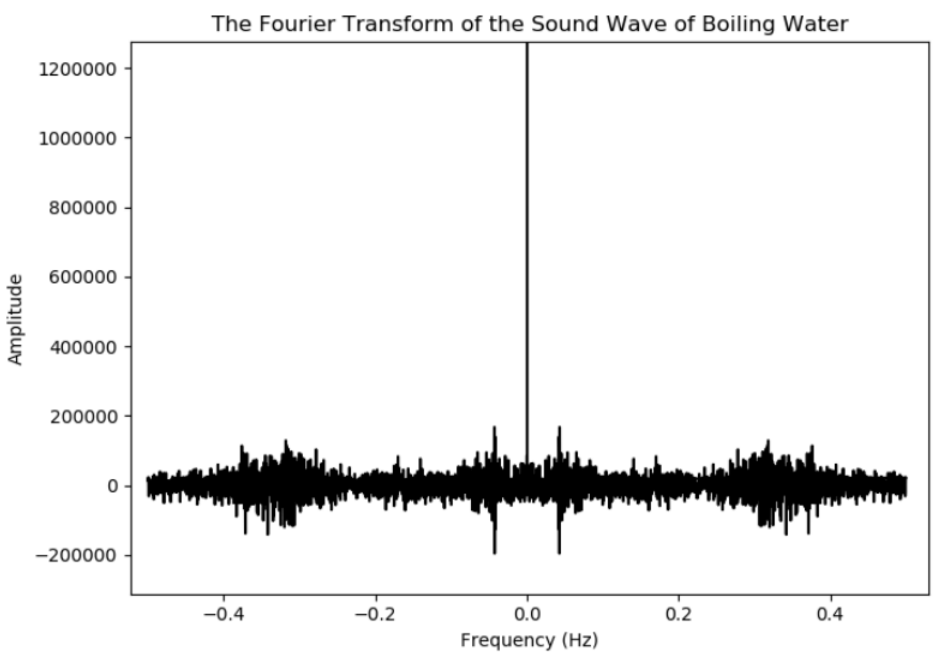

Figure 33. Fourier transform of boiling water

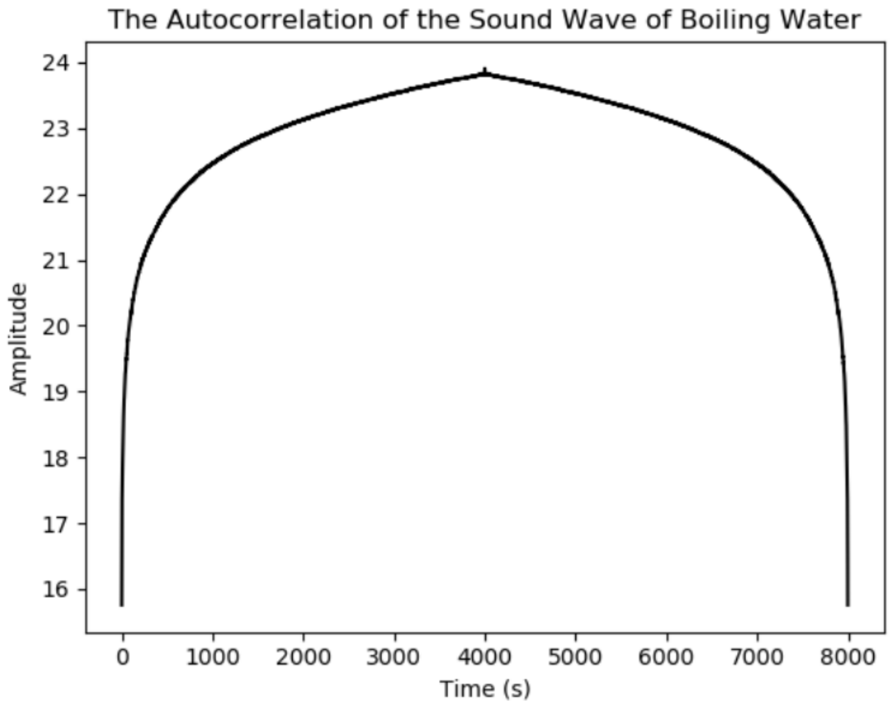

Figure 34. Autocorrelation plot of boiling water.

7. CONCLUSION

The purpose of this assignment is to understand how Fourier transforms are prevalent in signal processing. No matter the type of signal, Fourier transforms are useful in decoding them to retrieve information. From pieces of music to the sound of boiling water, Fourier transforms can be utilized on them all.