STAT5444: Homework #1

Due on September 26, 2014 at 3:10pm

Professor Scott Leman 12:20 MWF

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Problem 1

$x_i \sim N(\mu, \sigma^2)$ for i = 1, ..., N Assuming σ^2 is a known parameter.

$p(\mu|X)$ where $X = \{x_1, ..., x_N\}$

Part 1:
$L(\mu|X) = \prod_{i=1}^{N} \frac{1}{\sqrt{2\pi}\sigma} e^{\frac{-1}{2}\frac{(x_i-\mu)}{\sigma^2}}$

Part 2:
$p(\mu) \propto 1$

$p(\mu|X) = L(\mu|X)p(\mu)$

$= \prod_{i=1}^{N} \frac{1}{\sqrt{2\pi}\sigma} e^{\frac{-1}{2}\frac{(x_i-\mu)}{\sigma^2}} * 1$

$\propto e^{\frac{-1}{2}\frac{\sum_{i=1}^{N}(x_i-\mu)}{\sigma^2}}$

$= e^{\frac{-1}{2}\frac{\sum_{i=1}^{N}(x_i^2-2x_i\mu+\mu^2)}{\sigma^2}}$

$\propto e^{\frac{-1}{2}\frac{(-2\sum_{i=1}^{N}x_i\mu+N\mu^2)}{\sigma^2}}$

$= e^{\frac{-1}{2}\frac{(-2\sum_{i=1}^{N}x_i\mu+N\mu^2)}{\sigma^2}\frac{\frac{1}{N}}{\frac{1}{N}}}$

$= e^{\frac{-1}{2}\frac{(-2\mu\bar{x}+\mu^2)}{\frac{\sigma^2}{N}}}$

$= e^{\frac{-1}{2}(\mu^2[\frac{N}{\sigma^2}]-2\mu[\frac{N\bar{x}}{\sigma^2}])}$

Answers:

$p(\mu|X) = e^{\frac{-1}{2}(\mu^2[\frac{N}{\sigma^2}]-2\mu[\frac{N\bar{x}}{\sigma^2}])}$

$Var(\mu|X) = [\frac{N}{\sigma^2}]^{-1} = \frac{\sigma^2}{N}$

$E(\mu|X) = \frac{N\bar{x}}{\sigma^2} * [\frac{N}{\sigma^2}]^{-1} = \bar{x}$

Problem 2

$X \sim Bin(N, p)$ so that $p(X = x) = (\begin{matrix} N \\ x \end{matrix})p^x(1-p)^{N-x}$

Reference prior: $p \sim Beta(\frac{1}{2}, \frac{1}{2})$

pdf for $z \sim Beta(\alpha, \beta)$

$p(x) = \frac{1}{B(\alpha,\beta)}x^{\alpha-1}(1-x)^{\beta-1}$

$B(\alpha, \beta) = \int_0^1 z^{\alpha-1}(1-z)^{\beta-1}$

Posterior distribution will be a beta distribution.

$p(p|X) \propto L(p|X)p(p)$

$\propto [(\begin{matrix} N \\ x \end{matrix})p^x(1-p)^{N-x}] * \frac{1}{B(\alpha,\beta)}p^{\alpha-1}(1-p)^{\beta-1}$

$\propto [p^x(1-p)^{N-x}] * p^{\alpha-1}(1-p)^{\beta-1}$

$= p^{x+\alpha-1}(1-p)^{N-x+\beta-1}$

$= p^{\alpha^*-1}(1-p)^{\beta^*-1}$

$\alpha^* = x + \alpha \rightarrow \alpha^* = x + \frac{1}{2}$

$\beta^* = N - x + \beta \rightarrow \beta^* = N - x + \frac{1}{2}$

Answers:

$p(p|X) \propto p^{\alpha^*-1}(1-p)^{\beta^*-1}$

$\alpha^* = x + \alpha \rightarrow \alpha^* = x + \frac{1}{2}$

$\beta^* = N - x + \beta \rightarrow \beta^* = N - x + \frac{1}{2}$

If samples are I.I.D then the posterior parameters are:
$\alpha^* = \sum_{i=1}^{Z} x + \alpha \rightarrow \alpha^* = \sum_{i=1}^{Z} xx + \frac{1}{2}$

$\beta^* = N - \sum_{i=1}^{Z} x + \beta \rightarrow \beta^* = N - \sum_{i=1}^{Z} xx + \frac{1}{2}$

Problem 3

$E[Y] = E[XB + \epsilon] = XB$ since ϵ has a expected value of zero

$p(\beta|Y, X) = L(\beta|Y, X) * P(\beta)$

$= e^{\frac{-1}{2}(Y-XB)^T\Sigma^{-1}(Y-XB)}$

$= e^{\frac{-1}{2}(Y-XB)^T(\sigma^2 I)^{-1}(Y-XB)}$

$= e^{\frac{-1}{2\sigma^2}(Y-XB)^T(Y-XB)}$

$= e^{\frac{-1}{2\sigma^2}(Y^T-B^TX^T)(Y-XB)}$

$= e^{\frac{-1}{2\sigma^2}(Y^TY-)(Y-XB)}$

$= e^{\frac{-1}{2\sigma^2}(Y^TY-B^TX^TY-Y^TXB+B^TX^TXB)}$

$\propto e^{\frac{-1}{2\sigma^2}(-2B^TX^TY+B^TX^TXB)}$

$Var(\beta|X, Y) = \sigma^2(X^TX)^{-1}$

$E(\beta|X, Y) = \sigma^2(X^TX)^{-1} * \frac{X^TY}{\sigma^2} = (X^TX)^{-1}X^TY$

Answers:

$p(\beta|Y, X) \propto e^{\frac{-1}{2\sigma^2}(-2B^TX^TY+B^TX^TXB)}$

$Var(\beta|X, Y) = \sigma^2(X^TX)^{-1}$

$E(\beta|X, Y) = \sigma^2(X^TX)^{-1} * \frac{X^TY}{\sigma^2} = (X^TX)^{-1}X^TY$

Part 2:

$\hat{\beta} = (X^TX)^{-1}X^TY$

$Var(\hat{\beta}) = Var((X^TX)^{-1}X^TY)$

$Var(\hat{\beta}) = [(X^TX)^{-1}X^T] * Var(Y) * [(X^TX)^{-1}X^T]^T$

$Var(\hat{\beta}) = [(X^TX)^{-1}X^T] * Var(Y) * [X(X^TX)^{-1}]$

$Var(\hat{\beta}) = [(X^TX)^{-1}X^T]\Sigma[X(X^TX)^{-1}]$

$Var(\hat{\beta}) = [(X^TX)^{-1}X^T]\sigma^2 I[X(X^TX)^{-1}]$

$Var(\hat{\beta}) = \sigma^2[(X^TX)^{-1}X^TX(X^TX)^{-1}]$

$Var(\hat{\beta}) = \sigma^2(X^TX)^{-1}$

Problem 3 continued on next page. . .

$E(\hat{\beta}) = (X^T X)^{-1} X^T E(Y) = (X^T X)^{-1} X^T X B = B$

Answers:

$Var(\hat{\beta}) = \sigma^2 (X^T X)^{-1}$

$E(\hat{\beta}) = B$

The variances are the same, but the expected values are different. The Posterior distribution's expected value is equivalent to $\hat{\beta}$. This makes sense since the MLE will approach β. In conclusion, they both vary equivalent just centered around the ground truth β or the posterior truth $\hat{\beta}$.

Problem 4

Blood Type in 0.01 of population is innocent. $P(E|I) = 0.01$

Blood is found with no innocent explanation. $P(E|\tilde{}I) = 1$

$P(E|I) = 0.01 = P(I|E)$ (Fallacy)

$P(\tilde{}I|E) = 1 - P(I|E) = 0.99$

$P(\tilde{}I|E) = \frac{P(E|\tilde{}I)P(\tilde{}I)}{P(E|\tilde{}I)P(\tilde{}I)+P(E|I)P(I)} = 0.99$

$P(\tilde{}I|E) = \frac{1*(1-P(I))}{1*(1-P(I))+P(E|I)P(I)} = 0.99$

$P(\tilde{}I|E) = \frac{(1-P(I))}{(1-P(I))+0.01P(I)} = 0.99$

$P(\tilde{}I|E) = \frac{1-P(I)}{1-0.99P(I)} = 0.99$

$P(\tilde{}I|E) = \frac{1-P(I)}{1-0.99P(I)} = 0.99$

$P(\tilde{}I|E) = 1 - P(I) = 0.99 - 0.9801 * P(I)$

$0.01 = 0.0199 * P(I) \to P(I) = 0.5025$

Answer:

$P(I) = 0.5025$

The answer is slightly over 0.5, thus his claim is incorrect.

Problem 5

$\Lambda = log(\frac{p}{1-p}) \rightarrow e^{\Lambda} = \frac{p}{1-p} \rightarrow e^{\Lambda} - e^{\Lambda} * p = p \rightarrow e^{\Lambda} = p(1+e^{\Lambda}) \rightarrow p = \frac{e^{\Lambda}}{(1+e^{\Lambda})}$

$g(p) = \Lambda = log(\frac{p}{1-p})$ and $g^{-1}(\Lambda) = p = \frac{e^{\Lambda}}{(1+e^{\Lambda})}$

Part 1:

$P_p(p) \propto 1$

$P_{\Lambda}(\Lambda) = P_p(g^-1(\Lambda)) \mid \frac{d}{d\Lambda} g^-1(\Lambda) \mid$

$P_{\Lambda}(\Lambda) = 1* \mid \frac{d}{d\Lambda} \frac{e^{\Lambda}}{1+e^{\Lambda}} \mid$

$P_{\Lambda}(\Lambda) = 1* \mid \frac{(1+e^{\Lambda})e^{\Lambda} - e^{2\Lambda}}{(e^{\Lambda}+1)^2} \mid$

$P_{\Lambda}(\Lambda) = 1* \mid \frac{e^{\Lambda} + e^{2\Lambda} - e^{2\Lambda}}{(e^{\Lambda}+1)^2} \mid$

$P_{\Lambda}(\Lambda) = \mid \frac{e^{\Lambda}}{(e^{\Lambda}+1)^2} \mid$

Answer:

$P_{\Lambda}(\Lambda) = \frac{e^{\Lambda}}{(e^{\Lambda}+1)^2}$

Part 2:

$P_{\Lambda}(\Lambda) \propto 1$

$P_p(p) = P_{\Lambda}(g^-1(p)) \mid \frac{d}{dp} g^-1(p) \mid$

$P_p(p) = 1* \mid \frac{d}{dp} log(\frac{p}{1-p}) \mid$

$P_p(p) = \mid \frac{1-p}{p} \frac{d}{dp} \frac{p}{1-p} \mid$

$P_p(p) = \mid \frac{1-p}{p} \frac{(1-p)-(-1)p}{(1-p)^2} \mid$

$P_p(p) = \mid \frac{1-p}{p} \frac{1}{(1-p)^2} \mid$

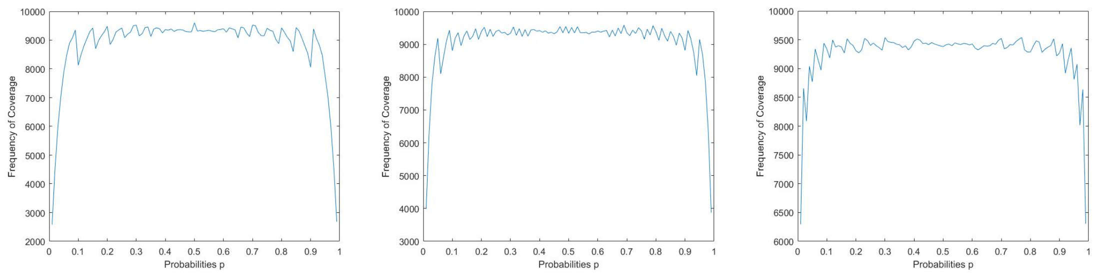
$P_p(p) = \mid \frac{1}{(1-p)p} \mid$

Answer:

$P_p(p) = \frac{1}{(1-p)p}$

What is the implied prior distribution on p?

Problem 6



(a) N = 30 (b) N = 50 (c) N = 100

Figure 1: Confidence Interval Coverage

Conclude with your thoughts on the experiment. Are you surprised?
As the number of trials increases the coverage is hitting 95 percent asymptotically. This may be due to the discrete nature of the draws. Towards the end of the probability spectrum [0, 1] there are a lot less frequencies of hitting the 95 percent coverage section. It seems much more difficult to acquire higher coverage on those sections.

Matlab code:

```
N = 30;
frequency = zeros(99, 1);
temp = 1;
for p=0.01:0.01:0.99
    for i=1:10000
        x = binornd(N, p);
        p_hat = x/N;
        sigma_hat = sqrt((p_hat*(1-p_hat))/N);
        upper_bound = p_hat + 1.96* sigma_hat;
        lower_bound = p_hat - 1.96* sigma_hat;
        if p >= lower_bound && p <= upper_bound
            frequency(temp, 1) = frequency(temp, 1) + 1;
        end
    end
    temp = temp + 1;
end
p = 0.01:0.01:0.99;
plot(p, frequency)
xlabel('Probabilities p')
ylabel('Frequency of Coverage')
```