AMath 118 final project

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May 2022

1 The function

$$f(x) := 1 + \sum_{j=1}^{n} \frac{b_j}{(d_j - x)}$$

2 Finding the starting root

To find an initial point, I need to find a function with the same poles at the same interval and zero that lies right to the zero of the original function of the interval of interest. The approximate function needs to be smaller to have a larger zero and have the same denominators of the interval of interest which would give the same pole at the interval.

For example:

$$f(x) = \frac{1}{(2-x)} + \frac{3}{(4-x)} + \frac{1}{(5-x)} + \frac{1}{(7-x)} + 1$$

The approximation of this function in second interval $\{4, 5\}$ is:

$$r(x) = \frac{3}{(4-x)} + \frac{1}{(5-x)} + \frac{1}{(2-4)}$$

In this equation, 4 and 5 are the poles and since x will be somewhere between 4 and 5, $\frac{1}{(2-x)}$ will be smaller than zero. When x is 4, the fraction $\frac{1}{(2-x)}$ will be the largest in magnitude. Hence the approximated function should also plus the fraction $\frac{1}{(2-x)}$ to stay below the original function. If the root of interest is in the third interval, then we simply deduct the first two terms with x as the first pole of their next term, illustrated as the function below.

$$r(x) = \frac{1}{(2-4)} + \frac{3}{(4-5)} + \frac{1}{(5-x)} + \frac{1}{(7-x)}$$

This isn't a problem if the root of interest is in the first interval. Since every d_j after d_{j+1} is larger and $d_i > 0$, the fractions that come after this interval will be larger than zero. Hence deleting the terms after the interval will only make the function smaller which is desired. In this case the approximate function will be monotonic and the zero of which can be easily found.

$$r(x) = \frac{1}{(2-x)} + \frac{3}{(4-x)} = 0$$

$$\frac{1(4-x)+3(2-x)}{(2-x)(4-x)} = 0$$

$$4 - x + 6 - 3x = 0$$

$$x = 2.5$$

The deducted 1 is a safety net if the original function only have 1 interval and nothing can be deducted except 1. The general formula of the approximated function is:

$$r(x) := \sum_{j=1}^{i} \frac{b_j}{(d_j - d_{j+1})} + \frac{b_i}{(d_i - x)} + \frac{b_{i+1}}{(d_{i+1} - x)}$$

In this function, the term $\sum_{j=1}^{i} \frac{b_j}{(d_j - d_{j+1})}$ can be considered as a constant C. Set this function to 0 to find its root. This function has an asymptote for y = 0 if there is no constant term, but since the constant term in this function is negative, the whole graph will move downwards and the function will intersect with the x-axis on the left of the first pole d_i. Hence, if this function is quadratic, that is if there is constant term in this function, we would want the larger zero that's larger than the first pole d_i.

$$r(x) = C + \frac{b_i}{(d_i - x)} + \frac{b_{i+1}}{(d_{i+1} - x)} = 0$$

$$\frac{C \cdot (d_{i+1} - x) \cdot (d_i - x) + b_i \cdot (d_{i+1} - x) + b_{i+1} \cdot (d_i - x)}{(d_{i+1} - x) \cdot (d_i - x)} = 0$$

$$C \cdot (d_{i+1} - x) \cdot (d_i - x) + b_i \cdot (d_{i+1} - x) + b_{i+1} \cdot (d_i - x) = 0$$

$$Cx^2 - ((d_{i+1} + d_i)C + (b_i + b_{i+1}))x + (d_{i+1}b_i + b_{i+1}d_i + d_i d_{i+1}C) = 0$$

Since the coefficient of x^2 is negative, the larger zero is the equation:

$$x = \frac{-b - \sqrt{(b^2 - 4ac)}}{2a}$$

because 2a is negative and the numerator, $-b - \sqrt{(b^2 - 4ac)}$, needs to be a negative number with the largest magnitude, where a = C, b = $((d_{i+1} + d_i)C + (b_i + b_{i+1}))$, and c = $(d_{i+1}b_i + b_{i+1}d_i + d_i d_{i+1}C)$

3 finding coefficients of the approximation function g(x) and h(x)

Split the function f(x) into φ(x) and ψ(x) as follows:

$$f(x) := 1 + \sum_{j=1}^{n} \frac{b_j}{(d_j - x)}$$

$$\varphi(x) := 1 + \sum_{j=1}^{i} \frac{b_j}{(d_j - x)} \quad \psi(x) := \sum_{j=1}^{n} \frac{b_j}{(d_j - x)}$$

And approximate these two function with g(x) and h(x) up to first derivative as shown below:

$$g(x) = \alpha + \frac{\beta}{x} \quad h(x) = \frac{\gamma}{(\delta - x)}$$

$$\varphi(x) = g(x) \quad \psi(x) = h(x)$$

$$\varphi'(x) = g'(x) \quad \psi'(x) = h'(x)$$

$$1 + \sum_{j=1}^{i} \frac{b_j}{(d_j - x)} = \alpha + \frac{\beta}{x} \quad \sum_{j=1}^{n} \frac{b_j}{(d_j - x)} = \frac{\gamma}{(\delta - x)}$$

$$\sum_{j=1}^{i} \frac{b_j}{(d_j - x)^2} = \frac{\beta}{x^2} \quad \sum_{j=1}^{i} \frac{b_j}{(d_j - x)^2} = \frac{\gamma}{(\delta - x)^2}$$

After we plug in the numbers and arguments into the φ(x), ψ(x) and their derivative function, we will get 4 scalars and lets name them m n k l.

$$m = \alpha + \frac{\beta}{x} \quad n = \frac{\gamma}{(\delta - x)}$$

$$k = \frac{\beta}{x^2} \quad l = \frac{\gamma}{(\delta - x)^2}$$

$$\beta = kx^2 \quad \alpha = m - \frac{\beta}{x}$$

$$\frac{n}{l} = (\delta - x) \quad \delta = \frac{n}{l} + x$$

$$\gamma = n(\delta - x)$$

From these equations we can obtain the value for all the coefficients and find the root of the function g(x) + h(x).

$$0 = \alpha + \frac{\beta}{x} + \frac{\gamma}{(\delta - x)}$$

$$0 = \frac{\alpha x(\delta - x) + \beta(\delta - x) + \gamma x}{x(\delta - x)}$$

$$0 = -\alpha x^2 + \alpha\delta x + \beta\delta - \beta x + \gamma x$$

$$\alpha x^2 + (\beta - \alpha\delta - \gamma)x - \beta\delta = 0$$

Similar to the logic of finding the zero of the approximated function in the part 2, we used the formula below to find the root that lies in the interval of interest.

$$x = \frac{-b - \sqrt{(b^2 - 4ac)}}{2a}$$

$$a = \alpha \quad b = (\beta - \alpha\delta - \gamma) \quad c = -\beta\delta$$

This is the new x and we use this x to start the next iteration of approximation until the relative error is smaller than our user defined tolerance.

$$|\frac{z_{i+1}-z_i}{z_{i+1}} \leq tol|$$