STAT 5444: Homework #2

Due on October 17, 2016 at 3:10pm

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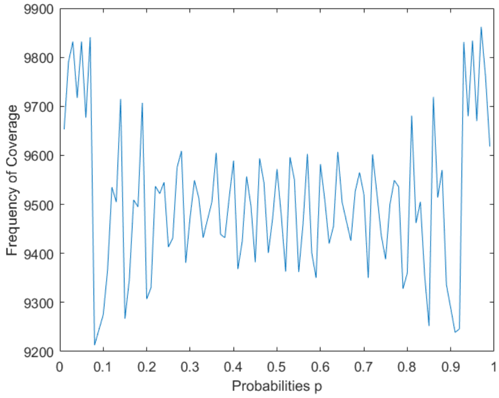
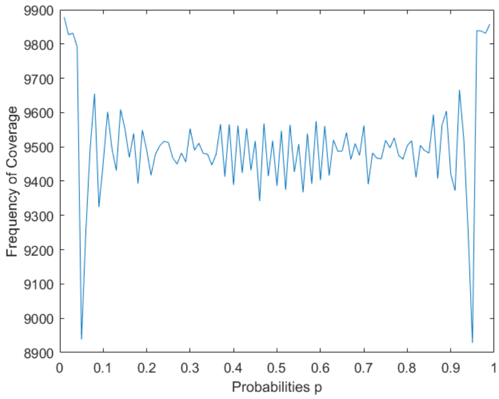
Problem 1

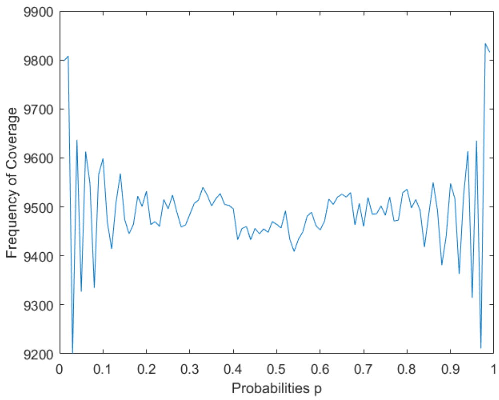
N = 30



N = 50



N = 100



For low probabilities of p or extremely high probabilities the coverage is within the 95 percent more so in this example. There is a bit more variance in hitting the 95 percent coverage mark in this Bayesian approach. The beta prior provides support for the areas with low probability p or high probability p which aids in acquiring coverage in those areas. This is particularly useful for ensuring a centered credible interval for all probabilities p.

```
alpha = 0.5;
beta = 0.5;
frequency = zeros(99, 1);
```

```
N = 100;
totalX = zeros(10000, 10000, 1);
temp = 1;
for p=0.01:0.01:0.99
    p
    x = binornd(N, p, 10000, 1);
    alphaStar = x + alpha;
    betaStar = beta + N - x;
    for j=1:10000
        totalX(:, j) = betarnd(alphaStar, betaStar);
    end
    aSort = sort(totalX, 2);
    upper_bound = aSort(:, 9750) >= p;
    lower_bound = aSort(:, 250) <= p;
    frequency(temp, 1) = sum(upper_bound == lower_bound);
    temp = temp + 1;
end
p = 0.01:0.01:0.99;
plot(p, frequency)
xlabel('Probabilities p')
ylabel('Frequency of Coverage')
```

Problem 2

$$
\begin{aligned}
p(\phi|X) &= \int p(\mu|X\phi)p(\phi)p(\mu)d\mu \\
p(\phi|X) &= \int \prod_{i=1}^{N} \frac{\phi^2}{\sqrt{2\pi}} e^{\frac{\phi}{2}(x-\mu)^2} d\mu \\
p(\phi|X) &\propto \phi^{\frac{N}{2}-1} \int e^{\frac{\phi}{2}(x-\mu)^2} d\mu \\
p(\phi|X) &\propto \phi^{\frac{N}{2}-1} \int e^{\frac{\phi}{2}((\sum_{i=1}^{N} x_i^2)-2\mu\sum_{i=1}^{N} x_i + N\mu^2)} d\mu \\
p(\phi|X) &\propto \phi^{\frac{N}{2}-1} e^{\frac{-\phi}{2}(\sum_{i=0}^{N} x^2)} \int e^{\frac{\phi}{2}(N\mu^2 - 2\mu\sum_{i=1}^{N} x_i)} d\mu
\end{aligned}
$$

To acquire the pdf we must have this

$$
\begin{aligned}
V &= (\phi N)^{-1} = \frac{\phi^{-1}}{N} \\
E &= \frac{\phi^{-1}}{N} * \sum_{i=1}^{N} x_i \phi = \bar{x} \\
pdfNormal &= \int e^{\frac{-1}{2V}(u^2 - 2\bar{x}\mu + \bar{x}^2)}
\end{aligned}
$$

Thus, we integrate to 1

$$p(\phi|X) \propto \phi^{\frac{N}{2}-1} e^{\frac{-\phi}{2}(\sum_{i=0}^{N} x^2)} \int e^{\frac{\phi}{2}(N\mu^2 - 2\mu \sum_{i=1}^{N} x_i)} d\mu * \frac{\frac{1}{\sqrt{2\pi V} e^{\frac{-1}{2}\frac{\bar{x}^2}{V}}}}{\frac{1}{\sqrt{2\pi V} e^{\frac{-1}{2}\frac{\bar{x}^2}{V}}}}$$

$$p(\phi|X) \propto \phi^{\frac{N}{2}-1} e^{\frac{-\phi}{2}(\sum_{i=0}^{N} x^2)} \sqrt{2\pi V} e^{\frac{1}{2}\frac{\bar{x}^2}{V}}$$

$$p(\phi|X) \propto \phi^{\frac{N}{2}-1} e^{\frac{-\phi}{2}(\sum_{i=0}^{N} x^2)} \sqrt{2\pi \frac{\phi^{-1}}{N}} e^{\frac{1}{2}\frac{\bar{x}^2}{\frac{\phi^{-1}}{N}}}$$

$$p(\phi|X) \propto \phi^{\frac{N}{2}-1} e^{\frac{-\phi}{2}(\sum_{i=0}^{N} x^2)} \phi^{\frac{-1}{2}} e^{\frac{1}{2}N\bar{x}^2\phi}$$

$$p(\phi|X) \propto \phi^{\frac{N-1}{2}-1} e^{\frac{-\phi}{2}(\sum_{i=0}^{N} x^2)} e^{\frac{-1}{2}(-N\bar{x}^2\phi)}$$

$$p(\phi|X) \propto \phi^{\frac{N-1}{2}-1} e^{\frac{-\phi}{2}(\sum_{i=0}^{N} x^2 - N\bar{x}^2)}$$

$$p(\phi|X) \propto \phi^{\frac{N-1}{2}-1} e^{\frac{-\phi}{2}(\sum_{i=0}^{N} x^2 - \sum_{i=0}^{N} \bar{x}^2)}$$

The follow theorem is used:

$$\begin{aligned}
E(x-\bar{x})^2 &= E(x^2 - 2x\bar{x} + \bar{x}^2) \\
&= E(x^2 - 2E(x)^2 + E(x)^2) \\
&= E(x^2) - E(x)^2 \\
N * E(x-\bar{x})^2 &= \sum_{i=0}^{N} (x-\bar{x})^2 \\
N * E(x^2) - N * E(x)^2 &= (\sum_{i=0}^{N} x^2 - \sum_{i=0}^{N} \bar{x}^2)
\end{aligned}$$

Thus, we can do the following

$$\begin{aligned}
p(\phi|X) &\propto \phi^{\frac{N-1}{2}-1} e^{\frac{-\phi}{2}(\sum_{i=0}^{N} x^2 - \sum_{i=0}^{N} \bar{x}^2)} \\
p(\phi|X) &\propto \phi^{\frac{N-1}{2}-1} e^{\frac{-\phi}{2}(\sum_{i=0}^{N}(x-\bar{x})^2)} \\
s^2 &= \frac{\sum (x_i - \bar{x})^2}{N-1} \\
p(\phi|X) &\propto \phi^{\frac{N-1}{2}-1} e^{\frac{-\phi}{2}(s^2(N-1))} \\
p(\phi|X) &- Gamma[\frac{N-1}{2}, \frac{(s^2(N-1))}{2}]
\end{aligned}$$

Part 2:

Problem 2 continued on next page...

$$P(\widetilde{x}|X) = \int P(\widetilde{x}|X\phi)P(\phi)d\phi$$

$$P(\widetilde{x}|X) = \int \frac{\phi^{\frac{1}{2}}}{\sqrt{2\pi}(1+\frac{1}{N})} e^{\frac{-\phi}{2}\frac{(\widetilde{x}-\bar{x})^2}{(1+\frac{1}{N})}} \phi^{\frac{N-1}{2}-1} e^{\frac{-\phi}{2}(s^2(N-1))}$$

$$P(\widetilde{x}|X) \propto \int \phi^{\frac{N}{2}-1} e^{\frac{-\phi}{2}\frac{(\widetilde{x}-\bar{x})^2}{(1+\frac{1}{N})}} e^{\frac{-\phi}{2}(s^2(N-1))}$$

$$P(\widetilde{x}|X) \propto \int \phi^{\frac{N}{2}-1} e^{\frac{-\phi}{2}(\frac{(\widetilde{x}-\bar{x})^2}{(1+\frac{1}{N})}+(s^2(N-1)))}$$

$$P(\widetilde{x}|X) \propto \int \phi^{\frac{N}{2}-1} e^{\frac{-\phi}{2}(\frac{(\widetilde{x}-\bar{x})^2}{(1+\frac{1}{N})}+(s^2(N-1)))} * \frac{\frac{\Gamma(N/2)}{\left[\frac{1}{2}*(\frac{(\widetilde{x}-\bar{x})^2}{(1+\frac{1}{N})}+(s^2(N-1)))\right]^{N/2}}}{\frac{\Gamma(N/2)}{\left[\frac{1}{2}*(\frac{(\widetilde{x}-\bar{x})^2}{(1+\frac{1}{N})}+(s^2(N-1)))\right]^{N/2}}}$$

$$P(\widetilde{x}|X) \propto \frac{\Gamma(N/2)}{\left[\frac{1}{2}*(\frac{(\widetilde{x}-\bar{x})^2}{(1+\frac{1}{N})}+(s^2(N-1)))\right]^{N/2}}$$

$$P(\widetilde{x}|X) \propto \left[\frac{(\widetilde{x}-\bar{x})^2}{(1+\frac{1}{N})}+(s^2(N-1))\right]^{-N/2}$$

$$P(\widetilde{x}|X) \propto \left[\frac{(\widetilde{x}-\bar{x})^2}{(1+\frac{1}{N})}+(s^2(N-1))\right]^{-N/2}$$

$$P(\widetilde{x}|X) \propto \left[1+\frac{(\widetilde{x}-\bar{x})^2}{(1+\frac{1}{N})(s^2(N-1))}\right]^{-N/2}$$

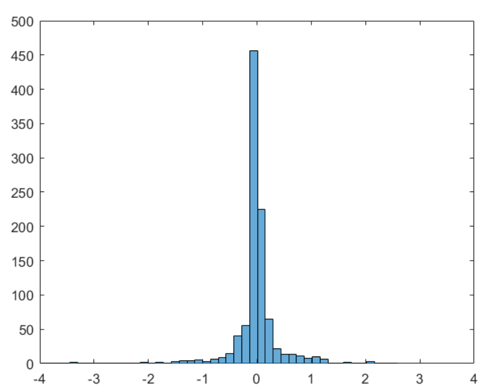
$$P(\widetilde{x}|X) \propto \left[1+\frac{(\widetilde{x}-\bar{x})^2}{(1+\frac{1}{N})(s^2(N-1))}\right]^{\frac{-(N-1)+1}{2}}$$

$$\widetilde{x}|X - t(\bar{x}, s^2(1+\frac{1}{N}, N-1)$$

Problem 3

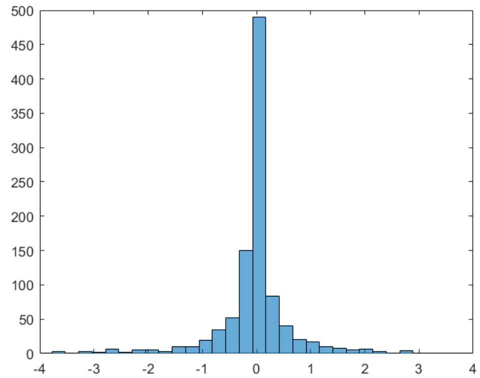
$$p(x|\mu,\sigma^2,\gamma) \propto (\phi/\gamma)^{1/2} exp(-\frac{\phi}{2}\frac{(x-\mu)^2}{\gamma}$$
$$p(\phi) \propto \phi^{\alpha-1}e^{-\beta\phi}$$
$$p(x|\mu,\gamma) \propto \int p(x|\mu,\sigma^2,\gamma)p(\phi)d\phi$$
$$p(x|\mu,\gamma) \propto \int \frac{\phi^{\frac{1}{2}}}{\gamma} e^{\left[\frac{-\phi}{2}\frac{(x-\mu)^2}{\gamma}\right]} * \phi^{\alpha-1}e^{-\beta\phi}$$
$$p(x|\mu,\gamma) \propto \int \frac{\phi^{\frac{1}{2}+\alpha-1}}{\gamma} e^{\left[\frac{-\phi}{2}\frac{(x-\mu)^2}{\gamma}-\beta\phi\right]}$$
$$p(x|\mu,\gamma) \propto \int \frac{\phi^{\frac{1}{2}+\alpha-1}}{\gamma} e^{-\phi\left[\frac{(x-\mu)^2}{2\gamma}+\beta\right]}$$
$$p(x|\mu,\gamma) \propto \int \frac{\phi^{\frac{1}{2}+\alpha-1}}{\gamma} e^{-\phi\left[\frac{(x-\mu)^2}{2\gamma}+\beta\right]} * \frac{\frac{\Gamma(\frac{1}{2}+\alpha)}{\left[\frac{(x-\mu)^2}{2\gamma}+\beta\right]^{\frac{1}{2}+\alpha}}}{\frac{\Gamma(\frac{1}{2}+\alpha)}{\left[\frac{(x-\mu)^2}{2\gamma}+\beta\right]^{\frac{1}{2}+\alpha}}}$$
$$p(x|\mu,\gamma) \propto \frac{\Gamma(\frac{1}{2}+\alpha)}{\left[\frac{(x-\mu)^2}{2\gamma}+\beta\right]^{\frac{1}{2}+\alpha}}$$
$$p(x|\mu,\gamma) \propto \frac{1}{\left[\frac{(x-\mu)^2}{2\gamma}+\beta\right]^{\frac{1}{2}+\alpha}}$$
$$p(x|\mu,\gamma) \propto \frac{1}{\left[\frac{(x-\mu)^2}{2\gamma}+1\right]^{\frac{1}{2}+\frac{1}{2}}}, \alpha=\frac{1}{2}, \beta=1$$

Part 3:
$\mu = 0, \gamma = 0.5, min = -4.18, max = 3.1$

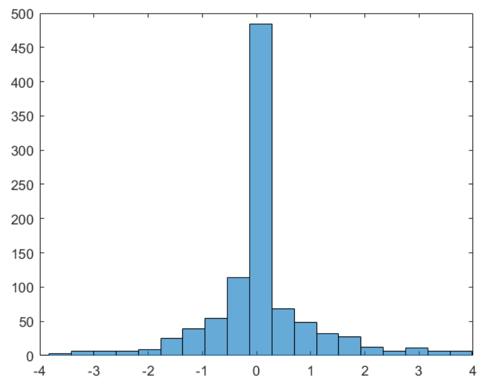


$\mu = 0, \gamma = 1, min = -9.74, max = 7.094$

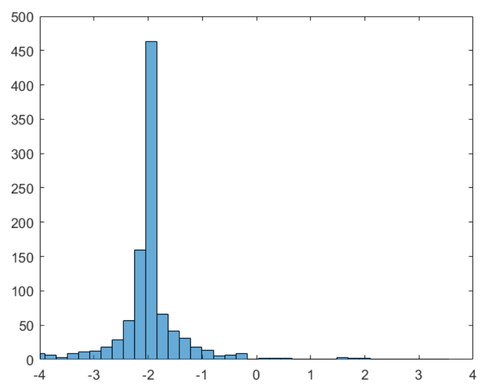
Problem 3 continued on next page. . .



$\mu = 0, \gamma = 2, min = -18.51, max = 11.33$



$\mu = -2, \gamma = 1, min = -6.18, max = 6.51$



```
mu = 0;
gamma = 0.5;
post = normrnd(mu, sigma*gamma);
figure(1), histogram(post, 50)
axis([-4 4 0 500])
min(post)
max(post)
```

Problem 4

MAP is not invariant to transformations. Using a normal distributed MAP estimator with no prior the MAP estimator is the value 0. Applying a transformation of $\tau(\theta) = \theta^2$ will result in the MAP estimator being the value 0.7. When we apply the transformation to the original value 0 it will not yield the value 0.7. Thus, the MAP is not invariant to transformations.

Problem 5

Part 1:
In previous derivations we have the following posterior: $\mu - N(\bar{x}, \frac{\sigma^2}{N})$
To determine if the predictive distribution is normal we can apply the prediction distribution function. Since it ends being an integral of a normal * normal it will end up being a normal just be rules of conjugate priors. Proof Below

$$P(\widetilde{x}|X,\sigma^2) = \int \frac{1}{\sqrt{2\pi}\sigma} * e^{\frac{-1}{2}\frac{(\widetilde{x}-\mu)^2}{\sigma^2}} * \frac{1}{\sqrt{2\pi}\frac{\sigma}{\sqrt{N}}} * e^{\frac{-1}{2}\frac{(\mu-\bar{x})^2}{\frac{\sigma^2}{N}}} d\mu$$

$$P(\widetilde{x}|X,\sigma^2) \propto \int e^{\frac{-1}{2}\frac{(\widetilde{x}-\mu)^2}{\sigma^2}} * e^{\frac{-1}{2}\frac{(\mu-\bar{x})^2}{\frac{\sigma^2}{N}}} d\mu$$

$$P(\widetilde{x}|X,\sigma^2) \propto \int e^{\frac{-1}{2}\frac{\widetilde{x}^2-2\widetilde{x}\mu+\mu^2}{\sigma^2}} * e^{\frac{-1}{2}\frac{\mu^2-2\mu\bar{x}+\bar{x}^2}{\frac{\sigma^2}{N}}} d\mu$$

$$P(\widetilde{x}|X,\sigma^2) \propto e^{\frac{-1}{2}\frac{\widetilde{x}^2}{\sigma^2}} \int e^{\frac{-1}{2}(\mu^2\left[\frac{1}{\sigma^2}+\frac{N}{\sigma^2}\right]-2\mu\left[\frac{\bar{x}N}{\sigma^2}+\frac{\widetilde{x}}{\sigma^2}\right])} d\mu$$

$$V* = (\frac{N+1}{\sigma^2})^{-1} = \frac{\sigma^2}{N+1}$$

$$E* = \left[\frac{\bar{x}N}{\sigma^2} + \frac{\widetilde{x}}{\sigma^2}\right]\frac{\sigma^2}{N+1} = \frac{\widetilde{x}+\bar{x}N}{N+1}$$

$$NormalDist = \int e^{\frac{-1}{2}\frac{(\mu-\mathrm{E}*)^2}{V*}}$$

$$P(\widetilde{x}|X,\sigma^2) \propto e^{\frac{-1}{2}\frac{\widetilde{x}^2}{\sigma^2}} e^{\frac{1}{2}(\frac{\widetilde{x}+\bar{x}}{N+1})^2(\frac{N+1}{\sigma^2})} d\mu$$

$$P(\widetilde{x}|X,\sigma^2) \propto e^{\frac{-1}{2}\frac{\widetilde{x}^2}{\sigma^2})} e^{\frac{1}{2}(\frac{\widetilde{x}^2+2\widetilde{x}\bar{x}N+(\bar{x}N)^2}{N+1})^2(\frac{N+1}{\sigma^2})} d\mu$$

$$P(\widetilde{x}|X,\sigma^2) \propto e^{\frac{-1}{2}(\widetilde{x}^2\left[\frac{1}{\sigma^2}-\frac{1}{(N+1)\sigma^2}\right]-2\widetilde{x}\left[\frac{\bar{x}N}{(N+1)^2}*\frac{N+1}{\sigma^2}\right])} d\mu$$

Thus, we can see that the following intermediate steps above is in fact normally distributed. This formalizes to the posterior provided in the problem.
Part 2:

$$E[\widetilde{x}|X\sigma^2] = E[E[\widetilde{x}|\mu X\sigma^2]]|X\sigma^2]$$
$$E[\widetilde{x}|X\sigma^2] = E(\mu|X\sigma^2) = \bar{x}$$
$$V[\widetilde{x}|X\sigma^2] = V[E[\widetilde{x}|\mu X\sigma^2]|X\sigma^2] + E[V[\widetilde{x}|\mu X\sigma^2]|X\sigma^2]V[\widetilde{x}|X\sigma^2] \quad = V[\mu|X\sigma^2) + E(\sigma^2|X\sigma^2) = \frac{\sigma^2}{N} + \sigma^2$$