# 自動機與形式語言 Homework 5

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## (1)

First, recall the computable function that appears in the solution to HW4:

| Function-All-or-Nothing | |
|---|---|
| **Input**: | A TM description $\lfloor \mathcal{M} \rfloor$ and a word $w$ |
| **Task**: | Output another TM description $\lfloor \mathcal{M}_w \rfloor$ such that:<br>• If $\mathcal{M}$ accepts $w$, $\mathcal{M}_w$ accepts every word.<br>• If $\mathcal{M}$ does not accept $w$, $\mathcal{M}_w$ does not accept any word. |

Let Accept-Epsilon $= \{\lfloor \mathcal{M} \rfloor \mid \mathcal{M}$ accepts $\epsilon\}$. Now I show HALT $= \{\lfloor \mathcal{M} \rfloor \$ w \mid \mathcal{M}$ accepts $w\} \leq_T$ Accept-Epsilon by giving the following algorithm:

> On input $\lfloor \mathcal{M} \rfloor \$ w$ of HALT,
>
> - Compute $\lfloor \mathcal{M}_w \rfloor$ with Function-All-or-Nothing$(\lfloor \mathcal{M} \rfloor, w)$.
> - Suppose $\mathcal{N}$ decides Accept-Epsilon. Check whether $\lfloor \mathcal{M}_w \rfloor \in$ Accept-Epsilon by running $\mathcal{N}$ on $\lfloor \mathcal{M}_w \rfloor$. Output `True` if so, otherwise output `False`.

The correctness of this reduction is clear as follows:

- If $\lfloor \mathcal{M} \rfloor \$ w \in$ HALT, $\mathcal{M}$ accepts $w$. Thus $\lfloor \mathcal{M}_w \rfloor$ accepts every word, including $\epsilon$. So $\lfloor \mathcal{M}_w \rfloor \in$ Accept-Epsilon.
- If $\lfloor \mathcal{M} \rfloor \$ w \notin$ HALT, $\mathcal{M}$ does not accept $w$. Thus $\lfloor \mathcal{M}_w \rfloor$ does not accept any word, which means it does not accept $\epsilon$. So $\lfloor \mathcal{M}_w \rfloor \notin$ Accept-Epsilon.

Since HALT is undecidable and HALT $\leq_T$ Accept-Epsilon, we know Accept-Epsilon is undecidable.

## (2)

Suppose $L_1, L_2 \in \mathbf{NP}$. By definition, there are NTMs $\mathcal{M}_1, \mathcal{M}_2$ such that $\mathcal{M}_1$ decides $L_1$ in $O(n^{k_1})$ time, and $\mathcal{M}_2$ decides $L_2$ in $O(n^{k_2})$ time. Construct NTMs $\mathcal{M}_\cup, \mathcal{M}_\cap$ as follows:

- $\mathcal{M}_\cup =$ "On input $w$, run $\mathcal{M}_1$ on $w$ and $\mathcal{M}_2$ on $w$. If at least one of them accept, accept; otherwise, reject."
- $\mathcal{M}_\cap =$ "On input $w$, run $\mathcal{M}_1$ on $w$ and $\mathcal{M}_2$ on $w$. If both of them accept, accept; otherwise, reject."

It's obvious that $\mathcal{M}_\cup$ decides $L_1 \cup L_2$ and $\mathcal{M}_\cap$ decides $L_1 \cap L_2$.
Since $\mathcal{M}_\cup$ and $\mathcal{M}_\cap$ both call $\mathcal{M}_1$ and $\mathcal{M}_2$ exactly once, we know $\mathcal{M}_\cup$ and $\mathcal{M}_\cap$ both run in $O(n^{k_1}) + O(n^{k_2}) = O(n^k)$ time, where $k = \max(k_1, k_2)$.
It then follows that $L_1 \cup L_2, L_1 \cap L_2 \in \mathbf{NP}$.

## (3)

By definition of $\mathbf{coNP} = \{L \mid \Sigma^* - L \in \mathbf{NP}\}$, if $\Sigma^* - L \in \mathbf{coNP}$, then $L = \Sigma^* - (\Sigma^* - L) \in \mathbf{NP}$.
The second argument (if $L \in \mathbf{coNP}$ then $\Sigma^* - L \in \mathbf{NP}$) is immediate by definition.

## (4)

Suppose $\mathbf{NP} \subseteq \mathbf{coNP}$.
Now for any $L \in \mathbf{coNP}$, we have $\Sigma^* - L \in \mathbf{NP} \subseteq \mathbf{coNP}$, which implies $L = \Sigma^* - (\Sigma^* - L) \in \mathbf{NP}$.
So $\mathbf{coNP} \subseteq \mathbf{NP}$, hence $\mathbf{NP} = \mathbf{coNP}$.

## (5)

Suppose $\mathsf{SAT} \in \mathbf{coNP}$. Then $\Sigma^* - \mathsf{SAT} \in \mathbf{NP}$.
Let $\mathcal{M}$ be the polynomial-time NTM that decides $\Sigma^* - \mathsf{SAT}$.

Now for any $L \in \mathbf{NP}$, we have $L \leq_p \mathsf{SAT}$, since $\mathsf{SAT}$ is $\mathbf{NP}$-hard. So there is a polynomial-time computable function $f : \Sigma^* \to \Sigma^*$ such that

$$w \in L \Leftrightarrow f(w) \in \mathsf{SAT}. \tag{1}$$

Clearly (1) equivalent to

$$w \in \Sigma^* - L \Leftrightarrow f(w) \in \Sigma^* - \mathsf{SAT}, \tag{2}$$

which implies $\Sigma^* - L \leq_p \Sigma^* - \mathsf{SAT}$.

Now construct the NTM $\mathcal{N}$ = "On input $w$, output the result of $\mathcal{M}$ on $f(w)$," which decides $\Sigma^* - L$ by (2). We see that $\mathcal{N}$ runs in polynomial time since $\mathcal{M}$ and $f$ both run in polynomial time.
Therefore $\Sigma^* - L \in \mathbf{NP}$, or $L \in \mathbf{coNP}$.

The above shows $\mathbf{NP} \subseteq \mathbf{coNP}$, and hence $\mathbf{NP} = \mathbf{coNP}$ by Problem (4).