自動機與形式語言 Homework 1

B03902086 李鈺昇

1

We know that $x \sim_n y \iff x \equiv y \bmod n \iff x - y = kn, k \in \mathbb{Z}$.
Assume $n \neq 0$. To show that $\sim_n$ is an equivalence relation, it suffices to check the following:

- Reflexive: $\forall x \in \mathbb{Z}, x - x = 0 = 0 \times n$, so $x \sim_n x$.
- Symmetric: $\forall x, y \in \mathbb{Z}$,
  $x \sim_n y$
  $\iff x - y = kn, k \in \mathbb{Z}$
  $\iff y - x = -kn$, and $-k \in \mathbb{Z}$
  $\iff y \sim_n x$.
- Transitive: $\forall x, y, z \in \mathbb{Z}$,
  $x \sim_n y$ and $y \sim_n z$
  $\implies x - y = k_1 n, y - z = k_2 n, k_1, k_2 \in \mathbb{Z}$
  $\implies x - z = (x - y) + (y - z) = (k_1 + k_2)n$, and $k_1 + k_2 \in \mathbb{Z}$
  $\implies x \sim_n z$.

There are n equivalence classes, since all the possible remainders modulo n are $\{0, 1, ..., n-1\}$, and $x \sim_n y$ iff $[x]_\sim = [y]_\sim$, which means two integers falls into the same class iff their remainders modulo n are the same.

2

All the following k $\in \mathbb{Z}$.

- 0. If $x \equiv 0 \bmod 3$, $x - 0 = 3k, 2x = 6k = 3(2k) + 0, 2x \equiv 0 \bmod 3$.
- 1. If $x \equiv 0 \bmod 3$, $x - 0 = 3k, 2x + 1 = 6k + 1 = 3(2k) + 1, 2x + 1 \equiv 1 \bmod 3$.
- 2. If $x \equiv 1 \bmod 3$, $x - 1 = 3k, 2x = 6k + 2 = 3(2k) + 2, 2x \equiv 2 \bmod 3$.
- 0. If $x \equiv 1 \bmod 3$, $x - 1 = 3k, 2x + 1 = 6k + 3 = 3(2k + 1) + 0, 2x + 1 \equiv 0 \bmod 3$.
- 1. If $x \equiv 2 \bmod 3$, $x - 2 = 3k, 2x = 6k + 4 = 3(2k + 1) + 1, 2x \equiv 1 \bmod 3$.
- 2. If $x \equiv 2 \bmod 3$, $x - 2 = 3k, 2x + 1 = 6k + 5 = 3(2k + 1) + 2, 2x + 1 \equiv 2 \bmod 3$.

3

Now assume $\sim$ is an equivalence relation over X.

- $[x]_\sim = [y]_\sim$ if and only if $x \sim y$.

  Proof On one hand, $[x]_\sim = [y]_\sim$
  $\implies y \in [y]_\sim = [x]_\sim$, by definition of $[\cdot]_\sim$ and $y \sim y$ (reflexivity)
  $\implies x \sim y$, by definition of $[\cdot]_\sim$.

On the other hand, $x \sim y$
$\implies y \sim x$, by symmetry
$\implies \forall z \in [x]_\sim, x \sim z$, and thus $y \sim z$ by transitivity, which implies $z \in [y]_\sim$. Hence $[x]_\sim \subseteq [y]_\sim$. Similarly, $[y]_\sim \subseteq [x]_\sim$.
$\implies [x]_\sim = [y]_\sim$.

- If $[x]_\sim \neq [y]_\sim$, then $[x]_\sim \cap [y]_\sim = \emptyset$.

  Proof Suppose $[x]_\sim \neq [y]_\sim$. If $[x]_\sim \cap [y]_\sim \neq \emptyset, \exists z \in [x]_\sim \cap [y]_\sim$. Then $z \in [x]_\sim, z \in [y]_\sim$, hence $x \sim z$, $y \sim z$. By symmetry and transitivity, $x \sim y$.
  Now with the previous part of this lemma, we get $[x]_\sim = [y]_\sim$, a contradiction. So $[x]_\sim \cap [y]_\sim = \emptyset$.

4

Each member of X belongs to exactly one equivalence class of $\sim$.

Proof $\forall x \in X$, by reflexivity we know that $x \in [x]_\sim$. Now suppose $x \in A$ and $x \in B$, where A and B are (possibly identical) equivalence classes of $\sim$. Then $x \in A \cap B$, which means $A \cap B \neq \emptyset$.
By the contrapositive of the second part of Lemma 1.1, with A, B in place of $[x]_\sim, [y]_\sim$ respectively, we have $A = B$. Therefore x belongs to exactly one equivalence class, i.e. $[x]_\sim$.