Sample solution to HW 1

(1) and (2) are pretty standard, so their solutions are omitted. For (1), the number of equivalence classes in $\sim_n$ is $|n|$, where $|n|$ denotes the absolute number of n.

(3) Prove Lemma 1.1 in lecture note 1.

- $[x]_\sim = [y]_\sim$ if and only if $x \sim y$.

  ("if") Suppose $x \sim y$. By definition, for every $z \in [x]_\sim$, $z \sim x$. Thus, by transitivity and reflexivity, for every $z \in [x]_\sim$, $z \sim y$, i.e., $z \in [y]_\sim$. Therefore, $[x]_\sim \subseteq [y]_\sim$.

  In a similar manner, for every $z \in [y]_\sim$, $z \in [x]_\sim$, i.e., $[y]_\sim \subseteq [x]_\sim$. Thus, we get $[x]_\sim = [y]_\sim$.

  ("only if") Suppose $[x]_\sim = [y]_\sim$. By reflexivity, $x \sim x$, thus, $x \in [x]_\sim$. Since $[x]_\sim = [y]_\sim$, $x \in [y]_\sim$, and hence, by definition, $x \sim y$.

- If $[x]_\sim \neq [y]_\sim$, then $[x]_\sim \cap [y]_\sim = \emptyset$.

  Suppose $[x]_\sim \neq [y]_\sim$ and $[x]_\sim \cap [y]_\sim \neq \emptyset$. Let $z \in [x]_\sim \cap [y]_\sim$. By definition, $z \sim x$ and $z \sim y$. By reflexivity and transitivity, $x \sim y$. By the first bullet above, $[x]_\sim = [y]_\sim$, which contradicts our assumption that $[x]_\sim \neq [y]_\sim$.

(4) Prove Theorem 1.2 in lecture note 1.

By reflexivity of $\sim$, every element $x \in X$ belongs to $[x]_\sim$, thus, x belongs to at least one equivalence class. Moreover, by Lemma 1.1, every element $x \in X$ belongs to at most one equivalence class. Therefore, every element $x \in X$ belongs to exactly one equivalence class.