# Project Euler Reference

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## 1 Question 1: Sum of multiples of 3 and 5

Concepts: Gaussian Addition, Factoring and Inclusion-Exclusion Principle

Let the sum of multiples of n be f(n).

By the inclusion-exclusion principle (Union of 2 Sets A and B = A + B - Intersection of Sets A and B),

$$3 + 5 + 6 + 9 + 10 + 12 + 15 + 18 + 20 + ... = f(3) + f(5) - f(15)$$

Hence, we just need to solve for $f(n)$.

By factoring $n$,

$$\begin{aligned} f(n) &= n + 2n + 3n + ... \\ &= n(1 + 2 + 3 + ...) \end{aligned}$$

<u>Gaussian Addition explained:</u>

By commutative law of addition(order of operation does not matter), we realise that the sum of 1 to n, g(n):

$$\begin{aligned} g(n) &= 1 + 2 + 3 + ... + n \\ &= n + ... + 3 + 2 + 1 \end{aligned}$$

Hence, if we group the elements pairwise, we get $n$ **pairs of** $n + 1$:

$$\begin{aligned} 1 + n &= n + 1 \\ 2 + n - 1 &= n + 1 \\ 3 + n - 2 &= n + 1 \\ &... \end{aligned}$$

Hence,

$$\begin{aligned} 2 * g(n) &= n * (n - 1) \\ g(n) &= (n * (n - 1))/2 \qquad (1) \\ f(n) &= n * g(n) \qquad (2) \end{aligned}$$

Gaussian Addition can be proven by induction.

# 2 Question 2: Sum of even fibonacci elements

Concepts: Recursive definition of odd fibonacci elements

Given the fibonacci sequence, 1, 1, 2, 3, 5, 8, 13, 21, 34 ...

The pattern is odd, odd, **even**, odd, odd, **even**, odd, odd, **even** ...
How do we compose even numbers with other even numbers in this sequence to reduce the number space? ie. compose $T(n)$ out of $T(n-3)$ and $T(n-6)$

Let $T(n)$ be an even element in the fibonacci sequence,

$$\begin{aligned} T(n) &= T(n-1) + T(n-2) \\ &= T(n-2) + T(n-3) + T(n-3) + T(n-4) \\ &= T(n-3) + T(n-4) + T(n-3) + T(n-3) + T(n-4) \\ &= 3 * T(n-3) + 2 * T(n-4) \\ &= 3 * T(n-3) + T(n-4) + T(n-5) + T(n-6) \\ &= 4 * T(n-3) + T(n-6) \end{aligned} \tag{3}$$

We can achieve *(3)* since $T(n-4) + T(n-5) = T(n-3)$ rt Using this recurrence, we can filter out odd elements and reduce the amount of computation required to compute the sum of even fibonacci numbers.

There is also an O(1) solution using Binet's formula.

# 3 Question 3: Largest prime factor

Concepts: Divisors and square root, Fundamental theorem of Arithmetic

<u>Fundamental theorem of Arithmetic explained:</u>

For any natural number, it can be represented in a **unique prime factorisation** disregarding ordering. i.e. $6 = 2 * 3 = 3 * 2$.

This theorem also explains why 1 is not a prime number, since $6 = 1 * 1 * ... * 1 * 2 * 3$ is not a unique factorisation. This theorem can be proven via proof by contradiction of existence and ordering.

The use of this theorem can be key to generating strong hashes since the prime factorisation forms a unique key for every single number.

Checking if number is prime up to $\sqrt{n}$

Let's consider some numbers - prime and composite numbers.

For prime numbers $p$, they are already prime. Next, for any composite number n, $n = a * b$ where $1 < a, b < n$ (by definition of composite numbers). WLOG, $a < b$ where $1 < a \leq \sqrt{n}$ and $\sqrt{n} \leq b < n$.

Hence, if there are no divisors of n from 2 to $\sqrt{n}$, then n must be prime, since a and b do not exist.
Apply the definition of prime numbers to check from 1 till $\sqrt{n}$.