自動機與形式語言 Homework 2

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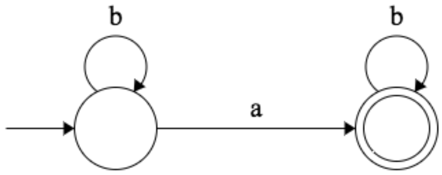
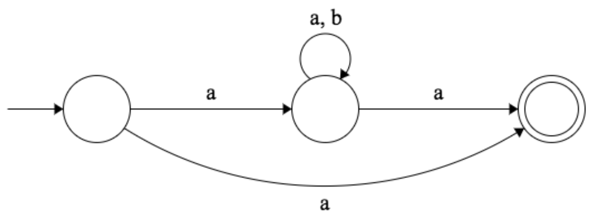
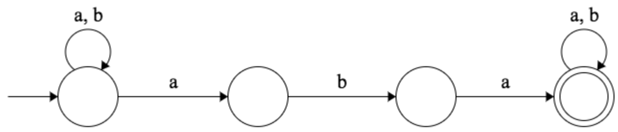
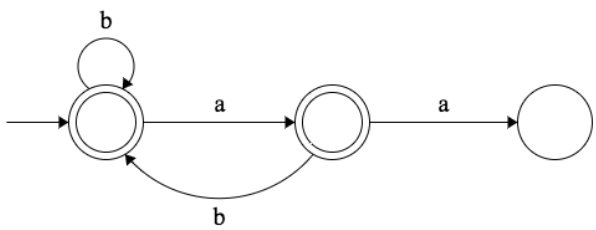
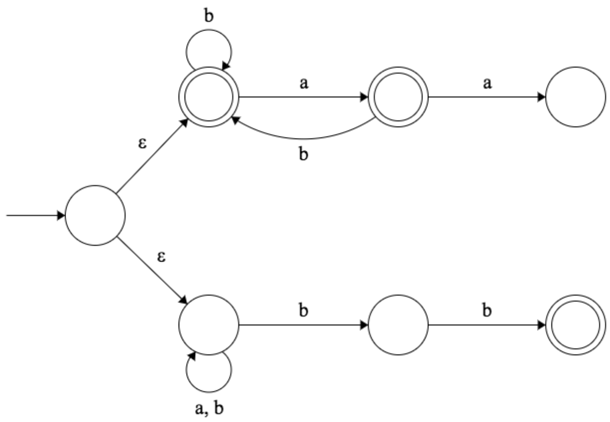
(1)(2)

(a)		b^*ab^*
(b)		$a\Sigma^*a \cup a$
(c)		$\Sigma^*aba\Sigma^*$
(d)		$(ab \cup b)^*(\epsilon \cup a)$
(e)		$(ab \cup b)^*(\epsilon \cup a) \cup \Sigma^*bb$

(3)

(a)

Regular. Regex: $b^*ab^*ab^*ab^*$

(b)

Regular. Regex: $(b^*ab^*ab^*)^*$

(c)

Regular. Regex: $(\Sigma\Sigma)^*$

(d)

Nonregular.
Suppose L is regular, and the pumping length is p. Take $s = a^p b a^p \in L$. Clearly $|s| = 2p+1 \geq p$, and pumping lemma says that there exists x, y, z such that $s = xyz, |y| > 0, |xy| \leq p, xy^i z \in L\ \forall i \geq 0$. Thus $y \neq \epsilon$ and y consists only of a's.
If we examine $xy^0z = xz$, then there would be fewer a's on the left side of b ($|y| > 0$), implying that $xz \notin L$. Hence L is NOT regular.

(e)

Nonregular.
Suppose L is regular, and the pumping length is p. Take $s = a^q \in L$, where $q \geq p$ is a chosen prime (possible since primes are unbounded). Clearly $|s| = q \geq p$, and pumping lemma says that there exists x, y, z such that $s = xyz, |y| > 0, |xy| \leq p, xy^i z \in L\ \forall i \geq 0$. Thus $y \neq \epsilon$ and y consists only of a's.

Now let $r = |x| + |z|$ and consider three cases:

- $r = 0$
  In this case, $x = z = \epsilon$ and $|y| = q$. Examine $xy^q z = y^q$, whose length is q^2, not a prime (q is a prime so $q > 1$). So $xy^q z \notin L$.

- $r = 1$
  Examine $xy^0 z = xz$, whose length is $|xz| = |x| + |z| = r = 1$, not a prime. So $xy^0 z \notin L$.

- $r > 1$
  Examine $xy^r z$, whose length is $|x| + r|y| + |z| = (|y| + 1)r$, not a prime ($|y| > 0$ so $|y| + 1 > 1$). So $xy^r z \notin L$.

In each case, pumping lemma fails, so L is NOT regular.

(4)

(a)

- Reflexivity: Fix one $u \in \Sigma^*$. Now $\forall w \in \Sigma^*$, since trivially $uw = uw$, we immediately have $u \sim_L u$.

- Symmetry: $\forall u, v \in \Sigma^*$,
  $u \sim_L v$
  $\Longrightarrow \forall w \in \Sigma^*, uw \in L$ iff $vw \in L$
  $\Longrightarrow \forall w \in \Sigma^*, vw \in L$ iff $uw \in L$
  $\Longrightarrow v \sim_L u$.

- Transitivity: $\forall x, y, z \in \Sigma^*$,
  $x \sim_L y$ and $y \sim_L z$
  $\Longrightarrow \forall w \in \Sigma^*, xw \in L$ iff $yw \in L$ and $yw \in L$ iff $zw \in L$
  $\Longrightarrow \forall w \in \Sigma^*, xw \in L$ iff $zw \in L$
  $\Longrightarrow x \sim_L z$.

(b)

Suppose $u \sim_L v$. Take $w = \epsilon \in \Sigma^*$, then $uw = u$ and $vw = v$.
By definition we know $u \in L$ iff $v \in L$, which implies either $u, v \in L$ or $u, v \notin L$.

(c)

Assume $\mathcal{A} = \langle \Sigma, Q, q_0, F, \delta \rangle$.
Suppose $u, v \in \Sigma^*$ with $\mathcal{A}(u) = \mathcal{A}(v) = q$ for some $q \in Q$. Then we consider the two possible cases:

- $q \in F$
  In this case, both u and v are accepted by $\mathcal{A}$, so $u, v \in L$.

- $q \notin F$
  In this case, both u and v are rejected by $\mathcal{A}$, so $u, v \notin L$.

Thus $u \in L$ iff $v \in L$, that is, $u \sim_L v$.

(d)

Let $n = |Q|$. Suppose $\#(\sim_L) > n$, then there are at least $n + 1$ words $s_1, s_2, \ldots, s_{n+1}$ such that $s_i \not\sim_L s_j, i \neq j$. From (c) we can conclude that for $u, v \in \Sigma^*$, if $u \not\sim_L v$, then $\mathcal{A}(u) \neq \mathcal{A}(v)$. So $s_1, \ldots, s_{n+1}$ satisfies $\mathcal{A}(s_i) \neq \mathcal{A}(s_j), i \neq j$, which means that their are $n + 1$ distinct states. But $\mathcal{A}$ has only n states, a contradiction. So $\#(\sim_L)$ is finite, and more specifically $\#(\sim_L) \leq n = |Q|$.

(5)

(a)

Let $M = \{1, \ldots, m\}$. For each $i \in M$, take one arbitrary word $s_i \in C_i$. Then let $S = \{i | s_i \in L\} \subseteq M$. Now I shall show that $L = \bigcup_{i \in S} C_i$.

On one hand, pick any $s \in L$, clearly $s \in C_j$ for some $j \in M$, since $\sim_L$ partitions $\Sigma^* \ni s$. Also $s_j \in C_j$, hence $s_j \sim_L s$. By (4)(b) and $s \in L$, we know $s_j \in L$. So $j \in S$ by definition of S, hence $s \in C_j \subseteq \bigcup_{i \in S} C_i$.
Since the above holds for all $s \in L$, we have $L \subseteq \bigcup_{i \in S} C_i$.

On the other hand, pick any $s \in \bigcup_{i \in S} C_i$. Then $s \in C_j$ for some $j \in S$. By $j \in S$ and my definition of S, $s_j \in L$. We already know $s_j \in C_j$, so again $s_j \sim_L s$. With $s_j \in L$ and (4)(b), $s \in L$.
Since the above holds for all $s \in \bigcup_{i \in S} C_i$, we have $\bigcup_{i \in S} C_i \subseteq L$.

Now rewrite $S = \{i_1, \ldots, i_k\}$ where $k = |S|$. Hence $L = \bigcup_{i \in S} C_i = C_{i_1} \cup \cdots \cup C_{i_k}$. This completes the proof.

(b)

$\forall w_1, w_2 \in C_i, w_1 \sim_L w_2$, which by definition guarantees that $w_1 x \in L$ iff $w_2 x \in L$ for all $x \in \Sigma^*$. This implies $w_1 a x \in L$ iff $w_2 a x \in L$ for all $x \in \Sigma^*$. Again by definition, $w_1 a \sim_L w_2 a$, and thus $[w_1 a] = [w_2 a]$.
Because the above holds for any i, the proof is done.

(c)

Claim For $s \in \Sigma^*, \mathcal{A}(s) = p_j$ if and only if $s \in C_j, 1 \leq j \leq m$.

- *Proof:* The proof is by induction on $|s|$.
  Base case, $|s| = 0$: So $s = \epsilon, \mathcal{A}(s) = q_0 = p_j$, where $s = \epsilon \in C_j$ as in the definition of $\mathcal{A}$.
  Now suppose the claim holds for all s with $|s| = n, n \geq 0$. Then for any s with $|s| = n + 1$, say $s = s_1 s_2 \cdots s_{n+1}$, we know the claim holds for $s_1 \cdots s_n$. So we can assume $\mathcal{A}(s_1 \cdots s_n) = p_i$ and $s_1 \cdots s_n \in C_i$ for some $1 \leq i \leq m$. As in the definition of δ, take $w = s_1 \cdots s_n \in C_i$ and $a = s_{n+1}$, then $\delta(p_i, s_{n+1}) = p_j$ and $[s] = [s_1 \cdots s_n s_{n+1}] = C_j$ for some $1 \leq j \leq m$. Thus $\mathcal{A}(s) = \delta(\mathcal{A}(s_1 \cdots s_n), s_{n+1}) = \delta(p_i, s_{n+1}) = p_j$ and $s \in C_j$. This induction step completes the proof.

Now we can work on the two directions of the problem.

- $L(\mathcal{A}) \subseteq L$
  For any $s = s_1 s_2 \ldots s_n \in L(\mathcal{A})$, there is a run $p_{j_0} s_1 p_{j_1} \ldots s_{n-1} p_{j_{n-1}} s_n p_{j_n}$ with $\epsilon \in C_{j_0}$ and $\mathcal{A}(s) = p_{j_n} \in F$. With the above claim, we can conclude $s =\in C_{j_n}$. And $p_{j_n} \in F = \{p_{i_1}, \ldots, p_{i_k}\}$, so $j_n \in \{i_1, \ldots, i_k\}$, hence $s \in C_{i_1} \cup \cdots \cup C_{i_k} = L$.

- $L \subseteq L(\mathcal{A})$
  Suppose $s \in L = C_{i_1} \cup \cdots \cup C_{i_k}$. Then $s \in C_i$ for some $i \in \{i_1, \ldots, i_k\}$. With the claim, $\mathcal{A}(s) = p_i \in \{p_{i_1}, \ldots, p_{i_k}\} = F$. Hence $\mathcal{A}$ accepts s, or $s \in L(\mathcal{A})$.

This completes the proof.

(d)

If L is regular, then some DFA $\mathcal{A}$ recognizes it, and thus $\#(\sim_L) \leq |Q| < \infty$ by (4.d), where Q is the set of states of $\mathcal{A}$.
On the other hand, if $\#(\sim_L) = m < \infty$, then by the previous (5.abc) we can construct a DFA $\mathcal{A}$ with m states such that $L(\mathcal{A}) = L$, implying L is regular.