自動機與形式語言 Homework 3

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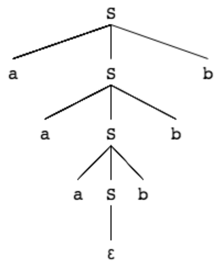
(1)

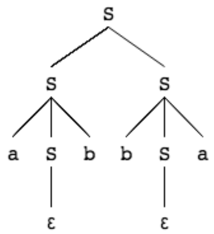
(i) In $\mathcal{G}$



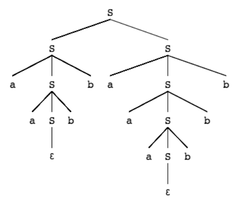
(ii) Not in $\mathcal{G}$

The numbers of a's and b's differ. See Corollary 1 in Problem (5).

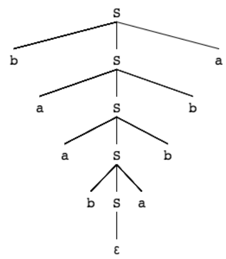
(iii) In $\mathcal{G}$



(iv) In $\mathcal{G}$



(v) In $\mathcal{G}$



(2)

(i) $\mathcal{G} = \langle \{a, b\}, \{S\}, R, S \rangle$

$R: \quad S \to aS \mid aSb \mid a$

(ii) $\mathcal{G} = \langle \{a, b\}, \{S\}, R, S \rangle$

$R: \quad S \to Sb \mid aSb \mid b$

(iii) $\mathcal{G} = \langle \{a, b\}, \{S\}, R, S \rangle$

$R: \quad S \to aaSb \mid \epsilon$

(iv) $\mathcal{G} = \langle \{a, b, \$\}, \{S\}, R, S \rangle$

$R: \quad S \to aSa \mid bSb \mid \$$

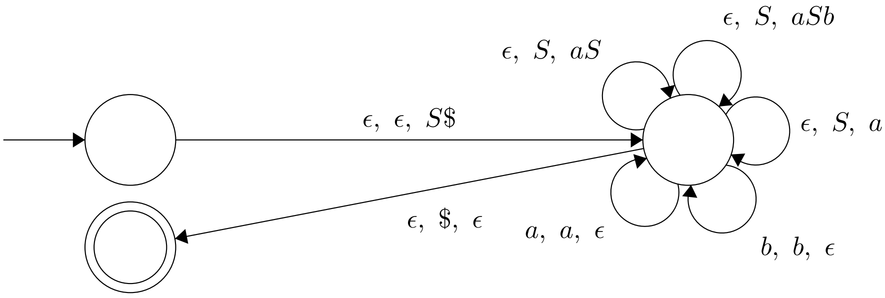
(v) $\mathcal{G} = \langle \{a, b\}, \{S, A, B, C\}, R, S \rangle$

$$
\begin{aligned}
R: \quad & S \to A \mid B \mid bC \mid aCbCaC \\
& A \to aA \mid aAb \mid a \\
& B \to Bb \mid aBb \mid b \\
& C \to aC \mid bC \mid \epsilon
\end{aligned}
$$

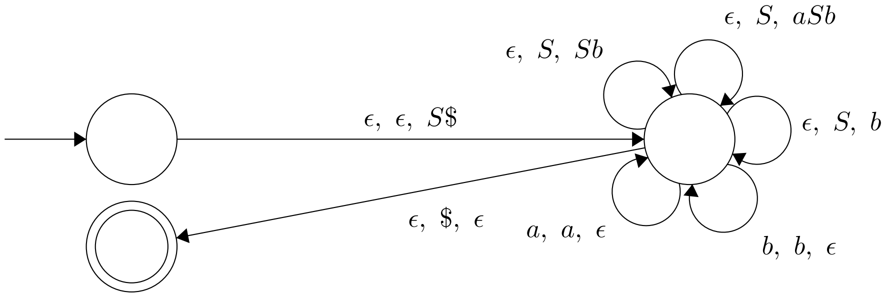
(3)

In the following PDAs, I use $\xrightarrow{a,u,v}$ to mean a transition that reads input $a \in \Sigma \cup \{\epsilon\}$, pops $u \in \Gamma^*$, and pushes $v \in \Gamma^*$, where the order of popping and pushing is from right to left.

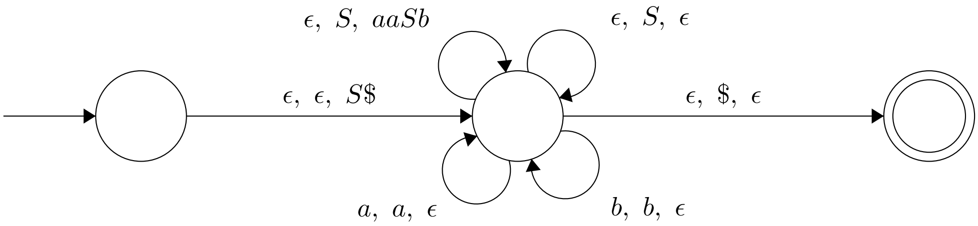
(i)


$\epsilon,\ S,\ aSb$
$\epsilon,\ S,\ aS$
$\epsilon,\ \epsilon,\ S\$$
$\epsilon,\ S,\ a$
$\epsilon,\ \$,\ \epsilon$
$a,\ a,\ \epsilon$
$b,\ b,\ \epsilon$

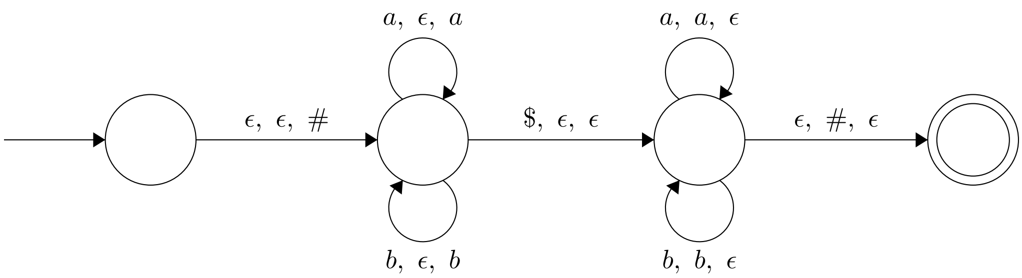
(ii)


$\epsilon,\ S,\ aSb$
$\epsilon,\ S,\ Sb$
$\epsilon,\ \epsilon,\ S\$$
$\epsilon,\ S,\ b$
$\epsilon,\ \$,\ \epsilon$
$a,\ a,\ \epsilon$
$b,\ b,\ \epsilon$

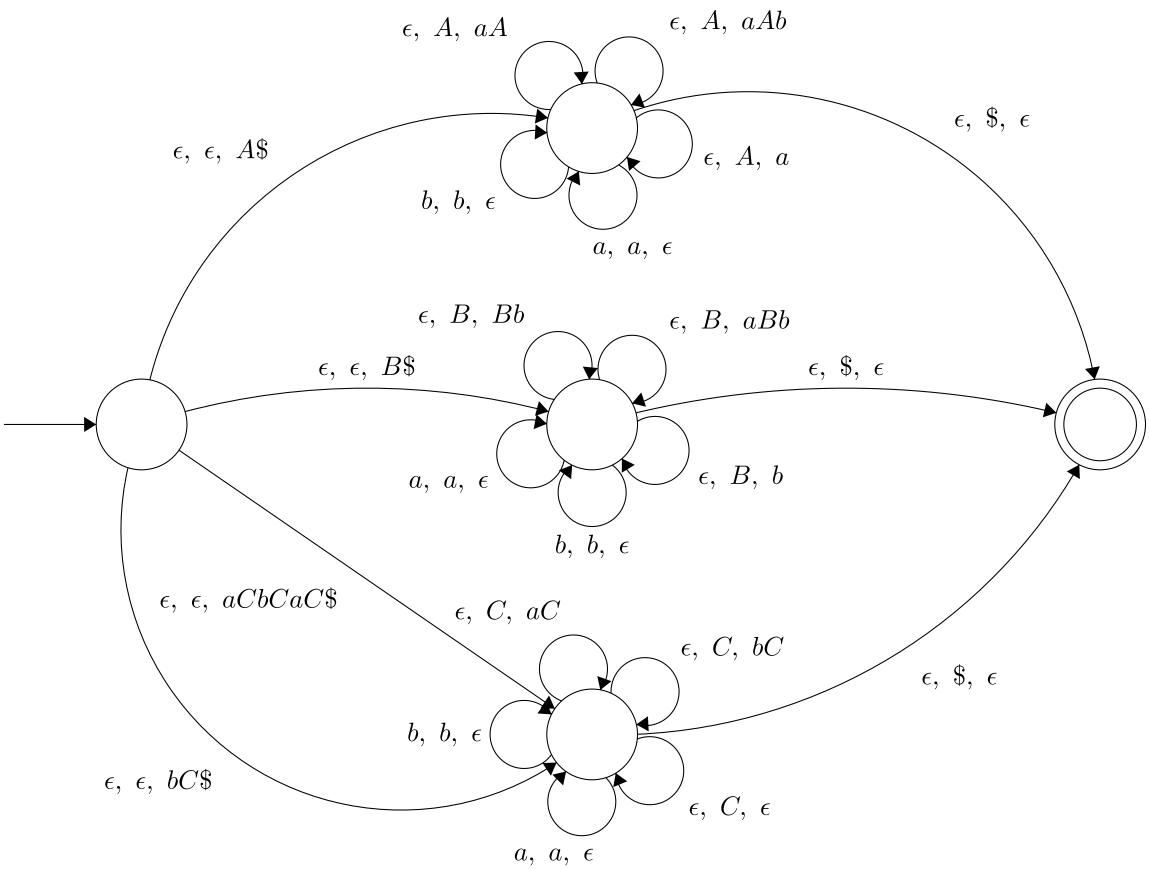
(iii)


$\epsilon,\ S,\ aaSb$
$\epsilon,\ S,\ \epsilon$
$\epsilon,\ \epsilon,\ S\$$
$\epsilon,\ \$,\ \epsilon$
$a,\ a,\ \epsilon$
$b,\ b,\ \epsilon$

(iv)


$a,\ \epsilon,\ a$
$a,\ a,\ \epsilon$
$\epsilon,\ \epsilon,\ \#$
$\$,\ \epsilon,\ \epsilon$
$\epsilon,\ \#,\ \epsilon$
$b,\ \epsilon,\ b$
$b,\ b,\ \epsilon$

(v)



(4)

(i)

Suppose L_1 is a CFL, and the pumping length is p. Take $u = a^p b^p c^p \in L_1$. Clearly $|u| = 3p \geq p$, and pumping lemma ensures the existence of $s, x, y, z, t \in \{a, b, c\}^*$ such that $u = sxyzt, |xz| > 0, sx^i yz^i t \in L_1\ \forall i \geq 0$.
If either x or z contains at least two kinds of symbols, sx^2yz^2t will not be in the form $a^*b^*c^*$, hence not in L_1.
On the other hand, if x and z each contains at most one kind of symbol, then at most two kinds of symbols appear in xz. We can check the three possible cases:

- There is no a in xz
  $sx^0yz^0t \notin L_1$ since it has the same number of a but fewer b or fewer c.
- There is no b in xz
  If there is a in xz, then $sx^2yz^2t \notin L_1$ since it has the same number of b but more a. Otherwise there is c in xz (since $|xz| > 0$), then $sx^0yz^0t \notin L_1$ since it has the same number of b but fewer c.
- There is no c in xz
  $sx^2yz^2t \notin L_1$ since it has the same number of c but fewer a or fewer b.

In either case, u can't be pumped, violating the pumping lemma. Thus L_1 is not a CFL.

(ii)

Suppose L_2 is a CFL, and the pumping length is p. Take $u = a^p b^{2p} c^{3p} \in L_2$. Clearly $|u| = 6p \geq p$, and pumping lemma ensures the existence of $s, x, y, z, t \in \{a, b, c\}^*$ such that $u = sxyzt, |xz| > 0, sx^i yz^i t \in L_2\ \forall i \geq 0$.
If either x or z contains at least two kinds of symbols, sx^2yz^2t will not be in the form $a^*b^*c^*$, hence not in L_2.
On the other hand, if x and z each contains at most one kind of symbol, then at most two kinds of symbols appear in xz. It's clear that we need to add k a, $2k$ b, $3k$ c simultaneously to maintain the $1:2:3$ ratio. But sx^2yz^2t adds at most two kinds of symbols, so $sx^2yz^2t \notin L_2$.
In either case, u can't be pumped, violating the pumping lemma. Thus L_2 is not a CFL.

(iii)

Suppose L_3 is a CFL, and the pumping length is p. Pick any prime $q \geq p$. Take $u = a^q \in L_3$. Clearly $|u| = q \geq p$, and pumping lemma ensures the existence of $s, x, y, z, t \in a^*$ such that $u = sxyzt, |xz| > 0, sx^i yz^i t \in L_3 \ \forall i \geq 0$. We examine $v = sx^{q+1}yz^{q+1}t$, whose length $|v| = |sxyzt| + |x^q z^q| = q + q|xz| = q(|xz| + 1)$. Since $|xz| > 0$, we have $|xz| + 1 \geq 2$, and so $|v|$ is not a prime. Hence $v \notin L_3$, u can't be pumped, violating the pumping lemma. Thus L_3 is not a CFL.

(5)

For $w \in \{a, b, S\}^*$, define $\alpha(w)$ to be the number of a's in w, and $\beta(w)$ to be the number of b's in w. Let $L = \{w \in \{a, b\}^* \mid \alpha(w) = \beta(w)\}$.
The desired equality $L = L(\mathcal{G})$ is proved by the following two lemmas.

Lemma 1. $L(\mathcal{G}) \subseteq L$

Proof. For the CFG $\mathcal{G}$, name $S \to SS \mid \epsilon$ as Rule 1 and $S \to aSb \mid bSa$ as Rule 2 (each of Rule 1 and Rule 2 has two rules).
I shall show that for any $w \in L(\mathcal{G})$, $\alpha(w) = \beta(w)$. Because $w \in L(\mathcal{G})$, there is a derivation $S \overset{*}{\Longrightarrow} w$. Starting from the start variable S, $\alpha(S) = \beta(S) = 0$. Next consider each rule applied for $u \Longrightarrow v$, given $\alpha(u) = \beta(u)$.

- The rule belongs to Rule 1
  One S is replaced with either SS or ϵ, adding no a or b. So $\alpha(v) = \alpha(u)$ and $\beta(v) = \beta(u)$. Thus $\alpha(v) = \beta(v)$.
- The rule belongs to Rule 2
  One S is replaced with either aSb or bSa, adding one a and one b. So $\alpha(v) = \alpha(u) + 1$ and $\beta(v) = \beta(u) + 1$. Thus $\alpha(v) = \beta(v)$.

In either case, the equation $\alpha(\cdot) = \beta(\cdot)$ is maintained after applying the rule. So we can conclude that $\alpha(w) = \beta(w)$. □

Corollary 1. *If $w \in \{a, b\}^*$ contains different numbers of a's and b's, then $w \notin L(\mathcal{G})$.*

Lemma 2. $L \subseteq L(\mathcal{G})$

Proof. I shall show that for any $w \in L$, $w \in L(\mathcal{G})$. Let's prove it by induction on $\alpha(w)$.

- Basis: $w \in L, \alpha(w) = 0$. Then $\beta(w) = 0$, and thus $w = \epsilon$. Using the rule $S \to \epsilon$, we have the derivation $S \Longrightarrow \epsilon$, hence $w \in L(\mathcal{G})$. So the lemma holds for $\alpha(w) = 0$.
- Induction hypothesis: Suppose the lemma holds for w with $\alpha(w) \leq n$, for some $n \geq 0$.
- Induction step: For any $w \in L$ with $\alpha(w) = n + 1$, we know $\beta(w) = n + 1$, implying $|w| = 2n + 2$. Write $w = w_0 w_1 \cdots w_{2n} w_{2n+1}$. Consider w_0 and w_{2n+1}:
  - $w_0 \neq w_{2n+1}$
    Let $w' = w_1 \cdots w_{2n}$ so that $w = w_0 w' w_{2n+1}$. Because $w_0 \neq w_{2n+1}$, it's obvious $\alpha(w') = \beta(w') = n$. By the induction hypothesis, there is a derivation $S \overset{*}{\Longrightarrow} w'$.
    Now if $w_0 = a, w_{2n+1} = b$, then $S \Longrightarrow aSb \overset{*}{\Longrightarrow} aw'b = w$. Otherwise, $w_0 = b, w_{2n+1} = a$, and $S \Longrightarrow bSa \overset{*}{\Longrightarrow} bw'a = w$.
  - $w_0 = w_{2n+1}$
    Define a function $f(i) \equiv \alpha(w_0 \cdots w_i) - \beta(w_0 \cdots w_i)$ for $0 \leq i \leq 2n + 1$. Because $w_0 \cdots w_i$ and $w_0 \cdots w_{i+1}$ differs only by a single a or a single b, only $\alpha(\cdot)$ or $\beta(\cdot)$ differs by ± 1. So we know $f(i + 1) - f(i) = \pm 1$ for $0 \leq i \leq 2n$.
    Because $\alpha(w) = \beta(w)$, $f(2n + 1) = 0$. Now if $w_0 = w_{2n+1} = a$, then $f(0) = 1$ and $f(2n) = -1$. Otherwise, $w_0 = w_{2n+1} = b$, then $f(0) = -1$ and $f(2n) = 1$. In either case, $f(0)f(2n) = -1 < 0$, so there must be some j such that $0 < j < 2n$ and $f(j) = 0$ (like the discrete version Intermediate Value Theorem).

Let $u = w_0 \cdots w_j, v = w_{j+1} \cdots w_{2n+1}$, so that $w = uv$. It's obvious that $2 \leq |u|, |v| \leq 2n$ by $0 < j < 2n$. Because $f(j) = f(2n+1) = 0$, we know $\alpha(u) = \beta(u)$ and $\alpha(v) = \beta(v)$. By the induction hypothesis, $u, v \in L(\mathcal{G})$, and thus there are derivations $S \overset{*}{\Longrightarrow} u$ and $S \overset{*}{\Longrightarrow} v$. So we can derive w by $S \Longrightarrow SS \overset{*}{\Longrightarrow} uS \overset{*}{\Longrightarrow} uv = w$.

In either case, there is a derivation from S to w, so $w \in L(\mathcal{G})$.

The proof is done by induction. $\square$