# Advanced Integrated Circuit Design Homework 1

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## I. Wiener-Hopf Filter

The optimal weight is obtained as

$$\bar{w}_{opt} = \bar{\bar{R}}^{-1} \cdot \bar{P}, \tag{1}$$

where $\bar{\bar{R}} = \mathrm{E}\{\bar{u}(n) \cdot \bar{u}^\dagger(n)\}$, $\bar{P} = \mathrm{E}\{\bar{u}(n) \cdot d^*(n)\}$, $d(n)$ is the desired output at the nth iteration, $\bar{w}(n)$ is the tap weights at the nth iteration and $\bar{u}(n)$ is the input at the nth iteration.

In this simulation, I modify $W$ from $2.9$, $3.1$ to $3.3$ and the corresponding tap weights and mean-square-erroe(MSE) are listed in Table I. It is observed as $W$ becomes larger, the MSE increases. Furthermore, tap weights are symmetric about the sixth weight (i.e. $w_5$).

TABLE I
TAP WEIGHTS AND MSE OF WIENER-HOPF FILTER.

| | $W=2.9$ | $W=3.1$ | $W=3.3$ |
|---|---|---|---|
| $w_0$ | $-0.0025$ | $-0.0043$ | $-0.0096$ |
| $w_1$ | $0.0032$ | $0.0100$ | $0.0270$ |
| $w_2$ | $-0.0119$ | $-0.0320$ | $-0.0731$ |
| $w_3$ | $0.0580$ | $0.1105$ | $0.1970$ |
| $w_4$ | $-0.2566$ | $-0.3674$ | $-0.5180$ |
| $w_5$ | $1.1112$ | $1.2038$ | $1.3460$ |
| $w_6$ | $-0.2562$ | $-0.3669$ | $-0.5175$ |
| $w_7$ | $0.0581$ | $0.1105$ | $0.1969$ |
| $w_8$ | $-0.0112$ | $-0.0313$ | $-0.0724$ |
| $w_9$ | $0.0037$ | $0.0104$ | $0.0273$ |
| $w_{10}$ | $-0.0019$ | $-0.0038$ | $0.0090$ |
| MSE | $0.0030$ | $0.0034$ | $0.0040$ |

## II. Least Mean Squares Filter

Since $\bar{\bar{R}}$ and $\bar{P}$ are hardly avalible in the real-world applications, we modify the weight updating function of least mean squares (LMS) filter as

$$\bar{w}(n+1) = \bar{w}(n) + \mu \bar{e}(n) \cdot \bar{u}(n), \tag{2}$$

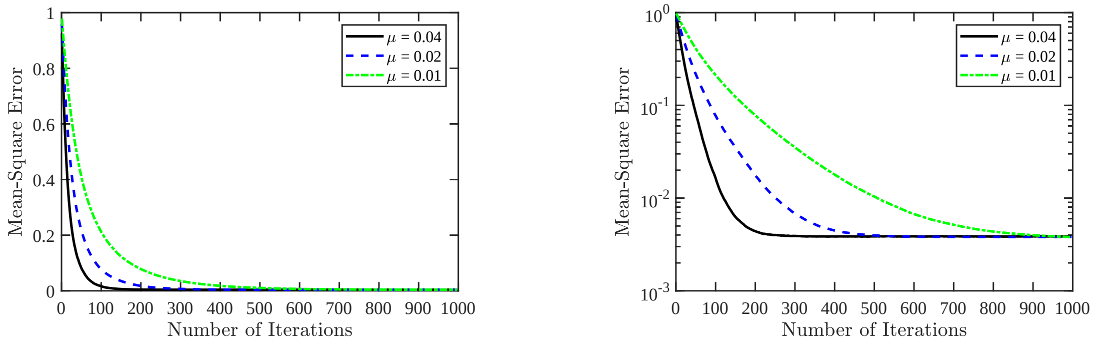
where $\mu$ is the step size and $e(n) = d(n) - \bar{w}^\dagger(n) \cdot \bar{u}(n)$ is the error at the nth iteration,.

We conduct $200$ simulations to demonstrate the learning curve of the tap weight and MSE.

### A. Learning Curves of MSE

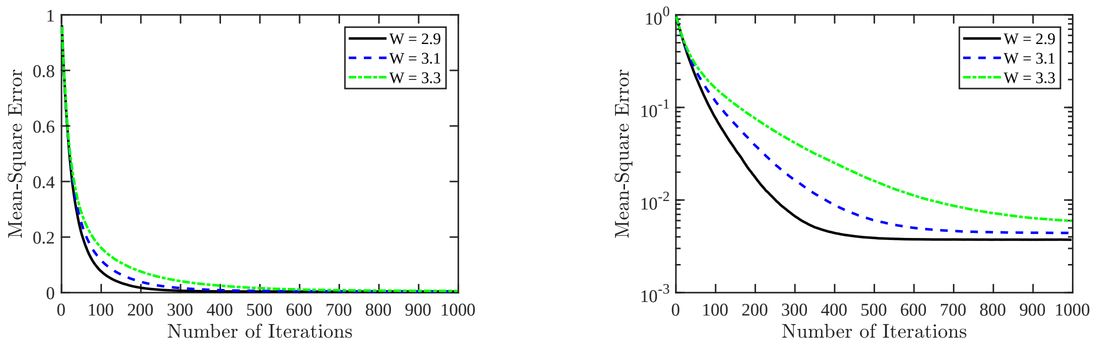
*1) Variation at $\mu$:* Fig. 1. shows the MSE converges more quickly when $\mu$ is bigger. However, the final MSE is about the same, to be more specific, $0.0039$, $0.0038$, $0.0038$, respectively.

*2) Variation at $W$:* Fig. 2. shows the MSE converges more quickly and final MSE is smaller when $W$ is smaller. The final MSE are $0.0037$, $0.0045$, $0.0060$, respectiely. This result is obvious since as $W$ increasing, the magnitude of the impulse response of the channel(i.e. raised cosine function) becomes larger, which means higher intersymbol interference.




(a) (b)

Fig. 1. learning curve of MSE in LMS filter at $W = 2.9$, (a)linear scale and (b)logarithmic scale. ——— : $\mu = 0.04$ ; – – – : $\mu = 0.02$; – · – : $\mu = 0.01$.




(a) (b)

Fig. 2. learning curve of MSE in LMS filter at $\mu = 0.02$, (a)linear scale and (b)logarithmic scale. ——— : $W = 2.9$ ; – – – : $W = 3.1$; – · – : $W = 3.3$.

### *B. Learning Curves of tap weights*

Fig. 3. shows the learning waves of tap weights of LMS filter and the ideal value calculated by Wiener-Hopf Filter at $W = 2.9$ and $\mu = 0.02$. It is observed that all the tap weights will finally converges to the ideal value after about 500 iterations. In addition, only $w_3$ to $w_7$ have evident learning process and the final value is symmetric to $w_5$ just as the result of Wiener-Hopf Filter.

The final tap weights and MSE of Wiener-Hopf filter and LMS filter are listed in Table II, which indicates there is little difference between results from Wiener-Hopf filter and LMS filter.

TABLE II
TAP WEIGHTS AND MSE OF WIENER-HOPF FILTER AND LMS FILTER

| | $w_0$ | $w_1$ | $w_2$ | $w_3$ | $w_4$ | $w_5$ | $w_6$ | $w_7$ | $w_8$ | $w_9$ | $w_{10}$ | MSE |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| Wiener-Hopf filter | $-0.0025$ | $0.0032$ | $-0.0119$ | $0.0580$ | $-0.2586$ | $1.1112$ | $-0.2562$ | $0.0581$ | $-0.0112$ | $0.0037$ | $-0.0019$ | $0.0030$ |
| LMS filter | $-0.0004$ | $0.0029$ | $-0.0133$ | $0.0587$ | $-0.2560$ | $1.1113$ | $-0.2556$ | $0.0583$ | $-0.0136$ | $0.0027$ | $-0.0003$ | $0.0038$ |

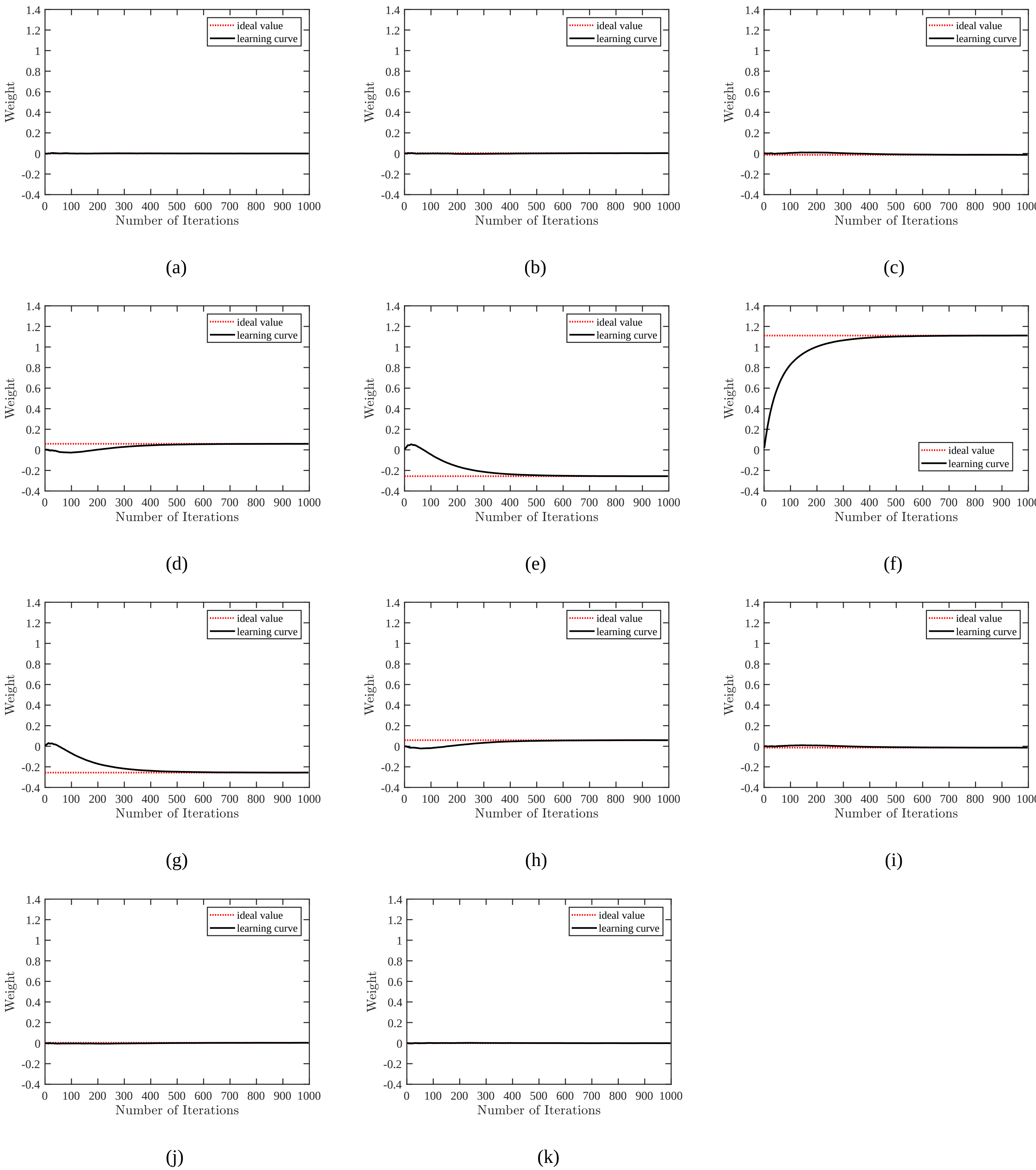



Fig. 3. learning curve of tap weights of LMS filter and ideal value by Wiener-Hopf filter at $W = 2.9$ and $\mu = 0.02$, (a) - (k) : $w_0$ - $w_{10}$. ——— : learning curve ; — — — : ideal value by Wiener-Hopf filter.

## III. Normalized Least Mean Squares Filter

Since LMS filter is senstive to the magnitude of input, normalized LMS (NLMS) filter can be immune to such problem. The weight updating function of NLMS filter is as below

$$\bar{w}(n+1) = \bar{w}(n) + \frac{\mu \bar{e}(n) \cdot \bar{u}(n)}{\bar{u}^{\dagger}(n) \cdot \bar{u}(n)}. \tag{3}$$

### *A. Learning Curves of MSE*

*1) Variation at $\mu$:* Fig. 4. shows the learning curve of MSE at $W = 2.9$ and $\mu = 0.04, 0.02, 0.01$, respectively. It is observed MSE converges more quickly when $\mu$ is bigger in both NLMS and LMS filter. However, LMS filter outperforms NLMS filter in both converge time and final MSE. Also, Fig. 4(b) shows the learning curve of MSE in NLMS filter converges about exponentially. The final MSE are $[0.0190, 0.0833, 0.2299]$ and $[0.0038, 0.0037, 0.0038]$, respectively. This is because normalization will decrease the step size. Therefore, the converge process of NLMS filter cannot reach the optimal state for limited number of inputs.

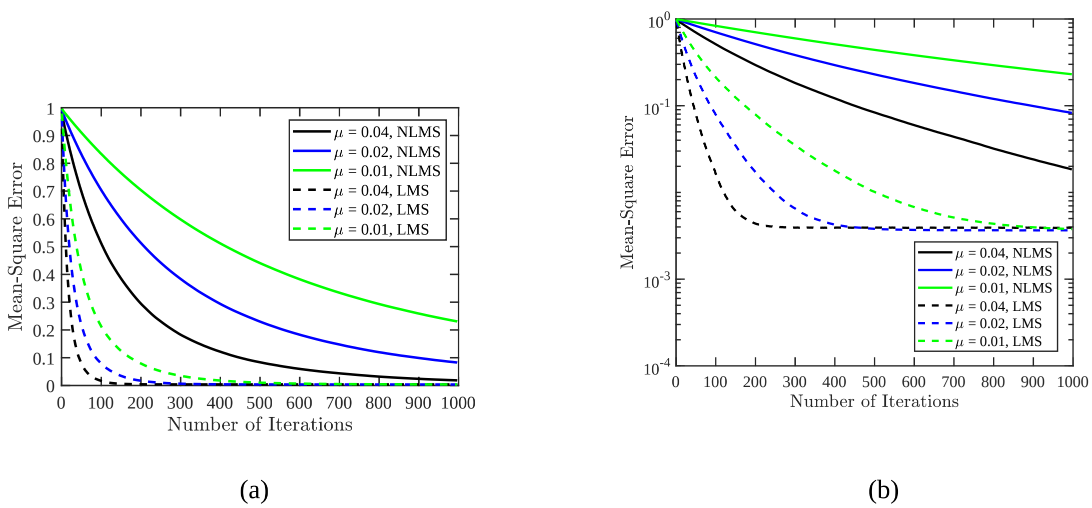
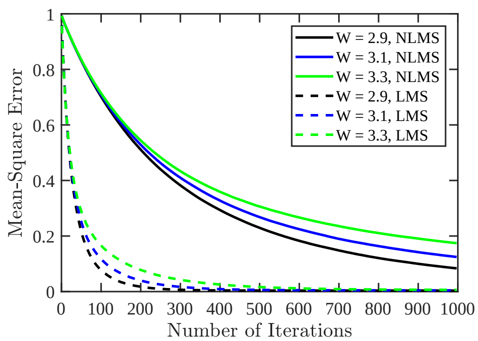



Fig. 4. learning curve of MSE in NLMS filter at $W = 2.9$, (a)linear scale and (b)logarithmic scale. ——— : $\mu = 0.04$, NLMS ; ——— : $\mu = 0.02$, NLMS; ——— : $\mu = 0.01$, NLMS; – – – : $\mu = 0.04$, LMS ; – – – : $\mu = 0.02$, LMS; – – – : $\mu = 0.01$, LMS.

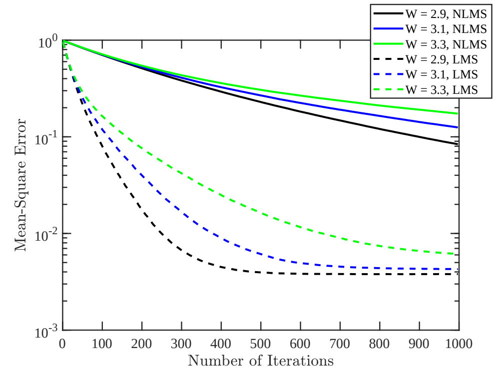
*2) Variation at $W$:* Fig. 5. shows the learning curve of MSE at $\mu = 0.02$ and $W = 2.9, 3.1, 3.3$, respectively. It is observed MSE converges more quickly when $W$ is smaller in both NLMS and LMS filter. However, LMS filter outperforms NLMS filter in both converge time and final mse. Also, Fig. 5(b) shows the learning curve of MSE in NLMS filter converges about exponentially. The final MSE are $[0.0842, 0.1251, 0.1735]$ and $[0.0037, 0.0045, 0.0060]$, respectively. This is because normalization will decrease the magnitude of the step. Therefore, for limited number of inputs, the converge process cannot reach the optimal.

### *B. Learning Curves of tap weights*

Fig. 6. shows the learning waves of tap weights of NLMS filter and the ideal value calculated by Wiener-Hopf Filter at $W = 2.9$ and $\mu = 0.02$. It is observed only $w_5$ has evident learning process, however, it doesn't reach the optimal value




(a)




(b)

Fig. 5. learning curve of MSE in NLMS filter at $\mu = 0.02$, (a)linear scale and (b)logarithmic scale. ——— : $W = 2.9$, NLMS ; ——— : $W = 3.1$, NLMS; ——— : $W = 3.3$, NLMS; $---$ : $W = 2.9$, LMS ; $---$ : $W = 3.1$, LMS; $---$ : $W = 3.3$, LMS.

because of the limited number of input (iterations). All the final weights are symmetric to $w_5$ just as the result of Wiener-Hopf filter.

The final tap weights and MSE of Wiener-Hopf filter and NLMS filter are listed in Table III, which indicates there is obvious difference between results from Wiener-Hopf filter and NLMS filter from $w_3$ to $w_7$ and MSE.

TABLE III
TAP WEIGHTS AND MSE OF WIENER-HOPF FILTER AND NLMS FILTER

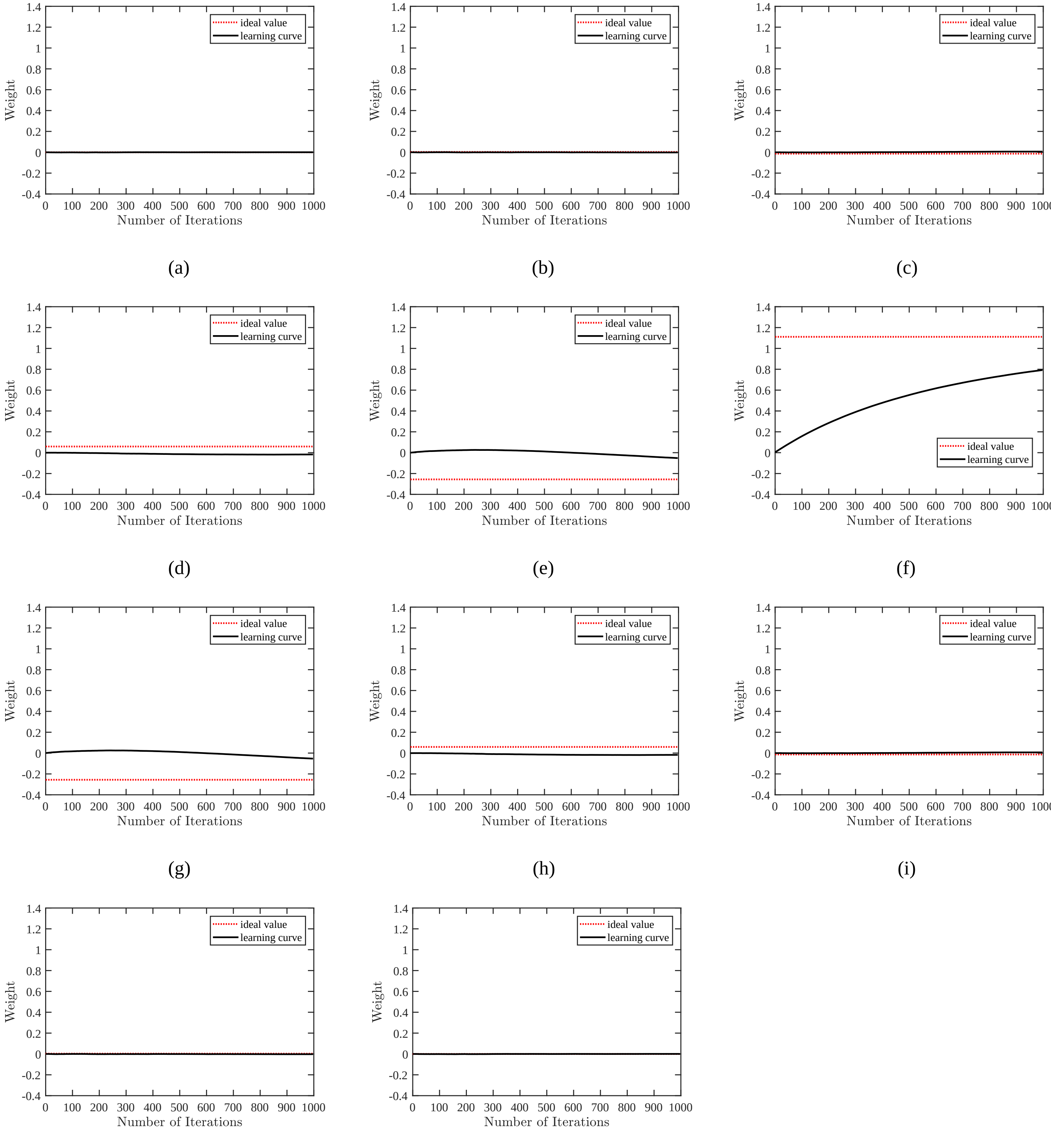
| | $w_0$ | $w_1$ | $w_2$ | $w_3$ | $w_4$ | $w_5$ | $w_6$ | $w_7$ | $w_8$ | $w_9$ | $w_{10}$ | MSE |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| Wiener-Hopf filter | $-0.0025$ | 0.0032 | $-0.0119$ | 0.058 | $-0.2586$ | 1.1112 | $-0.2562$ | 0.0581 | $-0.0112$ | 0.0037 | $-0.0019$ | 0.003 |
| NLMS filter | 0.0002 | $-0.0017$ | 0.0064 | $-0.017$ | $-0.052$ | 0.7927 | $-0.0533$ | $-0.0171$ | 0.0067 | $-0.0022$ | 0.0004 | 0.0838 |

### C. Misadjustment

In all of above simulation cases, LMS filter outperforms NLMS filter in both converge time and final MSE. Therefore, I want to compare their performance at some harsh conditions.

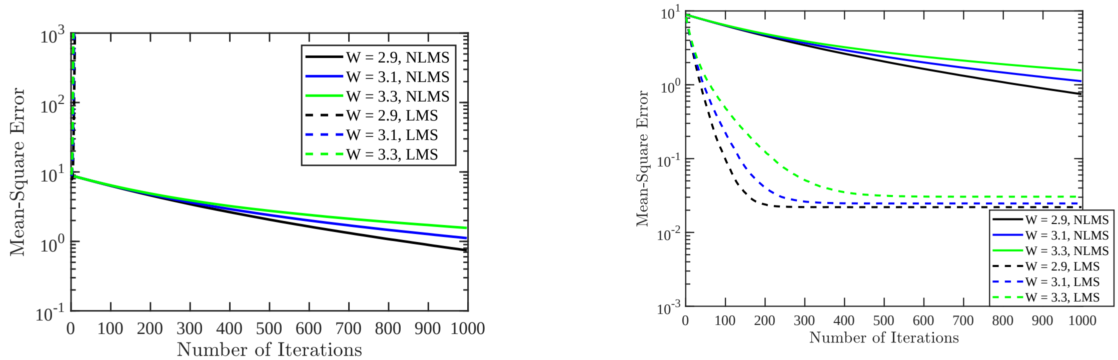
*1) Higher Input Power:* I multiply each entry of $s(n)$ by three. Fig. 7.(a) shows the learning curve of MSE at $\mu = 0.02$ and $W = 2.9, 3.1, 3.3$, respectively. It is observed the MSE of LMS filter diverges at the begining and never goes back. However, NLMS filter can still function accurately. This is obvious because NLMS filter will conduct normalization toward input sequence. If we modify $\mu$ for LMS to $0.005$, the learning curve of MSE will converge as shown in Fig. 7.(b). Therefore, the step size of LMS is input-dependent.

*2) Higher Noise Power or ISI:* Fig. 8.(a) shows the learing curve of MSE at the noise covariance = $0.01$ and (b) shows the learing curve of MSE when W becomes higher. Both figures show that LMS filter outperforms NLMS filter, however, the difference becomes smaller.



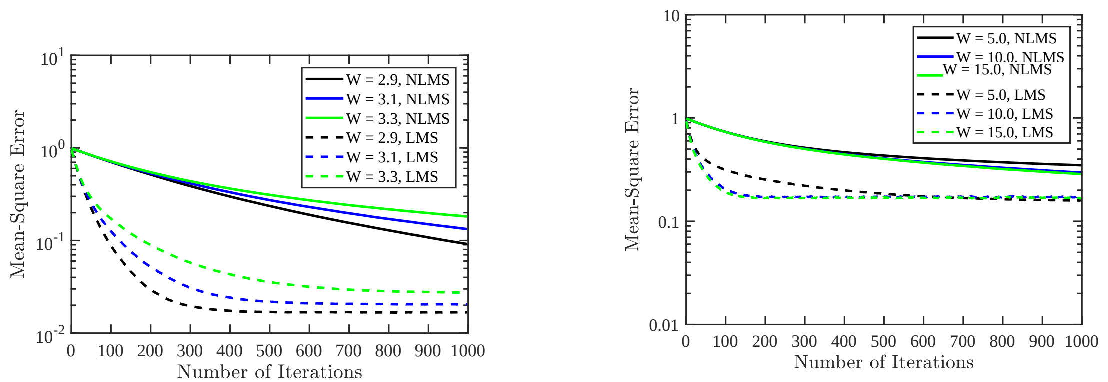

(a) (b) (c) (d) (e) (f) (g) (h) (i) (j) (k)

Fig. 6. learning curve of tap weights of NLMS filter and ideal value by Wiener-Hopf filter at $W = 2.9$ and $\mu = 0.02$, (a) - (k) : $w_0$ - $w_{10}$. ——— : learning curve of NLMS filter; – – – : ideal value by Wiener-Hopf filter.




(a) (b)

Fig. 7. learning curve of MSE in NLMS filter at (a)($\mu_{LMS}$, $\mu_{NLMS}$) = $(0.02, 0.02)$ and (b)($\mu_{LMS}$, $\mu_{NLMS}$) = $(0.005, 0.02)$. ——— : $W = 2.9$, NLMS ; ——— : $W = 3.1$, NLMS; ——— : $W = 3.3$, NLMS; − − − : $W = 2.9$, LMS ; − − − : $W = 3.1$, LMS; − − − : $W = 3.3$, LMS.




(a) (b)

Fig. 8. learning curve of MSE in NLMS filter at $\mu = 0.02$ (a)noise covariance = 0.01 ——— : $W = 2.9$, NLMS ; ——— : $W = 3.1$, NLMS; ——— : $W = 3.3$, NLMS; − − − : $W = 2.9$, LMS ; − − − : $W = 3.1$, LMS; − − − : $W = 3.3$, LMS and (b) noise covariance = 0.001, ——— : $W = 5.0$, NLMS ; ——— : $W = 10.0$, NLMS; ——— : $W = 15.0$, NLMS; − − − : $W = 5.0$, LMS ; − − − : $W = 10.0$, LMS; − − − : $W = 15.0$, LMS

### D. Maximum of $\mu$

To get the maximum of $\mu$, we first conduct expectation toward (2) as

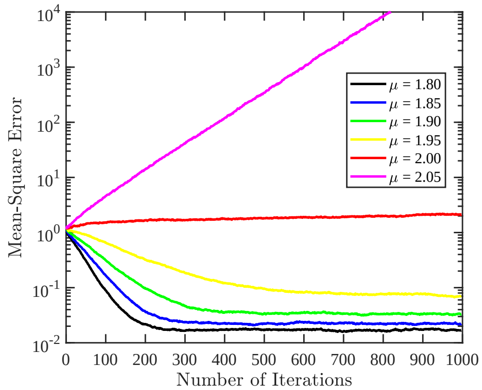
$$
\begin{aligned}
&\mathrm{E}\{\bar{w}(n+1)\} = \mathrm{E}\{\bar{w}(n)\} + \mu \mathrm{E}\{\bar{e}(n) \cdot \bar{u}(n)\} \\
&= \mathrm{E}\{\bar{w}(n)\} + \mu(\bar{P} - \bar{\bar{R}} \cdot \mathrm{E}\{\bar{w}(n)\}) \\
&= (\bar{\bar{I}} - \mu\bar{\bar{R}})\mathrm{E}\{\bar{w}(n)\} + \mu\bar{\bar{R}}\bar{w}_{opt} \\
&= (\bar{\bar{I}} - \mu\bar{\bar{R}})(\mathrm{E}\{\bar{w}(n)\} - \bar{w}_{opt}) + \bar{w}_{opt} \qquad (4)
\end{aligned}
$$

We can conduct eigenvalue decomposition upon $\bar{\bar{R}}$ and expressed as

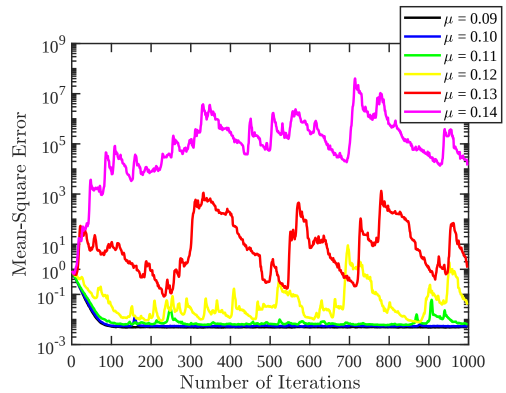
$$\bar{\bar{R}} = \bar{\bar{Q}} \cdot \bar{\bar{\Sigma}} \cdot \bar{\bar{Q}}^{-1}. \qquad (5)$$

We further define

$$\bar{V}(n) = \mathrm{E}\{\bar{w}(n)\} - \bar{w}_{opt} = \bar{\bar{Q}} \cdot \bar{U}(n). \qquad (6)$$




(a)




(b)

Fig. 9. learning curve of MSE at $W = 2.9$ with varying $\mu$. (a)NLMS filter, : $\mu = 1.80$; : $\mu = 1.85$; : $\mu = 1.90$; : $\mu = 1.95$; : $\mu = 2.00$; : $\mu = 2.05$ and (b) LMS filter, : $\mu = 0.09$; : $\mu = 0.10$; : $\mu = 0.11$; : $\mu = 0.12$; : $\mu = 0.13$; : $\mu = 0.14$

By (5) and (6), we can modify (4) as

$$\bar{V}(n+1) = (\bar{\bar{I}} - \mu\bar{\bar{R}})\bar{V}(n). \tag{7}$$

By some calculation, we can get

$$\begin{aligned}\bar{U}(n+1) &= \bar{\bar{Q}}^T \cdot \bar{V}(n+1) = \bar{\bar{Q}}^T \cdot (\bar{\bar{I}} - \mu\bar{\bar{R}})\bar{\bar{Q}} \cdot \bar{U}(n) \\ &= (\bar{\bar{I}} - \mu\bar{\bar{Q}}^T \cdot \bar{\bar{R}} \cdot \bar{\bar{Q}})\bar{U}(n) = (\bar{\bar{I}} - \mu\bar{\bar{\Sigma}})\bar{U}(n).\end{aligned} \tag{8}$$

Therefore,

$$\bar{U}(n) = (\bar{\bar{I}} - \mu\bar{\bar{\Sigma}})^n \bar{U}(0). \tag{9}$$

To make $\bar{V}(n) = \bar{U}(n)$ equal to zero vector, we will require

$$|1 - \mu\lambda_1| < 1 \leftrightarrow 0 < \mu < \frac{2}{\lambda_1}, \tag{10}$$

where $\lambda_1$ is the largest eigenvalue of $\bar{\bar{R}}$.

In our simulations of NLMS, we find that $\lambda_1 \leq 1$, so we can choose $\mu$ at most 2. The learning curves of MSE at $W = 2.9$ with varying $\mu$ in LMS and NLMS filter are depicted in Fig. 9.

Fig. 9.(a) shows when $\mu$ is equal to 2, the MSE of NLMS filter will keep the same value. However, when $\mu$ is larger than 2, the learning curve of MSE sarts to diverge about exponentially. When $\mu$ is smaller than 2, the learning curve of MSE will converge. It proves the correctness of my induction. Fig. 9.(b) shows $\mu$ for LMS filter cannot exceed 0.10 or it will have some fluctuation and may not converge. The effective range of $\mu$ in LMS filter is more difficult to analyze and will depend on many factors, such as input power and SINR. However, NLMS filter shows better immunity to such problems at the expense of convergence speed. If the number of input is limited, NLMS may have worse MSE compared to LMS filter.